UNIVERSITÉ SORBONNE PARIS NORD
Laboratoire d'Informatique de Paris-Nord

T H E S E

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITE SORBONNE PARIS NORD

Discipline : Informatique

présentée et soutenue publiquement par

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Caractérisation structurelle de quelques problèmes dans les graphes de cordes et d'intervalles

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Résumé

Caractérisation structurelle de quelques problèmes dans les graphes de cordes et d'intervalles

Étant donnée une famille d'ensembles non vides $\mathcal{S} = \{S_i\}$, le *graphe d'intersection de la famille* $\mathcal{S}$ est celui pour lequel chaque sommet représent un ensemble S_i de tel façon que deux sommets sont adjacents si et seulement si leurs ensembles correspondants ont une intersection non vide. Un graphe est dit *graphe de cordes* s'il existe une famille de cordes $L = \{C_v\}_{v \in G}$ dans un cercle tel que deux sommets sont adjacents si les cordes correspondantes se croisent. Autrement dit c'est le graphe d'intersection de la famille de cordes L. Ils existent différentes caractérisations des graphes de cordes qui utilisent certaines opérations dont notamment la complémentation locale ou encore la décomposition split. Cependant on ne connaît pas encore aucune caractérisation structurelle des graphes de cordes par sous-graphes induits interdits minimales. Dans cette thèse nous donnons une caractérisation des graphes de cordes par sous-graphes induits interdits dans le cas où le graphe original est un graphe split. Une matrice binaire possède la *propriété des unités consécutives en lignes (C1P)* s'il existe une permutation de ses colonnes de sorte que les 1's dans chaque ligne apparaissent consécutivement. Dans cette thèse nous développons des caractérisations par sous-matrices interdites de matrices binaires avec C1P pour lesquelles les lignes sont 2-coloriables sous une certaine condition d'adjacence et nous caractérisons structurellement quelques sous-classes auxiliaires de graphes de cordes qui découlent de ces matrices.

Étant donnée une classe de graphes Π, une Π-*complétion* d'un graphe $G = (V, E)$ est un graphe $H = (V, E \cup F)$ tel que H appartient à Π. Une Π-complétion H de G est minimale si $H' = (V, E \cup F')$ n'appartient pas à Π pour tout F' sous-ensemble propre de F. Une Π-complétion H de G est minimum si pour toute Π-complétion $H' = (V, E \cup F')$ de G la cardinalité de F est plus petite ou égale à la cardinalité de F'. Dans cette thèse nous étudions le problème de complétion minimale d'un graphe d'intervalles propre quand le graphe d'entrée est un graphe d'intervalles quelconque. Nous trouvons des conditions nécessaires qui caractérisent une complétion minimale dans ce cas particulier, puis nous laissons quelques conjectures à considérer dans un futur.

Mots clés : graphes, cordes, intervalles, sous-graphes interdits, complétion minimale.

Resumen

Caracterización estructural de algunos problemas en grafos círculo y de intervalos

Dada una familia de conjuntos no vacíos $\mathcal{S} = \{S_i\}$, se define el *grafo de intersección* de la familia $\mathcal{S}$ como el grafo obtenido al representar con un vértice a cada conjunto S_i de forma tal que dos vértices son adyacentes sí y sólo si los conjuntos correspondientes tienen intersección no vacía. Un grafo se dice *círculo* si existe una familia de cuerdas $L = \{C_v\}_{v \in G}$ en un círculo de modo que dos vértices son adyacentes si las cuerdas correspondientes se intersecan. Es decir, es el grafo de intersección de la familia de cuerdas L. Existen diversas caracterizaciones de los mismos mediante operaciones como complementación local o descomposición split. Sin embargo, no se conocen aún caracterizaciones estructurales de los grafos círculo por subgrafos inducidos minimales prohibidos. En esta tesis, damos una caracterización de los grafos círculo por subgrafos inducidos prohibidos, restringido a que el grafo original sea split. Una matriz de 0's y 1's tiene la *propiedad de los unos consecutivos (C1P) para sus filas* si existe una permutación de sus columnas de forma tal que para cada fila, todos sus 1's se ubiquen de manera consecutiva. En esta tesis desarrollamos caracterizaciones por submatrices prohibidas de matrices de 0's y 1's con la C1P para sus filas que además son 2-coloreables bajo una cierta relación de adyacencia, y caracterizamos estructuralmente algunas subclases de grafos círculo auxiliares que se desprenden de estas matrices.

Dada una clase de grafos Π, una Π-*completación* de un grafo $G = (V, E)$ es un grafo $H = (V, E \cup F)$ tal que H pertenezca a Π. Una Π-completación H de G es minimal si $H' = (V, E \cup F')$ no pertenece a Π para todo F' subconjunto propio de F. Una Π-completación H de G es mínima si para toda Π-completación $H' = (V, E \cup F')$ de G, se tiene que el tamaño de F es inferior o igual al tamaño de F'. En esta tesis estudiamos el problema de completar de forma minimal a un grafo de intervalos propios, cuando el grafo de input es de intervalos. Hallamos condiciones necesarias que caracterizan una completación minimal en este caso, y dejamos algunas conjeturas para considerar en el futuro.

Palabras clave: grafos, círculo, subgrafos prohibidos, completación, minimal.

Abstract

Structural characterization of some problems on circle and interval graphs

Given a family of nonempty sets $\mathcal{S} = \{S_i\}$, the *intersection graph of the family* $\mathcal{S}$ is the graph with one vertex for each set S_i, such that two vertices are adjacent if and only if the corresponding sets have nonempty intersection. A graph is *circle* if there is a family of chords in a circle $L = \{C_v\}_{v \in G}$ such that two vertices are adjacent if the corresponding chords cross each other. In other words, it is the intersection graph of the chord family L. There are diverse characterizations of circle graphs, many of them using the notions of local complementation or split decomposition. However, there are no known structural characterization by minimal induced forbidden subgraphs for circle graphs. In this thesis, we give a characterization by induced forbidden subgraphs of those split graphs that are also circle graphs. A $(0,1)$-matrix has the *consecutive-ones property (C1P) for the rows* if there is a permutation of its columns such that the 1's in each row appear consecutively. In this thesis, we develop characterizations by forbidden subconfigurations of $(0,1)$-matrices with the C1P for which the rows are 2-colorable under a certain adjacency relationship, and we characterize structurally some auxiliary circle graph subclasses that arise from these special matrices.

Given a graph class Π, a Π-*completion* of a graph $G = (V, E)$ is a graph $H = (V, E \cup F)$ such that H belongs to Π. A Π-completion H of G is minimal if $H' = (V, E \cup F')$ does not belong to Π for every proper subset F' of F. A Π-completion H of G is minimum if for every Π-completion $H' = (V, E \cup F')$ of G, the cardinal of F is less than or equal to the cardinal of F'. In this thesis, we study the problem of completing minimally to obtain a proper interval graph when the input is an interval graph. We find necessary conditions that characterize a minimal completion in this particular case, and we leave some conjectures for the future.

Keywords: graphs, circle, forbidden subgraphs, completion, minimal.

Contents

Acknowledgements

I've never been good with this things, I'll probably forget people or neglect the proper ways. Please know that, in my mind, the order is arbitrary and lacks of importance.

To my love and partner, Guido. To my mother and my father. To Pablo, the brother my parents did not give me.

To my advisors, Willy and Mario, for all the help, trust and support. To Martín Safe, for all his teachings. To Flavia, for her generosity.

To the members of the jury, for the good predisposition and the patience to read this very long thesis. To Dominique Ledoux and Pablo Solernó, who made this collaboration possible.

To the music, my outlet and a key part of my life. To my music brothers, Nacho, Emma, Sami, and all the beautiful people I've meet through music all these years. To the teachers, Flor, Pablo, Edgardo. To all the friends, the recent ones, those who've returned to my life and those brave ones that have stuck around: Emi, Pri, Germán, Alessandra, Melina, Leti, Lucas, Juan. To my office and doctoral colleagues, for the endless catharsis. To my cats, for the sweet company while writing. To aunt Cora, although isn't my aunt, she's my most auntie-aunt.

To all of you, a big ''thank you" for everything you've done for me during this long five years. For the infinite tolerance and patience, for the coffees and the beers, the long talks, advices and teachings. For all the hugs and support. For helping me grow a little bit every day, and for believing in me when I wasn't able to do so.

A general introduction

Structural graph theory studies characterizations and decompositions of particular graph classes, and uses these results to prove theoretical properties from such graph classes as well as to derive various algorithmic consequences. Typical topics in this area are graph minors and treewidth, modular decomposition and clique-width, characterization of graph families by forbidden configurations, among others.

This thesis consists on two parts, in each of which we focus on the study of two distinct topics in structural graph theory: characterization by forbidden induced subgraphs and characterization of minimal and minimum completions.

Part I: Characterization by forbidden induced subgraphs

Given a family of nonempty sets $\mathcal{S} = \{S_i\}$, the *intersection graph of the family* $\mathcal{S}$ is the graph with one vertex for each set S_i, such that two vertices are adjacent if and only if the corresponding sets have nonempty intersection. A graph is *circle* if there is a family of chords in a circle $L = \{C_v\}_{v \in G}$ such that two vertices are adjacent if the corresponding chords cross each other. In other words, it is the intersection graph of the chord family L. There are diverse characterizations of circle graphs, many of them using the notions of local complementation or split decomposition. In spite of having many diverse characterizations, there is no known complete characterization of circle graphs by minimal forbidden induced subgraphs. Current research on this direction focuses on finding partial characterizations of this graph class. In other words, some characterizations by minimal forbidden induced subgraphs for circle graphs are known when the graph we consider in the first place also belongs to another certain subclass, such as P_4-tidy graphs, linear-domino graphs, diamond-free graphs, to give some examples. In this thesis, we give a characterization by induced forbidden subgraphs of those split graphs that are also circle graphs. The motivation to study this particular graph class comes from chordal graphs, which are those graphs that contain no induced cycle of length greater than 3. Chordal graphs are a widely studied and interesting graph class, which is also a subset of perfect graphs. They may be recognized in polynomial time, and several problems that are hard on other classes of graphs such as graph coloring may be solved in polynomial time when the input is chordal. This is why the question of finding a list of forbidden subgraphs for the class of circle graphs when the graph is also chordal arises naturally. In turn, split graphs are those graphs whose vertex set can be split into a complete set and an independent set, and they are a subclass of chordal graphs. Moreover, split graphs are those chordal graphs whose complement is also a chordal graph. Thus, studying how to characterize circle graphs by forbidden induced subgraphs when the graph is split seemed a good place to start in order to find such a characterization for chordal circle graphs.

A $(0,1)$-matrix has the *consecutive-ones property (C1P) for the rows* if there is a permutation of its columns such that the 1's in each row appear consecutively. In order to characterize those split graphs that are circle, we develop characterizations by forbidden subconfigurations of $(0, 1)$-matrices with the C1P for which the rows admit a color assignment of two distinct colors under a certain adjacency relationship. This leads to structurally characterize some auxiliar circle graph subclasses that arise from these special matrices.

Part II: The Π-completion problem

For a graph property Π, the Π-graph modification problem is defined as follows. Given a graph G and a graph property Π, we need to delete (or add or edit) a subset of vertices (or edges) so that the resulting graph has the property Π. As graphs can be used to represent diverse real world and theoretical structures, it is not difficult to see that a modification problem can be used to model a large number of practical applications in several different fields. In particular, many fundamental problems in graph theory can be expressed as graph modification problems. For instance, the Connectivity problem is the problem of finding the minimum number of vertices or edges that disconnect the graph when removed from it, or the Maximum Induced Matching problem can be seen as the problem of removing the smallest set of vertices from the graph to obtain a collection of disjoint edges.

A particular graph modification problem is the Π-completion. Given a graph class Π, a Π-completion of a graph $G = (V, E)$ is a graph $H = (V, E \cup F)$ such that H belongs to Π. A Π-completion H of G is minimal if $H' = (V, E \cup F')$ does not belong to Π for every proper subset F' of F. A Π-completion H of G is minimum if for every Π-completion $H' = (V, E \cup F')$ of G, the cardinal of F is less than or equal to the cardinal of F'.

The problem of calculating a minimum completion in an arbitrary graph to a specific graph class has been rather studied. Unfortunately, minimum completions of arbitrary graphs to specific graph classes, such as cographs, bipartite graphs, chordal graphs, etc., have been showed to be NP-hard to compute [29, 7, 36]. For this reason, current research on this topic is focused on finding minimal completions of arbitrary graphs to specific graph classes in the most efficient way possible from the computational point of view. And even though the minimal completion problem is and has been rather studied, structural characterizations are still unknown for most of the problems for which a polynomial algorithm to find such a completion has been given. Studying the structure of minimal completions may allow to find efficent recognition algorithms. In particular, minimal completions from an arbitrary graph to interval graphs and proper interval graphs have been studied in [8, 33]. In this thesis, we study the problem of completing minimally to obtain a proper interval graph when the input is an interval graph. We find necessary conditions that characterize a minimal completion in this particular case, and we leave some conjectures for the future.

Part I

Split circle graphs

Chapter 1

Introduction

Circle graphs [15] are intersection graphs of chords in a circle. In other words, a graph is circle if there is a family of chords $L = \{C_v\}_{v \in G}$ in a circle such that two vertices in G are adjacent if and only if the corresponding chords cross each other. These graphs were defined by Even and Itai [15] to solve a problem stated by Knuth, which consists in solving an ordering problem with the minimum number of parallel stacks without the restriction of loading before unloading is completed. It was proven that this problem can be translated into the problem of finding the chromatic number of a circle graph. For its part, in 1985, Naji [28] characterized circle graphs in terms of the solvability of a system of linear equations, yielding a $\mathcal{O}(n^7)$-time recognition algorithm for this class.

The *local complement* of a graph G with respect to a vertex $u \in V(G)$ is the graph $G * u$ that arises from G by replacing the induced subgraph $G[N(u)]$ by its complement. Two graphs G and H are *locally equivalent* if and only if G arises from H by a finite sequence of local complementations. Circle graphs were characterized by Bouchet [5] in 1994 by forbidden induced subgraphs under local complementation. Inspired by this result, Geelen and Oum [21] gave a new characterization of circle graphs in terms of *pivoting*. The result of pivoting a graph G with respect to an edge uv is the graph $G \times uv = G * u * v * u$.

Circle graphs are a superclass of *permutation* graphs. Indeed, permutation graphs can be defined as those circle graphs having a circle model such that a chord can be added in such a way that this chord meets every chord belonging to the circle model. On the other hand, permutation graphs are those comparability graphs whose complement graph is also a comparability graph [14]. Since comparability graphs have been characterized by forbidden induced subgraphs [17], such a characterization implies a forbidden induced subgraphs characterization for the class of permutation graphs.

In spite of all these results, there are no known characterizations for the entire class of circle graphs by forbidden induced subgraphs. Some partial characterizations of circle graphs have been given. In other words, there are some characterizations of circle graphs by forbidden minimal induced subgraphs when these graphs also belong to a certain subclass, such as P_4-tidy graphs, Helly circle graphs, linear-domino graphs, among others. In Chapter 4 we give a brief introduction to these known results.

In order to extend these results, we considered the problem of characterizing by minimal induced forbidden subgraphs those circle graphs that are also split graphs. The motivation to study circle graphs restricted to this particular graph class came from chordal graphs, which are defined as those graphs that contain no induced cycles of length greater than 3. Chordal graphs

–which is a subset of perfect graphs– is a very widely studied graph class, for which there are several interesting characterizations. They may be recognized in polynomial time, and several problems that are hard on other classes of graphs –such as graph coloring– may be solved in polynomial time when the input is chordal. Another interesting property of chordal graphs, is that the treewidth of an arbitrary graph may be characterized by the size of the cliques in the chordal graphs that contain it. Block graphs are a particular subclass of chordal graphs, and are also circle. However, not every chordal graph is a circle graph. All these reasons lead to consider chordal graphs as a natural restriction to study a partial characterization of circle graphs by forbidden induced subgraphs. Something similar happens with split graphs, which is an interesting subclass of chordal graphs. More precisely, split graphs are those chordal graphs for which its complement is also a chordal graph. In Chapter 2 we give an example of a chordal graph that is neither circle nor split. Hence, studying those split graphs that are also circle is a good first step towards a characterization of those chordal graphs that are also circle.

We started by considering a split graph H such that H is minimally non-circle. Since comparability graphs are a subclass of circle graphs, in particular H is not a comparability graph. Notice that permutation graphs are those comparability graphs for which their complement is also a comparability graph. It is easy to prove that permutation graphs are precisely those circle graphs having a circle model with an equator. Using the list of minimal forbidden subgraphs of comparability graphs (see Figures 2.1 and 2.2) and the fact that H is also a split graph, we conclude that H contains either a tent, a 4-tent, a co-4-tent or a net as a subgraph (See Figure 2.3). In Chapter 2, given a split graph $G = (K, S)$ and a subgraph T that can be either a tent, a 4-tent or a co-4-tent, we define partitions of K and S according to the adjacencies and prove that these partitions are well defined.

A $(0, 1)$-matrix has the *consecutive-ones roperty (C1P) for the rows* if there is a permutation of its columns such that the ones in each row appear consecutively. In order to characterize those circle graphs that contain a tent, a 4-tent, a co-4-tent or a net as a subgraph, we first address the problem of characterizing those matrices that can be ordered with the C1P for the rows and for which there is a particular color assignment for every row, having exactly 2 colors to do so. Such a color assignment is defined in Chapter 3, considering the fullfillment of some special properties which are purely based on the partial ordering relationship that must hold between the neighbourhoods of the vertices in the independent partition of a split graph. These properties are contemplated in the definition of admissibillity.

In Chapter 3, we define and characterize 2-nested matrices by minimal forbidden submatrices. This characterization leads to a minimal forbidden induced subgraph characterization for the associated graph class, which is a subclass of split and circle graphs. In order to do this, we define the concept of enriched matrix, which are those $(0, 1)$-matrices for which some rows are labeled with a letter L (standing for *left*) or R (standing for *right*) or LR (standing for *left-right*), and some of these labeled rows may also be colored with either red or blue each. In the first sections of Chapter 3, we define and characterize the notions of admissibility, LR-orderable and partially 2-nested. This notions allowed to define what is a ''valid pre-coloring" and characterize those enriched matrices with valid pre-colorings that admit an LR-ordering, which is the property of having a lineal ordering Π of the columns such that, when ordered according to Π, the non-LR-rows and the complements of the LR-rows have the C1P, those rows labeled with L start in the first column and those rows labeled with R end in the last column. This leads to a characterization of 2-nested matrices by forbidden induced submatrices. 2-nested matrices are those partially 2-nested matrices for which the given 2-coloring of the rows can be extended to a total proper 2-coloring of all the matrix, while maintaining certain properties. Chapter 3 is crucial in order to determine which are the forbidden induced subgraphs for those circle graphs that are also split.

In chapter 4, we address the problem of characterizing the forbidden induced subgraphs of a circle graph that contains either a tent, 4-tent, co-4-tent or a net as an induced subgraph. In each section we see a case of the theorem, proving a characterization theorem and finishing with the guidelines to draw a circle model for each case.

1.1 Known characterizations of circle graphs

Recall that a graph is circle if it is the intersection graph of a family of chords in a circle. The characterization of the entire class of circle graphs by forbidden minimal induced subgraphs is still an open problem. However, some partial characterizations are known. In this section, we state some of the known characterizations for circle graphs, including those that are partial, and give the necessary definitions to understand these results.

A *double occurrence word* is a finite string of symbols in which each symbol appears precisely twice. Let $\Gamma = (\pi_1, \pi_2, \dots, \pi_{2n})$ be a double occurrence word. The *shift operation on* Γ transforms Γ into $(\pi_{2n}, \pi_1, \pi_2, \dots, \pi_{2n-1})$. The *reverse operation* transforms Γ into $\overline{\Gamma} = (\pi_{2n}, \pi_{2n-1}, \dots, \pi_2, \pi_1)$. With each double occurrence word Γ we associate a graph $G[\Gamma]$ whose vertices are the symbols in Γ and in which two vertices are adjacent if and only if the corresponding symbols appear precisely once between the two occurrences of the other. Clearly, a graph is circle if and only if it is isomorphic to $G[\Gamma]$ for some double occurrence word. Those graphs that are isomorphic to $G[\Gamma]$ for some double occurrence Γ are also called alternance graphs. A graph G is overlap interval if there exists a bijective function $f : V \to I(f(v) = I_v)$ where $I = \{I_v\}_{\{}I \in V(G)\}$ is a family of intervals on the real line, such that $uv \in E$ if and only if I_u and I_v overlap; i.e., $I_u \cap I_v \neq \emptyset$, $I_u \nsubseteq I_v$ and $I_v \nsubseteq I_u$. It is well known that circle graphs and overlap interval graphs are the same class (see [20]). Moreover, circle graphs are also equivalent to alternance graphs.

Given a double alternance word Γ, we denote by $\overline{\Gamma}$ the word that arises by traversing Γ from right to left, for instance, if $\Gamma = abcadcd$, then $\overline{\Gamma} = dcdacba$. Given a graph G and a vertex v of G. The local complement of G at v, denoted by $G * v$, is the graph that arises from G by replacing $N(v)$ by its complementary graph. Two graphs G and H are *locally equivalent* if and only if G arises from H by a finite sequence of local complementations. This operation is strongly linked with circle graphs; namely, *if* G *is a circle graph, then* $G * v$ *is a circle graph*. This is because, if a represents the vertex v in Γ and $\Gamma = AaBaC$ where A, B and C are subwords of Γ, then $G[AaBaC]$ is a double alternance model for $G * v$. Bouchet proved the following theorem.

Theorem 1.1. *[5] Let* G *be a graph.* G *is a circle graph if and only if any graph locally equivalent to* G *has no induced subgraph isomorphic to* W_5, W_7, *or* BW_3 *(see Figure 1.1).*

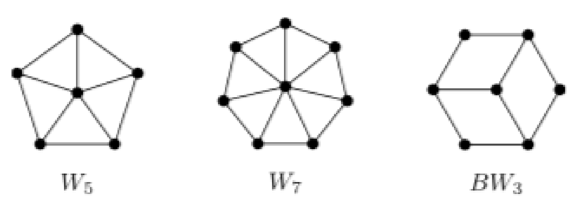


Figure 1.1 – The graphs W_5, W_7 and BW_3.

Bouchet also proved the following property of circle graphs. Let $G = (V, E)$ and let $A = \{A_{vw} : v, w \in V\}$ be an antisymmetric integer matrix [4]. For $W \subseteq V$, we denote $A[W] = \{A_{vw} : v, w \in W\}$. The matrix A satisfies the property α if the following property (related to unimodularity) holds: $\det(A[W]) \in \{-1, 0, 1\}$ for all $W \subseteq V$. Graph G is unimodular if there is an orientation of G such that the resulting digraph satisfies property α. Bouchet proved that every circle graph admits such an orientation [4]. Moreover, it was also Bouchet who proved that, if G is a bipartite graph such that its complement is circle, then G is a circle graph [6]. In [16], the authors give a new and shorter prove for this result.

Inspired by Theorem 1.1, Geelen and Oum gave a new characterization of circle graphs in terms of *pivoting* [21]. The result of pivoting a graph G with respect to an edge uv is the graph $G \times uv = G * u * v * u$, where $*$ stands for local complementation. A graph G' is *pivot equivalent* to G if G' arises from G by a sequence of pivoting operations. They proved, with the aid of a computer, that G is a circle graph if and only if each graph that is pivot equivalent to G contains none of 15 prescribed induced subgraphs.

Let G_1 and G_2 be two graphs such that $|V(G_i)| \geq 3$, for each $i = 1, 2$, and assume that $V(G_1) \cap V(G_2) = \emptyset$. Let v_i be a distinguished vertex of G_i, for each $i = 1, 2$. The *split composition* of G_1 and G_2 with respect to v_1 and v_2 is the graph $G_1 \circ G_2$ whose vertex set is $V(G_1 \circ G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$ and whose edge set is $E(G_1 \circ G_2) = E(G_1 \setminus \{v_1\}) \cup E(G_2 \setminus \{v_2\}) \cup \{uv : u \in N_{G_1}(v_1) \text{ and } v \in N_{G_2}(v_2)\}$. The vertices v_1 and v_2 are called the *marker vertices*. We say that G has a *split decomposition* if there exist two graphs G_1 and G_2 with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 \circ G_2$ with respect to some pair of marker vertices. If so, G_1 and G_2 are called the factors of the split decomposition. Those graphs that do not have a split decomposition are called *prime graphs*. The concept of split decomposition is due to Cunningham [9].

Circle graphs turned out to be closed under this decomposition [4] and in 1994 Spinrad presented a quadratic-time recognition algorithm for circle graphs that exploits this peculiarity [34]. Also based on split decomposition, Paul [31] presented an $\mathcal{O}((n + m)\alpha(n + m))$-time algorithm for recognizing circle graphs, where α is the inverse of the Ackermann function.

In [11] De Fraysseix presented a characterization of circle graphs, which leads to a novel interpretation of circle graphs as the intersection graphs of induced paths of a given graph. A *cocycle* of a graph G with vertex set V is the set of edges joining a vertex of V_1 to a vertex of V_2 for some bipartition (V_1, V_2) of V. A *cocyclic-path* is an induced path whose set of edges constitutes a cocycle. A *cocyclic-path intersection graph* is a simple graph with vertex set being a family of cocyclic-paths of a given graph, two vertices being adjacent if and only if the corresponding cocyclic-paths have an edge in common. Notice that the definition is restricted to those graphs covered by cocyclic-paths any two of which have at most a common edge. Fraysseix proved the following characterization of circle graphs as cocyclic-path intersection graphs.

Theorem 1.2. *[11] Let G be a graph. G is a circle graph if and only if G is a cocyclic-path intersection graph.*

A *diamond* is the complete graph with 4 vertices minus one edge. A claw is the graph with 4 vertices that has 1 vertex with degree 3 and a maximum independent set of size 3. *Prisms* are the graphs that arise from the cycle C_6 by subdividing the edges that link the triangles.

A graph is *Helly circle* if it has a circle model whose chords are all different and every subset of pairwise intersecting chords has a point in common. A characterization by minimal forbidden induced subgraphs for Helly circle graphs, inside circle graphs, was conjectured in [13] and was proved some years later in [10]. Notice that this characterization does not solve the general characterization of Helly circle graphs by forbidden subgraphs.

Theorem 1.3. *[10] Let* G *be a circle graph.* G *is Helly circle if and only if* G *is diamond-free.*

A graph G is *domino* if each of its vertices belongs to at most two cliques. In addition, if each of its edges belongs to at most one clique, G is *linear-domino*. Linear-domino graphs coincide with {claw,diamond}-free graphs.

There are no known characterizations for the class of circle graphs by minimal forbidden induced subgraphs. In order to obtain some results in this direction, this problem was addressed by attempting to characterize circle graphs by minimal forbidden induced subgraphs when given a graph that belongs to a certain graph class. This is known as a partial structural characterization. Some results in this direction are the following.

Theorem 1.4. *[3] Let* G *be a linear domino graph. Then,* G *is a circle graph if and only if* G *contains no induced prisms.*

The proof given in [3] is based on the fact that circle graphs are closed under split decomposition [4]. As a corollary of the above theorem, the following partial characterization of Helly circle graphs is obtained.

Corollary 1.5. *[3] Let* G *be a claw-free graph. Then,* G *is a Helly circle graph if and only if* G *contains no induced prism and no induced diamond.*

A graph is *cograph* if it is P_4-free. A graph is *tree-cograph* if it can be constructed from trees by disjoint union and complement operations. Let A be a P_4 in some graph G. A *partner of* A *in* G is a vertex v in $G \setminus A$ such that $A + v$ induces at least two P_4's. A graph G is P_4-tidy if any P_4 has at most one partner.

Theorem 1.6. *[3] Let* G *be a* P_4*-tidy graph. Then,* G *is a circle graph if and only if* G *contains no* W_5*, net+*K_1*, tent+*K_1*, or tent-with-center as induced subgraph.*

Theorem 1.7. *[3] Let* G *be a tree-cograph. Then,* G *is a circle graph if and only if* G *contains no induced (bipartite-claw)+*K_1 *and no induced co-(bipartite-claw).*

1.2 Basic definitions and notation

Let $A = (a_{ij})$ be a $n \times m$ $(0,1)$-matrix. We denote $a_{i.}$ and $a_{.j}$ the ith row and the jth column of matrix A. From now on, we associate each row $a_{i.}$ with the set of columns in which $a_{i.}$ has a 1. For example, the intersection of *two rows* $a_{i.}$ *and* $a_{j.}$ is the subset of columns in which both rows have a 1. Let $l_i = \min\{j\colon\ a_{ij} = 1\}$ and $r_i = \max\{j\colon\ a_{ij} = 1\}$ for each $i \in \{1, \dots, n\}$. Two rows $a_{i.}$ *and* $a_{k.}$ *are disjoint* if there is no j such that $a_{ij} = a_{kj} = 1$. We say that $a_{i.}$ is *contained* in $a_{k.}$ if for each j such that $a_{ij} = 1$ also $a_{kj} = 1$. We say that $a_{i.}$ *and* $a_{k.}$ *are nested* if $a_{i.}$ is contained in $a_{k.}$ or $a_{k.}$ is contained in $a_{i.}$. We say that a row $a_{i.}$ *is empty* if every entry of $a_{i.}$ is 0, and we say that $a_{i.}$ *is nonempty* if there is at least one entry of $a_{i.}$ equal to 1. We say that two nonempty rows *overlap* if they are non-disjoint and non-nested. Finally, we say that $a_{i.}$ *and* $a_{k.}$ *start* (resp. *end*) *in the same column* if $l_i = l_k$ (resp. $r_i = r_k$), and we say $a_{i.}$ *and* $a_{k.}$ *start (end) in different columns* otherwise.

We say a $(0,1)$-matrix A has the *consecutive-ones property for the rows* (for short, C1P) if there is permutation of the columns of A such that the 1's in each row appear consecutively. Any such permutation of the columns of A is called a *consecutive-ones ordering* for A. In [35], Tucker characterized all the minimal forbidden submatrices for the C1P, later known as *Tucker matrices*. For the complete list of Tucker matrices, see Figure 1.2.

$$M_{I}(k)=\begin{pmatrix}110...00\\011...00\\.....\\.....\\.....\\000...11\\100...01\end{pmatrix}\qquad M_{II}(k)=\begin{pmatrix}011...111\\110...000\\011...000\\.....\\.....\\000...110\\111...101\end{pmatrix}\qquad M_{III}(k)=\begin{pmatrix}110...000\\011...000\\.....\\.....\\000...110\\011...101\end{pmatrix}$$

$$M_{IV}=\begin{pmatrix}110000\\001100\\000011\\010101\end{pmatrix}\qquad M_{V}=\begin{pmatrix}11000\\00110\\11110\\10011\end{pmatrix}$$

Figure 1.2 – Tucker matrices $M_I(k) \in \{0,1\}^{k\times k}$, $M_{III}(k) \in \{0,1\}^{k\times(k+1)}$ with $k \geq 3$, and $M_{II}(k) \in \{0,1\}^{k\times k}$ with $k \geq 4$

Let A and B be $(0,1)$-matrices. We say that B *is a subconfiguration of* A if there is a permutation of the rows and the columns of B such that B with this permutation results equal to a submatrix of A. Given a subset of rows R of A, we say that R *induces a matrix* B if B is a subconfiguration of the submatrix of A given by selecting only those rows in R.

All graphs in this work are simple, undirected, with no loops and no multiple edges. The pair (K,S) is a *split partition* of a graph G if $\{K,S\}$ is a partition of the vertex set of G and the vertices of K (resp. S) are pairwise adjacent (resp. nonadjacent), and we denote it $G=(K,S)$. A graph G is a *split graph* if it admits some split partition. Let G be a split graph with split partition (K,S), $n=|S|$, and $m=|K|$. Let $s_1,\dots,s_n$ and $v_1,\dots,v_m$ be linear orderings of S and K, respectively. Let $A=A(S,K)$ be the $n\times m$ matrix defined by $A(i,j)=1$ if s_i is adjacent to v_j and $A(i,j)=0$, otherwise.

Chapter 2

Preliminaries

Let us consider a split graph $G = (K, S)$ and suppose that G is minimally non-circle. Equivalently, any proper induced subgraph of H is circle. If G is not circle, then in particular G is not a permutation graph. Permutation graphs are exactly those comparability graphs whose complement graph is also a comparability graph [14]. Comparability graphs have been characterized by forbidden induced subgraphs in [17].

Theorem 2.1 ([17]). *A graph is a comparability graph if and only if it does not contain as an induced subgraph any graph in Figure 2.1 and its complement does not contain as an induced subgraph any graph in Figure 2.2.*

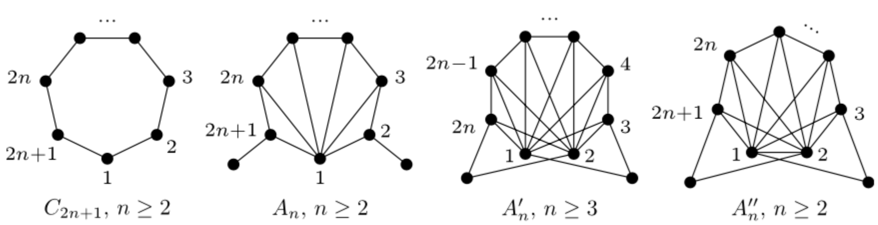


Figure 2.1 – Forbidden subgraphs for comparability graphs.

This characterization of comparability graphs leads to a forbidden induced subgraph characterization for the class of permutation graphs. Hence, since comparability graphs is a subclass of circle graphs, in particular G is not a comparability graph. Using the list of minimal forbidden subgraphs for comparability graphs given in Figures 2.1 and 2.2 and the fact that G is also a split graph, we conclude that G contains either a tent, a 4-tent, a co-4-tent or a net as an induced subgraph (See Figure 2.3).

As previously mentioned, the motivation to study circle graphs restricted to split graphs came from chordal graphs. Remember that split graphs are those chordal graphs for which its complement is also a chordal graph. Let us consider the graph A''_n for $n = 3$ depicted in Figure 2.1.

This is a chordal graph since A''_3 contains no cycles of length greater than 3. Moreover, it is easy to see that A''_3 is not a split graph. This follows from the fact that the maximum clique has size 4, and the removal of any such clique leaves out a non-independent set of vertices. The same holds for any clique of size smaller than 4. Furthermore, if we apply local complement of the

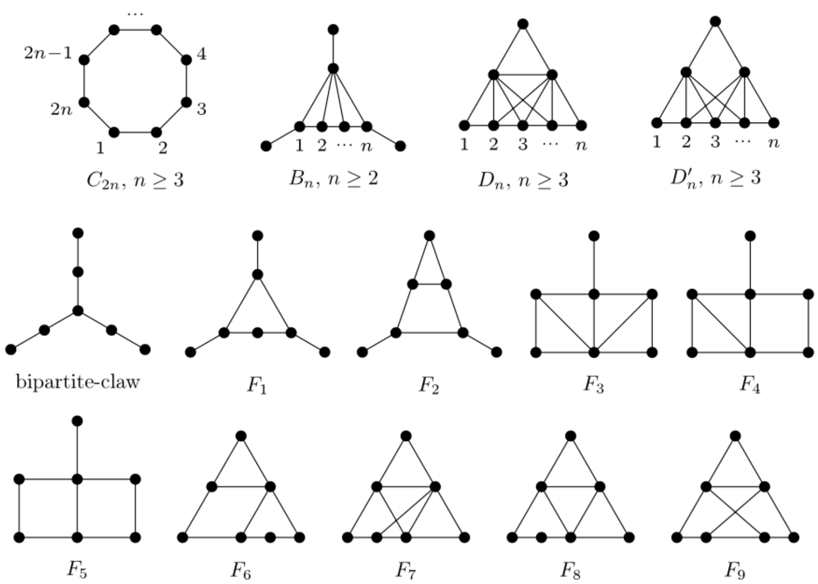


Figure 2.2 – Forbidden subgraphs for comparability graphs.

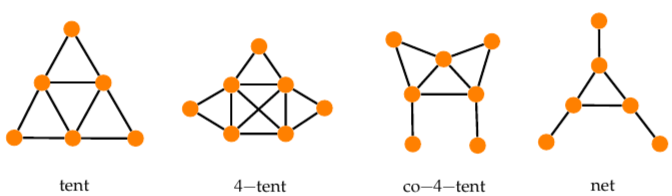
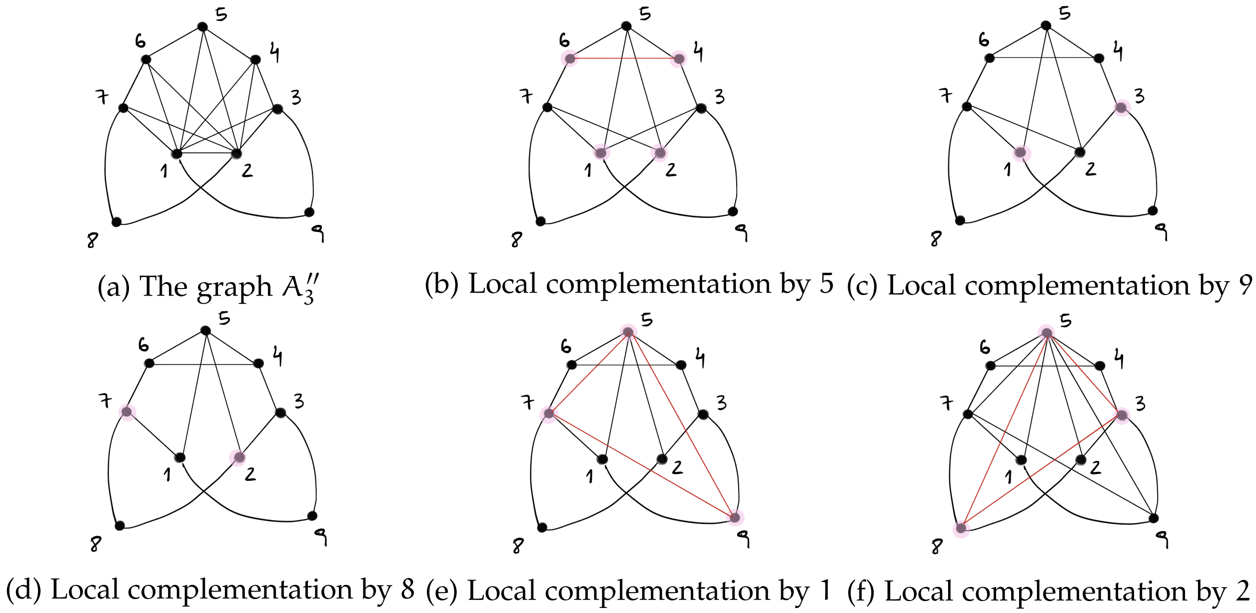


Figure 2.3 – Forbidden subgraphs for comparability $\cap$ split graphs.

graph sequentially on the vertices 5, 9, 8, 1 and 2, then we find W_5 induced by the subset {5, 3, 4, 6, 7, 8}. For more detail on this, see Figure 2.4. It follows from the characterization given by Bouchet in 1.1 that A_3'' is not a circle graph.

This shows an example of a graph that is neither circle nor split, but is chordal. In particular, it follows from this example (which is minimally non-circle) that whatever list of forbidden subgraphs found for split circle graphs is not enough to characterize those chordal graphs that are also circle. Therefore, studying split circle graphs is a good first step towards characterizing those chordal graphs that are also circle.

Throughout the following sections, we will define some subsets in both K and S depending on whether G contains an induced tent, 4-tent or co-4-tent T as an induced subgraph. We will prove that these subsets induce a partition of both K and S. In each case, the vertices in the complete partition K are split into subsets according to the adjacencies with the independent vertices of T, and the vertices in the independent partition S are split into subsets according to the adjacencies with each partition of K. These partitions will be useful in Chapter 3, in order to give motivation for the matrix theory developed in that chapter, and in Chapter 4, when we give the proof of the characterization by forbidden induced subgraphs for split circle graphs. Notice that we do not consider the case in which G contains an induced net in order to define the partitions of K and



(a) The graph A_3'' (b) Local complementation by 5 (c) Local complementation by 9

(d) Local complementation by 8 (e) Local complementation by 1 (f) Local complementation by 2

Figure 2.4 – Sequence of local complementations applied to A_3''.

S, for it will be explained in detail in Section 4.4 that this case can be reduced using the cases in which G contains a tent, a 4-tent and a co-4-tent.

In Figures 2.6 and 2.5, we define two graph families that will be central throughout the sequel. These matrices are necessary to state the main result of this part, which is the following characterization by forbidden induced subgraphs for those split graphs that are also circle.

Theorem 4.1 (continuing from p. 85)**.** *Let* $G = (K, S)$ *be a split graph. Then,* G *is a circle graph if and only if* G *is* $\{\mathcal{T}, \mathcal{F}\}$*-free.*

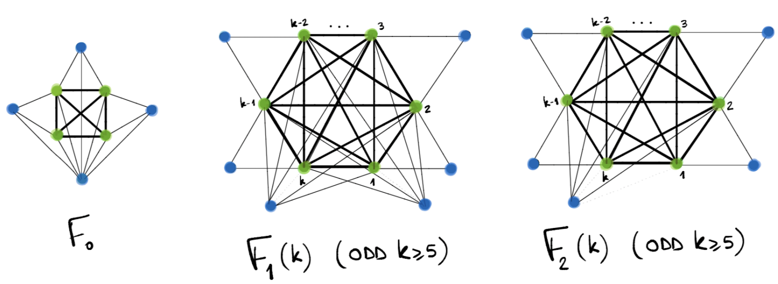


Figure 2.5 – The graphs in the family $\mathcal{F}$.

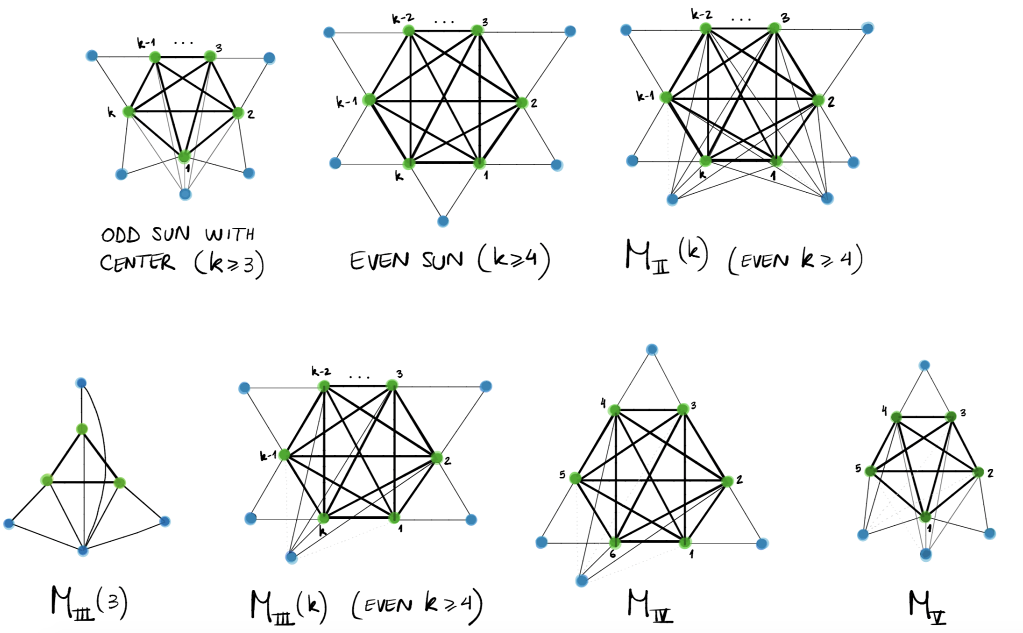


Figure 2.6 – The graphs in the family $\mathcal{T}$.

2.1 Partitions of S and K for a graph containing an induced tent

Let $G = (K, S)$ be a split graph where K is a clique and S is an independent set. Let T be an induced subgraph of G isomorphic to tent. Let $V(T) = \{k_1, k_3, k_5, s_{13}, s_{35}, s_{51}\}$ where $k_1, k_3, k_5 \in K$, $s_{13}, s_{35}, s_{51} \in S$, and the neighbors of s_{ij} in T are precisely k_i and k_j.

We introduce sets $K_1, K_2, \dots, K_6$ as follows.

— For each $i \in \{1, 3, 5\}$, let K_i be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely $s_{(i-2)i}$ and $s_{i(i+2)}$ (where subindices are modulo 6).

— For each $i \in \{2, 4, 6\}$, let K_i be the set of vertices of K whose only neighbor in $V(T) \cap S$ is $s_{(i-1)(i+1)}$ (where subindices are modulo 6).

See Figure 2.7 for a graphic idea of this.

We say a vertex v *is complete to the set of vertices* X if v is adjacent to every vertex in X, and we say v *is anticomplete to* X if v is nonadjacent to every vertex in X. We say by abuse of language that v *is adjacent to* X if there is at least one vertex x in X such that v is adjacent to x. Let v in S. We denote $N_i(v)$ to the neighbourhood of the vertex v restricted to K_i. Given two vertices v_1 and v_2 in S, if either $N(v_1) \subseteq N(v_2)$ or $N(v_2) \subseteq N(v_1)$, then we say that v_1 *and* v_2 *are nested*. In particular, given $i \in \{1, \dots, 6\}$, if either $N_i(v_1) \subseteq N_i(v_2)$ or $N_i(v_2) \subseteq N_i(v_1)$, then we say that v_1 *and* v_2 *are nested in* K_i. Aditionally, if $N(v_1) \subseteq N(v_2)$, then we say that v_1 *is contained in* v_2.

Lemma 2.2. *If* G *is a circle graph,* $\{K_1, K_2, \dots, K_6\}$ *is a partition of* K.

Proof. Every vertex of K is adjacent to precisely one or two vertices of $V(T) \cap S$, for if not we find a tent $\vee K_1$ or a tent with center as induced subgraphs of G, which are not circle graphs. □

Let $i, j \in \{1, \dots, 6\}$ and let S_{ij} be the set of vertices of S that are adjacent to some vertex in K_i

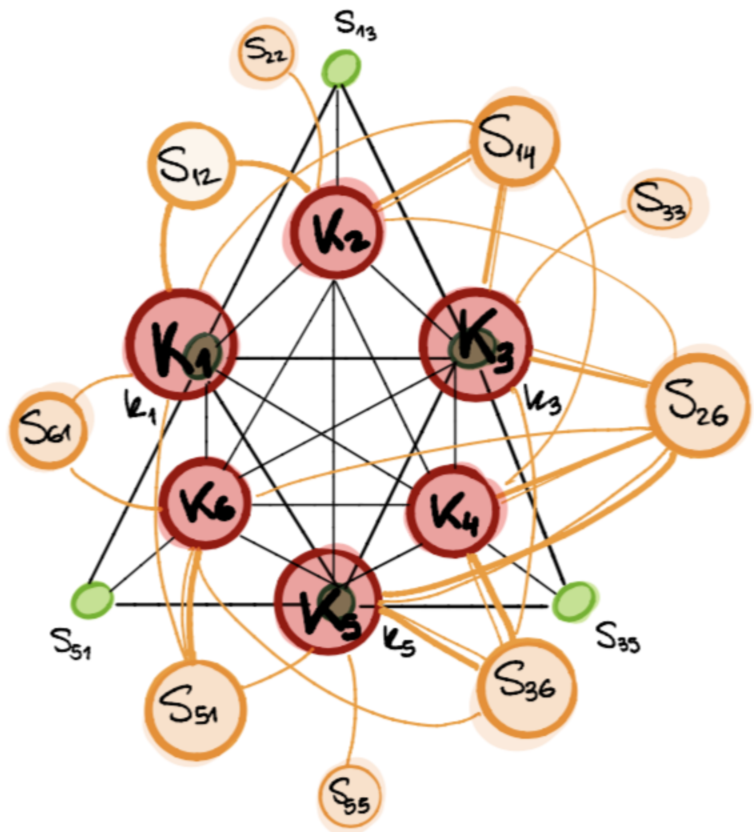


Figure 2.7 – Tent T and the split graph G according to the given extensions

and some vertex in K_j, are complete to K_{i+1},$K_{i+2}, \dots, K_{j-1}$, and are anticomplete to K_{j+1},$K_{j+2}, \dots, K_{i-1}$ (where subindices are modulo 6).

The following claims are necessary to prove Lemma 2.8.

Claim 2.3. *If* G *is a circle graph, then there is no vertex* v *in* S *such that* v *is simultaneously adjacent to* K_1, K_3 *and* K_5. *Moreover, there is no vertex* v *in* S *adjacent to* K_2, K_4 *and* K_6 *such that* v *is anticomplete to any two of* K_j, *for* $j \in \{1, 3, 5\}$.

Let v is S and w_i in K_i for each $i \in \{1, 3, 5\}$, such that v is adjacent to each w_i. Hence, there is a tent with center induced by $\{w_1, w_3, w_5, s_{13}, s_{35}, s_{51}, v\}$, thus G is not circle, which is a contradiction.

To prove the second statement, let w_i in K_i such that v is adjacent to w_i for every $i \in \{2, 4, 6\}$. Suppose that v is anticomplete to K_3 and K_5. Thus, we find a 4-sun induced by the set $\{w_2, w_3, w_5, w_6, s_{13}, s_{35}, s_{51}, v\}$ which is a non-circle graph. If instead v is anticomplete to K_1 and K_3, then a 4-sun is induced by $\{w_1, w_3, w_4, w_6, s_{13}, s_{35}, s_{51}, v\}$, and if v is anticomplete to K_1 and K_5, then a 4-sun is induced by the set $\{w_1, w_2, w_4, w_5, s_{13}, s_{35}, s_{51}, v\}$. □

Claim 2.4. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+3}, *then* v *is complete to* K_j *for either* $j \in \{i+1, i+2\}$ *or* $j \in \{i-1, i-2\}$.

We assume without loss of generality that K_j is nonempty for every $j \in \{1, \dots, 6\}$, thus let w_j in K_j, for each $j \in \{1, \dots, 6\}$.

If v is anticomplete to K_j for every $j \in \{i-2, i-1, i+1, i+2\}$, then we find an induced net $\vee K_1$.

Let us assume for simplicity that i is even, since the proof is analogous if i is odd. If v is adjacent to w_{i+1} in K_{i+1} and v is anticomplete to K_{i+2}, then in particular v is anticomplete to K_{i-1}, for if not we find a tent with center. Thus, we find $M_{III}(3)$ induced by the set $\{s_{(i-1)(i+3)}, s_{(i+3)(i-1)}, v, w_{i-1}, w_{i+1}, w_{i+2}, w_{i+3}\}$.

If instead v is adjacent to w_{i+2} in K_{i+2} and v is anti-complete to K_{i+1}, then v is anticomplete to K_{i-1} for if not we find a tent with center. Thus, we find $M_{III}(3)$ induced by $\{s_{(i+1)(i+3)}, s_{(i+3)(i-1)}, v, w_i, w_{i+3}, w_{i-1}, w_{i+1}\}$.

Notice that the same argument holds if v is adjacent but not complete to either K_{i+1} or K_{i+2}, for we find the same subgraphs . □

Claim 2.5. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+2}*, then either* v *is complete to* K_{i+1}*, or* v *is complete to* K_j *for* $j \in \{i-1, i-2, i-3\}$*.*

Once more, we assume without loss of generality that K_j is nonempty, for all $j \in \{1, \dots, 6\}$. Given the simmetry of the odd-indexed and even-indexed sets K_j, we may also separate in two cases without losing generality: if v is adjacent to K_1 and K_3 and if v is adjacent to K_2 and K_4.

Suppose first that v is adjacent to K_1 and K_3. By Claim 2.3, v is anticomplete to K_5. If v is nonadjacent to some vertex w_2 in K_2, then the set $\{s_{35}, v, s_{51}, w_1, w_3, w_5, w_2\}$ induces a tent with center. Hence, v is complete to K_2.

Suppose now that v is adjacent to K_2 and K_4. First, notice that v is complete to either K_1 or K_5, for if not we find a 4-sun induced by $\{s_{13}, s_{51}, s_{35}, v, w_2, w_1, w_5, w_4\}$. Suppose that v is complete to K_1. If v is not complete to K_3, then v is complete to K_5 and K_6, for if not there is $M_{III}(3)$ induced by $\{s_{13}, s_{51}, v, w_1, w_3, w_4, w_j\}$ for $j = 5, 6$. □

Remark 2.6. As a consequence of the previous claims we also proved that, if G is a circle graph, then:

— For each $i \in \{1, 2, \dots, 6\}$, the sets $S_{i,i-2}$ are empty, for if not, there is a vertex v in S such that v is adjacent to K_1, K_3 and K_5 (Claim 2.3). Moreover, the same holds for $S_{i(i-2)}$, for each $i \in \{1, 3, 5\}$.

— For each $i \in \{2, 4, 6\}$, the sets $S_{i(i+2)}$ are empty since every vertex v in S such that v is adjacent to K_i and K_{i+2} is necessarily complete to either K_{i-1} or K_{i+3} (Claim 2.5).

Claim 2.7. *If* G *is a circle graph, then for each* $i \in \{1, 3, 5\}$*, every vertex in* $S_{i(i+3)}$ *and* $S_{(i+3)i}$ *is complete to* K_i*.*

We will prove this claim without loss of generality for $i = 1$. As denoted in the previous claims, let w_3 in K_3 and w_5 in K_5.

Let v in S_{14}. Toward a contradiction, let w_{11} and w_{12} in K_1 such that v is nonadjacent to w_{11} and v is adjacent to w_{12}, and let w_4 in K_4 such that v is adjacent to w_4. In this case, we find F_0 induced by the set $\{s_{13}, s_{35}, v, w_{11}, w_{12}, w_3, w_4, w_5\}$.

Analogously, if v in S_{41}, then F_0 is induced by $\{s_{35}, s_{51}, v, w_{11}, w_{12}, w_3, w_4, w_5\}$. □

The following Lemma is a straightforward consequence of Claims 2.3 to 2.7.

Lemma 2.8. *If* G *is a circle graph, then all the following assertions hold:*

— $\{S_{ij}\}_{i,j \in \{1,2,\dots,6\}}$ *is a partition of* S*.*

— *For each* $i \in \{1,3,5\}$, $S_{i(i-1)}$ *and* $S_{i(i-2)}$ *are empty.*
— *For each* $i \in \{2,4,6\}$, $S_{i(i-1)}$ *and* $S_{i(i+2)}$ *are empty.*
— *For each* $i \in \{1,3,5\}$, $S_{i(i+3)}$ *and* $S_{(i+3)i}$ *are complete to* K_i.

$i \backslash j$	1	2	3	4	5	6
1	✓	✓	✓	✓	∅	∅
2	∅	✓	✓	∅	✓	✓
3	∅	∅	✓	✓	✓	✓
4	✓	✓	∅	✓	✓	∅
5	✓	✓	∅	∅	✓	✓
6	✓	∅	✓	✓	∅	✓

Figure 2.8 – The nonempty partitions of S in the tent case.

2.2 Partitions of S and K for a graph containing an induced 4-tent

Let $G = (K, S)$ be a split graph where K is a clique and S is an independent set. Let T be a 4-tent induced subgraph of G. Let $V(T) = \{k_1, k_2, k_4, k_5, s_{12}, s_{24}, s_{45}\}$ where $k_1, k_2, k_4, k_5 \in K$, $s_{12}, s_{24}, s_{45} \in S$, and the neighbors of s_{ij} in T are precisely k_i and k_j.

We introduce sets $K_1, K_2, \dots, K_6$ as follows.

— Let K_1 be the set of vertices of K whose only neighbor in $V(T) \cap S$ is s_{12}. Analogously, let K_3 be the set of vertices of K whose only neighbor in $V(T) \cap S$ is s_{24}, and let K_5 be the set of vertices of K whose only neighbor in $V(T) \cap S$ is s_{45}.
— For each $i \in \{2,4\}$, let K_i be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely s_{ji} and s_{ik}, for $i = 2$, $j = 1$ and $k = 2$ or $i = 4$, $j = 2$ and $k = 5$.
— Let K_6 be the set of vertices of K that are anticomplete to $V(T) \cap S$.

The following Lemma is straightforward.

Lemma 2.9. *If* G *is a circle graph, then* $\{K_1, K_2, \dots, K_6\}$ *is a partition of* K.

Proof. Every vertex in K is adjacent to precisely one, two or no vertices of $V(T) \cap S$, for if not we find a 4-tent $\vee K_1$. □

Let $i, j \in \{1, \dots, 6\}$ and let S_{ij} defined as in the previous section. We denote $S_{[ij}$ (resp. $S_{ij]}$) to the set of vertices in S that are adjacent to K_j and complete to $K_i, K_{i+1}, \dots, K_{j-1}$ (resp. adjacent to K_i and complete to $K_{i+1}, \dots, K_{j-1}, K_j$). We denote $S_{[ij]}$ to the set of vertices in S that are complete to $K_i, \dots, K_j$.

In particular, we consider separately those vertices adjacent to K_6 and complete to $K_1, K_2, \dots, K_5$: we denote $S_{[16}$ to the set that contains these vertices, and S_{16} to the subset of vertices of S that are adjacent but not complete to K_1. Furthermore, we consider the set S_{65} as those vertices in S that are adjacent but not complete to K_5.

Claim 2.10. *If* v *in* S *fullfils one of the following conditions:*

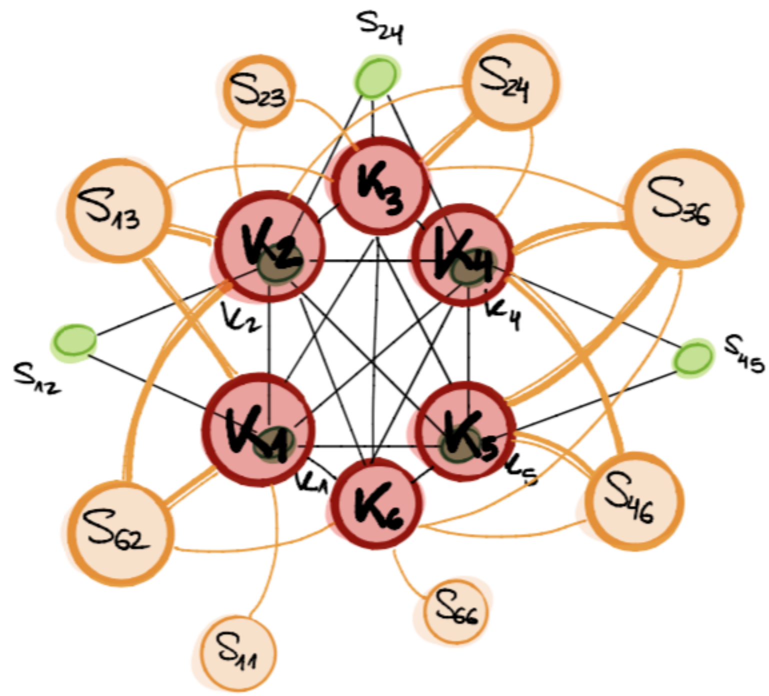


Figure 2.9 – Some of the possible extensions of the 4-tent graph.

— *v is adjacent to K_i and K_{i+2} and is anticomplete to K_{i+1}, for $i = 1, 3$*
— *v is adjacent to K_1 and K_4 and is anticomplete to K_2*
— *v is adjacent to K_2 and K_5 and is anticomplete to K_4*

Then, there is an induced tent in G.

If v is adjacent to K_1 and K_3 and is anticomplete to K_2, then we find a tent induced by $\{s_{12}, s_{24}, v, k_1, k_2, k_3\}$. If instead v is adjacent to K_3 and K_5 and is anticomplete to K_4, then the tent is induced by $\{s_{45}, s_{24}, v, k_3, k_4, k_5\}$.

If v is adjacent to K_1 and K_4 and is anticomplete to K_2, then we find a tent induced by $\{s_{12}, s_{24}, v, k_1, k_2, k_4\}$.

Finally, if v is adjacent to K_2 and K_5 and is anticomplete to K_4, then the tent is induced by the set $\{s_{45}, s_{24}, v, k_2, k_4, k_5\}$. □

As a direct consequence of the previous claim, we will assume without loss of generality that the subsets S_{31}, S_{41}, S_{52} and S_{53} of S are empty.

Claim 2.11. *If G is a circle graph, then S_{51} is empty. Moreover, if $K_3 \neq \emptyset$, then S_{42} is empty.*

Suppose there is a vertex v in S_{51}, let k_1 in K_1 and k_5 in K_5 be vertices adjacent to v. Thus, we find a 4-sun induced by the set $\{s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5, v\}$.

If $K_3 \neq \emptyset$, suppose v in S_{42}, and let k_2 in K_2, k_4 in K_4 be vertices adjacent to v. Notice that, by definition, v is complete to K_5 and K_1, and anticomplete to K_3. Then, we find M_V induced by the set $\{s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5, k_3, v\}$. □

We want to prove that $\{S_{ij}\}$ is indeed a partition of S, analogously as in the tent case. Towards this purpose, we state and prove the following claims.

Claim 2.12. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+2} *and anticomplete to* K_j *for* $j < i$ *and* $j > i+2$*, then:*

— *If* $i \equiv 0 \pmod 3$*, then* v *is complete to* K_{i+1} *and* K_{i+2}*.*
— *If* $i \equiv 1 \pmod 3$*, then* v *is complete to* K_i *and* K_{i+1}*.*
— *If* $i \equiv 2 \pmod 3$*, then* v *lies in* S_{24}*.*

Let v in S adjacent to some vertices k_1 in K_1 and k_3 in K_3, such that v is anticomplete to K_4, K_5 and K_6. By the previous Claim, we know that v is complete to K_2 for if not there is an induced tent. Moreover, suppose that v is not complete to K_1. Let k_2 in K_2, k_4 in K_4 and let k_1' in K_1 be a vertex nonadjacent to v. Then, we find F_0 induced by $\{s_{12}, s_{24}, v, k_1, k_1', k_2, k_3, k_4\}$. The proof is analogous for v adjacent to K_3 and K_5, and anticomplete to K_1, K_2 and K_6.

Let v in S be a vertex adjacent to k_4 in K_4 and k_6 in K_6, such that v is anticomplete to K_1, K_2 and K_3 (it is indistinct if $K_3 = \emptyset$). Suppose there is a vertex k_5 in K_5 nonadjacent to v. In this case, we find a net $\vee K_1$ induced by $\{s_{24}, s_{45}, v, k_2, k_4, k_5\}$. Moreover, suppose that v is not complete to K_4. Let k_4' in K_4 nonadjacent to v. Thus, we find F_0 induced by $\{s_{24}, s_{45}, v, k_2, k_4', k_4, k_5, k_6\}$. The proof is analogous for v adjacent to K_6 and K_2, and anticomplete to K_3, K_4 and K_5.

Finally, we know that in the third statement either $i = 2$ or $i = 5$. If $i = 5$, then v is a vertex adjacent to K_5 and K_1 such that v is anticomplete to K_2, K_3 (if nonempty) and K_4. Hence, as a direct consequence of the proof of Claim 2.11, we find a 4-sun. Hence, there is no such vertex v adjacent to K_5 and K_1 and thus necessarily $i = 2$. Let v in S adjacent to k_2 in K_2 and k_4 in K_4 such that v is anticomplete to K_5, K_6 and K_1 (it is indistinct if $K_6 = \emptyset$). If $K_3 \neq \emptyset$, let k_3 in K_3 and suppose that v is nonadjacent to K_3. Then, we find $M_{III}(4)$ induced by $\{s_{12}, s_{24}, s_{45}, v, k_1, k_2, k_4, k_5, k_3\}$. □

Claim 2.13. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+3} *and* v *is anticomplete to* K_j *for* $j < i$ *and* $j > i+3$*, then:*

— *If* $i \equiv 0 \pmod 3$*, then* v *is complete to* K_{i+1} *and* K_{i+2}*.*
— *If* $i \equiv 1 \pmod 3$*, then* v *lies in* $S_{14]}$*.*
— *If* $i \equiv 2 \pmod 3$*, then* v *lies in* $S_{25]}$*.*

Proof. Suppose first that $i \equiv 0 \pmod 3$. Let v in S such that v is adjacent to some vertices k_3 in K_3 and k_6 in K_6 and v is anticomplete to K_1 and K_2. Let k_1 in K_1 and k_2 in K_2 be any two vertices. If there are vertices k_4 in K_4 and k_5 in K_5 such that k_4 and k_5 are both nonadajcent to v, then we find M_{IV} induced by the set $\{s_{12}, s_{24}, v, s_{45}, k_1, k_2, k_6, k_3, k_5, k_4\}$. If instead v is adjacent to a vertex k_5 in K_5 and v is nonadjacent to a vertex k_4 in K_4, then we find a tent with center induced by $\{s_{24}, v, s_{45}, k_1, k_3, k_4, k_5\}$. Conversely, if v is adjacent to k_4 in K_4 and is nonadjacent to some k_5 in K_5, then we find a net $\vee K_1$ induced by the set $\{s_{12}, s_{24}, v, s_{45}, k_2, k_6, k_5, k_4\}$. The proof is analogous by symmetry for v in S_{63}.

Let us see now the case $i \equiv 1 \pmod 3$, thus either $i = 1$ or $i = 4$. If $i = 4$, then v is adjacent to K_4 and K_1 and v is anticomplete to K_2 and K_3 (if nonempty). Thus, by Claim 2.10 we may discard this case. Let v in S such that v is adjacent to some vertices k_1 in K_1, k_4 in K_4 and v is nonadjacent to a vertex k_5 in K_5. Suppose that v is not complete to K_2 and K_3. Whether $K_3 = \emptyset$ or not, if there is a vertex k_2 in K_2 that is nonadjacent to v, then we find a net $\vee K_1$ induced by $\{s_{24}, s_{45}, v, k_1, k_5, k_2, k_4\}$. If $K_3 \neq \emptyset$, we find a net $\vee K_1$ by replacing the vertex k_2 in the previous set for any vertex in K_3 that is nonadjacent to v. Let us see that v is also complete to K_4. If this is not true, then there is a vertex k_4' in K_4 nonadjacent to v. However, we find F_0 induced by $\{s_{24}, s_{45}, v, k_1, k_2, k_4, k_4', k_5\}$.

The proof for the third statement is analogous by symmetry. □

Claim 2.14. *If G is a circle graph, v in S is adjacent to K_i and K_{i+4} and v is anticomplete to K_{i-1}, then:*
— *If $i \equiv 0 \pmod 3$, then v lies in $S_{64]}$.*
— *If $i \equiv 1 \pmod 3$ and $K_3 \neq \emptyset$, then v lies in S_{15}.*
— *If $i \equiv 2 \pmod 3$, then v lies in $S_{[26}$.*

Proof. Suppose first that $i \equiv 0 \pmod 3$. In this case, either $i = 3$ or $i = 6$. If $i = 3$, then v is adjacent to K_3 and K_1 and is anticomplete to K_2. By Claim 2.10, this case is discarded for there is an induced tent. Hence, necessarily $i = 6$. Equivalently, v is adjacent to K_6 and K_4 and v is anticomplete to K_5. Suppose there is a vertex k_2 in K_2 nonadjacent to v. Then, we find a net $\vee K_1$ induced by $\{k_2, k_5, k_6, v, s_{45}, s_{24}, k_4\}$, and thus v must be complete to K_2. Furthermore, suppose v is not complete to K_1, thus there is a vertex k_1 in K_1 nonadjacent to v. Since v is complete to K_2, we find $M_{III}(4)$ induced by the set $\{k_1, k_2, k_4, k_5, k_6, s_{12}, s_{24}, s_{45}, v\}$. If $K_3 \neq \emptyset$, then v is complete to K_3, for if not we find a net $\vee K_1$ induced by $\{k_3, k_5, k_6, v, s_{24}, s_{45}, k_4\}$. Finally, if v is not complete to K_4, then there is a vertex k_4' in K_4 nonadjacent to v. In this case, we find F_0 induced by $\{k_1, k_2, k_4, k_4', k_5, s_{24}, s_{45}, v\}$ and this finishes the proof of the first statement.

Suppose now that $i \equiv 1 \pmod 3$. Thus, either $i = 1$ or $i = 4$. By hypothesis, $K_3 \neq \emptyset$. Suppose that $i = 4$, thus v is adjacent to K_4 and K_2 and v is anticomplete to K_3. By Claim 2.11, if $K_3 \neq \emptyset$, then v is anticomplete to K_1 and K_5, for if not we find an induced M_V. Hence, if $K_3 \neq \emptyset$, then we find $M_{III}(4)$ induced by $\{k_1, k_2, k_4, k_5, s_{12}, s_{24}, s_{45}, v\}$. Thus, if $i = 4$, then necessarily $K_3 = \emptyset$. Let $i = 1$. Suppose that v is adjacent to K_1 and K_5 and that v is anticomplete to K_6. If v is nonadjacent to some vertices k_2 in K_2 and k_4 in K_4, then we find a 4-sun induced by $\{k_1, k_2, k_4, k_5, s_{12}, s_{24}, s_{45}, v\}$. If v is not complete to k_2 in K_2, then we find a tent induced by $\{k_2, k_4, k_5, v, s_{45}, s_{24}\}$. The same holds for K_4 by replacing the vertex k_5 for some vertex k_1 in K_1 adjacent to v and s_{45} by s_{12}. Notice that it was not necessary for the argument that $K_6 \neq \emptyset$.

Finally, suppose that $i \equiv 2 \pmod 3$. By Claim 2.10 we can discard the case where $i = 5$, thus we may assume that $i = 2$. However, the proof for $i = 2$ is analogous to the proof of the first statement.

□

Claim 2.15. *Let v in S such that v is adjacent to at least one vertex in each nonempty K_i, for all $i \in \{1, \ldots, 6\}$.*

If G is a circle graph, then the following statements hold:
— *The vertex v is complete to K_2 and K_4, regardless of whether K_3 and K_6 are empty or not.*
— *If $K_3 \neq \emptyset$, then v is complete to K_3.*
— *If $K_6 \neq \emptyset$, then either v is complete to K_1 or v is complete to K_5.*

Proof. Let k_i in K_i be a vertex adjacent to v, for each $i = 1, 2, 4, 5$, which are always nonempty sets. If v is not complete to K_2, then there is a vertex k_2' in K_2 nonadjacent to v. Thus, we find a tent induced by $\{s_{12}, v, s_{24}, k_2', k_4, k_1\}$. The same holds if v is not complete to K_4, and thus the first statement holds. Notice that, in fact, this holds regardless of K_3 or K_6 being empty or not.

Suppose now that $K_3 \neq \emptyset$, and that there is a vertex k_3 in K_3 such that v is nonadjacent to k_3. Then, we find M_V induced by $\{s_{12}, s_{45}, v, s_{24}, k_2, k_1, k_5, k_1, k_3\}$.

Finally, let us suppose that $K_6 \neq \emptyset$ and toward a contradiction, let k_1' in K_1 and k_5' in K_5 be two vertices nonadjacent to v, and k_6 in K_6 adjacent to v. Then, we find $M_{III}(4)$ induced by $\{s_{12}, s_{24}, s_{45}, v, k_1', k_2, k_4, k_5', k_6\}$. Notice that this holds even if $K_3 = \emptyset$. □

As a consequence of Claims 2.10 to 2.15, we have the following Lemma.

Lemma 2.16. *If G is a circle graph, then all the following assertions hold:*

- *—* $\{S_{ij}\}_{i,j\in\{1,2,\ldots,6\}}$ *is a partition of* S.
- *—* *For each* $i \in \{2,3,4,5\}$, S_{i1} *is empty.*
- *—* *For each* $i \in \{3,4,5\}$, S_{i2} *is empty.*
- *—* *The subsets* S_{43}, S_{53} *and* S_{54} *are empty.*
- *—* *The following subsets coincide:* $S_{13} = S_{[13}$, $S_{14} = S_{14]}$, $S_{25} = S_{[25}$, $S_{26} = S_{[26}$, $S_{35} = S_{35]}$, $S_{46} = S_{[46}$, $S_{62} = S_{62]}$ *and* $S_{64} = S_{64]}$.

$i \backslash j$	1	2	3	4	5	6
1	$\checkmark$	$\checkmark$	$\color{orange}\checkmark$	$\color{orange}\checkmark$	$\checkmark$	$\checkmark$
2	$\emptyset$	$\checkmark$	$\checkmark$	$\checkmark$	$\color{orange}\checkmark$	$\color{orange}\checkmark$
3	$\emptyset$	$\emptyset$	$\checkmark$	$\checkmark$	$\color{orange}\checkmark$	$\checkmark$
4	$\emptyset$	$\emptyset$	$\emptyset$	$\checkmark$	$\checkmark$	$\color{orange}\checkmark$
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\checkmark$	$\checkmark$
6	$\checkmark$	$\color{orange}\checkmark$	$\checkmark$	$\color{orange}\checkmark$	$\checkmark$	$\checkmark$

Figure 2.10 – The orange checkmark is for those sets S_{ij} complete to either K_i or K_j.

2.3 Partitions of S and K for a graph containing an induced co-4-tent

Let $G = (K, S)$ be a split graph where K is a clique and S is an independent set, and suppose that G contains no induced tent or 4-tent. Let T be a co-4-tent induced subgraph of G. Let $V(T) = \{k_1, k_3, k_5, s_{13}, s_{35}, s_1, s_5\}$ where $k_1, k_3, k_5 \in K$, s_{13}, s_{35}, s_1, s_5 in S such that the neighbors of s_{ij} in T are precisely k_i and k_j and the neighbor of s_i in T is precisely k_i.

We introduce sets $K_1, K_2, \ldots, K_{15}$ as follows.

- — Let K_1 be the set of vertices of K whose only neighbors in $V(T) \cap S$ are s_1 and s_{13}. Analogously, let K_5 be the set of vertices of K whose only neighbors in $V(T) \cap S$ are s_5 and s_{35}, and let K_3 be the set of vertices of K whose only neighbors in $V(T) \cap S$ are s_{13} and s_{35}. Let K_{13} be the set of vertices of K whose only neighbors in $V(T) \cap S$ are s_1 and s_5, K_{14} be the set of vertices of K whose only neighbors in $V(T) \cap S$ are s_{13} and s_5 and K_{15} be the set of vertices of K whose only neighbors in $V(T) \cap S$ are s_1 and s_{35}.
- — Let K_2 be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely s_1, s_{13} and s_{35}, and let K_4 be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely s_5, s_{13} and s_{35}. Let K_9 be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely s_1, s_{13} and s_5, and let K_{10} be the set of vertices of K whose neighbors in $V(T) \cap S$ are precisely s_1, s_{35} and s_5.
- — Let K_6 be the set of vertices of K whose only neighbor in $V(T) \cap S$ is precisely s_{35}, and let K_8 be the set of vertices of K whose only neighbor in $V(T) \cap S$ is precisely s_{13}. Let K_{11} be the set of vertices of K whose only neighbor in $V(T) \cap S$ is precisely s_1, and let K_{12} be the set of vertices of K whose only neighbor in $V(T) \cap S$ is precisely s_5.
- — Let K_7 be the set of vertices of K that are anticomplete to $V(T) \cap S$.

***Remark** 2.17.* If $K_{13} \neq \emptyset$, then there is an induced 4-sun in G. If $K_{14} \neq \emptyset$, $K_{15} \neq \emptyset$, $K_9 \neq \emptyset$ or $K_{10} \neq \emptyset$, then we find an induced tent. Moreover, if $K_{11} \neq \emptyset$ or $K_{12} \neq \emptyset$, then we find an induced

4-tent in G. Hence, in virtue of the previous chapters, we will assume that $K_9, \ldots, K_{15}$ are empty sets.

The following Lemma is straightforward.

Lemma 2.18. *Let* $G = (K, S))$ *be a split graph that contains no induced tent or 4-tent. If* G *is a circle graph, then* $\{K_1, K_2, \ldots, K_8\}$ *is a partition of* K.

Proof. Every vertex in K is adjacent to precisely one, two, three or no vertices of $V(T) \cap S$, for if it is adjacent to every vertex in $V(T) \cap S$, then we find a co-4-tent $\vee K_1$. Moreover, by the previous remark, the only possibilities are the sets $K_1, \ldots, K_8$. $\square$

Let $i, j \in \{1, \ldots, 8\}$ and let S_{ij} defined as in the previous sections.

Remark 2.19. If $K_4 = \emptyset$, then there is a split decomposition of G. Let us consider the subset K_5 on one hand, and on the other hand a vertex $u \notin G$ such that u is complete to K_5 and is anti-complete to $V(G) \setminus K_5$. Let G_1 and G_2 be the subgraphs induced by the vertex subsets $V_1 = V(G) \setminus S_{55}$ and $V_2 = \{u\} \cup K_5 \cup S_{55}$, respectively. Hence, G is the result of the split composition of G_1 and G_2 with respect to K_5 and u. The same holds if $K_2 = \emptyset$ considering the subgraphs induced by the vertex subsets $V_1 = V(G) \setminus S_{11}$ and $V_2 = \{u\} \cup K_1 \cup S_{11}$, where in this case u is complete to K_1 and is anti-complete to $V(G) \setminus K_1$.

If we consider H a minimally non-circle graph, then H is a prime graph, for if not one of the factors should be non-circle and thus H would not be minimally non-circle. Hence, in order characterize those circle graphs that contain an induced co-4-tent, we will assume without loss of generality that G is a prime graph, and therefore $K_2 \neq \emptyset$ and $K_4 \neq \emptyset$.

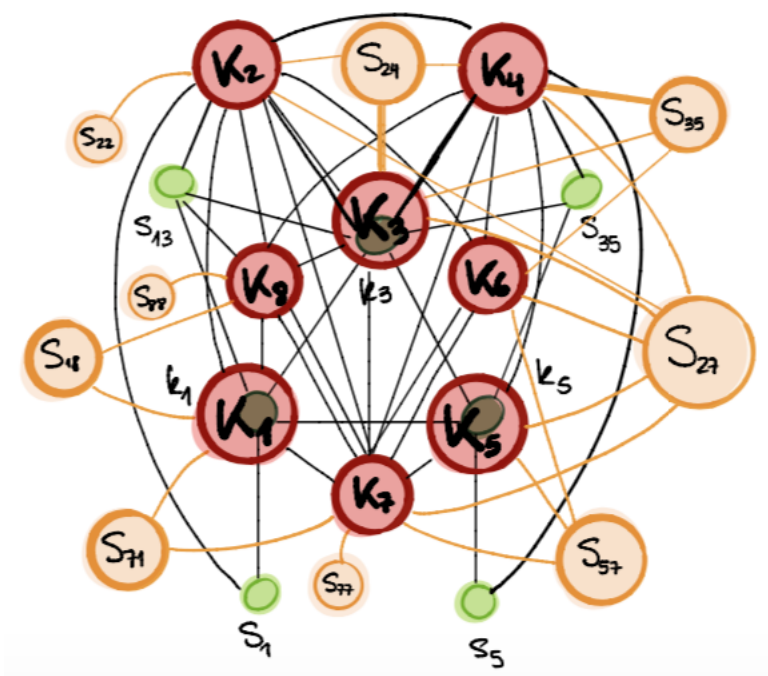


Figure 2.11 – The partition of K and some of the subsets of S according to the adjacencies with T.

Claim 2.20. *If v in S fullfils one of the following conditions:*
— *v is adjacent to K_1 and K_5 and is not complete to K_3 (resp. K_2 or K_4)*
— *v is adjacent to K_1 and K_4 and is not complete to K_2*
— *v is adjacent to K_2 and K_5 and is not complete to K_4*
— *v is adjacent to K_i and K_{i+2} and is not complete to K_{i+1}, for $i = 1, 3$*
— *v is adjacent to K_2 and K_4 and is not complete to K_1 and K_5*
— *v is adjacent to K_1, K_3 and K_5 and is not complete to K_2 and K_4*
Then, G contains either a tent or a 4-tent.

Proof. If v is adjacent to K_1 and K_5 and is not complete to K_3, then we find a tent induced by $\{s_{13}, s_{35}, v, k_1, k_3, k_5\}$. If instead it is not complete to K_2, then we find a tent induced by $\{s_1, s_{35}, v, k_1, k_2, k_5\}$. It is analogous by symmetry if v is not complete to K_4. If v is adjacent to K_1 and K_4 and is not complete to K_2, then the tent is induced by $\{s_1, s_{35}, v, k_1, k_2, k_4\}$. It is analogous by symmetry if v is adjacent to K_2 and K_5 and is not complete to K_4. If v is adjacent to K_1 and K_3 and is not complete to K_2, then we find a tent induced by $\{s_1, s_{35}, v, k_1, k_2, k_3\}$. It is analogous by symmetry if $i = 3$. If v is adjacent to K_2 and K_4 and is not complete to K_1 and K_5, then we find a 4-tent induced by the set $\{s_1, s_5, v, k_1, k_2, k_4, k_5\}$. Finally, if v is adjacent to K_1, K_3 and K_5 and is not complete to K_2 and K_4, then there are tents induced by the sets $\{s_{13}, s_5, v, k_3, k_4, k_5\}$ and $\{s_{35}, s_1, v, k_1, k_2, k_3\}$. ☐

It follows from the previous claim that the following subsets are empty: S_{51}, S_{52}, S_{53}, S_{41}, S_{31}, S_{24}.

Moreover, the following subsets coincide: $S_{54} = S_{54]}$, $S_{42} = S_{42]}$, $S_{43} = S_{[43]}$, $S_{32} = S_{32]}$.

Claim 2.21. *If there is a vertex v in S such that v belongs to either S_{61}, S_{71}, S_{81}, S_{56}, S_{57}, S_{58}, S_{67}, S_{68} or S_{78}, then there is an induced tent or a 4-tent in G.*

Proof. If v in S_{61}, then we find a tent induced by the set $\{s_1, s_{35}, v, k_1, k_2, k_6\}$. If v in S_{71}, then we find a 4-tent induced by the set $\{s_1, s_{35}, v, k_7, k_1, k_2, k_3\}$. If v in S_{81}, then we find a 4-tent induced by the set $\{s_1, s_{35}, v, k_8, k_1, k_2, k_4\}$. If v in S_{56}, then we find a 4-tent induced by the set $\{s_{13}, s_5, v, k_3, k_4, k_5, k_6\}$. It is analogous for v in S_{57}, swaping k_6 for k_7. If v in S_{58}, then we find a 4-tent induced by the set $\{s_{35}, s_1, v, k_1, k_2, k_5, k_8\}$. If v in S_{67}, then we find a 4-tent induced by the set $\{s_{13}, s_{35}, v, k_1, k_2, k_6, k_7\}$. If v in S_{68}, then we find a 4-tent induced by the set $\{s_{13}, s_{35}, v, k_1, k_5, k_6, k_7\}$. Finally, If v in S_{78}, then we find a 4-tent induced by the set $\{s_{13}, s_5, v, k_4, k_5, k_7, k_8\}$. ☐

As a direct consequence of the previous claims, we will assume without loss of generality that the following subsets are empty: S_{5i} for $i = 1, 2, 3, 6, 7, 8$, S_{4i} for $i = 1, 2$, S_{6i} for $i = 1, 7, 8$, S_{7i} for $i = 1, 8$, S_{81}, S_{31} and S_{24}.

Claim 2.22. *If G is a circle graph that contains no induced tent or 4-tent, then $S_{64} = \emptyset$, $S_{54} = S_{54]}$ and $S_{65} = S_{65]}$. Moreover, if $K_6 \neq \emptyset$, then $S_{54} = S_{[54]}$.*

Proof. Let v in S_{64}, k_i in K_i for $i = 1, 4, 5, 6$ such that v is adjacent to k_1, k_4 and k_6 and is nonadjacent to k_5. Hence, we find $M_{II}(4)$ induced by the vertex set $\{k_1, k_4, k_5, k_6, v, s_5, s_{13}, s_{35}\}$. Hence, $S_{64} = \emptyset$. Notice that this also implies that, if $K_6 \neq \emptyset$, then every vertex in S_{54} or S_{65} is complete to K_5. Suppose now that v lies in S_{54} and is not complete to K_4. Thus, there is a vertex k_4 in K_4 such that v is adjacent to k_4. Let k_1 in K_1 and k_5 in K_5 such that k_1 and k_5 are adjacent to v. Hence, we find a tent induced by $\{k_1, k_4, k_5, v, s_{13}, s_{35}\}$. ☐

As a consequence of the previous claim, we will assume througout the proof that $S_{54} = \emptyset$. This follows from the fact that the vertices in S_{54} are exactly those vertices in S_{76} that are complete to K_6 and K_7, since the endpoints of both subsets coincide. The same holds for the vertices in S_{65}, which are those vertices in S_{76} that are complete to K_7.

We want to prove that $\{S_{ij}\}$ is indeed a partition of S. Towards this purpose, we need the following claims.

Claim 2.23. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+2} *and anticomplete to* K_j *for* $j < i$ *and* $j > i+2$*, then* v *is complete to* K_{i+1}.

Proof. Once discarded the subsets of S that induce a tent or 4-tent and those that are empty, the remaining cases are $i = 4, 8$.

Let v in S adjacent to k_4 in K_4 and k_6 in K_6, and suppose there is a vertex k_5 in K_5 nonadjacent to v. Then, we find a net $\vee K_1$ induced by $\{k_6, k_4, k_5, k_3, v, s_5, s_{13}\}$.

If instead v in S is adjacent to k_8 in K_8 and k_2 in K_2 and is nonadjacent to some k_1 in K_1, then we find a net $\vee K_1$ induced by $\{k_8, k_1, k_5, k_2, v, s_1, s_{35}\}$. □

Claim 2.24. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+3} *and anticomplete to* K_j *for* $j < i$ *and* $j > i+3$*, then:*

— *If* $i \equiv 0 \pmod 3$*, then* v *lies in* S_{36}.
— *If* $i \equiv 1 \pmod 3$*, then* v *lies in* $S_{[14}$.
— *If* $i \equiv 2 \pmod 3$*, then* v *lies in* $S_{25]}$ *or* S_{83}.

Proof. Suppose first that $i \equiv 0 \pmod 3$. Equivalently, either $i = 3$ or $i = 6$. Let v in S such that v is adjacent to some vertices k_3 in K_3 and k_6 in K_6. If there is a vertex k_4 in K_4 nonadjacent to v and a vertex k_5 in K_5 adjacent to v, then we find a tent induced by $\{k_3, k_4, k_5, v, s_{13}, s_5\}$. If instead k_5 is nonadjacent to v, then we find a 4-tent induced by $\{k_3, k_6, k_4, k_5, v, s_5, s_{13}\}$. If instead v is nonadjacent to k_4 and is adjacent to k_5, then we find a 4-tent induced by $\{k_1, k_4, k_5, k_6, v, s_5, s_{13}\}$. Hence, v is complete to K_4 and K_5. If $i = 6$, since v is anticomplete to K_2, then we find a tent induced by $\{k_6, k_1, k_1, v, s_1, s_{35}\}$.

Let us prove the second statement. If $i \equiv 1 \pmod 3$, then either $i = 1$, $i = 4$ or $i = 7$. First, we need to see that v is complete to K_{i+1} and K_{i+2}. If $i = 4, 7$, then $K_7 \neq \emptyset$. If $i = 4$, then there are vertices $k_4 \in K_4$ and $k_7 \in K_7$ adjacent to v. Suppose that v is nonadjacent to some vertex in K_5. Then, we find a net $\vee K_1$ induced by $\{k_3, k_4, k_5, k_7, v, s_5, s_{13}\}$. If instead there is a vertex $k_6 \in K_6$ nonadjacent to v, then there is a net $\vee K_1$ induced by $\{k_1, k_4, k_6, k_7, v, s_{35}, s_{13}\}$. It is analogous by symmetry if $i = 7$. However, by Claim 2.22, S_{47} and S_{72} are empty sets. Suppose now that $i = 1$, let k_1 in K_1 and k_4 in K_4 be vertices adjacent to v and k_3 in K_3 nonadjacent to v. Then, we find $M_{II}(4)$ induced by $\{k_1, k_4, k_5, k_3, v, s_{35}, s_{13}, s_5\}$. It is analogous if v is nonadjacent to some vertex in K_2. Notice that, if v is not complete to K_1, we find a 4-tent induced by $\{k_1, k_1', k_4, k_5, v, s_5, s_1\}$.

Finally, suppose that $i \equiv 2 \pmod 3$. Hence, either $i = 2, 5, 8$. Suppose $i = 2$. Let k_2 in K_2 and k_5 in K_5 be vertices adjacent to v, and let k_3 in K_3 and k_4 in K_4. If k_4 is nonadjacent to v, then we find a 4-tent induced by $\{k_1, k_2, k_4, k_5, v, s_5, s_1\}$. Hence, v is complete to K_4. If instead v is nonadjacent to k_3, then we find $M_{II}(2)$ induced by $\{k_1, k_2, k_3, k_5, v, s_1, s_{35}, s_{13}\}$ and therefore v lies in S_{25}.

Suppose now that $i = 5$. Notice that, in this case, there is no vertex v adjacent to K_5 and K_8 such that v is anticomplete to $K_1, \dots, K_4$, since in that case we find a tent induced by $\{k_5, k_8, k_4, v, s_5, s_{13}\}$. Hence, we discard this case.

Suppose that $i = 8$. Let k_3 in K_3 and k_8 in K_8 adjacent to v, and let k_1 in K_1 and k_2 in K_2. If both k_1 and k_2 are nonadjacent to v, then we find a 4-tent induced by $\{k_8, k_1, k_2, k_3, v, s_1, s_{35}\}$. Hence, either v is complete to K_1 or K_2. If k_1 is nonadjacent to v, then we find a net $\vee K_1$ induced by $\{k_8, k_1, k_2, k_5, v, s_1, s_{35}\}$. If instead k_2 is nonadjacent to v, then we find a tent induced by $\{k_1, k_2, k_5, v, s_1, s_{35}\}$, and therefore v lies in S_{83}. □

Claim 2.25. *If* G *is a circle graph and* v *in* S *is adjacent to* K_i *and* K_{i+4} *and anticomplete to* K_j *for* $j < i$ *and* $j > i + 4$, *then either* v *lies in* S_{15} *(or* S_{51} *if* $K_6, K_7, K_8 = \emptyset$*), or* S_{26} *or* S_{84}.

Proof. Notice that, if v is adjacent to k_3 in K_3 and k_7 in K_7 and nonadjacent to k_2 in K_2, then v is complete to K_1 for if not we find 4-tent induced by $\{k_7, k_1, k_2, k_3, v, s_1, s_{35}\}$. However, if k_1 in K_1 is adjacent to v, then we find a tent induced by $\{k_1, k_2, k_3, v, s_1, s_{35}\}$. Hence, we discard this case. Suppose $i = 4$. If v is adjacent to k_4 in K_4 and k_8 in K_8 and is nonadjacent to k_3 in K_3 and k_5 in K_5, then we find $M_{II}(4)$ induced by $\{k_8, k_3, k_4, k_5, v, s_5, s_{13}, s_{35}\}$. However, if k_5 is adjacent to v, then we find a tent induced by $\{k_3, k_5, k_8, v, s_{13}, s_{35}\}$. Suppose $i = 5$. Let k_5 in K_5 and k_1 in K_1 are adjacent to v and let k_4 in K_4 nonadjacent to v. Thus, we find a tent induced by $\{k_1, k_4, k_5, v, s_5, s_{13}\}$. Suppose $i = 6$. If k_6 in K_6 and k_2 in K_2 are adjacent to v, and k_4 in K_4 and k_5 in K_5 are nonadjacent to v, then we find a 4-tent induced by $\{k_4, k_5, k_6, k_2, v, s_5, s_{13}\}$. Suppose $i = 7$. Let k_7 in K_7 and k_3 in K_3 adjacent to v, and k_4 in K_4 and k_5 in K_5 nonadjacent to v. Thus, we find a 4-tent induced by $\{k_7, k_3, k_4, k_5, v, s_5, s_{13}\}$. Suppose $i = 8$. Let k_8 in K_8 and k_4 in K_4 adjacent to v and let k_j in K_j for $j = 1, 2, 3$. If k_j is nonadjacent to v, then we find $M_{II}(4)$ induced by $\{k_5, k_4, k_8, k_j, v, s_5, s_{13}, s_{35}\}$, for each $j = 1, 2, 3$. Hence, v lies in S_{84}. Suppose $i = 1$. Let k_1 in K_1 and k_5 in K_5 be adjacent to v, and k_j in K_j for $i = 2, 3, 4$. If v is nonadjacent to k_j, then we find a tent induced by $\{k_1, k_3, k_5, v, s_{35}, s_{13}\}$. Hence, if $K_6, K_7, K_8 = \emptyset$, then v lies in S_{15} or S_{51}, and if $K_j \neq \emptyset$ for any $j = 6, 7, 8$, then v lies in S_{15}. Finally, suppose $i = 2$. Let k_2 in K_2 and k_6 in K_6 adjacent to v, and let k_j in K_j for $j = 3, 4, 5$. If v is nonadjacent to both k_4 and k_5, then we find a 4-tent induced by $\{k_2, k_4, k_5, k_6, v, s_5, s_{13}\}$. Thus, either v is complete to K_4 or K_5. If v is complete to K_5 and not complete to K_4, then we find a tent induced by $\{k_2, k_4, k_5, v, s_5, s_{13}\}$. If instead v is complete to K_4 and not complete to K_5, then we find a net $\vee K_1$ induced by $\{k_1, k_6, k_4, k_5, v, s_5, s_{13}\}$. If k_3 is nonadjacent to v, then we find $M_{II}(k)$ induced by $\{k_1, k_3, k_2, k_6, v, s_1, s_{35}, s_{13}\}$. Hence, v lies in S_{26}. □

Claim 2.26. *If* G *is a circle graph, then the sets* S_{i1} *for* $i = 2, 3, 4$, S_{ij} *for* $j = 2, 3, 4, 5$ *and* $i = j + 1, \ldots, 7$, S_{i7} *for* $i = 3, 4$ *and* S_{i8} *for* $i = 2, 3, 4$ *are empty, unless* v *in* $S_{[32]}$ *or* $S_{21} = S_{[21]}$.

Proof. Let v in S_{i1} for $i = 2, 3, 4$. If $i = 2$ and v is not complete to every vertex in K, then there is either a vertex k_1 in K_1 or a vertex in k_2 in K_2 that are nonadjacent to v. Suppose there is such a vertex k_2, and let k_1' in K_1 adjacent to v. Thus, we find a tent induced by $\{v, s_1, s_{35}, k_1', k_2, k_3\}$. Similarly, we find a tent if there is a vertex in K_1 nonadjacent to v. If $i = 3$, then we find a tent induced by $\{v, s_{13}, s_{35}, k_3', k_5, k_3\}$ where $k_3, k_3' \in K_3$, k_3 is adjacent to v and k_3' is nonadjacent to v. Similarly, we find a tent if $i = 4$ considering two analogous vertices k_4 and k_4' in K_4.

Let v in S_{i2} for $i = 3, 4, 5, 6, 7$ and let us assume in the case where $i = 3$ that v is not complete to every vertex in K. Thus, there is a vertex k_3(or maybe a vertex k_2 in K_2 if $i = 3$, which is indistinct to this proof) in K_3 that is nonadjacent to v. If $i = 3, 4, 5, 6$, then there are vertices k_1 in K_1 and k_5 in K_5 (resp. k_6 in K_6 if $i = 6$) such that k_1 and k_5 (resp. k_6) are adjacent to v. Hence, we find a tent induced by $\{v, s_{13}, s_{35}, k_1, k_3, k_5(k_6)\}$. If instead $i = 7$, then we find a 4-tent induced by $\{v, s_{13}, s_{35}, k_1, k_3, k_5, k_7\}$, where k_l in K_l is adjacent to v for $l = 1, 7$ and k_n in K_n is nonadjacent to v for $n = 3, 5$.

Let v in S_{i7} for $i = 3, 4$. In either case, there are vertices k_1 in K_1 and k_2 in K_2 nonadjacent to v, and vertices k_l in K_l for $l = 4, 5, 7$ adjacent to v. Thus, we find F_0 induced by $\{v, s_{13}, s_{35}, k_1, k_2, k_4, k_5, k_7\}$.

Finally, let v in S_{i8} for $i = 2, 3, 4$. Suppose first that $i = 3, 4$ or that v is not complete to K_2, thus there is a vertex k_2 in K_2 nonadjacent to v. In that case, we find a tent induced by $\{v, s_{13}, s_{35}, k_2, k_5, k_8\}$. If instead v in S_{28} and is complete to K_2, then we find $M_{II}(4)$ induced by $\{v, s_1, s_{13}, s_{35}, k_1, k_2, k_5, k_8\}$. □

***Remark* 2.27.** It follows from the previous proof that $S_{32} = S_{[32]}$ and $S_{21} = S_{[21]}$.

Claim 2.28. *If G is a circle graph, $K_6 \neq \emptyset$ and $K_8 \neq \emptyset$, then $S_{15} = \emptyset$. Moreover, if $K_8 = \emptyset$, then $S_{15} = S_{[15}$, and if $K_6 = \emptyset$, then $S_{15} = S_{15]}$.*

Proof. Let v in S_{15}, and k_6 in K_6 and k_8 in K_8 be vertices nonadjacent to v. Since there are vertices k_1 in K_1, k_2 in K_2 and k_5 in K_5 adjacent to v, then we find F_0 induced by $\{v, s_{13}, s_{35}, k_8, k_1, k_2, k_5, k_6\}$.

The proof is analogous if $K_8 = \emptyset$ (resp. if $K_6 = \emptyset$) considering two vertices k_{11} and k_{12} in K_1 (k_{51}, k_{52} in K_5) such that v is adjacent to k_{11} (resp. k_{51}) and is nonadjacent to k_{22} (resp. k_{52}). □

Claim 2.29. *Let v in S_{ij} such that v is adjacent to at least one vertex in each nonempty K_l, for every $l \in \{1, \ldots, 8\}$. If G is a circle graph, then the following statements hold:*

- *The vertex v is complete to K_2, K_3 and K_4.*
- *If $K_j \neq \emptyset$ for some $j = 6, 8$, then then v is complete to K_5. Moreover, v is either complete to K_i or K_j.*

Proof. The first statement follows as a direct consequence of Claim 2.20: if v is adjacent to K_1, K_3 and K_5, then v is complete to K_2 and K_4. Moreover, v is complete to K_3.

To prove the second statement, suppose first that $K_6 \neq \emptyset$ and $K_7, K_8 = \emptyset$. Let us see that v is complete to K_5. Suppose there is a vertex k_5 in K_5 such that v is nonadjacent to k_5, and let k_i in K_i adjacent to v for each $i = 1, 4, 6$. We find $M_{II}(4)$ induced by $\{v, s_{13}, s_{35}, s_5, k_1, k_4, k_5, k_6\}$. The proof is analogous if $K_8 \neq \emptyset$ and $K_7, K_6 = \emptyset$.

Let us suppose now that v is not complete to K_1 and K_6. We find F_0 induced by $\{v, s_{13}, s_{35}, k_{11}, k_{12}, k_3, k_{61}, k_{62}\}$, where k_{1j} in K_1, k_{6j} in K_6 for each $j = 1, 2$ and v is adjacent to k_{i1} and is nonadjacent to k_{i2} for each $i = 1, 6$. The proof is analogous if $K_8 \neq \emptyset$ and $K_7, K_6 = \emptyset$ and if $K_6, K_8 \neq \emptyset$, independently on whether $K_7 = \emptyset$ or not.

Let us suppose now that $K_6, K_8 = \emptyset$. If $K_7 = \emptyset$, then

□

By simplicity, we will also consider that every vertex in $S_{[32]}$ and $S_{[21]}$ lies in $S_{76]}$, and that in particular, if $K_7 \neq \emptyset$, then such vertices are complete to K_7. This follows from Claim 2.29 and Remark 2.27. As a consequence of Claims 2.20 to 2.29, we have the following Lemma.

Lemma 2.30. *Let $G = (K, S)$ be a split graph that contains no induced tent or 4-tent. If G is a circle graph, then all the following assertions hold:*

- *$\{S_{ij}\}_{i,j \in \{1,2,\ldots,8\}}$ is a partition of S.*
- *For each $i \in \{2, 3, 4, 5, 6, 7, 8\}$, S_{i1} is empty.*
- *For each $i \in \{3, 4, 5, 6, 7\}$, S_{i2} is empty.*
- *For each $i \in \{4, 5, 6, 7\}$, S_{i3} is empty, and S_{56} is also empty.*
- *For each $i \in \{3, 4, 5, 6\}$, S_{i7} is empty.*

— *For each* $i \in \{2,3,4,5,6,7\}$, S_{i8} *is empty.*
— *The subsets* S_{64}, S_{54} *and* S_{56} *are empty.*
— *The following subsets coincide:* $S_{1i} = S_{[1i}$ *for* $i = 3,4,8$; $S_{16} = S_{16]}$, $S_{25} = S_{25]}$, $S_{27} = S_{[27}$, $S_{35} = S_{35]}$, $S_{46} = S_{[46}$, $S_{82} = S_{82]}$ *and* $S_{85} = S_{[85}$ *(as the case may be, according to whether* $K_i \neq \emptyset$ *or not, for* $i = 6,7,8$*).*

Since $S_{18} = S_{[18}$, we will consider these vertices as those in S_{87} that are complete to K_7 and $S_{18} = \emptyset$. Moreover, those vertices that are complete to $K_1, \ldots, K_6, K_8$ and are adjacent to K_7 will be considered as in $S_{76]}$, thus S_{87} is the set of independent vertices that are complete to $K_1, \ldots, K_7$ and are adjacent but not complete to K_8. Therefore, in this case we have the following table:

$i \setminus j$	1	2	3	4	5	6	7	8
1	✓	✓	✓	✓	$\emptyset$	✓	✓	$\emptyset$
2	$\emptyset$	✓	✓	$\emptyset$	✓	✓	✓	$\emptyset$
3	$\emptyset$	$\emptyset$	✓	✓	✓	✓	$\emptyset$	$\emptyset$
4	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓	$\emptyset$	$\emptyset$
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$	$\emptyset$	$\emptyset$
6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$	$\emptyset$
7	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓	✓	$\emptyset$
8	$\emptyset$	✓	✓	✓	✓	✓	✓	✓

Figure 2.12 – The orange checkmarks denote those subsets S_{ij} that are either complete to K_i or K_j.

Chapter 3

2-nested matrices

In this chapter, we will define and characterize nested and 2-nested matrices, which are of fundamental importance to describe each portion of a circle model for those split graphs that are also circle. The results in this chapter are crucial for the proof of the main result in the next chapter, which gives a complete characterization of split circle graphs by minimal forbidden induced subgraphs.

In order to give some motivation for the definitions on this chapter, let us consider the split graph $G = (K, S)$ represented in Figure 3.1. Since G contains an induced tent T, we can consider the partitions $K_1, K_2 \dots, K_6$ and $\{S_{ij}\}_{1 \leq i,j \leq 6}$ as defined in Section 2.1.

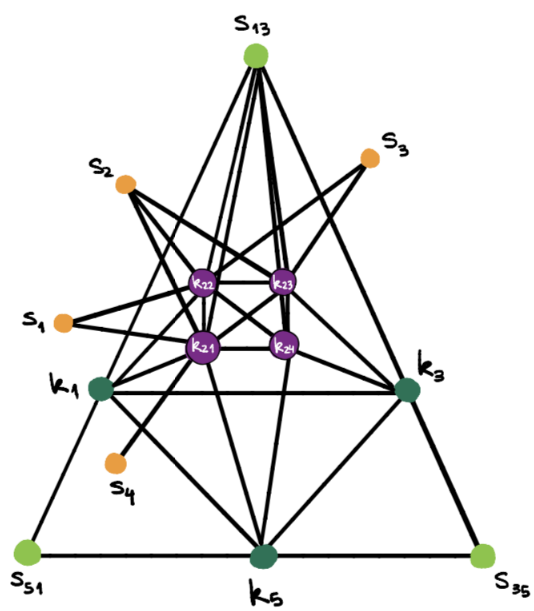


Figure 3.1 – Example 1: a split circle graph G.

Notice that every vertex in the complete partition of $G \setminus T$ lies in K_2, for the only adjacency of these vertices with regard to S is the vertex s_{13}. Thus, $K_2 = \{k_{21}, k_{22}, k_{23}, k_{24}\}$. Moreover, the orange vertices are the only independent vertices in $G \setminus T$ and these vertices are adjacent only to vertices in K_2, thus they all lie in S_{22}. Furthermore, the graph G is also a circle graph. Indeed, we would like to give a circle model for G. The tent is a prime graph, and as such, T admits a unique circle model. Hence, let us begin by considering a circle model as the one presented in Figure 3.2, having only the chords that represent the subgraph T. We will consider the arcs and chords of a model described clockwise. For example, in Figure 3.2 the arc k_1k_3 is the portion of the circle that lies between k_1 and k_3 when traversing the circumference clockwise.

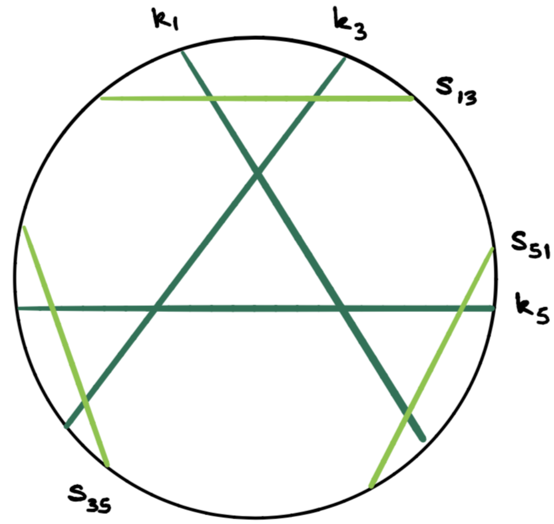


Figure 3.2 – A circle model for the tent graph T.

To place the chords corresponding to each vertex in S_{22}, first we need to place the chords that represent every vertex in K_2. This follows from the fact that a chord representing a vertex in K_2 has one endpoint between the arc k_1k_3 and the other endpoint between the arc $s_{51}s_{35}$, and a chord representing a vertex in S_{22} has either both endpoints inside the arc k_1k_3 or both endpoints inside the arc $s_{51}s_{35}$, always intersecting chords representing vertices in K_2. Thus, in order to place the chords corresponding to each vertex of K_2, we need to establish an ordering of the vertices in K_2 that respects the partial ordering relationship given by the neighbourhoods of the vertices in S_{22}. For example, since $N(s_1) \subseteq N(s_2)$, it follows that an ordering of the chords in K_2 that allows us to give a circle model must contain one of the following subsequences: (k_{21}, k_{22}, k_{23}) or (k_{22}, k_{21}, k_{23}) or (k_{23}, k_{21}, k_{22}) or (k_{23}, k_{22}, k_{21}). Moreover, since $N(s_2) \cap N(s_3) \neq \emptyset$ and $N(s_2)$ and $N(s_3)$ are not nested, then the chords corresponding to s_2 and s_3 must be drawn in distinct portions of the circle model, for they represent independent vertices and thus the chords cannot intersect. The vertex s_4 is adjacent only to k_{21}, thus $N(s_4)$ is contained in both $N(s_1)$ and $N(s_2)$ and is disjoint with $N(s_3)$. Hence, the chord that represents s_4 may be placed indistinctly in any of the two portions of the circle corresponding to the partition S_{22}.

Therefore, when considering the placement of the chords, we find ourselves in front of two important decisions: (1) in which order should we place the chords corresponding to the vertices in K_2 so that we can draw the chords of those independent vertices adjacent to K_2, and (2) in which portion of the circle model should we place both endpoints of the chords corresponding to vertices in S_{22}. We give a circle model for G in Figure 3.3

Yet in this small example of a split graph that is circle, it becomes evident that there is a property that must hold for every pair of independent vertices that have both of its endpoints

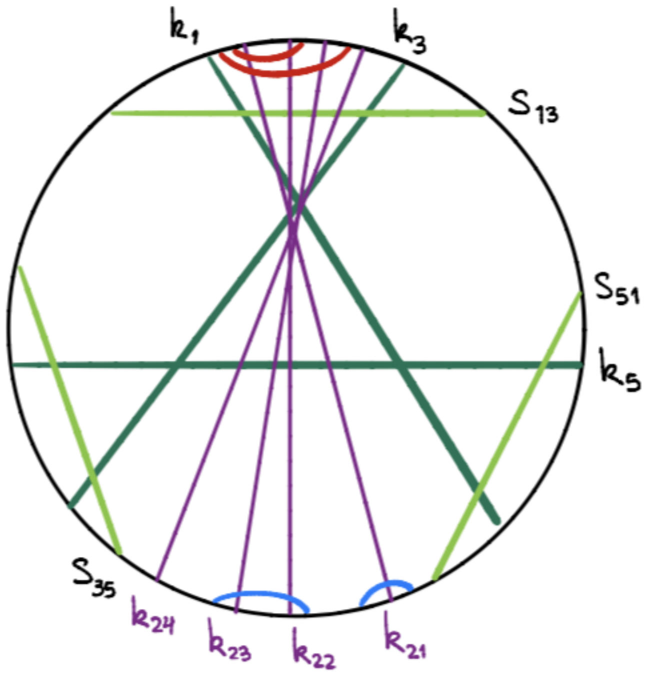


Figure 3.3 – A circle model for the split graph G.

placed within the same arc of the circumference. This led to the definition of nested matrices, which was the first step in order to translate some of these problems to having certain properties in the adjacency matrix $A(S, K)$ (See Section 1.2 for more details on the definition of $A(S, K)$).

Definition 3.1. *Let* A *be a* $(0, 1)$*-matrix. We say* A *is* nested *if there is a consecutive-ones ordering for the rows and every two rows are disjoint or nested.*

Definition 3.2. *A split graph* $G = (K, S)$ *is* nested *if and only if* $A(S, K)$ *is a nested matrix.*

Theorem 3.3. *A* $(0, 1)$*-matrix is nested if and only if it contains no* 0*-gem as a submatrix (See Figure 3.4).*

Proof. Since no Tucker matrix has the C1P and the rows of the 0-gem are neither disjoint nor nested, no nested matrix contains a Tucker matrix or a 0-gem as submatrices. Conversely, as each Tucker matrix contains a 0-gem as a submatrix, every matrix containing no 0-gem as a submatrix is a nested matrix. □

$$\begin{pmatrix} 110 \\ 011 \end{pmatrix}$$

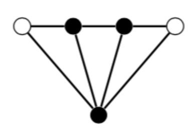


Figure 3.4 – The 0-gem matrix and the associated gem graph.

Let us consider the matrix $A(S, K_2)$ that corresponds to the example given in Figure 3.1, where the rows are given by s_1, s_2, s_3 and s_4, and the columns are k_{21}, k_{22}, k_{23} and k_{24}.

$$A(S, K_2) = \begin{pmatrix} 1100 \\ 1110 \\ 0110 \\ 1000 \end{pmatrix}$$

Notice that the existance of a C1P for the columns of the matrix $A(S, K_2)$ is a necessary condition to find an ordering of the vertices in K_2 that is compatible with the partial ordering given by containment for the vertices in S_{22}. Moreover, if the matrix $A(S, K_2)$ is nested, then any two independent vertices are either nested or disjoint. In other words, if $A(S, K_2)$ is nested, then we can draw every chord corresponding to an independent vertex in $G \setminus T$ in the same arc of the circumference. However, this is not the case in the previous example, for the vertices s_1 and s_3 are neither disjoint nor nested, and thus they cannot be drawn in the same portion of the circle model. Hence, $A(S, K)$ is not a nested matrix, and thus the notion of nested matrix is not enough to determine whether there is a circle model for a given split graph or not.

Let us see one more example. Consider H to be the split graph presented in Figure 3.5. Notice that this graph is equal to G plus three new independent vertices.

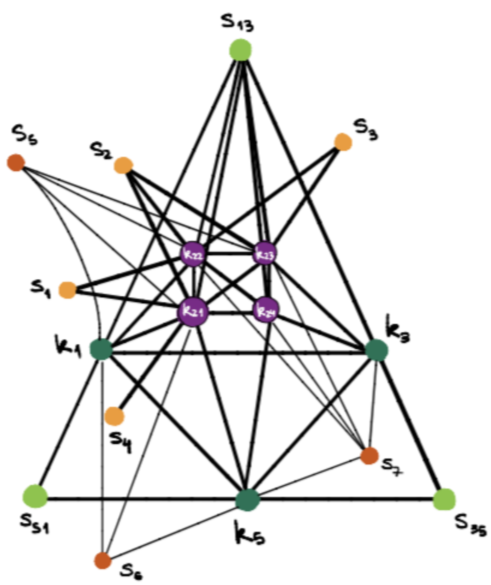


Figure 3.5 – Example 2: the split circle graph H.

Moreover, unlike s_1, s_2, s_3 and s_4, the chords that represent these new independent vertices s_5, s_6 and s_7 have only one of its endpoints in the arcs corresponding to the area of the circle designated for K_2, this is, in the arcs k_1k_3 and $s_{51}s_{35}$. Furthermore, each of these new vertices has a unique possible placement for each endpoint of their corresponding chord. If we consider the rows given by the vertices $s_1, \ldots, s_7$ and the columns given by $k_{21}, \ldots, k_{24}$, then the adjacency matrix $A(S, K_2)$ in this example is as follows:

$$A(S, K_2) = \begin{pmatrix} 1100 \\ 1110 \\ 0110 \\ 1000 \\ 1110 \\ 1000 \\ 0111 \end{pmatrix}$$

As in the previous example, $A(S, K_2)$ is not a nested matrix. Furthermore, notice that in this case not every adjacency of each independent vertex $s_1, \dots, s_7$ is depicted in this matrix, since s_5, s_6 and s_7 all are adjacent to at least one vertex in $K \setminus K_2$.

Let us concentrate in the placement of the endpoints of the chords representing s_5, s_6 and s_7 that lie between the arcs k_1k_3 and $s_{51}s_{35}$. Notice that the "nested or disjoint" property must still hold, and not only for those vertices in K_2. More precisely, since s_5 is adjacent to k_{24}, k_{23} and k_1 and s_1 is nonadjacent to k_1 and adjacent to k_{23} and k_{24}, then necessarily s_1 must be contained in s_5. Something similar occurs with s_7 and s_3, whereas s_6 and s_3 are disjoint.

There is one situation in this example that did not occur in the previous one. Since s_6 is adjacent to k_{21}, k_1 and k_5, then the chord corresponding to the vertex k_{21} is forced to be placed first within every chord corresponding to K_2. This follows from the fact that a chord that represents s_6 has a unique possible placement inside the arc $s_{51}s_{35}$, for we need k_{21} to be the first chord of K_2 that comes right after s_{51}. Moreover, this is confirmed by the fact that s_5 is adjacent to k_1 and k_{21}, thus the chord corresponding to the vertex k_{21} must be drawn first when considering the ordering given by the neighbourhoods of those independent vertices that have at least one endpoint lying in k_1k_3. It follows from the previous that k_{21} being the first vertex in the ordering is a necessary condition when searching for a consecutive-ones ordering for the matrix $A(S, K_2)$. See Figure 3.6, where we give a circle model for the graph H.

The previously described situations must also hold for each partition K_i of K. We translated the problem of giving a circle model to the fullfilment of some properties for each of the matrices $A(S, K_i)$, where $K = \bigcup_i K_i$ and these partitions depend on whether G contains an induced tent, 4-tent or co-4-tent. This led to the definition of enriched matrices, which allowed us to model some of the above mentioned properties, and also others that came up when considering split graphs containing a 4-tent and a co-4-tent.

Definition 3.4. *Let A be a $(0, 1)$-matrix. We say A is an* enriched matrix *if all of the following conditions hold:*

1. *Each row of A is either unlabeled or labeled with one of the following labels: L or R or LR. We say that a row is an* LR-row (resp. L-row, R-row) *if it is labeled with LR (resp. L, R).*
2. *Each row of A is either uncolored or colored with either blue or red.*
3. *The only colored rows may be those labeled with L or R, and those LR-rows having a 0 in every column.*
4. *The LR-rows having a 0 in every column are all colored with the same color.*

The underlying matrix of A *is the $(0, 1)$-matrix that coincides with A that has neither labels nor colored rows.*

We will denote the color assignment for a row with a colored bullet at the right side of the matrix.

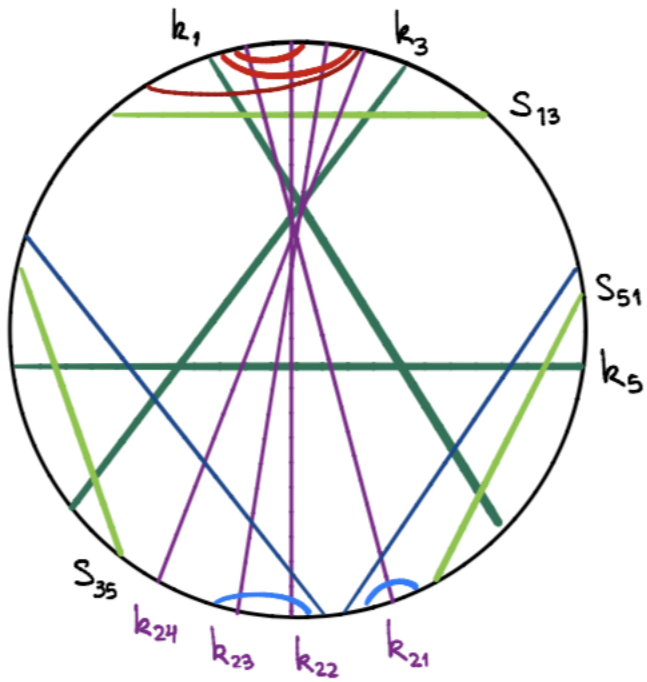


Figure 3.6 – A circle model for the split graph H.

The color assignment for some of the rows represents in which arc of the circle corresponding to K_i we must draw one or both endpoints when considering the placement of the chords. Some of the independent vertices have a unique possible placement, and some of them can be –a priori– drawn in either two of the arcs corresponding to K_i. Moreover, the labeling of the rows explains ''from which direction does the chord come from'' if we are standing in a particular portion of the circle. For example, the following is the matrix $A(S, K_2)$ for the graph represented in Figure 3.6 considered as an enriched matrix –taking into account all the information on the placement of the chords:

$$A(S, K_2) = \begin{array}{c} \\ \\ \\ \\ \mathbf{L} \\ \mathbf{L} \\ \mathbf{R} \end{array} \begin{pmatrix} 1100 \\ 1110 \\ 0110 \\ 1000 \\ 1110 \\ 1000 \\ 0111 \end{pmatrix} \begin{array}{c} \\ \\ \\ \\ \bullet \\ \bullet \\ \bullet \end{array}$$

Definition 3.5. *Let A be an enriched matrix. We say A is* LR-orderable *if there is a linear ordering Π for the columns of A such that each of the following assertions holds:*

— *Π is a consecutive-ones ordering for every non-LR row of A.*

— *The ordering Π is such that the ones in every nonempty row labeled with L (resp. R) start in the first column (resp. end in the last column).*

— *Π is a consecutive-ones ordering for the complements of every LR-row of A.*

Such an ordering is called an LR-ordering. *For each row of* A *labeled with L or LR and having a* 1 *in the first column of* Π*, we define its* L-block (with respect to Π) *as the maximal set of consecutive columns of* Π *starting from the first one on which the row has a 1.* R-blocks *are defined on an entirely analogous way. For each unlabeled row of* A*, we say its* U-block (with respect to Π) *is the set of columns having a* 1 *in the row. The blocks of* A *with respect to* Π *are its L-blocks, its R-blocks and its U-blocks.*

Definition 3.6. *Let* A *be an enriched matrix. We say an* L-block (resp. R-block, U-block) is colored *if there is a 1-color assignment for every entry of the block.*

A block bi-coloring for the blocks of A *is a color assignment with either red or blue for some L-blocks, U-blocks and R-blocks of* A*. A block bi-coloring is* total *if every L-block, R-block and U-block of* A *is colored, and is* partial *if otherwise.*

Notice that for every enriched matrix, the only colored rows are those labeled with L or R and those empty LR-rows. Moreover, for every LR-orderable matrix, there is an ordering of the columns such that every row labeled with L (resp. R) starts in the first column (resp. ends in the last column), and thus all its 1's appear consecutively. Thus, if an enriched matrix is also LR-orderable, then the given coloring induces a partial block bi-coloring (see Figure 3.7), in which every empty LR-row remains the same, whereas for every nonempty colored labeled row, we color all its 1's with the color given in the definition of the matrix.

$$A = \begin{array}{l} \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10001 \\ 11001 \\ 01100 \\ {\color{red}111}00 \\ 00000 \\ 00{\color{blue}111} \end{pmatrix} \begin{array}{l} \\ \\ \\ {\color{red}\bullet} \\ {\color{blue}\bullet} \\ {\color{blue}\bullet} \end{array} \qquad B = \begin{array}{l} \mathbf{LR} \\ \mathbf{L} \\ \mathbf{R} \\ \\ \end{array} \begin{pmatrix} 10101 \\ {\color{red}11}000 \\ 000{\color{blue}11} \\ 00110 \end{pmatrix} \begin{array}{l} \\ {\color{red}\bullet} \\ {\color{blue}\bullet} \\ \\ \end{array}$$

Figure 3.7 – Example: An enriched LR-orderable matrix A, where the column ordering given from left to right is a consecutive-ones ordering. B is an enriched non-LR-orderable matrix.

We now define 2-nested matrices, which will allow us to address and solve both the problem of ordering the columns in each adjacency matrix $A(S, K_i)$ of a split graph for each partition $K_i \subset K$, and the problem of deciding if there is a feasible distribution of the independent vertices adjacent to K_i between the two portions of the circle corresponding to K_i. This allows to give a circle model for the given graph. We give a complete characterization of these matrices by forbidden subconfigurations at the end of this chapter.

Definition 3.7. *Let* A *be an enriched matrix. We say* A *is* 2-nested *if there exists an LR-ordering* Π *of the columns and an assignment of colors red or blue to the blocks of* A *such that all of the following conditions hold:*

1. *If an LR-row has an L-block and an R-block, then they are colored with distinct colors.*
2. *For each colored row* r *in* A*, any of its blocks is colored with the same color as* r *in* A*.*

$$A = \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10001 \\ 11001 \\ 01100 \\ 11100 \\ 00000 \\ 00111 \end{pmatrix} \begin{array}{l} \\ \\ \\ \bullet \\ \bullet \\ \bullet \end{array}$$

Figure 3.8 – Example of a total block bi-coloring of the blocks of the matrix in Figure 3.7, considering the columns ordered from left to right. Moreover, A is 2-nested considering this LR-ordering and total block bi-coloring.

3. *If an L-block of an LR-row is properly contained in the L-block of an L-row, then both blocks are colored with different colors.*
4. *Every L-block of an LR-row and any R-block are disjoint. The same holds for an R-block of an LR-row and any L-block.*
5. *If an L-block and an R-block are not disjoint, then they are colored with distinct colors.*
6. *Each two U-blocks colored with the same color are either disjoint or nested.*
7. *If an L-block and a U-block are colored with the same color, then either they are disjoint or the U-block is contained in the L-block. The same holds replacing L-block for R-block.*
8. *If two distinct L-blocks of non-LR-rows are colored with distinct colors, then every LR-row has an L-block. The same holds replacing L-block for R-block.*
9. *If two LR-rows overlap, then the L-block of one and the R-block of the other are colored with the same color.*

An assignment of colors red and blue to the blocks of A that satisfies all these properties is called a (total) block bi-coloring.

Remark 3.8. We will give some insight on which properties we are modeling with Definition 3.7, which are necessary conditions that each matrix $A(S, K_i)$ must fullfil in order to give a circle model for any split graph containing a tent, 4-tent or co-4-tent.

The LR-rows represent independent vertices that have both endpoints in the arcs corresponding to K_i. The difference between these rows and those that are unlabeled, is that one endpoint of the chords must be placed in one of the arcs corresponding to K_i and the other endpoint must be placed in the other arc corresponding to K_i. Hence, the first property ensures that, when deciding where to place the chord corresponding to an LR-row, if the ordering indicates that the chord intersects some of its adjacent vertices in one arc and the other in the other arc, then the distinct blocks corresponding to the row must be colored with distinct colors.

With the second property, we ensure that the colors that are pre-assigned are respected, since they correspond to independent vertices with a unique possible placement.

The third property refers to the ordering given by containement for the vertices. We will further on see that every LR-row represents vertices that are adjacent to almost every vertex in the complete partition K of G. Hence, when dividing the LR-rows into blocks, we need to ensure that each of its block is not properly contained in the neighbourhoods of vertices that are nonadjacent to at least one partition of K. Something similar must hold for L-rows (resp. R-rows) and U-rows, and L-rows (resp. R-rows) and LR-rows. This is modeled by properties 7 and 8.

The properties 4, 5, 6 and 9 refer to the previously discussed ''nested or disjoint'' property that we need to ensure in order to give a circle model for G.

This chapter is organized as follows. In Section 3.1 we give some more definitions which are necessary to state a characterization of 2-nested matrices. In Section 3.2 we define and characterize admissible matrices, which give necessary conditions for a matrix to admit a total block bicoloring. In Section 3.3 we define and characterize LR-orderable and partially 2-nested matrices, and then we prove some properties of LR-orderings in admissible matrices. Finally, in Section 3.4 we prove Theorem 3.12, which characterizes 2-nested matrices by forbidden subconfigurations.

3.1 A characterization for 2-nested matrices

In this section, we begin by giving some definitions and examples that are necessary to state Theorem 3.12, which is presented at the end of this section and is the main result of this chapter. The proof of Theorem 3.12 will be given in Section 3.4.

Definition 3.9. *Let A be an enriched matrix. The dual matrix of A is defined as the enriched matrix $\tilde{A}$ that coincides with the underlying matrix of A and for which every row of A that is labeled with L (resp. R) is now labeled with R (resp. L) and every other row remains the same. Also, the color assigned to each row remains as in A.*

$$A = \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10001 \\ 11001 \\ 01100 \\ \textcolor{red}{111}00 \\ 00000 \\ 00\textcolor{blue}{111} \end{pmatrix} \begin{array}{l} \\ \\ \\ \textcolor{red}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array} \qquad \tilde{A} = \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{R} \\ \mathbf{LR} \\ \mathbf{L} \end{array} \begin{pmatrix} 10001 \\ 11001 \\ 01100 \\ \textcolor{red}{111}00 \\ 00000 \\ 00\textcolor{blue}{111} \end{pmatrix} \begin{array}{l} \\ \\ \\ \textcolor{red}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array}$$

Figure 3.9 – Example: A and its dual matrix.

The 0-gem, 1-gem and 2-gem are the following enriched matrices:

$$\begin{pmatrix} 110 \\ 011 \end{pmatrix}, \qquad \begin{pmatrix} 10 \\ 11 \end{pmatrix}, \qquad \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \end{array} \begin{pmatrix} 110 \\ 101 \end{pmatrix}$$

respectively.

Definition 3.10. *Let* A *be an enriched matrix. We say that* A contains a gem *(resp.* doubly-weak gem*) if it contains a* 0*-gem (resp. a* 2*-gem) as a subconfiguration. We say that* A contains a weak gem *if it contains a* 1*-gem such that, either the first is an L-row (resp. R-row) and the second is a U-row, or the first is an LR-row and the second is a non-LR-row. We say that a* 2*-gem is* badly-colored *if the entries in the column in which both rows have a* 1 *are in blocks colored with the same color.*

Let r be an LR row of A. We denote with $\bar{r}$ to *the complement of* r, this is, the row that has a 1 in each coordinate of r that has a 0, and has a 0 in each coordinate of r that has a 1.

Definition 3.11. *Let* A *be an enriched matrix and let* Π *be a LR-ordering. We define* A^* *as the enriched matrix that arises from* A *by:*

- *Replacing each LR-row by its complement.*
- *Adding two distinguished rows: both rows have a* 1 *in every column, one is labeled with L and the other is labeled with R.*

In Figures 3.10, 3.11, 3.12, 3.13 and 3.14 we define some matrices, for they play an important role in the sequel. We will use green and orange to represent red and blue or blue and red, respectively. For every enriched matrix represented in the figures of this chapter, if a row labeled with L or R appears in black, then it may be colored with either red or blue indistinctly. Moreover, whenever a row is labeled with **L (LR)** (resp. **R (LR)**), then such a row may be either a row labeled with L or LR (resp. R or LR) indistinctly.

$$D_0 = \begin{array}{c}\mathbf{L}\\ \mathbf{L}\end{array}\begin{pmatrix}10\\01\end{pmatrix} \qquad D_1 = \begin{array}{c}\mathbf{L}\\ \mathbf{R}\end{array}\begin{pmatrix}1\\1\end{pmatrix} \qquad D_2 = \begin{array}{c}\mathbf{L}\\ \mathbf{R}\end{array}\begin{pmatrix}10\\10\end{pmatrix} \qquad D_3 = \begin{array}{c}\mathbf{L}\\ \mathbf{R}\\ \mathbf{LR}\end{array}\begin{pmatrix}100\\001\\010\end{pmatrix}$$

$$D_4 = \begin{array}{c}\mathbf{L}\\ \mathbf{L}\\ \mathbf{LR}\end{array}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad D_5 = \begin{array}{c}\mathbf{L}\\ \mathbf{R}\\ \mathbf{LR}\end{array}\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad D_6 = \begin{array}{c}\mathbf{L}\\ \mathbf{R}\\ \mathbf{LR}\end{array}\begin{pmatrix}10\\01\\00\end{pmatrix} \qquad D_7 = \begin{array}{c}\mathbf{L}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}100\\010\\001\end{pmatrix}$$

$$D_8 = \begin{array}{c}\mathbf{L}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}110\\101\\011\end{pmatrix} \qquad D_9 = \begin{array}{c}\mathbf{L}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}1110\\1100\\1001\end{pmatrix} \qquad D_{10} = \begin{array}{c}\mathbf{L}\\ \mathbf{R}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}1100\\0011\\1011\\1101\end{pmatrix}$$

$$D_{11} = \begin{array}{c}\mathbf{LR}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}100\\010\\001\end{pmatrix} \qquad D_{12} = \begin{array}{c}\mathbf{LR}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}101\\110\\011\end{pmatrix} \qquad D_{13} = \begin{array}{c}\mathbf{LR}\\ \mathbf{LR}\\ \mathbf{LR}\end{array}\begin{pmatrix}1100\\0110\\0011\end{pmatrix}$$

Figure 3.10 – The family of enriched matrices $\mathcal{D}$.

$$F_0 = \begin{pmatrix} 11100 \\ 01110 \\ 00111 \end{pmatrix} \qquad F_1(k) = \begin{pmatrix} 011...111 \\ 111...110 \\ 000...011 \\ 000...110 \\ ..... \\ ..... \\ ..... \\ 110...000 \end{pmatrix} \qquad F_2(k) = \begin{pmatrix} 0111...10 \\ 1100...00 \\ 0110...00 \\ ..... \\ ..... \\ ..... \\ 0000...11 \end{pmatrix}$$

$$F'_0 = \begin{array}{r} \textbf{L (LR)} \\ \\ \\ \end{array} \begin{pmatrix} 1100 \\ 1110 \\ 0111 \end{pmatrix} \qquad F''_0 = \begin{array}{r} \textbf{L} \\ \\ \textbf{R} \end{array} \begin{pmatrix} 110 \\ 111 \\ 011 \end{pmatrix} \qquad F'_1(k) = \begin{array}{r} \\ \textbf{L (LR)} \\ \\ \\ \\ \textbf{L (LR)} \end{array} \begin{pmatrix} 11\dots1111 \\ 11\dots1110 \\ 00\dots0011 \\ 00\dots0110 \\ \iddots \\ 10\dots0000 \end{pmatrix}$$

$$F'_2(k) = \begin{array}{r} \\ \textbf{L (LR)} \\ \\ \\ \\ \end{array} \begin{pmatrix} 111\dots10 \\ 100\dots00 \\ 110\dots00 \\ \ddots \\ 000\dots11 \end{pmatrix}$$

Figure 3.11 – The enriched matrices of the family $\mathcal{F}$.

The matrices $\mathcal{F}$ represented in Figure 3.11 are defined as follows: $F_1(k) \in \{0,1\}^{k\times(k-1)}$, $F_2(k) \in \{0,1\}^{k\times k}$, $F'_1(k) \in \{0,1\}^{k\times(k-2)}$ and $F'_2(k) \in \{0,1\}^{k\times(k-1)}$, for every odd $k \geq 5$. In the case of F'_0, $F'_1(k)$ and $F'_2(k)$, the labeled rows may be either L or LR indistinctly, and in the case of their dual matrices, the labeled rows may be either R or LR indistinctly.

The matrices $\mathcal{S}$ in Figure 3.12 are defined as follows. If k is odd, then $S_1(k) \in \{0,1\}^{(k+1)\times k}$ for $k \geq 3$, and if k is even, then $S_1(k) \in \{0,1\}^{k\times(k-2)}$ for $k \geq 4$. The remaining matrices have the same size whether k is even or odd: $S_2(k) \in \{0,1\}^{k\times(k-1)}$ for $k \geq 3$, $S_3(k) \in \{0,1\}^{k\times(k-1)}$ for $k \geq 3$, $S_5(k) \in \{0,1\}^{k\times(k-2)}$ for $k \geq 4$, $S_4(k) \in \{0,1\}^{k\times(k-1)}$, $S_6(k) \in \{0,1\}^{k\times k}$ for $k \geq 4$, $S_7(k) \in \{0,1\}^{k\times(k+1)}$ for every $k \geq 3$ and $S_8(2j) \in \{0,1\}^{2j\times(2j)}$ for $j \geq 2$. With regard to the coloring of the labeled rows, if k is even, then the first and last row of $S_2(k)$ and $S_3(k)$ are colored with the same color, and in $S_4(k)$ and $S_5(k)$ are colored with distinct colors.

$$S_1(2j) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{LR} \\ \mathbf{L} \end{array} \begin{pmatrix} 10\dots00 \\ 11\dots00 \\ \ddots \\ 00\dots11 \\ 00\dots01 \\ 11\dots11 \end{pmatrix} \qquad S_1(2j+1) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{LR} \end{array} \begin{pmatrix} 10\dots00 \\ 11\dots00 \\ \ddots \\ 00\dots11 \\ 00\dots01 \end{pmatrix} \qquad S_2(k) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{L} \end{array} \begin{pmatrix} 10\dots00 \\ 11\dots00 \\ \ddots \\ 00\dots11 \\ 11\dots10 \end{pmatrix}$$

$$S_3(k) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 10\dots00 \\ 11\dots00 \\ \ddots \\ 00\dots11 \\ 00\dots01 \end{pmatrix} \qquad S_4(k) = \begin{array}{c} \mathbf{LR} \\ \mathbf{L} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 11\dots11 \\ 10\dots00 \\ 11\dots00 \\ \ddots \\ 00\dots11 \\ 00\dots01 \end{pmatrix} \qquad S_5(k) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{LR} \\ \mathbf{L} \end{array} \begin{pmatrix} 10\dots00 \\ 11\dots00 \\ \ddots \\ 00\dots11 \\ 11\dots10 \\ 11\dots11 \end{pmatrix}$$

$$S_6(3) = \begin{array}{c} \mathbf{LR} \\ \mathbf{R} \\ \\ \end{array} \begin{pmatrix} 110 \\ 011 \\ 110 \end{pmatrix} \qquad S_6'(3) = \begin{array}{c} \mathbf{LR} \\ \mathbf{R} \\ \\ \end{array} \begin{pmatrix} 110 \\ 011 \\ 111 \end{pmatrix} \qquad S_6(k) = \begin{array}{c} \mathbf{LR} \\ \mathbf{R} \\ \\ \\ \\ \end{array} \begin{pmatrix} 111\dots110 \\ 011\dots111 \\ 110\dots000 \\ \ddots \\ 000\dots011 \end{pmatrix}$$

$$S_7(3) = \begin{array}{c} \mathbf{LR} \\ \mathbf{LR} \\ \\ \end{array} \begin{pmatrix} 11001 \\ 10011 \\ 11100 \end{pmatrix} \qquad S_7(2j) = \begin{array}{c} \mathbf{LR} \\ \mathbf{LR} \\ \\ \\ \\ \end{array} \begin{pmatrix} 1100\dots000 \\ 1000\dots001 \\ 0110\dots000 \\ \ddots \\ 0000\dots011 \end{pmatrix} \qquad S_8(2j) = \begin{array}{c} \mathbf{LR} \\ \\ \\ \\ \end{array} \begin{pmatrix} 100\dots001 \\ 110\dots000 \\ \ddots \\ 000\dots011 \end{pmatrix}$$

Figure 3.12 – The family of matrices $\mathcal{S}$ for every $j \geq 2$ and every odd $k \geq 5$

$$P_0(k,0) = \begin{array}{c} \mathbf{L} \\ \mathbf{LR} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 11000\ldots000 \\ 10011\ldots111 \\ 00110\ldots000 \\ \ddots \\ 00000\ldots011 \\ 00000\ldots001 \end{pmatrix}$$

$$P_0(k,l) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{LR} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 100\ldots0000\ldots0 \\ 110\ldots0000\ldots0 \\ \ddots \\ 000\ldots1100\ldots0 \\ 111\ldots1001\ldots1 \\ 000\ldots0011\ldots0 \\ \ddots \\ 000\ldots00\ldots011 \\ 000\ldots00\ldots001 \end{pmatrix}$$

$$P_1(k,0) = \begin{array}{c} \mathbf{L} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 1100\ldots000 \\ 1011\ldots111 \\ 1101\ldots111 \\ 00110\ldots000 \\ \ddots \\ 00000\ldots011 \\ 0000\ldots001 \end{pmatrix}$$

$$P_1(k,l) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 100\ldots0000\ldots0 \\ 110\ldots0000\ldots0 \\ \ddots \\ 000\ldots1100\ldots0 \\ 111\ldots1011\ldots1 \\ 111\ldots1101\ldots1 \\ 000\ldots0011\ldots0 \\ \ddots \\ 000\ldots00\ldots011 \\ 000\ldots00\ldots001 \end{pmatrix}$$

$$P_2(k,0) = \begin{array}{c} \mathbf{L} \\ \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 110000\ldots000 \\ 101111\ldots111 \\ 111011\ldots111 \\ 110111\ldots111 \\ 111001\ldots111 \\ 000011\ldots000 \\ \ddots \\ 00000\ldots011 \\ 00000\ldots001 \end{pmatrix}$$

$$P_2(k,l) = \begin{array}{c} \mathbf{L} \\ \\ \\ \\ \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 100\ldots00000\ldots0 \\ 110\ldots00000\ldots0 \\ \ddots \\ 000\ldots11000\ldots0 \\ 111\ldots10011\ldots1 \\ 111\ldots11101\ldots1 \\ 111\ldots11011\ldots1 \\ 111\ldots11001\ldots1 \\ 000\ldots00011\ldots0 \\ \ddots \\ 000\ldots000\ldots011 \\ 000\ldots000\ldots001 \end{pmatrix}$$

Figure 3.13 – The family of enriched matrices $\mathcal{P}$ for every odd k.

In the matrices $\mathcal{P}$, the integer l represents the number of unlabeled rows between the first row and the first LR-row. The matrices $\mathcal{P}$ described in Figure 3.13 are defined as follow: $P_0(k,0) \in \{0,1\}^{k\times k}$ for every $k \geq 4$, $P_0(k,l) \in \{0,1\}^{k\times(k-1)}$ for every $k \geq 5$ and $l > 0$; $P_1(k,0) \in \{0,1\}^{k\times(k-1)}$ for every $k \geq 5$, $P_1(k,l) \in \{0,1\}^{k\times(k-2)}$ for every $k \geq 6$, $l > 0$; $P_2(k,0) \in \{0,1\}^{k\times(k-1)}$ for every $k \geq 7$, $P_2(k,l) \in \{0,1\}^{k\times(k-2)}$ for every $k \geq 8$ and $l > 0$. If k is even, then the first and last row of every matrix in $\mathcal{P}$ are colored with distinct colors.

$$M'_2(k) = \begin{array}{r} \\ \mathbf{L} \\ \\ \\ \\ \mathbf{L} \end{array} \begin{pmatrix} 111\dots111 \\ 100\dots000 \\ 110\dots000 \\ \ddots \\ 000\dots110 \\ 111\dots101 \end{pmatrix} \qquad M''_2(k) = \begin{array}{r} \mathbf{R} \\ \mathbf{L} \\ \\ \\ \\ \mathbf{R} \\ \mathbf{L} \end{array} \begin{pmatrix} 111\dots111 \\ 100\dots000 \\ 110\dots000 \\ \ddots \\ 000\dots110 \\ 000\dots010 \\ 111\dots111 \end{pmatrix} \qquad M'_3(k) = \begin{array}{r} \mathbf{L} \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 100\dots000 \\ 110\dots000 \\ \ddots \\ 000\dots110 \\ 111\dots101 \end{pmatrix}$$

$$M''_3(k) = \begin{array}{r} \\ \\ \\ \\ \mathbf{R} \end{array} \begin{pmatrix} 110\dots00 \\ 011\dots00 \\ \ddots \\ 000\dots11 \\ 011\dots10 \end{pmatrix} \qquad M'_4 = \begin{array}{r} \mathbf{L} \\ \\ \\ \\ \end{array} \begin{pmatrix} 10000 \\ 01100 \\ 00011 \\ 10101 \end{pmatrix} \qquad M''_4 = \begin{array}{r} \mathbf{L} \\ \mathbf{R} \\ \\ \\ \end{array} \begin{pmatrix} 1000 \\ 0100 \\ 0011 \\ 1101 \end{pmatrix}$$

$$M'_5 = \begin{array}{r} \\ \\ \mathbf{R} \\ \\ \end{array} \begin{pmatrix} 1100 \\ 0011 \\ 1001 \\ 111 \end{pmatrix} \qquad M''_5 = \begin{array}{r} \mathbf{L} \\ \\ \\ \mathbf{L} \end{array} \begin{pmatrix} 1000 \\ 0110 \\ 1011 \\ 1110 \end{pmatrix}$$

Figure 3.14 – The enriched matrices in family $\mathcal{M}$: $M'_2(k)$, $M'_3(k)$, $M''_3(k)$, $M'''_3(k)$ for $k \geq 4$, and $M''_2(k)$ for $k \geq 5$.

$$M_0 = \begin{pmatrix} 1011 \\ 1110 \\ 0111 \end{pmatrix} \qquad M_{II}(4) = \begin{pmatrix} 0111 \\ 1100 \\ 0110 \\ 1101 \end{pmatrix} \qquad M_V = \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix} \qquad S_0(k) = \begin{pmatrix} 111...11 \\ 110...00 \\ 011...00 \\ ..... \\ ..... \\ ..... \\ 000...11 \\ 100...01 \end{pmatrix}$$

Figure 3.15 – The matrices M_0, $M_{II}(4)$, M_V and $S_0(k) \in \{0,1\}^{((k+1)\times k}$ for any even $k \geq 4$.

Now we are in conditions to state Theorem 3.12, which characterizes 2-nested matrices by forbidden subconfigurations and is the main result of this chapter. The proof for this theorem will be given at the end of the chapter.

Theorem 3.12. *Let A be an enriched matrix. Then, A is 2-nested if and only if A contains none of the following listed matrices or their dual matrices as subconfigurations:*

— M_0, $M_{II}(4)$, M_V *or* $S_0(k)$ *for every even* k *(See Figure 3.15)*

— *Every enriched matrix in the family* $\mathcal{D}$ *(See Figure 3.10)*

— *Every enriched matrix in the family* $\mathcal{F}$ *(See Figure 3.11)*

— *Every enriched matrix in the family* $\mathcal{S}$ *(See Figure 3.12)*

— *Every enriched matrix in the family* $\mathcal{P}$ *(See Figure 3.13)*

— *Monochromatic gems, monochromatic weak gems, badly-colored doubly-weak gems*

and A^* *contains no Tucker matrices and none of the enriched matrices in* $\mathcal{M}$ *or their dual matrices as subconfigurations (See Figure 3.14).*

Throughout the following sections we will give some definitions and characterizations that will allow us to prove this theorem. In Section 3.2 we will define and characterize the notion of admissibility, which englobes all the properties we need to consider when coloring the blocks of an enriched matrix. In Section 3.3, we give a characterization for LR-orderable matrices by forbidden subconfigurations. Afterwards, we define and characterize partially 2-nested matrices, which are those enriched matrices that admit an LR-ordering and for which the given pre-coloring of those labeled rows induces a partial block bi-coloring. These definitions and characterizations allow us to prove Lemmas 3.36 and 3.38, which are of fundamental importance for the proof of Theorem 3.12.

3.2 Admissibility

In this section we will define the notion of admissibility for an enriched $(0,1)$-matrix, which will allow us to characterize those enriched matrices for which there is a total block bi-coloring for A. In the next chapter, we will see that such a block bi-coloring is a necessary condition to give a circle model.

Notice that the existance of a block bi-coloring for an enriched matrix is a property that can be defined and characterized by subconfigurations and forbidden submatrices.

Let us consider the matrices defined in 3.10. The matrices in this family are all examples of enriched matrices that do not admit a total block bi-coloring as defined in Definition 3.7.

For example, let us consider D_0. In order to have a block bi-coloring for every block of D_0, it is necessary that D_0 admits an LR-ordering of its columns. In particular, in such an ordering every row labeled with L starts in the first column. Hence, if there is indeed an LR-ordering for D_0, then the existance of two distinct non-nested rows labeled with L is not possible. The same holds if both rows are labeled with R. We can use similar arguments to see that D_2, D_3, D_7 and D_{11} do not admit an LR-ordering.

Let us consider the matrix D_1. In this case, we see that condition 5 does not hold for the enriched matrix D_1.

Consider now the matrix D_4. It follows from property 8 that if an enriched matrix has two distinct rows labeled with L and colored with distinct colors, then every LR-row has an L-block, and thus D_4 does not admit a total block-bi-coloring. Suppose now that D_4 is a submatrix of some enriched matrix and that the LR-row is nonempty. Notice that, if the LR-row has an L-block then it is properly contained in both rows labeled with L. It follows from this and property 3 of the definition of 2-nested that, that the L-block of the LR-row must be colored with a distinct color than the one given to each row labeled with L. However, each of these rows is colored with a distinct color, thus a total block-bi-coloring is not possible in that case.

If we consider the enriched matrix D_5, then it follows from property 4 that there is no possible LR-ordering such that the L-block of the LR-row does not intersect the L-row, and the same follows for the R-block of the LR-row and the R-row of D_5.

Let us consider an the enriched matrix in which we find D_6 as a subconfiguration.If the LR-row has an L-block, then it is contained in the L-row, and the same holds for the R-block of the LR-row and the R-row. By property 3, the L-block must be colored with a distinct color than the L-row, and the R-block must be colored with a distinct color than the R-row. Equivalently, the L-block and the R-block of the LR-row are colored with the same color. However, this is not possible by property 1. Similarly, we can see that D_8, D_9, D_{10}, D_{12} and D_{13} do not admit a total block bi-coloring, also having in mind that the property 9 must hold pairwise for LR-rows.

Definition 3.13. *Let A be an enriched matrix. We define the following list of properties:*

(Adm_1) *If two rows are labeled both with L or both with R, then they are nested.*

(Adm_2) *If two rows with the same color are labeled one with L and the other with R, then they are disjoint.*

(Adm_3) *If two rows with distinct colors are labeled one with L and the other with R, then either they are disjoint or there is no column where both have 0 entries.*

(Adm_4) *If two rows r_1 and r_2 have distinct colors and are labeled one with L and the other with R, then any LR-row with at least one non-zero column has nonempty intersection with either r_1 or r_2.*

(Adm_5) *If two rows r_1 and r_2 with distinct colors are labeled both with L or both with R, then for any LR-row r, r_1 is contained in r or r_2 is contained in r.*

(Adm_6) *If two non-disjoint rows r_1 and r_2 with distinct colors, one labeled with L and the other labeled with R, then any LR-row is disjoint with regard to the intersection of r_1 and r_2.*

(Adm_7) *If two rows with the same color are labeled one with L and the other with R, then for any LR-row r one of them is contained in r. Moreover, the same holds for any two rows with distinct colors and labeled with the same letter.*

(Adm_8) *For each three non-disjoint rows such that two of them are LR-rows and the other is labeled with either L or R, two of them are nested.*

(Adm_9) *If two rows r_1 and r_2 with distinct colors are labeled one with L and the other with R, and there are two LR-rows r_3 and r_4 such that r_1 is neither disjoint or contained in r_3 and r_2 is neither disjoint or contained in r_4, then r_3 is nested in r_4 or viceversa.*

(Adm_{10}) *For each three LR-rows, two of them are nested.*

For each of the above properties, we will characterize the set of minimal forbidden subconfigurations with the following Lemma.

Lemma 3.14. *For any enriched matrix A, all of the following assertions hold:*

1. *A satisfies 1 if and only if A contains no D_0 or its dual matrix as a subconfiguration.*
2. *A satisfies 2 if and only if A contains no D_1 or its dual matrix as a subconfiguration.*

3. A *satisfies 3 if and only if* A *contains no* D_2 *or its dual matrix as a subconfiguration.*
4. A *satisfies 4 if and only if* A *contains no* D_2, D_3 *or their dual matrices as subconfigurations.*
5. A *satisfies 5 if and only if* A *contains no* D_0, D_4 *or their dual matrices as subconfigurations.*
6. A *satisfies 6 if and only if* A *contains no* D_5 *or its dual matrix as a subconfiguration.*
7. A *satisfies 7 if and only if* A *contains no* D_0, D_1, D_4, D_6 *or their dual matrices as subconfigurations.*
8. A *satisfies 8 if and only if* A *contains no* D_7, D_8, D_9 *or their dual matrices as subconfigurations.*
9. A *satisfies 9 if and only if* A *contains no* D_5, D_9, D_{10} *or its dual matrix as a subconfiguration.*
10. A *satisfies 10 if and only if* A *contains no* D_{11}, D_{12}, D_{13} *or their dual matrices as subconfigurations.*

Proof. First, we will find every forbidden subconfiguration given by statement 1.

Let f_1 and f_2 be two rows labeled with the same letter, and suppose they are not nested. Thus, there is a column in which f_1 has a 1 and f_2 has a 0, and another column in which f_2 has a 1 and f_1 has a 0. Since the color of each row is irrelevant in the definition, we find D_0 as a forbidden subconfiguration in A.

Let us find now every forbidden subconfiguration given by statement 2. Let f_1 and f_2 be rows labeled with distinct letters and colored with the same color. If f_1 and f_2 are not disjoint, then there is a column in which both rows have a 1. In this case, we find D_1 as a forbidden subconfiguration in A.

For statement 3, let f_1 and f_2 be two rows labeled with distinct letters and colored with distinct colors, and suppose they are not disjoint and there is a column j_1 such that both rows have a 0 in column j_1. Thus, there is a column $j_2 \neq j_1$ such that both rows have a 1 in column j_2. If f_1 and f_2 have the same color, then we find D_1 as a subconfiguration. Hence, D_2 is a forbidden subconfiguration in A.

With regard to statement 4, let f_1 and f_2 be two rows labeled with distinct letters and colored with distinct colors. Let f_3 be a non-zero LR-row. Suppose that f_3 is disjoint with both f_1 and f_2. Hence, there is a column l_1 such that f_1 and f_2 have a 0 and f_3 has a 1. Moreover, either there are two distinct columns j_1 and j_2 such that the column j_i has a 1 in row f_i and a 0 in the other rows, for $i = 1, 2$, or there is a column l_2 such that f_1 and f_2 both have a 1 in column l_2 and f_3 has a 0. If the last statement holds, we find D_2 as a subconfiguration considering only the submatrix given by the rows f_1 and f_2. If instead there are two distinct columns j_1 and j_2 as described above, then we find D_3 as a minimal forbidden subconfiguration in A.

For statement 5, let f_1 and f_2 be two rows labeled with L and colored with distinct colors, and let r be an LR-row. If f_1 and f_2 are not nested, then we find D_0. Suppose that f_1 and f_2 are nested. If neither f_1 or f_2 are contained in r, then there is a column j in which f_1 and f_2 have a 1 and r has a 0. Thus, D_4 is a forbidden subconfiguration in A.

For statement 6, let f_1 and f_2 be two non-disjoint rows colored with distinct colors, f_1 labeled with L and f_2 labeled with R. Since they are non-disjoint, there is at least one column j in which both rows have a 1. Suppose that for every such column j, there is an LR-row f having a 1 in that column. Then, we find D_5 as a subconfiguration in A.

For statement 7, let f be an LR-row and let f_1 and f_2 be two rows labeled with L and R respectively, and colored with the same color. If f_1 and f_2 are not disjoint, then we find D_1. Suppose that f_1 and f_2 are disjoint. If neither f_1 is contained in f nor f_2 is contained in f, then there are columns $j_1 \neq j_2$ such that f_i has a 1 and f has a 0, for $i = 1, 2$. Thus, we find D_6 as a subconfiguration of A. If instead f_1 and f_2 are both labeled with L and colored with distinct colors, and neither is contained in f, then either they are not nested –in which case we find D_0– or we find D_4 in A.

Suppose that A satisfies 8. Let f_1 be a row labeled with L, and f_2 and f_3 two distinct LR-rows such that none of them are nested in the others. Thus, we have three possibilities. If there are three columns j_i $i = 1, 2, 3$ such that f_i has a 1 and the other rows have a 0, then we find D_7 as a subconfiguration of A. If instead there are three rows j_i, $i = 1, 2, 3$ such that f_i and f_{i+1} have a 1 and f_{i+2} has a 0 in j_i (mod 3), then we find D_8 as a subconfiguration. The remaining possibility, is that there are 4 columns j_1, j_2, j_3, j_4 such that f_1 and f_2 have a 1 and f_3 has a 0 in j_1, f_1 has a 1 and f_2 and f_3 have a 0 in j_2, f_3 has a 1 and f_1 and f_2 have a 0 in j_3, and f_2 and f_3 have a 1 and f_1 has a 0 in j_4. Moreover, since all three rows are pairwise non-disjoint, either there is a fifth column for which f_1 and f_3 have a 1 and f_2 has a 0 (in which case we find D_8), or f_2 has a 1 and f_1 and f_3 have a 0 (in which case we have D_7), or all three rows have a 1 in such column. In this case, we find D_9 has a subconfiguration of A.

For statement 9, let f_1 and f_2 be two rows labeled with L and R, respectively, and colored with distinct colors. Let f_3 and f_4 be two LR-rows such that f_1 is neither disjoint or contained in f_3 and f_2 is neither disjoint or contained in f_4. If f_1 is also not contained in f_4 or f_2 is not contained in f_3, then we find D_9. Thus, suppose that f_1 is contained in f_4 and f_2 is contained in f_3. Moreover, we may assume that for any column such that f_1 and f_3 have a 1, f_2 has a 0, (and analogously for f_2 and f_4 having a 1 and f_1), for if not we find D_5. Hence, there is a column j_1 in A having a 1 in f_1 and f_4 and having a 0 in f_3 and f_2, and another column j_2 having a 1 in f_2 and f_3 and having a 0 in f_1 and f_4. Moreover, since f_1 and f_3 are not disjoint and f_2 and f_4 are not disjoint (and f_1 is nested in f_4 and f_2 is nested in f_3), then there are columns j_3 and j_4 such that f_1, f_3 and f_4 have a 1 and f_2 has a 0 in j_3 and f_2, f_3 and f_4 have a 1 and f_1 has a 0. Therefore, we find D_{10} as a subconfiguration of A.

It follows by using a similar argument as in the previous statements that, if A satisfies 10, then that there are no D_{11}, D_{12} or D_{13} in A.

□

Corollary 3.15. *Every enriched matrix* A *that admits a total block bi-coloring contains none of the matrices in $\mathcal{D}$. Equivalently, if* A *admits a total block bi-coloring, then every property listed in 3.13 hold.*

Another example of families of enriched matrices that do not admit a total block bi-coloring are $\mathcal{S}$ and $\mathcal{P}$, which are the matrices shown in Figures 3.12 and 3.13, respectively. Therefore, since the existance of a total block bi-coloring is a property that must hold for every subconfiguration of an enriched matrix, if an enriched matrix A admits a total block bi-coloring, then it is a necessary condition that A contains none of the matrices in $\mathcal{S}$ or $\mathcal{P}$. With this in mind, we give the following definition, which is also a characterization by forbidden subconfigurations.

Definition 3.16. *Let* A *be an enriched matrix. We say* A *is* admissible *if and only if* A *is $\{\mathcal{D}, \mathcal{S}, \mathcal{P}\}$-free.*

3.3 Partially 2-nested matrices

This section is organized as follows. First, we give some definitions and examples that will help us obtain a characterization of LR-orderable matrices by forbidden subconfigurations, which were defined in 3.5. Afterwards, we define and characterize partially 2-nested matrices, which are those enriched matrices that admit an LR-ordering and for which the given pre-coloring of those labeled rows of A induces a partial block bi-coloring.

Definition 3.17. *A* tagged matrix *is a matrix A, each of whose rows are either uncolored or colored with blue or red, together with a set of at most two distinguished columns of A. The distinguished columns will be refered to as* tag columns.

Definition 3.18. *Let A be an enriched matrix. We define the* tagged matrix of A *as a tagged matrix, denoted by A_{tag}, whose underlying matrix is obtained from A by adding two columns, c_L and c_R, such that: (1) the column c_L has a 1 if f is labeled L or LR and 0 otherwise, (2) the column c_R has a 1 if f is labeled R or LR and 0 otherwise, and (3) the set of distinguished columns of A_{tag} is $\{c_L, c_R\}$. We denote A^*_{tag} to the tagged matrix of A^*. By simplicity we will consider column c_L as the first and column c_R as the last column of A_{tag} and A^*_{tag}.*

$$
A = \begin{array}{c} \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array}
\begin{pmatrix} 10001 \\ 11001 \\ 01100 \\ \color{red}{111}00 \\ 00000 \\ 00\color{blue}{111} \end{pmatrix}
\begin{array}{c} \\ \\ \\ \color{red}{\bullet} \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \end{array}
\qquad
A_{\text{tag}} = \begin{array}{c} c_L \quad\quad c_R \\ \begin{pmatrix} 1100011 \\ 1110011 \\ 0011000 \\ 1\color{red}{111}000 \\ 1000001 \\ 000\color{blue}{111}1 \end{pmatrix} \end{array}
\begin{array}{c} \\ \\ \\ \\ \color{red}{\bullet} \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \end{array}
\qquad
A^*_{\text{tag}} = \begin{array}{c} c_L \quad\quad c_R \\ \begin{pmatrix} 0011100 \\ 0001100 \\ 0011000 \\ 1\color{red}{111}000 \\ 000\color{blue}{111}1 \end{pmatrix} \end{array}
\begin{array}{c} \\ \\ \\ \color{red}{\bullet} \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \end{array}
$$

Figure 3.16 – Example of a matrix A and the matrices A_{tag} and A^*_{tag}

The following remarks will allow us to give a simpler proof for the characterization of LR-orderable matrices.

***Remark* 3.19.** If A^*_{tag} has the C1P, then the distinguished rows force the tag columns c_L and c_R to be the first and last columns of A_{tag}, respectively.

***Remark* 3.20.** An admissible matrix A is LR-orderable if and only if the tagged matrix A^*_{tag} has the C1P for the rows.

Theorem 3.21. *An admissible matrix A is LR-orderable if and only if the tagged matrix A^*_{tag} does not contain any Tucker matrices, nor $M'_2(k)$, $M''_2(k)$, $M'_3(k)$, $M''_3(k)$ for $k \geq 3$, M'_4, M''_4, M'_5, M''_5 as subconfigurations.*

Proof. $\Rightarrow$) This follows from the last remark.

$\Leftarrow$) Suppose that the tagged matrix A_{tag} does not contain any of the above listed submatrices as subconfigurations, and still the C1P does not hold for the rows of A_{tag}.

Hence, there is a Tucker matrix M such that M is a submatrix of A_{tag}.

Suppose without loss of generality that, if M intersects only one tag column, then this tag column is c_L, since the analysis is symmetric if assumed otherwise and gives as a result in each case the dual matrix.

Case (1) *Suppose first that M intersects one or both of the distinguished rows.* Thus, the underlying matrix of M (i.e., the matrix without the tags) is either M_V, *or* $M_I(3)$, *or* $M_{II}(k)$ *for some* $k \geq 3$. We consider each case separately.

$$M'_2(k) = \begin{pmatrix} \mathbf{0}11\dots111 \\ \mathbf{1}10\dots000 \\ \ddots \\ \mathbf{0}00\dots110 \\ \mathbf{1}11\dots101 \end{pmatrix} \qquad M''_2(k) = \begin{pmatrix} \mathbf{0}11\dots11\mathbf{1} \\ \mathbf{1}10\dots00\mathbf{0} \\ \ddots \\ \mathbf{0}00\dots10\mathbf{1} \\ \mathbf{1}11\dots11\mathbf{0} \end{pmatrix} \qquad M'_3(k) = \begin{pmatrix} \mathbf{1}10\dots000 \\ \mathbf{0}11\dots000 \\ \ddots \\ \mathbf{0}00\dots110 \\ \mathbf{0}11\dots101 \end{pmatrix}$$

$$M''_3(k) = \begin{pmatrix} 110\dots00\mathbf{0} \\ 011\dots00\mathbf{0} \\ \ddots \\ 000\dots11\mathbf{0} \\ 011\dots10\mathbf{1} \end{pmatrix} \qquad M'_4 = \begin{pmatrix} \mathbf{1}10000 \\ \mathbf{0}01100 \\ \mathbf{0}00011 \\ \mathbf{0}10101 \end{pmatrix} \qquad M''_4 = \begin{pmatrix} \mathbf{1}1000\mathbf{0} \\ \mathbf{0}0100\mathbf{1} \\ \mathbf{0}0011\mathbf{0} \\ \mathbf{0}1101\mathbf{0} \end{pmatrix}$$

$$M'_5 = \begin{pmatrix} 1100\mathbf{0} \\ 0011\mathbf{0} \\ 1001\mathbf{1} \\ 1111\mathbf{0} \end{pmatrix} \qquad M''_5 = \begin{pmatrix} \mathbf{1}1000 \\ \mathbf{0}0110 \\ \mathbf{0}1011 \\ \mathbf{1}1110 \end{pmatrix}$$

Figure 3.17 – The tagged matrices of the family $\mathcal{M}$

Case (1.1) $M_V = \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix}$

In this case, the distinguished row is $(1,1,1,1,0)$ and thus the last column is a tag column. Hence $M = M'_5$, which results in a contradiction.

Case (1.2) $M_I(3) = \begin{pmatrix} 110 \\ 011 \\ 101 \end{pmatrix}$

If $(1,1,0)$ is a distinguished row, then we find D_0 as a forbidden submatrix given by the second and third rows. It is symmetric if the distinguished row is either the second or the third row, and therefore this case is not possible.

Case (1.3) $M_{II}(k) = \begin{pmatrix} 011...111 \\ 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 111...101 \end{pmatrix}$

In this case, the distinguished rows can be only the first and the last row.

Suppose only the first row $(0,1,\dots,1)$ of M is a distinguished row. Thus, the first column is a tag column.

Hence, $M'_2(k)$ is a submatrix of A_{tag}, and this results in a contradiction. The same holds if instead the last row is the sole distinguished row.

Finally, suppose both the first and the last row are distinguished. If this is the case, then the columns 1 and $k-1$ are tag columns.

Suppose first that $M = M_{II}(4)$. Since every row is a labeled row, then every row is colored. Moreover, the first and second row have distinct colors, for if not we find D_1 as a submatrix. The same holds for the second and third row, and also for the third and fourth row. However, this implies that the second and third row induce D_2, hence this case is not possible.

If instead $M = M_{II}(k)$ for $k \geq 5$, then $M''_2(k)$ is a submatrix of A_{tag}, and thus we reached a contradiction.

Case (2) *Suppose that* M *does not intersect any distinguished row.*

If M does not have any tag column, then M is a submatrix of A. Thus, A does not have the C1P and we conclude that M is a Tucker matrix.

Suppose that instead one of the columns in M is a tag column.

Case (2.1) $M_I(k) = \begin{pmatrix} 110...00 \\ 011...00 \\ ..... \\ ..... \\ ..... \\ 000...11 \\ 100...01 \end{pmatrix}$ for some $k \geq 3$.

Notice that, if any of the columns is a tag column, then we find D_0 as a submatrix, which results in A not being admissible and thus reaching a contradiction.

Case (2.2) $M_{II}(k) = \begin{pmatrix} 011...111 \\ 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 111...101 \end{pmatrix}$ for some $k \geq 4$

As in the previous case, some of the columns are not elegible for being tag columns. If there is only one tag column, the only remaining possibilities for tag columns are column 1 or column $k-1$, for in any other case we find D_0 as a submatrix. Analogously, if instead M intersects both tag columns, then these columns are also columns 1 and $k-1$.

However, if c_L is either column 1 or column $k-1$, then $M''_2(k)$ is a submatrix of A_{tag}. Notice that we can reorder the columns of $M_{II}(k)$ to have the same disposition of the rows by taking column $k-1$ as the first column. Analogously, if c_R is either column 1 or $k-1$, then we find the dual matrix of $M'_2(k)$ as a submatrix.

Finally, suppose that both columns are tag columns. Notice that the first and second rows are colored with distinct colors, for if not we find D_1 as a submatrix. The same holds for the last two rows of M. Hence, if $k = 4$, then we find D_2 as a submatrix given by the second and third rows. If instead $k > 5$, then $M''_2(k)$ is a submatrix of A_{tag}, which results once more in a contradiction.

Case (2.3) $M_{III}(k) = \begin{pmatrix} 110...000 \\ 011...000 \\ ..... \\ ..... \\ 000...110 \\ 011...101 \end{pmatrix}$ for some $k \geq 3$.

In this case, the only possibilities for tag columns are column 1, column $k-1$ and column k, for if not we find D_0 as a submatrix. Once more, it is easy to see that we can reorder the columns in such a way to have the same disposition of the rows with column $k-1$ or column k replacing column 1.

Suppose first that the tag column is the first column. In that case, we find $M'_3(k)$ as a submatrix of M, which also results in a contradiction since M is admissible.

If instead the tag column is column k, then we use an analogous reasoning to find $M''_3(k)$ as a submatrix and thus reaching a contradiction.

Suppose now that both the first column and the last column of M are tag columns.

Since M is admissible, this case is not possible for the first and last row induce D_1 or D_2 as submatrices, depending on whether the rows are colored with the same color or with distinct colors, respectively.

Case (2.4) $M_{IV} = \begin{pmatrix} 110000 \\ 001100 \\ 000011 \\ 010101 \end{pmatrix}$

In this case, the only elegible columns for being tag columns are column 1, column 3 and column 5, since if any other column is a tag column, we find D_0 as a submatrix, thus contradicting the hypothesis of pre-admissibility for M and thus for A. Furthermore, the election of the tag column is symmetric since there is a reordering of the rows that allows us to obtain the same matrix if the tag column is either column 1, column 3 or column 5, disregarding the election of the column. Hence, we have two possibilities: when column 1 is the sole tag column of M, and when the two tag columns are columns 1 and 3. If column 1 is the only tag column, then we find M'_4 as a submatrix. If instead the columns 1 and 3 are both tag columns, then the first row and the second row are colored with the same color, for if not there is $S_3(3)$ as a submatrix and this is not possible since M is admissible. Thus, in this case we find M''_4 as a submatrix.

Case (2.5) $M_V = \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix}$

Once more and using the same argument, the only elegible columns for being tag columns are columns 2, 3 or 5. Moreover, if the second column is the sole tag column, then there is a reordering of the rows such that the matrix obtained is the same as the matrix when the third column is the tag column. If column 5 is the only tag column, then we find M'_5 as in Case 1. 1. If instead column 2 is the only tag column, then the first and second rows have the same color, for if not we find $S_2(3)$ as a submatrix of M, and thus we have $M = M''_5$. Finally, if columns 2 and 5 are both tag columns, then the first and last row induce D_2 as a submatrix, disregarding the coloring of the rows and thus this case is also not possible.

This finishes every possible case, and therefore we have reached a contradiction by assuming that A_{tag} does not contain any of the listed submatrices and still the C1P does not hold for A_{tag}. □

When giving the guidelines to draw a circle model for any split graph $G = (K, S)$, not only is important that the adjacency matrix of each partition of K results admissible and LR-orderable. We also need to ensure that there is an LR-ordering that satisfies a certain property when considering how to split every LR-row into its L-block and its R-block. The following definition states necessary conditions for the LR-ordering that we need to consider to obtain a circle model. We will call this a *suitable LR-ordering*. The lemma that follows ensures that, if a matrix A is admissible and LR-orderable, then we can always find a suitable LR-ordering for the columns of A.

Definition 3.22. *An LR-ordering Π is* suitable *if the L-blocks of those LR-rows with exactly two blocks are disjoint with every R-block, the R-blocks of those LR-rows with exactly two blocks are disjoint with the L-blocks and for each LR-row the intersection with any U-block is empty with either its L-block or its R-block.*

Theorem 3.23. *If A is admissible, LR-orderable and contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$, then there is at least one suitable LR-ordering.*

Proof. Let A be an admissible LR-orderable matrix. Toward a contradiction, suppose that every LR-ordering is non-suitable. If Π is an LR-ordering of A, since Π is non-suitable, then either (1) there is a U-block u such that u is not disjoint with the L-block and the R-block of f_1, or (2) there is an LR-row f_1 such that its L-block is not disjoint with some R-block. In both cases, there is no possible reordering of the columns to obtain a suitable LR-ordering.

Notice that, since A is admissible, the LR-rows can be split into a two set partition such that the LR-rows in each set are totally ordered. Moreover, any two LR-rows for which the L-block of one intersects the R-block of the other are in distinct sets of the partition and thus the columns may be reordered by moving the portion of the block that one of the rows has in common with the other all the way to the right (or left). Hence, if two such blocks intersect and there is no possible LR-reordering of the columns, then there is at least one non-LR row blocking the reordering. Throughout the proof and by simplicity, we will say that a row or block a *is chained to the left (resp. to the right) of another row or block* b if a and b overlap and a intersects b in column $l(b)$ (resp. $r(b)$).

Case (1) *Let a_1 be the L-block of f_1 and b_1 be the R-block of f_1. Suppose first there is a U-block u such that u intersects both a_1 and b_1.*

Let $j_1 = r(a_1) + 1$, this is, the first column in which f_1 has a 0, $j_2 = r(a_1)$ and $j_3 = l(b_1)$ in which both rows f_1 and u have a 1. Since it is not possible to rearrange the columns to obtain a suitable LR-ordering, in particular, there are two columns $j_4 < j_2$ and $j_5 > j_3$ in which u has 0, one before and one after the string of 1's of u. Moreover, there is at least one row f_2 distinct to f_1 and u blocking the reordering of the columns j_1, j_2 and j_3.

Case (1.1) Suppose f_2 is the only row blocking the reordering. Notice that f_2 is neither disjoint nor nested with u and there is at least one column in which f_1 has a 0 and f_2 has a 1. We may assume without loss of generality that this is column j_1. Suppose f_2 is unlabeled. The only possibility is that f_2 overlaps with u, a_1 and b_1, for if not we can reorder the columns to obtain a suitable LR-ordering. In that case, we find M_0 in A. If instead f_2 is labeled with either L or R, then we find $S_6'(3)$ in A considering columns j_4, j_2, j_1, j_3, j_5 and both tag columns. If f_2 is an

LR-row and f_2 is the only row blocking the reordering, then either the L-block of f_2 is nested in the L-block of f_1 and the R-block of f_2 contains the R-block of f_1, or viceversa. However, in that case we can move the portion of the L-block of f_1 that intersects u to the right and thus we find a suitable LR-ordering, therefore this case is not possible.

Case (1.2) Suppose now there is a sequence of rows $f_2, \dots, f_k$ for some $k \geq 3$ blocking the reordering such that f_i and f_{i+1} overlap for each $i \in \{2, \dots, k\}$. Moreover, there is either –at least– one row that overlaps a_1 or b_1. We may assume without loss of generality that f_2 is such a row and that f_2 and b_1 overlap. Suppose that f_2 and f_3 are unlabeled rows. Notice that, either all the rows are chained to the left of f_2 or to the right. Furthermore, since A contains no M_0 and we assumed that b_1 and f_2 overlap, if f_i is chained to the left of f_2, then f_i is contained in b_1 for every $i \geq 3$, and if f_i is chained to the right of f_2, then f_i is contained in u for every $3 \leq i < k$. In either case, we find $M_{II}(4)$ considering the columns j_2, j_1, j_3 and j_5. Suppose that f_2 is the only labeled row in the sequence and that f_2 is labeled with R. If u and f_2 overlap, then we find $S_6'(3)$ as in the previous paragraphs. Thus, we assume u is nested in f_2. Since the sequence of rows is blocking the reordering, the rows $f_3, \dots, f_k$ are chained one to one to the right and $f_k = u$, therefore we find $S_6(k)$ as a subconfiguration. The only remaining possibility is that there are two labeled rows in the sequence blocking the reordering. Since there are no D_1 or $S_3(3)$, then either these two rows are labeled with the same letter and nested, or they are labeled one with L and the other with R and are disjoint. We may assume without loss of generality that f_2 and f_k are such labeled rows.

If f_2 and f_k are both labeled with L, then necessarily one is nested in the other, for Π is an LR-ordering. In that case, one has a 0 in column j_1 and the other has a 1, for if not we can reorder the columns moving j_1 –and maybe some other columns in which f_1 has a 0– to the right. Hence, in this case we find $S_5(k)$ as a subconfiguration of the submatrix given by considering the rows $f_1, f_2, \dots, f_k$. It is analogous if f_2 and f_k are labeled with R.

If instead f_2 and f_k are labeled one with L and the other with R, then we have two possibilities. Either $f_2, \dots, f_{k-1}$ are nested in a_1, or f_2 is chained to the right of u and f_3 is chained to the left. In either case, if f_2 or f_3 have a 1 in some column in which f_1 has a 0 and u has a 1, then we find $S_6'(3)$. If instead f_3 is nested in a_1 and f_2 is nested in b_1, then we find M_V as a subconfiguration considering the columns j_4, j_2, j_1, j_3 and j_5.

Case (2) *Suppose now that there is a row f_2 such that the L-block a_1 of f_1 and the R-block b_2 of f_2 are not disjoint.* Notice that, by definition of R-block, f_2 is either labeled with R or LR. Once more, we consider $j_1 = r(a_1) + 1$ the first column in which f_1 has a 0.

Since a_1 and b_2 intersect, there is a column $j_2 < j_1$ such that a_1 and b_2 both have a 1 in column j_2.

Case (2.1) Suppose first that there is exactly one row f_3 blocking the possibility of reordering the columns to obtain a suitable LR-ordering. Notice that, for a row to block the reordering of the columns, such row must have a 1 in j_2 and at least one column with a 0. We have three possible cases:

Case (2.1.1) Suppose first that f_3 is unlabeled. If f_2 is labeled with LR and f_3 does not intersect the L-block of f_2, then we can move to the R-block of f_1 those columns in which f_3 has 0 and a_1 has 1. If f_3 intersects the L-block of f_2, then this is precisely as in the previous case. Thus, we assume f_2 is labeled with R. If f_3 is not nested in either f_1 nor f_2, then there is a column j_3 in which f_3 and f_2 have a 1 and f_1 has a 0, and a column j_4 in which f_3 and f_1 have a 1 and f_2 has a 0. In that case, we find $S_6(3)$ considering the columns j_1, j_2, j_3, j_4 and both tag columns. If f_3 is nested in f_2, then we can reorder by moving to the right all the columns in which a_1 and f_2 both have 1 and mantaining those columns in which f_3 has a 1 together. If instead f_3 is nested in f_1, then we find $S'_6(3)$ as a subconfiguration.

Case (2.1.2) Suppose now that f_3 is labeled with L. If f_2 is labeled with R, then f_2 and f_3 are colored with distinct colors, for if not we find D_1. Thus, we find D_5 as a subconfiguration in the submatrix given by f_1, f_2, f_3. Moreover, notice that, if f_3 is also labeled with R, then it is possible to move all those columns of a_1 that have a 1 and intersect f_2 (and f_3) in order to obtain a suitable LR-ordering and thus f_3 did not block the reordering. If instead f_2 is an LR-row, then we find either D_7, D_8 or D_9, depending on where is the string of 0's in row f_3. Also notice that it is indistinct in this case if f_3 is labeled with R.

Case (2.1.3) Suppose f_3 is labeled with LR. If f_2 is an LR-row, since A is admissible, then either f_3 is nested in f_1 or f_3 is nested in f_2 (we may assume this since it is analogous if f_3 contains f_1 or f_2: we will see that f_3 is not blocking the reordering). If f_3 is nested in f_2, then we can move the part of the L-block a_1 that intersects b_2 all the way to the right and then we have a suitable reordering. It is analogous if f_3 is nested in f_1. If f_2 is labeled with R, then we may assume that f_2 is not nested in f_3, for if not we have a similar situation as in the previous paragraphs. The same holds if f_1 and f_3 are nested LR-rows. We know that the L-block a_3 of f_3 intersects the R-block $b_2 = r_2$. Hence, in the column $j_3 = r(a_3) + 1$ the row f_3 has a 0 and f_2 has a 1, and in the column $j_4 = l(b_2) - 1$ the row f_3 has a 1 and f_2 has a 0. Moreover, since f_1 and f_3 are not nested, then there is a column greater than j_2 in which f_1 has a 0 and f_2 and f_3 have a 1. In this case, we find D_8 as a subconfiguration.

Case (2.2) Suppose now that it is not possible to reorder the columns to obtain a suitable LR-ordering, since there is a sequence of rows $f_3, \dots, f_k$, with $k > 3$, blocking –in particular– the reordering of the columns $j_1 = r(a_1) + 1$ and $j_2 = r(a_1)$.

We may assume that the sequence of rows is either chained to the right –and thus f_k is labeled with R– or to the left –and thus f_k is labeled with L, for if not we find M_V as in the first case. Suppose that f_2 is labeled with R. If the sequence $f_3, \dots, f_k$ is chained to the left, then we find $S_4(k)$ as a subconfiguration. If instead the sequence $f_3, \dots, f_k$ is chained to the right, then we find $S_1(k)$. Suppose now that f_2 is an LR-row. Since the L-block of f_1 and the R-block of f_2 intersect, then these rows are not nested. Whether the sequence is chained to the right or to the left, we may assume that f_3 is nested in a_1 and is disjoint with a_2. Let k be the number of 0's between the L-block and the R-block of f_2. Depending on whether k is odd or even, we find $S_0(k)$ or $S_8(k)$, respectively, as a subconfiguration of the submatrix given by considering the rows $f_1, f_2, \dots, f_{k+3}$.

This finishes the proof.

□

Definition 3.24. *Let* A *be an enriched matrix. We say* A *is* partially 2-nested *if the following conditions hold:*

- *A is admissible, LR-orderable and contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$.*
- *Each pair of non-LR-rows colored with the same color are either disjoint or nested in A.*
- *If an L-block (resp. R-block) of an LR-row is colored, then any non-LR row colored with the same color is either disjoint or contained in such L-block (resp. R-block).*
- *If an L-block (resp. R-block) of an LR-row f_1 is colored and there is a distinct LR-row f_2 for which its L-block (resp. R-block) is also colored with the same color, then f_1 and f_2 are nested in A.*

Remark *3.25.* Notice that the second statement of the definition of partially 2-nested implies that there are no monochromatic gems or monochromatic weak gems in A, since A is admissible and thus any two labeled non-LR-rows do not contain D_1 as a subconfiguration. Moreover, the third statement implies that there are no monochromatic weak gems in A. Furthermore, the last statement implies that there are no badly-colored doubly-weak gems in A.

The following Corollary is a consequence of the previous remark and Theorem 3.21.

Corollary 3.26. *An admissible matrix A is partially 2-nested if and only if A contains no M_0, $M_{II}(4)$, M_V, monochromatic gems nor monochromatic weak gems nor badly-colored doubly-weak gems and the tagged matrix A^*_{tag} does not contain any Tucker matrices, $M'_2(k)$, $M''_2(k)$, $M'_3(k)$, $M''_3(k)$ for $k \geq 3$, M'_4, M''_4, M'_5, M''_5.*

3.4 A characterization of 2-nested matrices

We begin this section stating and proving a lemma that characterizes when a partial 2-coloring can be extended to a total proper 2-coloring, for every partially 2-colored connected graph G. Then, we give the definition and some properties of the auxiliary matrix $A+$, which will help us throughout the proof of Theorem 3.12 at the end of the section.

Lemma 3.27. *Let G be a connected graph with a partial proper 2-coloring of the vertices. Then, the partial 2-coloring can be extended to a total proper 2-coloring of the vertices of G if and only if all of the following conditions hold:*

— *There are no even induced paths such that the only colored vertices of the path are its endpoints, and they are colored with the same color*
— *There are no odd induced paths such that the only colored vertices of the path are its endpoints, and they are colored with distinct colors*
— *There are no induced uncolored odd cycles*
— *There are no induced odd cycles with exactly one colored vertex*
— *There are no induced cycles of length 3 with exactly on uncolored vertex*

Proof. The if case is trivial.

On the other hand, for the only if part, suppose all of the given statements hold. Notice that, since G has a given proper partial 2-coloring of the rows, then there are no adjacent vertices pre-colored with the same color.

Let H be the induced uncolored subgraph of G. We will prove this by induction on the number of vertices of H.

For the base case, this is to say when $|H| = 1$, let v in H. If v cannot be colored, since v is the only uncolored vertex in G, then there are two vertices x_1 and x_2 such that x_1 and x_2 have distinct colors. Thus, the set $\{x_1, v, x_2\}$ either induces an odd path in G of length 3 with the endpoints colored with distinct colors, or an induced C_3 with exactly one uncolored vertex, which results in a contradiction.

For the inductive step, suppose that we can extend the partial 2-coloring of G to a proper 2-coloring if $|V(H)| \leq n$.

Suppose that $|V(H)| = n + 1$. If H is not connected, then for any isolated vertex we have the same situation as in the base case. Hence, we assume H is connected. Let v in H such that $N(v) \cap V(G - H)$ is maximum. Every vertex w in $N(v) \cap V(G - H)$ must be colored with the same color, for if not we find either a C_3 with exactly on uncolored vertex or an odd induced path with its endpoints colored with distinct colors. Suppose that such a color is red. Thus, we can color v with blue. We will see that the graph G' defined as $G' = (G - H) \cup \{v\}$ fullfils every listed property. It is straightforward that there are no uncolored odd cycles, for there were no odd uncolored cycles in H. Furthermore, using the same argument, we see that there are no induced odd cycles with exactly one colored vertex, for this would imply that there is an odd uncolored cycle C in H such that v is a vertex of C.

Since every statement of the list holds for G when H is uncolored, if there was an even induced path P such that the only colored vertices are its endpoints and they are colored with the same color, then the only possibility is that one of the endpoints of P is v. Let v_1 be the uncolored vertex of P such that v_1 is adjacent to v. Since $N(v) \cap V(G-H)$ is maximum, then there is a vertex w in $N(v) \cap V(G-H)$ such that w is nonadjacent to v_1. Hence, there is an odd induced path P' in the pre-colored G given by $< P, w >$ such that the only colored vertices of P' are its endpoints and they are colored with the same color, which results in a contradiction.

The same argument holds if there is an odd induced path in $H - \{v\}$.

Finally, there are no C_3 with exactly one uncolored vertex, for in that case we would have an odd cycle in the pre-colored G with exactly one colored vertex, and this results once more in a contradiction.

□

Let A be an enriched matrix, and let A_{LR} be the enriched submatrix of A given by considering every LR-row of A. We now give a useful property for this enriched submatrix when A is admissible.

Lemma 3.28. *If A is admissible, then A_{LR} contains no $F_1(k)$ or $F_2(k)$, for every odd $k \geq 5$.*

Proof. Toward a contradiction, suppose that A_{LR} contains either $F_1(k)$ or $F_2(k)$ in A_{LR} as subconfiguration, for some odd $k \geq 5$. Moreover, since $k \geq 5$, we find the following enriched submatrix in A as a subconfiguration:

$$\begin{array}{c} \textbf{LR} \\ \textbf{LR} \\ \textbf{LR} \end{array} \begin{pmatrix} 1100 \\ 0110 \\ 0011 \end{pmatrix}$$

Since these three rows induce D_{13}, this is not possible. It follows from the same argument that there is no $F_2(k)$ in A_{LR}. Therefore, if A_{LR} contains no D_{13}, then A_{LR} contains no $F_1(k)$ or $F_2(k)$, for all odd $k \geq 5$.

□

Remark 3.29. It follows from Lemma 3.28 that, if A is admissible, then there is a partition of the LR-rows of A into two subsets S_1 and S_2 such that every pair of rows in each subset are either nested or disjoint. Moreover, since A contains no D_{11} as a subconfiguration, every pair of LR-rows that lie in the same subset S_i are nested, for each $i = 1, 2$. Equivalently, the LR-rows in each subset S_i are totally ordered by inclusion, for each $i = 1, 2$.

Let A be an admissible matrix, let S_1 and S_2 be a partition of the LR-rows of A such that every pair of rows in S_i is nested, for each $i = 1, 2$. Since there is no D_0, there is a row m_L such that m_L is labeled with L and contains every L-block of those rows in A that are labeled with L. Analogously, we find a row m_R such that every R-block of a row in A labeled with R is contained in m_R. Moreover, there are two rows m_1 in S_1 and m_2 in S_2 such that every row in S_i is contained in m_i, for each $i = 1, 2$. This property allows us to well define the following auxiliary matrix, which will be helpful throughout the proof of Theorem 3.12.

$$B = \begin{array}{c} \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10000 \\ 11001 \\ 11111 \\ 01100 \\ 11100 \\ 00000 \\ 00011 \end{pmatrix} \qquad B+ = \begin{array}{c} \mathbf{L} \\ \mathbf{L} \\ \mathbf{R} \\ \mathbf{L} \\ \mathbf{R} \\ \\ \mathbf{L} \\ \mathbf{R} \end{array} \begin{pmatrix} 1000000 \\ 1100010 \\ 0000110 \\ 1110001 \\ 0001101 \\ 0110000 \\ 1110000 \\ 0001100 \end{pmatrix}$$

Figure 3.18 – Example of an enriched admissible matrix B and $B+$. The last two columns of $B+$ are c_{r_2} and c_{r_3}.

Definition 3.30. *Let A be an enriched matrix and let Π be a suitable LR-ordering of A. The enriched matrix $A+$ is the result of applying the following rules to A:*

- *Every empty row is deleted.*
- *Each LR-row f with exactly one block is replaced by a row labeled with either L or R, depending on whether it has an L-block or an R-block.*
- *Each LR-row f with exactly two blocks, is replaced by two uncolored rows, one having a 1 in precisely the columns of its L-block and labeled with L, and another having a 1 in precisely the columns of its R-block and labeled with R. We add a column c_f with 1 in precisely these two rows and 0 otherwise.*
- *If there is at least one row labeled with L or R in A, then each LR-row f whose entries are all 1's is replaced by two uncolored rows, one having a 1 in precisely the columns of the maximum L-block and labeled with L, and another having a 1 in precisely the complement of the maximum L-block and labeled with R. We add a column c_f with 1 in precisely these two rows and 0 otherwise.*

Notice that every non-LR-row remains the same.

Remark 3.31. Let A be a partially 2-nested matrix. Since A is admissible, LR-orderable and contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$, then by Theorem 3.23 we know that there exists a suitable LR-ordering Π. Hence, whenever we consider defining the matrix $A+$ for such a matrix A, we will always use a suitable LR-ordering Π to do so.

Let us consider $A+$ as defined in 3.30 according to a suitable LR-ordering Π. Suppose there is at least one LR-row in A. Recall that, since A is admissible, the LR-rows may be split into two disjoint subsets S_1 and S_2 such that the LR-rows in each subset are totally ordered by inclusion. This implies that there is an inclusion-wise maximal LR-row m_i for each S_i, $i = 1, 2$. If we assume that m_1 and m_2 overlap, then either the L-block of m_1 is contained in the L-block of m_2 and the R-block of m_1 contains the R-block of m_2, or viceversa. Hence, if there is at least one LR-row in A, since Π is suitable and A contains no D_1, D_4 or uncolored rows labeled with either L or R, then the following holds:

- There is an inclusion-wise maximal L-block b_L in $A+$ such that every R-block in $A+$ is disjoint with b_L.

— There is an inclusion-wise maximal R-block b_R in $A+$ such that every L-block in $A+$ is disjoint with b_L.

Therefore, when defining $A+$ we replace each LR-row having two strings of 1's by two distinct rows, one labeled with L and the other labeled with R, such that the new row labeled with L does not intersect with any row labeled with R and the new row labeled with R does not intersect with any row labeled with L.

We denote $A + \setminus C_f$ to the submatrix induced by considering every non-c_f column of $A+$. Notice that A differs from $A+$ only in its LR-rows, which are either deleted or replaced in $A+$ by labeled uncolored rows. The following is a straightforward consequence of this.

Lemma 3.32. *If A is admissible and LR-orderable, then $A + \setminus C_f$ is admissible and LR-orderable.*

Let us consider an enriched $(0,1)$-matrix A. From now on, for each row f in A that is colored, we consider its blocks colored with the same color as f in A.

Definition 3.33. *A 2-color assignment for the blocks of an enriched matrix A is a* proper 2-coloring *if A is admissible, the L-block and R-block of each LR-row of A are colored with distinct colors, and A contains no monochromatic gems, weak monochromatic gems or badly-colored doubly-weak gems as subconfigurations.*

Given a 2-color assignment for the blocks an enriched matrix A, we say it is a proper 2-coloring of $A+$ *if it is a proper 2-coloring of A.*

Remark 3.34. Let A be an enriched matrix. If A is admissible, then the given pre-coloring of the blocks is a (partial) proper 2-coloring. This follows from the fact that every pre-colored row is either labeled with L or R, of is an empty LR-row, thus there are no monochromatic gems, monochromatic weak gems or badly-colored weak gems in A for they would induce D_1.

In Figure 3.18 we give an example of the matrix B with a pre-coloring that is a proper 2-coloring, since B is admissible and contains no monochromatic gems, monochromatic weak gems or badly-colored doubly-weak gems (there is no pre-colored nonempty LR-row).

In Figure 3.19, we show two distinct coloring extensions for the pre-coloring of B, and how each of these colorings induce a coloring for $B+$. The first one –represented by $B^{(1)}$– is a proper 2-coloring of A, whereas the second one represented by $B^{(2)}$ is not. This follows from the fact that the first LR-row and the first L-row of $B^{(2)}$ induce a monochromatic weak gem.

The following is a straightforward consequence of Remark 3.25.

Lemma 3.35. *Let A be an enriched matrix. If A is partially 2-nested, then the given pre-coloring of A is a proper partial 2-coloring. Moreover, if A is partially 2-nested and admits a total 2-coloring, then A with such 2-coloring is partially 2-nested.*

Lemma 3.36. *Let A be an enriched matrix. Then, A is* 2-nested *if A is partially 2-nested and the given partial block bi-coloring of A can be extended to a total proper 2-coloring of A.*

Proof. Let A be an enriched matrix that is partially 2-nested and for which the given pre-coloring of the blocks can be extended to a total proper 2-coloring of A. In particular, this induces a total block bi-coloring for A. Indeed, we want to see that a proper 2-coloring induces a total block bi-coloring for A. Notice that the only pre-colored rows may be those labeled with L or R and those empty LR-rows.

Let us see that each of the properties that define 2-nested hold.

$$B = \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10000 \\ 11001 \\ 11111 \\ 01100 \\ 11100 \\ 00000 \\ 00011 \end{pmatrix} \qquad B^{(1)} = \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10000 \\ 11001 \\ 11111 \\ 01100 \\ 11100 \\ 00000 \\ 00011 \end{pmatrix} \qquad B^{(2)} = \begin{array}{r} \mathbf{LR} \\ \mathbf{LR} \\ \mathbf{LR} \\ \\ \mathbf{L} \\ \mathbf{LR} \\ \mathbf{R} \end{array} \begin{pmatrix} 10000 \\ 11001 \\ 11111 \\ 01100 \\ 11100 \\ 00000 \\ 00011 \end{pmatrix}$$

$$B+^{(1)} = \begin{array}{r} \mathbf{L} \\ \mathbf{L} \\ \mathbf{R} \\ \mathbf{L} \\ \mathbf{R} \\ \\ \mathbf{L} \\ \mathbf{R} \end{array} \begin{pmatrix} 1000000 \\ 1100010 \\ 0000110 \\ 1110001 \\ 0001101 \\ 0110000 \\ 1110000 \\ 0001100 \end{pmatrix} \qquad B+^{(2)} = \begin{array}{r} \mathbf{L} \\ \mathbf{L} \\ \mathbf{R} \\ \mathbf{L} \\ \mathbf{R} \\ \\ \mathbf{L} \\ \mathbf{R} \end{array} \begin{pmatrix} 1000000 \\ 1100010 \\ 0000110 \\ 1110001 \\ 0001101 \\ 0110000 \\ 1110000 \\ 0001100 \end{pmatrix}$$

Figure 3.19 – Example of a proper and a non-proper 2-coloring extension for the admissible matrix B and the respective induced colorings for $B+$. The last two colums of $B+$ are c_{r_2} and c_{r_3}.

1. Since A is an enriched matrix and the only rows that are not pre-colored are the nonempty LR-rows and those that correspond to U-blocks, then there is no ambiguity when considering the coloring of the blocks of a pre-colored row (Prop. 2 of 2-nested).

2. If A is partially 2-nested, then in particular is admissible, LR-orderable and contains no M_0, $M_{II}(4)$ or M_V. Thus, by Theorem 3.23, there is a suitable LR-ordering Π for the columns of A. We consider A ordered according to Π from now on. Since Π is suitable, then every L-block of an LR-row and an R-block of a non-LR-row are disjoint, and the same holds for every R-block of an LR-row and an L-block of a non-LR-row (Prop. 4 of 2-nested).

3. Since A is admissible, thus there are no subconfigurations as in $\mathcal{D}$. Moreover, since A is partially 2-nested, by Corollary 3.26 there are no monochromatic gems or weak gems and no badly-colored doubly-weak gems induced by pre-colored rows. It follows from this and the fact that the LR-ordering is suitable, that Prop. 8 of 2-nested holds.

4. The pre-coloring of the blocks of A can be extended to a total proper 2-coloring of A. This induces a total block bi-coloring for A, for which we can deduce the following assertions:
   — Since there is a total proper 2-coloring of A, in particular the L-block and R-block of each LR-row are colored with distinct colors. (Prop. 1 of 2-nested).
   — Each L-block and R-block corresponding to distinct LR-rows with nonempty intersection are also colored with distinct colors since there are no badly-colored doubly-weak gems in A (Prop. 9 of 2-nested).
   — Since A is admissible, every L-block and R-block corresponding to distinct non-LR-rows are colored with different colors since there is no D_1 in A (Prop. 5 of 2-nested)
   — Since there are no monochromatic weak gems in A, an L-block of an LR-row and an L-block of a non-LR row that contains the L-block must be colored with distinct colors. Furthermore, if any L-block and a U-block are not disjoint and are colored with the same color, then the U-block is contained in the L-block. (Prop. 3 and 7 of 2-nested)
   — There is no monochromatic gem in A, then each two U-blocks colored with the same color are either disjoint or nested. (Prop. 6 of 2-nested)

□

Lemma 3.37. *Let A be an enriched matrix. If A admits a suitable LR-ordering, then A contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$.*

Proof. The result follows trivially if A contains no LR-rows, since A admits an LR-ordering, thus if we consider A without its LR-rows, that submatrix has the C1P and hence it contains no Tucker matrices. Toward a contradiction, suppose that A contains either M_0, $M_{II}(4)$, M_V or $S_0(k)$ for some even $k \geq 4$. Since there is no $M_I(k)$ for every $k \geq 3$, then in particular there is no M_0 or $S_0(k)$ where at most one of the rows is an LR-row. Moreover, it is easy to see that, if we reorder the columns of M_0, then there is no possible LR-ordering in which every L-block and every R-block are disjoint. Similarly, consider $S_0(4)$, whose first row has a 1 in every column. We may assume that the last row is an LR-row for any other reordering of the columns yields an analogous situation with one of the rows. However, whether the first row is unlabeled or not, the first and the last row prevent a suitable LR-ordering. The reasoning is analogous for any even $k > 4$.

Suppose that A contains M_V, and let f_1,f_2, f_3 and f_4 be the rows of M_V depicted as follows:

$$M_V = \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{pmatrix} 11000 \\ 00110 \\ 11110 \\ 10011 \end{pmatrix}$$

If the first row is an LR-row, then either f_3 or f_4 is an LR-row, for if not we find $M_I(3)$ in A^*, which is not possible since there is an LR-ordering in A. The same holds if the second row is an LR-row. If f_3 is an LR-row, then f_4 is an LR-row, for if not f_4 must have a consecutive string of 1′s, thus, if f_4 is an unlabeled row, then it intersect both blocks of f_3, and if f_4 is an R-row, then its R-block intersects the L-block of f_3. However, if we move the columns so that the L-block of f_3 does not intersect the R-block of f_4, then we either cannot split f_1 into two blocks such that one starts one the left and the other ends on the right, of we cannot maintain a consecutive string of 1′s in f_2. It follows analogously if we assume that f_4 is an LR-row, thus f_1 is not an LR-row. By symmetry, we assume that f_2 is also a non-LR-row, and thus the proof is analogous if only f_3 and f_4 may be LR-rows.

Suppose A contains $M_{II}(4)$. Let us denote f_1, f_2, f_3 and f_4 to the rows of $M_{II}(4)$ depicted as follows:

$$M_{II}(4) = \begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{pmatrix} 0111 \\ 1100 \\ 0110 \\ 1101 \end{pmatrix}$$

If f_2 is an LR-row, then necessarily f_3 or f_4 are LR-rows, for if not we find $M_I(3)$ in A^*. If only f_2 and f_3 are LR-rows, then we find $M_{II}(4)$ in A^*. If instead only f_2 and f_4 are LR-rows, then –as it is– whether f_1 is an R-row or an unlabeled row, the block of f_1 intersects the L-block and the R-block of f_4 (and also the L-block of f_2). The only possibility is to move the second column all the way to the right and split f_2 into two blocks and give the R-block of f_4 length 2. However in this case, it is not possible to move another column and obtain an ordering that keeps all the 1′s consecutive for f_3 and f_1 not intersecting both blocks of f_4 simultaneously. Thus, f_1 is also an LR-row. However, for any ordering of the columns, either it is not possible to simultaneously split the string of 1′s in f_1 and keep the L-block of f_2 starting on the left, or it is not possible to simultaneously maintain the string of 1′s in f_3 consecutive and the L-block of f_1 disjoint with the R-block of f_4. It follows analogously if both f_3 and f_4 are LR-rows. Hence, f_2 is a non-LR-row, and by symmetry, we may assume that f_3 is also a non-LR-row. Suppose now that f_1 is an LR-row. If f_4 is not an LR-row, then there is no possible way to reorder the columns and having a consecutive string of 1′s for the rows f_2, f_3 and f_4 simultaneously, unless we move the fourth column all the way to the left. However in that case, either f_4 is an L-row and its L-block intersects the R-block of f_1 of it is an unlabeled row that intersects both blocks of f_1. Moreover, the same holds if f_4 is an LR-row, with the difference that in this case the R-block of f_4 intersects the L-block of f_1 or the string of 1′s in f_2 and f_3 is not consecutive.

□

Lemma 3.38. *Let A be an enriched matrix. If A is 2-nested, then A is partially 2-nested and the total block bi-coloring induces a proper total 2-coloring of A.*

Proof. If A is 2-nested, then in particular there is an LR-ordering Π for the columns. Moreover, by properties 4 and 7, such an ordering is suitable.

Suppose first there is a monochromatic gem in A. Such a gem is not induced by two unlabeled rows since in that case property 6 of the definition of 2-nested matrix would not hold. Hence, such a gem is induced by at least one labeled row. Moreover, if one is a labeled row and the other is an unlabeled row, then property 7 would not hold. Thus, both rows are labeled. By property 5, if the gem is induced by two non-disjoint L-block and R-block, then it is not monochromatic, disregarding on whether they correspond to LR-rows or non-LR-rows. Hence, exactly one of the rows is an LR-row. However, by property 4, an L-block of an LR-row and an R-block of a non-LR-row are disjoint, thus they cannot induce a gem.

Suppose there is a monochromatic weak gem in A, thus at least one of its rows is a labeled row. It is not possible that exactly one of its rows is a labeled row and the other is an unlabeled row, since property 7 holds. Moreover, these rows do not correspond to rows labeled with L and R, respectively, for properties 4 and 5 hold. Furthermore, both rows of the weak gem are LR-rows, since if exactly one is an LR-row, then properties 3, 4 and 7 hold and thus it is not possible to have a weak gem. However, in that case, property 5 guarantees that this is also not possible.

Finally, there is no badly-colored doubly-weak gem since properties 4, 5 and 9 hold.

Now, let us see that A is admissible. Since there is an LR-ordering of the columns, there are no D_0, D_2, D_3, D_6, D_7, D_8 or D_{11} in A. Moreover, by property 5, there is no D_1. As we have previously seen, there are no monochromatic gems or monochromatic weak gems. Hence, it is easy to see that if there is a total block bi-coloring, then A contains none of the matrices in $\mathcal{S}$ or $\mathcal{P}$ as a subconfiguration. Suppose there is D_4. By property 8, if there are two L-blocks of non-LR-rows colored with distinct colors, then every LR-row has a nonempty L-block, and in this case such an L-block is contained in both rows labeled with L. However, by property 3, the L-block of the LR-row is properly contained in the L-blocks of the non-LR-rows, thus it must be colored with a distinct color than the color assigned to each L-block of a non-LR-row, and this leads to a contradiction. By property 4, there is no D_5. Let us suppose there is D_9 given by the rows f_1, f_2 and f_3, were f_1 is labeled with L and f_2 and f_3 are LR-rows. Suppose that f_1 is colored with red. Since the L-block of f_2 is contained in f_1, by property 3, then the L-block of f_2 is colored with blue. The same holds for the L-block of f_3. However, f_2 and f_3 are not nested, thus by property 9 the L-blocks of f_2 and f_3 are colored with distinct colors, which results in a contradiction.

Let us suppose there is D_{10} given by the rows f_1, f_2, f_3 and f_4, were f_1 is labeled with L and colored with red, f_2 is labeled with R and colored with blue, and f_3 and f_4 are LR-rows. Since the L-block of f_3 is properly contained in f_1, then by property 3, it is colored with blue. By property 1, the R-block of f_3 is colored with red. Using a similar argument, we assert that the R-block of f_4 is colored with red and the L-block of f_4 is colored with blue. However, f_3 and f_4 are non-disjoint and non-nested, thus the L-block of f_3 and the R-block of f_4 are colored with distinct colors, which results in a contradiction.

By Lemma 3.37, since there is a suitable LR-ordering, then A contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$.

Finally, by property 9 and the fact that there is an LR-ordering, there are no D_{12} nor D_{13}.

Therefore A is partially 2-nested.

Finally, we will see that the total block bi-coloring for A induces a proper total 2-coloring of A. Since every property of 2-nested holds, then it is straightforward that there are no monochromatic gems or monochromatic weak gems or badly-colored weak gems in A. For more details on this, see Remark 3.25 and Lemma 3.36 since the same arguments are detailed there. Moreover, since property 1 of 2-nested holds, the L-block and R-block of the same LR-row are colored with distinct colors. Therefore, it follows that a total block bi-coloring of A induces a proper total 2-coloring of A.

□

The following corollary is a straightforward consequence of the previous.

Corollary 3.39. *Let A be an enriched matrix. If A is partially 2-nested and B is obtained from A by extending its partial coloring to a total proper 2-coloring, then B is 2-nested if and only if for each LR-row its L-block and R-block are colored with distinct colors and B contains no monochromatic gems, monochromatic weak gems or badly-colored doubly-weak gems as subconfigurations.*

We are now ready to give the proof for the main result of this chapter.

Theorem 3.12 (continuing from p. 51)**.** *Let A be an enriched matrix. Then, A is 2-nested if and only if A contains none of the following listed matrices or their dual matrices as subconfigurations:*
— *M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even k (See Figure 3.15)*
— *Every enriched matrix in the family $\mathcal{D}$ (See Figure 3.10)*
— *Every enriched matrix in the family $\mathcal{F}$ (See Figure 3.11)*
— *Every enriched matrix in the family $\mathcal{S}$ (See Figure 3.12)*
— *Every enriched matrix in the family $\mathcal{P}$ (See Figure 3.13)*
— *Monochromatic gems, monochromatic weak gems, badly-colored doubly-weak gems*
and A^* contains no Tucker matrices and none of the enriched matrices in $\mathcal{M}$ or their dual matrices as subconfigurations (See Figure 3.14).

The proof is organized as follows. The if case follows immediately using Lemma 3.38 and the characterizations of admissibility, LR-orderable and partially 2-nested given in the previous sections. For the only if case, we have two possible cases: (1) either there are no labeled rows in A, or (2) there is at least one labeled row in A (either L, R or LR). In each case, we define an auxiliary graph $H(A)$ that is partially 2-colored according to the pre-coloring of the blocks of A. Toward a contradiction, we suppose that $H(A)$ is not bipartite. Using the characterization given in Lemma 3.27, we know there is one of the 5 possible kinds of paths or cycles, we analyse each case and reach a contradiction. A complete proof of case (1) has been published in [30].

Proof. Suppose A is 2-nested. In particular, A is partially 2-nested with the given pre-coloring and the block bi-coloring induces a total proper 2-coloring of A. Thus, by Corollary 3.26, A is admissible and contains no M_0, $M_{II}(4)$, M_V, $S_0(k)$ for every even $k \geq 4$, monochromatic gems, monochromatic weak gems or badly-colored doubly-weak gems as subconfigurations, and A^*_{tag} contains no Tucker matrices, M_4', M_4'', M_5', M_5'', $M_2'(k)$, $M_2''(k)$, $M_3'(k)$, $M_3''(k)$, $M_3'''(k)$, for any $k \geq 4$ as subconfigurations. In particular, since A is admissible, there is no D_{13} induced by any three LR-rows.

Moreover, notice that every pair of consecutive rows of any of the matrices F_0, $F_1(k)$, and $F_2(k)$ for all odd $k \geq 5$ induces a gem, and there is an odd number of rows in each matrix. Thus, if one of these matrices is a submatrix of A_{tag}, then there is no proper 2-coloring of the blocks. Therefore, A contains no F_0, $F_1(k)$, and $F_2(k)$ for any odd $k \geq 5$ as submatrices. A similar argument holds for F_0', $F_1'(k)$, $F_2'(k)$, changing 'gem' for 'weak gem' whenever one of the two rows considered is a labeled row.

Conversely, suppose A is not 2-nested. Henceforth, we assume that A is admissible.

If A is not partially 2-nested, then either A contains M_0, $M_{II}(4)$, M_V, $S_0(k)$ for some even $k \geq 4$, or there is a submatrix M in A^*_{tag} such that M represents the same configuration as one of the forbidden submatrices for partially 2-nested stated above, and thus M is a subconfiguration of A^*_{tag}.

Henceforth, we assume that A is partially 2-nested. . If A is partially 2-nested but is not 2-nested, then the pre-coloring of the rows of A (which is a proper partial 2-coloring of A since A is admissible) cannot be extended to a total proper 2-coloring of A.

Case (1) *There are no labeled rows in* A*.*

We define the auxiliary graph $H(A) = (V, E)$ where the vertex set $V = \{w_1, \dots, w_n\}$ has one vertex for each row in A, and two vertices w_i and w_k in V are adjacent if and only if the rows $a_{i.}$ and $a_{k.}$ are neither disjoint nor nested. By abuse of language, w_i will refer to both the vertex w_i in $H(A)$ and the row $a_{i.}$ of A. In particular, the definitions given in the introduction apply to the vertices in $H(A)$; i.e., we say two vertices w_i and w_k in $H(A)$ are *nested* (resp. *disjoint*) if the corresponding rows $a_{i.}$ and $a_{k.}$ are nested (resp. disjoint). And two vertices w_i and w_k in $H(A)$ *start* (resp. *end*) *in the same column* if the corresponding rows $a_{i.}$ and $a_{k.}$ start (resp. end) in the same column. It follows from the definition of 2-nested matrices that A is a 2-nested matrix if and only if there is a bicoloring of the auxiliary graph $H(A)$ or, equivalently, if $H(A)$ is bipartite (i.e., $H(A)$ does not have contain cycles of odd length), since there are no labeled rows in A,and thus there are no pre-colored vertices in H.

Let Π be a linear ordering of the columns such that the matrix A does not contain any F_0, $F_1(k)$ and $F_2(k)$ for every odd $k \geq 5$ or Tucker matrices as subconfigurations. Due to Tucker's Theorem, since there are no Tucker submatrices in A, the matrix A has the C1P.

Toward a contradiction, suppose that the auxiliary graph $H(A)$ is not bipartite. Hence there is an induced odd cycle C in $H(A)$.

Suppose first that $H(A)$ has an induced odd cycle $C = w_1, w_2, w_3, w_1$ of length 3, and suppose without loss of generality that the first rows of A are those corresponding to the cycle C. Since w_1 and w_2 are adjacent, both begin and end in different columns. The same holds for w_2 and w_3, and w_1 and w_3. We assume without loss of generality that the vertices start in the order of the cycle, in other words, that $l_1 < l_2 < l_3$.

Since w_1 starts first, it is clear that $a_{2l_1} = a_{3l_1} = 0$, thus the column $a_{.l_1}$ of A is the same as the first column of the matrix F_0.

Since A has the C1P and w_1 and w_2 are adjacent, then $a_{1l_2} = 1$. As stated before, w_2 starts before w_3 and thus $a_{3l_2} = 0$. Hence, column $a_{.l_2}$ is equal to the second column of F_0.

The third column of F_0 is $a_{.l_3}$, for w_3 is adjacent to w_1 and w_2, hence it is straightforward that $a_{1l_3} = a_{2l_3} = a_{3l_3} = 1$.

To find the next column of F_0, let us look at column $a_{.(r_1+1)}$. Notice that $r_1 + 1 > l_3$. Since w_1 is adjacent to w_2 and w_3, and w_2 and w_3 both start after w_1, then necessarily $a_{2(r_1+1)} = a_{3(r_1+1)} = 1$, and thus $a_{.(r_1+1)}$ is equal to the fourth column of F_0.

Finally, we look at the column $a_{.(r_2+1)}$. Notice that $r_2 + 1 > r_1 + 1$. Since A has the C1P, $a_{1(r_2+1)} = 0$ and $r_2 + 1 > r_1 + 1$, then $a_{1(r_2+1)} = 0$ and $a_{3(r_2+1)} = 1$, which is equal to last column of F_0. Therefore we reached a contradiction that came from assuming that there is a cycle of length 3 in $H(A)$.

Suppose now that $H(A)$ has an induced odd cycle $C = w_1, \dots, w_k, w_1$ of length $k \geq 5$. We assume without loss of generality that the first k rows of A are those in C and that A is ordered according to the C1P.

***Remark* 3.40.** Let w_i, w_j be vertices in $H(A)$. If w_i and w_j are adjacent and w_i starts before w_j, then $a_{ir_i} = a_{jr_i} = 1$ and $a_{i(r_i+1)} = 0$, $a_{j(r_i+1)} = 1$.

***Remark* 3.41.** If $l_{i-1} > l_i$ and $l_{i+1} > l_i$ for some $i = 3, \dots, k-1$, then for all $j \geq i+1$, w_j is nested in w_{i-1}. The same holds if $l_{i-1} < l_i$ and $l_{i+1} < l_i$. Since $l_{i-1} > l_i$ and $l_{i+1} > l_i$, then w_{i-1} and w_{i+1} are not disjoint, thus necessarily w_{i+1} is nested in w_{i-1}. It follows from this argument that this holds for $j \geq i+1$.

Notice that w_2 and w_k are nonadjacent, hence they are either disjoint or nested. Using this fact and Remark 3.40, we split the proof into two cases.

Case (1.1) *w_2 and w_k are nested* We may assume without loss of generality that w_k is nested in w_2, for if not, we can rearrange the cycle backwards as w_1, w_k, $w_{k-1}, \dots, w_2, w_1$. Moreover, we will assume without loss of generality that both w_2 and w_k start before w_1. First, we need the following Claim.

Claim 3.42. *If w_2 and w_k are nested, then w_i is nested in w_2, for $i = 4, \dots, k-1$.*

Suppose first that w_1 and w_3 are disjoint, and toward a contradiction suppose that w_2 and w_4 are disjoint. In this case, $l_4 < l_3 < r_4 < l_2 < r_3 < r_2$. The contradiction is clear if $k = 5$. If instead $k > 5$ and w_5 starts before w_4, then $r_i < l_3$ for all $i > 5$, which contradicts the assumption that w_k is nested in w_2. Hence, necessarily w_5 is nested in w_3 and w_5 and w_2 are disjoint. This implies that $l_3 < l_5 < r_4 < r_5 < l_2$ and once more, $r_i < l_2$ for all $i > 5$, which contradicts the fact that w_k is nested in w_2.

Suppose now that w_3 is nested in w_1. Toward a contradiction, suppose that w_4 is not nested in w_2. Thus, w_2 and w_4 are disjoint since they are nonadjacent vertices in $H(A)$. Notice that, if w_3 is nested in w_1, then $l_2 < l_3$ and $r_2 < r_3$. Furthermore, since w_4 is adjacent to w_3 and nonadjacent to w_2, then $l_3 < r_2 < l_4 < r_3 < r_4$. This holds for every odd $k \geq 5$.

If $k = 5$, since w_5 is nested in w_2, then $r_5 < r_2 < l_4$, which results in a contradiction for w_4 and w_5 are adjacent.

Suppose that $k > 5$. If w_2 and w_i are disjoint for all $i = 5, \dots, k-1$, then w_{k-1} and w_k are nonadjacent for w_k is nested in w_2, which results in a contradiction. Conversely, if w_i and w_2 are not disjoint for some $i > 3$, then they are adjacent, which also results in a contradiction that came from assuming that w_2 and w_4 are disjoint. Therefore, since w_4 is nested in w_2, w_2 and w_i are nonadjacent and w_i is adjacent to w_{i+1} for all $i > 4$, then necessarily w_i is nested in w_2, which finishes the proof of the Claim.

Claim 3.43. *Suppose that w_2 and w_k are nested. Then, if w_3 is nested in w_1, then $l_i > l_{i+1}$ for all $i = 3, \dots, k-1$. If instead w_1 and w_3 are disjoint, then $l_i < l_{i+1}$ for all $i = 3, \dots, k-1$.*

Recall that, by the previous Claim, since w_i is nested in w_2 for all $i = 4, \dots, k$, in particular w_4 is nested in w_2. Moreover, since w_3 and w_4 are adjacent, notice that, if w_3 is nested in w_1, then $l_3 > l_4$, and if w_1 and w_3 are disjoint, then $l_3 < l_4$.

It follows from Remark 3.41 that, if $l_5 > l_4$, then w_i is nested in w_3 for all $i = 5, \dots, k$, which contradicts the fact that w_1 and w_{k-1} are adjacent. The proof of the first statement follows from applying this argument successively.

The second statement is proven analogously by applying Remark 3.41 if $l_5 < l_4$, and afterwards successively for all $i > 4$.

If w_1 and w_3 are disjoint, then we obtain $F_2(k)$ first, by putting the first row as the last row, and considering the submatrix given by columns $j_1 = l_1 - 1$, $j_2 = l_3$, …, $j_i = l_{i+1}$, …, $j_k = r_1 + 1$ (using the new ordering of the rows). If instead w_3 is nested in w_1, then we obtain $F_1(k)$ by taking the submatrix given by the columns $j_1 = l_1 - 1$, $j_2 = r_k$, …, $j_i = l_{k-i+2}$, …, $j_{k-1} = r_3$.

Case (1.2) *w_2 and w_k are disjoint*

We assume without loss of generality that $l_2 < l_1$ and $l_k > l_1$.

Claim 3.44. *If w_2 and w_k are disjoint, then $l_i < l_{i+1}$ for all $i = 2, \dots, k-1$.*

Notice first that, in this case, w_i is nested in w_1, for all $i = 3, \dots, k-1$. If not, then using Remark 3.41, we notice that it is not possible for the vertices $w_1, \dots, w_k$ to induce a cycle. This implies, in particular, that w_3 is nested in w_1 and thus $l_2 < l_3$. Furthermore, using this and the same remark, we conclude that $l_i < l_{i+1}$ for all $i = 2, \dots, k-1$, therefore proving Claim 3.44.

In this case, we obtain $F_2(k)$ by considering the submatrix given by the columns $j_1 = l_1 - 1$, $j_2 = l_3$, ..., $j_i = l_{i+1}$, ..., $j_k = r_1 + 1$.

Case (2) *There is at least one labeled row in A.*

We wish to extend the partial pre-coloring given for A. By Corollary 3.39, if B is obtained by extending the pre-coloring of A and B is 2-nested, then neither two blocks corresponding to the same LR-row are colored with the same color, nor there are monochromatic gems, monochromatic weak gems or badly-colored doubly-weak gems in B. Let us consider the auxiliary matrix $A+$, defined from a suitable LR-ordering Π of the columns of A. Notice that, if there is at least one labeled row in A, then there is at least one labeled row in $A+$ and these labeled rows in $A+$ correspond to rows of A that are labeled with either L, R, or LR.

Let $H = H(A+)$ be the graph whose vertices are the rows of $A+$. We say a vertex is an *LR-vertex (resp. non-LR vertex)* if it corresponds to a block of an LR-row (resp. non-LR row) of A. The adjacencies in H are as follows:

- Two non-LR vertices are adjacent in H if the underlying uncolored submatrix of A determined by these two rows contains a gem or a weak gem as a subconfiguration.
- Two LR-vertices corresponding to the same LR-row in A are adjacent in H.
- Two LR-vertices v_1 and v_2 corresponding to distinct LR-rows are adjacent if v_1 and v_2 are labeled with the same letter in $A+$ and the LR-rows corresponding to v_1 and v_2 overlap in A.
- An LR-vertex v_1 and a non-LR vertex v_2 are adjacent in H if the rows corresponding to v_1 and v_2 are not disjoint and v_2 is not contained in v_1.

The vertices of H are partially colored with the pre-coloring given for the rows of A.

Notice that every pair of vertices corresponding to the same LR-row f induces a gem in $A+$ that contains the column c_f, and two adjacent LR-vertices v_1 and v_2 in H do not induce a any kind of gem in $A+$, except when considering both columns c_{r_1} and c_{r_2}.

The following Claims will be useful throughout the proof.

Claim 3.45. *Let C be a cycle in $H = H(A+)$. Then, there are at most 3 consecutive LR-vertices labeled with the same letter. The same holds for any path P in H.*

Let v_1, v_2 and v_3 be 3 consecutive LR-vertices in H, all labeled with the same letter. Notice that any subset in H of LR-vertices labeled with the same letter in $A+$ corresponds to a subset of the same size of distinct LR-rows in A. By definition, two LR-vertices are adjacent in H only if they are labeled with the same letter and the corresponding rows in A contain a gem, or equivalently, if they are not nested. Moreover, notice that once the columns of A are ordered according to Π, these rows have a 1 in the first non-tag column and a 1 in the last non-tag column. Hence, if there were 4 consecutive LR-vertices v_1, v_2, v_3 and v_4 in the cycle C of H and all of them are labeled with the same letter, then v_1 and v_2 are not nested, v_2 and v_3 are not nested and v_1 and v_3 must be nested. Thus, since v_2 and v_4 and v_1 and v_4 are also nested, then v_4 either contains v_1 and v_2

or is nested in both. In either case, since v_3 and v_4 are not nested, then v_1 and v_3 are not nested and this results in a contradiction. □

Claim 3.46. *There are at most 6 uncolored labeled consecutive vertices in* C*. The same holds for any path* P *in* H*.*

This follows from the previous claim and the fact that every pair of uncolored labeled vertices labeled with distinct rows are adjacent only if they correspond to the same LR-row in A. □

If A is not 2-nested, then the partial 2-coloring given for H cannot be extended to a total proper 2-coloring of the vertices. Notice that the only pre-colored vertices are those labeled with either L or R, and those LR vertices corresponding to an empty row, which we are no longer considering when defining $A+$. According to Lemma 3.27 we have 5 possible cases.

Case (2.1) *There is an even induced path* $P = v_1, v_2, \dots, v_k$ *such that the only colored vertices are* v_1 *and* v_k*, and they are colored with the same color.*

We assume without loss of generality througout the proof that v_1 is labeled with L, since it is analogous otherwise by symmetry.

If $v_2, \dots, v_{k-1}$ are unlabeled rows, then we find either $S_2(k)$ or $S_3(k)$ which is not possible since A is admissible.

Suppose there is at least one LR-vertex in P. Recall that, an LR-vertex and a non-LR-vertex are adjcent in H only if the rows in $A+$ are both labeled with the same letter and the LR-row is properly contained in the non-LR-row.

Suppose that every LR-vertex in P is nonadjacent with each other. Let v_i be the first LR-vertex in P, and suppose first that $i = 2$. Since v_2 is an LR-vertex and is adjacent to v_1, then v_2 is labeled with L and $v_2 \subsetneq v_1$. Hence, since we are assuming there are no adjacent LR-vertices in P and $k \geq 4$, then v_3 is not an LR-vertex, thus it is unlabeled since we are considering a suitable LR-ordering to define $A+$. Let $v_3, \dots, v_j$ be the maximal sequence of consecutive unlabeled vertices in P that starts in v_3. Thus, $v_l \subseteq v_1$ for every $3 \leq l \leq j$.

Notice that there are no other LR-vertices in P: toward a contradiction, let v_j be the next LR-vertex in P. If v_j is labeled with L, since v_3 is nested in v_1, then v_j is adjacent to v_1, which is not possible. It is analogous if v_j is labeled with R. Thus, v_l is unlabeled for every $3 \leq l \leq k-1$. Moreover, the vertex v_k is labeled with L, for if not we find D_1 in A induced by v_1 and v_k and this is not possible since A is admissible. However, in that case we find $S_5(k)$.

Hence, if v_i is an isolated LR-vertex (i.e., nonadjacent to other LR-vertices), then $i > 2$. It follows that v_2 is an unlabeled vertex. Notice that a similar argument as in the previous paragraph proves that there are no more LR-vertices in P: since v_{i+1} is nested in v_{i-1}, it follows that any other LR-vertex is adjacent to v_{i-1}. Suppose first that v_i is labeled with L and let $v_2, \dots, v_{i-1}$ be the maximal sequence of unlabeled vertices in P that starts in v_2.

Since v_i is the only LR-vertex in P, if v_k is labeled with L, then necessarily $i = k-1$ for if not v_k is adjacent to v_{i-1}. However, since in that case $v_k \supsetneq v_{k-1} = v_i$ and v_k is nonadjacent to every other vertex in P, then we find $S_5(k)$. Analogously, if v_k is labeled with R, since $v_j \subseteq v_{i-1}$ for every $j > i$, then v_k is adjacent to v_{i-1} which leads to a contradiction.

Suppose now that v_i is labeled with R and remember that $i > 2$. Furthermore, v_j is unlabeled for every $j > i$. Moreover, v_j is nested in v_{i-1} for every $j > i$, for if not v_k would be adjacent to v_i. However, in that case v_k is adjacent to v_{i-1}, whether labeled with R or L, and this results in a contradiction.

Notice that we have also proven that, when considering an admissible matrix and a suitable LR-ordering to define H, there cannot be an isolated LR-vertex in such a path P, disregarding of the parity of the length of P. This last part follows from the previous and the fact that, if the length is 3 and P has one LR-vertex, since the endpoints are colored with distinct colors, then we find D_4 if the endpoints are labeled with the same letter and D_5 if the endpoints are labeled one with L and the other with R. Moreover, the ordering would not be suitable, which is a necessary condition for the well definition of $A+$, and thus of H. If the length of P is odd and greater than 3, then the arguments are analogous as in the even case. The following Claim is a straightforward consequence of the previous.

Claim 3.47. *If there is an isolated LR-vertex in P, then it is the only LR-vertex in P. Moreover, there are no two nonadjacent LR-vertices in P. Equivalently, every LR-vertex in P lies in a sequence of consecutive LR-vertices.*

We say a subpath Q of P is an *LR-subpath* if every vertex in Q is an LR-vertex. We say an LR-subpath Q in P is *maximal* if Q is not properly contained in any other LR-subpath of P.

We say that two LR-vertices v_i and v_j are *consecutive* in the path P (resp. in the cycle C) if either $j = i + 1$ or v_l is unlabeled for every $l = i + 1, \dots, j - 1$.

It follows from Claims 3.45, 3.46 and 3.47 that there is one and only one maximal LR-subpath in P. Thus, we have one subcase for each possible length of such maximal LR-subpath of P, which may be any integer between 2 and 6, inclusive.

Case (2.1.1) Let v_i and v_{i+1} be the two adjacent LR-vertices that induce the maximal LR-subpath. Suppose first that both are labeled with L and that $i = 2$. Since v_2 is an LR-vertex, v_2 is nested in v_1 and v_3 contains v_1. Moreover, v_4 is labeled with R, for if not v_4 is also adjacent to v_2. This implies that the R-block of the LR-row corresponding to v_2 contains v_4 in A, for if not we find D_6. However, either the R-block of v_2 intersects the L-block of v_3 –which is not possible since we are considering a suitable LR-ordering–, or v_3 is disjoint with v_4 since the LR-rows corresponding to v_2 and v_3 are nested, and thus we find D_6. Hence, $k > 4$.

By Claim 3.47 and since there is no other LR-vertex in the maximal LR-subpath, there are no other LR-vertices in P. Equivalently, $v_4, \dots, v_{k-1}$ are unlabeled vertices. Moreover, this sequence of unlabeled vertices is chained to the right, since if it was chained to the left, then every left endpoint of v_j for $j = 4, \dots, k-1$ would be greater than $r(v_1)$ and thus v_k results adjacent to v_2. Hence, we find $P_0(k-1, 0)$ in A as a subconfiguration of the submatrix given by considering the rows corresponding to v_1, v_2, $v_4, \dots, v_k$, which is not possible since A is admissible. The proof is analogous if $i > 2$, with the difference that we find $P_0(k-1, i)$ in A. Furthermore, the proof is analogous if v_i and v_{i+1} are labeled with distinct letters.

Case (2.1.2) Let $Q = < v_i, v_{i+1}, v_{i+2} >$ be the maximal LR-subpath of P. Suppose first that not every vertex in Q is labeled with the same letter.

If v_i is labeled with R, since there is a sequence of unlabeled vertices between v_1 and v_i, then v_{i+1} is labeled with R. This follows from the fact that if not, v_{i+1} would be adjacent to either v_1 or some vertex in the unlabeled chain. The same holds for v_{i+1} and thus we are in the previous situation. Hence, v_i is labeled with L and we have the following claim.

Claim 3.48. *For every maximal LR-subpath of P, the first vertex is labeled with L.*

Suppose v_i and v_{i+1} are both labeled with L and v_{i+2} is labeled with R. Notice that, if $i = 2$, then v_2 is labeled with L, v_4 is labeled with R and v_3 may be labeled with either L or R.

Since v_{i+1} and v_{i+2} are labeled with distinct letters, then they correspond to the same LR-row in A. Notice that v_i is contained in v_{i+1}. Thus, since v_i and v_{i+1} are adjacent, the R-block corresponding to v_i in A contains v_{i+2}. Therefore, we find $P_0(k,i)$ or $P_1(k,i)$ as a subconfiguration of the submatrix induced by considering all the rows of P.

If instead v_{i+1} and v_{i+2} are both labeled with R, then v_i and v_{i+1} are the two blocks of the same LR-row in A. Hence, since v_{i+1} and v_{i+2} are adjacent and v_k is nonadjacent to v_{i+1}, then v_{i+1} contains v_{i+2} and thus the L-block of the LR-row corresponding to v_{i+2} contains v_i. Once again, we find either $P_0(k,i)$ or $P_1(k,i)$.

Suppose now that all vertices in Q are labeled with the same letter and suppose first that $i = 2$. Since v_1 and v_2 are adjacent, then every vertex in Q is labeled with L. Notice that $k > 4$ since v_5 is uncolored and the endpoints of P are colored with the same color. Since v_2 is adjacent to v_1, then $v_1 \subsetneq v_3$ and $v_4 \subsetneq v_3$. Since k is even and $k > 4$, then v_5 is an unlabeled vertex. Moreover, for every unlabeled vertex v_j such that $j > 4$, $l(v_j) > r(v_1)$ and $r(v_j) \leq r(v_3)$, for if not v_j and v_3 would be adjacent. However, v_k is not labeled with L for in that case it would be adjacent to v_3. Furthermore, if v_k is labeled with R, then we find D_8, which is not possible since we assumed A to be admissible.

Suppose now that $i > 2$. In this case, there is a sequence of unlabeled vertices between v_1 and v_i. If every vertex in Q is labeled with L, since v_1 and v_i are nonadjacent (and thus v_1 is nested in v_i) and v_{i+1} is nonadjacent with v_{i-1}, then $v_i \subsetneq v_{i+1}$, $v_{i+2} \subsetneq v_{i+1}$. It follows that v_j is contained between $r(v_i)$ and $r(v_{i+1})$ for every $j > i+2$ and therefore v_k is adjacent either to v_{i+1} or v_i, which results in a contradiction.

If every vertex in Q is labeled with R, then $v_{i+1} \subsetneq v_i$ and $v_{i+1} \subsetneq v_{i+2}$ for if not v_{i+1} would be adjacent to v_{i-1} and v_{i+2}. Hence, if $i+2 = k-1$, then v_k would be adjacent also to v_{i+1}. Hence, there is at least one unlabeled vertex v_j with $j > i+2$. Moreover, for every such vertex v_j holds that $l(v_j) < l(v_i)$ and $r(v_j) > l(v_{i+1})$. Hence, if v_k is labeled with R, then v_k is adjacent to v_{i+1}. If instead v_k is labeled with L, then we find D_8 as a subconfiguration in the submatrix of A induced by v_k and the LR-rows corresponding to v_i and v_{i+1}.

Case (2.1.3) Let $Q = < v_i, v_{i+1}, v_{i+2}, v_{i+3} >$ be the maximal LR-subpath of P. Notice that either 2 are labeled with L and 2 are labeled with R, or 1 is labeled with L and 3 are labeled with R, or viceversa. Moreover, by Claim 3.48 we know that v_i is labeled with L. Every vertex v_j such that $1 < j < i$ or $i+3 < j < k$ is an unlabeled vertex.

Suppose first that v_i is the only vertex in Q labeled with L. Thus, v_{i+1} is the R-block of the LR-row in A corresponding to v_i. Hence, either $v_{i+1} \subsetneq v_{i+2}$ or viceversa. Notice that there is at least one unlabeled vertex v_j between v_{i+3} and v_k, for if not v_k is adjacent to v_{i+1} or v_{i+2}. Moreover, either v_j is contained in $v_{i+2} \setminus v_{i+3}$ or in $v_{i+3} \setminus v_{i+2}$ for every $j > i+4$. In any case, v_k results adjacent to either v_{i+2} or v_{i+3}, which results in a contradiction.

Hence, at least v_i and v_{i+1} are labeled with L. Suppose that v_{i+2} is labeled with R –and thus v_{i+3} is labeled with R. Notice that, if $v_{i+3} \supsetneq v_{i+2}$, then there is no possibility for v_k for, if v_k is labeled with R, then v_k is adjacent to v_{i+2} and if v_k is labeled with L, then v_k is adjacent to v_i and v_{i+1}. However, the same holds if $v_{i+2} \supsetneq v_{i+3}$ since there is at least one unlabeled vertex v_j with $j > i+3$ and thus for every such vertex holds $l(v_j) > l(v_{i+2})$ and therefore this case is not possible.

Finally, suppose that v_i, v_{i+1} and v_{i+2} are labeled with L and thus v_{i+3} is labeled with R. Thus, v_k is labeled with R and is nested in v_{i+3}. Moreover, there is a chain of unlabeled vertices v_j between v_{i+3} and v_k such that v_j is nested in v_{i+3} for every $j > i+4$. Furthermore, $v_i \subsetneq v_{i+1}$ and $v_i \subseteq v_{i+2} \subsetneq v_{i+1}$: if $i = 2$, then $v_2 \subsetneq v_1$ and since v_3 and v_4 are nonadjacent to v_1, then $v_3, v_4 \supseteq v_1$. If instead $i > 2$, then for every unlabeled vertex v_j between v_1 and v_i, $r(v_j) < r(v_i)$, except for $j = i-1$ for which holds $r(v_{i-1}) > r(v_i)$. Hence, since v_{i+1} and v_{i+2} are nonadjacent to every such vertex, then $v_j \subset v_{i+1}, v_{i+2}$ for $1 < j < i$. We find $P_0(k-3, i)$ in A since the R-block corresponding to v_i is contained in v_{i+3} and thus the R-block intersects the chain of vertices between v_{i+3} and v_k.

We have the following as a consequence of the previous arguments.

Claim 3.49. *Let v_i and v_{i+1} be the first LR-vertices that appear in P. If v_{i+1} is also labeled with L, then $v_i \subsetneq v_{i+1}$. Moreover, if v_{i+2} is also an LR-vertex that is labeled with L, then $v_{i+2} \subsetneq v_{i+1}$.*

Case (2.1.4) Let $Q =< v_i, \dots, v_{i+4} >$ be the maximal LR-subpath of P. By Claim 3.48, v_i is labeled with L. Moreover, either (1) v_i and v_{i+1} are labeled with L and v_{i+2}, v_{i+3} and v_{i+4} are labeled with R, or (2) v_i, v_{i+1} and v_{i+2} are labeled with L and v_{i+3} and v_{i+4} are labeled with R. It follows from Claim 3.49 that $v_i \subsetneq v_{i+1}$.

Let us suppose the first statement. If $v_{i+3} \subsetneq v_{i+4}$, then there is at least one unlabeled vertex in P between v_{i+4} and v_k, for if not v_k would be adjacent to v_{i+2}. Since every vertex v_j for $i+5 < j \leq k$ is contained in $v_{i+4} \setminus v_{i+3}$, it follows that v_k is adjacent to v_{i+2} and thus this is not possible. Hence, $v_{i+3} \supsetneq v_{i+4}$. Furthermore, $v_{i+3} \supsetneq v_{i+2}$, and since v_k is nonadjacent to v_{i+2}, then $v_{i+2} \supsetneq v_{i+4}$. Since there is a sequence of unlabeled vertices between v_{i+4} and v_k, then we find $P_2(k, i-2)$ if v_{i+4} is nested in the R-block of v_{i+2}, or $P_0(k-3, i-2)$ otherwise.

Suppose now (2), this is, v_i, v_{i+1} and v_{i+2} are labeled with L and v_{i+3} and v_{i+4} are labeled with R. By Claim 3.49, $v_i \subsetneq v_{i+1}$ and $v_{i+2} \subsetneq v_{i+1}$. Furthermore, since v_k is nonadjacent to v_{i+3}, it follows that $v_{i+3} \supsetneq v_{i+4}$. In this case, we find $P_2(k, i-2)$ if v_{i+4} is nested in the R-block of v_{i+2}, or $P_0(k-3, i-2)$ otherwise.

Case (2.1.5) Suppose by simplicity that the length of P is 8 (the proof is analogous if $k > 8$), and thus let $Q =< v_2, \dots, v_7 >$ be the maximal LR-subpath of P of length 6. Notice that v_8 is labeled with R and colored with the same color as v_1. Hence, v_2, v_3 and v_4 are labeled with L and v_5, v_6 and v_7 are labeled with R. By Claim 3.49, $v_2 \subsetneq v_3$ and $v_4 \subsetneq v_3$. It follows that $v_2 \subsetneq v_4$, since v_1 and v_4 are nonadjacent. Using an analogous argument, we see that $v_5 \subsetneq v_6$, $v_6 \supsetneq v_5, v_7$ and $v_7 \subsetneq v_5$ for if not it would be adjacent to v_8. Since consecutive LR-vertices are adjacent, the LR-rows corresponding to v_{i+3} and v_{i+4} are not nested, and the same holds for the LR-rows in A of v_3 and v_2. Since A is admissible, the LR-rows of v_6 and v_3 are nested. This implies that the L-block of the LR-row corresponding to v_6 contains the L-block of v_4 and v_2. Moreover, since the LR-rows of v_7 and v_5 are nested, the LR-rows of v_6 and v_7 are not and v_7 is contained in v_6, then the L-block of v_7 contains the L-block of v_6. Hence, v_7 contains v_5 and thus v_8 results adjacent to v_5, which is a contradiction.

Case (2.2) *There is an odd induced path $P =< v_1, v_2, \dots, v_k >$ such that the only colored vertices are v_1 and v_k, and they are colored with distinct colors.*

Throughout the previous case proof we did not take under special consideration the parity of k, with one exception: when $k = 5$ and the maximal LR-subpath has length 2. In other words, notice that for every other case, we find the same forbidden submatrices of admissibility with the appropriate coloring for those colored labeled rows.

Suppose that $k = 5$, the maximal LR-subpath has length 2, and suppose without loss of generality that v_2 and v_3 are the LR-vertices (it is analogous otherwise by symmetry). If both are labeled with L, then v_2 is contained in v_3 and thus the R-block of v_2 properly contains the R-block of v_3. Moreover, since v_4 is unlabeled and adjacent to v_5 –which should be labeled with R since the LR-ordering is suitable–, it follows that there is at least one column in which the R-block of the LR-row corresponding to v_3 has a 0 and v_5 has a 1. Furthermore, there exists such a column in which also the R-block of v_2 has a 1. Since v_1 and v_2 are adjacent, then $v_2 \subsetneq v_1$ and thus there is also a column in which v_2 has a 0, v_3 has a 1 and v_1 has a 1. Moreover, there is a column in which v_1, v_2 and v_3 have a 1 and v_5 and the R-blocks of v_2 and v_3 all have a 0, and an analogous column in which v_1, v_2 and v_3 have a 0 and v_5 and the R-blocks of v_2 and v_3 have a 1. It follows that there is D_{10} in A which is not possible since A is admissible. If instead v_2 is labeled with L and v_3 is labeled with R, then v_2 and v_3 are the L-block and R-block of the same LR-row r in A, respectively. We can find a column in A in which v_1 and r have a 1 and the other rows have a 0, a column in which only v_1 has a 1, a column in which only v_4 has a 1 (notice that v_4 is unlabeled), and a column in which r, v_4 and v_5 have a 1 and v_1 has a 0. It follows that there is $P_0(4,0)$ in A, which results in a contradiction.

Case (2.3) *There is an induced uncolored odd cycle* C *of length* k.

If every vertex in C is unlabeled, then the proof is analogous as in case 1, where we considered that there are no labeled vertices of any kind.

Suppose there is at least one LR-vertex in C. Notice that there no labeled vertices in C corresponding to rows in A labeled with L or R, which are the only colored rows in $A+$.

Suppose $k = 3$. If 2 or 3 vertices in C are LR-vertices, then there is either D_7, D_8, D_9, D_{11}, D_{12}, D_{13} or $S_7(3)$. If instead there is exactly one LR-vertex and since every uncolored vertex corresponds either to an unlabeled row or to an LR-row, then we find F'_0 in A.

Suppose that $k \geq 5$ and let $C = v_1, v_2, \dots, v_k$ be an uncolored odd cycle of length k. Suppose first that there is exactly one LR-vertex in C. We assume without loss of generality by symmetry that v_1 is such LR-vertex and that v_1 is labeled with L in $A+$.

Hence, either v_j is nested in v_1, or v_j is disjoint with v_1, for every $j = 3, \dots, k-1$. If v_j is nested in v_1 for every $j = 3, \dots, k-1$, since v_k is adjacent to v_1 and nonadjacent to v_j for every $j = 3, \dots, k-1$, then $l(v_k) < l(v_{k-2}) < l(v_{k-3}) < \dots < l(v_2) < r(v_1)$ and $r(v_k) > r(v_1)$. Hence, we either find $F_1(k)$ or $F'_1(k)$ induced by the columns $l(v_{k-1}), \dots, r(v_k)$.

If instead v_j is disjoint with v_1 for all $j = 3, \dots, k-1$, then v_j is nested in v_k for every $j = 3, \dots, k-2$. In this case, we find $F_2(k)$ or $F'_2(k)$ induced by the columns $l(v_k) - 1, \dots, r(v_{k-1})$.

Now we will see what happens if there is more than one LR-vertex in C. First we need the following Claim.

Claim 3.50. *If* v *and* w *in* C *are two nonadjacent consecutive LR-vertices, then there is one sense of the cycle for which there is exactly one unlabeled vertex between* v *and* w.

If $k = 5$, then we have to see what happens if v_1 and v_4 are such vertices and v_5 is an LR-vertex. We are assuming that v_2 and v_3 are unlabeled since by hypothesis v_1 and v_4 are consecutive LR-vertices in C. Suppose that v_1 and v_4 are labeled with L and for simplicity assume that $v_1 \subsetneq v_4$. Thus, v_5 is labeled with L, for if not v_5 can only be adjacent to v_1 or v_4 and not both. Moreover, since v_5 is nonadjacent to v_2, then v_5 is contained in v_1 and v_4. In this case, we find $F_2(5)$ as a subconfiguration in A.

If instead v_1 is labeled with L and v_4 is labeled with R, then v_5 is the L-block of the LR-row corresponding to v_4. In this case, we find $S_7(4)$ as a subconfiguration of A_{tag}.

Let $k > 5$, and suppose without loss of generality that v_1 and v_4 are such LR-vertices. Thus, by hypothesis, v_2 and v_3 are unlabeled vertices. Suppose first that v_1 and v_4 are labeled with L and $v_1 \subsetneq v_4$. Then $l(v_2) < l(v_3)$. If v_j is unlabeled for every $j > 4$, then v_j is nested in v_3 and thus v_k cannot be adjacent to v_1. Moreover, for every $j > 4$, v_j is not an LR-vertex labeled with L either. Suppose to the contrary that v_5 is an LR-vertex labeled with L. Since v_5 is adjacent to v_4 and the LR-rows corresponding to v_1 and v_4 are nested, then v_5 is also adjacent to v_1, which is not possible since we are assuming that $k > 5$. If instead $j > 5$, since there is a sequence of unlabeled vertices between v_4 and v_j, then $r(v_j) > l(v_3)$ and thus it is adjacent to v_3. By an analogous argument, we may assert that v_j is not an LR-vertex for every $j > 4$. The proof is analogous if $v_1 \supsetneq v_4$.

Thus, let us suppose now that v_1 is labeled with L and v_4 is labeled with R. If v_5 is the L-block of the LR-row corresponding to v_4, since v_2 and v_5 are nonadjacent, then $r(v_5) < l(v_2)$ and hence $v_5 \subsetneq v_1$. Moreover, v_6 is not an LR-vertex for in that case v_6 must be labeled with L and thus v_6 is also adjacent to v_1. Furthermore, since at least v_6 is an unlabeled vertex, then every LR-vertex v_j in C with $j > 4$ is labeled with L, for if not v_j is either adjacent to v_4 or nonadjacent to v_6 (or the maximal sequence of unlabeled vertices in C that contains v_6). Thus, we may assume that there no other LR-vertices in C, perhaps with the exception of v_k. However, if v_k is an LR-vertex labeled with L, since it is adjacent to v_1, then it is also adjacent to v_5. And if v_k is unlabeled, then v_k is adjacent to v_2, v_3 or v_4 (v_k must contain this vertices so that it results nonadjacent to them, but v_4 is the limit since v_4 is labeled with R and thus it ends in the last column).

Analogously, if v_5 is unlabeled, then v_k is nonadjacent to v_1 since it must be contained in v_3. Finally, if v_5 is an LR-vertex labeled with R, then it is contained in v_4. Thus, the only possibility is that v_{k-1} is an LR-vertex labeled with R and v_7 is the L-block of the corresponding LR-row. However, since A is admissible, either v_6 is nested in v_5 or v_6 is nested in v_4. In the first case, it results also adjacent to v_4 and in the second case it results nonadjacent to v_5, which is a contradiction. Notice that the arguments are analogous if the number of unlabeled vertices in both senses of the cycle is more than 2. Therefore, there is one sense of the cycle in which there is exactly one unlabeled vertex between any two nonadjacent consecutive LR-vertices of C. □

This Claim follows from the previous proof.

Claim 3.51. *If C is an odd uncolored cycle in H, then there are at most two nonadjacent LR-vertices.*

Suppose that v_1 and v_i are consecutive nonadjacent LR-vertices, where $i > 2$. It follows from Claim 3.50 that $i = 3$ or $i = k-1$. We assume the first without loss of generality, and suppose that v_1 is labeled with L. Suppose there is at least one more LR-vertex nonadjacent to both v_1 and v_3, and let v_j be the first LR-vertex that appears in C after v_3. It follows from Claim 3.50 that $j = 5$. If v_1 and and v_3 are labeled with distinct letters, since v_4 is an unlabeled vertex, then v_4 is contained in v_2, and thus v_5 cannot be labeled with L or R for, in either case, it would be adjacent to v_2. Thus, every LR-vertex in C must be labeled with the same letter. Let us assume for simplicity that $k = 5$ (the proof is analogous for every odd $k > 5$) and that $v_1 \subset v_3$. Since v_5 is nonadjacent to v_3, then the corresponding LR-rows are nested. The same holds for v_1 and v_3. Moreover, v_5 contains both v_1 and v_3, and the R-block of v_3 contains the R-block of v_1. Furthermore, since v_1 and v_5 are adjacent, the R-block of the LR-row corresponding to v_1 contains the R-block of the LR-row corresponding to v_5 and thus the R-block of v_3 also contains the R-block of v_5, which results in v_3 and v_5 being adjacent and thus, in a contradiction that came from assuming that there is are at least three nonadjacent LR-vertices in C. □

We now continue with the proof of the case. Notice first that, as a consequence of the previous claim and Claim 3.46, either there are exactly two nonadjacent LR-vertices in C or every LR-vertex is contained in a maximal LR-subpath of length at most 6.

Case (2.3.1) Suppose there are exactly two LR-vertices in C and that they are nonadjcent. Let v_1 and v_3 be such LR-vertices. Suppose without loss of generality that $v_1 \subset v_3$. Hence, every vertex that lies between v_3 and v_1 is nested in v_2, since they are all unlabeled vertices by hypothesis. Thus, if v_1 and v_3 are both labeled with L, then we find $F_1(k)$ contaned in the submatrix induced by the columns $r(v_1), \dots, r(v_2)$.If instead v_1 is labeled with L and v_3 is labeled with R, then we find $F_2(k)$ contained in the same submatrix.

Case (2.3.2) Suppose instead that v_1 and v_2 are the only LR-vertices in C. If v_1 and v_2 are the L-block and R-block of the same LR-row, then we find $S_8(k-1)$ in A. If instead they are both labeled with L, then every other vertex v_j in C is unlabeled and v_j is nested in v_1 or v_2 for every $j > 3$, depending on whether $v_1 \subsetneq v_2$ or viceversa. Suppose that $v_1 \subsetneq v_2$. If there is a column in which both v_3 and the R-block of v_1 have a 1, then we find $S_8(k-1)$ in A. If there is not such a column, then we find $F_2(k)$ in A.

Case (2.3.3) Suppose that the maximal LR-subpath Q in C has length 3, and suppose $Q =< v_1, v_2, v_3 >$. If v_1, v_2 and v_3 are labeled with the same letter, then either $v_2 \subsetneq v_1, v_3$ or $v_2 \supsetneq v_1, v_3$, and since v_1 and v_3 are nonadjacent if $k > 3$, either $v_3 \subsetneq v_1$ or $v_1 \subsetneq v_3$. Suppose without loss of generality that all three LR-vertices are labeled with L, $v_2 \subsetneq v_1, v_3$ and $v_1 \subsetneq v_3$. In this case, there is a sequence of unlabeled vertices between v_3 and v_1 such that the column index of the left endpoints of the vertices decreases as the vertex path index increases. As in the previous case, if there is a column such that the R-block of v_2 and v_4 have a 1, then we find $S_8(k-1)$ in A contained in the submatrix induced by the columns $r(v_1), \dots, l(v_1)$. If instead there is not such column, then we find $F_2(k)$ contained in the same submatrix.

If v_1 and v_2 are labeled with L and v_3 is labeled with R, then there is a sequence of unlabeled vertices $v_4, \dots, v_k$ such that the column index of the left endpoints of such vertices decreases as the path index increases. Moreover, since v_k is adjacent to v_1 and nonadjacent to v_2, then $v_1 \supsetneq v_2$. Hence, we find $S_7(k-1)$ contained in the submatrix of A induced by the columns $r(v_1), \dots, l(v_1)$.

Case (2.3.4) Suppose that the maximal LR-subpath Q in C has length 4 and that $Q =< v_1, v_2, v_3, v_4 >$. Suppose that v_1 and v_2 are labeled with L and v_3 and v_4 are labeled with R. If $v_1 \subsetneq v_2$, then v_k cannot be adjacent to v_1. Thus $v_2 \subsetneq v_1$ and $v_4 \supsetneq v_3$. Since there is a chain of unlabeled vertices and its left endpoints decrease as the cycle index increases, then we find $S_7(k-1)$ considering the submatrix induced by every row in A. If instead v_1 is labeled with L and the other three LR-vertices are labeled with R, then first let us notice that v_2 is the R-block of v_1, the LR-rows of v_2 and v_4 are nested and $v_3 \subsetneq v_2, v_4$. Moreover, $v_2 \subsetneq v_4$, for if not v_k would not be adjacent to v_1. Thus, the left endpoint of the chain of unlabeled vertices between v_4 and v_1 decreases as the cycle index increases. Hence, if $k = 5$, then we find $S_7(3)$ induced by the LR-rows corresponding to v_3 and v_4 and the unlabeled row corresponding to v_5. Suppose that $k > 5$. Since $v_3 \subsetneq v_2$ and v_2 is the R-block of v_1, then the L-block of the LR-row corresponding to v_3 contains both v_1 and the L-block of v_4. We find $S_7(k-3)$ in A as a subconfiguration of the submatrix induced by the rows $v_3, v_4, \dots, v_{k-1}$.

Case (2.3.5) Suppose now that $Q =< v_1, \dots, v_5 >$ is the longest LR-subpath in C, and suppose that v_1 and v_2 are labeled with L and that the remaining rows in Q are labeled with R. Since v_1 is adjacent to v_k, then $v_1 \supsetneq v_2$ and $v_5 \supsetneq v_4, v_3$. Since the LR-rows corresponding to v_3 and v_5 are nested, then v_2 is contained in the L-block corresponding to v_5, and since $v_4 \subsetneq v_5$, the R-block of v_1 is also contained in v_5. Thus, we find $S_7(k-3)$ in A as a subconfiguration considering the LR-rows corresponding to v_1 and v_5 and $v_6, \dots, v_k$. The proof is analogous if Q has length 6, and thus this case is finished.

Case (2.4) *There is an induced odd cycle $C = v_1, v_2, \dots, v_k, v_1$ with exactly one colored vertex.* We assume without loss of generality that v_1 is the only colored vertex in the cycle C, and that v_1 is labeled with L. Notice that, if there are no LR-vertices in C, then the proof is analogous as in the case in which we considered that there are no labeled vertices of any kind. Hence, we assume there is at least one LR-vertex in C.

Claim 3.52. *If there is at least one LR-vertex v_i in C and $i \neq 2$, then v_i is the only LR-vertex in C.*

Let v_i be the LR-vertex in C with the minimum index, and suppose first that v_i is labeled with L. Since $i \neq 2$ and v_1 is a non-LR row in A, then $v_i \supseteq v_1$, for if not they would be adjacent. Moreover, $v_l \subset v_i$ for every $l < i-1$. Toward a contradiction, let v_j the first LR-vertex in C with $j > i$ and suppose v_j is labeled with L. Notice that the only possibility for such vertex is $j = i+1$. This follows from the fact that, if v_{i+1} is unlabeled, then v_{i+1} is contained in v_{i-1}, and the same holds for every unlabeled vertex between v_i and v_j. Hence, if there was other LR-vertex v_j labeled with L such that $j > i+1$, then it would be adjacent to v_{i+1} which is not possible. Then, $j = i+1$ and thus v_j contains v_l for every $l \leq i$. However, v_k and v_1 are adjacent, and since v_k must be an unlabeled vertex, then v_k is not disjoint with v_i, which results in a contradiction.

Suppose that instead v_j is labeled with R. Using the same argument, we see that, if $j > i+1$, then every unlabeled vertex between v_i and v_j is contained in v_{i-1} and thus it is not possible that v_j results adjacent to v_{j-1} if it is unlabeled. Hence, $j = i+1$. Moreover, there must be at least one more LR-vertex labeled with R since if not, it is not possible for v_1 and v_k to be adjacent. Thus, v_{k-1} must be labeled with R and v_k is the L-block of the LR-row corresponding to v_{k-1}. Furthermore, v_{k-1} is contained in v_1. We find $F_2(k)$ in A as a subconfiguration in the submatrix induced by considering every row. Therefore, v_i is the only LR-vertex in C. □

The following is a straightforward consequence of the previous proof and the fact that, if v_i is the first LR-vertex in C and $i > 2$, then every unlabeled vertex that follows v_i is nested in v_{i-1}, thus if v_1 is adjacent to v_k then v_k must be nested in v_2.

Claim 3.53. *If v_i in C is an LR-vertex and $i \neq 2$, then $i = 3$.*

It follows from Claim 3.46 that there are at most 6 consecutive LR-vertices in such a cycle C. Let $Q =< v_i, \dots, v_j >$ be the maximal LR-subpath and suppose that $|Q| = 5$ and v_1 is labeled with L. Notice that, if v_i is labeled with R, then v_{j-1} and v_j are labeled with L. Moreover, since there is a sequence of unlabeled vertices between v_1 and v_i and v_{j-1} is nonadjacent to v_2, then v_{j-1} is contained in v_1 and thus it results adjacent to v_1, which is not possible. Then, necessarily v_i is labeled with L and thus v_j is labeled with R. Moreover, if $i > 2$, then v_i contains v_1 and every unlabeled vertex between v_1 and v_{i-1}, and if $i = 2$, then $v_2 \subsetneq v_1$. In either case, v_{i+1} contains v_i. Hence, at most v_{i+2} is labeled with L and there are no other LR-vertices labeled with L for they would be adjacent to v_i or v_{i+1}. In particular, the last vertex of the cycle v_k is not labeled with L, thus, since it is uncolored, v_k is an unlabeled vertex. However, v_k is adjacent to v_1, and this results in a contradiction. Therefore, it is easy to see that it is not possible to have more than 4 consecutive LR-vertices in C. Furthermore, in the case of $|Q| = 4$, either v_i and v_{i+3} are labeled with L and v_{i+1} and v_{i+2} are labeled with R, or v_i and v_{i+1} are labeled with R and v_{i+3} is labeled with L.

Claim 3.54. *Suppose v_2 is an LR-vertex and let v_i be another LR-vertex in C. Then, either $i = k$ or $i \in \{3, 4, 5\}$. Moreover, in this last case, v_j is an LR-vertex for every $2 \leq j \leq i$.*

Notice first that, if v_2 is an LR-vertex, then by definition of H, v_2 is labeled with L and $v_2 \subsetneq v_1$. If $i = 3$ or $i = k$, then we are done. Suppose that $i \neq k$ and there is a sequence of unlabeled vertices v_j between v_2 and v_i, where $j = 3, \dots, i-1$. Hence, since $v_2 \subsetneq v_1$, then $v_j \subseteq v_1$ for $j = 3, \dots, i-1$. In that case, v_i is labeled with the same letter than v_1 and v_2. Moreover, since $i \neq k$, v_1 and v_i are nonadjacent and thus $v_i \supseteq v_1$ which is not possible since $v_{i-1} \subseteq v_1$. The contradiction came for assuming that there is a sequence of unlabeled vertices between v_2 and v_i and that $v_i \neq v_k$. Hence, if $i \neq 3, k$, then every vertex between v_2 and v_i is an LR-vertex. Since we know that the maximal LR-subpath in C has length at most 4 and v_2 is an LR-vertex, then necessarily v_i must be either v_3,v_4 or v_5. □

We now split the proof in two cases.

Case (2.4.1) *v_2 is an LR-vertex.*

Suppose first that v_2 is the only LR-vertex in C. By definition of H, v_2 is labeled with L and $v_2 \subsetneq v_1$. Since there are no other LR-vertices in C, then $v_j \subseteq v_1$ for every $j < k$. In this case, we find $F'_1(k)$ as a subconfiguration contained in the submatrix of A induced by the columns $r(v_2), \ldots, r(v_k)$.

Suppose now that there is exactly one more LR-vertex v_i with $i > 2$. If $i \neq 3$, then by the previous claim we know that $i = k$. If v_k is labeled with L, then we find $F'_1(k)$ contained in the submatrix of A induced by the columns $r(v_2), \ldots, r(v_k)$. If instead v_k is labeled with R, then $r(v_1) > l(v_k)$ and this is not possible since the LR-ordering used to define A+ is suitable. Suppose that $i = 3$. If v_3 is labeled with L, then $v_3 \supseteq v_1$, and if v_3 is labeled with R, then v_3 is the R-block corresponding to the same LR-row of v_2 in A. In either case, since every other vertex v_j in C is unlabeled, then $l(v_j) > r(v_1)$ for every $j < k$. Thus, if v_3 is labeled with L, then we find $F'_2(k)$ contained in the submatrix of A induced by the columns $r(v_1), r(v_2), r(v_3), \ldots, r(v_k)$. If instead v_3 is labeled with R, then we find $S_1(k)$ in A contained in the submatrix induced by the columns $r(v_1), r_(v_{k-1}), \ldots, r(v_3)$.

Suppose that there are exactly two LR-vertices distinct than v_2. As a consequence of Claim 3.54, we see that these vertices are necessarily v_3 and v_4. If v_3 and v_4 are LR-vertices and are both labeled with L, then v_3 and v_4 correspond to two distinct LR-rows that are not nested. Moreover, since $v_2 \subsetneq v_1$, then $v_3 \supseteq v_1$ and thus $v_1 \subseteq v_4 \subsetneq v_3$. Thus, since v_5 is unlabeled and there is at least one column for which the R-blocks of v_2, v_3 and v_5 have 1, 0 and 1, respectively, we find $F_1(k)$ contained in the submatrix of A induced by columns 1 to $k-1$.

If instead v_3 or v_4 (or both) are labeled with R, then v_3 corresponds to the same LR-row in A as v_2. This follows from the fact that, if v_3 and v_4 correspond to the same LR-row in A, then v_3 is labeled with L and v_4 is labeled with R. Hence, since $v_3 \subseteq v_1$, v_k cannot be adjacent to v_1 and thus this is not possible. However, if v_3 is the R-block of the LR-row corresponding to v_2, then we find D_9 in A induced by the three rows corresponding to v_1, v_2 and v_3, and v_4.

Suppose that there are exactly three LR-vertices other than v_2. Hence, these vertices are v_3, v_4 and v_5. Recall that v_1 and v_2 are labeled with L, and that two LR-vertices labeled with distinct letters are adjacent only if they correspond to the same LR-row in A. In any case, v_5 is labeled with R. However, since v_1 is labeled with L and $v_3 \supseteq v_1$, then v_k results either adjacent to v_3, v_4 or v_5, which is a contradiction.

Case (2.4.2) *v_2 is not an LR-vertex.*

By Claim 3.53, if there is an LR-vertex v, then there are no other LR-vertices and $v = v_3$.

In either case, since there is a exactly one LR-vertex in C (we are assuming that there is at least one LR-vertex for if not the proof is as in Case 1.), then v_2 contains v_j for every $j > 3$. If v_3 is labeled with L, then there is $F'_2(k)$ as a subconfiguration in A of the submatrix given by columns $r(v_1), \ldots, l(v_2)$. If instead v_3 is labeled with R, then we find $S_1(k)$ as a subconfiguration in the same submatrix.

Case (2.5) *There is an induced 3-cycle with exactly one uncolored vertex.*

Let $C_3 = v_1, v_2, v_3, v_1$. We assume without loss of generality that v_1 and v_3 are the colored vertices. Since A+ is defined by considering a suitable LR-ordering and v_1 and v_3 are adjacent colored vertices, then v_1 and v_3 are labeled with distinct letters, for if not, the underlying uncolored matrix induced by these rows either induce D_0 or not induce any kind of gem. Moreover, v_1 and v_3 are colored with distinct colors since A is admissible and thus there is no D_1. Furthermore, v_2 is unlabeled for if not it cannot be adjacent to both v_1 and v_3, since in that case v_2 should be nested in both v_1 and v_3. However, we find F_0'' as a submatrix of A, and this is a contradiction.

This finishes the proof, since we have reached a contradiction by assuming that A is partially 2-nested but not 2-nested.

□

Chapter 4

Characterization by forbidden subgraphs for split circle graphs

The main result of this chapter is Theorem 4.1, which uses the matrix theory developed in the previous chapter.

We will denote $\mathcal{T}$ to the family of graphs obtained by considering all the odd-suns with center and those graphs whose adjacency matrix $A(S, K)$ represents the same configuration as a Tucker matrix distinct to $M_I(k)$ for every odd $k \geq 3$ or $M_{III}(k)$ for every odd $k \geq 5$. We will denote $\mathcal{F}$ to the family of graphs obtained by considering those graphs whose adjacency matrix $A(S, K)$ represents the same configuration as either F_0, $F_1(k)$ or $F_2(k)$ for some odd $k \geq 5$. For a representation of these graphs, see Figures 2.6 and 2.5.

Theorem 4.1. *Let $G = (K, S)$ be a split graph. Then, G is a circle graph if and only if G is $\{\mathcal{T}, \mathcal{F}\}$-free (See Figures 4.1 and 4.2).*

This chapter is organized as follows. In Sections 4.1, 4.2, 4.3 and 4.4 we address the problem of characterizing those split graphs that are also circle. In each of these sections, we consider a split graph G that contains a subgraph T, where T is either a tent, a 4-tent or a co-4-tent, and each of these is a case of the proof of Theorem 4.1. Using the partitions of K and S described in Chapter 2, we define one enriched $(0, 1)$-matrix for each partition K_i of K and four auxiliary non-enriched $(0, 1)$-matrices that will help us give a circle model for G. At the end of each section, we prove that G is circle if and only if these enriched matrices are 2-nested and the four non-enriched matrices are nested, giving the guidelines for a circle model in each case.

The first case, adressed in Section 4.1, consists of considering a split graph G that contains a tent as an induced subgraph. This is the simplest case, given the symmetry between most of the partitions of K and S and since the enriched matrices $\mathbb{A}_1, \ldots, \mathbb{A}_6$ that are defined in Section 4.1.1 do not have any LR-rows. In the second case, adressed in Section 4.2, we consider a split graph G that contains no tent but contains a 4-tent as an induced subgraph. The main difference with the previous section is that the enriched matrix $\mathbb{B}_6$ defined in Section 4.2.1 may have some LR-rows. In Section 4.3 we consider a split graph G that contains no tent or 4-tent, but contains a co-4-tent as an induced subgraph. In this case, the main obstacles are that the co-4-tent is not a prime graph and that the enriched matrix $\mathbb{C}_7$ defined in Section 4.3.1 may have some LR-rows. Finally, in Section 4.4 we explain in detail how to reduce the case in which G contains a net as an induced subgraph using the previous cases.

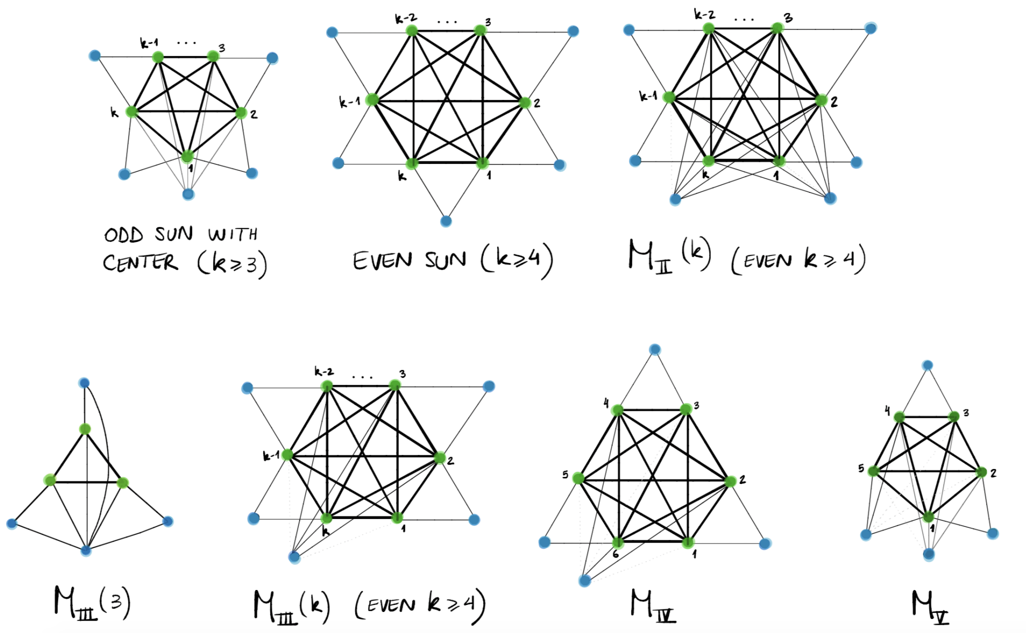


Figure 4.1 – The graphs in the family $\mathcal{T}$.

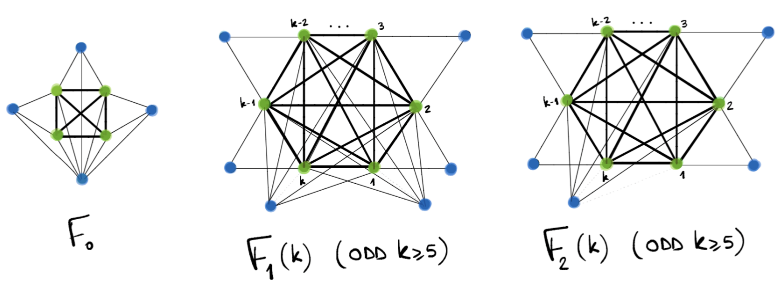


Figure 4.2 – The graphs in the family $\mathcal{F}$.

4.1 Split circle graphs containing an induced tent

In this section we will address the first case of the proof of Theorem 4.1, which is the case where G contains an induced tent. This section is subdivided as follows. In Section 4.1.1, we use the partitions of K and S given in Section 2.1 to define the matrices $\mathbb{A}_i$ for each $i = 1, 2, \ldots, 6$ and prove some properties that will be useful further on. In Subsection 4.1.2, the main results are the necessity of the 2-nestedness of each $\mathbb{A}_i$ for G to be a circle graph and the guidelines to give a circle model for a split graph G containing an induced tent in Theorem 4.6.

All the graphs stated in Theorem 4.1 are non-circle graphs. Also notice that the net $\vee K_1$, the 4-tent $\vee K_1$ and the co-4-tent $\vee K_1$ are the graphs whose adjacency matrix $A(S, K)$ represents the same configuration as $M_{III}(3)$, F_0 and $M_{II}(4)$, respectively.

4.1.1 Matrices $\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_6$

Let $G = (K, S)$ and T as in Section 2.1. For each $i \in \{1, 2, \ldots, 6\}$, let $\mathbb{A}_i$ be an enriched $(0, 1)$-matrix having one row for each vertex $s \in S$ such that s belongs to S_{ij} or S_{ji} for some $j \in \{1, 2, \ldots, 6\}$, and one column for each vertex $k \in K_i$ and such that the entry corresponding to the row s and the column k is 1 if and only if s is adjacent to k in G. For each $j \in \{1, 2, \ldots, 6\} - \{i\}$, we mark those rows corresponding to vertices of S_{ji} with L and those corresponding to vertices of S_{ij} with R.

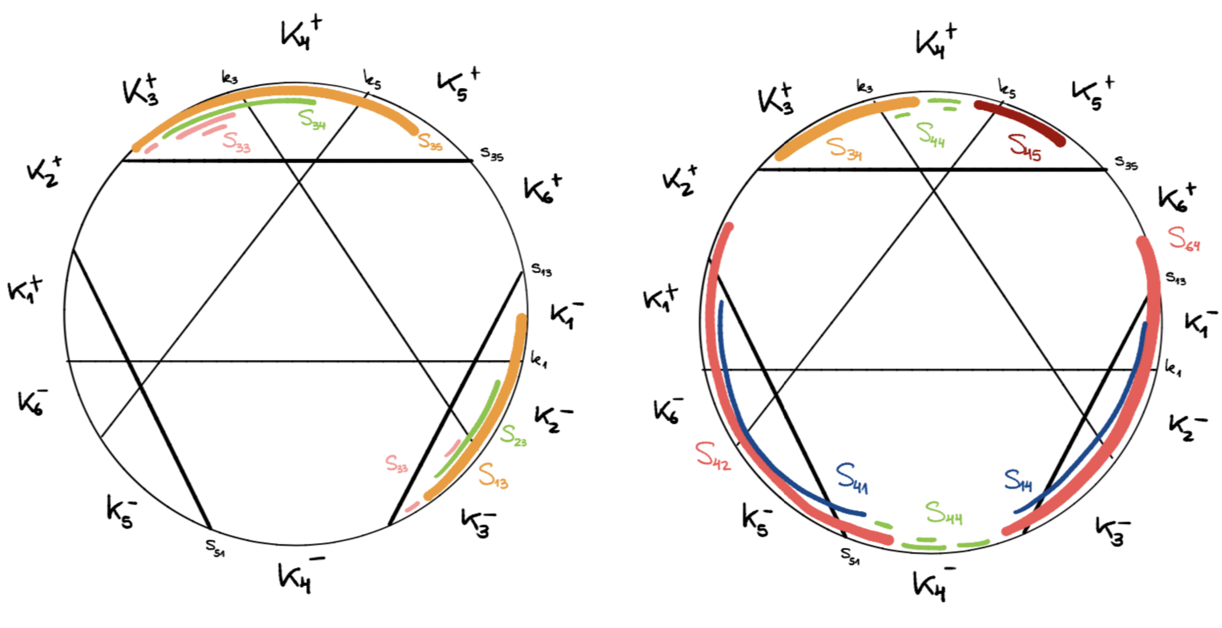
Moreover, we color some of the rows of $\mathbb{A}_i$ as follows.

— If $i \in \{1, 3, 5\}$, then we color each row corresponding to a vertex $s \in S_{ij}$ for some $j \in \{1, 2, \ldots, 6\} - \{i\}$ with color red and each row corresponding to a vertex $s \in S_{ji}$ for some $j \in \{1, 2, \ldots, 6\} - \{i\}$ with color blue.

— If $i \in \{2, 4, 6\}$, then we color each row corresponding to a vertex $s \in S_{ij} \cup S_{ji}$ for some $j \in \{1, 2, \ldots, 6\}$ with color red if $j = i + 1$ or $j = i - 1$ (modulo 6) and with color blue otherwise.

Example:

$$\mathbb{A}_3 = \begin{array}{r} \\ S_{34}\ \mathbf{R} \\ S_{35}\ \mathbf{R} \\ S_{33} \\ S_{13}\ \mathbf{L} \\ S_{23}\ \mathbf{L} \end{array} \begin{array}{c} K_3 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{c} \\ \color{red}{\bullet} \\ \color{red}{\bullet} \\ \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \end{array} \qquad \mathbb{A}_4 = \begin{array}{r} \\ S_{34}\ \mathbf{L} \\ S_{45}\ \mathbf{R} \\ S_{44} \\ S_{14}\ \mathbf{L} \\ S_{64}\ \mathbf{L} \\ S_{41}\ \mathbf{R} \\ S_{42}\ \mathbf{R} \end{array} \begin{array}{c} K_4 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{c} \\ \color{red}{\bullet} \\ \color{red}{\bullet} \\ \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \\ \color{blue}{\bullet} \end{array}$$



(a) $\mathbb{A}_3$ (b) $\mathbb{A}_4$

Figure 4.3 – Sketch model of G with some of the chords associated to rows in $\mathbb{A}_3$ and $\mathbb{A}_4$, respectively.

The following results are useful througout the next subsection.

Claim 4.2. *Let v_1 in S_{ij} and v_2 in S_{ik}, for $i, j, k \in \{1, 2, \ldots, 6\}$ such that $i \neq j, k$. If $\mathbb{A}_i$ is admissible for each $i \in \{1, 2, \ldots, 6\}$, then the following assertions hold:*

— *If $j \neq k$, then v_1 and v_2 are nested in K_i. Moreover, if $j = k$, then v_1 and v_2 are nested in both K_i and K_j.*

— *For each $i \in \{1, 2, \ldots, 6\}$, there is a vertex v_i^* in K_i such that for every $j \in \{1, 2, \ldots, 6\} - \{i\}$ and every s in S_{ij}, the vertex s is adjacent to v_i^*.*

Let v_1, v_2 in S_{ij}, for some $i, j \in \{1, \ldots, 6\}$. Toward a contradiction, suppose without loss of generality that v_1 and v_2 are not nested in K_i, since by symmetry the proof is analogous in K_j. Since v_1 and v_2 are both adjacent to at least one vertex in K_i, then there are vertices w_1, w_2 in K_i such that w_1 is adjacent to v_1 and nonadjacent to v_2, and w_2 is adjacent to v_2 and nonadjacent to v_1. Moreover, since v_1 and v_2 lie in S_{ij} and $i \neq j$, then by definition of $\mathbb{A}_i$ the corresponding rows are labeled with the same letter and colored with the same color. Therefore, we find D_0 induced by the rows corresponding to v_1 and v_2, and the columns w_1 and w_2, which results in a contradiction for $\mathbb{A}_i$ is admissible. The proof is analogous if $j \neq k$. Moreover, the second statement of the claim follows from the previous argument and the fact that there is a C1P for the columns of $\mathbb{A}_i$. □

4.1.2 Split circle equivalence

In this subsection, we will use the matrix theory developed in Chapter 3 to characterize the forbidden induced subgraphs that arise in a split graph that contains an induced tent when this graph is not a circle graph. We will start by proving that, given a split graph G that contains an induced tent, if G is a circle graph, then the matrices $\mathbb{A}_i$ for each $i = 1, 2, \ldots, 6$ are 2-nested.

Lemma 4.3. *If $\mathbb{A}_i$ is not 2-nested, for some $i \in \{1, \ldots, 6\}$, then G contains an induced subgraph of the families $\mathcal{T}$ or $\mathcal{F}$.*

Proof. We will prove each case assuming that either $i = 3$ or $i = 4$, since the matrices $\mathbb{A}_i$ are analogous when i is odd or even, and thus the proof depends solely on the parity of i.

The proof is organized as follows. First, we will assume that $\mathbb{A}_i$ is not admissible. In that case, $\mathbb{A}_i$ contains one of the forbidden subconfigurations stated in Theorem 3.16. Once we reach a contradiction, we will assume that $\mathbb{A}_i$ is admissible but not LR-orderable, thus $\mathbb{A}_i$ contains one of the forbidden subconfigurations in Theorem 3.21, once again reaching a contradiction. The following steps are to assume that $\mathbb{A}_i$ is LR-orderable but not partially 2-nested, and finally that $\mathbb{A}_i$ is partially 2-nested but not 2-nested. We will use the characterizations given in Corollary 3.26 and Theorem 3.12 for each case, respectively.

Recall that for each vertex k_i of the tent, k_i lies in K_i by definition and thus $K_i \neq \emptyset$ for every $i = 1, 3, 5$. Notice that, if G is circle, then in particular, for each $i = 1, \ldots, 6$, $\mathbb{A}_i$ contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$ since these matrices are the adjacency matrices of non-circle graphs.

Case (1) *Suppose first that $\mathbb{A}_i$ is not admissible.* By Theorem 3.16 and since $\mathbb{A}_i$ contains no LR-rows, then $\mathbb{A}_i$ contains either D_0, D_1, D_2 or $S_2(k)$, $S_3(k)$ for some $k \geq 3$.

Case (1.1) $\mathbb{A}_i$ *contains* D_0.

Let v_0 and v_1 in S be the vertices whose adjacency is represented by the first and second row of D_0, respectively, and let k_{i1} and k_{i2} in K_i be the vertices whose adjacency is represented by the first and second column of D_0, respectively.

Notice that both rows of D_0 are labeled with the same letter, and the coloring given to each row is indistinct. We assume without loss of generality that both rows are labeled with L, due to the symmetry of the problem.

Case (1.1.1) *Suppose first that* $i = 3$. In this case, v_1 and v_2 lie in S_{34} or S_{35}. Hence, there are vertices k_{31} and k_{32} in K_3 such that v_j is adjacent to k_{3j} and is nonadjacent to $k_{3(j+1)}$ (induces modulo 2). By Claim 4.2 there is a vertex k_4 in K_4 (resp. k_5 in K_5) adjacent to every vertex in S_{34} (resp. S_{35}). Thus, if both v_1 and v_2 lie in S_{35}, since s_{51} is adjacent to every vertex in K_5 by definition, then we find a net $\vee K_1$ induced by $\{k_5, k_{31}, k_{32}, v_1, v_2, s_{51}, k_1\}$. If instead both v_1 and v_2 lie in S_{34}, then we find a tent with center induced by $\{k_4, k_{31}, k_{32}, v_1, v_2, s_{35}, s_{13}\}$.

Thus, let us suppose that v_1 in S_{34} and v_2 in S_{35}. Let k_4 in K_4 such that v_1 is adjacent to k_4. Recall that v_2 is complete to K_4. Let k_5 in K_5 such that v_2 is adjacent to k_5, and let k_1 be any vertex in K_1. Since v_2 in S_{34} and v_1 in S_{35}, then v_1 and v_2 are nonadjacent to k_1, and also v_1 is nonadjacent to k_5. Hence, we find a 4-sun induced by the set $\{s_{13}, s_{51}, v_1, v_2, k_1, k_{31}, k_4, k_5\}$.

Case (1.1.2) *Suppose now that* $i = 4$. Thus, the vertices v_1 and v_2 belong to either S_{34}, S_{14} or S_{64}. Suppose v_1 in S_{34} and v_2 in S_{14}, and let k_1 in K_1 and k_3 in K_3 such that v_1 is adjacent to k_3. Since v_2 is complete to K_3, then v_2 is adjacent to k_3, and both v_1 and v_2 are nonadjacent to k_1. Hence, we find co-4-tent $\vee K_1$ induced by $\{s_{13}, s_{35}, v_1, v_2, k_3, k_{41}, k_{42}, k_1\}$. The same holds if v_2 lies in S_{64}.

If instead v_1 and v_2 lie in S_{34}, then we find a net $\vee K_1$ induced by the set $\{k_3, k_{41}, k_{42}, v_1, v_2, s_{13}, k_1\}$.

Finally, if v_1 and v_2 lie in $S_{14} \cup S_{64}$, then we find a tent $\vee K_1$ induced by $\{k_3, k_1, k_{41}, k_{42}, v_1, v_2, s_{35}\}$, where k_1 is a vertex in K_1 adjacent to v_1 and v_2.

Case (1.2) $\mathbb{A}_i$ *contains* D_1.

As in the previous case, let v_1 and v_2 in S be the vertices whose adjacency is represented by the first and second row of D_1, respectively, and let k_i in K_i be the vertex whose adjacency is represented by the column of D_1.

Notice that both rows of D_1 are labeled with distinct letters and are colored with the same color. We assume without loss of generality that v_1 is labeled with L and v_2 is labeled with R. Moreover, if i is odd, then it is not possible to have two such vertices corresponding to rows in $\mathbb{A}_i$ labeled with distinct letters and colored with the same color.

Case (1.2.1) *Let us suppose that* $i = 4$. In this case, either v_1 in S_{34} and v_2 in S_{45}, or v_1 in $S_{14} \cup S_{64}$ and v_2 in $S_{41} \cup S_{42}$.

If v_1 in S_{34} and v_2 in S_{45}, then we find a 4-sun induced by $\{v_1, v_2, s_{13}, s_{51}, k_1, k_3, k_4, k_5\}$, where k_3 in K_3 is adjacent to v_1 and nonadjacent to v_2, k_4 in K_4 is adjacent to both v_1 and v_2, k_5 in K_5 is adjacent to v_2 and nonadjacent to v_1, and k_1 in K_1 is nonadjacent to both v_1 and v_2.

Suppose that v_1 lies in S_{14} and v_2 lies in S_{41}. In this case, we find a tent $\vee K_1$ induced by $\{v_1, v_2, s_{35}, k_1, k_3, k_4, k_5\}$, where k_1, k_3, k_4 and k_5 are vertices analogous as those described in the previous paragraph. The same holds if v_1 in S_{64} or v_2 in S_{42}.

Case (1.3) D_2 *in* $\mathbb{A}_i$.

Let v_1 and v_2 in S be the vertices whose adjacency is represented by the first and second row of D_2, respectively, and let k_{i1} and k_{i2} in K_i be the vertices whose adjacency is represented by the first and second column of D_2, respectively.

Both rows of D_2 are labeled with distinct letters and colored with distinct colors, for the ''same color'' case is covered since we proved that there is no D_1 as a submatrix of $\mathbb{A}_i$. We assume without loss of generality that v_1 is labeled with L and v_2 is labeled with R.

Case (1.3.1) *Suppose that* $i = 4$. Thus, v_1 in S_{34} and v_2 in $S_{41} \cup S_{42}$. In this case we find a tent with center induced by $\{v_1, v_2, s_{13}, k_1, k_3, k_{41}, k_{42}\}$, where k_1 in K_1 is adjacent to v_2 and nonadjacent to

v_1 and k_3 in K_3 is adjacent to v_1 and nonadjacent to v_2. We find the same forbidden subgraph if v_2 in S_{41} or S_{42}.

Case (1.3.2) *Suppose that* $i = 3$. In this case, v_1 in $S_{13} \cup S_{23}$, and v_2 in $S_{34} \cup S_{35}$.

Suppose first that $K_2 \neq \emptyset$. If v_1 in S_{23} and v_2 in S_{34}, then we find co-4-tent $\vee K_1$ induced by $\{v_1, v_2, s_{13}, s_{35}, k_2, k_4, k_{31}, k_{32}\}$, where k_2 in K_2 is adjacent to v_1 and nonadjacent to v_2, and k_4 in K_4 is adjacent to v_2 and nonadjacent to v_1. If instead v_2 in S_{35}, then we find once more a co-4-tent $\vee K_1$ induced by the same set of vertices with the exception of k_4 and adding a vertex k_5 in K_5 adjacent to v_2 and nonadjacent to v_1. The same forbidden subgraph can be found if v_1 in S_{13}, if $K_2 \neq \emptyset$.

If instead $K_2 = \emptyset$, then necesarily v_1 in S_{13}. If v_2 in S_{35}, then we find a tent with center induced by the subset $\{v_1, v_2, s_{51}, k_1, k_5, k_{31}, k_{32}\}$, where k_1 in K_1 is adjacent to v_1 and nonadjacent to v_2, and k_5 in K_5 is adjacent to v_2 and nonadjacent to v_1. If v_2 in S_{34}, then we find $M_{III}(4)$ induced by $\{v_1, v_2, s_{51}, s_{13}, k_1, k_4, k_5, k_{31}, k_{32}\}$, where k_1 in K_1 is adjacent to v_1 and nonadjacent to v_2, k_4 in K_4 is adjacent to v_2 and nonadjacent to v_1, and k_5 in K_5 is nonadjacent to both v_1 and v_2.

Case (1.4) *There is* $S_2(j)$ *as a submatrix of* $\mathbb{A}_i$, *with* $j \geq 3$. Let $v_1, v_2, \ldots, v_j$ be the vertices in S represented by the rows of $S_2(j)$, and let $k_{i1}, \ldots, k_{i,j-1}$ be the vertices in K_i that are represented by columns 1 to $j-1$ of $S_2(j)$. Notice that v_1 and v_j are labeled with the same letter, and depending on whether j is odd or even, then v_1 and v_j are colored with distinct colors or with the same color, respectively. We assume without loss of generality that v_1 and v_j are both labeled with L.

Case (1.4.1) *Suppose first that* j *is odd.* If $i = 3$, then there are no vertices v_1 and v_j labeled with the same letter and colored with distinct colors as in $S_2(j)$. Hence, suppose that $i = 4$. In this case, v_1 in S_{34} and v_j in $S_{14} \cup S_{64}$. Let k_3 in K_3 be a vertex adjacent to both v_1 and v_j, and let k_1 in K_1 adjacent to v_j. Thus, we find $F_1(j+2)$ induced by the subset $\{s_{13}, s_{35}, v_1, \ldots, v_j, k_1, k_3, k_{i1}, \ldots, k_{i,j-1}\}$.

Case (1.4.2) *Suppose* j *is even.* We split this in two cases, depending on the parity of i. If $i = 3$, then v_1 and v_j lie in $S_{13} \cup S_{23}$. Suppose that v_1 in S_{13} and v_j in S_{23}. Let k_2 in K_2 adjacent to v_1 and v_j. Hence, we find $F_1(j+2)$ induced by the subset $\{v_1, \ldots, v_j, k_2, k_{i2}, \ldots, k_{i,j-1}, s_{35}\}$. The same holds if both v_1 and v_j lie in S_{23}. If instead v_1 and v_j both lie in S_{13}, then we find $F_1(j+2)$ induced by the same subset but replacing k_2 for a vertex k_1 in K_1 adjacent to both v_1 and v_j.

Suppose now that $i = 4$. In this case, v_1 and v_j lie in $S_{14} \cup S_{64}$. In either case, there is a vertex k_1 in K_1 that is adjacent to both v_1 and v_j. Hence, we find $F_1(j+1)$ induced by $\{v_1, \ldots, v_j, k_1, k_{i1}, \ldots, k_{i,j-1}, s_{35}\}$.

Case (1.5) *There is* $S_3(j)$ *as a submatrix of* $\mathbb{A}_i$, *for some* $j \geq 3$. Let $v_1, v_2, \ldots, v_j$ be the vertices in S represented by the rows of $S_3(j)$, and let $k_{i1}, \ldots, k_{i(j-1)}$ be the vertices in K_i that are represented by columns 1 to $j-1$ of $S_3(j)$. Notice that v_1 and v_j are labeled with distinct letters, and as in the previous case, depending on whether j is odd or even, v_1 and v_j are either colored with distinct colors or with the same color, respectively. We assume without loss of generality that v_1 is labeled with L and v_j is labeled with R.

Case (1.5.1) *Suppose first that* j *is odd.* If $i = 3$, then v_1 lies in $S_{34} \cup S_{35}$, and v_j lies in $S_{13} \cup S_{23}$. If v_1 lies in S_{34} and v_j lies in S_{23}, then we find $F_1(j+2)$ induced by $\{v_1, \ldots, v_j, k_2, k_4, k_{i1}, \ldots, k_{i(j-1)}, s_{35}, s_{13}\}$, where k_4 in K_4 is adjacent to v_1 and nonadjacent to v_j, and k_2 in K_2 adjacent to v_j and nonadjacent to v_1. If v_1 lies in S_{34} and v_j lies in S_{13}, then we find $F_1(j+2)$ induced by $\{v_1, \ldots, v_j, k_1, k_4, k_{i1}, \ldots, k_{i(j-1)}, s_{35}, s_{13}\}$, with k_1 in K_1 adjacent to v_j and nonadjacent to v_1. If instead v_1 lies in S_{35} and v_j lies in S_{23}, then we find $F_1(j+2)$ induced by $\{v_1, \ldots, v_j, k_2, k_5, k_{i1}, \ldots, k_{i(j-1)}, s_{35}, s_{13}\}$, with k_5 in K_5 adjacent to v_1 and nonadjacent to v_j.

Suppose that $i = 4$. In this case, v_1 in S_{34} and v_j in $S_{41} \cup S_{42}$. In either case, we find a $j+1$-sun induced by $\{v_1, \ldots, v_j, k_{i1}, \ldots, k_{i(j-1)}, k_1, k_3, s_{13}\}$, with k_1 in K_1 adjacent to v_j and nonadjacent to v_1, and k_3 in K_3 adjacent to v_1 and nonadjacent to v_j.

Case (1.5.2) *Suppose now that* j *is even.* If $i = 3$, then there no two rows in $\mathbb{A}_3$ labeled with distinct letters and colored with the same color. Hence, let $i = 4$. In this case, either v_1 in S_{34} and v_j in S_{45}, or v_1 in $S_{14} \cup S_{64}$ and v_j in $S_{41} \cup S_{42}$.

If v_1 in S_{34} and v_j in S_{45}, then we find a $(j+2)$-sun induced by $\{v_1, \ldots, v_j, k_1, k_3, k_5, k_{i1}, \ldots, k_{i(j-1)}, s_{13}, s_{51}\}$, where k_1 in K_1 is nonadjacent to both v_1 and v_j, k_3 in K_3 is adjacent to v_1 and nonadjacent to v_j, and k_5 in K_5 is adjacent to v_j and nonadjacent to v_1.

If instead v_1 in $S_{14} \cup S_{64}$ and v_j in $S_{41} \cup S_{42}$, then we find a j-sun induced by $\{v_1, \ldots, v_j, k_1, k_{i1}, \ldots, k_{i(j-1)}\}$, with k_1 in K_1 adjacent to both v_1 and v_j.

Since we have reached a contradiction for every forbidden submatrix of admissibility, then the matrix $\mathbb{A}_i$ is admissible.

Case (2) *$\mathbb{A}_i$ is admissible but not LR-orderable.*

Then it contains a Tucker matrix, or one of the following submatrices: M'_4, M''_4, M'_5, M''_5, $M'_2(k)$, $M''_2(k)$, $M'_3(k)$, $M''_3(k)$, or their corresponding dual matrices, for any $k \geq 4$.

We will assume throughout the rest of the proof that, for each pair of vertices x and y that lie in the same subset S_{ij} of S, there are vertices k_i in K_i and k_j in K_j such that both x and y are adjacent to k_i and k_j. This is given by Claim 4.2.

Suppose there is $M_I(j)$ as a submatrix of $\mathbb{A}_i$. Let $v_1, \ldots, v_j$ be the vertices of S represented by rows 1 to j of $M_I(k)$, and let $k_{i1}, \ldots, k_{ij}$ be the vertices in K represented by colums 1 to j. Thus, if j is even, then we find either a j-sun induced by $\{v_1, \ldots, v_j, k_{i1}, \ldots, k_{ij}\}$, and if j is odd, then we find a j-sun with center induced by the subset $\{v_1, \ldots, v_j, k_{i1}, \ldots, k_{ij}, s_{i,i+2}\}$.

For any other Tucker matrix, we find the homonym forbidden subgraph induced by the subset $\{v_1, \ldots, v_j, k_{i1}, \ldots, k_{ij}\}$.

Suppose that $\mathbb{A}_i$ contains one of the following submatrices: M'_4, M''_4, M'_5, M''_5, $M'_2(k)$, $M''_2(k)$, $M'_3(k)$, $M''_3(k)$, or their corresponding dual matrices, for any $k \geq 4$. Let M be such a submatrix. In this case, we have the following remark.

Notice that, for any tag column c of M that denoted which vertices are labeled with L, there is a vertex k' in either K_{i-1} or K_{i-2} such that the vertices represented by a labeled row in c are adjacent in G to k'. If instead the tag column c denoted which vertices are labeled with R, then we find an analogous vertex k'' in either K_{i+1} or K_{i+2}.

Depending on whether there is one or two tag columns in M, we find the homonym forbidden subgraph induced by the vertices in S and K represented by the rows and non-tagged columns of M plus one or two vertices k' and k'' as described in the previous remark.

Case (3) *$\mathbb{A}_i$ is LR-orderable but not partially 2-nested.* Thus, since there are no LR-rows in $\mathbb{A}_i$, then there is either a monochromatic gem or a monochromatic weak gem in $\mathbb{A}_i$.

Let v_1 and v_2 in S the independent vertices represented by the rows of the monochromatic gem. Notice that both rows are labeled rows, since every unlabeled row in $\mathbb{A}_i$ is uncolored. It follows from this that a monochromatic gem or a monochromatic weak gem is induced only by two rows labeled with L or R, and thus both are the same case.

Case (3.1) If $i = 3$, since both vertices need to be colored with the same color, then v_1 in S_{34} and v_2 in S_{35}. In that case, we find D_0 in $\mathbb{A}_i$ since both rows are labeled with the same letter, which results in a contradiction for we assumed that $\mathbb{A}_i$ is admissible. The same holds if both vertices belong to either S_{34} or S_{35}.

Case (3.2) If instead $i = 4$, then we have three possibilities. Either v_1 in S_{14} and v_2 in S_{64}, or v_1 in S_{34} and v_2 in S_{45}, or v_1 in S_{14} and v_2 in S_{41}. The first case is analogous to the $i = 3$ case stated above. For the second and third case, since both rows are labeled with distinct letters, then we find D_1 as a submatrix of $\mathbb{A}_i$. This results once more in a contradiction, for $\mathbb{A}_i$ is admissible.

Therefore, $\mathbb{A}_i$ is partially 2-nested.

Case (4) *$\mathbb{A}_i$ is partially 2-nested but not 2-nested.*

Hence, for every proper 2-coloring of the rows of $\mathbb{A}_i$, there is either a monochromatic gem or a monochromatic weak gem. Notice that, in such a gem, there is at least one unlabeled row for there are no LR-rows in $\mathbb{A}_i$ and we have just proven that $\mathbb{A}_i$ is partially 2-nested. We consider the columns of the matrix $\mathbb{A}_i$ ordered according to an LR-ordering. Let us suppose without loss of generality that there is a monochromatic gem, since the case in which one of the rows is labeled with L or R and the other is unlabeled is analogous. Let v_j and v_{j+1} be the rows that induce such a gem, and suppose that the gem induced by v_j and v_{j+1} is colored with red.

Since there is no possible 2-coloring for which these two rows are colored with distinct colors, then there is at least one distinct row v_{j-1} colored with blue that forces v_j to be colored with red. If v_{j-1} is unlabeled, then v_{j-1} and v_j are neither disjoint or nested. If v_{j-1} is labeled with L or R, then v_j and v_{j-1} induce a weak gem.

If v_{j-1} forces the coloring only on v_j, let v_{j+2} be a row such that v_{j+2} forces v_{j+1} to be colored with red. Suppose first that v_{j+2} forces the coloring only to the row v_{j+1}. Hence, there is a submatrix as the following in $\mathbb{A}_i$:

$$\begin{array}{l} v_{j-1} \\ v_j \\ v_{j+1} \\ v_{j+2} \end{array} \begin{pmatrix} 11000 \\ 01100 \\ 00110 \\ 00011 \end{pmatrix} \begin{array}{l} {\color{blue}\bullet} \\ {\color{red}\bullet} \\ {\color{red}\bullet} \\ {\color{blue}\bullet} \end{array}$$

If there are no more rows forcing the coloring of v_{j-1} and v_{j+2}, then this submatrix can be colored blue-red-blue-red. Since this is not possible, there are rows $v_l, \dots, v_{j-2}$ and $v_{j+3}, \dots, v_k$ such that every row forces the coloring of the next one -and only that row- including v_{j-1}, v_j, v_{j+1} and v_{j+2}. Moreover, if this is the longest chain of vertices with this property, then v_l and v_k are labeled rows, for if not, we can proper color again the rows and thus extending the pre-coloring, which would be a contradiction. Hence, we find either $S_2(k-l+1)$ or $S_3(k-l+1)$ in $\mathbb{A}_i$, and this also results in a contradiction, for $\mathbb{A}_i$ is admissible.

Suppose now that v_{j-1} forces the red color on both v_j and v_{j+1}. Thus, if v_{j-1} is unlabeled, then v_{j-1} is neither nested nor disjoint with both v_j and v_{j+1}. Since v_j and v_{j+1} are neither disjoint nor nested, either $v_j[r_j] = v_{j+1}[r_j] = 1$ or $v_j[l_j] = v_{j+1}[l_j] = 1$. Suppose without loss of generality that $v_j[r_j] = v_{j+1}[r_j] = 1$. Since v_{j-1} is neither disjoint or nested with v_j, then either $v_{j-1}[l_j] = 1$ or $v_{j-1}[r_j] = 1$, and the same holds for $v_{j-1}[l_{j+1}] = 1$ or $v_{j-1}[r_{j+1}] = 1$.

If $v_{j-1}[l_j] = 1$, then $v_{j-1}[l_{j+1}] = 1$ and $v_j[l_{j+1}] = 1$, and thus we find F_0 induced by $\{v_{j-1}, v_j, v_{j+1}, l_{j-1}, l_{j+1}-1, l_{j+1}, r_j, r_j+1\}$, which results in a contradiction.

Analogously, if $v_{j-1}[r_j] = 1$, then $v_{j-1}[l_{j+1}] = 1$ and $v_{j-1}[l_j] = 1$, and thus we find F_0 induced by $\{v_{j-1}, v_j, v_{j+1}, l_j, l_{j+1}, r_j, r_j+1, r_{j-1}\}$.

If instead v_{j-1} is labeled with L or R, then the proof is analogous except that we find F'_0 instead of F_0 as a subconfiguration in $\mathbb{A}_i$.

Therefore, we have reached a contradiction in every case and thus $\mathbb{A}_i$ is 2-nested.

□

Let $G = (K, S)$ and T as in Section 2.1, and the matrices $\mathbb{A}_i$ for each $i = 1, 2, \dots, 6$ as in the previous subsection.

Suppose $\mathbb{A}_i$ is 2-nested for each $i = 1, 2, \dots, 6$. Let χ_i be a coloring for every matrix $\mathbb{A}_i$. Hence, every row in each matrix $\mathbb{A}_i$ is colored with either red or blue, and this is a proper 2-

coloring extension of the given precoloring (or equivalently, a block bi-coloring), and there is an LR-ordering Π_i for each $i = 1, 2, \dots, 6$.

Let Π be the ordering of the vertices of K given by concatenating the LR-orderings Π_1, Π_2, …, Π_6. Let $A = A(S, K)$ and consider the columns of A ordered according to Π.

For each vertex s in S_{ij}, if $i \leq j$, then the R-block corresponding to s in $\mathbb{A}_i$ and the L-block corresponding to s in $\mathbb{A}_j$ are colored with the same color. Thus, we consider the row corresponding to s in A colored with that color. Notice that, if $i < l < j$, then v is complete to each K_l. Thus, when defining $\mathbb{A}_l$ we did not consider such vertices since they do not interfere with the possibility of having an LR-ordering of the columns, for such a vertex would have a 1 in each column of $\mathbb{A}_l$.

If instead $i > j$, then the R-block corresponding to s in $\mathbb{A}_i$ and the L-block corresponding to s in $\mathbb{A}_j$ are colored with distinct colors. Moreover, notice that the row corresponding to s in A has both an L-block and an R-block. Thus, we consider its L-block colored with the same color assigned to s in $\mathbb{A}_j$ and the R-block colored with the same color assigned to s in $\mathbb{A}_j$. Notice that the distinct coloring in $\mathbb{A}_i$ and $\mathbb{A}_j$ makes sense, since we are describing vertices whose chords must have one of its endpoints drawn in the K_i^+ portion of the circle and the other endpoint in the K_j^- portion of the circle. Throughout the following, we will denote s_i to the row corresponding to s in $\mathbb{A}_i$.

Let $s \in S$. Hence, s lies in S_{ij} for some $i, j \in \{1, 2, \dots, 6\}$. Notice that, a row representing a vertex s in S_{ii} is entirely colored with the same color. Moreover, this is also true for a row representing s in S_{ij} such that $i < j$. However, if s in S_{ij} and $i > j$, then s_i and s_j are colored with distinct colors.

Definition 4.4. *We define the $(0,1)$-matrix $\mathbb{A}_r$ as the matrix obtained by considering only those rows representing vertices in $S \setminus \bigcup_{i=1}^{6} S_{ii}$ and adding two distinct columns c_L and c_R such that the entry $\mathbb{A}_r(s,k)$ is defined as follows:*

- *If $i < j$ and s_i is colored with red, then the entry $\mathbb{A}_r(s,k)$ has a 1 if s is adjacent to k and a 0 otherwise, for every k in K, and $\mathbb{A}_r(s, c_R) = \mathbb{A}_r(s, c_L) = 0$.*
- *If $i > j$ and s_i is colored with red, then the entry $\mathbb{A}_r(s,k)$ has a 1 if s is adjacent to k and a 0 otherwise, for every k in $K_i \cup \dots K_6$, and $\mathbb{A}_r(s, c_R) = 1$, $\mathbb{A}_r(s, c_L) = 0$. Analogously, if $i > j$ and instead s_j is colored with red, then the entry $\mathbb{A}_r(s,k)$ has a 1 if s is adjacent to k and a 0 otherwise, for every k in $K_1 \cup \dots K_j$, and $\mathbb{A}_r(s, c_R) = 0$, $\mathbb{A}_r(s, c_L) = 1$.*

The matrix $\mathbb{A}_b$ is defined in an entirely analogous way, changing red for blue in the definition.

We define the $(0,1)$-matrix $\mathbb{A}_{r-b}$ as the submatrix of A obtained by considering only those rows corresponding to vertices s in S_{ij} with $i > j$ for which s_i is colored with red. The matrix $\mathbb{A}_{b-r}$ is defined as the submatrix of A obtained by considering those rows corresponding to vertices s in S_{ij} with $i > j$ for which s_i is colored with blue.

Lemma 4.5. *Suppose that $\mathbb{A}_i$ is 2-nested for every $= 1, 2, \dots, 6$. If $\mathbb{A}_r$, $\mathbb{A}_b$, $\mathbb{A}_{r-b}$ or $\mathbb{A}_{b-r}$ are not nested, then G contains F_0 as a minimal forbidden induced subgraph for the class of circle graphs.*

Proof. Suppose first that $\mathbb{A}_r$ is not nested. Then, there is a 0-gem. Since $\mathbb{A}_i$ is 2-nested for every $= 1, 2, \dots, 6$, in particular there are no monochromatic gems in each $\mathbb{A}_i$. Let f_1 and f_2 be two rows that induce a 0-gem in $\mathbb{A}_r$ and let v_1 in S_{ij} and v_2 in S_{lm} be the vertices corresponding to such rows in G. Notice that, in each case the proof will be analogous whenever two rows overlap and the corresponding two vertices lie in the same subset.

The rows in $\mathbb{A}_r$ represent vertices in the following subsets of S: S_{34}, S_{45}, S_{35}, S_{36}, S_{25}, S_{26}, S_{42}, S_{52}, S_{51}, S_{61}, S_{64} or S_{63}. Notice that $S_{36} = S_{[36}$, $S_{25} = S_{25]}$.

Case (1) v_1 in S_{34}. Thus, v_2 in S_{35} since $\mathbb{A}_4$ is admissible. We find F_0 induced by $\{v_1, v_2, s_{13}, k_1, k_{31}, k_{32}, k_4, k_5\}$. It follows analogously if v_1 in S_{45}, for in this case the only possibility is v_2 in S_{35} since S_{25} is complete to K_5.
Case (2) v_1 in $S_{35} \cup S_{36}$. Since S_{36} is complete to K_3, S_{25} is complete to K_5 and $\mathbb{A}_6$ is admissible, the only possibility is v_1 in S_{36} and v_2 in $S_{25} \cup S_{26}$. We find F_0 induced by $\{v_1, v_2, s_{13}, k_1, k_2, k_3, k_5, k_6\}$ if v_2 in S_{25} or $\{v_1, v_2, s_{13}, k_1, k_2, k_3, k_{61}, k_{62}\}$ if v_2 in S_{26}.
Case (3) v_1 in $S_{25} \cup S_{26}$. In this case, the only possibility is that v_1 in S_{25} and v_2 in S_{26}, since $\mathbb{A}_2$ and $\mathbb{A}_6$ are admissible. We find F_0 induced by $\{v_1, v_2, s_{13}, k_1, k_{21}, k_{22}, k_5, k_6\}$.

Thus, $\mathbb{A}_r$ is nested. Let us suppose that $\mathbb{A}_b$ is not nested. The rows in $\mathbb{A}_b$ represent vertices in the following subsets of S: S_{12}, S_{13}, S_{23}, S_{14}, S_{42}, S_{52}, S_{51}, S_{61}, S_{64} or S_{63}. Notice that $S_{14} = S_{[14}$.
Case (1) v_1 in S_{13}. Thus, v_2 in $S_{12} \cup S_{23}$. We find F_0 induced by $\{v_1, v_2, s_{51}, k_5, k_{11}, k_{12}, k_2, k_3\}$. The proof is analogous by symmetry if v_1 in S_{23}. Notice that there is no 0-gem induced by S_{12} and S_{23} since $\mathbb{A}_2$ is admissible.
Case (2) v_1 in S_{23}. Since S_{14} is complete to K_1, the only possibility is v_2 in S_{63}. We find F_0 induced by $\{v_1, v_2, s_{35}, k_6, k_2, k_{31}, k_{32}, k_5\}$.
Case (3) v_1 in S_{14}. In this case, the only possibility is that v_1 in $S_{63} \cup S_{64}$, since $\mathbb{A}_4$ is admissible. We find F_0 induced by $\{v_1, v_2, s_{35}, k_6, k_1, k_3, k_4, k_5\}$ if v_2 in S_{63} and induced by $\{v_1, v_2, s_{35}, k_6, k_1, k_3, k_{41}, k_{42}\}$ if v_2 in S_{64}.

Suppose now that $\mathbb{A}_{b-r}$ is not nested. The rows in $\mathbb{A}_{b-r}$ represent vertices in the following subsets of S: S_{41}, S_{42}, S_{51}, S_{52} or S_{61}. Suppose that v_1 in S_{41} and v_2 in S_{42}. Thus, we find F_0 induced by $\{v_1, v_2, s_{13}, k_{41}, k_{42}, k_1, k_2, k_3\}$. The proof is analogous if the vertices lie in $S_{51} \cup S_{52}$. Suppose that v_1 in S_{61}, thus v_2 in $S_{51} \cup S_{41}$. We find F_0 induced by $\{v_1, v_2, s_{13}, k_{11}, k_{12}, k_3, k_5, k_6\}$ and therefore $\mathbb{A}_{b-r}$ is nested.

Suppose that $\mathbb{A}_{r-b}$ is not nested. The rows in $\mathbb{A}_{r-b}$ represent vertices in S_{63} or S_{64}. If v_1 in S_{63} and v_2 in S_{64}, then we find F_0 induced by $\{v_1, v_2, s_{51}, k_5, k_{61}, k_{62}, k_3, k_4\}$. It follows analogously if one of both lie in S_{63} or one or both lie in S_{64} changing k_3 and k_4 for some analogous k_{31}, k_{32} in K_3 or k_{41}, k_{42} in K_4, respectively.

This finishes the proof and therefore the four matrices are nested. $\square$

Theorem 4.6. *Let* $G = (K, S)$ *be a split graph containing an induced tent. Then,* G *is a circle graph if and only if* $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_6$ *are 2-nested and* $\mathbb{A}_r$, $\mathbb{A}_b$, $\mathbb{A}_{b-r}$ *and* $\mathbb{A}_{r-b}$ *are nested.*

Proof. Necessity is clear by the previous lemmas. Suppose now that each of the matrices $\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_6$ is 2-nested, and that the matrices $\mathbb{A}_r$, $\mathbb{A}_b$, $\mathbb{A}_{b-r}$ and $\mathbb{A}_{r-b}$ are nested. Let Π_i be an LR-ordering for the columns of $\mathbb{A}_i$ for each $i = 1, 2, \dots, 6$, and let Π be the ordering obtained by concatenation of Π_i for all the vertices in K. Consider the circle divided into twelve pieces as in Figure 4.3a. For each $i \in \{1, 2, \dots, 6\}$ and for each vertex $k_i \in K_i$ we place a chord having one end in K_i^+ and the other end in K_i^-, in such a way that the ordering of the endpoints of the chords in K_i^+ and K_i^- is Π_i.

Let us see how to place the chords for every subset S_{ij} of S.

Notice that, by Lemma 4.5 for every subset S_{ij} such that $i \neq j$, all the vertices in S_{ij} are nested according to the ordering Π. In other words, the vertices in each S_{ij} are totally ordered by inclusion. Moreover, it is also a consequence of Lemma 4.5 and Claim 4.2, that if $i \geq k$ and $j \leq l$, then every vertex in S_{ij} is contained in every vertex of S_{kl}.

Furthermore, let $i \in \{1, 3, 5\}$. Notice that, since $S_{i-1,i}$ is labeled with L in $\mathbb{A}_i$, $S_{i,i+1}$ is labeled with R in $\mathbb{A}_i$, any row in each of these subsets is colored with red and $\mathbb{A}_i$ is admissible and LR-orderable, then there is no vertex in K_i such that the corresponding column has value 1 in two distinct vertices in $S_{i-1,i}$ and $S_{i,i+1}$, respectively. Equivalently, the vertex set $N_{K_i}(S_{i-1,i}) \cap N_{K_i}(S_{i,i+1})$ is empty.

A similar situation occurs with the vertices in $S_{i-2,i+1}$ and $S_{i+1,i-2}$ for each $i \in \{2,4,6\}$, for the vertices in each subset are labeled with R and L respectively, and since $\mathbb{A}_{i-2}$ is 2-nested, then the rows corresponding to vertices in $S_{i-2,i+1}$ end in the last column of $\mathbb{A}_{i-2}$ and the vertices corresponding to $S_{i+1,i-2}$ start in the first column of $\mathbb{A}_{i-2}$. Furthermore, this implies that the sets $N_{K_{i-2}}(S_{i-2,i+1})$ and $N_{K_{i-2}}(S_{i+1,i-2})$ are disjoint. The same holds for $N_{K_{i+1}}(S_{i-2,i+1})$ and $N_{K_{i+1}}(S_{i+1,i-2})$.

We will place the chords according to the ordering Π given for every vertex in K. For each subset S_{ij}, we order its vertices with the inclusion ordering of the neighbourhoods in K and the ordering Π. When placing the chords corresponding to the vertices of each subset, we do it from lowest to highest according to the previously stated ordering given for each subset.

Hence, we first place the chords of every subset $S_{i,i+1}$.

— If $i = 1,2,5$, then we place one endpoint in K_i^- and the other endpoint in K_{i+1}^-.
— If $i = 3,4$, then we place one endpoint in K_i^+ and the other endpoint in K_{i+1}^+.
— If $i = 6$, then we place one endpoint in K_6^- and the other endpoint in K_1^+.

Afterwards, we place the chords that represent vertices in $S_{i-1,i+1}$.

— If $i = 2$, then we place one endpoint in K_1^- and the other endpoint in K_3^-.
— If $i = 4$, then we place one endpoint in K_3^+ and the other endpoint in K_5^+.
— If $i = 6$, then we place one endpoint in K_5^- and the other endpoint in K_1^+.

We denote a_i^- and a_i^+ to the placement in the circle given to the chords of K_i corresponding to the first and last column of $\mathbb{A}_i$, respectively. We denote $s_{i,i+2}^+$ to the placement of the chord corresponding to the vertex $s_{i,i+2}$ of the tent T, which lies between a_{i-1}^+ and a_i^-, and $s_{i,i+2}^+$ to the placement of the chord of the vertex $s_{i,i+2}$ that lies between a_{i+1}^+ and a_{i+2}^-.

For each $i \in \{1,2,\dots,6\}$, we give the placement of the chords corresponding to the vertices in $S_{i-1,i+2}$:

— For $i = 1$, we place one endpoint in K_6^+, and the other endpoint between s_{13}^- and the chord corresponding to a_4^- in K_4^-.
— For $i = 2$, we place one endpoint between the chord corresponding to a_6^+ in K_6^+ and s_{13}^+, and the other endpoint in K_4^-.
— For $i = 3$, we place one endpoint in K_2^+, and the other endpoint between s_{35}^- and the chord corresponding to a_6^- in K_6^+.
— For $i = 4$, we place one endpoint between the chord corresponding to a_2^+ in K_2^+ and s_{35}^+, and the other endpoint in K_6^+.
— For $i = 5$, we place one endpoint in K_4^-, and the other endpoint between s_{51}^- and the chord corresponding to a_2^- in K_2^+.
— For $i = 6$, we place one endpoint between the chord corresponding to a_4^+ in K_4^- and s_{51}^+, and the other endpoint in K_2^+.

Finally, for the vertices in $S_{i-2,i+2}$, we place the chords as follows:

— For $i = 2$, we place one endpoint in K_6^+ and the other endpoint in K_4^-.
— For $i = 4$, we place one endpoint in K_2^+ and the other endpoint in K_6^+.
— For $i = 6$, we place one endpoint in K_4^- and the other endpoint in K_2^+.

This gives a circle model for the given split graph G. □

4.2 Split circle graphs containing an induced 4-tent

In this section we will address the second case of the proof of Theorem 4.1, which is the case where G contains an induced 4-tent. The difference between this case and the tent case, is that one of the matrices that we need to define contains LR-rows, which does not happen in the tent case. This section is subdivided as follows. In Subsection 4.2.1, we define the matrices $\mathbb{B}_i$ for each $i = 1, 2, \dots, 6$ and demonstrate some properties that will be useful further on. In subsections 4.2.2 and 4.2.3, we prove the necessity of the 2-nestedness of each $\mathbb{B}_i$ for G to be a circle graph, and give the guidelines to draw a circle model for a split graph G containing an induced 4-tent in Theorem 4.20.

4.2.1 Matrices $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_6$

Let $G = (K, S)$ and T as in Section 2.2. For each $i \in \{1, 2, \dots, 6\}$, let $\mathbb{B}_i$ be an enriched $(0, 1)$-matrix having one row for each vertex $s \in S$ such that s belongs to S_{ij} or S_{ji} for some $j \in \{1, 2, \dots, 6\}$ and one column for each vertex $k \in K_i$ and such that such that the entry corresponding to row s and column k is 1 if and only if s is adjacent to k in G. For each $j \in \{1, 2, \dots, 6\} - \{i\}$, we mark those rows corresponding to vertices of S_{ji} with L and those corresponding to vertices of S_{ij} with R. Those vertices in $S_{[15]}$ and $S_{[16}$ are labeled with LR.

As in the previous section, some of the rows of $\mathbb{B}_i$ are colored. However, since we do not have the same symmetry as in the tent case, we will give a description of every matrix separately, for each $i \in \{1, \dots, 6\}$ (See Figure 4.4).

Notice that, since S_{25}, S_{26}, S_{52} and S_{62} are complete to K_2, then they are not considered for the definition of the matrix $\mathbb{B}_2$. The same holds for S_{13} with regard to $\mathbb{B}_1$, S_{63} with regard to $\mathbb{B}_3$, S_{41}, S_{46}, S_{14} and S_{64} with regard to $\mathbb{B}_4$, and S_{35} with regard to $\mathbb{B}_5$. Also notice that we considered S_{16} and $S_{[16}$ as two distinct subsets of S. Moreover, every vertex in $S_{[16}$ is labeled with LR and every vertex in S_{16} is labeled with L. Furthermore, every row that represents a vertex in $S_{[15]}$ is an empty LR-row in $\mathbb{B}_6$. Since we need $\mathbb{B}_6$ to be an enriched matrix, by definition of enriched matrix every row corresponding to a vertex in $S_{[15]}$ must be colored with the same color. We will give more details on this in Subsection 4.2.3.

Remark 4.7. Claim 4.2 remains true if G contains an induced 4-tent. The proof is analogous as in the tent case.

4.2.2 Split circle equivalence

In this subsection, we will prove a result analogous to Lemma 4.3. In this case, the matrices $\mathbb{B}_i$ contain no LR-rows, for each $i \in \{1, \dots, 5\}$, hence the proof is very similar to the one given in Subsection 4.1.2 for the tent case.

Lemma 4.8. *If $\mathbb{B}_i$ is not 2-nested, for some $i \in \{1, \dots, 5\}$, then G contains one of the forbidden subgraphs in $\mathcal{T}$ or $\mathcal{F}$.*

Proof. Relying on the symmetry between some of the sets $K_1, \dots, K_5$, we will only prove the statement for $i = 1, 2, 3$. The proof is organized analogously as in Lemma 4.3. As in Lemma 4.3, notice that, if G is circle, then in particular, for each $i = 1, \dots, 6$, $\mathbb{B}_i$ contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$ since these matrices are the adjacency matrices of non-circle graphs.
Case (1) *$\mathbb{B}_i$ is not admissible*

$$\mathbb{B}_1 = \begin{array}{r} \\ S_{12}\ \mathbf{R} \\ S_{11} \\ S_{14}\ \mathbf{R} \\ S_{15}\ \mathbf{R} \\ S_{16}\ \mathbf{R} \\ S_{61}\ \mathbf{L} \end{array} \begin{array}{c} K_1 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \textcolor{red}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array} \qquad \mathbb{B}_2 = \begin{array}{r} \\ S_{12}\ \mathbf{L} \\ S_{22} \\ S_{23}\ \mathbf{R} \\ S_{24}\ \mathbf{R} \end{array} \begin{array}{c} K_2 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \textcolor{red}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array} \qquad \mathbb{B}_3 = \begin{array}{r} \\ S_{35}\ \mathbf{R} \\ S_{36}\ \mathbf{R} \\ S_{13}\ \mathbf{L} \\ S_{33} \\ S_{34}\ \mathbf{R} \\ S_{23}\ \mathbf{L} \end{array} \begin{array}{c} K_3 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array}$$

$$\mathbb{B}_4 = \begin{array}{r} \\ S_{45}\ \mathbf{R} \\ S_{44} \\ S_{24}\ \mathbf{L} \\ S_{34}\ \mathbf{L} \end{array} \begin{array}{c} K_4 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \textcolor{red}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array} \qquad \mathbb{B}_5 = \begin{array}{r} \\ S_{45}\ \mathbf{L} \\ S_{55} \\ S_{56}\ \mathbf{R} \\ S_{25}\ \mathbf{L} \\ S_{15}\ \mathbf{L} \\ S_{65}\ \mathbf{L} \end{array} \begin{array}{c} K_5 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \textcolor{red}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \end{array} \qquad \mathbb{B}_6 = \begin{array}{r} \\ S_{61}\ \mathbf{R} \\ S_{64}\ \mathbf{R} \\ S_{65}\ \mathbf{R} \\ S_{36}\ \mathbf{L} \\ S_{46}\ \mathbf{L} \\ S_{66} \\ S_{62}\ \mathbf{R} \\ S_{63}\ \mathbf{R} \\ S_{26}\ \mathbf{L} \\ S_{56}\ \mathbf{L} \\ S_{16}\ \mathbf{L} \\ S_{[15]}\ \mathbf{LR} \\ S_{[16}\ \mathbf{LR} \end{array} \begin{array}{c} K_6 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{l} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \\ \\ \end{array}$$

Figure 4.4 – The matrices $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$, $\mathbb{B}_4$, $\mathbb{B}_5$ and $\mathbb{B}_6$.

It follows from Theorem 3.16 and the fact that $\mathbb{B}_i$ contains no LR-rows that $\mathbb{B}_i$ contains some submatrix D_0, D_1, D_2, $S_2(k)$ or $S_3(k)$ for some $k \geq 3$.

Let v_1 and v_2 in S be the vertices whose adjacency is represented by the first and second row of D_j, for each $j = 0, 1, 2$, and let k_{i1} and k_{i2} in K_i be the vertices whose adjacency is represented by the first and second column of D_j respectively, for each $j = 0, 2$, and k_i in K_i is the vertex whose adjacency is represented by the column of D_1.

Case (1.1) *$\mathbb{B}_i$ contains* D_0

We assume without loss of generality that both rows are labeled with L.

Case (1.1.1) *Suppose that* $i = 1$. Since the coloring is indistinct, the vertices v_1 and v_2 may belong to one or two of the following subclasses: S_{61}, S_{12}, S_{14}, S_{15}, S_{16}. Suppose first that v_1 and v_2 both lie in S_{61}, thus $K_6 \neq \emptyset$. If $N_{K_6}(v_1)$ and $N_{K_6}(v_2)$ are non-disjoint, then we find a tent induced by $\{v_1, v_2, s_{12}, k_{11}, k_{12}, k_6\}$, where k_6 in K_6 is adjacent to both v_1 and v_2. If instead there is no such vertex k_6, then there are vertices k_{61} and k_{62} in K_6 such that k_{61} is adjacent to v_1 and nonadjacent to v_2, and k_{62} is adjacent to v_2 and nonadjacent to v_1. Then, we find M_{IV} induced by $\{v_1, v_2, s_{12}, s_{24}, k_{11}, k_2, k_4, k_{61}, k_{62}, k_{12}\}$.

Suppose that v_1 and v_2 lie in S_{12}. If $N_{K_2}(v_1)$ and $N_{K_2}(v_2)$ are non-disjoint, then we find net$\vee K_1$ induced by $\{v_1, v_2, s_{24}, k_{11}, k_2, k_4, k_{12}\}$. We find the same subgraph very similarly if v_1 and v_2 lie both in S_{14} or in S_{15} and neither v_1 nor v_2 is complete to K_5. If v_1 in S_{12} and v_2 in $S_{14]} \cup S_{15} \cup S_{16}$, then we find $M_{II}(4)$ induced by $\{k_{11}, k_{12}, k_2, k_4, v_1, v_2, s_{12}, s_{24}\}$.

If v_1 in $S_{14]}$ and v_2 in $S_{15} \cup S_{16}$, then we find tent with center induced by $\{k_{11}, k_{12}, k_2, k_4, v_1, v_2, s_{12}\}$. Moreover, we find the same subgraph if v_1 and v_2 in S_{15} and only v_1 is complete to K_5 and if v_1 and v_2 in S_{16} or in S_{15} and are both complete to K_5.

If instead $N_{K_2}(v_1)$ and $N_{K_2}(v_2)$ are disjoint, then we find M_{IV} induced by $\{k_{11}, k_{22}, k_{12}, k_{21}, k_5, k_4, v_1, v_2, s_{45}, s_{24}\}$.

Case (1.1.2) *Suppose that* $i = 2$. If v_1 and v_2 lie in S_{12}, and $N_{K_1}(v_1)$ and $N_{K_1}(v_2)$ are disjoint, then we find M_{IV} as in the previous case, induced by $\{v_1, v_2, s_{24}, s_{45}, k_{11}, k_{21}, k_{12}, k_{22}, k_5, k_4\}$. If instead $N_{K_1}(v_1)$ and $N_{K_1}(v_2)$ are non-disjoint, then we find net*vee*K_1 induced by $\{v_1, v_2, s_{24}, k_{21}, k_{22}, k_1, k_4\}$. Similarly, we find the same subgraphs if v_1 and v_2 lie in $S_{23} \cup S_{24}$.

Case (1.1.3) *Suppose that* $i = 3$. If v_1 and v_2 lie in S_{34} and $N_{K_4}(v_1)$ and $N_{K_4}(v_2)$ are disjoint, then we find M_V induced by $\{v_1, v_2, s_{24}, s_{45}, k_{41}, k_{31}, k_{32}, k_{42}, k_5\}$. If $N_{K_4}(v_1)$ and $N_{K_4}(v_2)$ are non-disjoint, then we find a net $\vee K_1$ induced by $\{v_1, v_2, s_{45}, k_{31}, k_{32}, k_4, k_5\}$. Similarly, we find the same subgraphs if v_1 and v_2 in S_{23}.

Suppose that v_1 and v_2 lie in $S_{36} \cup S_{35}$. If $N_{K_5}(v_1)$ and $N_{K_5}(v_2)$ are not disjoint, then we find a tent induced by $\{v_1, v_2, s_{24}, k_5, k_{31}, k_{32}\}$. If $N_{K_5}(v_1)$ and $N_{K_5}(v_2)$ are disjoint, then we find a 4-sun induced by $\{v_1, v_2, s_{45}, s_{24}, k_{31}, k_{32}, k_{51}, k_{52}\}$.

If v_1 in S_{34} and v_2 in $S_{35} \cup S_{36}$, then we find $M_{II}(4)$ induced by $\{v_1, v_2, s_{24}, s_{45}, k_{31}, k_{32}, k_4, k_5\}$. The proof is analogous if v_1 and v_2 in $S_{23} \cup S_{13}$ or $S_{34} \cup S_{35}$.

Case (1.2) *$\mathbb{B}_i$ contains* D_1

Case (1.2.1) *Suppose that* $i = 1$. In this case, v_1 lies in S_{61} and v_2 lies in $S_{14} \cup S_{15} \cup S_{16}$. If v_1 in S_{61} and v_2 in $S_{14} \cup S_{15}$ is not complete to K_5, then we find $F_2(5)$ induced by $\{v_1, s_{12}, s_{24}, s_{45}, v_2, k_6, k_1, k_2, k_4, k_5\}$. If v_2 lies in S_{15} but is complete to K_5, then by definition of $\mathbb{B}_1$, v_2 is not complete to K_1. Let k_{11} in K_1 be a vertex nonadjacent to v_2 and let k_{12} in K_1 be the vertex represented by the column of D_1. Thus, v_1 and v_2 are adjcent to k_{12}. If v_1 is also adjacent to k_{11}, then we find F_0 induced by $\{v_1, v_2, s_{12}, k_6, k_{11}, k_{12}, k_2, k_4\}$. If instead v_1 is nonadjacent to k_{11}, then we find a net $\vee K_1$ induced by $\{v_1, v_2, s_{12}, k_6, k_{11}, k_{12}, k_4\}$. The same forbidden subgraph arises when considering a vertex v_2 in S_{16} such that there is a vertex k_6 in K_6 adjacent to v_1 and nonadjacent to v_2. Suppose now that v_2 in S_{16} and v_2 is nested in v_1 with regard to K_6. If v_1 is adjacent to k_{11}

and k_{12}, then we find a tent with center induced by {v_1, v_2, s_{12}, s_{24}, s_{45}, k_6, k_{11}, k_{12}, k_2, k_4, k_5}. If instead v_1 is nonadjacent to k_{11}, then we find M_V induced by {s_{24}, v_1, v_2, s_{12}, k_2, k_4, k_6, k_{12}, k_{11}}.
Case (1.2.2) If $i = 2$, then there are no vertices labeled with distinct letters and colored with the same color.
Case (1.2.3) *Suppose that* $i = 3$, We have two possibilities: either v_1 lies in $S_{35} \cup S_{36}$ and v_2 lies in S_{13}, or v_1 lies in S_{23} and v_2 lies in S_{34}. If v_1 lies in $S_{35} \cup S_{36}$ and v_2 lies in S_{13}, then we find F_0 induced by {v_1, v_2, s_{24}, k_1, k_2, k_3, k_4, k_5}. If v_1 lies in S_{34} and v_2 lies in S_{23}, then we find $F_2(5)$ induced by {v_1, v_2, s_{12}, s_{45}, s_{24}, k_1, k_2, k_3, k_4, k_5}.
Case (1.3) $\mathbb{B}_i$ *contains* D_2
Case (1.3.1) *Let* $i = 1$. In this case, v_1 in S_{12} and v_2 in S_{61}, hence we find $M_{III}(4)$ induced by {s_{24}, v_1, v_2, s_{12}, k_4, k_2, k_{11}, k_6, k_{12}}.
Case (1.3.2) *Suppose that* $i = 2$. In this case, v_1 in S_{12} and v_2 lies in $S_{23} \cup S_{24}$. We find $M_{II}(4)$ induced by {v_1, v_2, s_{24}, s_{12}, k_1, k_{21}, k_4, k_{22}}.
Case (1.3.3) Finally, *let* $i = 3$. We have two possibilities. If v_1 lies in $S_{35} \cup S_{36}$ and v_2 lies in S_{23}, then we find $M_{III}(4)$ induced by {v_1, v_2, s_{12} s_{24}, k_1, k_2, k_{31}, k_5, k_{32}}. If v_1 in S_{13} and v_2 lies in S_{34}, then we find $M_{III}(4)$ induced by {v_1, v_2, s_{24}, s_{45}, k_1, k_{31}, k_4, k_5, k_{32}}.
Case (1.4) *Suppose there is* $S_2(j)$ *in* $\mathbb{B}_i$ *for some* $j \geq 3$. Let $v_1, v_2, \dots, v_j$ be the vertices corresponding to the rows in $S_2(j)$ and $k_{i1}, k_{i2}, \dots, k_{i(j-1)}$ be the vertices corresponding to the columns in $S_3(j)$. Thus, v_1 and v_j are labeled with the same letter.
Case (1.4.1) *Let* $i = 1$, and suppose first that j is odd. Hence, v_1 and v_j are colored with distinct colors. If v_1 in S_{12} and v_j in $S_{14} \cup S_{15} \cup S_{16}$, then we find $F_1(j+2)$ induced by {v_1, …, v_j, s_{12}, s_{24}, k_4, k_2, k_{11}, …, k_{1j}}. Conversely, if v_j in S_{12} and v_1 in $S_{14} \cup S_{15} \cup S_{16}$, then we find $F_2(j)$ induced by {v_1, …, v_j, k_4, k_{11}, …, k_{1j}}.

Suppose instead that j is even, hence v_1 and v_j are colored with the same color. If v_1 and v_j lie in $S_{14} \cup S_{15} \cup S_{16}$, since there is no D_0, then Claim 4.2 and Claim 4.2 hold and thus v_1 and v_j are nested in K_4. Hence, we find $F_1(j+1)$ induced by {v_1, …, v_j, s_{12}, k_4, k_{11}, …, k_{1j}}. We find the same forbidden subgraph if v_1 and v_j lie both in S_{61} by changing k_6 for k_4.
Case (1.4.2) *Let* $i = 2$. Since there are no vertices labeled with the same letter and colored with distinct colors, then it is not possible to find $S_2(j)$ for any odd j. If instead j is even, then either v_1 and v_j lie in S_{12} or v_1 and v_j lie in S_{23}. If v_1 and v_j lie in S_{12}, then we find $F_2(j+1)$ induced by {v_1, …, v_j, s_{24}, k_1, k_{21}, …, k_{2j}}. We find the same forbidden subgraph if v_1 and v_j lie in S_{23} or S_{24} by changing k_1 for k_4 and S_{24} for s_{12}.
Case (1.4.3) *Suppose that* $i = 3$, and suppose first that $j \geq 3$ is odd. If v_1 in $S_{35} \cup S_{36}$ and v_j in S_{34}, then we find $F_2(j)$ induced by {v_1, …, v_j, k_5, k_{31}, …, k_{3j}}. If instead v_1 in S_{34} and v_j in $S_{35} \cup S_{36}$, then we find $F_1(j+2)$ induced by {v_1, …, v_j, s_{45}, s_{24}, k_5, k_4, k_{31}, …, k_{3j}}. We find the same forbidden subgraphs if v_1 in S_{13} and v_j in S_{23} by changing k_1 for k_5, and if v_1 in S_{23} and v_j in S_{13} by changing k_4 for k_2 and k_5 for k_1.

Suppose that j is even. If v_1 and v_j lie in $S_{35} \cup S_{36}$, then it follows from Claim 4.2 that they are nested in K_5, hence we find $F_1(j+1)$ induced by {v_1, …, v_j, s_{24}, k_5, k_{31}, …, k_{3j}}. If v_1 and v_j lie in S_{13} we find the same forbidden subgraph by changing k_5 for k_1. It follows analogously for v_1 and v_j lying both in S_{34} or S_{23}.
Case (1.5) *Suppose there is* $S_3(j)$ *in* $\mathbb{B}_i$ *for some* $j \geq 3$. Let $v_1, v_2, \dots, v_j$ be the vertices corresponding to the rows in $S_3(j)$ and $k_{i1}, k_{i2}, \dots, k_{i(j-1)}$ be the vertices corresponding to the columns in $S_3(j)$. Thus, v_1 and v_j are labeled with the distinct letters.
Case (1.5.1) *Let* $i = 1$, and suppose that j is odd. In this case, v_1 in S_{12} and v_j in S_{61}, and we find $F_2(j+2)$ induced by {v_1, …, v_j, s_{12}, s_{24}, k_4, k_2, k_{11}, …, $k_{1(j-1)}$, k_6}. If instead j is even, then v_1 in

$S_{14} \cup S_{15} \cup S_{16}$ and v_j in S_{61}, and we find $F_2(j+1)$ induced by $\{v_1, \ldots, v_j, s_{12}, k_4, k_{11}, \ldots, k_{1(j-1)}, k_6\}$.

Case (1.5.2) *Let* $i = 2$. If j is even, then there are no vertices labeled with the same letter and colored with distinct colors in $S_3(j)$.

If instead j is odd, then v_1 in S_{12} and v_j in $S_{23} \cup S_{24}$. In this case, we find $F_1(j+2)$ induced by $\{v_1, \ldots, v_j, s_{12}, s_{24}, k_1, k_{21}, \ldots, k_{2(j-1)}, k_4\}$.

Case (1.5.3) *Suppose that* $i = 3$. Let j be odd. If v_1 lies in $S_{35} \cup S_{36}$ and v_j in S_{23}, then we find $F_2(j+2)$ induced by $\{v_1, \ldots, v_j, s_{12}, s_{24}, k_5, k_{31}, \ldots, k_{3(j-1)}, k_2, k_1\}$. If instead v_1 in S_{13} and v_j in S_{34}, then we find $F_2(j+2)$ induced by $\{v_1, \ldots, v_j, s_{45}, s_{24}, k_1, k_{31}, \ldots, k_{3(j-1)}, k_4, k_5\}$.

If instead j is even, then v_1 in $S_{35} \cup S_{36}$ and v_j in S_{13}. In this case we find $F_2(j+1)$ induced by $\{v_1, \ldots, v_j, s_{24}, k_5, k_{31}, \ldots, k_{3(j-1)}, k_1\}$.

Notice that $\mathbb{B}_i$ has no LR-rows, thus there are no $S_1(j)$, $S_4(j)$, $S_5(j)$, $S_6(j)$, $S_7(j)$, $S_8(j)$, $P_0(k,l)$, $P_1(k,l)$ or $P_2(k,l)$ as subconfigurations. Hence, $\mathbb{B}_i$ is admissible for each $i = 1, 2, 3$, and thus it follows for $i = 4, 5$ for symmetry.

Furthermore, it follows by the same argument as in the tent case that it is not possible that $\mathbb{B}_i$ is admissible but not LR-orderable.

Case (2) *Suppose that* $\mathbb{B}_i$ *is LR-orderable and is not partially 2-nested.*

Since there are no LR-rows in $\mathbb{B}_i$ for each $i = 1, 2, 3$, if $\mathbb{B}_i$ is not partially 2-nested, then there is either a monochromatic gem or a monochromatic weak gem in $\mathbb{B}_i$ as a subconfiguration. Remember that every colored row in $\mathbb{B}_i$ is a row labeled with L or R, hence both rows of a monochromatic gem or weak gem are labeled rows. However, this is not possible since in each case we find either D_0 or D_1, and this results in a contradiction for we showed that $\mathbb{B}_i$ is admissible and therefore $\mathbb{B}_i$ is partially 2-nested.

Case (3) *Suppose that* $\mathbb{B}_i$ *is partially 2-nested and is not 2-nested.*

If $\mathbb{B}_i$ is partially 2-nested and is not 2-nested, then, for every proper 2-coloring of the rows of $\mathbb{B}_i$, there is a monochromatic gem or a monochromatic wek gem indued by at least one unlabeled row. This proof is also analogous as in the tent case (See Lemma 4.3 for details).

□

4.2.3 The matrix $\mathbb{B}_6$

In this subsection we will demostrate a lemma analogous to Lemma 4.8 but for the matrix $\mathbb{B}_6$. In other words, we will use the matrix theory developed in Chapter 3 in order to characterize the $\mathbb{B}_6$ matrix when the split graph G that contains an induced 4-tent is also a circle graph. Although the result is the same –we will find all the forbidden subgraphs for the class of circle graphs given when $\mathbb{B}_6$ is not 2-nested–, the most important difference between this matrix and the matrices $\mathbb{B}_i$ for each $i = 1, 2, \ldots, 5$, is that $\mathbb{B}_6$ contains LR-rows.

First, we will define how to color those rows that correspond to vertices in $S_{[15]}$, since we defined $\mathbb{B}_6$ as an enriched matrix and these rows are the only empty LR-rows in $\mathbb{B}_6$. Remember that all the empty LR-rows must be colored with the same color. Hence, if there is at least one red row labeled with L or one blue row labeled with R (resp. blue row labeled with L or red row labeled with R), then we color every LR-row in $S_{[15]}$ with blue (resp. with red). This will give a 1-color assignment to each empty LR-row only if G is a circle graph.

Lemma 4.9. *Let* G *be a split graph that contains an induced* 4*-tent and such that* G *contains no induced tent, and let* $\mathbb{B}_6$ *as defined in the previous section. If* $S_{[15]} \neq \emptyset$ *and one of the following holds:*

— *There is at least one red row* f_1 *and one blue row* f_2*, both labeled with L (resp. R)*

— *There is at least one row* f_1 *labeled with L and one row* f_2 *labeled with R, both colored with red (resp. blue).*

Then, we find either $F_1(5)$ *or* 4*-sun as an induced subgraph of* G.

Proof. We assume that $\mathbb{B}_6$ contains no D_0, for we will prove this in Lemma 4.10.

Let v_1 be a vertex corresponding to a red row labeled with L, v_2 be the vertex corresponding to a blue row labeled with L, and w in $S_{[15]}$. Thus, v_1 in $S_{36} \cup S_{46}$ and v_2 in $S_{56} \cup S_{26} \cup S_{16}$. In either case, we find $F_1(5)$ induced by $\{k_2, k_4, k_5, k_6, v_1, v_2, w, s_{24}, s_{45}\}$ or $\{k_1, k_2, k_4, k_6, v_1, v_2, w, s_{12}, s_{24}\}$, depending on whether v_2 in S_{56} or in $S_{26} \cup S_{16}$. Suppose now that v_1 is a vertex corresponding to a red row labeled with L and v_2 is a vertex corresponding to a red row labeled with R. Thus, v_1 in $S_{36} \cup S_{46}$ and v_2 in $S_{61} \cup S_{64} \cup S_{65}$. If v_2 in S_{61}, then there is a 4-sun induced by $\{k_1, k_2, k_4, k_6, v_1, v_2, s_{12}, s_{24}\}$. If instead v_2 in $S_{64} \cup S_{65}$, then we find a tent with center induced by $\{k_6, k_1, k_4, k_5, v_1, v_2, w\}$. This finished the proof since the other cases are analogous by symmetry. □

In order to prove the following lemma, we will assume without loss of generality that $S_{[15]} = \emptyset$.

Lemma 4.10. *Let* $G = (K, S)$ *be a split graph containing an induced* 4*-tent such that* G *contains no induced tent, and let* $B = \mathbb{B}_6$*. If* B *is not 2-nested, then* G *contains an induced subgraph of the families* $\mathcal{T}$ *or* $\mathcal{F}$*.*

Proof. We will assume proven Lemma 4.8. This is, we assume that the matrices $\mathbb{B}_1, \dots, \mathbb{B}_5$ are 2-nested. In particular, it follows that any pair of vertices v_1 in S_{ij} and v_2 in S_{ik} such that $i \neq 6$ and $j \neq k$ are nested in K_i. Moreover, there is a vertex $v*_i$ in K_i adjacent to both v_1 and v_2.

Throughout the proof, we will refer indistinctly to a row r (resp. a column c) and the vertex in the independent (resp. complete) partition of G whose adjacency is represented by the row (resp. column). The structure of the proof is analogous as in Lemmas 4.3 and 4.8. The only difference is that, in this case B admits LR-rows by definition, and thus we have to consider all the forbidden subconfigurations for every characterization in each case.

Case (1) *Suppose that* B *is not admissible.*

Hence, B contains at least one of the matrices $D_0, D_1, \dots, D_{13}, S_1(j), S_2(j), \dots, S_8(j)$ for some $j \geq 3$ or $P_0(j, l)$, $P_1(j, l)$ for some $l \geq 0, j \geq 5$ or $P_2(j, l)$, for some $l \geq 0, j \geq 7$.

Case (1.1) B *contains* D_0. Let v_1 and v_2 be the vertices represented by the first and second row of D_0 respectively, and k_{61}, k_{62} in K_6 represented by the first and second column of D_0, respectively.

Case (1.1.1) Suppose first that both vertices are colored with the same color. Since the case is symmetric with regard of the coloring, we may assume that both rows are colored with red, hence either v_1 and v_2 lie in $S_{61} \cup S_{64} \cup S_{65}$, or v_1 and v_2 lie in $S_{36} \cup S_{46}$. If v_1 and v_2 lie in S_{61} and k_1 in K_1 is adjacent to both v_1 and v_2, then we find a net $\vee K_1$ induced by $\{k_{61}, k_{62}, k_1, k_2, v_1, v_2, s_{12}\}$. We find the same forbidden subgraph if either v_1 and v_2 lie in $S_{64} \cup S_{65}$ changing k_1 for some k_4 in K_4 adjacent to both v_1 and v_2, k_2 for some k_5 in K_5 nonadjacent to both v_1 and v_2 and s_{12} for s_{45}. We also find the same subgraph if v_1 and v_2 lie in $S_{36} \cup S_{46}$, changing k_1 for some k_4 in K_4 adjacent to both v_1 and v_2 and s_{12} for s_{24}. If instead v_1 in S_{61} and v_2 in $S_{64} \cup S_{65}$, since by definition every vertex in S_{65} is adjacent but not complete to K_5, then there are vertices k_4 in K_4 and k_5 in K_5 such that v_1 is nonadjacent to both, and v_2 is adjacent to k_4 and is nonadjacent to k_5. Thus, we find $F_2(5)$ induced by $\{k_{62}, k_1, k_2, k_4, k_5, v_1, v_2, s_{45}, s_{12}, s_{24}\}$.

Case (1.1.2) Suppose now that both rows are colored with distinct colors. By symmetry, assume without loss of generality that v_1 is colored with red and v_2 is colored with blue. Hence, v_1 lies in $S_{62} \cup S_{63}$, and v_2 lies in $S_{61} \cup S_{64} \cup S_{65}$. If v_2 in S_{61}, then there is a vertex k_4 in K_4 nonadjacent

to v_1 and v_2. Hence, we find $M_{III}(4)$ induced by $\{k_{61}, k_{62}, k_1, k_2, k_4, v_1, v_2, s_{12}\ s_{24}\}$. If instead v_2 in S_{64} or S_{65}, then we find $M_{III}(4)$ induced by $\{k_{61}, k_{62}, k_2, k_4, k_5, v_1, v_2, s_{24}, s_{45}\}$.
Case (1.2) B *contains* D_1. Let v_1 and v_2 be the vertices that represent the rows of D_1, and let k_6 in K_6 be the vertex that represents the column of D_1. Suppose without loss of generality that both rows are colored with red, hence v_1 in $S_{36} \cup S_{46}$ and v_1 in $S_{61} \cup S_{64} \cup S_{65}$. Notice that we are assuming there is no D_1 in $\mathbb{B}_4$, thus, if v_2 is not complete to K_4, then there is a vertex k_4 in K_4 adjacent to v_1 and nonadjacent to v_2. If v_2 in S_{61}, then we find a 4-sun induced by $\{k_6, k_1, k_2, k_4, v_1, v_2, s_{12}, s_{24}\}$. If v_2 in S_{64} is not complete to K_4, then we find a tent induced by $\{k_6, k_2, k_4, v_1, v_2, s_{24}\}$. If instead v_2 in $S_{64} \cup S_{65}$ is complete to K_4, then we find a $M_{II}(4)$ induced by $\{k_2, k_4, k_5, k_6, v_1, v_2, s_{24}, s_{45}\}$.
Case (1.3) B *contains* D_2. Let v_1 and v_2 be the first and second row of D_2, and let k_{61} and k_{62} be the vertices corresponding to first and second column of D_2, respectively. By symmetry we suppose without loss of generality that v_1 is colored with blue and v_2 is colored with red. Thus, v_1 lies in $S_{56} \cup S_{26} \cup S_{16}$ and v_2 lies in $S_{61} \cup S_{64} \cup S_{65}$. If v_1 in S_{56} and v_2 in S_{61}, then we find a 5-sun with center induced by $\{k_{61}, k_{62}, k_1, k_2, k_4, k_5, v_1, v_2, s_{12}, s_{24}, s_{45}\}$. If instead v_1 in $S_{26} \cup S_{16}$, since v_1 is not complete to K_1 and we assume that $\mathbb{B}_1$ is admissible, then there is a vertex k_1 in K_1 adjacent to v_2 and nonadjacent to v_1, for if not we find D_1 in $\mathbb{B}_1$. We find a tent induced by $\{k_{61}, k_1, k_2, v_1, v_2, s_{12}\}$. The same holds if v_1 in S_{56} and v_2 in S_{65}, for $\mathbb{B}_5$ is admissible and v_2 is adjacent but not complete to K_5. Moreover, if v_1 in S_{56} and v_2 in S_{64}, then we find a tent induced by $\{k_{61}, k_4, k_5, v_1, v_2, s_{45}\}$. Finally, if v_1 in $S_{26} \cup S_{16}$ and v_2 in $S_{64} \cup S_{65}$, then there are vertices k_1 in K_1 and k_5 in K_5 such that k_1 is nonadjacent to v_1 and adjacent to v_2, and k_5 is nonadjacent to v_2 and adjacent to v_1. Hence, we find $F_1(5)$ induced by $\{k_1, k_2, k_4, k_5, v_1, v_2, s_{12}, s_{24}, s_{45}\}$.

Remark 4.11. If G is circle, then $S_{26} \cup S_{16} \neq \emptyset$ if $S_{64} \cup S_{65} = \emptyset$, and viceversa.

Case (1.4) B *contains* D_3. Let v_1 and v_2 be the vertices corresponding to the rows of D_3 labeled with L and R, respectively, w be the vertex corresponding to the LR-row, and k_{61}, k_{62} and k_{63} in K_6 be the vertices corresponding to the columns of D_3. Notice that an uncolored LR-row in B represents a vertex in $S_{[16}$.

Remark 4.12. We consider all the vertices above described. If there is a vertex k_i in K_i for some $i \in \{1, \dots, 5\}$ such that v_1 and v_2 are both adjacent to k_i, since w is complete to K_i, then we find a net $\vee K_1$ induced by $\{k_{61}, k_{62}, k_{63}, k_i, v_1, v_2, w\}$.

Case (1.4.1) *Suppose first that* v_1 *and* v_2 *are colored with distinct colors.* If v_1 is colored with blue and v_2 is colored with red, then v_1 in $S_{56} \cup S_{26} \cup S_{16}$ and v_2 in $S_{61} \cup S_{64} \cup S_{65}$.

It follows by symmetry and the previous remark that we only need to see what happens if v_1 in S_{61} and v_2 in either S_{56} or S_{26}. If v_2 in S_{56}, then we find $M_{III}(6)$ induced by $\{k_{61}, k_1, k_2, k_4, k_5, k_{62}, k_{63}, v_1, v_2, s_{12}, s_{24}, s_{45}, w\}$. If instead v_2 in S_{26}, then we find $M_{III}(4)$ induced by $\{k_{63}, k_{61}, k_1, k_2, k_{62}, v_1, v_2, s_{12}, w\}$.

Conversely, if v_1 is colored with red and v_2 is colored with blue, then v_1 in $S_{36} \cup S_{46}$ and v_2 in $S_{62} \cup S_{63}$. In this case, we find $M_{III}(4)$ induced by $\{k_{62}, k_2, k_4, k_{61}, k_{63}, v_1, v_2, w, s_{24}\}$.
Case (1.4.2) *Suppose now that* v_1 *and* v_2 *are colored with the same color.* Hence, v_1 in $S_{36} \cup S_{46}$ and v_2 in $S_{61} \cup S_{64} \cup S_{65}$. We may assume from Remark 4.12 that there no vertex k_i in K_i adjacent to both v_1 and v_2 for every $i \in \{1, \dots, 5\}$. Hence, v_2 in S_{61}. We find $F_2(5)$ induced by $\{k_{62}, k_1, k_2, k_4, k_{61}, v_1, v_2, w, s_{12}, s_{24}\}$.

We have the following remark as a consequence of the previous statements.

Remark 4.13. If G is circle and $S_{36} \cup S_{46} \neq \emptyset$, then $S_{61} = \emptyset$, and viceversa.

Moreover, if G is circle, $S_{36} \cup S_{46} \neq \emptyset$ and $\mathbb{B}_4$ is admissible, then $S_{61} \cup S_{64} \cup S_{65} = \emptyset$. The same holds for the subsets $S_{56} \cup S_{26} \cup S_{16}$ and $S_{62} \cup S_{63}$.

Case (1.5) B *contains* D_4. Let v_1 and v_2 be the vertices represented by the rows labeled with L, w be the vertex represented by the LR-row and k_6 in K_6 corresponding to the only column of D_4. Suppose without loss of generality that v_1 is colored with red and v_2 is colored with blue. Thus, v_1 lies in $S_{61} \cup S_{64} \cup S_{65}$ and v_2 lies in $S_{62} \cup S_{63}$. In either case, we find $F_1(5)$: if v_1 in S_{61}, then it is induced by $\{k_6, k_1, k_2, k_4, v_1, v_2, w, s_{12}, s_{24}\}$, and if v_1 in S_{64} or S_{65}, then it is induced by $\{k_6, k_2, k_4, k_5, v_1, v_2, w, s_{24}, s_{45}\}$.

Case (1.6) B *contains* D_5. Let v_1 and v_2 be the vertices representing the rows labeled with L and R, respectively, w be the vertex corresponding to the LR-row, and k_6 in K_6 corresponding to the column of D_5. Suppose without loss of generality that v_1 is colored with blue and v_2 is colored with red.

Remark 4.14. If x_1 in S_{ij} and x_2 in S_{jk}, then we may assume that there are vertices k_{j1} and k_{j2} in K_j such that x_1 is adjacent to k_{j1} and is nonadjacent to k_{j2} and x_2 is adjacent to k_{j2} and is nonadjacent to k_{j1}, for if not $\mathbb{B}_i$ is not admissible, for $i \in \{1, \ldots, 5\}$.

By the previous remark, notice that, if v_1 in $S_{26} \cup S_{16}$ and v_2 in S_{61}, then there is a tent induced by $\{k_6, k_1, k_2, v_1, v_2, s_{12}\}$, where k_1 is a vertex nonadjacent to v_1. The same holds if v_1 in S_{56} and v_2 in S_{65}, where the tent is induced by $\{k_6, k_4, k_5, v_1, v_2, s_{45}\}$, with k_5 in K_5 adjacent to v_1 and nonadjacent to v_2. Finally, if v_1 in S_{56} and v_2 in S_{61}, then we find a 5-sun with center induced by $\{k_5, k_6, k_1, k_2, k_4, v_1, v_2, w, s_{12}, s_{24}, s_{45}\}$.

Remark 4.15. If G contains no induced tent, we may assume that, if $S_{56} \neq \emptyset$, then $S_{65} = \emptyset$, and viceversa. Moreover, if $S_{26} \cup S_{16} \neq \emptyset$, then $S_{61} = \emptyset$, and viceversa.

Suppose that B contains D_6. Let v_1 and v_2 be the vertices represented by the rows labeled with L and R, respectively, w be the vertex corresponding to the LR-row, and k_{61} and k_{62} in K_6 corresponding to the first and second column of D_6, respectively. Suppose without loss of generality that v_1 and v_2 are both colored with red. In this case, v_1 lies in $S_{36} \cup S_{46}$, v_2 lies in $S_{61} \cup S_{64} \cup S_{65}$ and w lies in $S_{[16}$. However, by Remark 4.13 this is not possible since we are assuming that $\mathbb{B}_i$ is admissible for every $i = 1, 2, \ldots, 5$.

Case (1.7) B *contains* D_7 *or* D_{11}. Thus, there is a vertex k_i in some K_i with $i \neq 6$ such that k_i is adjacent to the three vertices corresponding to every row of D_7, thus we find a net $\vee K_1$. The same holds if there is D_{11}.

Case (1.8) B *contains* D_8 *or* D_{12}. In that case, there is an induced tent.

Case (1.9) B *contains* D_9 *or* D_{13}. It is straightforward that in this case we find F_0.

Case (1.10) B *contains* D_{10}. Let v_1 and v_2 be the vertices represented by the rows labeled with L and R, respectively, w_1 and w_2 be the vertices represented by the LR-rows and $k_{61}, \ldots, k_{64}$ in K_6 be the vertices corresponding to the columns of D_{10}. Suppose without loss of generality that v_1 is colored with red and v_2 is colored with blue. Hence, v_1 lies in $S_{36} \cup S_{46}$ and v_2 lies in $S_{62} \cup S_{63}$. Let k_2 in K_2 adjacent to v_2 and nonadjacent to v_1 and let k_4 in K_4 adjacent to v_1 and nonadjacent to v_2. Hence, we find $F_1(5)$ induced by $\{v_1, v_2, w_1, w_2, s_{24}, k_2, k_4, k_{62}, k_{63}\}$.

Case (1.11) *Suppose that* B *contains* $S_1(j)$

Case (1.11.1) If $j \geq 4$ is even, let $v_1, v_2, \ldots, v_j$ be the vertices represented by the rows of $S_1(j)$, where v_1 and v_j are labeled both with L or both with R, v_{j-1} is a vertex corresponding to the LR-row, and $k_{61}, \ldots, k_{6(j-1)}$ in K_6 the vertices corresponding to the columns. Suppose without loss of generality that v_1 and v_j are labeled with L. It follows that either v_1 and v_j lie in $S_{36} \cup S_{46}$, or v_1 and v_j lie in $S_{62} \cup S_{63}$ or v_1 lies in $S_{56} \cup S_{26} \cup S_{16}$ and v_j lies in $S_{36} \cup S_{46}$. In either case, there

is k_5 in K_5 adjacent to both v_1 and v_j. Moreover, k_5 is also adjacent to v_{j-1}. Thus, this vertex set induces a $j-1$-sun with center.
Case (1.11.2) If j is odd, since $S_1(j)$ has $j-2$ rows (thus there are $v_1, \dots, v_{j-2}$ vertices), then the subset of vertices given by $\{v_1, \dots, v_{j-2}, k_{61}, \dots, k_{6(j-2)}, k_5\}$ induces an even $j-1$-sun.
Case (1.12) *Suppose that* B *contains* $S_2(j)$.

Let v_1 and v_j be the vertices corresponding to the labeled rows, $k_{61}, \dots, k_{6(j-1})$ in K_6 be the vertices corresponding to the columns of $S_2(j)$, and suppose without loss of generality that v_1 and v_j are labeled with R.
Case (1.12.1) Suppose first that j is odd, v_1 is colored with red and v_j is colored with blue. Thus, v_1 in $S_{61} \cup S_{64} \cup S_{65}$ and v_j in $S_{62} \cup S_{63}$, or viceversa. If v_1 in S_{61}, then let k_i in K_i for $i = 1, 2, 4$ such that k_1 is adjacent to v_1 and v_j, k_2 is adjacent to v_j and nonadjacent to v_1, and k_4 is nonadjacent to both v_1 and v_j. We find $F_2(j+2)$ induced by $\{k_4, k_2, k_1, k_{61}, \dots, k_{6(j-1}), v_1, \dots, v_j, s_{12}, s_{24}\}$. If v_1 in $S_{64} \cup S_{65}$, then we find $F_2(j)$ induced by $\{k_5, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j\}$, with k_5 in K_5 adjacent to v_1 and nonadjacent to v_j.

Conversely, suppose v_1 in $S_{62} \cup S_{63}$ and v_j in $S_{61} \cup S_{64} \cup S_{65}$. If v_j lies in $S_{64} \cup S_{65}$, then $F_2(j+2)$ is induced by $\{k_2, k_4, k_5, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{24}, s_{45}\}$, with k_i in K_i for $i = 2, 4, 5$ such that k_2 is adjacent to v_1 and v_k, k_4 is adjacent to v_j and nonadjacent to v_1, and k_5 is nonadjacent to both v_1 and v_j. If instead v_j in S_{61}, then it is induced by $\{k_4, k_2, k_1, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{12}, s_{24}\}$.
Case (1.12.2) Suppose now that j is even, and thus both v_1 and v_j are colored with the same color. Suppose without loss of generality that are both colored with red, and thus v_1 and v_j lie in $S_{61} \cup S_{64} \cup S_{65}$. If v_1 and v_j in S_{61}, then we find $F_2(j+1)$ induced by $\{k_2, k_1, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{12}\}$. We find the same forbidden subgraph if v_1 and v_j lie in S_{64} or S_{65}, by changing s_{12} for s_{45}, and k_1 and k_2 for k_4 and k_5, where k_5 is nonadjacent to both v_1 and v_j and k_4 is adjacent to both. If only v_1 lies in S_{61}, then we find $F_2(j+3)$ induced by $\{k_1, k_2, k_4, k_5, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{12}, s_{24}, s_{45}\}$, with k_i in K_i for $i = 1, 2, 4, 5$. If only v_j lies in S_{61}, then we find $F_2(5)$ induced by $\{k_1, k_2, k_4, k_5, k_{62}, v_1, v_j, s_{12}, s_{24}, s_{45}\}$, with k_i in K_i for $i = 1, 2, 4, 5$.
Case (1.13) *Suppose that* B *contains* $S_3(j)$. Let v_1 and v_j be the vertices corresponding to the labeled rows, $k_{61}, \dots, k_{6(j-1})$ in K_6 be the vertices corresponding to the columns of $S_3(j)$.
Case (1.13.1) Suppose first that j is odd, and suppose that v_1 is labeled with L and colored with blue and v_j is labeled with R and colored with red. In this case, v_1 in $S_{56} \cup S_{26} \cup S_{16}$ and v_j in $S_{61} \cup S_{64} \cup S_{65}$. If v_1 in S_{56}, then we find a $(j+3)$-sun if v_j in S_{61}, induced by $\{k_1, k_2, k_4, k_5, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{45}, s_{12}, s_{24}\}$. If v_j in $S_{64} \cup S_{65}$, then we find a $(j+1)$-sun induced by $\{k_4, k_5, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{45}\}$. Moreover, if v_j in S_{61} and v_1 in $S_{26} \cup S_{16}$, then we find a $(j+1)$-sun induced by $\{k_1, k_{61}, \dots, k_{6(j-1)}, k_2, v_1, \dots, v_j, s_{12}\}$. Finally, if v_j in $S_{64} \cup S_{65}$ and v_1 in $S_{26} \cup S_{16}$, then we find $F_1(5)$ induced by $\{k_1, k_2, k_4, k_5, v_1, v_j, s_{24}, s_{45}, s_{12}\}$.
Case (1.13.2) Suppose now that j is even, and suppose without loss of generality that v_1 and v_j are both colored with red. Thus, v_1 in $S_{61} \cup S_{64} \cup S_{65}$ and v_j in $S_{36} \cup S_{46}$. If v_1 in S_{61}, then we find $(j+2)$-sun induced by $\{k_4, k_2, k_1, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{12}, s_{24}\}$. If instead v_1 in $S_{64} \cup S_{65}$, then we find j-sun induced by $\{k_4, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j\}$.
Case (1.14) B *contains* $S_4(j)$.

Let v_1, v_2 and v_j be the labeled rows and $k_{61}, \dots, k_{6(j-1})$ in K_6 be the vertices corresponding to the columns of $S_4(j)$. Suppose without loss of generality that v_1 is the vertex corresponding to the row labeled with LR, v_2 corresponding to the row labeled with L, v_j labeled with R. Notice that v_1 lies in $S_{[16}$.
Case (1.14.1) Suppose j is even, thus v_2 and v_j are colored with the same color. Suppose without loss of generality that they are both colored with red. Hence, v_2 in $S_{36} \cup S_{46}$ and v_j in $S_{61} \cup S_{64} \cup S_{65}$. If v_j lies in $S_{64} \cup S_{65}$, then we find a $(j-1)$-sun with center induced by $\{k_4, k_{61}, \dots, k_{6(j-1)}, v_1,$

$2, \dots, v_j\}$. If instead v_j in S_{61}, then we find a $(j+1)$-sun with center induced by $\{k_{61}, \dots, k_{6(j-1)}$, k_1, k_2, k_4, v_1, $2, \dots, v_j$, s_{12}, $s_{24}\}$.

Case (1.14.2) Suppose j is odd, thus assume without loss of generality that v_2 is colored with red and v_j is colored with blue. Hence, v_2 in $S_{36} \cup S_{46}$ and v_j in $S_{62} \cup S_{63}$. We find a j-sun with center induced by $\{k_4$, $k_{61}, \dots, k_{6(j-1)}$, k_2, v_1, $2, \dots, v_j$, $s_{24}\}$.

Case (1.15) B *contains* $S_5(j)$.

Let v_1, v_{j-1} and v_j be the labeled rows and $k_{61}, \dots, k_{6(j-2)}$ in K_6 be the vertices corresponding to the columns of $S_4(j)$. Suppose without loss of generality v_2 and v_j are labeled with L and that v_{j-1} is the vertex corresponding to the row labeled with LR.

Case (1.15.1) Suppose j is even, hence v_1 and v_j lie in $S_{36} \cup S_{46}$. In this case we find $F_1(j+1)$ induced by $\{k_2$, k_4, $k_{61}, \dots, k_{6(j-2)}$, $v_1, \dots, v_{j-1}, v_j$, $s_{24}\}$.

Case (1.15.2) Suppose j is odd, and suppose that v_1 is colored with red and v_j is colored with blue. Thus, v_1 in $S_{36} \cup S_{46}$ and v_j in $S_{56} \cup S_{26} \cup S_{16}$. If v_j in S_{56}, then we find $F_1(j)$ induced by $\{k_4$, k_5, $k_{61}, \dots, k_{6(j-2)}$, $v_1, \dots, v_{j-1}, v_j$, $s_{45}\}$. If instead v_j lies in $S_{26} \cup S_{16}$, then we find $F_1(j+2)$ induced by $\{k_1$,, k_2, k_4, $k_{61}, \dots, k_{6(j-2)}$, $v_1, \dots, v_{j-1}, v_j$, s_{24}, $s_{12}\}$.

Case (1.16) B *contains* $S_6(j)$.

Case (1.16.1) Suppose first that B contains $S_6(3)$ or $S_6'(3)$, and let v_1, v_2 and v_3 be the vertices that respresent the LR-row, the R-row and the unlabeled row, respectively. Independently on where does v_2 lie in, there is vertex v in $K \setminus K_6$ such that v is adjacent to v_1 and v_2 and nonadjacent to v_3, then we find an induced tent with center.

Case (1.16.2) If B contains $S_6(j)$ for some even j, then we find $F_1(j)$ induced by every row and column of $S_6(j)$. If instead j is odd, then we find $M_{II}(j)$ induced by every row and column of $S_6(j)$ and a vertex k_i in some K_i with $i \neq 6$. We choose such a vertex k_i adjacent to v_2, and thus since v_1 in $S_{[16}$, v_1 is also adjacent to k_i and $v_3, \dots, v_j$ are nonadjacent to k_i for they represent vertices in S_{66}.

Case (1.17) B *contains* $S_7(j)$.

Suppose B contains $S_7(3)$. It is straightforward that the rows and columns induce a co-4-tent $\vee K_1$. Furthermore, if $j > 3$, then j is even. The rows and columns of $S_7(j)$ induce a j-sun.

Case (1.18) B *contains* $S_8(2j)$.

If $j = 2$, then we can find an tent induced by the last three columns and the last three rows. If instead $j > 2$, then we find a $(2j-1)$-sun with center induced by every unlabeled row, every column but the first and one more column –which will be the center– representing any vertex in K_1, since $K_1 \neq \emptyset$.

Case (1.19) B *contains* $P_0(j, l)$.

Let $v_1, \dots, v_j$ be the vertices represented by the rows of $P_0(j, l)$ and $k_{61}, \dots, k_{6j}$ be the vertices in K_6 represented by the columns. The rows corresponding to v_1 and v_j are labeled with L and R, respectively, and the row corresponding to v_{l+2} is an LR-row.

Case (1.19.1) Suppose first that $l = 0$. If j is even, then v_1 and v_j are colored with the same color. Suppose without loss of generality that both are colored with red, thus v_1 lies in $S_{36} \cup S_{46}$ and v_j lies in $S_{62} \cup S_{63}$. In that case, there are vertices k_i in K_i for $i = 2, 4$ such that k_2 is adjacent to v_j and nonadjacent to v_1 and k_4 is adjacent to v_1 and nonadjacent to v_j. We find $F_2(j+1)$ induced by $\{k_2$,, k_4, $k_{62}, \dots, k_{6j}$, $v_1, \dots, v_j$, $s_{24}\}$

If instead j is odd, then v_1 and v_j are colored with the same colors. Suppose without loss of generality that they are both colored with red. Hence, v_1 lies in $S_{36} \cup S_{46}$ and v_j lies in $S_{61} \cup S_{64} \cup S_{65}$. In either case, we find $F_2(j+2)$ induced by $\{k_1$,, k_2, k_4, $k_{62}, \dots, k_{6j}$, $v_1, \dots, v_j$, s_{24}, $s_{12}\}$ if v_j lies in S_{61}, and induced by $\{k_2$,, k_4, k_5, $k_{62}, \dots, k_{6j}$, $v_1, \dots, v_j$, s_{24}, $s_{45}\}$ if v_j lies in $S_{64} \cup S_{65}$.

Case (1.19.2) Suppose that $l > 0$. The proof is very similar to the case $l = 0$. If j is odd, then v_1 and v_j are colored with the same color. If it is red, then we find $F_2(j+2)$ induced by $\{k_1,, k_2, k_4, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{24}, s_{12}\}$ if v_j lies in S_{61}, and we find $F_2(j)$ induced by $\{k_4, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j\}$ if v_j lies in $S_{64} \cup S_{65}$.

If instead j is even, then v_1 and v_j are colored with distinct colors. Then, we find $F_2(j+1)$ induced by $\{k_2, k_4, k_{61}, \dots, k_{6(j-1)}, v_1, \dots, v_j, s_{24}\}$.

Case (1.20) B *contains* $P_1(j, l)$.

Let $v_1, \dots, v_j$ be the vertices represented by the rows of $P_1(j, l)$ and $k_{61}, \dots, k_{6(j-1)}$ be the vertices in K_6 represented by the columns. The rows corresponding to v_1 and v_j are labeled with L and R, respectively, and the rows corresponding to v_{l+2} and v_{l+3} are LR-rows.

Case (1.20.1) Suppose first that $l = 0$. If j is odd, then v_1 and v_j are colored with the same color. We assume without loss of generality that they are colored with red. Thus, v_1 lies in $S_{36} \cup S_{46}$ and v_j lies in $S_{61} \cup S_{64} \cup S_{65}$. In either case, v_1 is anticomplete to K_1. Hence, we find $F_1(j)$ induced by every row and column of $P_1(j, 0)$ and an extra column that represents a vertex in K_1 adjacent to v_j, v_2 and v_3 and nonadjacent to v_i, for $1 \leq i \leq j-1$, $i \neq j, 2, 3$. If instead j is even, then we assume that v_1 and v_j are colored with red and blue, respectively. Thus, v_1 lies in $S_{36} \cup S_{46}$ and v_j lies in $S_{62} \cup S_{63}$. We find $F_1(j+1)$ induced by every row and every column of $P_1(j, 0)$, the row corresponding to s_{24} and two columns corresponding to vertices k_2 in K_2 and k_4 in K_4 such that k_2 is adjacent to v_j, v_2 and v_3 and is nonadjacent to v_i, and k_4 is adjacent to v_1, v_2 and v_3 and is nonadjacent to v_i, for each $1 \leq i \leq j-1$, $i \neq j, 2, 3$.

Case (1.20.2) Suppose $l > 0$. The proof is analogous to the previous case if j is even. If instead j is odd, then v_1 lies in $S_{36} \cup S_{46}$ and v_j lies in $S_{61} \cup S_{64} \cup S_{65}$. If v_j in S_{61}, then we find $F_1(j+2)$ induced by $\{k_4, k_{61}, \dots, k_{6(j-2)}, k_1, k_2, v_1, \dots, v_j, s_{12}, s_{24}\}$. If instead $v_j \notin S_{61}$, then we find $F_1(j)$ induced by every row and every column of $P_1(j, l)$ and one more column representing a vertex in K_4 adjacent to every vertex represented by a labeled row.

Case (1.21) B *contains* $P_2(j, l)$.

Let $v_1, \dots, v_j$ be the vertices represented by the rows of $P_2(j, l)$ and $k_{61}, \dots, k_{6(j-1)}$ be the vertices in K_6 represented by the columns. The rows corresponding to v_1 and v_j are labeled with L and R, respectively, and the rows corresponding to v_{l+2}, v_{l+3}, v_{l+4} and v_{l+5} are LR-rows.

Suppose $l = 0$. If j is even, then we find $F_1(j-1)$ induced by $\{k_{62}, k_{65}, \dots, k_{6(j-1)}, v_1, v_2, v_5, \dots, v_j, s_{24}\}$. The same subgraph arises if $l > 0$.

Suppose now that j is odd, thus v_1 and v_j are colored with the same color. We can assume without loss of generality that v_1 lies in $S_{36} \cup S_{46}$ and v_j lies in $S_{61} \cup S_{64} \cup S_{65}$. If $v_j \notin S_{61}$, then we find $F_1(j-2)$ induced by $\{k_{61}, k_{62}, k_{65}, \dots, k_{6(j-1)}, v_1, v_2, v_5, \dots, v_j, k_4\}$, where k_4 in K_4 is adjacent to v_1, v_2, v_5 and v_j. The same subgraph arises if $l > 0$. If v_j in S_{61}, then there are vertices k_i in K_i, for $i = 1, 2, 4$ such that k_1 is adjacent to v_j and is nonadjacent to v_1, k_2 is nonadjacent to both and k_4 is adjacent to v_1 and nonajcent to v_j. If $l = 0$, we find $M_{II}(j)$ induced by $\{k_{62}, k_{63}, k_{65}, \dots, k_{6(j-1)}, v_1, v_2, v_5, \dots, v_j, k_1, k_2, k_4, s_{12}, s_{24}\}$. If instead $l > 0$, then we find $F_1(j)$ induced by $\{k_{61}, k_{62}, k_{64}, \dots, k_{6(j-1)}, k_1, k_2, k_4, v_1, v_2, v_3, v_6, \dots, v_j, s_{12}, s_{24}\}$.

Therefore, B is admissible.

Case (2) Suppose now that B is admissible but not LR-orderable, thus B^*_{tag} contains either a Tucker matrix, or M'_4, M''_4, M'_5, M''_5, $M'_2(k)$, $M''_2(k)$, $M'_3(k)$, $M''_3(k)$, $M'''_3(k)$ for some $k \geq 4$.

Toward a contradiction, it suffices to see that B^*_{tag} does not contain any Tucker matrix, for in the case of the matrices listed in Figure 3.17, each labeled column can be replaced by a column that represents a vertex that belongs to the same subclasses considered in the analysis for a Tucker matrix with at least one LR-row, and since some of the rows may be non-LR-rows, then that case can be reduced to a particular case.

Let M be a Tucker matrix contained in B^*_{tag}. Thoughout the proof, when we refer to an LR-row in M, we refer to the row in B, this is, the complement of the row that appears in M.

Case (2.1) Suppose first that B^*_{tag} contains $M_I(j)$, for some $j \geq 3$. Let $v_1, \dots, v_j$ be the vertices corresponding to the rows of $M_I(j)$, and $k_{61}, \dots, k_{6j}$ in K_6 be the vertices corresponding to the columns.

Remark 4.16. If two non-LR-rows in $M_I(j)$ are labeled with the same letter, then they induce D_0. Moreover, any pair of consecutive non-LR-rows labeled with distinct letters induce D_1 or D_2. This follows from the fact that B is admissible. Hence, there are at most two non-LR-rows in $M_I(j)$ and such rows are non-consecutive and labeled with distinct letters. Furthermore, since B is admissible, it is easy to see that there are at most two LR-rows in $M_(j)$, for if not such rows induce D_{11}, D_{12} or D_{13}.

Case (2.1.1) Suppose first that $j = 3$ and that v_1 is the only LR-row in $M_I(j)$.

If rows v_2 and v_3 are unlabeled, then we find a net $\vee K_1$ induced by $\{v_1, v_2, v_3, k_{61}, k_{62}, k_{63}, k_l\}$, where k_l is any vertex in $K_l \neq K_6$. The same holds if either v_2 or v_3 are labeled rows, by accordingly replacing k_l for some l such that k_l is nonadjacent to both v_2 and v_3 (there are no labeled rows complete to each partition $K_i \neq K_6$ of K). By the previous remark, if both v_2 and v_3 are labeled rows, then they are labeled with distinct letters. Thus, we find F_0 induced by $\{v_1, v_2, v_3, k_{61}, k_{62}, k_{63}, k_1, k_5\}$, where k_1 in K_1 is adjacent to v_2 and nonadjacent to v_3 and k_5 in K_5 is adjacent to v_3 and nonadjacent to v_2, or viceversa. Such vertices exist since we assumed $\mathbb{B}_i$ admissible for every $i \in \{1, \dots, 5\}$.

If instead v_1 and v_2 are LR-rows, then we find a tent by considering any vertex k_l in K_l for some $l \in \{1, \dots, 5\}$ such that v_3 is nonadjacent to k_l. The tent is induced by the set $\{v_1, v_2, v_3, k_{61}, k_{63}, k_l\}$. Every other case is analogous by symmetry. Moreover, if v_1, v_2 and v_3 are LR-rows, then there is a vertex k_l in K_l with $l \neq 6$ such that v_1, v_2 and v_3 are adjacent to k_l, hence we find a net $\vee K_1$ induced by $\{v_1, v_2, v_3, k_{61}, k_{62}, k_{63}, k_l\}$.

Case (2.1.2) Suppose now that $j \geq 4$, and let us suppose first that there is exactly one LR-row in $M_I(j)$. Thus, we may assume that v_1 is the only LR-row in $M_I(j)$. Notice first that, if j is odd, then we find $F_2(j)$ in B induced by the vertices represented by every row and column. Hence, we may assume that j is even. By Remark 4.16, there are at most two labeled rows in $M_I(j)$ and such rows are labeled with distinct letters.

If either there are no labeled rows or there is exactly one labeled row, then we find $M_{III}(j)$ induced by $\{v_1, \dots, v_j, k_{61}, \dots, k_{6j}, k_l\}$, where k_l is any vertex in some $K_l \neq K_6$ that is nonadjacent to the only labeled row.

Suppose there are two labeled rows in $M_I(j)$. If there are two labeled rows v_i and v_l, then it suffices to see what happens if v_i belongs to $S_{36} \cup S_{46}$ and v_l belongs to either S_{61}, $S_{64} \cup S_{65}$ or $S_{62} \cup S_{63}$. If v_l belongs to S_{61}, then there is a vertex k_2 in K_2 nonadjacent to both v_i and v_l, and thus we also find $M_{III}(j)$ induced by the same vertex set as before. If instead v_l lies in $S_{64} \cup S_{65}$, then there are vertices k_2 in K_2 and k_4 in K_4 such that k_4 is adjacent to both v_l and v_i. Hence, if $|l - i|$ is even, then we find an $(l - i)$-sun. If instead $|l - i|$ is odd, then we find a $(l - i)$-sun with center, where the center is given by the LR-vertex v_1. Using a similar argument, if v_l lies in $S_{62} \cup S_{63}$, then we find an even sun or an odd sun with center considering the same vertex set as before plus s_{24}.

Suppose now that v_1 and v_2 are LR-rows. If $j \geq 4$ is even and every row v_i with $i > 2$ is unlabeled (or is at most one is a labeled row), then we find $M_{II}(j)$ induced by $\{v_1, \dots, v_j, k_{61}, k_{63}, \dots, k_{6j}, k_l\}$, where k_l is any vertex in some $K_l \neq K_6$ such that each v_i is nonadjacent to k_l for every $i \geq 3$. Moreover, if $j \geq 4$ is odd, then we find $F_1(j)$ induced by $\{v_1, \dots, v_j, k_{61}, k_{63}, \dots, k_{6j}\}$. The same holds if there is exactly one labeled row since we can always choose when necessary a vertex in some K_l with $l \neq 6$ that is nonadjacent to such labeled vertex.

Let us suppose there are exactly two labeled rows v_i and v_l. By Remark 4.16, these rows are non-consecutive and are labeled with distinct letters. As in the previous case, v_i belongs to $S_{36} \cup S_{46}$ and v_l belongs to either S_{61} or $S_{64} \cup S_{65}$. If v_l belongs to S_{61}, then there is a vertex k_2 in K_2 nonadjacent to both v_i and v_l, and thus we find $\{v_1, \dots, v_j, k_{61}, k_{63}, \dots, k_{6j}, k_2\}$. If instead v_l lies in $S_{64} \cup S_{65}$, then we find k_4 in K_4 adjacent to both v_i and v_l and thus we find either an even sun or an odd sun with center as in the previous case. Using a similar argument, if v_l lies in $S_{62} \cup S_{63}$, then we find an even sun if $l-i$ is even or an odd sun with center if $l-i$ is odd.

Finally, suppose v_1 and v_i are LR-rows, where $i > 2$. If $j = 4$, then we find a 4-sun induced by every row and every column, hence, suppose that $j > 5$. In that case, we find a tent contained in the subgraph induced by $\{v_1, v_2, v_3\}$ if $i = 3$ and $\{v_1, v_{j-1}, v_j\}$ if $i = j-1$. Thus, let $3 < i < j-1$. However, in that case we find $M_{II}(i)$ induced by $\{v_1, v_2, \dots, v_i, k_{62}, \dots, k_{6(j-2)}, k_{6j}\}$. Therefore, there is no $M_I(j)$ in B^*_{tag}.

Case (2.2) Suppose that B^*_{tag} contains $M_{II}(j)$. Let $v_1, \dots, v_j$ be the vertices corresponding to the rows, and $k_{61}, \dots, k_{6j}$ in K_6 the vertices representing the columns. If j is odd and there are no labeled rows, then we find $F_1(j)$ by considering $\{v_1, \dots, v_j, k_{61} \dots, k_{6(j-1)}\}$. Moreover, if there are no LR-rows and j is odd, then we find $M_{II}(j)$ as a subgraph. Hence, we assume from now on that there is at least one LR-row.

Remark 4.17. As in the previous case, there are at most two rows labeled with L or R in $M_{II}(k)$, for any three LR-rows induce an enriched submatrix that contains either D_0, D_1 or D_2. Moreover, since B is admisssible, then there are at most three LR-rows.

If v_i and v_l with $1 < i < l < j$ are two rows labeled with either L or R, then they are labeled with distinct letters for if not we find D_0. Moreover, they are not consecutive since in that case we find either D_1 or D_2. Thus, since v_i belongs to $S_{36} \cup S_{46}$ and v_l belongs to either S_{61} or $S_{64} \cup S_{65}$ or $S_{62} \cup S_{63}$, one of the following holds:

— If v_l in S_{61}, then we find a $(l-i+2)$-sun if $l-i$ is even or a $(l-i+2)$-sun with center if $l-i$ is odd (the center is k_{6j}) induced by $\{v_i, \dots, v_l, s_{12}, s_{24}, k_{6(i+1)} \dots, k_{6l}, k_1, k_2, k_4, k_{6j}\}$.

— If v_l in $S_{64} \cup S_{65}$ (resp. $S_{62} \cup S_{63}$), then we find a $(l-i)$-sun if $l-i$ is even or a $(l-i)$-sun with center if $l-i$ is odd (the center is k_{6j}) induced by $\{v_i, \dots, v_l, k_{6(i+1)} \dots, k_{6l}, k_4, k_{6j}\}$ (resp. k_1, k_2).

Furthermore, suppose v_1 and v_i are rows labeled with either L or R, where $1 < i \leq j$. If $i = 2, j$, then they are labeled with distinct letters for if not we find D_0. Moreover, they are colored with distinct colors for if not we find D_1. If instead $2 < i < j$, then they are labeled with the same letter for if not we find D_1 or D_2.

As a consequence of the previous remark we may assume without loss of generality that, if there are rows labeled with either L or R, then these rows are either v_j and v_{j-1}, v_1 and v_j or v_{j-2} and v_j for every other case is analogous. Moreover, if v_j and v_{j-1} (resp. v_1) are labeled rows, then we may assume they are colored with distinct colors.

Case (2.2.1) Suppose there is exactly one LR-row and suppose first that v_1 is the only LR-row. If every non-LR row is unlabeled or v_{j-2} and v_j are labeled rows, since they are labeled with the same letter (for if not we find D_1 or D_5 considering v_1, v_{j-2} and v_j), then we find $M_{III}(j)$ induced by $\{k_l, v_1, \dots, v_j, k_{61}, \dots, k_{6j}\}$, where k_l is any vertex in $K_l \neq K_6$. Moreover, if v_{j-1} is a labeled row, then we find either a $(j-1)$-sun or a $(j-1)$-sun with center, depending on whether j is even or odd, induced by $\{v_1, \dots, v_{j-1}, k_l, k_{61}, \dots, k_{6(j-2)}, k_{6j}\}$, thus we finished this case.

If v_2 is an LR-row, then we find $M_{II}(j-1)$ or $F_1(j-1)$ (depending on whether j is odd or even) induced by every column of B and the rows v_2 to v_j. It does not depend on whether there are or not rows labeled with L or R.

Suppose v_i is an LR-row for some $2 < i < j-1$. Let r_i be the first column in which v_i has a 0 and c_i be column in which v_j has a 0, then we find a tent induced by columns k_{61}, $k_{6(r_i)}$ and $k_{6(c_i)}$ and the rows v_1, v_i and v_j.

If v_{j-1} is an LR-row, then we find $M_{II}(j-1)$ induced by $\{v_1, \ldots, v_{j-1}, k_{61}, \ldots, k_{6(j-2)}, k_{6j}\}$.

If v_j is an LR-row and either every other row is unlabeled or there is exactly one labeled row, then we find $M_{III}(j)$ induced by $\{k_l, v_1, \ldots, v_j, k_{61}, \ldots, k_{6j}\}$, where k_l is any vertex in $K_l \neq K_6$ such that the vertex representing the only labeled row is nonadjacent to k_l. Suppose there are two labeled rows. It follows from Remark 4.17 that such rows are either v_1 and v_2 or v_1 and v_i for some $2 < i < j$. However, if v_i is a labeled row for some $1 < i < j-1$, then we find either an even sun or an odd sun with center analgously as we have in Remark 4.17. If instead v_{j-1} and v_1 are labeled rows, then they are labeled with the same letter and thus we are in the same situation as if there were no labeled rows in B since we can find a vertex that results nonadjacent to both v_1 and v_{j-1}.

Case (2.2.2) Suppose there are two LR-rows. If v_1 and v_2 are LR-rows, then we find $M_{II}(j-1)$ as we have in the case where only v_2 is an LR-row. Suppose v_1 and v_3 are LR-rows. If $j = 4$, then we find $M_{II}(j)$ induced by $\{v_1, \ldots, v_4, k_{61}, k_{62}, k_{64}, k_l\}$ where k_l in $K_l \neq K_6$. Such a vertex exists, since v_2 and v_4 are either unlabeled rows or are rows labeled with the same letter, for if they were labeled with distinct letters we would find D_0 or D_1. Thus, there is a vertex that is nonadjacent to both v_2 and v_4 and is adjacent to v_1 and v_3. If $j > 4$, then we find a tent induced by rows v_3, v_{j-1} and v_j and columns $j-2$, $j-1$ and j. Moreover, if v_i is an LR-row for $1 < 2 < j-1$ and v_{j-1} and v_j are non-LR-rows, then we find a tent induced by the rows v_i, v_{j-1} and v_j and the columns $j-2$, $j-1$ and j.

Thus, it remains to see what happens if v_1 and v_{j-1} and v_1 and v_j are LR-rows. If v_1 and v_{j-1} are LR-rows, then we find $M_{II}(j)$ induced by all the rows of $M_{II}(j)$ and every column except for column $j-1$, which is replaced by some vertex k_l in $K_l \neq K_6$ (since in this case, if there are two labeled rows, then they must be v_i for some $1 < i < j-1$ and v_j, thus they are labeled with the same letter, hence there is a vertex k_l nonadjacent to both). Finally, if v_1 and v_j are LR-rows, then we find a j-sun or a j-sun with center, depending on whether j is even or odd, contained in the subgraph induced by $\{v_1, \ldots, v_j, k_{61}, \ldots, k_{6j}, k_l\}$, where k_l in $K_l \neq K_6$ is nonadjacent to every non-LR row (same argument as before). Therefore, there is no $M_{II}(j)$ in B^*_{tag}.

Case (2.3) Suppose that B contains $M = M_{III}(j)$, let $v_1, \ldots v_j$ be the rows of M and $k_{61}, \ldots, k_{6(j+1)}$ be the columns of M. If there are no LR-rows, then we find $M_{III}(j)$, hence we assume there is at least one LR-row. As in the previous cases, since B is admissible, there are at most two LR-rows in M.

Notice that every pair of rows v_i and v_l with $1 \leq 1 < i, l < j-1$ are not labeled with the same letter, since they induce D_0. Once more, if such rows are labeled with distinct letters, then they are not consecutive for in that case we would find D_1 or D_2. Furthermore, if such v_i and v_l are labeled rows, then we find either an even sun or an odd sun with center. Moreover, if $i = 1, j-1$ and $l = j$, then v_i and v_l are not both labeled rows, for the same arguments holds. Hence, if there are two labeled rows, then such rows must be v_j and v_i for some i such that $2 < i < j-1$.

Case (2.3.1) There is exactly one LR-row. Suppose first that v_1 is an LR-row. In this case, we find $M_{II}(j)$ induced by $\{v_1, \ldots, v_j, k_{62}, \ldots, k_{6(j+1)}\}$. If v_i is an LR-row, for some $1 \leq i < j-1$, then we find $M_{II}(j-i+1)$ induced by $\{v_i, \ldots, v_j, k_{6(i+1)}, \ldots, k_{6(j+1)}\}$.

If v_{j-1} is an LR-row, then we also find $M_{II}(j)$, induced by $\{v_1, \ldots, v_j, k_{62}, \ldots, k_{6(j-1)}, k_{6(j+1)}\}$.

If instead v_j is an LR-row, then we find an even j-sun or an odd j-sun with center $k_{6(j+1)}$.
Case (2.3.2) Suppose now there are two LR-rows v_i and v_l. If $1 \leq i < l < j-1$ and v_i and v_l are not consecutive rows, then we find a tent induced by the rows v_i, v_l and v_j, and columns k_s in $K_s \neq K_6$ adjacent to both v_i and v_l and nonadjacent to v_j, and k_{6i} (or $k_{6(i+1)}$ if $i = 1$) and k_{6l} (or $k_{6(l+1)}$ if $l = j-1$). The same subgraph contains an induced tent if $l = i+1$ and $i > 1$. If instead $i = 1$ or $i = j-1$ and $l = i+1$, then we find F_0 (or $M_{III}(3)$ if $j = 3$) induced by $\{v_i, v_{i+1}, k_{6i}, k_{6(i+1)}, k_{6(i+2)}, k_6(j+1), k_s\}$ with k_s in $K_s \neq K_6$ adjacent to both v_i and v_{i+1}.

Finally, if v_1 and v_j are LR-rows, then we find $M_{III}(j)$ induced by every row $v_1, \ldots, v_j$ and column $k_{61}, \ldots, k_6(j+1)$. If instead v_i and v_j are LR-rows with $i > 1$, then we find M_V induced by $\{v_i, v_j, v_1, v_{j-1}, k_{61}, k_{62}, k_{6i}, k_{6(i+1)}, k_{6j}\}$, therefore there is no $M_{III}(j)$ in B^*_{tag}.
Case (2.4) Suppose that B contains $M = M_{IV}$, let $v_1, \ldots, v_4$ be the rows of M and $k_{61}, \ldots, k_{66}$ be the columns of M. If there are no labeled rows, then we find M_{IV} as a subgraph, and since B is admissible and any three rows are not pairwise nested, then there are at most two LR-rows, hence we assume there are exactly either one or two LR-rows.

If the row v_i is an LR-row for $i = 1, 2, 3$, then we find M_V induced by $\{v_2, v_3, v_4, k_{62}, \ldots, k_{66}\}$. Moreover, if only v_4 is an LR-row, then we find M_{IV} induced by all the rows and columns of M. Thus, we assume there are exactly two LR-rows. If v_1 and v_4 are LR-rows, then we find M_V induced by $\{v_1, v_2, v_3, v_4, k_{61}, k_{63}, \ldots, k_{66}\}$. The same holds if v_i and v_4 are LR-rows, with $i = 2, 3$. Finally, if v_1 and v_2 are LR-rows, then we find a tent induced by $\{v_1, v_2, v_4, k_{62}, k_{64}, k_{65}\}$. It follows analogously by symmetry if v_1 and v_3 or v_2 and v_3 are LR-rows, therefore there is no M_{IV} in B^*_{tag}.
Case (2.5) Suppose that B contains $M = M_V$, let $v_1, \ldots, v_4$ be the rows of M and $k_{61}, \ldots, k_{65}$ be the columns of M. Once more, if there are no LR-rows, then we find M_V as a subgraph, thus we assume there is at least one LR-row. Moreover, since any three rows are not pairwise nested, there are at most two LR-rows.
Case (2.5.1) If v_1 is the only LR-row, then we find a tent induced by $\{v_1, v_3, v_4, k_{61}, k_{63}, k_{65}\}$. The same holds if v_2 is the only LR-row.

If v_3 is the only LR-row and every other row is unlabeled or are all labeled with the same letter, then we find M_{IV} induced by$\{v_1, v_2, v_3, v_4, k_{61}, \ldots, k_{65}, k_l\}$ where k_l in $K_l \neq K_6$ adjacent only to v_3. Suppose there are at least two rows labeled with either L or R. Notice that, if v_1 and v_2 are labeled, then they are labeled with distinct letters for if not they contain D_0. Moreover, v_1 (resp. v_2) and v_4 cannot be both labeled, for in that case they contain either D_0 or D_1 or D_2. Hence, there are at most two rows labeled with either L or R, and they are necessarily v_1 and v_2. In that case, there is a vertex k_l in some $K_l \neq K_6$ such that v_2 and v_3 are adjacent to k_l and v_4 is nonadjacent to k_l, thus we find a tent induced by v_2, v_3, v_4, k_l, k_{64} and k_{65}.

If v_4 is the only LR-row and every other row is unlabeled or are (one, two or) all labeled with the same letter, then we find M_V induced by $\{v_1, v_2, v_3, v_4, k_{61}, \ldots, k_{64}, k_l\}$ where k_l in $K_l \neq K_6$ adjacent only to v_4.
Case (2.5.2) Suppose there are exactly two LR-rows. If v_1 and v_2 are such LR-rows, then we find a tent induced by $\{v_1, v_2, v_3, k_{62}, k_{63}, k_{65}\}$, thus we discard this case. If instead v_1 and v_3 are LR-rows and every other row is unlabeled or (one or) all are labeled with the same letter, then we find M_V induced by every row and column plus a vertex k_l in some $K_l \neq K_6$ such that both v_2 and v_4 are nonadjacent to k_l. Moreover, since v_2 and v_4 are neither disjoint or nested and there is a column in which both rows have a 0, then they are not labeled with distinct letters, disregarding of the coloring, for in that case we find D_1 or D_2.

If exactly v_1 and v_4 are LR-rows and every other row is unlabeled or are (one or) all labeled with the same letter, then we find a tent induced by every row and column plus a vertex k_l in some $K_l \neq K_6$ such that both v_2 and v_4 are nonadjacent to k_l. Once more, v_2 and v_3 are not labeled with distinct letters since in that case we find either D_1 or D_2.

If exactly v_3 and v_4 are LR-rows and every other row is unlabeled or either v_1 or v_2 is labeled with L or R, then we find M_{IV} induced by every row and column plus a vertex k_l in some $K_l \neq K_6$ such that both v_1 and v_2 are nonadjacent to k_l. Once more, v_1 and v_2 are not labeled with the same letter for they would induce D_0, neither they are labeled with distinct letters since in that case we find either D_1 or D_2.

If v_1, v_2 and v_3 are LR-rows, since there is a vertex $k_l \in K_l$ with $l \neq 6$ such that v_4 is nonadjacent to k_l, then we find a tent induced by $\{v_1, v_2, v_4, k_{61}, k_{64}, k_l\}$. Analogously, if v_1, v_2 and v_4 are LR-rows and v_3 is not, then the tent is induced by $\{v_1, v_2, v_3, k_{61}, k_{64}, k_{65}\}$. The same holds if all 4 rows are LR-rows, where the tent is induced by $\{v_1, v_2, v_4, k_{62}, k_{63}, k_{65}\}$. Finally, if v_2, v_3 and v_4 are LR-rows, since there is a vertex $k_l \in K_l$ with $l \neq 6$ such that v_1 is nonadjacent to k_l, then we find M_V induced by $\{v_1, v_2, v_3, v_4, k_{61}, k_{62}, k_{63}, k_{65}, k_l\}$.

Case (3) Therefore, we may assume that B is admissible and LR-orderable but is not partially 2-nested. Since there are no uncolored labeled rows and those colored rows are labeled with either L or R and do not induce any of the matrices $\mathcal{D}$, then in particular no pair of pre-colored rows of B induce a monochromatic gem or a monochromatic weak gem, and there are no badly-colored gems since every LR-row is uncolored, therefore B is partially 2-nested.

Case (4) Finally, let us suppose that B is partially 2-nested but is not 2-nested. As in the previous cases, we consider B ordered with a suitable LR-ordering. Let B' be a matrix obtained from B by extending its partial pre-coloring to a total 2-coloring. It follows from Lemma 3.39 that, if B' is not 2-nested, then either there is an LR-row for which its L-block and R-block are colored with the same color, or B' contains a monochromatic gem or a monochromatic weak gem or a badly-colored doubly weak gem.

If B' contains a monochromatic gem where the rows that induce such a gem are not LR-rows, then the proof is analogous as in the tent case. Thus, we may assume that at least one of the rows is an LR-row.

Case (4.1) *Let us suppose first that there is an LR-row w for which its L-block w_L and R-block w_R are colored with the same color.* If these two blocks are colored with the same color, then there is either one odd sequence of rows $v_1, \dots, v_j$ that force the same color on each block, or two distinct sequences, one that forces the same color on each block.

Case (4.1.1) Let us suppose first that there is one odd sequence $v_1, \dots, v_j$ that forces the color on both blocks. If $k = 1$, then notice this is not possible since we are coloring B' using a suitable LR-ordering. If there is not a suitable LR-ordering, then B is not admissible or LR-orderable, which results in a contradiction. Thus, let $j > 1$ and assume without loss of generality that v_1 intersects w_L and v_j intersects w_R. Moreover, we assume that each of the rows in the sequence $v_1, \dots, v_j$ is colored with a distinct color and forces the coloring on the previous and the next row in the sequence. If $v_1, \dots, v_j$ are all unlabeled rows, then we find an even $(j+1)$-sun. If instead v_1 is an L-row, then w_L is properly contained in v_1. Thus, $v_2, \dots, v_{j-1}$ are not contained in v_1, since at least v_j intersects w_R. If v_j is unlabeled or labeled with R, then we find an even $(j+1)$-sun. If instead v_j is labeled with L, since j is odd, then we find $S_1(j+1)$ in B which is not possible since we are assuming B admissible.

Case (4.1.2) Suppose now that there are two independent sequences $v_1, \dots, v_j$ and $x_1, \dots, x_l$ that force the same color on w_L and w_R, respectively. Suppose without loss of generality that w_L and w_R are colored with red. If $j = 1$ and $l = 1$, then we find D_6, which is not possible. Hence, we assume that either $j > 1$ or $l > 1$. Suppose that $j > 1$ and $l > 1$. In this case, there is a labeled row in each sequence, for if not we can change the coloring for each row in one of the sequences and thus each block of w can be colored with distinct colors. We may assume that v_j

is labeled with L and x_l is labeled with R (for the LR-ordering used to color B' is suitable and thus there is no R-row intersecting w_L, and the same holds for each L-block and w_R). As in the previous paragraphs, we assume that each row in each sequence forces the coloring on both the previous and the next row in its sequence. In that case, $v_2, \dots, v_j$ is contained in w_L and $x_2, \dots, x_l$ is contained in w_R. Moreover, w represents a vertex in $S_{[16}$, v_j lies in $S_{46} \cup S_{36}$ or $S_{16} \cup S_{26} \cup S_{56}$ and x_l lies in $S_{61} \cup S_{64} \cup S_{65}$ or $S_{62} \cup S_{63}$ (depending on whether they are colored with red or blue, respectively). Suppose first that they are both colored with red, thus v_j lies in $S_{46} \cup S_{36}$ and x_l lies in $S_{61} \cup S_{64} \cup S_{65}$. In this case j and l are both even. If x_l lies in $S_{64} \cup S_{65}$, since there is a k_i in some $K_i \neq K_6$ adjacent to both v_j and x_l, then we find $F_2(j+l+1)$ contained in the submatrix induced by each row and column on which the rows in w and both sequences are not null and the column representing k_i. If instead x_l lies in S_{61}, we find $F_2(k+l+3)$ contained in the same submatrix but adding three columns representing vertices k_i in K_i for $i = 1, 2, 4$. The same holds if v_j and x_l are both blue. Suppose now that v_j is colored with red and v_l is colored with blue. Thus, j is even and l is odd. In this case, we find $F_2(j+l+2)$ contained in the submatrix induced by the row that represents s_{24}, two columns representing any two vertices in K_2 and K_4 and each row and column on which the rows in w and both sequences are not null. The proof is analogous if either $j = 1$ or $l = 1$.

Hence, we may assume there is either a monochromatic weak gem in which one of the rows is an LR-row or a badly-colored doubly-weak gem in B', for the case of a monochromatic gem or a monochromatic weak gem where one of the rows is an L-row (resp. R-row) and the other is unlabeled is analogous to the tent case.

Case (4.2) *Let us suppose there is a monochromatic weak gem in* B', and let v_1 and v_2 be the rows that induce such gem, where v_2 is an LR-row. Suppose first that v_1 is a pre-colored row. Suppose without loss of generality that the monochromatic weak gem is induced by v_1 and the L-block of v_2 and that v_1 and v_2 are both colored with red. We denote v_{2L} to the L-block of v_2. If v_1 is labeled with R, then v_2 is the L-block of some LR-row r in B and v_1 is the R-block of itself. However, since the LR-ordering we are considering to color B' is suitable, then the L-block of an LR-row has empty intersection with the R-block of a non-LR row and thus this case is not possible.

If v_1 is labeled with L, since they induce a weak gem, then v_{2L} is properly contained in v_1. Since v_1 is a row labeled with L in B, then v_1 is a pre-colored row. Moreover, since v_{2L} is colored with the same color as v_1, then there is either a blue pre-colored row, or a sequence of rows $v_3, \dots, v_j$ where v_j forces the red coloring of v_{2L}. In either case, there is a pre-colored row in that sequence that forces the color on v_{2L}, and such row is either labeled with L or with R.

Suppose first that such row is labeled with L. If v_3 is a the blue pre-colored row that forces the red coloring on v_{2L}, then v_{2L} is properly contained in v_3. However, in that case we find D_4 which is not possible since B is admissible. Hence, we assume $v_3, \dots, v_{j-1}$ is a sequence of unlabeled rows and that v_j is a labeled row such that this sequence forces v_{2L} to be colored with red, and each row in the sequence forces the color on both its predecesor and its succesor. If $j-3$ is even, then v_j is colored with blue, and if $j-3$ is odd then v_j is colored with red. In either case, we find $S_5(j)$ contained in the submatrix induced by rows $v_1, v_2, v_3, \dots, v_j$.

If instead the row v that forces the coloring on v_{2L} is labeled with R, since the LR-ordering used to color B is suitable, then the intersection between v_{2L} and v is empty. Hence, $v \neq v_3$, thus we assume that $v_3, \dots, v_{j-1}$ are unlabeled rows and $v_j = v$. If $j-3$ is odd, then v_j is colored with red, and if $j-3$ is even, then v_j is colored with blue. In either case we also find $S_5(j)$, which is not possible since B is admissible.

Suppose now that v_1 is an unlabeled row. Notice that, since v_1 and v_2 induce a weak gem, then v_1 is not nested in v_2.

Hence, either the coloring of both rows is forced by the same sequence of rows or the coloring of v_1 and v_2 is forced for each by a distinct sequence of rows. As in the previous cases, we assume that the last row of each sequence represents a pre-colored labeled row.

Suppose first that both rows are forced to be colored with red by the same row v_3. Thus, v_3 is a labeled row pre-colored with blue. Moreover, since v_3 forces v_1 to be colored with red, then v_1 is not contained in v_3 and thus there is a column k_{61} in which v_1 has a 1 and v_3 has a 0.

We may also assume that v_2 has a 0 in such a column since v_1 is also not contained in v_2. Moreover, since v_3 forces v_2 to be colored with red, then v_3 is labeled with the same letter than v_2 and v_3 is not contained in v_2, thus we can find a column k_{62} in which v_2 has a 0 and v_1 and v_3 both have a 1. Furthermore, since v_3 and v_2 are both labeled with the same letter and the three rows have pairwise nonempty intersection, then there is a column k_{63} in which all three rows have a 1. Since v_3 is a row labeled with either L or R in B, then there are vertices $k_l \in K_l$, $k_m \in K_m$ with $l \neq m$, $l, m \neq 6$ such that v_3 is adjacent to k_l and nonadjacent to k_m. Moreover, since v_2 is an LR-row, then v_2 is adjacent to both k_l and k_m and v_j is nonadjacent to k_l and k_m. Hence, we find F_0 induced by $\{v_3, v_1, v_2, k_l, k_{61}, k_{63}, k_{62}, k_m\}$.

Suppose instead there is a sequence of rows $v_3, \dots, v_j$ that force the coloring of both v_1 and v_2, where $v_3, \dots, v_{j-1}$ are unlabeled rows and v_j is labeled with either L or R and is pre-colored.

We have two possibilities: either v_j is labeled with L or with R.

If v_j is labeled with L and v_j forces the coloring of v_2, then we have the same situation as in the previous case. Thus we assume v_j is nested in v_2. In this case, since v_j and v_2 are labeled with L, the vertices $v_3, \dots, v_{j-1}$ are nested in v_2 and thus they are chained from right to left. Moreover, since v_1 and v_2 are colored with the same color, then there is an odd index $1 \leq l \leq j-1$ such that v_1 contains $v_3, \dots, v_l$ and does not contain $v_{l+1}, \dots, v_j$. Hence, we find $F_1(l+1)$ considering the rows $v_1, v_2, \dots, v_{l+1}$.

Suppose now that v_j is labeled with R. Since B' is colored using a suitable LR-ordering, then v_j and v_2 have empty intersection, thus there is a sequence of unlabeled rows $v_3, \dots, v_{j-1}$, chained from left to right. Notice that it is possible that $v_1 = v_3$. Suppose first that j is even. If $v_1 = v_3$, then there is an odd number of unlabeled rows between v_1 and v_j. In this case we find a $(j-2)$-sun contained in the subgraph induced by rows $v_2, v_1 = v_3, v_4, \dots, v_j$. If instead $v_1 \neq v_3$, then v_1 and v_3 and v_1 and v_5 both induce a 0-gem, and thus we find a $(j-2)$-sun in the same subgraph. If j is odd, then there is an even number of unlabeled rows between v_2 and v_j. Once more, we find a $(j-1)$-sun contained in the subgraph induced by rows $v_2, v_3, \dots, v_j$.

Notice that these are all the possible cases for a weak gem. This follows from the fact that, if there is a pre-colored labeled row that forces the coloring upon v_1 then it forces the coloring upon v_2 and viceversa. Moreover, if there is a sequence of rows that force the coloring upon v_2, then one of these rows of the sequence also forces the coloring upon v_1, and viceversa. Furthermore, since the label of the pre-colored row of the sequence determines a unique direction in which the rows overlap in chain, then there is only one possibility in each case, as we have seen in the previous paragraphs. It follows that the case in which there is a sequence forcing the coloring upon each v_1 and v_2 can be reduced to the previous case.

Case (4.3) *Suppose there is a badly-colored doubly-weak gem in* B'. Let v_1 and v_2 be the LR-rows that induce the doubly-weak gem. Since the suitable LR-ordering determines the blocks of each LR-row, then the L-block of v_1 properly contains the L-block of v_2 and the R-block of v_1 is properly contained in the R-block of v_2, or viceversa. Moreover, the R-block of v_1 may be empty. Let us denote v_{1L} and v_{2L} (resp. v_{1R} and v_{2R}) to the L-blocks (resp. R-blocks) of v_1 and v_2.

There is a sequence of rows that forces the coloring on both LR-rows simultaneously or there are two sequences of rows and each forces the coloring upon the blocks of v_1 and v_2, respectively.

Whenever we consider a sequence of rows that forces the coloring upon the blocks of v_1 and v_2, we will consider a sequence in which every row forces the coloring upon its predecessor and its succesor, a pre-colored row is either the first or the last row of the sequence, the first row of the sequence forces the coloring upon the corresponding block of v_1 and the last row forces the coloring upon the corresponding block of v_2. It follows that, in such a sequence, every pair of consecutive unlabeled rows overlap. We can also assume that there are no blocks corresponding to LR-rows in such a sequence, for we can reduce this to one of the cases.

Suppose first there is a sequence of rows $v_3, \dots, v_j$ that forces the coloring upon both LR-rows simultaneously. We assume that v_3 intersects v_1 and v_j intersects v_2.

If $v_3, \dots, v_j$ forces the coloring on both L-blocks, then we have four cases: (1) either $v_3, \dots, v_j$ are all unlabeled rows, (2) v_3 is the only pre-colored row, (3) v_j is the only pre-colored row or (4) v_3 and v_j are the only pre-colored rows. In either case, if $v_3, \dots, v_j$ is a minimal sequence that forces the same color upon both v_{1L} and v_{2L}, then j is odd.

Case (4.3.1) Suppose $v_3, \dots, v_j$ are unlabeled. If $j = 3$, then we find $S_7(3)$ contained in the submatrix induced by v_1, v_2 and v_3. Suppose $j > 3$, thus we have two possibilities. If $v_2 \cap v_3 \neq \emptyset$, since j is odd, then we find a $(j-1)$-sun contained in the submatrix induced by considering all the rows $v_1, v_2, v_3, \dots, v_j$. If instead $v_2 \cap v_3 = \emptyset$, then we find $F_2(j)$ contained in the same submatrix.

Case (4.3.2) Suppose v_3 is the only pre-colored row. Since v_3 is a pre-colored row and forces the color red upon the L-block of v_1, then v_3 contains v_{1L} and v_3 is colored with blue. If $v_4 \cap v_{1L} \neq \emptyset$, then we find F_0 in the submatrix given by considering the rows v_1, v_3, v_4, having in mind that there is a column representing some k_i in $K_i \neq K_6$ in which the row corresponding to v_1 has a 1 and the rows corresponding to v_3 and v_4 both have 0. This follows since v_4 is unlabeled and thus represents a vertex that lies in S_{66}, and v_3 is pre-colored and labeled with L or R and, thus it represents a vertex that is not adjacent to every partition K_i of K. If instead $v_4 \cap v_{1L} = \emptyset$, then we find $F_2(j-2)$ contained in the submatrix induced by the rows $v_1, v_2, \dots, v_{j-2}$ if $v_2 \cap v_{2R} = \emptyset$, and induced by the rows $v_1, v_2, v_5, \dots, v_j$ if $v_2 \cap v_{2R} \neq \emptyset$.

Case (4.3.3) Suppose v_j is the only pre-colored row. In this case, v_j properly contains v_{2L} and we can assume that the rows $v_4, \dots, v_{j-1}$ are contained in v_{1L}. If $v_3 \cap v_2 \neq \emptyset$, then we find an even $(j-1)$-sun in the submatrix induced by the rows $v_2, v_3, \dots, v_j$. If instead $v_3 \cap v_2 = \emptyset$, then we find $F_2(j)$ in the submatrix given by rows $v_1, \dots, v_j$.

Case (4.3.4) Suppose that v_3 and v_j are the only pre-colored rows. Thus, we can assume that v_j properly contains v_{2L} and v_3 properly contains v_{2L}, thus v_3 properly contains v_{2L}. Hence, we find D_9 induced by the rows v_1, v_2 and v_3 which is not possible since B is admissible.

The only case we have left is when $v_3, \dots, v_j$ forces the coloring upon v_{1L} and v_{2R}. This follows from the fact that, if $v_3, \dots, v_j$ forces the color upon v_{2L} and $v_{1R} \neq \emptyset$, then this case can be reduced to case (4.3.3).

Hence, either (1) $v_3, \dots, v_j$ are unlabeled rows, (2) v_3 is the only pre-colored row, or (3) v_3 and v_j are the only pre-colored rows. Notice that in either case, j is even and thus for (1) we find $S_8(j)$, which results in a contradiction since B is admissible. Moreover, in the remaining cases, v_3 properly contains v_{1L} and v_{2L}. Since v_1 and v_2 overlap, we find D_9 which is not possible for B is admissible.

This finishes the proof.

□

Let $G = (K, S)$, T as in Section2.2 and the matrices $\mathbb{B}_i$ for $i = \{1 \dots, 6\}$ as defined in the previous subsection. Suppose $\mathbb{B}_i$ is 2-nested for each $i \in \{1, 2, \dots, 6\}$. Let χ_i be a proper 2-coloring for $\mathbb{B}_i$ for each $i \in \{1, \dots, 5\}$ and χ_6 be a proper 2-coloring for $\mathbb{B}_6$. Moreover, there is a suitable LR-ordering Π_i for each $i \in \{1, 2, \dots, 6\}$.

Let Π be the ordering of the vertices of K given by concatenating the orderings $\Pi_1, \Pi_2, \dots, \Pi_6$, as defined in Subsection 4.1.2. Let $s \in S$. Hence, s lies in S_{ij} for some $i, j \in \{1, 2, \dots, 6\}$. Notice that there are at most two rows r_1 in $\mathbb{B}_i$ and r_2 in $\mathbb{B}_j$ both representing s. Also notice that the row r_l represents the adjacencies of s with regard to K_l for each $l = i, j$, and if $i > j$, then r_i and r_j are colored with distinct colors.

Definition 4.18. *We define the $(0,1)$-matrices $\mathbb{B}_r$, $\mathbb{B}_b$, $\mathbb{B}_{r-b}$ and $\mathbb{B}_{b-r}$ as in the previous subsection, considering only those independent vertices that are not in $S_{[16}$.*

Notice that the only nonempty subsets S_{ij} with $i > j$ that we are considering are those with $i = 6$. Hence, the rows of $\mathbb{B}_{r-b}$ are those representing vertices in $S_{61} \cup S_{64} \cup S_{65}$ and the rows of $\mathbb{B}_{b-r}$ are those representing vertices in $S_{62} \cup S_{63}$.

Lemma 4.19. *Suppose that $\mathbb{B}_i$ is 2-nested for each $i = 1, 2 \dots, 6$. If $\mathbb{B}_r$, $\mathbb{B}_b$, $\mathbb{B}_{r-b}$ or $\mathbb{B}_{b-r}$ are not nested, then G contains F_0, $F_1(5)$ or $F_2(5)$ as forbidden induced subgraphs for the class of circle graphs.*

Proof. Notice that the only partial rows considered in $\mathbb{B}_r$ and $\mathbb{B}_b$ may be those in $S_{62} \cup S_{63}$ and $S_{61} \cup S_{64} \cup S_{65}$, respectively. Thus, if the partial row coincides with the row in $\mathbb{B}_6$ or $\mathbb{B}_1$, then we can consider the matrices $\mathbb{B}_r$ and $\mathbb{B}_b$ without these rows since the compatiblity with the rest of the rows was already considered when analysing if $\mathbb{B}_6$ and $\mathbb{B}_1$ are 2-nested or not.

Suppose first that $\mathbb{B}_r$ is not nested. Thus, there is a 0-gem. Let f_1 and f_2 be two rows that induce a gem in $\mathbb{B}_r$ and v_1 in S_{ij} with $i < j$ and v_2 in S_{lm} with $l < m$ be the corresponding to vertices in G. Suppose without loss of generality that f_1 starts before f_2, thus $i \geq l$. Since $\mathbb{B}_i$ is 2-nested for every $i \in \{1, 2, \dots, 5, 6\}$, in particular there are no monochromatic gems in each $\mathbb{B}_i$. Moreover, if $j = l$, then we find D_1 in K_i or K_j, respectively.

Notice that every row in $\mathbb{B}_r$ represents a vertex that belongs to one of the following subsets of S: S_{12}, S_{13}, S_{35}, S_{36}, S_{45}, S_{62} or S_{63}. Analogously, every row in $\mathbb{B}_b$ represents a vertex belonging to either S_{23}, S_{24}, S_{34}, S_{14}, S_{25}, S_{15}, S_{16}, S_{61}, S_{64} or S_{65}.
Case (1) Suppose first that $i = l$. We have two cases:
Case (1.1) v_1, v_2 in $S_{12} \cup S_{13}$. Suppose without loss of generality that both vertices lie in S_{12} since the proof is analogous otherwise. Let k_{ii} in K_i such that v_i is adjacent to k_{ii} and v_{i+1} is nonadjacent to k_{ii} for $i = 1, 2 \pmod 2$. Notice that v_1 and v_2 are labeled with R in $\mathbb{B}_1$ and are labeled with L in $\mathbb{B}_2$. Moreover, since $\mathbb{B}_1$ and $\mathbb{B}_2$ are admissible, then there are vertices k_{12} in K_1 and k_{21} in K_2 adjacent to both v_1 and v_2, for if not we find D_0 in each matrix. Moreover, there is a vertex k_4 in K_4 nonadjacent to both. We find F_0 induced by $\{v_1, v_2, s_{24}, k_{11}, k_{12}, k_{21}, k_{22}, k_4\}$.

The proof is analogous if v_1 and v_2 in $S_{45} \cup S_{46}$, where F_0 is induced by $\{v_1, v_2, s_{24}, k_2, k_{41}, k_{42}, k_5, k_6\}$ or $\{v_1, v_2, s_{24}, k_2, k_{41}, k_{42}, k_{51}, k_{52}\}$, depending on whether only one lies in S_{46} or both lie in S_{46}. If v_1 in $S_{45} \cup S_{46}$ and v_2 in $S_{62} \cup S_{63}$ is the vertex represented by a partial row in $\mathbb{B}_r$, then it is not possible that these rows induce a gem since they do not intersect. Thus, we assume that v_1 in $S_{12} \cup S_{13}$. We find F_0 induced by $\{v_1, v_2, s_{24}, k_{11}, k_{12}, k_{21}, k_{22}, k_4\}$ if v_1 in S_{12} (thus necessarily v_2 in S_{62} since they induce a 0-gem). If instead v_1 in S_{13}, since v_1 is complete to K_1, then one of the columns of the 0-gem is induced by the column c_L. Thus, there is a vertex k_6 in K_6 adjacent to v_2 and nonadjacent to v_1. Hence, we find F_0 induced by $\{v_1, v_2, s_{24}, k_6, k_1, k_2, k_3, k_4\}$.
Case (1.2) v_1, v_2 in $S_{35} \cup S_{36}$. Suppose that v_1 in S_{35} and v_2 in S_{36}. Let k_2 in K_2 nonadjacent to both. There are vertices k_{31}, k_{32} in K_3 such that k_{31} is adjacent only to v_1 and k_{32} is adjacent to both. Moreover, there are vertices k_5 in K_5 and k_6 in K_6 such that k_5 is adjacent to both and k_6 is

adjacent only to v_2. We find F_0 induced by $\{v_1, v_2, s_{24}, k_{31}, k_{32}, k_5, k_6, k_2\}$. The proof is analogous if both lie in S_{35} changing k_6 for other vertex in K_5 adjacent only to v_2 (exists since both rows induce a gem), and if both lie in S_{36} we can find two vertices k_{61} and k_{62} in K_6 to replace k_5 and k_6 in the previous subset. Notice that, if instead v_1 in $S_{35} \cup S_{36}$ and v_2 in $S_{45} \cup S_{46}$ we also find F_0 changing k_{32} for some vertex k_4 in K_4 in the same subset. This is the only case we had to see in which $j = m$. Furthermore, the partial rows corresponding to $S_{62} \cup S_{63}$ cannot induce a gem with a row corresponding to a vertex in $S_{35} \cup S_{36}$ since we aer assuming that $\mathbb{B}_3$ is admissible.
Case (2) Suppose now that $i < l$. Since $j \neq l$ and both rows induce a gem, then $i < l < j < m$. Thus, the only possibility is v_1 in S_{35} and v_2 in S_{46}. In this case we find F_0 induced by $\{v_1, v_2, s_{24}, k_2, k_3, k_4, k_5, k_6\}$.

Hence $\mathbb{B}_r$ is nested. Suppose now that $\mathbb{B}_b$ is not nested, and let v_1 in S_{ij} with $i < j$ and v_2 in S_{lm} with $l < m$ two vertices for which its rows in $\mathbb{B}_b$ induce a 0-gem. Once more, we assume that $i \leq l$.
Case (1) Suppose that the gem is induced by two rows corresponding to two vertices v_1 and v_2 such that v_2 is a partial row, thus v_2 in $S_{64} \cup S_{65}$. Notice that the 0-gem may be induced by the column c_L.
Case (1.1) v_2 in S_{64}.
Case (1.1.1) v_1 in $S_{24} \cup S_{34} \cup S_{14}$. We find F_0 induced by $\{v_1, v_2, s_{45}, k_1, k_2, k_{41}, k_{42}, k_5\}$. Notice that, since S_{64} is complete to K_4, the 0-gem cannot be induced by v_2 and a vertex v_1 in S_{14} complete to K_1, since we are considering that every vertex in S_{14} is also complete to K_4 (for if not we have previously shown a forbidden subgraph).
Case (1.1.2) v_1 in $S_{15} \cup S_{25} \cup S_{16}$. In this case we find $F_1(5)$ induced by $\{v_1, v_2, s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5\}$ if v_1 in S_{15} is not complete to K_1. If instead v_1 in S_{15} is complete to K_1, then it is not complete to K_5 (for we split those vertices that are adjacent to $K_1, \dots, K_5$ into two disjoint subsets, $S_{[15]}$ and S_{15}). Moreover, one of the columns that induce the 0-gem is the column c_L. Thus, there are vertices k_6 in K_6, k_{51} and k_{52} in K_5 such that v_2 is adjacent to k_6 and is nonadjacent to k_{51} and k_{52} and v_1 is adjacent to k_{51} and is nonadjacent to k_6 and k_{52}. Hence, we find F_0 induced by $\{v_1, v_2, s_{45}, k_6, k_2, k_4, k_{51}, k_{52}\}$.
Case (1.2) v_2 in S_{65}. In this case, v_1 in $S_{25} \cup S_{15} \cup S_{16}$. Since these rows induce a gem and v_2 has a 1 in every column corresponding to $K_1, \dots, K_4$, there are vertices k_1 in K_1 and k_5 in K_5 such that v_1 is adjacent to k_1 and v_2 is nonadjacent to k_1, and v_1 is nonadjacent to k_5 and v_2 is adjacent to k_5. Thus, we find $F_1(5)$ induced by $\{v_1, v_2, s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5\}$.
Case (2) Suppose now that $i = l$.
Case (2.1) v_1, v_2 in $S_{23} \cup S_{24} \cup S_{25}$. Suppose first that both lie in S_{24}. In that case we find F_0 induced by $\{v_1, v_2, s_{12}, k_{21}, k_{22}, k_{41}, k_{42}, k_1\}$. If instead one of both lie in S_{23}, then we change k_{41} for some analogous k_3 in K_3, and if one of both lie in S_{25} we change k_{42} for some analogous k_5 in K_5.
Case (2.2) v_1, v_2 in S_{34}. In this case, we find F_0 induced by $\{v_1, v_2, s_{45}, k_{31}, k_{32}, k_{41}, k_{42}, k_5\}$.
Case (2.3) v_1, v_2 in $S_{14} \cup S_{15} \cup S_{16}$. Remember that S_{15} are those independent vertices that are not complete to K_5 and S_{16} are those independent vertices that are not complete to K_1.
Case (2.3.1) If both lie in S_{14}, then we find F_0 induced by $\{v_1, v_2, s_{24}, k_{11}, k_{12}, k_{41}, k_{42}, k_5\}$.
Case (2.3.2) If v_1 in S_{14} and v_2 in S_{15}, then we find $F_1(5)$ induced by $\{v_1, v_2, s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5\}$. The same holds if instead v_2 in S_{16} or if both lie in S_{15}. Moreover, we find the same subgraph induced by the same subset if v_1 in S_{15} and v_2 in S_{16}, since there is a vertex in K_5 that is nonadjacent to v_1.
Case (2.3.3) If both lie in S_{16}, then we find F_0 induced $\{v_1, v_2, s_{12}, k_{11}, k_{12}, k_2, k_4, k_6\}$.

Case (3) Suppose now that $j = m$. The case where v_1, v_2 in $S_{14} \cup S_{24} \cup S_{34}$ is analogous as Case 1. Let v_1 in S_{15} and v_2 in S_{25}. We find $F_1(5)$ induced by $\{v_1, v_2, s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5\}$.
Case (4) Suppose that $i < l$, thus $i < l < j < m$. In this case, v_1 in S_{14} and v_2 in S_{25}. We find $F_1(5)$ induced by $\{v_1, v_2, s_{12}, s_{24}, s_{45}, k_1, k_2, k_4, k_5\}$.

Hence $\mathbb{B}_b$ is nested. Suppose that $\mathbb{B}_{b-r}$ is not nested, thus let v_1 and v_2 in $S_{62} \cup S_{63}$ two vertices whose rows induce a 0-gem. If both lie in S_{62}, then we find F_0 induced by $\{v_1, v_2, s_{24}, k_{61}, k_{62}, k_{21}, k_{22}, k_4\}$. If instead one or both lie in S_{63}, we find the same subgraph changing k_{22} for some analogous k_3 in K_3.

Finally, suppose that $\mathbb{B}_{r-b}$ is not nested, and let v_1 and v_2 in $S_{61} \cup S_{64} \cup S_{65}$ be two vertices whose rows induce a 0-gem. If both lie in S_{61}, then we find F_0 induced by $\{v_1, v_2, s_{12}, k_{61}, k_{62}, k_{11}, k_{12}, k_2\}$. Similarly, we find F_0 induced by $\{v_1, v_2, s_{45}, k_{61}, k_2, k_4, k_{51}, k_{52}\}$ if v_1 in S_{64} and v_2 in S_{65} or if both lie in S_{64}, changing k_{51} for an analogous vertex k'_4 in K_4. If instead v_1 in S_{61} and v_2 in $S_{64} \cup S_{65}$, then we find $F_2(5)$ induced by $\{v_1, v_2, s_{12}, s_{24}, s_{45}, k_{61}, k_1, k_2, k_4, k_5\}$.

□

Theorem 4.20. *Let $G = (K, S)$ be a split graph containing an induced 4-tent. Then, G is a circle graph if and only if $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_6$ are 2-nested and $\mathbb{B}_r$, $\mathbb{B}_b$, $\mathbb{B}_{r-b}$ and $\mathbb{B}_{b-r}$ are nested.*

Proof. Necessity is clear by the previous lemmas. Suppose now that each of the matrices $\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_6$ is 2-nested and the matrices $\mathbb{B}_r$, $\mathbb{B}_b$, $\mathbb{B}_{r-b}$ or $\mathbb{B}_{b-r}$ are nested. Let Π be the ordering for all the vertices in K obtained by concatenating each suitable LR-ordering Π_i for $i \in \{1, 2, \dots, 6\}$.

Consider the circle divided into twelve pieces as in Figure 4.5. For each $i \in \{1, 2, \dots, 6\}$ and for each vertex $k_i \in K_i$ we place a chord having one endpoint in K_i^+ and the other endpoint in K_i^-, in such a way that the ordering of the endpoints of the chords in K_i^+ and K_i^- is Π_i.

Let us see how to place the chords for each subset S_{ij} of S. First, some useful remarks.

Remark 4.21. The following assertions hold:

- By Lemma 4.19, all the vertices in S_{ij} are nested, for every pair $i, j = \{1, 2, \dots, 6\}$, $i \neq j$. This follows since any two vertices in S_{ij} are nondisjoint. Moreover, in each S_{ij}, all the vertices are colored with either one color (the same), or they are colored red-blue or blue-red. Hence, these vertices are represented by rows in the matrices $\mathbb{B}_{r-b}$ and $\mathbb{B}_{b-r}$ and therefore they must be nested since each of these matrices is a nested matrix.
- As a consequence of the previous and Claim 4.2, if $i \geq k$ and $j \leq l$, then every vertex in S_{ij} is nested in every vertex of S_{kl}.
- Also as a consequence of the previous and Lemma 4.19, if we consider only those vertices labeled with the same letter in some $\mathbb{B}_i$, then there is a total ordering of these vertices. This follows from the fact that, if two vertices v_1 and v_2 are labeled with the same letter in some $\mathbb{B}_i$, since $\mathbb{B}_i$ is –in particular– admissible, then they are nested in K_i. Moreover, if v_1 and v_2 are labeled with L in $\mathbb{B}_i$, then they are either complete to K_{i-1} or labeled with R in $\mathbb{B}_{i-1}$. Thus, there is an index j_l such that v_i is labeled with R in $\mathbb{B}_{j_l}$, for $l = 1, 2$. Therefore, we can find in such a way a total ordering of all these vertices.
- If v_1 and v_2 are labeled with distinct letters in some $\mathbb{B}_i$, then they are either disjoint in K_i (if they are colored with the same color) or $N_{K_i}(v_1) \cup N_{K_i}(v_2) = K_i$ (if they are colored with distinct colors), for there are no D_1 or D_2 in $\mathbb{B}_i$ for all $i \in \{1, 2, \dots, 6\}$.

Notice that, when we define the matrix $\mathbb{B}_6$, we pre-color every vertex in $S_{[15]}$ with the same color. Since, we are assuming $\mathbb{B}_6$ is 2-nested and thus in particular is admissible, the subset $S_{[15]} \neq \emptyset$ if and only if the vertices represented in $\mathbb{B}_6$ are either all vertices in $S_{66} \cup S_{[16}$ and

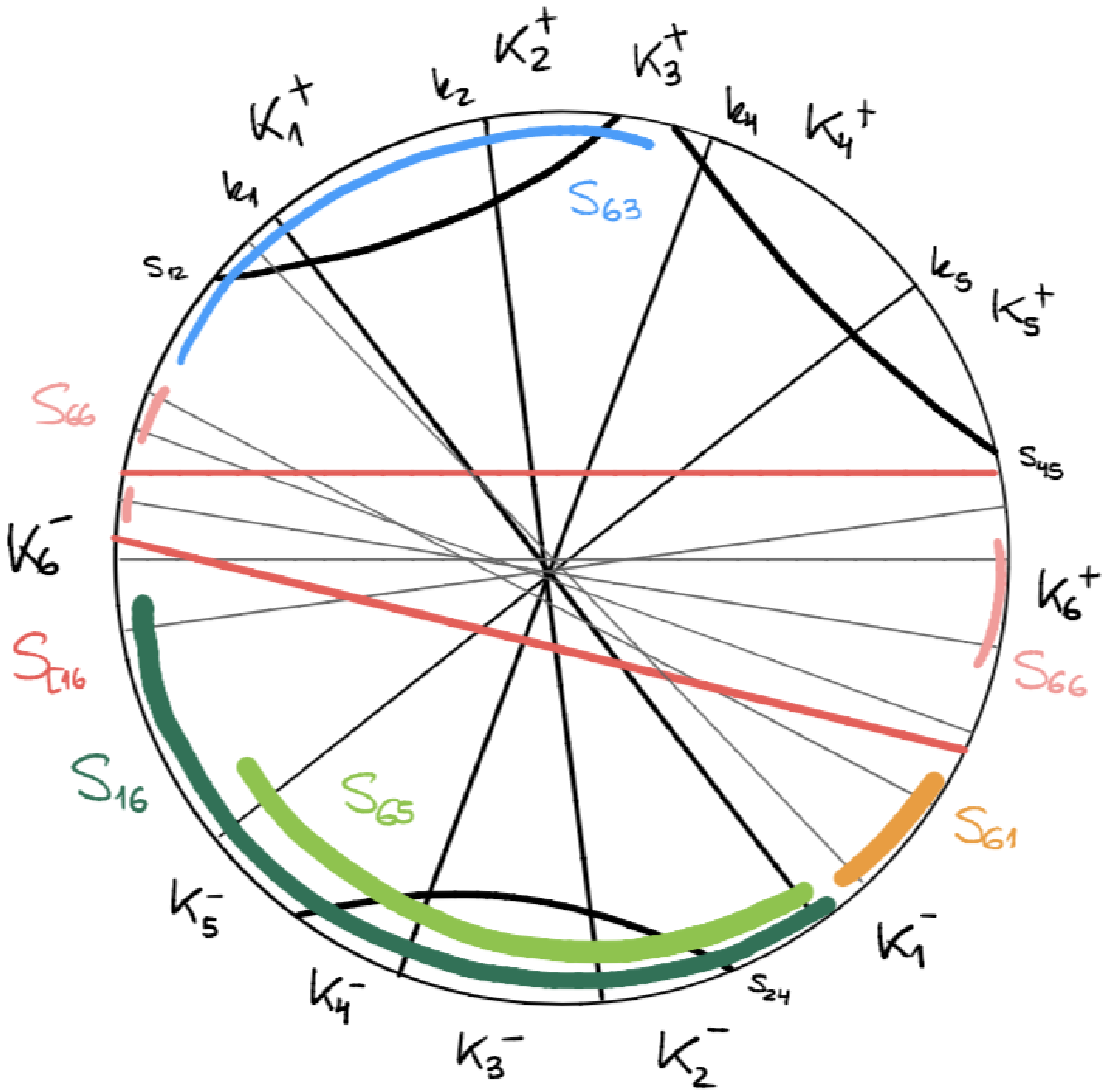


Figure 4.5 – Sketch model of G with some of the chords associated to the rows in $\mathbb{B}_6$.

vertices that are represented by labeled rows r, all of them colored and labeled with the same color and letter L or R.

Moreover, since $\mathbb{B}_6$ is admissible, the sets $N_{K_6}(S_{6i}) \cap N_{K_6}(S_{j6})$ are empty, for $i = 1, 4, 5$ and $j = 3, 4$. The same holds for the sets $N_{K_6}(S_{6i}) \cap N_{K_6}(S_{j6})$, for $i = 2, 3$ and $j = 2, 5, 1$.

If $S_{[16} = \emptyset$, then the placing of the chords that represent vertices with one or both endpoints in K_6 is very similar as in the tent case. Suppose that $S_{[16} \neq \emptyset$.

Before proceeding with the guidelines to draw the circle model, we have some remarks on the relationship between the vertices in S_{ij} with either $i = 6$ or $j = 6$, and those vertices in $S_{[16}$. This follows from the proof of Lemma 4.10:

Remark 4.22. Let G be a circle graph that contains no induced tent but contains an induced 4-

tent, and such that each matrix $\mathbb{B}_i$ is 2-nested for every $i = 1, 2, \dots, 6$. Then, all of the following statements hold:

— If $S_{26} \cup S_{16} \neq \emptyset$, then $S_{64} \cup S_{65} = \emptyset$, and viceversa.
— If $S_{36} \cup S_{46} \neq \emptyset$, then $S_{61} \cup S_{64} \cup S_{65} = \emptyset$, and viceversa.
— If $S_{56} \cup S_{26} \cup S_{16} \neq \emptyset$, then $S_{62} \cup S_{63} = \emptyset$, and viceversa.
— If $S_{56} \neq \emptyset$, then $S_{65} = \emptyset$.

Let v in $S_{ij} \neq S_{[16}$ and w in $S_{[16}$, with either $i = 6$ or $j = 6$. Suppose first that $i = j = 6$. Since $\mathbb{B}_6$ is 2-nested, the submatrix induced by the rows that represent v and w in $\mathbb{B}_6$ contains no monochromatic gems or monochromatic weak gems. If instead $i < j$, since $\mathbb{B}_6$ is admissible, then the submatrix induced by the rows that represent v and w in $\mathbb{B}_6$ contains no monochromatic weak gem, and thus we can place the endpoint of w corresponding to K_6 in the arc portion K_6^+ and the K_6 endpoint of v in K_6^-, or viceversa.

Remember that, since we are considering a suitable LR-ordering, there is an L-row m_L such that any L-row and every L-block of an LR-row are contained in m_L and every R-row and R-block of an LR-row are contained in the complement of m_L. Moreover, since we have a block bi-coloring for $\mathbb{B}_6$, then for each LR-row one of its blocks is colored with red and the other is colored with blue. Hence, for any LR-row, we can place one endpoint in the arc portion K_6^+ using the ordering given for the block that colored with red, and the other endpoint in the arc portion K_6^- using the ordering given for the block that is colored with blue.

Notice that, if $\mathbb{B}_6$ is 2-nested, then all the rows labeled with L (resp. R) and colored with the same color and those L-blocks (resp. R-blocks) of LR-rows are nested. In particular, the L-block (resp. R-block) of every LR-row contains all the L-blocks of those rows labeled with L (resp. R) that are colored with the same color. Equivalently, let r be an LR-row in $\mathbb{B}_6$ with its L-block r_L colored with red and its R-block r_R colored with blue, r' be a row labeled with L and r'' be a row labeled with R. Hence, if r_L, r' and r'' are colored with the same color, then r contains r' and $r \cap r'' = \emptyset$. This holds since we are considering a suitable LR-ordering and a total block bi-coloring of the matrix $\mathbb{B}_6$, thus it contains no D_0, D_1, D_2, D_8 or D_9.

Since every matrix $\mathbb{B}_r$, $\mathbb{B}_b$, $\mathbb{B}_{r-b}$ and $\mathbb{B}_{b-r}$ are nested, there is a total ordering for the rows in each of these matrices. Hence, there is a total ordering for all the rows that intersect that are colored with the same color, or with red-blue or with blue-red, respectively. Moreover, if v and w are two vertices in S such that they both have rows representing them in one of these matrices –hence, they are colored with the same color or sequence of colors–, then either v and w are disjoint or they are nested.

With this in mind, we give guidelines to build a circle model for G.

We place first the chords corresponding to every vertex in K, using the ordering Π. For each subset S_{ij}, we order its vertices with the inclusion ordering of the neighbourhoods in K and the ordering Π. When placing the chords corresponding to the vertices of each subset, we do it from lowest to highest according to the previously stated ordering given for each subset.

Notice that there are no other conditions besides being disjoint or nested outside each of the following subsets: S_{11}, S_{22}, S_{33}, S_{44}, S_{55}, S_{66}. For the subset S_{12}, we only need to consider if every vertex in $S_{12} \cup S_{11} \cup S_{22}$ are disjoint or nested. The same holds for the subsets S_{24}, S_{45}, considering $S_{22} \cup S_{44}$ and $S_{44} \cup S_{55}$, respectively.

Since each matrix $\mathbb{B}_i$ is 2-nested for every $i = 1, 2, \dots, 6$, if there are vertices in both S_{23} and S_{34}, then they are disjoint in K_3. The same holds for vertices in S_{62} and S_{63}, and S_{61} and $S_{14} \cup S_{15} \cup S_{16}$. This is in addition to every property seen in Remark 4.22.

First, we place those vertices in S_{ii} for each $i = 1, 2, \dots, 6$, considering the ordering given by inclusion. If v in S_{ii} and the row that represents v is colored with red, then both endpoints of the chord corresponding to v are placed in K_i^+. If instead the row is colored with blue, then both endpoints are placed in K_i^-.

For each v in $S_{ij} \neq S_{[16}$, if the row that represents v in $\mathbb{B}_i$ is colored with red (resp. blue) , then we place the endpoint corresponding to K_i in the portion K_i^+ (resp. K_i^-) . We apply the same rule for the endpoint corresponding to K_j.

Let us consider now the vertices in $S_{[15]}$. If G is circle, then all the rows in $\mathbb{B}_6$ are colored with the same color. Moreover, if $S_{[15]} \neq \emptyset$, then either every row labeled with L or R in $\mathbb{B}_6$ is labeled with L and colored with red or labeled with R and colored with blue, or viceversa. Suppose first that every row labeled with L or R in $\mathbb{B}_6$ is labeled with L and colored with red or labeled with R and colored with blue. In that case, every row representing a vertex v in $S_{[15]}$ is colored with blue, hence we place one endpoint of the chord corresponding to v in K_6^+ and the other endpoint in K_6^-. In both cases, the endpoint of the chord corresponding to v is the last chord of an independent vertex that appears in the portion of K_6^+ and is the first chord of an independent vertex that appears in the portion of K_6^-. We place all the vertices in $S_{[15]}$ in such a manner. If instead every row labeled with L or R in $\mathbb{B}_6$ is labeled with L and colored with blue or labeled with R and colored with red, then every row representing a vertex in $S_{[15]}$ is colored with red. We place the endpoints of the chord in K_6^- and K_6^+, as the last and first chord that appears in that portion, respectively.

Finally, let us consider now a vertex v in $S_{[16}$. Here we have two possibilities: (1) the row that represents v has only one block, (2) the row that represents the row that represents v has two blocks of 1's. Let us consider the first case. If the row that represents v has only one block, then it is either an L-block or an R-block. Suppose that it is an L-block. If the row in $\mathbb{B}_6$ is colored with red, then we place one endpoint of the chord as the last of K_6^- and the other endpoint in K_6^+, considering in this case the partial ordering given for every row that has an L-block colored with red in $\mathbb{B}_6$. If instead the row in $\mathbb{B}_6$ is colored with blue, then we place one endpoint of the chord as the first of K_6^+ and the other endpoint in K_6^-, considering in this case the partial ordering given for every row that has an L-block colored with blue in $\mathbb{B}_6$. The placement is analogous for those LR-rows that are an R-block.

Suppose now that the row that represents v has an L-block v_L and an R-block v_R. If v_L is colored with red, then v_R is colored with blue. We place one endpoint of the chord in K_6^+, considering the partial ordering given by every row that has an L-block colored with red in $\mathbb{B}_6$, and the other enpoint of the chord in K_6^-, considering the partial ordering given by every row that has an R-block colored with blue in $\mathbb{B}_6$. The placement is analogous if v_L is colored with blue.

This gives a circle model for the given split graph G.

□

4.3 Split circle graphs containing an induced co-4-tent

In this section we will address the last case of the proof of Theorem 4.1, which is the case where G contains an induced co-4-tent. This case is mostly similar to the 4-tent case, with one particular difference: the co-4-tent is not a prime graph, and thus there is more than one possible circle model for this graph.

This section is subdivided as follows. In Subsection 4.3.1, we define the matrices $\mathbb{C}_i$ for each $i = 1, 2, \ldots, 8$ and prove some properties that will be useful further on. In Subsection 4.3.2 we prove the necessity of the 2-nestedness of each $\mathbb{C}_i$ for G to be a circle graph and give the guidelines to draw a circle model for a split graph G containing an induced co-4-tent in Theorem 4.27.

4.3.1 Matrices $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_8$

Let $G = (K, S)$ and T as in Section 2.3. For each $i \in \{1, 2, \dots, 8\}$, let $\mathbb{C}_i$ be a $(0, 1)$-matrix having one row for each vertex $s \in S$ such that s belongs to S_{ij} or S_{ji} for some $j \in \{1, 2, \dots, 8\}$ and one column for each vertex $k \in K_i$ and such that such that the entry corresponding to row s and column k is 1 if and only if s is adjacent to k in G. For each $j \in \{1, 2, \dots, 8\} - \{i\}$, we label those rows corresponding to vertices of S_{ji} with L and those corresponding to vertices of S_{ij} with R, with the exception of those rows in $\mathbb{C}_7$ that represent vertices in $S_{76]}$ and $S_{[86]}$which are labeled with LR. Notice that we have considered those vertices that are complete to $K_1, \dots, K_5$ and K_8 and are also adjacent to K_6 and K_7 divided into two distinct subsets. Thus, S_{76} are those vertices that are not complete to K_6 and therefore the corresponding rows are labeled with R in $\mathbb{C}_6$ and with L in $\mathbb{C}_7$. As in the 4-tent case, there are LR-rows in $\mathbb{C}_7$. Moreover, there may be some empty LR-rows, which represent those independent vertices that are complete to $K_1, \dots, K_6$ and K_8 and are anticomplete to K_7. These vertices are all pre-colored with the same color, and that color is assigned depending on whether $S_{74} \cup S_{75} \cup S_{76} \neq \emptyset$ or $S_{17} \cup S_{27} \neq \emptyset$.

We color some of the remaining rows of $\mathbb{C}_i$ as we did in the previous sections, to denote in which portion of the circle model the chords have to be drawn. In order to characterize the forbidden induced subgraphs of G and using an argument of symmetry, we will only analyse the properties of the matrices $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, $\mathbb{C}_6$ and $\mathbb{C}_7$, since the matrices $\mathbb{C}_i$ $i = 4, 5, 8$ are symmetric to $\mathbb{C}_2$, $\mathbb{C}_3$ and $\mathbb{C}_6$, respectively.

We will consider 5 distinct cases, according to whether the subsets K_6, K_7 and K_8 are empty or not, for the matrices we need to define may be different in each case.

Using the symmetry of the subclasses K_6 and K_8, the cases we need to study are the following: (1) $K_6, K_7, K_8 \neq \emptyset$, (2) $K_6, K_7 \neq \emptyset$, $K_8 = \emptyset$, (3) $K_6, K_8 \neq \emptyset$, $K_7 = \emptyset$, (4) $K_6 \neq \emptyset$, $K_7, K_8 = \emptyset$, (5) $K_7 \neq \emptyset$, $K_6, K_8 = \emptyset$

In (1), the subsets are given as described in Table 2.12, and thus the matrices we need to analyse are as follows:

$$\mathbb{C}_1 = \begin{array}{l} \\ S_{12}\ \mathbf{L} \\ S_{11} \\ S_{16]}\ \mathbf{L} \\ S_{17}\ \mathbf{L} \end{array} \begin{array}{c} K_1 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \qquad \mathbb{C}_2 = \begin{array}{l} \\ S_{12}\ \mathbf{R} \\ S_{22} \\ S_{23}\ \mathbf{L} \\ S_{25]}\ \mathbf{L} \\ S_{26}\ \mathbf{L} \end{array} \begin{array}{c} K_2 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \qquad \mathbb{C}_3 = \begin{array}{l} \\ S_{13}\ \mathbf{R} \\ S_{34}\ \mathbf{L} \\ S_{33} \\ S_{35}\ \mathbf{L} \\ S_{36}\ \mathbf{L} \\ S_{23}\ \mathbf{R} \end{array} \begin{array}{c} K_3 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array}$$

$$\mathbb{C}_6 = \begin{array}{rcc} & K_6 & \\ \begin{array}{r} S_{66} \\ S_{26}\ \mathbf{R} \\ S_{36}\ \mathbf{R} \\ S_{[46}\ \mathbf{R} \\ S_{76}\ \mathbf{R} \\ S_{[86}\ \mathbf{R} \end{array} & \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} & \begin{array}{c} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \end{array} \end{array} \qquad \mathbb{C}_7 = \begin{array}{rcc} & K_7 & \\ \begin{array}{r} S_{17}\ \mathbf{R} \\ S_{[27}\ \mathbf{R} \\ S_{77} \\ S_{74]}\ \mathbf{L} \\ S_{75}\ \mathbf{L} \\ S_{76}\ \mathbf{L} \\ S_{87}\ \mathbf{R} \\ S_{[86]}\ \mathbf{LR} \\ S_{76]}\ \mathbf{LR} \end{array} & \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} & \begin{array}{c} \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \\ \\ \end{array} \end{array}$$

In (2), the matrices $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ are analogous. The subclasses $S_{[15}$ and $S_{[16}$ may be nonempty and are analogous to the subclasses $S_{[85}$ and $S_{[86}$, respectively. Moreover, the vertices in $S_{[16]}$ are analogous to those vertices in $S_{[86]}$, which are represented as empty LR-rows in $\mathbb{C}_7$.

$i \setminus j$	1	2	3	4	5	6	7
1	✓	✓	✓	✓	✓	✓	✓
2	∅	✓	✓	∅	✓	✓	✓
3	∅	∅	✓	✓	✓	✓	∅
4	∅	∅	∅	✓	✓	✓	∅
5	∅	∅	∅	∅	✓	∅	∅
6	∅	∅	∅	∅	∅	✓	∅
7	∅	∅	∅	✓	✓	✓	✓

Figure 4.6 – The nonempty subsets S_{ij} in case (2) $K_6, K_7 \neq \emptyset$, $K_8 = \emptyset$.

For its part, the matrices $\mathbb{C}_6$ and $\mathbb{C}_7$ are as follows:

$$\mathbb{C}_6 = \begin{array}{rcc} & K_6 & \\ \begin{array}{r} S_{66} \\ S_{26}\ \mathbf{R} \\ S_{36}\ \mathbf{R} \\ S_{[46}\ \mathbf{R} \\ S_{76}\ \mathbf{R} \\ S_{[16}\ \mathbf{R} \end{array} & \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} & \begin{array}{c} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{red}{\bullet} \\ \textcolor{red}{\bullet} \end{array} \end{array} \qquad \mathbb{C}_7 = \begin{array}{rcc} & K_7 & \\ \begin{array}{r} S_{17}\ \mathbf{R} \\ S_{[27}\ \mathbf{R} \\ S_{77} \\ S_{74]}\ \mathbf{L} \\ S_{75}\ \mathbf{L} \\ S_{76}\ \mathbf{L} \\ S_{[16]}\ \mathbf{LR} \\ S_{76]}\ \mathbf{LR} \end{array} & \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} & \begin{array}{c} \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \textcolor{blue}{\bullet} \\ \\ \\ \end{array} \end{array}$$

Therefore this case can be considered as a particular case of case (1).

If instead we are in case (3), then the matrices $\mathbb{C}_2$ and $\mathbb{C}_3$ are analogous as in (1). In this case there are no LR-vertices in any of the matrices.

For its part, the matrices $\mathbb{C}_1$ and $\mathbb{C}_6$ are as follows:

$i \backslash j$	1	2	3	4	5	6	8
1	✓	✓	✓	✓	$\emptyset$	✓	✓
2	$\emptyset$	✓	✓	$\emptyset$	✓	✓	$\emptyset$
3	$\emptyset$	$\emptyset$	✓	✓	✓	✓	$\emptyset$
4	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓	$\emptyset$
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$	$\emptyset$
6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$
8	$\emptyset$	✓	✓	✓	✓	✓	✓

Figure 4.7 – The nonempty subsets S_{ij} in case (3) $K_6, K_8 \neq \emptyset$, $K_7 = \emptyset$.

$$\mathbb{C}_1 = \begin{array}{r} \\ S_{12}\ \mathbf{L} \\ S_{11} \\ S_{16]}\ \mathbf{L} \end{array} \begin{array}{c} K_1 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{c} \\ \bullet \\ \\ \bullet \end{array} \qquad \mathbb{C}_6 = \begin{array}{r} \\ S_{66} \\ S_{26}\ \mathbf{R} \\ S_{36}\ \mathbf{R} \\ S_{46}\ \mathbf{R} \\ S_{[86}\ \mathbf{R} \end{array} \begin{array}{c} K_6 \\ \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

In case (4), the matrices $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ are analogous as in case (3). There is no matrix $\mathbb{C}_7$ and thus there are no LR-vertices. Notice that the subset S_{15} contains only vertices that are complete to K_1 and thus $S_{15} = S_{[15}$. Furthermore, this subset is equivalent to $S_{[85}$ in case (1). Moreover, in this case, the vertices in $S_{[16}$ in $\mathbb{C}_6$ are analogous as those vertices in $S_{[86}$ and thus the matrix $\mathbb{C}_6$ results analogous as in case (3). Also notice that those vertices in $S_{[16]}$ can be placed all having one endpoint in the arc $s_{13}s_{35}$ and the other in k_1k_3. It follows that $S_{54} = S_{[54]}$, thus these vertices are complete to K and hence $S_{[54]} = S_{[16]}$. Moreover, those vertices in S_{65} are complete to K_5 and thus we can consider $S_{65} = \emptyset$ and $S_{65} = S_{[16}$.

$i \backslash j$	1	2	3	4	5	6
1	✓	✓	✓	✓	✓	✓
2	$\emptyset$	✓	✓	$\emptyset$	✓	✓
3	$\emptyset$	$\emptyset$	✓	✓	✓	✓
4	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$
6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓

Figure 4.8 – The nonempty subsets S_{ij} in case (4) $K_6 \neq \emptyset$, $K_7, K_8 = \emptyset$.

Finally, let us consider case (5). When considering those vertices in S_{54}, it follows easily that $S_{54} = S_{54]}$ and thus these vertex subset is equivalent to those vertices in S_{75} (in case (1)) that are complete to K_7. Hence, we consider these vertices as in S_{75} and $S_{54} = \emptyset$. The subset S_{15} of vertices of S is split in three distinct subsets: $S_{15]}$, $S_{[15}$ and $S_{[15]}$. The rows representing vertices in $S_{15]}$ are pre-colored with blue and labeled with L, only in $\mathbb{C}_1$, and are equivalent to those vertices in $S_{16]}$

in case (1). For their part, the rows that represent $S_{[15}$ are pre-colored with red and labeled with R, and they appear only in $\mathbb{C}_5$. These rows are equivalent to those in $S_{[85}$ in case (1). Finally, the vertices in $S_{[15]}$ are represented by uncolored empty LR-rows in $\mathbb{C}_7$, resulting equivalent to those vertices in $S_{[86]}$ in case (1).

$i \backslash j$	1	2	3	4	5	7
1	✓	✓	✓	✓	✓	✓
2	$\emptyset$	✓	✓	$\emptyset$	✓	✓
3	$\emptyset$	$\emptyset$	✓	✓	✓	$\emptyset$
4	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	$\emptyset$
5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	✓	$\emptyset$
7	$\emptyset$	$\emptyset$	$\emptyset$	✓	✓	✓

Figure 4.9 – The nonempty subsets S_{ij} in case (5) $K_7 \neq \emptyset$, $K_6, K_8 = \emptyset$.

Therefore, it suffices to see what happens if $K_6, K_7, K_8 \neq \emptyset$, since the matrices defined in the cases (2) to (5) have the same rows or less that each of the corresponding matrices $\mathbb{C}_1, \dots, \mathbb{C}_8$ defined for case (1). In other words, the case $K_6, K_7, K_8 \neq \emptyset$ is the most general of all.

Let us suppose that $K_6, K_7, K_8 \neq \emptyset$. The Claims in Chapter 2 and the following prime circle model allow us to assume that some subsets of S are empty.

We denote S_{87} to the set of vertices in S that are complete to $K_1, \dots, K_6$, are adjacent to K_7 and K_8 but are not complete to K_8, and analogously S_{76} is the set of vertices in S that are complete to $K_1, \dots, K_5, K_8$, are adjacent to K_6 and K_7 but are not complete to K_6. Hence, $S_{76]}$ denotes the vertices of S that are complete to $K_1, \dots, K_6, K_8$ and are adjacent to K_7.

Remark 4.23*.* Claim 4.2 remains true if G contains an induced co-4-tent. The proof is analogous as in the tent case.

4.3.2 Split circle equivalence

In this subsection, we will show results analogous to Lemmas 4.8 and 4.10.

Lemma 4.24. *If $\mathbb{C}_1$, $\mathbb{C}_2, \dots, \mathbb{C}_8$ are not 2-nested, then* G *contains one of the forbidden subgraphs in $\mathcal{T}$ or $\mathcal{F}$.*

Proof. Using the argument of symmetry, we will prove this for the matrices $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, $\mathbb{C}_6$ and $\mathbb{C}_7$.

Let us suppose that one of the matrices $\mathbb{C}_i$ is not 2-nested. By Lemma 3.36, suppose that $\mathbb{C}_i$ is not partially 2-nested. The structure of the proof is analogous as in Lemmas 4.3, 4.8 and 4.10, and as in those lemmas we notice that, if G is circle, then in particular, for each $i = 1, \dots, 8$, $\mathbb{C}_i$ contains no M_0, $M_{II}(4)$, M_V or $S_0(k)$ for every even $k \geq 4$ since these matrices are the adjacency matrices of non-circle graphs.

Case (1) Suppose that one of the matrices $\mathbb{C}_i$ is not admissible, for some $i = 1, 2, 3, 6, 7$.

Case (1.1) Suppose first that $\mathbb{C}_1$ is not admissible. Hence, since $\mathbb{C}_1$ has no uncolored labeled rows, or any rows labeled with R or LR, then $\mathbb{C}_1$ contains either D_0 or $S_2(k)$. Suppose that $\mathbb{C}_1$ contains D_0. Let v_1 and v_2 in S_{12}, k_{11} and k_{12} in K_1 such that k_{1i} is adjacent to v_i and nonadjacent to v_{i+1} (mod 2), for $i = 1, 2$.

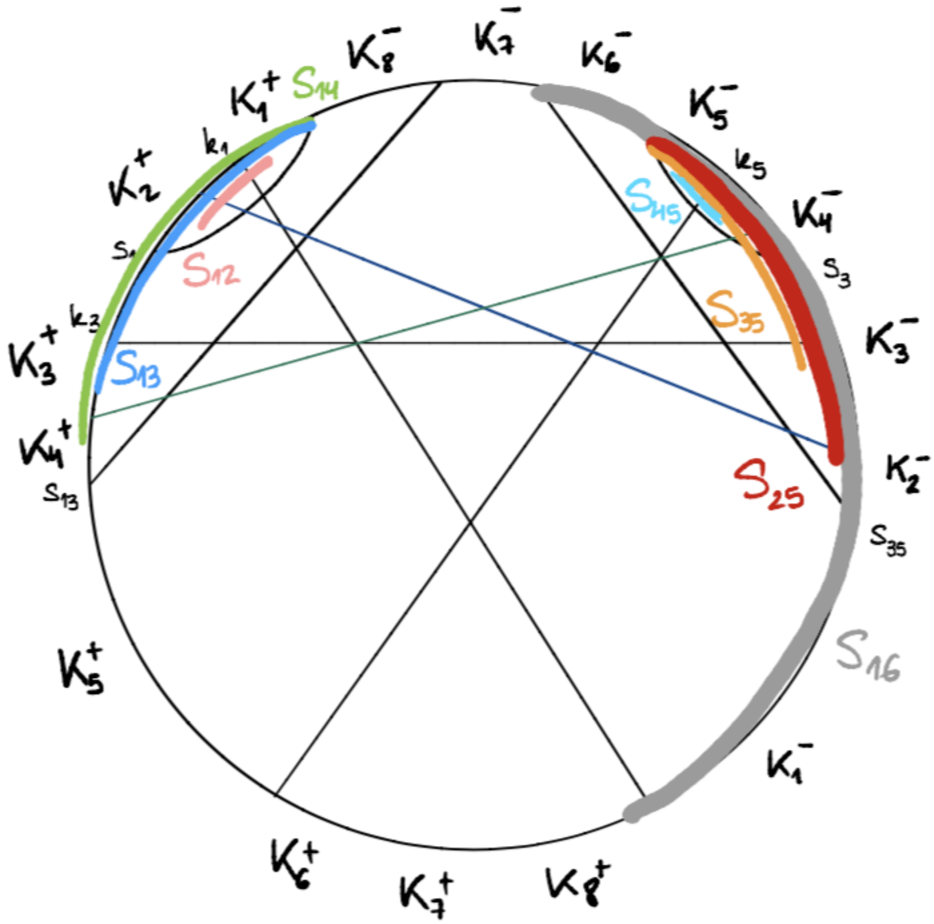


Figure 4.10 – A circle model for the co-4-tent graph.

Notice that, if v_1 and v_2 have empty intersection in K_2, then we find a 4-tent induced by $\{k_{21}$, k_{11}, k_{12}, k_{22}, v_1, v_2, $s_{35}\}$. The same holds for any two vertices v_1 and v_2 in $S_{12} \cup S_{16} \cup S_{17}$ (considering s_1 instead of s_{35}), hence we may assume that there is a vertex k_i in K_i –for $i = 2, 6, 7$ as appropriate– adjacent to both v_1 and v_2.

Thus, if both v_1 and v_2 lie in S_{12}, then we find a net $\vee K_1$ induced by $\{k_{11}$, k_{12}, k_2, k_3, v_1, v_2, $s_{35}\}$. If v_1 and v_2 both lie in or $S_{16]}$ or S_{17}, then we find a net $\vee K_1$ induced by $\{k_{11}$, k_{12}, k_2, k_8, v_1, v_2, $s_{35}\}$ (since $K_8 \neq \emptyset$, however the same holds using any vertex in K_7 nonadjacent to both v_1 and v_2). If v_1 in S_{12} and v_2 in $S_{16]}$, then we find $M_{II}(4)$ induced by $\{k_{11}$, k_2, k_5, k_{12}, v_1, v_2, s_{13}, $s_{35}\}$. If v_2 in S_{17} is analogous changing k_5 by k_6.

Suppose there is $S_2(j)$ as a subconfiguration of $\mathbb{C}_1$, and suppose j is even, thus v_1 and v_j lie both in S_{12} or both in $S_{16} \cup S_{17}$. If both lie in S_{12}, then we find $F_2(j+1)$ induced by $\{k_{11}, \ldots, k_{1(j-1)}$, k_2, k_3, $v_1, \ldots, v_j$, $s_{35}\}$. If instead both lie in $S_{16} \cup S_{17}$, then we find $F_1(j+1)$ induced by $\{k_{11}, \ldots, k_{1(j-1)}$, k_3, $v_1, \ldots, v_j$, $s_1\}$. Suppose j is odd, then v_1 in S_{12} and v_2 in $S_{16} \cup S_{17}$, or viceversa. In the first case, we find $F_1(j+2)$ induced by $\{k_{11}, \ldots, k_{1(j-1)}$, k_2, k_3, $v_1, \ldots, v_j$, s_1, $s_{35}\}$. In the second case, we find $F_2(j)$ induced by $\{k_{11}, \ldots, k_{1(j-1)}$, k_3, $v_1, \ldots, v_j\}$, therefore $\mathbb{C}_1$ is admissible.

Case (1.2) Suppose $\mathbb{C}_2$ is not admissible. Since $\mathbb{C}_2$ has no uncolored labeled rows, or LR rows, or blue rows labeled with R, or red rows labeled with L, then $\mathbb{C}_2$ contains either D_0, D_2, $S_2(j)$ for some j even or $S_3(j)$ for some j odd. Suppose there is D_0. Let v_1 and v_2 be the rows of D_0, and k_{21} and k_{22} in K_2 such that v_i is adjacent to k_{2i} and is nonadjacent to $k_{2(i+1)}$ (mod 2) for $i = 1, 2$. If v_1 and v_2 lie in S_{12}, then we know by the previous case that there is a vertex k_1 in K_1 adjacent to both. However, in this case we find a tent induced by $\{k_{21}, k_1, k_{22}, v_1, v_2, s_{35}\}$. The same holds if v_1 and v_2 lie in $S_{23} \cup S_{25} \cup S_{26}$, changing k_1 by k_3 and s_{35} by s_1, thus there is no D_0.

Suppose there is D_2, let v_1 and v_2 be the rows of D_2, one is labeled with L and the other is labeled with R. Thus, v_1 in S_{12} and v_2 in $S_{23} \cup S_{25} \cup S_{26}$, or viceversa. Let k_{21} and k_{22} in K_2 such that k_{21} is adjacent to both v_1 and v_2 and k_{22} is nonadjacent to v_1 and v_2. Then, we find $M_{II}(4)$ induced by $\{k_1, k_{21}, k_3, k_{22}, v_1, v_2, s_1, s_{35}\}$, and thus there is no D_2.

Suppose there is $S_2(j)$ for some even j. If v_1 and v_j lie in S_{12}, then we find $F_2(j+1)$ induced by $\{k_{21}, \dots, k_{2(j-1)}, k_1, v_1, \dots, v_j, s_{35}\}$. If instead v_1 and v_j lie in $S_{23} \cup S_{25} \cup S_{26}$, then we also find $F_2(j+1)$ induced by $\{k_{21}, \dots, k_{2(j-1)}, k_3, v_1, \dots, v_j, s_1\}$, and hence there is no $S_2(j)$.

Suppose there is $S_3(j)$ for some odd j. Thus, v_1 in S_{12} and v_2 in $S_{23} \cup S_{25} \cup S_{26}$, or viceversa. In that case, we find $F_2(j+2)$ induced by $\{k_{21}, \dots, k_{2(j-1)}, k_1, k_3, v_1, \dots, v_j, s_1, s_{35}\}$, and therefore $\mathbb{C}_2$ is admissible.

Case (1.3) Suppose $\mathbb{C}_3$ is not admissible. Since there are no LR-rows, or uncolored labeled rows, then there is either D_0, D_1, D_2, $S_2(j)$ or $S_3(j)$.

Suppose there is D_0, let v_1 and v_2 be the rows of D_0 and k_{31} and k_{32} in K_3 the columns of D_0. The vertices v_1 and v_2 lie in S_{13}, S_{34}, S_{35}, S_{36} or S_{23}. First notice that, in either case, if the intersection is empty in K_1 (resp. K_i for $i = 2, 3, 4, 5$), then we find a 4-tent induced by $\{k_{11}, k_{31}, k_{32}, k_{12}, v_1, v_2, s_{35}\}$ (resp. s_1, s_{13}, s_5).

If v_1 and v_2 both lie in S_{13}, then we find a tent induced by $\{k_1, k_{31}, k_{32}, v_1, v_2, s_{35}\}$. The same holds if both lie in S_{35} or S_{36}. If v_1 and v_2 lie in S_{34}, then we find net $\vee K_1$ induced by $\{k_{31}, k_4, k_{32}, k_5, v_1, v_2, s_5\}$. The same holds by symmetry if both lie in S_{23}. If v_1 in S_{13} and v_2 in S_{23}, then we find $M_{II}(4)$ induced by $\{k_1, k_2, k_{31}, k_{32}, v_1, v_2, s_1, s_{35}\}$. The same holds if v_1 in $S_{35} \cup S_{36}$ and v_2 in S_{23}, therefore there is no D_0.

Suppose there is D_1, let v_1 and v_2 be the rows of D_1 and k_3 in K_3 be the non-tag column of D_1. Suppose that v_1 in S_{13} and v_2 in S_{34}. Then, we find $F_1(5)$ induced by $\{k_1, k_3, k_4, k_5, v_1, v_2, s_5, s_{13}, s_{35}\}$. The same holds by symmetry if v_1 in $S_{35} \cup S_{36}$ and v_2 in S_{23}, thus there is no D_1.

Suppose there is D_2, let v_1 and v_2 be the rows of D_2, and k_{31} and k_{32} in K_3 be the columns of D_2. If v_1 in S_{13} and v_2 in $S_{35} \cup S_{36}$, then we find $M_{II}(4)$ induced by $\{k_1, k_5, k_{31}, k_{32}, v_1, v_2, s_{13}, s_{35}\}$. The other case is analogous, therefore there is no D_2.

Suppose there is $S_2(j)$ with j even. If v_1 and v_j in S_{13}, then we find $F_1(j+1)$ induced by $\{k_{31}, \dots, k_{3(j-1)}, k_1, v_1, \dots, v_j, s_{35}\}$. If instead v_1 and v_j in S_{34}, then we find $F_1(j)$ induced by $\{k_{31}, \dots, k_{3(j-1)}, k_4, k_5, v_1, \dots, v_j\}$. It is analogouos by symmetry if v_1 and v_j are colored with blue, thus there is no $S_2(j)$ with j even, hence suppose j is odd. If v_1 in S_{13} and v_j in S_{23}, then we find $F_2(j)$ induced by $\{k_{31}, \dots, k_{3(j-1)}, k_1, v_1, \dots, v_j\}$. If instead v_1 in S_{23} and S_{13}, then we find $F_1(j+2)$ induced by $\{k_{31}, \dots, k_{3(j-1)}, k_1, k_2, v_1, \dots, v_j, s_1, s_{35}\}$. It is analgous for the other cases.

Suppose there is $S_3(j)$. If j is even, then v_1 in S_{13} and v_j in S_{34}, or the analogous blue labeled rows. However, in that case we find $F_1(j+3)$ induced by $\{k_{31}, \dots, k_{3(j-1)}, k_1, k_4, k_5, v_1, \dots, v_j, s_1, s_5, s_{35}\}$. If instead j is odd, then v_1 in S_{13} and $S_{35} \cup S_{36}$ or the analogous labeled rows. In that case, we find $F_1(j+2)$ induced by $\{k_{31}, \dots, k_{3(j-1)}, k_1, k_5, v_1, \dots, v_j, s_{13}, s_{35}\}$, therefore $\mathbb{C}_3$ is admissible.

Case (1.4) Suppose $\mathbb{C}_6$ is not admissible. Since there are no LR-rows or uncolored labeled rows, or rows labeled with L, then there is either D_0 or $S_2(j)$. Suppose there is D_0, let v_1 and v_2 be the rows of D_0 and k_{61} and k_{62} in K_6 be the columns of D_0. If v_1 and v_2 lie in $S_{26} \cup S_{36} \cup S_{46}$, then we find a net $\vee K_1$ induced by $\{k_1, k_4, k_{61}, k_{62}, v_1, v_2, s_{13}\}$. Once more, if the intersection in K_4 is empty, then we find a 4-tent induced by $\{k_{41}, k_{61}, k_{62}, k_{42}, v_1, v_2, s_{13}\}$.

If v_1 and v_2 in $S_{76} \cup S_{[86}$, then we find a tent induced by $\{k_1, k_{61}, k_{62}, v_1, v_2, s_{35}\}$. If instead v_1 in $S_{26} \cup S_{36} \cup S_{46}$ and v_2 in $S_{76} \cup S_{[86}$, then we find $M_{II}(4)$ induced by $\{k_{61}, k_{62}, k_1, k_4, v_1, v_2, s_{13}, s_{35}\}$, thus there is no D_0.

Suppose there is $S_2(j)$. If j is even, then v_1 and v_j lie in $S_{26} \cup S_{36} \cup S_{46}$. In that case, we find $F_2(j+1)$ induced by $\{k_{61}, \dots, k_{6(j-1)}, k_1, k_4, v_1, \dots, v_j, s_{13}\}$. If instead j is odd, then v_1 in $S_{26} \cup S_{36} \cup S_{46}$ and v_2 in $S_{76} \cup S_{[86}$, or viceversa. In the first case, we find $F_2(j+1)$ induced by $\{k_{61}, \dots, k_{6(j-1)}, k_1, k_4, v_1, \dots, v_j, s_{13}, s_{35}\}$. In the second case, we find $F_2(j)$ induced by $\{k_{61}, \dots, k_{6(j-1)}, k_1, v_1, \dots, v_j\}$, and therefore $\mathbb{C}_6$ is admissible.

Case (1.5) Finally, suppose $\mathbb{C}_7$ is not admissible. Notice that, if there is D_8, then we find a tent, and if there is D_9, then we find F_0. Since there are no red labeled rows, then there is either D_0, D_1, D_6, D_7, $S_1(j)$, $S_2(j)$ with even j, $S_3(j)$ with even j, $S_4(j)$ with even j, $S_5(j)$ with even j, $S_6(j)$ or $S_7(j)$.

Suppose there is D_0, let v_1 and v_2 be the rows, and k_{71}, k_{72} in K_7 be the columns of D_0. If v_1 and v_2 lie in $S_{[74} \cup S_{75} \cup S_{76}$, then we find a net $\vee K_1$ induced by $\{k_{71}, k_{72}, k_4, k_6, v_1, v_2, s_{35}\}$. If instead v_1 and v_2 lie in $S_{17} \cup S_{[27} \cup S_{87}$, since v_1 and v_2 are not complete to K_8, then there is either a 4-tent (if there is no vertex in K_8 adjacent to both, induced by $\{k_{71}, k_{81}, k_{82}, k_{72}, v_1, v_2, s_{13}\}$), or a net $\vee K_1$ induced by $\{k_{71}, k_{72}, k_8, k_2, v_1, v_2, s_{13}\}$, therefore there is no D_0.

Suppose there is D_1, let v_1 and v_2 be the rows, and k_7 in K_7 be the non-tag column. Let v_1 in $S_{74]} \cup S_{75} \cup S_{76}$ (notice that v_1 is complete to K_8 and is not complete to K_6) and v_2 in $S_{17} \cup S_{[27} \cup S_{87}$ (is not complete to K_8 and is complete to K_6). Thus, we find $M_{II}(4)$ induced by $\{k_8, k_3, k_6, k_7, v_1, v_2, s_{13}, s_{35}\}$, hence there is no D_1. Suppose there is D_6, let v_1, v_2 and v_3 be the rows where v_3 is an LR-row, and k_{71} and k_{72} in K_7 be the columns of D_6. In that case, v_1 lies in $S_{74]} \cup S_{75} \cup S_{76}$, v_2 in $S_{17} \cup S_{[27} \cup S_{87}$ and v_3 in $S_{76]}$, hence we find a 4-tent induced by $\{k_{71}, k_8, k_6, k_{72}, v_1, v_2, v_3\}$, therefore there is no D_6. Suppose there is D_7, let v_1 be any row labeled with either L or R, and v_2 and v_3 LR-rows in $S_{76]}$. In either case, there is a vertex k_i in K_i with $i \neq 7$ such that v_1 is adjacent to k_i, and hence we find a net $\vee K_1$ induced by $\{k_{71}, k_{72}, k_{73}, k_i, v_1, v_2, v_3\}$, thus there is also no D_7.

Suppose there is $S_1(j)$, and suppose that j is even. Since v_1 and v_j correspond to rows labeled with either L or R, in either case v_1 and v_j are complete to K_4. Hence, we find an odd $(j-1)$-sun with center induced by $\{k_{71}, \dots, k_{7(j-2)}, k_4, v_1, \dots, v_j\}$. Moreover, if j is odd, then we find a $(j-1)$-sun induced by the same subset.

Suppose there is $S_2(j)$ where j is even. If v_1 and v_j are labeled with L, then they are both complete to K_6 and K_5. Analogously, if they are labeled with R, then they are both complete to K_8 and K_2. In the first case, we find $F_2(j+1)$ induced by $\{k_{71}, \dots, k_{7(j-1)}, k_5, k_6, v_1, \dots, v_j, s_{35}\}$. It is analogous if they are labeled with R.

Suppose there is $S_3(j)$ where j is even. However, we find a j-sun induced by $\{k_{71}, \dots, k_{7(j-1)}, k_5, v_1, \dots, v_j\}$ and thus it is not possible.

If there is $S_4(j)$ with even j, then we find a $j-1$-sun with center induced by $\{k_{71}, \dots, k_{7(j-2)}, k_5, v_1, \dots, v_j\}$.

If instead there is $S_5(j)$ with j even, then we find $F_2(j+1)$ induced by $\{k_{71}, \dots, k_{7(j-1)}, k_6, k_4, v_1, \dots, v_j, s_{35}\}$ if v_1 and v_j lie in $S_{74} \cup S_{75} \cup S_{76}$. It is analogous if v_1 and v_j lie in $S_{27} \cup S_{17} \cup S_{87}$ using k_8, k_2 and s_{13}.

Finally, if there is $S_6(j)$, then we find $M_{II}(j)$, and if there is $S_7(j)$ then we find a j-sun if j is even, and a j-sun with center if j is odd.

Therefore $\mathbb{C}_i$ is admissible for every $i = 1, 2, 3, 6, 7$.

Case (2) Let $C = \mathbb{C}_i$ and suppose that C is not LR-orderable, then C^*_{tag} contains either a Tucker matrix or M'_4, M''_4, M'_5, M''_5, $M'_2(k)$, $M''_2(k)$, $M'_3(k)$, $M''_3(k)$, $M'''_3(k)$ for some $k \geq 4$ (see Figure 3.17).

The proof of this case is analogous as in Lemma 4.10, since in most situations we only use the fact that C is admissible. Moreover, whenever we consider two labeled rows v and w labeled with distinct letters, we have at least two vertices k_6 in K_6 and k_8 in K_8 such that v is adjcent to k_6 and nonadjacent to k_8 and w is adjacent to k_8 and nonadjacent to k_6. Moreover, there is always a vertex k_4 in K_4 that is adjacent to both. This holds whether they are labeled with the same letter or not.

Case (3) Therefore, we may assume that $\mathbb{C}_i$ is admissible and LR-orderable but is not partially 2-nested. Since there are no uncolored labeled rows and those colored rows are labeled with either L or R and do not induce any of the matrices $\mathcal{D}$, then in particular no pair of pre-colored rows of $\mathbb{C}_i$ induce a monochromatic gem or a monochromatic weak gem, and there are no badly-colored gems since every LR-row is uncolored, therefore $\mathbb{C}_i$ is partially 2-nested.

Case (4) Finally, let us suppose that $C = \mathbb{C}_i$ is partially 2-nested but is not 2-nested. As in the previous cases, we consider C ordered with a suitable LR-ordering. Let C' be a matrix obtained from C by extending its partial pre-coloring to a total 2-coloring. It follows from Lemma 3.39 that, if C' is not 2-nested, then either there is an LR-row for which its L-block and R-block are colored with the same color, or C' contains a monochromatic gem or a monochromatic weak gem or a badly-colored doubly weak gem.

If C' contains a monochromatic gem where the rows that induce such a gem are not LR-rows, then the proof is analogous as in the tent case. Thus, we may assume that at least one of the rows is an LR-row and hence let $i = 7$.

Case (4.1) *Let us first suppose there is an LR-row w for which its L-block w_L and R-block w_R are colored with the same color.* If these two blocks are colored with the same color, then there is either one odd sequence of rows $v_1, \ldots, v_j$ that force the same color on each block, or two distinct sequences, one that forces the same color on each block.

Case (4.1.1) If there is one odd sequence $v_1, \ldots, v_j$ that forces the color on both blocks, then the proof is analogous as in 4.10.

Case (4.1.2) Suppose there are two independent sequences $v_1, \ldots, v_j$ and $x_1, \ldots, x_l$ that force the same color on w_L and w_R, respectively. Suppose without loss of generality that w_L and w_R are colored with red. If $j = 1$ and $l = 1$, then we find D_6, which is not possible. Hence, we assume that either $j > 1$ or $l > 1$. Suppose that $j > 1$ and $l > 1$, thus there is one labeled row in each sequence. We may assume that v_j is labeled with L and x_l is labeled with R, since LR-ordering used to color B' is suitable. As in the proof of Lemma 4.10, we assume throughout the proof that each row in each sequence forces the coloring on both the previous and the next row in its sequence. Thus in this case, $v_2, \ldots, v_j$ is contained in w_L and $x_2, \ldots, x_l$ is contained in w_R. Moreover, w represents a vertex in $S_{76]}$, v_j lies in $S_{74]} \cup S_{75} \cup S_{76}$ and x_l lies in $S_{[27} \cup S_{17} \cup S_{87}$, and thus both are colored with blue and j and l are both odd. If x_l lies in $S_{[27} \cup S_{17} \cup S_{87}$, since there is a k_4 in K_4 adjacent to both v_j and x_l, then we find $F_2(j + l + 1)$ contained in the submatrix induced by each row and column on which the rows in w and both sequences are not null and the column representing k_i. The proof is analogous if either $j = 1$ or $l = 1$.

Hence, we assume there is either a monochromatic weak gem in which one of the rows is an LR-row or a badly-colored doubly-weak gem in C', for the case of a monochromatic gem or a monochromatic weak gem where one of the rows is an L-row (resp. R-row) and the other is unlabeled is analogous to the tent case. Moreover, if an LR-row and an unlabeled row (or a row labeled with L or R) induce a monochromatic gem, then in particular these rows induce a monochromatic weak gem.

However, the proof follows analogously as in Lemma 4.10 and therefore, if G is a circle graph, then $\mathbb{C}_i$ is 2-nested for each $i = 1, 2, \dots, 8$.

□

Definition 4.25. *We define the matrices $\mathbb{C}_r$, $\mathbb{C}_b$, $\mathbb{C}_{r-b}$ and $\mathbb{C}_{b-r}$ as in Section 4.2.3. Similarly, we have the following Lemma for these matrices.*

Lemma 4.26. *Suppose that $\mathbb{C}_i$ is 2-nested for each $i = 1, 2 \dots, 8$. If $\mathbb{C}_r$, $\mathbb{C}_b$, $\mathbb{C}_{r-b}$ or $\mathbb{C}_{b-r}$ are not nested, then G contains F_0 as a minimal forbidden induced subgraph for the class of circle graphs.*

Proof. Suppose that $\mathbb{C}_r$ is not nested, and let v_1 and v_2 be the vertices represented by the rows that induce a 0-gem in $\mathbb{C}_r$. The rows in $\mathbb{C}_r$ represent vertices in the following subsets of S: S_{12}, $S_{[13}$, $S_{[14}$, S_{34}, $S_{74]}$, S_{75}, S_{76}, $S_{82]}$, S_{83}, S_{84}, S_{85}, $S_{[86}$, $S_{86]}$ or S_{87}. Notice that, by definition, these last two subsets are not complete to K_8.

Notice that the vertices in $S_{86]} \cup S_{87}$ do not induce 0-gems in $\mathbb{C}_r$.

Case (1) Suppose that v_1 in S_{12}. Since $S_{[13}$ and $S_{[14}$ are complete to K_1 and $S_{82]}$ is complete to K_2, the only possibility is that v_2 in S_{12}. In that case, we find F_0 induced by {v_1, v_2, s_{35}, k_{11}, k_{12}, k_{21}, k_{22}, k_4}, where k_{11} and k_{12} in K_1, k_{21} and k_{22} in K_2 and k_4 in K_4. We find the same forbidden subgraph if v_1 and v_2 lie both in S_{34}, with vertices k_{31}, k_{32} in K_3, k_{41} and k_{42} in K_4, k_5 in K_5 and s_5 instead of s_{35}.

Case (2) Let v_1 in $S_{[13} \cup S_{[14}$.

Case (2.1) If v_1 in $S_{[14}$, then v_2 lies in S_{34} or in $S_{82]} \cup S_{83} \cup S_{84}$ since every vertex in S_{12}, $S_{[13}$ is contained in every vertex of $S_{[14}$, and every vertex in $S_{[14}$ is properly contained in every vertex of $S_{74]} \cup S_{75} \cup S_{76} \cup S_{85} \cup S_{[86}$. If v_2 in S_{34}, then we find F_0 induced by {v_1, v_2, s_5, k_1, k_3, k_{41}, k_{42}, k_5}. If instead v_2 in $S_{82]} \cup S_{83} \cup S_{84}$, then we find F_0 induced by {v_1, v_2, s_{35}, k_8, k_1, k_2, k_4, k_5}, since there is a vertex k_4 in K_4 adjacent to v_1 and nonadjacent to v_2 and a vertex k_8 in K_8 adjacent to v_2 and nonadjacent to v_1 (which is represented in the 0-gem by the column c_L).

Case (2.2) If v_1 in $S_{[13}$, then v_2 lies in S_{34} or in $S_{82]} \cup S_{83}$. However, the first is not possible since $\mathbb{C}_3$ is admissible. The proof if v_2 lies in $S_{82]} \cup S_{83}$ follows analogously as in the previous subcase.

Case (3) Suppose v_1 in S_{34}. Since $\mathbb{C}_3$ is admissible and $S_{74]}$ is complete to K_4, then the only possibility is that v_2 in S_{84}. We find F_0 induced by {v_1, v_2, s_5, k_2, k_3, k_{41}, k_{42}, k_5}.

Suppose that $\mathbb{C}_b$ is not nested, and let v_1 and v_2 be the vertices represented by the rows that induce a 0-gem in $\mathbb{C}_b$. The rows in $\mathbb{C}_b$ represent vertices in the following subsets of S: S_{23}, $S_{25]}$, S_{26}, $S_{[27}$, $S_{16]}$, S_{17}, $S_{35]}$, S_{36}, S_{45}, $S_{[46}$, $S_{74]}$, S_{75}, S_{76}, $S_{82]}$, S_{83}, S_{84}, $S_{[85}$, $S_{[86}$, $S_{86]}$ or S_{87}.

Notice that the vertices in $S_{86]}$, S_{87}, $S_{82]}$, S_{83}, S_{84} do not induce 0-gems in $\mathbb{C}_r$. The same holds for those vertices in $S_{74]}$, S_{75} and S_{76}, however in this case this follows from the fact that $\mathbb{C}_7$ is admissible.

Case (1) Suppose v_1 in S_{23}. Since $\mathbb{C}_3$ is admissible, then v_2 lies in $S_{23} \cup S_{25]} \cup S_{26}$. If v_2 in S_{23}, then we find F_0 induced by {v_1, v_2, s_1, k_1, k_{21}, k_{22}, k_{31}, k_{32}}. If v_2 in $S_{25]}$ or S_{26}, then we find F_0 induced by the same subset changing k_{32} for some vertex in K_5 or K_6, respectively.

Case (2) Let v_1 in $S_{25]} \cup S_{26}$, thus v_2 in $S_{26} \cup S_{36} \cup S_{46}$. We assume that v_1 in $S_{25]}$, since the proof is analogous if v_1 in S_{26}. We find F_0 induced by {v_1, v_2, s_1, k_1, k_{21}, k_{22}, k_5, k_6} if v_2 in S_{26}. If instead v_2 in S_{36} or S_{46}, then the subset is the same with the exception of k_{22}, which is replaced by an analogous vertex in K_3 or K_4, respectively.

Case (3) Suppose v_1 in $S_{[27}$. Thus, v_2 in $S_{16]} \cup S_{17} \cup S_{86]} \cup S_{87}$. Since v_2 is never complete to K_8 and both vertices induce a 0-gem, we find F_0 induced by {v_1, v_2, s_{13}, k_1, k_2, k_6, k_7, k_8}.

Case (4) Suppose v_1 in $S_{16]}$. Thus, v_2 in S_{17}. Since $K_8 \neq \emptyset$, we find F_0 induced by {v_1, v_2, s_{13}, k_{11}, k_{12}, k_6, k_7, k_8}.

Case (5) Suppose v_1 in $S_{35]}$. Thus, v_2 in $S_{36} \cup S_{46}$. We find F_0 induced by $\{v_1, v_2, s_{13}, k_1, k_{31}, k_{32}, k_5, k_6\}$ if v_2 in S_{36}, and if v_2 in S_{46} we change k_{32} for an analogous vertex in K_4.
Case (6) Suppose v_1 in S_{17}. Thus, v_2 in $S_{86]} \cup S_{87}$. Since v_2 is not complete to K_8, then we find F_0 induced by $\{v_1, v_2, s_{13}, k_8, k_{11}, k_{12}, k_6, k_7\}$.

Suppose that $\mathbb{C}_{r-b}$ is not nested, and let v_1 and v_2 be the vertices represented by the rows that induce a 0-gem. The rows in $\mathbb{C}_{r-b}$ represent vertices in either $S_{86]}$ or S_{87}.

Suppose that v_1 in $S_{86]}$ and v_2 in S_{87}. Since none of the vertices is complete to K_8, $\mathbb{C}_8$ is admissible and these rows are R-rows in $\mathbb{C}_8$, then there is no D_0 and thhus there are three vertices k_{81}, k_{82} and k_{83} in K_8 such that k_{81} is nonadjacent to both v_1 and v_2, k_{83} is adjacent to both v_1 and v_2 and k_{82} is adjacent to v_1 and nonadjacent to v_2. We find F_0 induced by $\{v_1, v_2, s_{13}, k_{81}, k_{82}, k_{83}, k_6, k_7\}$. It follows analogously if both vertices lie in S_{87}, and if both lie in $S_{[86}$ only changing k_7 for an analogous k_{62} in K_6.

Suppose that $\mathbb{C}_{b-r}$ is not nested, and let v_1 and v_2 be the vertices represented by the rows that induce a 0-gem. The rows in $\mathbb{C}_{b-r}$ represent vertices in $S_{74]}$, S_{75}, S_{76}, $S_{82]}$, S_{83}, S_{84}, $S_{[85}$ or $S_{[86}$.
Case (1) Suppose that v_1 and v_2 in $S_{74]} \cup S_{75} \cup S_{76}$. In either case, v_1 and v_2 are not complete to K_6 by definition. Since $\mathbb{C}_6$ is admissible, thus there is no D_0 and there are vertices k_{61} and k_{62} in K_6 such that v_1 is nonadjacent to k_{61} and k_{62} and v_2 is adjacent to k_{61} and is nonadjacent to k_{62}. We find F_0 induced by $\{v_1, v_2, s_{35}, k_{71}, k_{72}, k_4, k_{61}, k_{62}\}$ if v_1 and v_2 lie in S_{76}. It follows analogously if v_1 or v_2 lie in $S_{74]} \cup S_{75}$ changing k_{61} for an analogous vertex k_5 in K_5.
Case (2) Suppose that v_1 and v_2 in $S_{82]} \cup S_{83} \cup S_{84} \cup S_{[85} \cup S_{[86}$. Since every vertex in $S_{[85}$ and $S_{[86}$ is complete to K_8, then none of these vertices induce a 0-gem in $\mathbb{C}_{b-r}$. Thus, v_1 and v_2 lie in $S_{82]} \cup S_{83} \cup S_{84}$. Moreover, since every vertex in $S_{82]}$ is complete to K_2, then it is not possible that both vertices lie in $S_{82]}$. Let k_{81} and k_{82} in K_8 such that v_1 is adjacent to both and v_2 is adjacent to k_{82} and is nonadjacent to k_{81}. Notice that in that case we are assuming that, if one of the vertices lies in $S_{82]}$, then such vertex is v_1. If v_2 in S_{83}, then we find F_0 induced by $\{v_1, v_2, s_{35}, k_{81}, k_{82}, k_2, k_3, k_5\}$. If instead v_2 in S_{84}, we find F_0 with the same subset only changing k_3 for some analogous k_4 in K_4.
Case (3) Suppose that v_1 in $S_{74]} \cup S_{75} \cup S_{76}$ and v_2 in $S_{82]} \cup S_{83} cup S_{84} \cup S_{[85} \cup S_{86}$. Notice that, if v_2 in $S_{82]} \cup S_{83} \cup S_{84}$, then v_2 is contained in v_1 and thus such vertices cannot induce a 0-gem in $\mathbb{C}_{b-r}$. Thus, v_2 in $S_{[85} \cup S_{[86}$. In this case, there is a vertex k_6 in K_6 that is nonadjacent to both v_1 and v_2 since none of these vertices is complete to $K : 6$ by definition and $\mathbb{C}_6$ is admissible. If v_1 in $S_{74]} \cup S_{75}$, then we find we find F_0 induced by $\{v_1, v_2, s_{35}, k_7, k_8, k_4, k_5, k_6\}$. If instead v_1 in S_{76} and v_1 and v_2 induce a 0-gem, then v_2 in $S_{[86}$. We find F_0 with the same subset as before, only changing k_5 for some analogous k_{62} in K_6.

This finishes the proof. □

The main result of this section is the following theorem, which follows directly from the previous lemmas.

Theorem 4.27. *Let $G = (K, S)$ be a split graph containing an induced co-4-tent. Then, G is a circle graph if and only if $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_8$ are 2-nested and $\mathbb{C}_r$, $\mathbb{C}_b$, $\mathbb{C}_{r-b}$ and $\mathbb{C}_{b-r}$ are nested.*

Proof. Necessity is clear by the previous lemmas. Suppose now that each of the matrices $\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_8$ is 2-nested and the matrices $\mathbb{C}_r$, $\mathbb{C}_b$, $\mathbb{C}_{r-b}$ or $\mathbb{C}_{b-r}$ are nested. Let Π be the ordering for all the vertices in K obtained by concatenating each suitable LR-ordering Π_i for $i \in \{1, 2, \dots, 8\}$.

Consider the circle divided into sixteen pieces as in Figure 4.10. For each $i \in \{1,2,\dots,8\}$ and for each vertex $k_i \in K_i$ we place a chord having one endpoint in K_i^+ and the other endpoint in K_i^-, in such a way that the ordering of the endpoints of the chords in K_i^+ and K_i^- is Π_i. Throughout the following, we will consider the circular ordering clockwise.

Let us see how to place the chords for each subset S_{ij} of S.

The vertices with exactly endpoint in K_7^- that are not LR-vertices in $\mathbb{C}_7$ are $S_{74]} \cup S_{75} \cup S_{76}$ and $S_{[27} \cup S_{17} \cup S_{87}$. Since $\mathbb{C}_7$ is admissible, the vertices in $S_{74]} \cup S_{75} \cup S_{76}$ and $S_{[27} \cup S_{17} \cup S_{87}$ do not intersect in K_7. Moreover, since there are no pre-colored red rows, then there are no vertices with exactly one endpoint in K_7^+. Furthermore, the vertices in $S_{76]}$ and $S_{[86]}$ are represented by LR-rows in $\mathbb{C}_7$. These last ones are exactly those empty LR-rows. Since $\mathbb{C}_7$ is 2-nested, then all of these vertices can be drawned in the circle model. It follows that, if $S_{[86]} \neq \emptyset$, then either $S_{74]} \cup S_{75} = \emptyset$ or $S_{[27} \cup S_{17} \cup S_{87} = \emptyset$. On the other hand, those nonempty LR-rows in $\mathbb{C}_7$ correspond to vertices in $S_{76]}$. Each of these vertices with two blocks in $\mathbb{C}_7$ have one endpoint in K_7^+, placed according to the ordering Π_7 of the nonempty columns of its red block, and the other endpoint placed in K_7^- according to the ordering Π_7 of the nonempty columns of its blue block. It follows analogously for those nonempty LR-vertices with exactly one block.

Notice that in $\mathbb{C}_1$ (resp. in $\mathbb{K}_5$ by symmetry) there are no R-rows (resp. L-rows). Since $\mathbb{C}_1$ is 2-nested, then all the vertices that have exactly one endpoint in K_1^- (resp. K_5^+) are nested and thus such endpoint can be placed without issues. The same holds for every vertex with both endpoints in K_1^- and K_5^+. Moreover, the only vertices with exactly one endpoint in K_1^+ may be those in S_{12}, for all the vertices in $S_{[13} \cup S_{[14}$ are nested and have the endpoint corresponding to K_1 placed between s_{14}^- and the first endpoint of a vertex in $S_{82]} \cup S_{83} \cup S_{84}$ (or s_{13}^- if this set is empty). The vertices in S_{12} are nested, and thus each endpoint of these vertices may be placed in the ordering given by Π_2 and Π_1, respectively, between s_1^+ and s_1^-.

The only vertices that have exactly one endpoint in K_2^+ are those in S_{12}. The vertices that have exactly one endpoint in K_2^- are those in $S_{23} \cup S_{25]} \cup S_{26}$. Since $\mathbb{C}_2$ is 2-nested and $\mathbb{C}_b$ is nested, then these vertices are all nested and thus we can place the chords according to the ordering Π_2. Those vertices in $S_{[27}$ have the endpoint corresponding to K_2 placed right after s_{35}^-, and before any of the chords with endpoint in K_1^-. The same holds by symmetry for those chords with exactly one endpoint in K_4^+ and K_4^-.

The vertices with exactly one endpoint in K_3^+ are S_{34} and $S_{[13} \cup S_{83}$. Since $\mathbb{C}_3$ is admissible, the vertices in S_{34} and $S_{[13} \cup S_{83}$ do not intersect in K_3. Moreover, since $\mathbb{C}_3$ is 2-nested and $\mathbb{C}_r$ and $\mathbb{C}_{b-r}$ are nested, then the vertices in $S_{[13} \cup S_{83}$ are nested and thus we can place both of its endpoints following the ordering given by Π_3. The vertices with exactly one endpoint in K_3^- are those in S_{23} (which we have already shown where to place) and those in $S_{35]} \cup S_{36}$. These last vertices are nested since $\mathbb{C}_b$ is nested and thus we place both its endpoints according to Π_3. Notice that, since $\mathbb{C}_3$ is admissible, then the vertices in S_{23} and $S_{35]} \cup S_{36}$ do not intersect in K_3.

Since $\mathbb{C}_{b-r}$ is nested, if $S_{[85} \neq \emptyset$, then $S_{74]} = \emptyset$, and viceversa. The same holds for $S_{[86}$ and $S_{74]} \cup S_{75}$. Moreover, if $S_{[85} \neq \emptyset$, then every vertex in $S_{[85}$ is nested in S_{75}, and if $S_{[86} \neq \emptyset$, then every vertex in $S_{[86}$ is nested in S_{76}. It follows analogously by symmetry for those vertices in $S_{[27} \cup S_{17} \cup S_{87} \cup S_{86] \cup S_{16]}}$.

Those vertices with exactly one endpoint in K_6^+ are those in $S_{76} \cup S_{[86}$. These vertices are nested since $\mathbb{C}_6$ is 2-nested and $\mathbb{C}_{b-r}$ is nested. Thus, if these subsets are nonempty, then $S_{74]} \cup S_{75} = \emptyset$. Therefore, we can place both its enpoints according to Π_6, one in K_6^+ and the other between s_{13}^- and s_{35}^+. The vertices that have exactly one endpoint in K_6^- are those in $S_{26} \cup S_{36} \cup S_{46}$, and since $\mathbb{C}_b$ is nested, then these vertices are all nested and therefore we place both its endpoints according to Π_6.

Finally, all the vertices represented by unlabeled rows in each $\mathbb{C}_i$ for $i = 1, 2, \dots, 8$ represent the vertices in S_{ii}. These vertices are entirely colored with either red or blue, and are either disjoint or nested with every other vertex colored with its color. Hence, we place both endpoints of the corresponding chord in K_i^+ if it is colored with red, and in K_i^- if it is colored with blue, according to the ordering Π_i given for K_i.

This gives the guidelines for a circle model for G.

□

4.4 Split circle graphs containing an induced net

Let $G = (K, S)$ be a split graph. If G is a minimally non-circle graph, then it contains either a tent, or a 4-tent, or a co-4-tent, or a net as induced subgraphs. In the previous sections, we have addressed the problem of having a split minimally-non-circle graph that contains an induced tent, 4-tent and co-4-tent, respectively. Let us consider a split graph G that contains no induced tent, 4-tent or co-4-tent, and suppose there is a net subgraph in G.

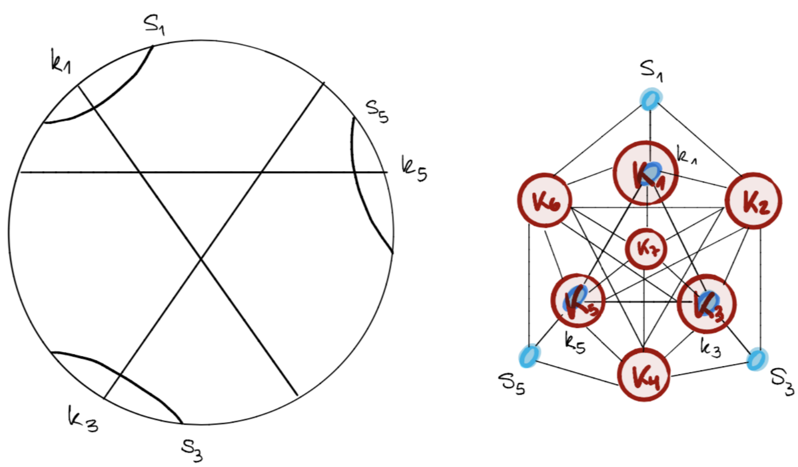


Figure 4.11 – A circle model for the net graph and the partitions of K.

We define K_i as the subset of vertices in K that are adjacent only to s_i if $i = 1, 3, 5$, and if $i = 2, 4, 6$ as those vertices in K that are adjacent to s_{i-1} and s_{i+1}. We define K_7 as the subset of vertices in K that are nonadjacent to s_1, s_3 and s_5. Let s in S. We denote $T(s)$ to the vertices that are false twins of s.

Remark 4.28. The net is not a prime graph. Moreover, if $K_i = \emptyset$, $K_j = \emptyset$ for any pair $i, j \in \{2, 4, 6\}$, then G is not prime. For example, if $K_2 = \emptyset$ and $K_4 = \emptyset$, then a split decomposition can be found considering the subgraphs $H_1 = K_3 \cup T(s_3)$ and $H_2 = G \setminus T(s_3)$.

Since in the proof we consider a minimally non-circle graph G, it follows from the previous remark that at least two of K_2, K_4 and K_6 must be nonempty so that G results prime. However, in that case we find a 4-tent as an induced subgraph. Therefore, as a consequence of this and the previous sections, we have now proven the characterization theorem given at the begining of the chapter.

Theorem 4.1 (continuing from p. 85)**.** *Let $G = (K, S)$ be a split graph. Then, G is a circle graph if and only if G is $\{\mathcal{T}, \mathcal{F}\}$-free (See Figures 4.1 and 4.2).*

Part II

Minimal completions

Chapter 1

Introduction

Given a graph G and a graph class Π, a graph modification problem consists in studying how to minimally add or delete vertices or edges from G such that the resulting graph belongs to the class Π.

As graphs can be used to represent various real world and theoretical structures, it is not difficult to see that these modification problems can model a large number of practical applications in several different fields. Some examples are: networks reliability; numerical algebra; molecular biology; computer vision; and relational databases. It is thus natural that such problems have been widely studied.

A graph class Π is a family of graphs having the property Π, for example, Π can be the property of being chordal, or planar, or perfect, etc.

The modification problem we studied is the Π-completion problem. A Π-completion of a graph $G = (V, E)$ is a supergraph $H = (V, E \cup F)$ such that H belongs to Π and $E \cap F = \emptyset$. In other words, we want to find a set of edges F such that, when added to G, the resulting graph belongs to the class Π. The edges in F are referred to as *fill edges*. A Π-completion is *minimum* if for any set of edges F' such that $H' = (V, E \cup F')$ belongs to Π, then $|F'| \geq |F|$. A Π-completion is *minimal* if for any proper subset $F' \subset F$, the supergraph $H' = (V, E \cup F')$ does not belong to Π.

The problem of calculating a minimum completion in an arbitrary graph to a specific graph class has been rather studied, since it has applications in areas such as molecular biology, computational algebra, and more specifically in those areas that involve modelling based in graphs where the missing edges are due to lack of data, for example in data clustering problems [19, 29]. Unfortunately, minimum completions of arbitrary graphs to specific graph classes, such as cographs, bipartite graphs, chordal graphs, etc., have been showed to be NP-hard to compute [29, 7, 36].

For this reason, current research on this topic is focused in finding minimal completions of arbitrary graphs to specific graph classes in the most efficient way possible from the computational point of view. And even though the minimal completion problem is and has been rather studied, structural characterizations are still unknown for most of the problems for which a polynomial algorithm to find such a completion has been given. Studying the structure of minimal completions may allow to find efficent recognition algorithms.

Minimal completions from an arbitrary graph to interval graphs and proper interval graphs have been studied in [8, 33]. In these particular cases, a minimal completion can be found in $\mathcal{O}(n^2)$ and $\mathcal{O}(n + m)$ respectively, but there are no results in the literature that refer to the complexity of the recognition problem in both cases.

The most well known motivation for Minimum Interval Modification problems, comes from molecular biology, and it is one of the main reasons why interval graphs started being studied in the first place. In a paper from 1959 [1], Benzer first gave strong evidences that the collection of DNA composing a bacterial gene was linear, just like the structure of the genes themselves in the chromosome. This linear structure could be represented as overlapping intervals on the real line, and therefore as an interval graph. However, mapping of the genetic structure is done by indirect observation. That is, such linear structure is not observed directly, but it is inferred by how various fragments of the original genome can be recombined. In order to study various properties of a certain DNA sequence, the original piece of DNA is fragmented into smaller pieces. This fragments are then cloned many times using various biological methods, and take the name of clones. In this process the position of each clone on the original stretch of DNA is lost, but since usually many copies of the same piece of DNA are fragmented in different ways, some clones will overlap. The problem of reconstructing the original arrangements of the clones in the original sequence is called physical mapping of DNA. Deciding whether two clones overlap or not is the critical part where errors may arise, since it is a process based on partial information. We know that once we decide an arrangement of these clones consistent with the overlapping, the resulting model should represent an interval graph. However, there might be some false positive or false negatives, due to erroneous interpretation of some data. Correcting the model to get rid of inconsistencies is then equivalent to remove or add edges to the graph representing the dataset, so that it becomes interval. Of course we want to change it as little as possible. Moreover, when all the clones have the same size, i.e., the DNA sequence has been fragmented in equal parts, the resulting graph should be not only interval, but proper interval.

It was shown in [23, 36, 18, 19] that the minimum Π-completion problem is NP-complete if Π is the family of chordal, interval, or proper interval graphs.

In the following sections we give some basic definitions and state some of the known structural characterizations for chordal, interval and proper interval graphs, which will be useful in the next chapter.

1.1 Basic definitions

A graph G is *chordal* if every cycle of length greater or equal to 4 has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

We say G is an *interval graph* if G admits an intersection model consisting of intervals in the real line. It has one vertex for each interval in the family and an edge between every pair of vertices represented by intervals that intersect. In particular, G is a *unit interval graph* if there is a model in which every interval has length 1, and G is a *proper interval graph* if G admits a model such that no interval is properly included in any other. Interval, unit interval and proper interval graphs are all subclasses of chordal graphs.

The neighbourhood of a vertex x in V is the set $N(x) = \{v \in V \mid v \text{ is adjacent to } x\}$. If $X \subseteq V$, we define $N_X(w) = \{v \in X \subseteq V \mid v \text{ is adjacent to } w\}$. When $X = V$ we will simply denote it $N(w)$.

Three independent vertices form an *asteroidal triple (AT)* if, for each two, there is a path P from one to the other such that P does not pass through a neighbor of the third one.

Let u and v in V be two nonadjacent vertices. A set $S \subseteq V$ is a *u, v-minimal separator* if u and v belong to distinct connected components in $G[V \setminus S]$, and S is minimal with this property. We say indistinctly that S is a minimal separator if such vertices u and v exist.

Let G and H be two graphs. We say that G *is* H-*free* if there is no subgraph isomorphic to H in G.

1.2 Known characterizations of interval and proper interval graphs

We now give a list of properties and characterization theorems that will be strongly used in the following chapter.

Lemma 1.1. *[24] Let* $G = (V, E)$ *be a graph, and* $S \subseteq V$*. Then,* S *is a minimal separator if and only if* $G[V \setminus S]$ *has at least two connected components* C_1, C_2 *such that* $N(C_1) = N(C_2) = S$.

Lemma 1.2. *[24] Let* $G = (V, E)$ *be a graph. If* a *and* b *are nonadjacent vertices in* G*, then there is a unique* a, b*-minimal separator* S *such that* $S \subseteq N(a)$.

Lemma 1.3. *[12] If* $G = (V, E)$ *is a chordal graph, then every minimal separator is a clique.*

Theorem 1.4. *[26]* G *is an interval graph if and only if* G *is chordal and AT–free.*

Theorem 1.5. *[22] The following properties are equivalent:*
— G *is a proper interval graph*
— G *is chordal and contains no claw, net or tent as induced subgraphs (See Figure 1.1)*
— G *is an interval graph and contains no claws*

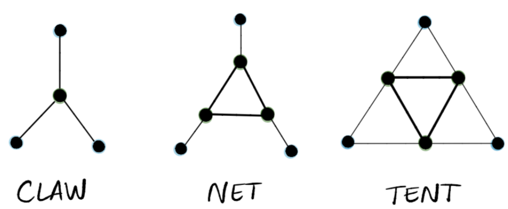


Figure 1.1 – Some of the forbidden induced subgraphs for proper interval graphs.

Theorem 1.6. *[32] The class of unit interval graphs coincides with the class of proper interval graphs.*

Chapter 2

Minimal completion of proper interval graphs

In this chapter, we study how to structurally characterize a minimal completion of an interval graph to a proper interval graph. In Section 2.1, we define and characterize some orderings for the vertices that are strongly based in the minimal separators of an interval graph. In Section 2.2, we define the types of edges that can be found in any completion of an interval graph. Afterwards, we state and prove a necessary condition for a minimal completion in this particular case.

2.1 Preliminaries

In this section, we will start giving some definitions and properties that will allow us to describe in the next section all the types of edges that can be found in a completion of an interval graph and state Theorem 2.14. These definitions and properties include a necessary condition regarding the ordering of the vertices for any proper interval graph.

The following property allows us to assume from now on that the graph G is connected.

Proposition 2.1. *[27] Let* $G = (V, E)$ *be a graph, let* $\mathcal{C}(G) = \{C_1, \ldots, C_k\}$ *be the set of all connected components of* G *and let* $H = (V, E \cup F)$ *be a* $\Pi-$*completion. Then,* H *is a minimal* $\Pi-$*completion of* G *if and only if* $H[C_i]$ *is a minimal* Π*-completion of* $G[C_i]$ *for every connected component* $C_i \in \mathcal{C}(G)$*.*

Definition 2.2. *Let* $G = (V, E)$ *a connected graph,* S *a minimal separator of* G*, and let* C_i *be a connected component of* $G[V \setminus S]$*. We define the* nucleus $A_i(S)$ *as the set of vertices* v *in* C_i *for which there is at least one vertex* s *in the separator* S *such that* v *and* s *are adjacent.*

In this regard, $A_i(S)$ *will refer as needed in each case by abuse of language both of the vertex set* $A_i(S)$ *and the induced subgraph* $G[A_i(S)]$*. Moreover, we will use* $A_i = A_i(S)$ *whenever it is clear which is the minimal separator.*

Proposition 2.3. *Let* $H = (V, E)$ *be a connected proper interval graph. Then, for every minimal separator* S *of* H*, the subgraph* $H[V \setminus S]$ *has exactly two connected components.*

Proof. By Lemma 1.1, there are at least two distinct connected components C_1, C_2 of $H[V \setminus S]$ such that $N(C_1) = N(C_2) = S$. Toward a contradiction, let C_3 be a nonempty connected component of $H[V \setminus S]$ such that $C_1 \neq C_3$ and $C_2 \neq C_3$.

Notice that, if we consider any three vertices x_i in C_i for each $i = 1, 2, 3$, then these vertices are nonadjacent. Since C_3 is nonempty and H is connected, there are vertices v_3 in C_3 and s in S such that v_3 is adjacent to s. Similarly, let v_1 in C_1 and v_2 in C_2 such that v_1 and v_2 are both adjacent to the vertex s. Hence, the set $\{v_1, v_2, v_3, s\}$ induces a claw and this contradicts the hypothesis of H being a proper interval graph. □

By proposition 2.3, we will assume from now on that, if H is a connected proper interval graph, then for every minimal separator S of H, the subgraph $H[V \setminus S]$ has *exactly* two connected components.

Proposition 2.4. *Let* $H = (V, E)$ *be a connected proper interval graph,* S *a minimal separator of* H *and let* A_i *be a nucleus of the separator* S*, for* $i = 1, 2$*. For every pair of vertices* v, w *in* A_i *with a common neighbour* s *in* S*, then* (v, w) *is an edge of* E*.*

Proof. Suppose to the contrary that v and w in A_1 are both adjacent to some vertex s in S, and that the edge (v, w) is not in E.

Since S is a minimal separator and H is connected, then A_2 is nonempty. Thus, let z in A_2 such that z is adjacent to s. Hence, the set $\{v, w, s, z\}$ induces a claw in H and this results in a contradiction. □

Corollary 2.5. *Under the previous hypothesis, if* $|S| = 1$*, then* A_i *is a clique for* $i = 1, 2$*.*

Proposition 2.6. *Let* $G = (V, E)$ *be an interval graph,* S *a minimal separator of* G *such that* $|S| > 1$*, and let* A *be a nucleus of the separator* S*.*

If s_1 *and* s_2 *in* S*, then* $N_A(s_1) \cap N_A(s_2)$ *is a nonempty set.*

Proof. Let s_1 and s_2 in S. Suppose there are two nonadjacent vertices v_1 and v_2 in A_1 such that s_1 is adjacent to v_1 and nonadjacent to v_2, and s_2 is adjacent to v_2 and nonadjacent to v_1. Since C_1 is connected, there is a simple path $\mathcal{P}$ in C_1 that joins v_1 and v_2. If there is a vertex in $\mathcal{P}$ nonadjacent to either s_1 or s_2, then we find a cycle of length greater or equal than 4. In particular, the same holds if $\mathcal{P} \cap (C_1 \setminus A_1)$ is nonempty because s_1 and s_2 are adjacent.

Hence, suppose that $\mathcal{P} \subseteq A_1$ and every vertex in $\mathcal{P}$ is adjacent to both s_1 and s_2. Since S is a minimal separator, there are vertices x_1 and x_2 in A_2 such that x_i is adjacent to s_i for $i = 1, 2$. In particular, since x_1 and x_2 are both in C_2 –which is a connected component of $G[V \setminus S]$–, there is a path $\mathcal{P}'$ joining x_1 and x_2 such that $\mathcal{P}'$ is entirely contained in C_2.

We claim that the set $\{x_1, v_1, v_2\}$ induces an AT. It is clear that x_1, v_1 and v_2 are three independent vertices. If x_1 is also adjacent to s_2, then we have the path $\mathcal{P} \subseteq A_1$ connecting v_1 and v_2, and the following paths:

$$\mathcal{P}_1 : x_1 \to s_1 \to v_1$$

$$\mathcal{P}_2 : x_1 \to s_2 \to v_2$$

The proof is analogous if $x_1 = x_2$. If instead x_1 is nonadjacent to s_2, then we have $\mathcal{P}$ joining v_1 and v_2, $\mathcal{P}_1$ defined as above joining x_1 and v_1, and the path:

$$\mathcal{P}_2 : x_1 \xrightarrow{\mathcal{P}'} x_2 \to s_2 \to v_2$$

and thus G is not an interval graph, which results in a contradiction. Hence, the vertices v_1 and v_2 are adjacent. However, since v_1 is adjacent to s_1, v_2 is adjacent to s_2, and s_1 is adjacent to s_2 for S is a minimal separator of a chordal graph, either v_1 is adjacent to s_2, or v_2 is adjacent to s_1 and therefore $N_{A_1}(s_1) \cap N_{A_1}(s_2)$ is nonempty. □

Corollary 2.7. *Under the hypothesis of Proposition 2.6, if* $N_{A_i}(s) = \{v\}$*, then* v *is complete to* S*.*

Proposition 2.8. *Let* $G = (V, E)$ *be a connected proper interval graph. If* S *is a minimal separator of* G *such that* $|S| > 1$*, then, for* $i = 1, 2$*, every nucleus* A_i *is a clique.*

Proof. We will prove this result for A_1. If $|A_1| = 1$, then the proposition holds.

If $|A_1| = 2$, then by Propositions 2.4 and 2.6, both vertices are adjacent.

Suppose that $|A_1| \geq 3$, and let v_1 and v_2 in A_1 be two nonadjacent vertices.

By definition of nucleus, there are vertices s_1 and s_2 in S such that s_i is adjacent to v_i, for each $i = 1, 2$. Since v_1 and v_2 are nonadjacent, by Proposition 2.4, $s_1 \neq s_2$, v_1 is nonadjacent to s_2 and v_2 is nonadjacent to s_1.

By Proposition 2.6, there are vertices w_1 in A_1 and w_2 in A_2 such that w_1 and w_2 are adjacent to both s_1 and s_2. It is clear that $w_1 \neq v_1$ and $w_1 \neq v_2$.

Since v_1 and w_1 are adjacent to s_1, by Proposition 2.4, v_1 is adjacent to w_1, and the same holds for v_2 and w_1. Therefore, the set $\{v_1, z_1, v_2, s_1, s_2, z_2\}$ induces a tent and this results in a contradiction, for the tent is a forbidden subgraph for proper interval graphs. □

Definition 2.9. *Let* G *be a graph,* S *a minimal separator of* G *and* A *a nucleus of* S*.* A nuclear ordering for A *is an ordering* $v_1, \ldots, v_k$ *of the vertices of* A *such that for every pair of vertices* v_i *and* v_j*, if* $i < j$*, then* $N_S(v_i) \subseteq N_S(v_j)$*.*

Notation: If σ *is a nuclear ordering for the nucleus* $A = \{v_1, \ldots, v_k\}$*, we denote* $v_1 <_\sigma v_2 <_\sigma \ldots <_\sigma v_k$*.*

Proposition 2.10. *Let* H *be a connected proper interval graph,* S *a minimal separator of* H *and* A *a nucleus of* S*. If* v_1 *and* v_2 *in* A*, then* $N_S(v_1) \cap N_S(v_2)$ *is nonempty. Moreover, there is a nuclear ordering* σ *for* A*.*

Proof. Let v_1 and v_2 in A. Let us see that either $N_S(v_1) \subseteq N_S(v_2)$ or $N_S(v_2) \subseteq N_S(v_1)$.

Toward a contradiction, suppose there is a vertex s_1 in $N_S(v_1)$ such that $s_1 \notin N_S(v_2)$, and a vertex s_2 in $N_S(v_2)$ such that $s_2 \notin N_S(v_1)$.

Since S and A are cliques –by Propositions 2.4 and 2.8– and H is chordal, v_1 and v_2 are adjacent and also s_1 is adjacent to s_2. Thus, the set $\{v_1, v_2, s_1, s_2\}$ induces a C_4 and this results in a contradiction.

Therefore, either $N_S(v_1) \subseteq N_S(v_2)$ or $N_S(v_2) \subseteq N_S(v_1)$, and since any two vertices in A are comparable, this induces a nuclear ordering in A. □

Corollary 2.11. *For each nucleus* A*, there is a vertex* $v \in A$ *such that* v *is complete to* S*.*

Proposition 2.12. *Let* H *be a proper interval graph and* S *a minimal separator of* H*. Then, there is a vertex ordering* $s_1, s_2, \ldots, s_m$ *for* S *such that*

$$N_{A_1}(s_1) \supseteq \ldots \supseteq N_{A_1}(s_m), \textit{ and}$$

$$N_{A_2}(s_1) \subseteq \ldots \subseteq N_{A_2}(s_m)$$

We call this a bi-ordering for S*, and we denote it regarding the nucleus corresponding each direction. For example, the previous would be denoted as* $s_1 \geq_{A_1} \ldots \geq_{A_1} s_m$ *and* $s_1 \leq_{A_2} \ldots \leq_{A_2} s_m$*.*

Proof. Suppose to the contrary that there is a minimal separator S of H such that every decreasing ordering of its vertices regarding A_1 is not an increasing ordering regarding A_2.

Let $s_1 \geq_{A_1} \ldots \geq_{A_1} s_m$ be a decreasing ordering of S regarding A_1. Suppose without loss of generality $s_1 \not\leq_{A_2} s_2$, and $s_2 <_{A_2} s_1 \leq_{A_2} s_3 \leq_{A_2} \ldots \leq_{A_2} s_m$.

Notice that, if $s_2 =_{A_2} s_1$, then the given ordering regarding A_1 holds for A_2, thus since the ordering is total between vertices in S, we may assume a strict ordering for A_2.

Moreover, if $s_1 =_{A_1} s_2$, then we can swap s_1 and s_2 in the ordering regarding A_1 and thus this new ordering results in a bi-ordering for S.

Suppose $s_1 >_{A_1} s_2$. Hence, there is a vertex x_1 in A_1 such that s_1 is adjacent to x_1 and s_2 is nonadjacent to x_1. Let x_2 in A_2 such that s_1 is adjacent to x_2 and s_2 is nonadjacent to x_2. We can find such a vertex for we are assuming $s_2 <_{A_2} s_1$. These four vertices induce a claw, and therefore this results in a contradiction since H is proper interval.

This argument holds for every pair of vertices in S for which the position given by the order in the other nucleus cannot be inverted. $\square$

2.2 A necessary condition

In this section, we will use the properties and definitions given in the previous section to define all the types of edges that may arise in a completion of an interval graph, and we will state and prove a necessary condition for any minimal completion to proper interval graphs when the input graphs is an interval graph, which is the main result of this chapter.

Definition 2.13. *Let G be an interval graph, H a completion of G to proper interval, and let $e = (v, w)$ in F be a fill edge.*

1. *We say e* is type I, *if there is a minimal separator S of H and a nucleus A such that v and w are both vertices in A.*
2. *We say e* is type II, *if e is not type I and there is at least one minimal separator S of H and a nucleus A for which v in S, w in A, such that if e is deleted, then there is no nuclear ordering in A.*
3. *We say e* is type III *if e is not type I, there is at least one minimal separator S of H and nucleus A for which v in S and w in A, and for each such minimal separator S and nucleus A, if e is deleted, then there is still a nuclear ordering in A.*
4. *We say e* is type IV, *if e is not type I and, for every minimal separator S, either both $v, w \in S$ or both $v, w \notin S$*

Notice that this definition induces a partition of the edges in F. Moreover, the definition of type IV edge can be restated as follows: e *is type IV if for every minimal separator S such that e and S intersect, then v and w are both vertices in S.*

Theorem 2.14. *Let $G = (V, E)$ be a connected interval graph and let $H = (V, E \cup F)$ be a completion of G to proper interval. If H is minimal, then every edge e in F is either type I or type II.*

Proof. Suppose H is minimal. We will see that every edge is either type I or type II. Toward a contradiction, suppose there is an edge e in F such that e is either a type III or type IV edge. If e is removed, then we will find a subset F' of F for which $H' = (V, E \cup F')$ is a completion of G to proper interval.

Case (1) Suppose the edge e is type III.

Since e is type III, there is a minimal separator S and a nucleus A_1 such that $e = (s, v)$, with s in S, v in A_1. We denote $F' = F \setminus \{e\}$.

If H is minimal and e in F is deleted, then the resulting graph $H' = H \setminus \{e\}$ is either not an interval graph, or H' contains an induced claw. Hence, by Theorems 1.4 and 1.5, we have three possible subcases:

1) The resulting subgraph H' contains an induced cycle C_n, with $n \geq 4$ (thus, H' is not a chordal graph), or
2) H' contains an AT (in this case, H' is chordal but H' is not an interval graph), or
3) H' is an interval graph but contains an induced claw (thus, H' is an interval graph and H' is not a proper interval graph).

Let $W \subset V$ a vertex subset, and $F \subset E$ an edge subset. We denote by $N_{W,F}(v)$ to those neighbours of the vertex v in W that are connected to v by edges in F.

Remark 2.15. Let σ_1 be a nuclear ordering for A_1 in H given by $v_1 \leq_{\sigma_1} v_2 \leq_{\sigma_1} \ldots \leq_{\sigma_1} v_t$, such that $v_j = v$ for some j in $\{1, \ldots, t\}$.

Let σ_2 be the -partial- ordering induced by σ_1 in the nucleus A_1 once the edge e is deleted, which we will refer to simply as the induced ordering and which we denote by $\leq_{\sigma_2}$.

Since e is type III, if e is deleted, then we can find a nuclear ordering for A_1. However, we cannot assert that the induced ordering is indeed a nuclear ordering.

A few observations:

— The inclusion $N_{S,F'}(v_j) \subseteq N_{S,F'}(v_{j+i})$ holds for every i in $\{1, \ldots, t-j\}$, thus, considering the edge set $E \cup F'$ we see that $v = v_j \leq_{\sigma_2} v_{j+1} \leq_{\sigma_2} \ldots \leq_{\sigma_2} v_t$ holds as for σ_1.

— Suppose $s \in N_{S,F}(v_j)$ and $s \notin N_{S,F}(v_i)$ for every $v_i \leq_{\sigma_1} v_j$. Then, the induced ordering σ_2 does not change for $v_1, \ldots, v_j$.

— Suppose instead that $s \in N_{S,F}(v_j) \cap N_{S,F}(v_i)$ for some $v_i <_{\sigma_1} v_j$, then we set k to be $\min\{i : s \in N_{S,F}(v_i)\}$. Notice that $k < j$. If e is deleted, then $s \notin N_{S,F'}(v_j)$. However, since $N_{S,F}(v_k) \subseteq N_{S,F}(v_j)$ and $s \in N_{S,F'}(v_k)$, then $N_{S,F'}(v_j) \subset N_{S,F'}(v_k)$ and hence we have that $N_{S,F}(v_k) = \ldots = N_{S,F}(v_j)$, since s is the only element removed from the neighbourhood of v_j.

Therefore, the induced ordering σ_2 must necessarily be

$$v_1 \leq_{\sigma_2} \ldots \leq_{\sigma_2} v_{k-1} \leq_{\sigma_2} v_j \leq_{\sigma_2} v_k \leq_{\sigma_2} \ldots \leq_{\sigma_2} v_t$$

Case (1.1) Suppose that if $e = (s, v)$ is deleted, then we find a cycle. Since S and A_1 are cliques and H is chordal, this cycle must have length 4 at the most. Moreover, it is induced by a set $\{v, s, w_1, s_1\}$ for some vertices w in A_1 and s_1 in S such that v is adjacent to w and s_1, and w is adjacent to s.

Since $s_1 \in N_{S,F}(v)$ and $s_1 \notin N_{S,F}(w)$, thus $w <_{\sigma_1} v$ and the inequality is *strict*. By Remark 2.15, if e is deleted, then the induced ordering σ_2 satisfies $N_{S,F'}(w) = N_{S,F'}(v)$. However, $s \in N_{S,F'}(w)$ which results in a contradiction.

Remark 2.16. For each minimal separator S, we can partition the vertices of the graph into 5 disjoint sets: $C_1 \setminus A_1, A_1, S, A_2$ and $C_2 \setminus A_2$ (see Figure 2.1).

Since S, A_1 and A_2 are cliques, the only way two independent vertices may belong to the same set is if they both lie in either $C_1 \setminus A_1$ or $C_2 \setminus A_2$.

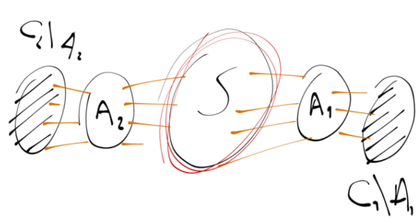


Figure 2.1 – Scheme of the partition of the graph H

Case (1.2) Suppose now that if $e = (s, v)$ is deleted, then there is an AT in the subgraph $H' = (V, E \cup F') = H \setminus \{e\}$ induced by some independent vertices w_1, w_2 and w_3.

Since there are no AT's in H (for H is an interval graph), there is a path $P_{1,2}$ in $H^{'}$ joining w_1 and w_2, such that there is a vertex w in $P_{1,2}$ adjacent to w_3 through the edge e. Hence, w is nonadjacent to w_3 in $H^{'}$. Thus, either $w = v$ and $w_3 = s$, or $w = s$ and $w_3 = v$.

Let us suppose first that $w = v$ and $w_3 = s$.

Claim 2.17. *Under the previous hypothesis, w_1 and w_2 are both in $C_1 \setminus A_1$.*

To prove this, we divide in cases according to the 5 partitions described in Remark 2.16.

First of all, since S is a clique and $w_3 = s$ lies in S, then $w_1 \notin S$ and $w_2 \notin S$. Furthermore, since A_1 and A_2 are cliques, the vertices w_1 and w_2 cannot belong to the same nucleus.

On one hand, we may assert that $w_1 \notin C_2$, for if this is the case, since w lies in C_1 and w is a vertex in $P_{1,2}$, then the path $P_{1,2}$ goes through the set S and thus, the path contains at least one neighbour of w_3 in S, which results in a contradiction for w is, by hypothesis, the only vertex adjacent to w_3 in the path $P_{1,2}$.

In an analogous way, we may assert that it is not possible to have w_1 in A_1 and w_2 in $C_1 \setminus A_1$, for we cannot find a path joining s and w_2 without going through neighbours of w_1 in A_1.

Therefore, the only remaining possibility is w_1 and w_2 in $C_1 \setminus A_1$. □

Let us study now the relationship between w and w_1, w_2. A couple of observations:

(1) There is no path joining w_1 and w_2 entirely contained in $C_1 \setminus A_1$, for if this was the case, then we can find an AT in H, which results in a contradiction since H is an interval graph.

(2) Since the set $\{w_1, w_2, w_3\}$ induces an AT in $H^{'}$ and w_3 is adjacent to w through e, the vertex w is nonadjacent to either w_1 or w_2 for if not, then we find a claw in H induced by $\{w_1, w_2, w, w_3\}$. Notice that this implies that the set $N_{A_1}(w_1) \cap N_{A_1}(w_2)$ is empty, since by definition every vertex in a nucleus is adjacent to at least one vertex in the separator, and thus the same argument holds.

Summing up the results in (1), (2) and Claim 2.17, w is nonadjacent to either w_1 or w_2, and thus there are vertices v_1 and v_2 in A_1 such that v_1 is adjacent to w_1 and is nonadjacent to w_2, and analogously v_2 is adjacent to w_2 and is nonadjacent to w_1. Notice that v_1 is adjacent to v_2 since they both lie in the same nucleus.

Suppose first that $w \neq v_1$ and $w \neq v_2$. Hence, the path $w_1 \to v_1 \to v_2 \to w_2$ joins w_1 and w_2 in H and contains no neighbour of w_3, therefore $\{w_1, w_2, w_3\}$ is an AT in H, which results in a contradiction.

Suppose now that $w \neq v_1$ and $w = v_2$. First of all, if $N_{A_1}(w_1) \cap N_{A_1}(s)$ is nonempty, then we can find a w_1, w_2-minimal separator such that e belongs to one of the nucleus as follows: Let $S' = N_{A_1}(w_1)$. Since there is no path connecting w_1 and w_2 entirely included in $C_1 \setminus A_1$, S' results in a minimal separator such that e lies in one of the nucleus, which is not possible since e is type III.

Hence, $N_{A_1}(w_1) \cap N_{A_1}(s)$ is empty. Let x be a vertex in $N_{A_1}(w_1)$ such that x is nonadjacent to s. Since x in A_1 and using the definition of nucleus, there is a vertex s_1 in S such that s_1 is adjacent to x and $s_1 \neq w_3$. Since $w = v$ is adjacent to $w_3 = s$ in H and x is nonadjacent to w_3, thus $w >_{\sigma_1} x$ and therefore w is adjacent to z for every z in $N_S(x)$. In particular, w is a neighbour of s_1 (see Figure 2.2).

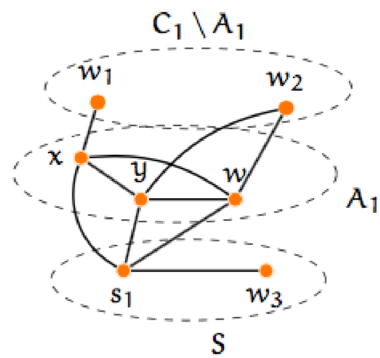


Figure 2.2 – $w_1, w_2 \in C_1 \setminus A_1$ nonadjacent; $w_3 = s$ and $w = v$.

Let w' in A_2 adjacent to s_1. We have the following paths:

$$P_1 : w' \to s_1 \to x \to w_1$$

$$P_2 : w' \to s_1 \to w \to w_2$$

$$P_3 : w_1 \to x \to w \to w_2$$

None of these paths goes through neighbours of the excluded vertex in each case, and $e \notin P_i$ for each $i = 1, 2, 3$. Therefore, $\{w', w_1, w_2\}$ induces an AT in H and this contradicts the hypothesis of completion.

Conversely, suppose that $w = s$ and $w_3 = v$. It is straightforward that w_1 and w_2 do not belong to A_1, for $w_3 \in A_1$ and A_1 is a clique. Moreover, if w_1 lies in $C_1 \setminus A_1$, then every path joining w_1 and w_2 goes through neighbours of w_3 in A_1, unless such a path is entirely contained in $C_1 \setminus A_1$, including both vertices w_1 and w_2. Moreover, notice that if there is a path joining w_1 and w_2 entirely contained in $C_1 \setminus A_1$, then we find an AT in H given by $\{w_1, w_2, w_3\}$, for we have a path joining w_1 and w_2 that does not contain the edge e and the paths in H' joining every other pair of vertices in the AT, which results in a contradiction.

Hence, if there is a path joining w_1 and w_2 that goes through s to avoid every other neighbour of w_3, then w_1 must lie in S and w_2 in C_2, for they do not belong to the clique A_1 and also they do not lie in $C_1 \setminus A_1$. Furthermore, $w_2 \notin C_2$ since any path joining w_2 and w_3 goes through neighbours of w_1 in S, therefore this case is not possible either.

Case (1.3) Suppose that we delete e and find an induced claw. Such a claw is induced by v, s and two more vertices w_1 and w_2.

Since v and s are nonadjacent in H', w_1 is nonadjacent to s and v, and w_2 is adjacent to v, s and w_1. If w_1 in $C_1 \setminus A_1$, then we can find a subset T of $N_{A_1}(w_1)$ such that T is a w_1, v-minimal separator. Since w_2 is adjacent to v, w_1 and s, then e is contained in one of the nucleus of T, which results in a contradiction since e is not type I.

The other possibility, is having a vertex w_2 in S adjacent to w_1, v and s, and w_1 in A_2 nonadjacent to s.

By Lemma 1.2, there is exactly one w_1, s–minimal separator T such that $T \subset N(w_1)$. Applying the definition of w_1, s–minimal separator and since $N_S(w_1) \subseteq N_S(s)$, then w_1 lies in one of the nucleus $A_1(T)$ and $s \in A_2(T)$. Furthermore, $v \notin T$ and w_2 in T, thus e is contained in the nucleus $A_1(T)$, for w_2 is adjacent to both v and s, and this contradicts the hypothesis of e not being a type I edge.

Therefore, since for every subcase 1, 2 and 3 the hypothesis of minimality does not hold, then the edge e is not type III.

Case (2) Suppose that the edge e is type IV.

Let S be a minimal separator such that $e = (s_1, s_2)$ for s_1 and s_2 in S. Suppose first that s_1 is not universal in H, thus there is a vertex v in V nonadjacent to s_1. By Lemma 1.2, there is exactly one v, s_1-minimal separator S' contained in $N(s_1)$. Suppose without loss of generality that v in $A_1(S')$ and s_1 in $A_2(S')$. Since s_2 in $N(s_1)$, hence s_2 in $A_2(S')$ or s_2 in S', which results in a contradiction since e is type IV. Therefore, s_1 is a universal vertex and the proof is analogous by symmetry for s_2.

Notice that, since s_1 and s_2 are universal vertices in H, for each minimal separator S, the sets $C_i(S) \setminus A_i(S)$ are empty for $i = 1, 2$.

If the edge e is deleted, then the resulting graph H' is not chordal and has two kinds of cycles: the ones induced by the vertices s_1, s_2, any vertex v_1 in A_1 and any vertex v_2 in A_2, and, if $|S| > 2$, the cycles induced by the vertices s_1, s_2, any vertex v in a nucleus A_i and some other vertex s_3 in S.

In the sequel, we will find a subset J of fill edges such that the proper subset $F \setminus (J \cup \{e\})$ of F results a completion of the original graph G to a proper interval graph, and thus contradicting the minimality of H.

Case (2.1) We will suppose first that S has exactly three elements s_1, s_2 and s_3, and once this is proved we will see the case $|S| = 2$.

Let B_i, B_j be a partition of the nucleus A_1. Thus, $|B_i| = i, |B_j| = j$ for some $i, j = 0, \ldots, |A_1|$ and $i + j = |A_1|$.

For each partition B_i, B_j of the vertices in the nucleus A_1, we denote $F_{i,j}$ to the edge subset $\{(s_1, b) : b \in B_j\} \cup \{(s_2, b) : b \in B_i\}$. Analogously, we define $F'_{i,j}$ for every partition D_i, D_j of the vertices in the nucleus A_2.

Let a_1 in A_1 and a_2 in A_2. Both vertices are adjacent to s_1 and s_2. When e is deleted, there is a C_4 in H' induced by the set $\{a_1, s_1, a_2, s_2\}$. Thus, there is either a partition B_i, B_j of A_1 for some $i, j = 0, \ldots, |A_1|$, $i + j = |A_1|$, such that $F_{i,j}$ is a subset of F, or there is a partition D_i, D_j of A_2 for some $i, j = 0, \ldots, |A_2|$, $i + j = |A_2|$, such that $F'_{i,j}$ is a subset of F. This follows, for if not, G would not be not chordal since s_1 and s_2 are universal vertices and thus, in particular, s_1 and s_2 are adjacent to every vertex in A_1 and A_2.

Suppose without loss of generality that there is a partition B_i, B_j of A_1 such that $F_{i,j}$ is a subset of F and $F_{i,j} \neq \emptyset$.

Furthermore, let a_2 in A_2, b_1 in B_i and b_2 in B_j. Since A_1 is a clique, the subset $\{b_1, b_2, s_2, a_2, s_1\}$ induces a cycle in $H \setminus (F_{i,j} \cup \{e\})$. Hence, the edge subset $F_1 = \{(b_1, b_2) : b_1 \in B_i, b_2 \in B_j\}$ is a subset of F.

Let B_i, B_j be a partition of the nucleus A_1 as stated above. For each partition B_i,B_j, we denote $X_{i,j}(A_1)$ to the subgraph of H resulting of deleting the edge e, every edge in F_1, and every edge in $F_{i,j}$. We denote $X_{i,j}(A_2)$ to the subgraph of H defined analogously by a partition D_i, D_j of the nucleus A_2.

For a graphic idea of this definition see Figure 2.3.

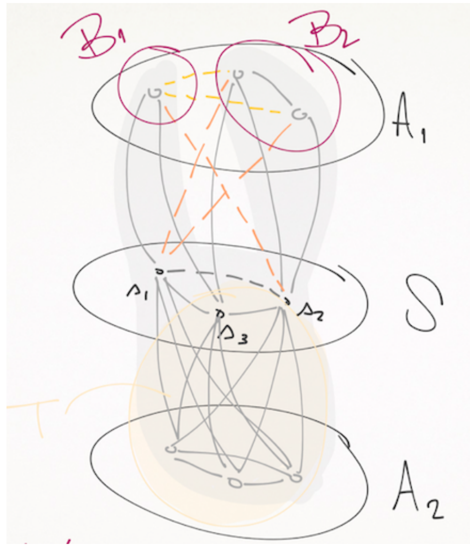


Figure 2.3 – An example of a subgraph $X_{i,j}(A_1)$.

As a consequence of the previous paragraphs, we have the following claim.

Claim 2.18. *Under the previous hypothesis, there is either a partition* B_i, B_j *of the nucleus* A_1 *or a partition* D_l, D_k *of the nucleus* A_2 *such that* G *is a subgraph of* $X_{i,j}(A_1)$ *or* $X_{l,k}(A_2)$.

Suppose without loss of generality that B_i, B_j is a partition of A_1 such that G is a subgraph of $X_{i,j}(A_1)$, and let $J = F_1 \cup F_{i,j} \cup \{e\}$ be the subset of every fill edge in H that was deleted to obtain $X_{i,j}(A_1)$.

Remark 2.19. There is no independent set of size 3 or more in $X_{i,j}(A_1)$.

Toward a contradiction, suppose there are independent vertices. Hence, the only possibility is v in A_1, w in A_2 and s in S. Remember that s_1 and s_2 are universal vertices, s is nonadjacent to both v and w. Thus, since the vertices s_1 and s_2 are complete to A_2 in the subgraph $X_{i,j}(A_1)$, then $s \neq s_2$ and $s \neq s_1$. On the other hand, let s in S such that $s \neq s_1$ and $s \neq s_2$. If there are vertices v_1 in A_1 and v_2 in A_2 such that s is nonadjacent to both v_1 and v_2, then we find a claw in H induced by $\{s_1, s, v_1, v_2\}$. Hence, s is complete in H to either A_1 or A_2. Since J does not contain any edges for which s is an endpoint, then s is complete in $X_{i,j}(A_1)$ to either A_1 or A_2. Therefore, it is not possible to find three independent vertices in $X_{i,j}(A_1)$. Moreover, this also proves that there are no AT's in $X_{i,j}(A_1)$.

If $i = 0$, then $j = |A_1|$ and it is easy to see by the previous remark that $X_{i,j}(A_1)$ is chordal, AT-free and claw-free. Since $\varnothing \neq J \subseteq F$, then $X_{i,j}(A_1)$ is a completion of G to proper interval graphs and this contradicts the hypothesis of H being minimal.

Suppose that $i > 0$ and $j > 0$. By hypothesis, there are three vertices s_1, s_2 and s_3 in S. If $N_{A_1}(s_3) \subseteq B_i$, since $\varnothing \neq B_i \neq A_1$, then we define the subset of fill edges

$$J_1 = F \setminus \{(s_1, v) \in F : v \in B_j\}$$

Notice that e in J_1. Let $H_1 = (V, E \cup J_1)$. By Remark 2.19, it is clear that the subgraph H_1 is AT-free and claw-free. Moreover, H_1 is chordal, for it is easy to see that either $N_{A_1}(s_1) \subseteq N_{A_1}(s_3) \subseteq N_{A_1}(s_2)$, or $N_{A_1}(s_3) \subseteq N_{A_1}(s_1) \subseteq N_{A_1}(s_2)$. Since J_1 is a proper subset of F, H is not a minimal completion of G and this results in a contradiction.

Analogously, if neither $B_i \not\subseteq N_{A_1}(s_3)$ and $B_j \not\subseteq N_{A_1}(s_3)$, then we define the subset of edges

$$J_2 = F \setminus \{(s_1, v) \in F : v \in B_j \setminus N_{A_1}(s_3)\}$$

We define the subgraph $H_2 = (V, E \cup J_2)$, and thus the same argument used for H_1 holds for H_2.

Case (2.2) If $|S| = 2$, then we claim that any graph $X_{i,j}(A_1)$ is a proper interval graph since it suffices to see that it is chordal and AT-free, thus we contradict the minimality.

Case (2.3) Finally, suppose that $|S| > 3$. If $i = 0$ and $j = |A_1|$, then we use the same argument as if $|S| = 3$. Suppose that $i > 0$ and $j > 0$.

Let X be the subset of S defined as

$$\{x \in S : x \neq s_1, x \neq s_2 \text{ and } x \text{ is not complete to } A_1\}$$

If $X = \emptyset$, then we define the subset of edges J as in the previous case.

Suppose that X is nonempty. Let s_3 in X be a vertex such that $N_{A_1}(s_3) \supseteq N_{A_1}(x)$, for every vertex x in X.

If $B_i \not\subseteq N_A(s_3)$ and $B_j \not\subseteq N_A(s_3)$, then we define the subgraph $H_2 = (V, E \cup J_2)$ as in the previous case with the subset of edges J_2.

If instead either $B_i \subseteq N_A(s_3)$ or $B_j \subseteq N_A(s_3)$, then we define the subgraph $H_1 = (V, E \cup J_1)$ as in the previous case with the subset of edges J_1.

In both cases, we find a proper subgraph H_i of H such that H_i is a completion of G to proper interval, and this results in a contradiction of the minimality.

Therefore, if the completion is minimal, then there are no type III or type IV edges.

□

Final remarks and future work

The main results in this thesis are Theorem 4.1 in Chapter 4 of Part I, and Theorem 2.14 in Chapter 2 of Part II. In Theorem 4.1, we give a characterization by minimal forbidden subgraphs for those split graphs that are circle, and in Theorem 2.14 we state and prove a necessary condition for a completion to proper interval graphs to be minimal when the input graph is an interval graph.

Part I

Chapters 2 and 3, were devoted to build the foundations and necessary tools to prove Theorem 4.1. More precisely, we define 2-nested matrices and then state and prove a characterization of these matrices by forbidden subconfigurations that allows us to represent and characterize the adjacency matrices of those split graphs studied in Chapter 4. Some of the results given in Chapter 3 have been published in [30], and the remaining results are being prepared in a manuscript to be submitted for publication. In Chapter 4 we address the problem of characterizing circle graphs when restricted to split graphs. In turn, this chapter is divided into 5 sections: an introduction to the known structural characterizations of circle graphs, and one section for each case of Theorem 4.1. This work resulted in a characterization by forbidden induced subgraphs for those split graphs that are also circle. For its part, this result will be shortly submitted for publication.

We leave some possible continuations of this work.

— We have found a characterization by forbidden induced subgraphs for those split graphs that are also circle. Are all the subgraphs given in Theorem 4.1 also minimally non-circle?

— Recall that split graphs are those chordal graphs for which its complement is also a chordal graph, and that the graph A''_n with $n = 3$ depicted in Figure 2.4 is a chordal graph that is neither circle nor a split graph. It follows from this example that Theorem 4.1 does not hold if we consider chordal graphs instead of split graphs, for there are more forbidden subgraphs that are not considered in the given list. However, Theorem 4.1 is indeed a good first step to characterize circle graphs by forbidden induced subgraphs within the class of chordal graphs, which remains as an open problem.

— Given that split graphs can be recognized in linear-time: is it possible to recognize a split circle graph in linear-time?

— Another possible continuation of this work would be studying the characterization of those circle graphs whose complement is also a circle graph.

— Characterize Helly circle graphs by forbidden induced subgraphs. The class of Helly circle graphs was characterized by forbidden induced subgraphs within circle graphs in [10]. Moreover, it would be interesting to find a decomposition analogous as the split decomposition is for circle graphs, this is, such that Helly circle graphs are closed under this decomposition.

Part II

In Chapter 2, we give some properties regarding the ordering of the vertices of an interval graph using minimal separators which hold both for interval and proper interval graphs, and we define a partition of the fill edges according to their relationship with the minimal separators of the graph. In the last part of this chapter, given a completion H to proper interval graphs of an interval graph G, we state and prove a necessary condition for H to be minimal.

With regard to the minimal completion problem studied in Chapter 2 of Part II, we have the following conjectures:

Conjecture 2.1. *We conjecture that the only if case of Theorem 2.14 holds. Furthermore, in that case the complexity of completing minimally to proper interval graphs when the input is an interval graph is polynomial.*

Conjecture 2.2. *The minimum completion to proper interval graphs when the input graph is interval is NP-complete.*

We would like to continue working on these conjectures in order to obtain a stronger result for an article.

Following a similar line as the one that led to the problem studied in Chapter 2, it remains as an open problem the characterization and complexity of minimum and minimal completions to proper circular-arc graphs, when the input graph is circular-arc.

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