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Abstract

In this thesis, we are interested in random walks related to Galton-Watson trees. Firstly, we study the λ-biased walk on a Galton-Watson tree, and deduce the scaling limit of its cover time, i.e. the time that every vertex is visited. Secondly, we study the capacity, in the sense of potential theory, of a branching random walk in $\mathbb{Z}^d$, and deduce its asymptotic for all dimensions $d \geq 3$. Thirdly, we study the spread of a branching random walk conditioned on rarely survival, and give its limiting behavior in the sense both of genealogy and of spatial distribution.

Résumé

Dans cette thèse, on s'intéresse aux marches aléatoires liées à l'arbre de Galton-Watson. Premièrement, on étudie la marche λ-biaisée sur un arbre de Galton-Watson, et en déduit la limite de son temps de recouvrement, c'est-à-dire la durée pour visiter chaque sommet dans l'arbre. Deuxièmement, on étudie la capacité, au sens de la théorie du potentiel, d'une marche aléatoire branchante dans $\mathbb{Z}^d$, et en déduit son asymptotique pour toutes les dimensions $d \geq 3$. Troisièmement, on étudie l'étendue d'une marche aléatoire branchante conditionnellement à une survie rare, et fournit son comportement limite au sens de la généalogie et de la distribution spatiale.

Contents

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Chapter 1

Introduction

This thesis mainly concerns branching random walks and biased random walks on Galton-Watson tree. We explore the fundamental concepts in Section 1.1, and introduce three distinct problems as follows. In Section 1.2, we study the cover time for biased random walks on supercritical Galton-Watson trees; in Section 1.3 we treat the capacity for critical branching random walks; and in Section 1.4 we consider the spatial spread of branching random walks conditioned on rarely survival. The following Chapter 3 to Chapter 6 are autonomous, presenting proofs for our results, based on [18], [21], [19], and [20], respectively.

1.1 Galton-Watson trees and branching random walks

Intuitively, a Galton-Watson tree is a discrete branching structure, where each particle at each generation splits into randomly many new particles independently, according to the same offspring distribution. Further, a branching random walk can be viewed as a random walk indexed by a Galton-Watson tree, in other words, the Galton-Watson tree describes the genealogy of individuals in a population, whose displacements are given by the random walk.

1.1.1 Planar trees and Galton-Watson trees

We define a planar tree as a set of integer sequences, $T \subset \cup_{n\geq 0}\mathbb{N}_+^n$, such that

- The root $\varnothing \in T$, where by convention we denote $\mathbb{N}_+^0 = \{\varnothing\}$.
- If a node $u = (u_1, \cdots, u_n) \in T$, then its parent $\overleftarrow{u} := (u_1, \cdots, u_{n-1}) \in T$.

- For each node $u = (u_1, \cdots, u_n) \in T$, there exists an integer $k_u(T) \geq 0$ called its number of children, such that for every $j \in \mathbb{N}, (u_1, \cdots, u_n, j) \in T$ if and only if $1 \leq j \leq k_u(T)$.

As conventional notations, we say one node $u = (u_1, \cdots, u_n) \in T$ is an ancestor of another one $u' = (u'_1, \cdots, u'_{n'}) \in T$, denoted by $u \prec u'$, if $n < n'$ and $u_i = u'_i$, $1 \leq i \leq n$. We also define the height (generation) of a node to be its length as a word, i.e. if $u = (u_1, \cdots, u_n)$, then $|u| = n$. Moreover, we denote by $\#T$ the total number of nodes, and by Z_n the total population of generation n.

On a tree T, we have an natural ordering known as the lexicographical order. Since each node of T belongs to $\cup_{n\geq 0}\mathbb{N}_+^n$, we can put them in lexicographical order as words, and explore the tree as a sequence of nodes

$$u_0 = \varnothing, u_1, u_2, \cdots$$

We remark that each node appears exactly once in this sequence if the tree is finite, thus if $\#T = n$, the sequence terminates at u_{n-1}.

Given a distribution μ on $\mathbb{N}$, we can construct a probability measure on the set of trees, denoted by P_μ, so that for all nodes u,

$$k_u \overset{i.i.d.}{\sim} \mu \text{ under } P_\mu.$$

This random object is called a Galton-Watson tree. To avoid trivial cases, we always assume that $\mu(0) + \mu(1) < 1$.

We present here two fundamental properties of Galton-Watson trees concerning the behavior of populations. In the following, we denote $m := E_\mu Z_1$ as the average number of children, and $f(x) := E_\mu\big[x^{Z_1}\big]$ as the generating function in a Galton-Watson tree.

Theorem 1.1.1 (Athreya and Ney [17, Theorem 1.5.1]). *Given that $\mu(0) + \mu(1) < 1$, $m < \infty$, let q be the smallest non-negative solution of $f(x) = x$, then the Galton-Watson tree with offspring distribution μ extincts with probability*

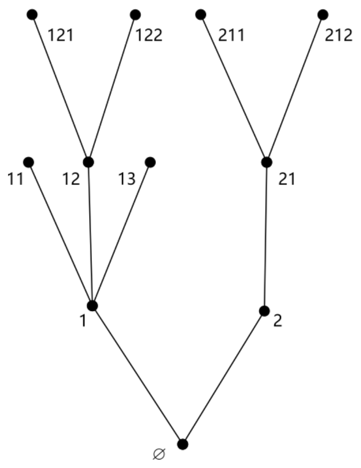
$$P_\mu(T \textit{ extincts}) = q.$$

In particular, the case $m > 1$ is called supercritical, where $q < 1$; the cases $m = 1$ and $m < 1$ are called critical and subcritical respectively, where $q = 1$.

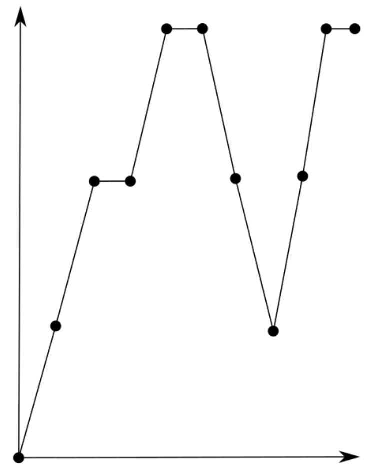
Theorem 1.1.2 (Kesten-Stigum [62]). *For $m < \infty$, $M_n := (\frac{Z_n}{m^n})_{n\geq 0}$ is a non-negative martingale converging almost surely to M_∞. If $m \in (1, \infty)$, then*

$$E_\mu\big[Z_1 \log_+ Z_1\big] < \infty \iff P_\mu(M_\infty > 0 \,|\, \textit{non-extinction}) = 1,$$

where we take $\log_+ x = \log(x \vee 1)$ to avoid $\log 0$.



tree T



height process $(H_i) = (|u_i|)$

Galton-Watson tree is a well-studied fundamental structure, see Athreya and Ney [17] for an overview. Moreover, we remark that the height process $(H_i) := (|u_i|)$ of a critical Galton-Watson tree has the following scaling limit,

Theorem 1.1.3 (Aldous [9])**.** *Let P_μ be the law of Galton-Watson trees with offspring distribution μ, where $E_\mu Z_1 = 1$, $\texttt{Var}(Z_1) = \sigma^2 < \infty$, and $\texttt{gcd}\{j : \mu(j) > 0\} = 1$. Denote by (H^n_i) the height process of the depth-first sequence (u_i), $H^n_i = |u_i|$, under $P_\mu(\cdot \mid \#T = n)$, then as $n \to \infty$, in the space of continuous functions $C[0,1]$,*

$$\left(\frac{\sigma}{\sqrt{n}} H^n_{\lfloor nt \rfloor} \right)_{0 \le t \le 1} \to (|B_t|)_{0 \le t \le 1} \textit{ in distribution,}$$

where (B_t) is the standard Brownian motion.

Further, this result induces a scaling limit for the Galton-Watson tree itself, and there are extensions on subcritical trees and Galton-Watson forests, see Duquesne and Le Gall [42].

1.1.2 Branching random walks

Consider each node on the Galton-Watson tree as a vertex, and add an edge between a node and its parent, then one can see T as a connected graph. If we attach a vector $\mathbf{d}_u$ in any vector space $\mathbb{V}$ (where we normally take $\mathbb{V} = \mathbb{R}^d$) to each directed edge $(\overleftarrow{u}, u)$, fix the position of the root at $V_\varnothing = 0$ and let $V_u = \sum_{u' \preceq u} \mathbf{d}_{u'}$, then $(V_u)_{u \in T}$ forms a spatial tree. Following the spirit that all displacements are i.i.d., we define a branching random walk as the random object constructed above such that

$$\left(k_u, (\mathbf{d}_{u'})_{\overleftarrow{u'}=u}\right) \text{ are i.i.d.}$$

Furthermore, we shall be interested in branching random walks with the auxiliary independence that

$$\mathbf{d}_{u'} \overset{i.i.d.}{\sim} \theta,$$

independent of offspring distributions (k_u). The corresponding probability measure is denoted by $P_{\mu,\theta}$.

The branching random walk have received extensive studies, see Shi [89] and references therein for an overview. A remarkable problem for the branching random walk is the position of the left-most particle,

$$M_n := \min\{V_u : |u| = n\}.$$

For this topic one studies the supercritical case $m > 1$, and the asymptotic behavior of M_n is mainly determined by its log Laplace transform

$$\psi(t) := \log \mathbb{E}\left[\sum_{|u|=1} e^{-tV_u}\right].$$

In general, M_n grows linearly,

Theorem 1.1.4 (Hammersley [54], Kingman [63] and Biggins [29]). *Assume that $\psi(0) = \log m > 0$ and $\psi(t) < \infty$ for some $t > 0$, then almost surely on the set of non-extinction,*

$$\lim_{n\to\infty} \frac{1}{n} M_n = \gamma,$$

where

$$\gamma := -\inf_{s>0} \frac{\psi(s)}{s} \in \mathbb{R}.$$

Moreover, the situations for $\psi(t) = \infty$, $\forall t > 0$ are discussed in Gantert [52], and the second order fluctuation for M_n is logarithmic,

Theorem 1.1.5 (Aïdékon [5]). *Let $\mathbb{P}$ denote the probability measure of a general branching random walk. In the boundary case,*

$$\mathbb{E}\left[\sum_{|u|=1} 1\right] > 1,\ \mathbb{E}\left[\sum_{|u|=1} e^{-V_u}\right] = 1,\ \mathbb{E}\left[\sum_{|u|=1} V_u e^{-V_u}\right] = 1,$$

given finite moment conditions on (V_u)*, there exists* $C > 0$ *such that for any* x*,*

$$\lim_{n\to\infty} \mathbb{P}\Big(M_n \geq \frac{3}{2}\log n + x \Big) = \mathbb{E}\big[e^{-Ce^x D_\infty} \big],$$

where D_∞ *is the martingal limit of* $D_n := \sum_{|u|=n} V_u e^{-V_u}$*.*

There are further results on the particles in the neighborhood of the extreme position, see Madaule [80] and references therein.

1.1.3 Randomly biased walks

We can construct a random walk on the Galton-Watson tree based on a branching random walk (V_u). The randomly biased walk is defined as a nearest-neighbor random walk (X_n) on a Galton-Watson tree, starting at $X_0 = \varnothing$, such that its transition probabilities $P(u,u') := \mathbb{P}(X_{n+1} = u' \mid X_n = u, (V_u)_{u\in T})$ are proportional to

$$\mathbf{1}_{\{\overleftarrow{u}=u'\}} e^{-V_u} + \mathbf{1}_{\{\overleftarrow{u'}=u\}} e^{-V_{u'}}.$$

The most studied problem for the biased random walk (X_n) is the behavior of its height $(|X_n|)$. It has been showed in Lyons and Pemantle [78] that, (X_n) is transient if $\inf_{t\in[0,1]} \psi(t) > 0$, and recurrent if $\inf_{t\in[0,1]} \psi(t) \leq 0$. In the transient case, $(|X_n|)$ grows with a polynomial speed,

Theorem 1.1.6 (Hammond [55])**.** *Let* (X_n) *denotes a biased random walk. If* $\psi(0) > 0$*,* $\inf_{t\in[0,1]} \psi(t) > 0$*, and the underlying branching random walk* (V_u) *has independent increments of the form* $P_{\mu,\theta}$*, with* $\theta \in [q_1, q_2]$ *for* $1 < q_1 < q_2$*, then there exists an explicit* $\gamma > 0$ *and an explicit non-degenerate law such that on the set of non-extinction,*

$$n^{-\gamma}|X_n| \text{ converges in distribution.}$$

Further, in the recurrent case $(|X_n|)$ grows much slower. The most studied situation is the critical case $\inf_{t\in[0,1]} \psi(t) = 0$, where exact behavior of $(|X_n|)$ depends on $\psi'(1)$ and $\kappa := \inf\{t > 1 : \psi(t) \geq 0\}$:

Theorem 1.1.7 (Hu and Shi [57])**.** *Let* (X_n) *denotes a biased random walk. If* $\psi(0) > 0$*,* $\inf_{t\in[0,1]} \psi(t) = 0$ *and* $\psi'(1) = 0$*, then on the set of non-extinction,*

$$\frac{1}{(\log n)^2}|X_n| \text{ converges to an explicit distribution.}$$

Theorem 1.1.8 (Aïdékon and de Raphélis [7] for (1.1.1); de Raphélis [35] for (1.1.2) and (1.1.3))**.** *Let (X_n) denotes a biased random walk. If $\psi(0) > 0$, $\inf_{t\in[0,1]} \psi(t) = 0$ and $\psi'(1) < 0$, given finite second moment for (e^{-V_u}), there exists an explicit constant $C > 0$ such that on the set of non-extinction,*

$$\left(\frac{1}{\sqrt{\sigma^2 n}}|X_{\lfloor nt\rfloor}|\right) \to |B| \text{ in distribution}, \qquad \text{if } \kappa > 2, \tag{1.1.1}$$

$$\left(\frac{C\sqrt{\log n}}{\sqrt{n}}|X_{\lfloor nt\rfloor}|\right) \to |B| \text{ in distribution}, \qquad \text{if } \kappa = 2, \tag{1.1.2}$$

$$\left(\frac{C}{n^{1-\frac{1}{\kappa}}}|X_{\lfloor nt\rfloor}|\right) \to H \text{ in distribution}, \qquad \text{if } \kappa \in (1,2), \tag{1.1.3}$$

where B is the standard Brownian motion on $\mathbb{R}$, and H is the height process of an explicit Lévy process.

We remark that all the results above deal with the supercritical case $m > 1$, where the Galton-Watson tree does not extinct with positive probability. In critical and subcritical cases, one can still study similar phenomenons conditioned on non-extinction, for discussions on this topic we refer to Ben Arous and Hammond [26]. There is also a systematic study on the range of (X_n), see Andreoletti and Chen [12] and references therein.

A special case of the biased random walk is when θ is deterministic. Indeed, if $\theta = \delta_{-\log\lambda}$, then it is called the λ-biased random walk, if further $\lambda = 1$, it is the simple random walk. On λ-biased random walks, finer results are possible concerning the speed of the biased random walk:

Theorem 1.1.9 (Lyons, Pemantle and Peres [79])**.** *Let $\lambda_c := \mathbb{E}[\mu q^{\mu-1}]$, then for any $\lambda \in (\lambda_c, m)$, on the set of non-extinction, the speed of the λ-biased random walk (X_n),*

$$l_\lambda := \lim_{n\to\infty} \frac{|X_n|}{n}$$

exists almost surely, and it is a positive constant.

This limit l_λ is explicitly calculated in Aïdékon [6], with boundary cases $\lambda \to m$ and $\lambda \leq \lambda_c$ discussed in Ben Arous, Hu, Olla and Zeitouni [27] and Ben Arous, Fribergh, Gantert and Hammond [25], respectively. We also have central limit theorems, see Peres and Zeitouni [83].

1.2 Cover time

In Chapter 3, we shall study the cover time of a Galton-Watson tree by a λ-biased random walk. In general, for a connected finite graph $G = (V, E)$

and a random walk (X_t) on it, its cover time is defined as

$$T^{cov}(G;X) = \inf\left\{t : \{X_s, 0 \le s \le t\} = V\right\}.$$

We remark that the random walk (X_t) can be either discrete or continuous. In fact, most theorems are valid for both cases, and one can easily adapt the proof of one case for the other. In the following we shall use the continuous setting $(X_t)_{t\in\mathbb{R}^+}$, where the walk jumps to adjacent positions in exponential times, for better compatability with the Ray-Knight theorem that we shall introduce later.

To study the cover time, one fundamental idea is to analyse the local times $(L^u(t))$, defined as

$$L^u(t) := \int_0^t \mathbf{1}_{\{X_s=u\}} ds,\ t \in \mathbb{R}^+.$$

Our problem can then be described as studying the minimum of local times,

$$T^{cov}(G;X) = \inf\{t \ge 0 : \min_{u\in V} L^u(t) > 0\}.$$

It is natural then, to study correlations of local times for upper and lower bounds of the cover time.

There are deep results when (X_t) is the (continuous-time) simple random walk on G. The first idea in general is to bound T^{cov} by the number of vertices $\#V = n$,

Theorem 1.2.1 (Feige [47], [48])**.** *Let $G = (V,E)$ be a connected graph with n vertices, let (X_t) be the (continuous-time) simple random walk starting at any fixed vertex on G, then as $n \to \infty$,*

$$(1 - o(1))n \log n \le \mathbb{E}T^{cov}(G;X) \le 4n^3/27.$$

Both bounds in this theorem are optimal, yet there is a large gap in between, since only the cardinality $\#V = n$ is far from enough to describe the connectivity of G. One can easily imagine that a straight line of n vertices should behave very differently from a complete graph of n vertices.

To improve this, for instance, one may involve the hitting time:

Theorem 1.2.2 (Matthews [81])**.** *Let $G = (V,E)$ be a connected graph of n vertices, let (X_t) be the simple random walk starting at any fixed vertex on G, as $n \to \infty$,*

$$\max_{S\subseteq G} \min_{u,v\in S} H(u,v)(\log(\#E) - 1) \le \mathbb{E}T^{cov}(G;X) \le \max_{u,v\in G} H(u,v)(1 + \log n),$$

where $H(u,v) = \mathbb{E}_u\tau_v + \mathbb{E}_v\tau_u$ with $\tau_u = \inf\{t \ge 0 : X_t = u\}$ being the hitting time of u.

Suppose all hitting times are of the same order, then this theorem fixes the scale of the cover time at $T^{cov}(G;X) \asymp H(u,v)\log n$. The next step is to seek a precise asymptotic. To this end, we introduce the main treatment for the cover time with the Discrete Gaussian Free Field (DGFF) and the Ray-Knight theorems.

1.2.1 Graphs as electrical networks

Before introducing the DGFF, we need some concepts on general random walks. We refer to Aldous and Fill [11] for a complete introduction.

Consider a graph $G(V,E)$ as an electrical network, consists of wires such that each edge in $(u,v) \in E$ is a resistor with resistance $R_{u,v}$.

We can arbitrarily choose two vertices $u_1, u_2 \in V$, assign to them voltages $V_{u_1} = 1, V_{u_2} = 0$, and then attach voltages V_u at all other vertices $u, v \in V \backslash \{u_1, u_2\}$ and currents $I_{u,v}$ along (oriented) edges $(u,v) \in E$, such that the following properties are satisfied:

- Kirchhoff's Law:

$$\sum_{v \in V} I_{u,v} = 0, \ \forall u \in V \backslash \{u_1, u_2\}.$$

- Ohm's Law:

$$V_u - V_v = R_{u,v} I_{u,v}, \ \forall (u,v) \in E.$$

We can then define effective resistances as the inverse of the total current,

$$R_{\text{eff}}(u_1, u_2) = \frac{1}{\sum_{u \sim u_1} I_{u_1,u}}.$$

Further, this electrical network is related to the random walk with transition probabilities $P(u,v)$ proportional to $\frac{1}{R_{u,v}}$, in that

Proposition 1.2.3 (Aldous and Fill [11, Proposition 3.10])**.** *Let $G = (V,E)$ be a finite connected graph with the above mentioned electrical network structure, let (X_t) be the (continuous-time) random walk with transition probabilities $P(u,v)$. Let π be the stationary distribution for (X_t), then there exists a universal constant $C > 0$ such that*

$$\pi(u)\mathbb{P}_u(\tau_v < \tau_u^+) = \frac{C}{R_{\mathit{eff}}(u,v)}, \ \forall u, v \in V,$$

where $\tau_v = \inf\{t \geq 0 : X_t = v\}$, and $\tau_u^+ = \inf\{t > 0 : X_t = u\}$.

Conversely, for any reversible random walk, i.e. (X_t) *with transition probabilities* $(P(u,v))$ *and stationary distribution* π *satisfying*

$$\pi(u)P(u,v) = \pi(v)P(v,u),\ \forall u \sim v \in V,$$

we can construct an electrical network such that transition probabilities $(P(u,v))$ *are proportional to inverse of resistances* $\frac{1}{R_{u,v}}$.

We remark that this electrical network is unique up to a constant factor. We can set a canonical electrical network with $C = 1$ in Proposition 1.2.3. Further, one can easily verify that the simple random walk, and more generally the randomly biased walk in Section 1.1.3 are reversible random walks, and we shall use their corresponding (canonical) electrical networks without further notifications.

We remark that, under the viewpoint of electrical networks, the effective resistance / conductance between points becomes a fundamental parameter. In our case, the study for cover time only requires resistances between two points on a tree, that is the sum of resistances along the path. An interesting case is the effective conductance between the root and generation n of a Galton-Watson tree: for this study, a canonical setting is to consider supercritical Galton-Watson trees $m > 1$, and on each edge at generation n one attaches a conductance independently distributed as $m^n\zeta$. Then the effective conductance C_n between the root and generation n satisfies

Theorem 1.2.4 (Chen, Hu and Lin [31]). *Assume that* ζ *and the offspring distribution* μ *has finite moments as follows: for* $X \sim \zeta$ *and* $Y \sim \mu$,

$$\mathbb{E}[X^3 + X^{-1} + Y^4] < \infty,$$

then on the set of non-extinction, $\frac{C_n}{\mathbb{E}[C_n]}$ *converges in distribution to an explicit measure. Moreover, there exist explicit constants* c_0, c_1, c_2 *such that*

$$\mathbb{E}[C_n] = \frac{1}{c_1 n} - \frac{c_2}{c_1^2}\frac{\log n}{n^2} - \frac{c_0}{c_1^2}\frac{1}{n^2} + O\left(\frac{(\log n)^2}{n^3}\right).$$

One may also take the conductance on each edge in the form of a branching random walk instead, in this case there are different asymptotic behaviors, see Rousselin [87].

1.2.2 Discrete Gaussian free field

Given $G = (V,E)$ and the electrical network on it, a Discrete Gaussian free field (DGFF) $(\eta_u)_{u\in V}$ (pinned at $u_0 \in V$) is then defined as a set of centered Gaussian variables indexed by vertices of G, such that

$$\mathbb{E}(\eta_u - \eta_{u'})^2 = R_{\text{eff}}(u,u'),\ \eta_{u_0} = 0.$$

We remark that its continuous analog is the Gaussian free field on $\mathbb{R}^d$, attaching a Gaussian distribution at every point in $\mathbb{R}^d$ and pinned at 0. On each bounded domain, the continuous Gaussian free field can be approximated by DGFF on the corresponding $(\frac{1}{n}\mathbb{Z})^d$ lattice. Gaussian Free fields are also related to quantum gravity in physics, and there are extensive researches on these topics. See Sznitman [91] for an overview.

What we are most interested in is the following Ray-Knight theorem that couples the DGFF of a graph to its local times, also known as Dynkin's isomorphism theorem:

Theorem 1.2.5 (Eisenbaum et al. [46]). *For any connected finite graph $G = (V, E)$, let (X_t) be the (continuous-time) simple random walk on G, and let $(\eta_u)_{u\in\mathbf{V}}$ be the DGFF on G. For any $t \in \mathbb{R}^+$,*

$$\left\{L^u(t) + \eta_u^2 : u \in V\right\} \stackrel{d}{=} \left\{(\eta_u + \sqrt{t})^2 : u \in V\right\}.$$

Recall that the cover time depends on minimum of local times, this theorem, first proved by Ding, Lee and Peres [39] and later improved by Zhai [93], further converts the problem to extreme values of the DGFF.

Theorem 1.2.6 (Ding, Lee and Peres [39], Zhai [93]). *Let $G = (V, E)$ be a connected graph, and let (X_t) be the (continuous-time) simple random walk starting at $u_0 \in V$. Denote by (η_u) the DGFF pinned at u_0, then there exist $c, C > 0$ such that, for any $s > 0$,*

$$\mathbb{P}\left(\left|T^{cov}(G; X) - M^2\#E\right| \geq \left(\sqrt{sR}M + sR\right)\#E\right) \leq Ce^{-cs},$$

where $M = \mathbb{E}(\max_{u\in V} \eta_u)$, $R = \max_{u,u'\in V} R_{eff}(u, u')$.

Note that this theorem gives the first-order estimate of the cover time for any graph, sharper estimates are then anticipated, starting with specific graphs. In fact, we consider trees as the most promising graphs for such estimates, because the electrical networks based on trees are simple, in that effective resistances are just sums of resistances along paths. Therefore, the covariances for the DGFF are explicit, and the DGFF can be reduced to a branching random walk, by assigning a Gaussian random variable with variance equal to the resistance along any edge.

1.2.3 Cover time on trees

With the help of the DGFF, one can then fully characterize local times on trees. For instance, for the simple random walk we have that

Lemma 1.2.7 (Bai [18, Lemma 2.3]). *Let $s, t \in \mathbb{R}^+$. If (X_t) is the (continuous-time) simple random walk, and $G = (V, E)$ is a tree, let $u, v \in V$ such that $v \prec u$, then*

$$L^u(t) \sim \mathtt{PG}\left(\frac{t}{|u|}, \frac{1}{|u|}\right),$$
$$(L^u(t)|L^v(t) = s) \sim \mathtt{PG}\left(\frac{s}{|u|-|v|}, \frac{1}{|u|-|v|}\right),$$

where $\mathtt{PG}(a, b)$ stands for the distribution of $\sum_{i=1}^{P} E_i$, where P and E_i are independent random variables such that $P \sim \mathtt{Poisson}(a)$ has Poisson distribution of expected value a, and $E_i \sim \mathtt{Exp}(b)$ has exponential distribution of expected value $\frac{1}{b}$.

By a second moment method based on local times, Ding and Zeitouni [40] gave the second order asymptotics with error $O((\log\log n)^8)$ for simple random walk on binary trees, it was then refined to $O(1)$ by Belius, Rosen and Zeitouni [23] using a Bessel estimate. Further, a scaling limit was established independently by Cortines, Louidor and Saglietti [32] and Dembo, Rosen and Zeitouni [37],

Theorem 1.2.8 (Cortines, Louidor and Saglietti [32], Dembo, Rosen and Zeitouni [37]). *Let T_n^{cov} denote the cover time of the first n generations of a complete binary tree by the continuous-time simple random walk, then as $n \to \infty$,*

$$\mathbb{P}\left(\frac{T_n^{cov}}{2^{n+1}n} - n\log 2 + \log n \le s\right) = \mathbb{E}\exp\left(-CZe^{-s}\right),$$

for some explicitly constructed constant $C > 0$ and distribution Z.

This scaling limit is based on, by studying correlations of local times, the observation that: At time close to the expected cover time, those vertices left uncovered form some almost independent clusters.

Our result extends this estimate to more general trees and random walks,

Theorem 1.2.9 (Bai [18]). *Let T_n^{cov} denote the cover time of the first n generations of a Galton-Watson tree T by the continuous-time λ-biased walk. Suppose $\lambda > 1, \mathbb{E}Z_1 > 1, \mathtt{Var}(Z_1) < \infty$. Let $\mathbf{P}_{\mathrm{w}}$ be the law of the λ-biased random walk. Then for P_μ-almost surely any tree T, conditioned on its first n generations T_n, for any $x \in \mathbb{R}$ and $n \to \infty$, when $\lambda > m$,*

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(\lambda-1)T_n^{cov}}{2\lambda^{n+1}\sum_{i=0}^{\infty}\frac{Z_i}{\lambda^i}} - n\log m - \log W \le x \middle| T_n\right) \to e^{-e^{-x}};$$

when $\lambda = m$*,*

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(m-1)T_n^{cov}}{2m^{n+1}\sum_{i=0}^n Z_i} - n\log m - \log W \le x \middle| T_n\right) \to e^{-e^{-x}};$$

when $1 < \lambda < m$*,*

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(\frac{m}{\lambda}-1)(\lambda-1)T_n^{cov}}{2Wm^{n+1}} - n\log m - \log W \le x \middle| T_n\right) \to e^{-e^{-x}}.$$

The proof is mainly inspired by [32], relying on an variant of the above-mentioned observation: For the λ-biased random walk, at time close to the expected cover time, the remaining uncovered vertices are almost independent. We shall establish this observation by a careful estimate of local times, and complete the proof with its help in Chapter 3.

The same cover time problem for randomly biased walks on Galton-Watson trees is left open. Indeed, on a Galton-Watson tree, there are short branches on the tree T_n (T cut at height n) that extincts in generations 1 to $n-1$, but they are negligible for the λ-biased walk, since covering them are much easier than visiting nodes in generation n. However, this is no longer true for the randomly biased walk. Further difficulties are anticipated in treating these anomalies.

1.3 Capacity of the range

In Chapter 4 and Chapter 5, we shall study the capacity of a branching random walk. For a discrete random walk (X_n) in $\mathbb{Z}^d$, its range is defined as

$$R[s,t] := \{X_s, X_{s+1}, \cdots, X_t\},\ s \le t.$$

Moreover, if $d \ge 3$, we can define its capacity (of the range) with respect to a symmetric probablity law η on $\mathbb{Z}^d$ as

$$\mathsf{cap}_\eta A := \sum_{x\in A} \mathsf{P}_x^\eta(\tau_A^+ = \infty),$$

where P_x^η is the law of a (discrete) random walk (S_n) started at x with transition probability η, and $\tau_A^+ := \inf\{n \ge 1 : S_n \in A\}$ is its first returning time to A. We remark that the continuous version for the size of the range $\#R[0,n]$ is clearly the Lebesgue measure, and that for the capacity is the Brownian capacity defined (in Mörters and Peres [82, Definition 8.17]) as follows: for any bounded set $F \subset \mathbb{R}^d$ $(d \ge 3)$,

$$\mathsf{cap}(F) := \left(\inf\left\{\iint G(x,y)\nu(dx)\nu(dy) : \nu \text{ is a probability measure on } F\right\}\right)^{-1},$$

where $G(x,y) = |x-y|^{2-d}$, or more generally any kernel of the form $G(x,y) = C|x-y|^{-\alpha}$. Moreover, there is an equivalent way to define capacity, namely that

$$\mathsf{cap}_\eta A = C_{d,\eta} \lim_{|x|\to\infty} |x|^{d-2} \mathsf{P}^\eta_x(\tau_A < \infty),$$

where the inverse of the scaling factor, $\left(C_{d,\eta}|x|^{d-2}\right)^{-1}$, is the expected time that a η-random walk starting from 0 hits x. The same definition works for the continuous setting, by changing the discrete random walk η to the Brownian motion.

The (size of) range of an ordinary random walk has been extensively studied since Dvoretzky and Erdős [43], we present here its asymptotic for the simple random walk. For an overview of later results on central limit theorems, large and moderate deviations of the range, see Asselah and Schapira [14].

Theorem 1.3.1 (Dvoretzky and Erdős [43]). *Let (X_n) be a simple random walk in $\mathbb{Z}^d$. As $n \to \infty$,*
if $d \geq 3$,

$$\frac{1}{n} \# R[0,n] \to \mathsf{P}_0(\tau_0^+ = \infty), \text{ almost surely;}$$

if $d = 2$,

$$\frac{\log n}{n} \# R[0,n] \to \pi, \text{ almost surely;}$$

if $d = 1$,

$$\frac{1}{\sqrt{n}} \# R[0,n] \to \sup_{0 \leq t \leq 1} B_t - \inf_{0 \leq t \leq 1} B_t \text{ in distribution,}$$

where (B_t) is a standard Brownian motion on $\mathbb{R}$.

Similar results for the capacity date back to Jain and Orey [58], with refinements to central limit theorems in the series of paper by Asselah, Schapira and Sousi [15], [16], [88] recently. The study for capacity is recently motivated by its connections to random interlacements. The model of random interlacements was introduced in Sznitman [90], serving as a model for scaling limits of traces of simple random walks. Intersection probabilities and potential theory then emerge naturally in the treatment, calling for sharp estimates of the capacity. For an overview of random interlacements, see Drewitz, Ráth and Sapozhnikov [41].

Theorem 1.3.2 (Jain and Orey [58] for (1.3.4) and (1.3.5), Chang [30] for (1.3.6)). *Let (X_n) be a simple random walk in $\mathbb{Z}^d$, let η be the one-step* *distribution* $X_1 - X_0$*, then as* $n \to \infty$*, if* $d \geq 5$*, there exist constants* $c_d > 0$ *such that*

$$\frac{1}{n}\mathtt{cap}_\eta R[0,n] \to c_d, \text{ almost surely;} \tag{1.3.4}$$

if $d = 4$*,*

$$\frac{\log n}{n}\mathtt{cap}_\eta R[0,n] \to \frac{\pi^2}{8}, \text{ almost surely;} \tag{1.3.5}$$

if $d = 3$*,*

$$\frac{1}{\sqrt{n}}\mathtt{cap}_\eta R[0,n] \to \frac{1}{3\sqrt{3}}\mathtt{cap}_{BM}(B[0,1]) \text{ in distribution,} \tag{1.3.6}$$

where (B_t) *here stands for the standard Brownian motion in* $\mathbb{R}^3$*.*

Theorem 1.3.3 (Asselah, Schapira and Sousi [15] for (1.3.7), [16] for (1.3.9), Schapira [88] for (1.3.8))**.** *In the same setting of Theorem 1.3.2, as* $n \to \infty$*, there exists a constant* $C_d > 0$ *such that if* $d \geq 6$*,*

$$\frac{\mathtt{cap}_\eta R[0,n] - \mathbb{E}\mathtt{cap}_\eta R[0,n]}{\sqrt{n}} \to \mathcal{N}(0, C_d) \text{ in distribution;} \tag{1.3.7}$$

if $d = 5$*,*

$$\frac{\mathtt{cap}_\eta R[0,n] - \mathbb{E}\mathtt{cap}_\eta R[0,n]}{\sqrt{n \log n}} \to \mathcal{N}(0, C_d) \text{ in distribution;} \tag{1.3.8}$$

if d=4, there exists an explicit distribution γ *such that*

$$\frac{(\log n)^2}{n}\big(\mathtt{cap}_\eta R[0,n] - \mathbb{E}\mathtt{cap}_\eta R[0,n]\big) \to \gamma \text{ in distribution;} \tag{1.3.9}$$

Further, one can see the critical branching random walk (conditioned on non-extinction) as a generalisation of an ordinary random walk, order the branching random walk by its lexicographical order, then we have results on its range,

Theorem 1.3.4 (Le Gall and Lin [73] for (1.3.10), [72] for (1.3.11); Zhu [94] for (1.3.12))**.** *Let* $P_{\mu,\theta}$ *be the law of a branching random walk, where* μ *is critical with finite variance, and* θ *is symmetric, irreducible with finite support. Let* (X_t) *denote the depth-first sequence under* $P_{\mu,\theta}(\cdot \mid \#T = n)$*. As* $n \to \infty$*, if* $d \geq 5$*, there exists* $c_{\mu,\theta} > 0$ *such that*

$$\frac{1}{n}R[0,n] \to c_{\mu,\theta} \text{ in probability;} \tag{1.3.10}$$

if $d = 4$,

$$\frac{\log n}{n} R[0,n] \to 8\pi^2\sigma^4 \text{ in probability,} \tag{1.3.11}$$

where $\sigma^2 = (\det M_\eta)^{1/4}$*, with* η *denoting the covariance matrix of* η*;*
if $d \leq 3$,

$$n^{-d/4} R[0,n] \to 2^{d/4} (\det M_\theta)^{1/2} \lambda_d(\operatorname{supp} \mathcal{I}) \text{ in distribution,} \tag{1.3.12}$$

where λ_d *is the Lebesgue measure, and* $\mathcal{I}$ *is the random measure on* $\mathbb{R}^d$ *known as Integrated Super-Brownian Excursion.*

Generalising the methods and estimates in [73], we managed to obtain the asymptotics for the capacity of a branching random walk,

Theorem 1.3.5 (Bai and Wan [21] for (1.3.13) and (1.3.14); Bai and Hu [19] for (1.3.15))**.** *Let* $P_{\mu,\theta}$ *be the law of a branching random walk, where* μ *is critical with finite variance, and* θ *is symmetric, irreducible with finite exponential moment. Let* η *be a irreducible distribution with mean* 0 *and finite* $(d+1)$*-th moment. Let* (X_t) *denote the depth-first sequence under* $P_{\mu,\theta}(\cdot \mid \#T = \infty)$*. As* $n \to \infty$*, if* $d \geq 7$*, there exists* $c_{\mu,\theta,\eta} > 0$ *such that*

$$\frac{1}{n} \mathtt{cap}_\eta R[0,n] \to c_{\mu,\theta,\eta} \text{ in probability;} \tag{1.3.13}$$

if $d = 6$ *and* μ *has finite* 5*-th moment, there exists* $c_{\mu,\theta,\eta} > 0$ *such that*

$$\frac{\log n}{n} \mathtt{cap}_\eta R[0,n] \to c_{\mu,\theta,\eta} \text{ in probability;} \tag{1.3.14}$$

if $3 \leq d \leq 5$*, then for any* $\epsilon > 0$*,* $P_{\mu,\theta}(\cdot \mid \#T = \infty)$*-almost surely,*

$$\mathtt{cap}_\eta R[0,n] = n^{-(d-2)/4 + o(1)},\ n \to \infty. \tag{1.3.15}$$

Remark 1.3.6. This theorem is formulated with the Galton-Watson tree conditioned to be infinite, in order to keep consistency in all dimensions. The model can be replaced by the Galton-Watson forest or the Galton-Watson conditioned to be large, with adjustments in the proof (see for instance Lemma 4.3.6). For convenience of the demonstration, in Chapter 4 and Chapter 5, we are going to use the Galton-Watson conditioned to be large and the Galton-Watson forest, respectively. In particular, as will be showed in Lemma 5.2.3, n subtrees in a Galton-Watson forest generally contain $\Theta(n^2)$ vertices, therefore its capacity is of the order $R[0,n^2]$ instead of $R[0,n]$.

Similar structures can be observed among the four theorems above, namely there is a critical dimension with $O(\frac{n}{\log n})$ behavior, and we have a linear increase in high dimensions and a fluctuation in low dimensions. The intuition behind this will be explained in the following subsections, focusing on the capacity for branching random walks.

1.3.1 High dimensions and the infinite model

Let us first look at the capacity for the simple random walk. The following subadditive property follows easily by definition,

$$\mathtt{cap}_\eta R[0,n] \leq \mathtt{cap}_\eta R[0,n/2] + \mathtt{cap}_\eta R[n/2,n].$$

Moreover, the simple random walk clearly has the following translational-invariant property, by which one can deduce its ergodicity

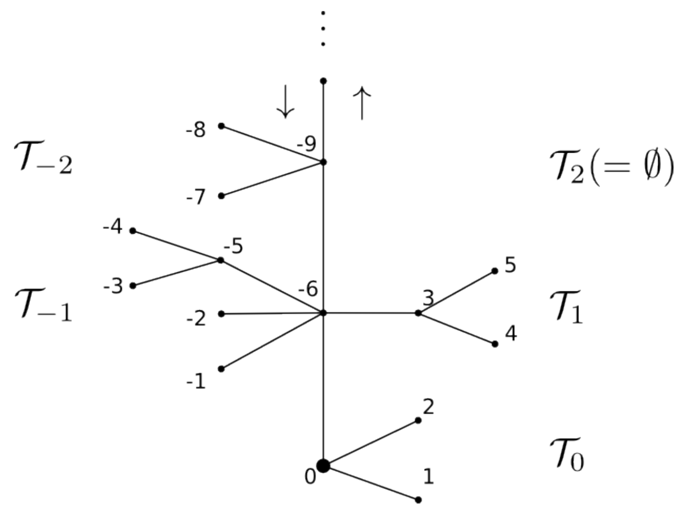
$$R[s,t] - X_s \overset{d}{=} R[0,t-s],\ \forall s \leq t. \tag{1.3.16}$$

One can thus conclude by Kingman's subadditive ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n} \mathtt{cap}_\eta R[0,n]$$

is a constant almost surely and in L^1. Therefore, one can easily show the linear growth in high dimensions by an lower bound of the expected value of the capacity.

As for branching random walks, the problem reduces to establishing a model with similar invariant properties as (1.3.16). Motivated by this, we construct the following model, which can be seen as the infinite version of Aldous [8, Section 2.6], or a generalization of Le Gall and Lin [73, Section 2.2].



We define a two-sided forest to be a sequence of trees with labels,

$$\mathcal{T} = ((0,\mathcal{T}_0),(1,\mathcal{T}_1),(1,\mathcal{T}_{-1}),(2,\mathcal{T}_2),(2,\mathcal{T}_{-2})\cdots),$$

where the roots $(\pm i, \varnothing)$ of $\mathcal{T}_i$ and $\mathcal{T}_{-i}$ $(i > 0)$ are identified (glued together) as one single point. The number of children $k_{(i,u)}(\mathcal{T})$, displacement from parent $\mathbf{d}_{(i,u)}(\mathcal{T})$ and spatial location of a vertex $V_{(i,u)}(\mathcal{T})$ are denoted as for ordinary trees. The roots $(\pm i, \varnothing)$ have the offspring distributions denoted by $k^{\pm}_{(i,\varnothing)}(\mathcal{T})$.

On the set of two-sided forests, we define the following probability measure $\mathbf{P}_{\mu,\theta}$ as a variant for that of a branching random walk $P_{\mu,\theta}$:

- For each $i \geq 0, u \neq \varnothing$,
$$k_{(i,u)}(\mathcal{T}) \overset{i.i.d.}{\sim} \mu,$$
$$k_{(0,\varnothing)}(\mathcal{T}) \sim \mu,$$
and for other nodes $(\pm i, \varnothing)$ $(i > 0)$,
$$\mathbf{P}_{\mu,\theta}(k^{+}_{(i,\varnothing)}(\mathcal{T}) = i, k^{-}_{(i,\varnothing)}(\mathcal{T}) = j) = \mu(i + j + 1).$$
- Displacements $\mathbf{d}_{(i,u)}(\mathcal{T})$ are i.i.d. and distributed as θ on each directed edge, starting at $V_{(0,\varnothing)}(\mathcal{T}) = 0$.

We remark that if we restrict μ to the geometric distribution, and forgets about branches at each instant, leaving only memories on the history from a site to infinity, the resulted model is a discrete snake. Its scaling limit is called a Brownian snake, introduced in Le Gall [70], with various interesting properties and applications mostly in connections with super-Brownian motion semilinear partial differential equations (see Le Gall [75] and [71]).

We shall explain in Chapter 4 that this model exhibits a combinatoric explanation, resulting in its translational invariance property,

Proposition 1.3.7 (Bai and Wan [21, Proposition 2.2])**.** *Let μ be a critical distribution on $\mathbb{N}$ with finite variance, let θ be a symmetric and irreducible distribution on $\mathbb{Z}^d$ with finite exponential moment. Then*

$$R[s,t] - X_s \overset{d}{=} R[0, t-s] \text{ under } \mathbf{P}_{\mu,\theta},\ \forall s \leq t.$$

Further, by showing that $\mathbf{P}_{\mu,\theta}$ is absolutely continuous with respect to ordinary branching random walks, we have linear growth for the capacity in high dimensions.

1.3.2 Critical dimension and low dimensions

The study of the critical dimension requires more precise estimates. To begin with, we define the discrete Green's function with respect to a random distribution η as

$$G_\eta(x,y) = G_\eta(x-y) = \sum_{n=0}^{\infty} \mathsf{P}_0^\eta(S_n = x-y).$$

Then we have that

Lemma 1.3.8 (Lawler [68, Theorem 4.3.5])**.** *Let η be an aperiodic and irreducible distribution on $\mathbb{Z}^d (d \geq 3)$ with mean 0, covariance matrix Γ_η and finite $(d+1)$-th moment. Then*

$$G_\eta(x) = \frac{C_{d,\eta}}{J_\eta(x)^{d-2}} + O(|x|^{1-d}),$$

where

$$C_{d,\eta} = \frac{\mathbf{\Gamma}(\frac{d}{2})}{(d-2)\pi^{d/2}\sqrt{\det \Gamma_\eta}},$$

$\mathbf{\Gamma}(\cdot)$ stands for the Gamma function, and

$$J_\eta(x) = \sqrt{x \cdot \Gamma_\eta^{-1} x}.$$

Further, the following lemma inspired by Lawler [67, Theorem 3.6.1] allows us to establish a relation between the capacity and Green's functions,

Lemma 1.3.9 (Bai and Hu [19])**.** *For $d \geq 3$, and any sequence $(X_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^d$ satisfying (1.3.16), then*

$$\frac{1}{n+1}\sum_{i=0}^{n} \mathbb{E}\left[\mathbf{1}_{\{X_0 \notin \{X_1,\cdots,X_{n-i}\}\}} \mathsf{P}_{X_0}^\eta(\tau^+_{\{X_{-i},\cdots,X_{n-i}\}} = \infty) \cdot \sum_{j=-i}^{n-i} G_\eta(X_0, X_j)\right] = 1.$$

By analysis of the Green's functions, one can show that in dimension $d = 6$, the sum $\sum_{j=-i}^{n-i} G_\eta(X_0, X_j)$ for the two-sided forest is concentrated around $C \log n$, with a weak dependence on the parameter i, thus this lemma enables us to deduce that

$$\begin{aligned}
\mathbf{E}_{\mu,\theta} \mathsf{cap}_\eta R[0,n] &= \sum_{i=0}^{n} \mathbb{E}\Big[\mathbf{1}_{\{X_i \notin \{X_{i+1},\cdots,X_n\}\}} \mathsf{P}_{X_i}^\eta(\tau^+_{\{X_0,\cdots,X_n\}} = \infty)\Big] \\
&= \sum_{i=0}^{n} \mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{\{X_0 \notin \{X_1,\cdots,X_{n-i}\}\}} \mathsf{P}_{X_0}^\eta(\tau^+_{\{X_{-i},\cdots,X_{n-i}\}} = \infty)\Big] \\
&\asymp \frac{n}{\log n}.
\end{aligned}$$

An estimation on the second moment with similar methods show that the capacity is concentrated around its expected value, and we convert from the two-sided forest to ordinary branching random walks by an argument of absolute continuity between them.

In low dimensions, a different approach is needed to show that the discrete setting converge to its continuous analog as in Theorem 1.3.1-Theorem 1.3.4. In fact, for the simple random walk, there is a coupling with the standard Brownian motion in that, for any $\epsilon > 0$, there exists $\gamma > 0$ such that (see for instance Lawler [66, Lemma 3.1])

$$\mathbb{P}\left(\max_{0\leq s\leq n} |X_{\lfloor ds\rfloor} - B_s| > n^{1/4+\epsilon}\right) < e^{-n^\gamma};$$

and for the range of branching random walks, the convergence of range is established through convergence of local times.

Currently we can only provide the order of the capacity in Theorem 1.3.5 using estimates of Green's functions and inequalities derived from lemmas like Lemma 1.3.9.

1.4 Rarely survived branching random walks

In Chapter 6 we consider the spread of the last generation of a branching random walk conditioned on rarely survival. Take a branching random walk $(V_u)_{u\in T}$ on $\mathbb{R}$, conditioned on survival at the n-th generation, their spatial locations $(V_u)_{|u|=n}$ can then be listed in increasing order as

$$V_n^{(1)} \leq \cdots \leq V_n^{(Z_n)},$$

where Z_n is the population of generation n. We study its spatial spread

$$R_n := V_n^{(Z_n)} - V_n^{(1)}$$

and gaps

$$g_n^i := V_n^{(i+1)} - V_n^{(i)},\ 1 \leq i \leq Z_n - 1.$$

The behavior of $(V_u)_{|u|=n}$ conditioned on survival is well-studied, for instance as we have introduced in Theorem 1.1.5. In our case, we study the behavior given the "rarely survival" condition $\{Z_n = k\}$ for k being a deterministic integer. This problem is mainly motivated by [85], [86], where Ramola, Majumdar and Schehr studied the range R_n and gaps (g_n^i) for the critical branching Brownian motion, which is the continuous analogy of our model with geometric μ of expected value 1 and Gaussian θ. We extend the result to general branching random walks,

Theorem 1.4.1 (Bai and Rousselin [20]). *Let $k \geq 2$, let $1 \leq i \leq k-1$. Consider a branching random walk $P_{\mu,\theta}$, where μ has finite variance, and θ has finite exponential moments. There are explicit positive constants C_1, C_2, C_3 such that, as $x \to \infty$,*

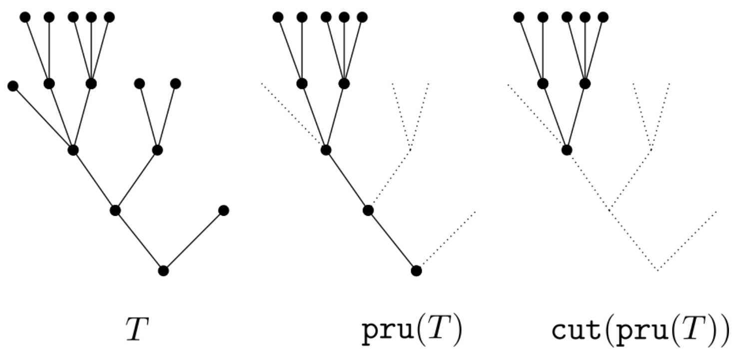
$$\lim_{n\to\infty} P_{\mu,\theta}(R_n > x \,|\, Z_n = k) = \begin{cases} (C_1 + o(1))x^{-2}, & \mathbb{E}[\mu] = 1, \\ \exp(-(C_2 + o(1))x), & \mathbb{E}[\mu] \neq 1, \end{cases}$$

$$\lim_{n\to\infty} P_{\mu,\theta}\big(g_n^i > x \,\big|\, Z_n = k\big) = \begin{cases} (C_1 C_3 + o(1))x^{-2}, & \mathbb{E}[\mu] = 1, \\ \exp(-(C_2 + o(1))x), & \mathbb{E}[\mu] \neq 1. \end{cases}$$

As for related works, a critical Galton-Watson tree conditioned on non-extinction typically satisfies $Z_n = \Theta(n)$, see Curien and Le Gall [34] for a detailed discussion. Moreover, see Abraham, Bouaziz, and Delmas [2] for the 'fat' case $Z_n \gg n$. For supercritical Galton-Watson trees, typically $Z_n = \Theta(m^n)$, see Berestycki et al. [28] for the case $Z_n < \epsilon m^n$.

1.4.1 The prune and cut operation

To study this problem, we first define the prune and cut operations on trees:



- For any tree T, we construct the pruned tree at height n by

$$\mathtt{pru}_n(T) := \{u \in T : \exists v \in T, |v| = n, u \preceq v\}.$$

  By convention, if $Z_n(T) = 0$, we take $\mathtt{pru}_n(T) = \{\varnothing\}$.

- Moreover, we define the cut operation by

$$\phi_n(T) = \bigwedge_{|u|=n,\, u\in T} u,\ h_n(T) = n - |\phi_n(T)|,\ \mathtt{cut}_n(T) = T[\phi_n(T)].$$

  where by convention, $\phi_n(T) = \varnothing$ if $Z_n(T) = 0$.

Clearly, these operations extend to branching random walks (i.e. trees with spatial displacements). Moreover, by construction we have that

$$R_n(T) = R_n(\mathtt{pru}_n T) = R_{H(\mathtt{cut}_n \mathtt{pru}_n T)}(\mathtt{cut}_n \mathtt{pru}_n T),$$

and the same thing applied to gaps. In other words, the study of R_n and g_n^i of a branching random walk reduces to those of the structure $\mathtt{cut}_n \mathtt{pru}_n T$ under $P_{\mu,\theta}(\cdot \mid Z_n = k)$.

We remark that the structure of $\mathtt{cut}_n \mathtt{pru}_n T$ conditioned on non-extinction along converges to the Yule tree, as showed in Curien and Le Gall [34].

1.4.2 Ratio Theorem

As indicated in the previous subsection, our main goal is to study the genealogy structure of $\mathtt{cut}_n \mathtt{pru}_n T$. The key component is then the ratio theorem, which provide transition probabilities for the Markov chain (Z_n):

Proposition 1.4.2 (Athreya and Ney [17, Section 1.7-1.11])**.** *Let μ be an offspring distribution such that*

$$\mu(0), \mu(1) > 0,\ \mu(0) + \mu(1) < 1,\ \mathbb{E}Z_1 < \infty.$$

Let $P_n(i,j)$ denote the transition probability $P_\mu(Z_{k+n} = j \mid Z_k = i)$ for large enough k.

1. *For any $j \geq 1$, there exists a sequence (π_j) such that*

$$\lim_{n\to\infty} \frac{P_n(1,j)}{P_n(1,1)} \nearrow \pi_j \in (0,\infty),$$

where $\nearrow$ means non-decreasing limit.

2. *For any $t \in \mathbb{Z}$, $i, j, k, l \geq 1$,*

$$\lim_{n\to\infty} \frac{P_{n+t}(i,j)}{P_n(k,l)} = \gamma^t q^{i-k} \frac{i\pi_j}{k\pi_l},$$

where q is the extinction probability, f is the generating function, and $\gamma = f'(q)$.

3. *If $\mathbb{E}Z_1 = 1, \sigma^2 := \mathtt{Var}\mu < \infty$, then for any $i, j \geq 1$,*

$$\lim_{n\to\infty} n^2 P_n(i,j) = \frac{2i\pi_j}{\sigma^2 \sum_{k=1}^\infty \pi_k (\mu(0))^k}.$$

4. *If* $\mathbb{E}Z_1 \neq 1$, $\sum_{j=1}^{\infty} j \log j \mu(j) < \infty$, *then for any* $i, j \geq 1$,

$$\lim_{n\to\infty} \gamma^{-n} P_n(i,j) = iq^{i-1} v_j,$$

where (v_j) *is determined by* $Q(s) = \sum_{j=0}^{\infty} v_j s^j, 0 \leq s < 1$, *with* Q *the unique solution of*

$$Q(f(s)) = \gamma Q(s)(0 \leq s < 1),\ Q(q) = 0,\ \lim_{s\to q} Q'(s) = 1.$$

Given this set of genealogical properties, we can show that under $P_\mu(\cdot \mid Z_n = k)$, the tree $\mathtt{cut}_n\mathtt{pru}_n T$ has its first branching $Z_1 = 2$ with high probability, and has no other branches until approaching the bottom of the tree. Further, we deduce that the law of $\mathtt{cut}_n\mathtt{pru}_n T$ converges as $n \to \infty$, and shall conclude on the desired spatial properties based on these observations.

Chapter 2

Introduction (en français)

Dans cette thèse, on s'intéresse essentiellement aux marches aléatoires branchantes et marches aléatoires biaisées sur l'arbre de Galton-Watson. On présente les concepts fondamentaux dans la section 2.1 et introduit trois problèmes distincts par la suite. Dans la section 2.2, on étudie le temps de recouvrement pour les marches aléatoires biaisées sur les arbres de Galton-Watson surcritiques; dans la section 2.3 on traite la capacité pour les marches aléatoires branchantes critiques; et dans la section 2.4 on considère l'étendue d'une marche aléatoire branchante conditionnellement à une survie rare. Les chapitres 3-6 sont autonomes, présentant des preuves de nos résultats, et se sont basés sur [18], [21], [19], et [20].

2.1 Arbres de Galton-Watson et marches aléatoires branchantes

Intuitivement, l'arbre de Galton-Watson est une structure de branchement discrète, où chaque particule à chaque génération donne naissance à de nouvelles particules indépendamment, selon la même loi de reproduction. De plus, une marche aléatoire branchante peut être considérée comme une marche aléatoire indexée par l'arbre de Galton-Watson, en d'autres termes, l'arbre de Galton-Watson décrit la généalogie des individus d'une population, dont les déplacements sont donnés par la marche aléatoire.

2.1.1 Arbres planaires and arbres de Galton-Watson

On définit un arbre planaire comme un ensemble des suites d'entiers, $T \subset \cup_{n\geq 0}\mathbb{N}_+^n$, tel que

- La racine $\varnothing \in T$, où par convention on note $\mathbb{N}_+^0 = \{\varnothing\}$.

- Si un nœud $u = (u_1, \cdots, u_n) \in T$, alors son parent $\overleftarrow{u} := (u_1, \cdots, u_{n-1}) \in T$.
- Pour chaque nœud $u = (u_1, \cdots, u_n) \in T$, il existe un entier $k_u(T) \geq 0$ appelé son nombre d'enfants, tel que pour tout $j \in \mathbb{N}, (u_1, \cdots, u_n, j) \in T$ si et seulement si $1 \leq j \leq k_u(T)$.

Par convention, on dit qu'un nœud $u = (u_1, \cdots, u_n) \in T$ est l'ancêtre d'un autre nœud $u' = (u'_1, \cdots, u'_{n'}) \in T$, noté par $u \prec u'$, si $n < n'$ et $u_i = u'_i$, $1 \leq i \leq n$. On définit également la hauteur (génération) d'un nœud comme sa longueur en tant que mot, i.e. si $u = (u_1, \cdots, u_n)$, alors $|u| = n$. De plus, on note $\#T$ le nombre total de nœuds, et Z_n la population totale de la génération n.

Dans un arbre T, on a un ordre lexicographique: Puisque chaque nœud de T appartient à $\cup_{n\geq 0}\mathbb{N}_+^n$, on peut les mettre dans l'ordre lexicographique comme des mots, et explorer l'arbre comme une suite de nœuds

$$u_0 = \varnothing, u_1, u_2, \cdots$$

On remarque que chaque nœud apparaît exactement une fois dans cette suite si l'arbre est fini, donc si $\#T = n$, la suite se termine à u_{n-1}.

Étant donnée une loi μ sur $\mathbb{N}$, on peut construire une mesure de probabilité sur l'ensemble des arbres, notée P_μ, telle que pour tous les nœuds u,

$$k_u \overset{i.i.d.}{\sim} \mu \text{ sous } P_\mu.$$

Cet objet aléatoire est appelé l'arbre de Galton-Watson. Pour éviter les cas triviaux, on suppose toujours que $\mu(0) + \mu(1) < 1$.

On présente ici deux propriétés fondamentales des arbres de Galton-Watson concernant le comportement des populations. Dans ce qui suit, on note $m := E_\mu Z_1$ comme le nombre moyen d'enfants, et $f(x) := E_\mu\left[x^{Z_1}\right]$ comme la fonction génératrice dans l'arbre de Galton-Watson.

Théorème 2.1.1 (Athreya and Ney [17, Théorème 1.5.1]). *Supposons que $\mu(0) + \mu(1) < 1$, $m < \infty$, soit q la plus petite solution positive de $f(x) = x$, alors l'arbre de Galton-Watson avec la loi de reproduction μ est fini avec probabilité*

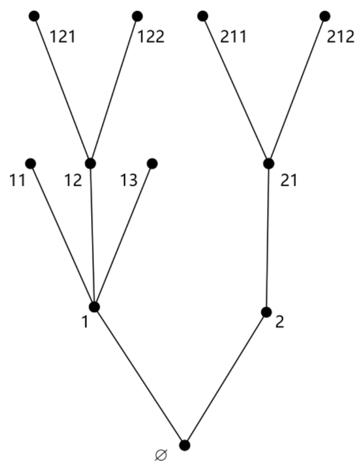
$$P_\mu(T \text{ est fini}) = q.$$

En particulier, le cas $m > 1$ est appelé surcritique, où $q < 1$; les cas $m = 1$ et $m < 1$ sont appelés, respectivement, critique et sous-critique, où $q = 1$.

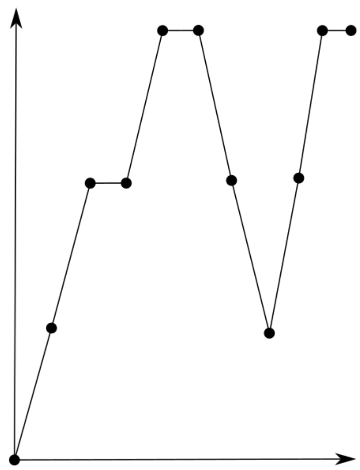
Théorème 2.1.2 (Kesten-Stigum [62]). *Pour $m < \infty$, $M_n := (\frac{Z_n}{m^n})_{n\geq 0}$ est une martingale positive, converge presque sûrement vers M_∞. Si $m \in (1, \infty)$, alors*

$$E_\mu\big[Z_1 \log_+ Z_1\big] < \infty \iff P_\mu(M_\infty > 0 \,|\, \textit{non-extinction}) = 1,$$

où $\log_+ x := \log(x \vee 1)$ *pour éviter* $\log 0$.



tree T



height process $(H_i) = (|u_i|)$

L'arbre de Galton-Watson est une structure fondamentale bien étudiée, voir Athreya et Ney [17].

De plus, on remarque que le processus de la hauteur $(H_i) := (|u_i|)$ de l'arbre critique de Galton-Watson a la limite suivante,

Théorème 2.1.3 (Aldous [9]). *Soit P_μ la loi des arbres de Galton-Watson avec loi de reproduction μ, où $E_\mu Z_1 = 1$, $\mathtt{Var}(Z_1) = \sigma^2 < \infty$, et $\mathtt{pgcd}\{j : \mu(j) > 0\} = 1$. On note par (H_i^n) le processus de la hauteur de la suite lexicographique (u_i), $H_i^n = |u_i|$, sous $P_\mu(\cdot \,|\, \#T = n)$, alors quand $n \to \infty$, dans l'espace des fonctions continues $C[0,1]$,*

$$\left(\frac{\sigma}{\sqrt{n}} H^n_{\lfloor nt \rfloor}\right)_{0\leq t\leq 1} \xrightarrow{loi} (|B_t|)_{0\leq t\leq 1},$$

où (B_t) est le mouvement brownien réel standard.

De plus, ce résultat induit une limite pour l'arbre de Galton-Watson lui-même, et il existe des extensions pour les arbres sous-critiques et les forêts de Galton-Watson, voir Duquesne et Le Gall [42].

2.1.2 Marches aléatoires branchantes

Si on considère chaque nœud de l'arbre de Galton-Watson comme un sommet, et ajoute une arête entre un nœud et son parent, alors on peut identifier T comme un graphe connexe. Si on attache un vecteur $\mathbf{d}_u$ dans un espace vecteur $\mathbb{V}$ (où on prend par défault $\mathbb{V} = \mathbb{R}^d$) à chaque arête dirigée $(\overleftarrow{u}, u)$, fixe la position de la racine à $V_\varnothing = 0$ et prend $V_u = \sum_{u' \preceq u} \mathbf{d}_{u'}$, alors $(V_u)_{u \in T}$ forme un arbre spatial. Sous l'hypothèse que tous les déplacements sont i.i.d., on définit une marche aléatoire branchante comme l'objet aléatoire construit ci-dessus tel que

$$\left(k_u, (\mathbf{d}_{u'})_{\overleftarrow{u'}=u}\right) \text{ sont i.i.d.}$$

En plus, on s'intéresse aux marches aléatoires branchantes avec l'indépendance supplémentaire que

$$\mathbf{d}_{u'} \overset{i.i.d.}{\sim} \theta,$$

indépendant de (k_u). La loi correspondante est notée par $P_{\mu,\theta}$.

Il y a des études approfondies pour les marches aléatoires branchantes, voir Shi [89] et les références citées. Un problème remarquable pour la marche aléatoire branchante est la position de la particule la plus à gauche,

$$M_n := \min\{V_u : |u| = n\}.$$

Pour cette direction on étudie surtout le cas surcritique $m > 1$, et le comportement asymptotique de M_n est essentiellement déterminé par sa transformation de log-Laplace

$$\psi(t) := \log \mathbb{E}\left[\sum_{|u|=1} e^{-tV_u}\right].$$

Typiquement, M_n est linéaire,

Théorème 2.1.4 (Hammersley [54], Kingman [63] et Biggins [29]). *Supposons que $\psi(0) = \log m > 0$ et $\psi(t) < \infty$ pour un certain $t > 0$, alors presque sûrement conditionnellement à la survie,*

$$\lim_{n\to\infty} \frac{1}{n} M_n = \gamma,$$

où

$$\gamma := -\inf_{s>0} \frac{\psi(s)}{s} \in \mathbb{R}.$$

De plus, le cas $\psi(t) = \infty, \forall t > 0$ a été étudié dans Gantert [52], et la fluctuation du second ordre de M_n est de l'ordre logarithmique,

Théorème 2.1.5 (Aïdékon [5]). *Supposons*

$$\mathbb{E}\left[\sum_{|u|=1} 1\right] > 1,\ \mathbb{E}\left[\sum_{|u|=1} e^{-V_u}\right] = 1,\ \mathbb{E}\left[\sum_{|u|=1} V_u e^{-V_u}\right] = 1,$$

et certains intégrabilités pour (V_u), alors il existe $C > 0$ telle que pour tout x,

$$\lim_{n\to\infty} \mathbb{P}\left(M_n \geq \frac{3}{2}\log n + x \right) = \mathbb{E}\left[e^{-Ce^x D_\infty}\right],$$

où D_∞ est la limite de la martingale dérivée $D_n := \sum_{|u|=n} V_u e^{-V_u}$.

Il existe une étude approfondie pour les particules au voisinage de la position extrême, voir Madaule [80] et les références citées.

2.1.3 Marches biaisées

On peut construire une marche aléatoire sur l'arbre de Galton-Watson basée sur une marche aléatoire branchante (V_u), appelée la marche biaisée. En fait, on définit une marche aléatoire aux plus proches voisins (X_n) sur l'arbre de Galton-Watson, issue de $X_0 = \varnothing$, telle que ses probabilités de transition $P(u, u') := \mathbb{P}(X_{n+1} = u' \,|\, X_n = u, (V_u)_{u\in T})$ soient proportionnelles à

$$\mathbf{1}_{\{\overleftarrow{u}=u'\}} e^{-V_u} + \mathbf{1}_{\{\overleftarrow{u'}=u\}} e^{-V_{u'}}.$$

Le sujet le plus étudié sur la marche biaisée (X_n) est le comportement de sa hauteur $(|X_n|)$. Il a été démontré dans Lyons and Pemantle [78] que, (X_n) est transient si $\inf_{t\in[0,1]} \psi(t) > 0$, et récurrent si $\inf_{t\in[0,1]} \psi(t) \leq 0$. Dans le cas transient, $(|X_n|)$ est linéaire, et on a:

Théorème 2.1.6 (Hammond [55]). *Supposons que $\psi(0) > 0$, $\inf_{t\in[0,1]} \psi(t) > 0$, et $(V_u) \overset{loi}{\sim} P_{\mu,\theta}$, avec $\theta \in [q_1, q_2]$ où $1 < q_1 < q_2$, alors il existe une constante $\gamma > 0$ explicite et une loi explicite non-dégénérée telle que conditionnellement à la survie,*

$$n^{-\gamma}|X_n| \text{ converge en loi.}$$

De plus, dans le cas récurrent, $(|X_n|)$ évolue plus lentement. Le cas critique $\inf_{t\in[0,1]} \psi(t) = 0$ est beaucoup étudié, où le comportement exact de $(|X_n|)$ dépend de $\psi'(1)$ et de $\kappa := \inf\{t > 1 : \psi(t) \geq 0\}$:

Théorème 2.1.7 (Hu and Shi [57]). *Supposons $\psi(0) > 0$, $\inf_{t\in[0,1]} \psi(t) = 0$ et $\psi'(1) = 0$, alors conditionnellement à la survie,*

$$\frac{1}{(\log n)^2}|X_n| \text{ converge vers une loi explicite.}$$

Théorème 2.1.8 (Aïdékon and de Raphélis [7] pour (2.1.1); de Raphélis [35] pour (2.1.2) et (2.1.3))**.** *Supposons* $\psi(0) > 0$*,* $\inf_{t\in[0,1]} \psi(t) = 0$ *et* $\psi'(1) < 0$*, et* (e^{-V_u}) *a un second moment fini, alors il existe une constante explicite* $C > 0$ *telle que en conditionnant à la survie,*

$$\left(\frac{1}{\sqrt{\sigma^2 n}}|X_{\lfloor nt\rfloor}|\right) \xrightarrow{loi} |B|, \qquad si\ \kappa > 2, \tag{2.1.1}$$

$$\left(\frac{C\sqrt{\log n}}{\sqrt{n}}|X_{\lfloor nt\rfloor}|\right) \xrightarrow{loi} |B|, \qquad si\ \kappa = 2, \tag{2.1.2}$$

$$\left(\frac{C}{n^{1-\frac{1}{\kappa}}}|X_{\lfloor nt\rfloor}|\right) \xrightarrow{loi} H, \qquad si\ \kappa \in (1,2), \tag{2.1.3}$$

où B *est le mouvement brownien réel standard, et* H *est le processus de la hauteur d'un processus de Lévy explicite.*

On remarque que tous les résultats ci-dessus sont pour le cas surcritique $m > 1$, où l'arbre de Galton-Watson est infini avec une probabilité positive. Dans les cas critique et sous-critique, on peut encore étudier les phénomènes similaires conditionnellement à la survie, voir Ben Arous et Hammond [26]. Il y a également une étude systématique sur le support de (X_n), voir Andreoletti and Chen [12] et les références citées.

Il y a un cas particulier de marche biaisée où θ est déterministe: Si on prend $\theta = \delta_{-\log\lambda}$, alors la marche est appelée marche λ-biaisée, si en plus $\lambda = 1$, on l'appelle la marche simple. Pour les marches λ-biaisées, il y a des résultats plus fins concernant sa vitesse:

Théorème 2.1.9 (Lyons, Pemantle and Peres [79])**.** *Soit* $\lambda_c := \mathbb{E}[\mu q^{\mu-1}]$*, alors pour tout* $\lambda \in (\lambda_c, m)$*, conditionnellement à la survie, la vitesse de la marche* λ*-biaisée* (X_n)*,*

$$l_\lambda := \lim_{n\to\infty} \frac{|X_n|}{n}$$

existe presque sûrement, et elle est une constante strictement positive.

Cette limite l_λ est explicitement calculée dans Aïdékon [6], avec les cas limite $\lambda \to m$ et $\lambda \le \lambda_c$ étudiés dans Ben Arous, Hu, Olla et Zeitouni [27] et Ben Arous, Fribergh, Gantert et Hammond [25], respectivement. On a également les théorèmes de limite centrale, voir Peres and Zeitouni [83].

2.2 Temps de recouvrement

Dans le chapitre 3, on va étudier le temps de recouvrement pour l'arbre de Galton-Watson par une marche λ-biaisée. De façon générale, pour un graphe

$G = (V, E)$ connexe fini et une marche aléatoire (X_t) sur G, son temps de recouvrement est défini par

$$T^{cov}(G; X) = \inf \{t : \{X_s, 0 \le s \le t\} = V\}.$$

On remarque que la marche aléatoire (X_t) peut être en temps discret ou continu, mais les deux cas sont similairs. En fait, la plupart des théorèmes ici sont valables pour les deux cas, et on peut facilement adapter la preuve d'un cas pour l'autre. Dans la suite, on considère le cas continu, $(X_t)_{t\in\mathbb{R}^+}$, qui saute entre les états en des temps exponentiels, pour la raison de compatibilité avec le théorème de Ray-Knight que l'on va présenter plus tard.

Pour étudier le temps de recouvrement, une idée fondamentale est d'analyser les temps locaux $(L^u(t))$, définis par

$$L^u(t) := \int_0^t \mathbf{1}_{\{X_s=u\}} ds,\ t \in \mathbb{R}^+.$$

Notre problème peut alors être décrit comme un problème sur le minimum des temps locaux,

$$T^{cov}(G; X) = \inf\{t \ge 0 : \min_{u\in V} L^u(t) > 0\}.$$

Il est donc naturel d'étudier les corrélations des temps locaux pour les bornes supérieure et inférieure du temps de recouvrement.

On prend d'abord le cas où (X_t) est une marche simple (à temps continu). La première idée est de contrôler T^{cov} par le nombre de sommets $\#V = n$,

Théorème 2.2.1 (Feige [47], [48]). *Soit $G = (V, E)$ un graphe connexe avec n sommets, et soit (X_t) la marche simple (à temps continu) issue d'un sommet fixé sur G, alors quand $n \to \infty$,*

$$(1 - o(1))n \log n \le \mathbb{E}T^{cov}(G; X) \le 4n^3/27.$$

Les deux bornes de ce théorème sont optimales, mais il y a un grand écart entre eux. Donc la cardinalité $\#V = n$ est loin d'être suffisante pour décrire le comportement de $T^{cov}(G; X)$. En fait, on peut facilement imaginer qu'une ligne droite avec n sommets se comporte très différemment d'un graphe complet de n sommets.

Pour l'améliorer, par exemple, on peut utiliser le temps d'atteinte:

Théorème 2.2.2 (Matthews [81]). *Soit $G = (V, E)$ un graphe connexe avec n sommets, et soit (X_t) la marche simple (à temps continu) issue d'un sommet fixé sur G, quand $n \to \infty$,*

$$\max_{S\subseteq G} \min_{u,v\in S} H(u, v)(\log(\#E) - 1) \le \mathbb{E}T^{cov}(G; X) \le \max_{u,v\in G} H(u, v)(1 + \log n),$$

où $H(u,v) = \mathbb{E}_u\tau_v + \mathbb{E}_v\tau_u$ *avec* $\tau_u = \inf\{t \geq 0 : X_t = u\}$ *le premier temps d'atteinte de* u.

Supposons que tous les temps d'atteinte sont du même ordre, alors ce théorème donne l'échelle du temps de recouvrement à $T^{cov}(G;X) \asymp H(u,v)\log n$. L'étape suivante est de trouver une asymptotique plus précise. On introduit alors le champ gaussien libre discret (DGFF) et les théorèmes de Ray-Knight.

2.2.1 Graphes comme réseaux électriques

Avant de présenter le DGFF, on a besoin de quelques notions pour les marches aléatoires générales. Voir Aldous and Fill [11] pour une présentation complète.

On va considérer un graphe $G(V,E)$ comme un réseau électrique, où à chaque arête $(u,v) \in E$ est associée une résistance $R_{u,v}$.

On peut choisir arbitrairement deux sommets $u_1, u_2 \in V$, leur attribuer les tensions $V_{u_1} = 1, V_{u_2} = 0$, puis déterminer les tensions V_u pour tous les autres sommets $u, v \in V\backslash\{u_1,u_2\}$ et les courants $I_{u,v}$ le long des arêtes (orientées) $(u,v) \in E$, tels que:

- Loi de Kirchhoff:
$$\sum_{v\in V} I_{u,v} = 0,\ \forall u \in V\backslash\{u_1,u_2\}.$$
- Loi d'Ohm:
$$V_u - V_v = R_{u,v}I_{u,v},\ \forall (u,v) \in E.$$

On peut alors définir les résistances effectives comme l'inverse du courant total,
$$R_{\text{eff}}(u_1,u_2) = \frac{1}{\sum_{u\sim u_1} I_{u_1,u}}.$$

De plus, ce réseau électrique est lié à la marche aléatoire avec probabilités de transition $P(u,v)$ proportionnelles à $\frac{1}{R_{u,v}}$:

Proposition 2.2.3 (Aldous and Fill [11, Proposition 3.10])**.** *Soit $G = (V,E)$ un graphe connexe fini avec la structure de réseau électrique, et soit (X_t) la marche aléatoire avec probabilités de transition $P(u,v)$ (à temps continu). On note par π la loi stationnaire pour (X_t), alors il existe une constante universelle $C > 0$ telle que*

$$\pi(u)\mathbb{P}_u(\tau_v < \tau_u^+) = \frac{C}{R_{\mathit{eff}}(u,v)},\ \forall u,v \in V,$$

où $\tau_v = \inf\{t \geq 0 : X_t = v\}$*, et* $\tau_u^+ = \inf\{t > 0 : X_t = u\}$*.*

À l'inverse, pour toutes les marches aléatoires réversibles, i.e. (X_n) *avec probabilités de transition* $(P(u,v))$ *et loi stationnaire* π *telles que*

$$\pi(u)P(u,v) = \pi(v)P(v,u), \, \forall u \sim v \in V,$$

on peut construire un réseau électrique tel que les probabilités de transition $(P(u,v))$ *sont proportionnels à l'inverse des résistances* $\frac{1}{R_{u,v}}$*.*

On remarque que ce réseau électrique est unique à un facteur multiplicatif près. On peut définir un réseau électrique canonique avec $C = 1$ dans la proposition 2.2.3. De plus, on peut facilement vérifier que la marche simple, et plus généralement la marche aléatoire biaisée dans la section 1.1.3 est réversible, et on va utiliser leurs réseaux électriques (canoniques) correspondants.

On remarque que, du point de vue des réseaux électriques, la résistance (conductance) effective entre les points est un paramètre fondamental. Dans notre cas, l'étude du temps de recouvrement ne nécessite que des résistances entre deux points d'un arbre, c'est-à-dire la somme des résistances le long du chemin entre eux. Un cas intéressant est la conductance effective entre la racine et la génération n de l'arbre de Galton-Watson: pour cette direction, une façon standard est de considérer les arbres de Galton-Watson surcritiques $m > 1$, et sur chaque arête à la génération n on attache une conductance distribuée indépendamment comme $m^n\zeta$. Alors la conductance effective C_n entre la racine et la génération n vérifie

Théorème 2.2.4 (Chen, Hu et Lin [31])**.** *Supposons que* ζ *et la loi de reproduction* μ *ont des moments finis: pour* $X \sim \zeta$ *et* $Y \sim \mu$*,*

$$\mathbb{E}[X^3 + X^{-1} + Y^4] < \infty,$$

alors conditionnellement à la survie, $\frac{C_n}{\mathbb{E}[C_n]}$ *converge en loi vers une mesure explicite. De plus, il existe des constantes explicites* c_0, c_1, c_2 *telles que*

$$\mathbb{E}[C_n] = \frac{1}{c_1 n} - \frac{c_2}{c_1^2}\frac{\log n}{n^2} - \frac{c_0}{c_1^2}\frac{1}{n^2} + O\left(\frac{(\log n)^2}{n^3}\right).$$

On peut également prendre la conductance sur chaque arête sous la forme d'une marche aléatoire branchante, dans ce cas, le comportement asymptotique est différent, voir Rousselin [87].

2.2.2 Champ gaussien libre discret

Étant donné $G = (V, E)$ et son réseau électrique, un champ gaussien libre discret(DGFF) $(\eta_u)_{u \in V}$ (enraciné en $u_0 \in V$) est alors défini par un ensemble de variables gaussiennes centrées indexées par les sommets de G, tel que

$$\mathbb{E}(\eta_u - \eta_{u'})^2 = R_{\text{eff}}(u, u'),\ \eta_{u_0} = 0.$$

On remarque que son analogue continu est le champ gaussien libre (GFF) sur $\mathbb{R}^d$, en attachant une loi gaussienne à chaque point de $\mathbb{R}^d$. Sur chaque domaine borné, le GFF continu peut être approché par le DGFF sur le réseau $(\frac{1}{n}\mathbb{Z})^d$. Le GFF est également liés à la gravité quantique, et il y a des recherches approfondies sur ces sujets, voir Sznitman [91].

Ce qui nous intéresse particulièrement est le théorème de Ray-Knight suivant qui couple le DGFF d'un graphe avec ses temps locaux, également connu comme théorème d'isomorphisme de Dynkin:

Théorème 2.2.5 (Eisenbaum et al. [46]). *Pour chaque graphe connexe fini $G = (V, E)$, soit (X_t) la marche simple (à temps continu) sur G, et soit $(\eta_u)_{u \in \mathbf{V}}$ le DGFF sur G. Alors pour tout $t \in \mathbb{R}^+$,*

$$\left\{L^u(t) + \eta_u^2 : u \in V\right\} \stackrel{d}{=} \left\{(\eta_u + \sqrt{t})^2 : u \in V\right\}.$$

Rappelons que le temps de recouvrement dépend du minimum des temps locaux, ce théorème, prouvé d'abord par Ding, Lee and Peres [39] et puis amélioré par Zhai [93], transforme le problème à l'étude des valeurs extrêmes du DGFF.

Théorème 2.2.6 (Ding, Lee and Peres [39], Zhai [93]). *Soit $G = (V, E)$ un graphe connexe fini, et soit (X_t) la marche simple (à temps continu) issue de $u_0 \in V$. On note par (η_u) le DGFF épinglé à u_0, alors il existe $c, C > 0$ telles que, pour tout $s > 0$,*

$$\mathbb{P}\left(\left|T^{cov}(G; X) - M^2 \# E\right| \geq \left(\sqrt{sR}M + sR\right)\# E\right) \leq Ce^{-cs},$$

où $M = \mathbb{E}(\max_{u \in V} \eta_u)$, $R = \max_{u,u' \in V} R_{\mathit{eff}}(u, u')$.

Puisque ce théorème donne le comportement au premier ordre du temps de recouvrement pour tous les graphes, une estimation plus précise est attendue, notamment par les graphes spécifiques. En fait, on considère l'arbre comme un graphe suffisamment simple pour une telle étude, car le réseau électrique basé sur un arbre est élémentaire: les résistances effectives ne sont que les sommes de résistances le long des chemins. Par conséquent, les covariances pour le DGFF sont explicites, et le DGFF peut être réduit à une marche aléatoire branchante, en attribuant une variable aléatoire gaussienne avec une variance égale à la résistance le long des arêtes.

2.2.3 Temps de recouvrement sur les arbres

À l'aide du DGFF, on peut alors caractériser explicitement les temps locaux sur les arbres. Par exemple, pour la marche aléatoire simple, nous avons

Lemme 2.2.7 (Bai [18, Lemma 2.3]). *Soit $s, t \in \mathbb{R}^+$. Si (X_t) est la marche simple (à temps continu), et $G = (V, E)$ est un arbre, soit $u, v \in V$ tels que $v \prec u$, alors*

$$L^u(t) \sim \mathtt{PG}\left(\frac{t}{|u|}, \frac{1}{|u|}\right),$$
$$(L^u(t)|L^v(t) = s) \sim \mathtt{PG}\left(\frac{s}{|u| - |v|}, \frac{1}{|u| - |v|}\right),$$

où $\mathtt{PG}(a, b)$ désigne la loi de $\sum_{i=1}^{P} E_i$, P et E_i sont indépendantes telles que $P \sim \mathtt{Poisson}(a)$ est une loi de Poisson de paramètre a, et $E_i \sim \mathtt{Exp}(b)$ est une loi exponentielle de paramètre b.

Par une méthode du second moment des temps locaux, Ding and Zeitouni [40] ont donné un estimation d'ordre $O((\log\log n)^8)$ pour la marche simple sur un arbre binaire, puis ce résultat est amélioré jusqu'à $O(1)$ par Belius, Rosen and Zeitouni [23] avec un argument basé sur l'estimation du processus de Bessel. De plus, une convergence en loi a été établie indépendamment par Cortines, Louidor et Saglietti [32] et par Dembo, Rosen et Zeitouni [37],

Théorème 2.2.8 (Cortines, Louidor et Saglietti [32], Dembo, Rosen et Zeitouni [37]). *Soit T_n^{cov} le temps de recouvrement des n premiers générations d'un arbre binaire complet par la marche simple à temps continu, alors quand $n \to \infty$,*

$$\mathbb{P}\left(\frac{T_n^{cov}}{2^{n+1}n} - n\log 2 + \log n \leq s\right) = \mathbb{E}\exp\left(-CZe^{-s}\right),$$

pour une constante $C > 0$ et une loi Z explicites.

Cette convergence est basée sur, en regardant les corrélations des temps locaux, l'observation suivante: en temps proche du temps de recouvrement, les sommets pas encore visités forment des clusters presque indépendants.

Notre résultat généralise cette estimation aux arbres de Galton-Watson et aux marches biaisées,

Théorème 2.2.9 (Bai [18]). *Soit T_n^{cov} le temps de recouvrement des n premières générations de l'arbre de Galton-Watson T par la marche λ-biaisée à temps continu. Supposons $\lambda > 1, \mathbb{E}Z_1 > 1, \mathtt{Var}(Z_1) < \infty$. Soit $\mathbf{P}_{\mathrm{w}}$ la loi de* *la marche λ-biaisée. Alors pour P_μ-presque sûrement chaque T, tout $x \in \mathbb{R}$ et $n \to \infty$, quand $\lambda > m$,*

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(\lambda-1)T_n^{cov}}{2\lambda^{n+1}\sum_{i=0}^{\infty}\frac{Z_i}{\lambda^i}} - n\log m - \log W \le x \middle| T\right) \to e^{-e^{-x}};$$

quand $\lambda = m$,

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(m-1)T_n^{cov}}{2m^{n+1}\sum_{i=0}^{n} Z_i} - n\log m - \log W \le x \middle| T\right) \to e^{-e^{-x}};$$

quand $1 < \lambda < m$,

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(\frac{m}{\lambda}-1)(\lambda-1)T_n^{cov}}{2Wm^{n+1}} - n\log m - \log W \le x \middle| T\right) \to e^{-e^{-x}}.$$

La preuve est principalement inspirée par [32], reposant sur une variante de l'observation déjà mentionnée ci-dessus: pour la marche aléatoire λ-biaisée, en temps proche du temps de recouvrement, les sommets restants sont presque indépendants. On établit cette observation par une estimation précise des temps locaux, et la preuve complète est donnée dans le chapitre 3.

Le même problème de temps de recouvrement pour les marches biaisées aléatoirement sur les arbres de Galton-Watson est ouvert. En effet, sur l'arbre de Galton-Watson, il y a de courtes branches sur l'arbre T_n (T coupées à la hauteur n) qui s'éteignent aux générations 1 à $n-1$. Elles sont négligeables pour la marche λ-biaisée, car les visiter est beaucoup plus facile que de visiter les nœuds de la génération n. Cependant, cela n'est plus vrai pour la marche à biais aléatoire.

2.3 Capacité d'image

Dans les chapitres 4 et 5, on va étudier la capacité d'une marche aléatoire branchante. Pour une marche aléatoire discrète (X_n) dans $\mathbb{Z}^d$, sa image est défini par

$$R[s,t] := \{X_s, X_{s+1}, \cdots, X_t\},\ s \le t.$$

De plus, si $d \ge 3$, on peut définir sa capacité (d'image) par rapport à une loi symétrique η sur $\mathbb{Z}^d$ comme

$$\mathsf{cap}_\eta A := \sum_{x \in A} \mathsf{P}_x^\eta(\tau_A^+ = \infty),$$

où P_x^η est la loi d'une marche aléatoire (S_n) issue de x avec probabilités de transition η, et $\tau_A^+ := \inf\{n \geq 1 : S_n \in A\}$ est son temps de retour à A. On remarque que la version continue pour l'image $\#R[0,n]$ est la mesure de Lebesgue, et celle pour la capacité est la capacité brownienne définie (dans Mörters et Peres [82, Definition 8.17]) comme suit: pour tous les ensembles boréliens bornés $F \subset \mathbb{R}^d$ $(d \geq 3)$,

$$\mathsf{cap}(F) := \left(\inf\left\{\iint G(x,y)\nu(dx)\nu(dy) : \nu \text{ est une mesure probabilité sur } F\right\}\right)^{-1},$$

où $G(x,y) = |x-y|^{2-d}$, ou plus généralement tout noyau de forme $G(x,y) = C|x-y|^{-\alpha}$. En fait, il existe une manière équivalente de définir la capacité,

$$\mathsf{cap}_\eta A = C_{d,\eta} \lim_{|x|\to\infty} |x|^{d-2}\mathsf{P}_x^\eta(\tau_A < \infty),$$

où l'inverse du facteur, $\left(C_{d,\eta}|x|^{d-2}\right)^{-1}$, est l'espérance qu'une η-marche issue de x visite 0. La même définition fonctionne pour le cas continu, en changeant la marche aléatoire discrète η en mouvement brownien.

L'image d'une marche aléatoire classique a été largement étudié depuis Dvoretzky et Erdős [43], on présente ici son comportement asymptotique pour la marche simple. Pour les résultats plus avancés, voir Asselah et Schapira [14].

Théorème 2.3.1 (Dvoretzky et Erdős [43]). *Soit (X_n) une marche simple dans $\mathbb{Z}^d$. Quand $n \to \infty$,*
si $d \geq 3$,

$$\frac{1}{n}\#R[0,n] \to \mathsf{P}_0(\tau_0^+ = \infty), \textit{ presque sûrement};$$

si $d = 2$,

$$\frac{\log n}{n}\#R[0,n] \to \pi, \textit{ presque sûrement};$$

si $d = 1$,

$$\frac{1}{\sqrt{n}}\#R[0,n] \xrightarrow{loi} \sup_{0\leq t\leq 1} B_t - \inf_{0\leq t\leq 1} B_t,$$

où (B_t) est un mouvement brownien réel standard.

Des résultats similaires pour la capacité remontent à Jain et Orey [58], avec ses raffinements aux théorèmes de limite centrale dans la série d'articles par Asselah, Schapira et Sousi [15], [16], [88]. L'étude de la capacité est récemment motivée par ses liens avec les entrelacements aléatoires. Le modèle de les entrelacements aléatoires a été introduit par Sznitman [90], motivé par les études des traces des marches simples. Les probabilités d'intersection et la théorie du potentiel émergent alors naturellement dans son étude, qui demandent des estimations très précises de la capacité, voir Drewitz, Ráth et Sapozhnikov [41].

Théorème 2.3.2 (Jain et Orey [58] pour (2.3.4) et (2.3.5), Chang [30] pour (2.3.6)). *Soit* (X_n) *une marche simple dans* $\mathbb{Z}^d$*, et soit* η *la loi de* $X_1 - X_0$*, alors quand* $n \to \infty$*, si* $d \geq 5$*, il existe une constante* $c_d > 0$ *telle que*

$$\frac{1}{n}\mathtt{cap}_\eta R[0,n] \to c_d, \text{ presque sûrement;} \tag{2.3.4}$$

si $d = 4$*,*

$$\frac{\log n}{n}\mathtt{cap}_\eta R[0,n] \to \frac{\pi^2}{8}, \text{ presque sûrement;} \tag{2.3.5}$$

si $d = 3$*,*

$$\frac{1}{\sqrt{n}}\mathtt{cap}_\eta R[0,n] \xrightarrow{loi} \frac{1}{3\sqrt{3}}\mathtt{cap}_{BM}(B[0,1]), \tag{2.3.6}$$

où (B_t) *ici est un mouvement brownien standard dans* $\mathbb{R}^3$*.*

Théorème 2.3.3 (Asselah, Schapira et Sousi [15] pour (2.3.7), [16] pour (2.3.9), Schapira [88] pour (2.3.8)). *Sous les même hypothèses que le théorème 2.3.2, il existe une constante* $C_d > 0$ *telle que si* $d \geq 6$*,*

$$\frac{\mathtt{cap}_\eta R[0,n] - \mathbb{E}\mathtt{cap}_\eta R[0,n]}{\sqrt{n}} \xrightarrow{loi} \mathcal{N}(0, C_d); \tag{2.3.7}$$

si $d = 5$*,*

$$\frac{\mathtt{cap}_\eta R[0,n] - \mathbb{E}\mathtt{cap}_\eta R[0,n]}{\sqrt{n \log n}} \xrightarrow{loi} \mathcal{N}(0, C_d); \tag{2.3.8}$$

si $d = 4$*, il existe une loi explicite* γ *telle que*

$$\frac{(\log n)^2}{n}\big(\mathtt{cap}_\eta R[0,n] - \mathbb{E}\mathtt{cap}_\eta R[0,n]\big) \xrightarrow{loi} \gamma. \tag{2.3.9}$$

De plus, on peut voir la marche aléatoire branchante critique (conditionnellement à la survie) comme une généralisation d'une marche aléatoire classique, En numérotant la marche aléatoire branchante par l'ordre lexicographique, on a des résultats sur l'image,

Théorème 2.3.4 (Le Gall et Lin [73] pour (2.3.10), [72] pour (2.3.11); Zhu [94] pour (2.3.12)). *Soit* $P_{\mu,\theta}$ *la loi d'une marche aléatoire branchante, où* μ *est critique avec second moment fini, et* θ *est symétrique, irréductible avec* *support fini. Soit* (X_t) *le suite lexicographique de la marche branchante sous* $P_{\mu,\theta}(\cdot \mid \#T = n)$. *Quand* $n \to \infty$, *si* $d \geq 5$, *il existe* $c_{\mu,\theta} > 0$ *telle que*

$$\frac{1}{n} R[0,n] \xrightarrow{(P)} c_{\mu,\theta}; \tag{2.3.10}$$

si $d = 4$,

$$\frac{\log n}{n} R[0,n] \xrightarrow{(P)} 8\pi^2\sigma^4, \tag{2.3.11}$$

où $\sigma^2 = (\det M_\eta)^{1/4}$, *avec* η *le matrice de covariance de* η*;*
si $d \leq 3$,

$$n^{-d/4} R[0,n] \xrightarrow{loi} 2^{d/4} (\det M_\theta)^{1/2} \lambda_d(\operatorname{supp} \mathcal{I}), \tag{2.3.12}$$

où λ_d *est la mesure de Lebesgue, et* $\mathcal{I}$ *est la mesure sur* $\mathbb{R}^d$ *connue comme l'excursion intégrée du super-mouvement brownien.*

En généralisant les méthodes et les estimations dans [73], on a réussi à obtenir les asymptotiques pour la capacité d'une marche aléatoire branchante,

Théorème 2.3.5 (Bai et Wan [21] pour (2.3.13) et (2.3.14); Bai et Hu [19] for (2.3.15))**.** *Soit* $P_{\mu,\theta}$ *la loi d'une marche branchante, où* μ *est critique avec variance finie, et* θ *est symétrique, irréductible à moment exponentiel fini. Soit* η *une loi irréductible avec espérance* 0 *et* $(d+1)$*-ème moment fini. Soit* (X_t) *la suite lexicographique de la marche branchante sous* $P_{\mu,\theta}(\cdot \mid \#T = \infty)$. *Quand* $n \to \infty$, *si* $d \geq 7$, *il existe* $c_{\mu,\theta,\eta} > 0$ *telle que*

$$\frac{1}{n} \mathtt{cap}_\eta R[0,n] \xrightarrow{(P)} c_{\mu,\theta,\eta}; \tag{2.3.13}$$

si $d = 6$ *et* μ *a un* 5*-ème moment fini, il existe* $c_{\mu,\theta,\eta} > 0$ *telle que*

$$\frac{\log n}{n} \mathtt{cap}_\eta R[0,n] \xrightarrow{(P)} c_{\mu,\theta,\eta}; \tag{2.3.14}$$

si $3 \leq d \leq 5$, *pour tout* $\epsilon > 0$, $P_{\mu,\theta}(\cdot \mid \#T = \infty)$*-presque sûrement,*

$$\mathtt{cap}_\eta R[0,n] = n^{-(d-2)/4 + o(1)}, \ n \to \infty. \tag{2.3.15}$$

Remarque 2.3.6. Ce théorème est formulé avec l'arbre de Galton-Watson conditionnellement à la survie pour la cohérence dans toutes les dimensions. Le modèle peut être remplacé par la forêt de Galton-Watson ou un Galton-Watson conditionnellement à être grand, avec des ajustements dans la preuve (voir par exemple lemme 4.3.6). Pour faciliter la démonstration, dans chapitres 4 et 5, on va utiliser l'arbre de Galton-Watson conditionnellement à être grand et la forêt de Galton-Watson, respectivement. En particulier, comme on le montrera dans lemme 5.2.3, n sous-arbres dans une forêt de Galton-Watson contiennent généralement $\Theta(n^2)$ sommets, donc sa capacité est comparable avec $R[0,n^2]$ au lieu de $R[0,n]$.

On peut observer des structures similaires pour les quatre théorèmes ci-dessus, à savoir qu'il existe une dimension critique avec un comportement $O(\frac{n}{\log n})$, et on a une croissance linéaire dans les grandes dimensions et une fluctuation dans les petites dimensions. L'intuition derrière cela sera expliquée dans les sous-sections suivantes, en se concentrant sur la capacité des marches aléatoires branchantes.

2.3.1 Grandes dimensions et modèle infini

Examinons d'abord la capacité de la marche simple. La propriété sous-additif suivante est facilement établie par définition,

$$\mathtt{cap}_\eta R[0,n] \leq \mathtt{cap}_\eta R[0,n/2] + \mathtt{cap}_\eta R[n/2,n].$$

De plus, la marche aléatoire simple a clairement la propriété d'invariance par translation suivante, par laquelle on peut déduire son ergodicité

$$R[s,t] - X_s \overset{d}{=} R[0,t-s],\, \forall s \leq t. \tag{2.3.16}$$

On peut donc conclure par le théorème ergodique sousadditif de Kingman que

$$\lim_{n\to\infty} \frac{1}{n} \mathtt{cap}_\eta R[0,n]$$

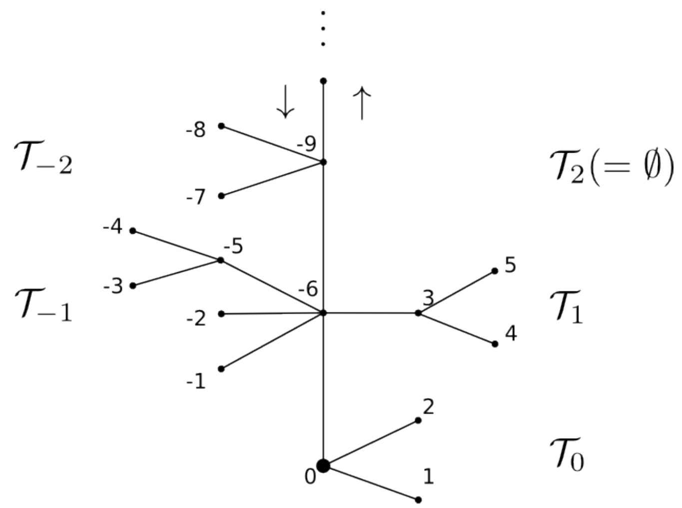
existe presque sûrement et dans L^1, et est égale à une constante. Par conséquent, on peut facilement montrer la croissance linéaire en grandes dimensions par une borne inférieure de son espérance.

Pour les marches aléatoires branchantes, le problème se réduit à trouver un modèle avec des propriétés d'invariance similaires à (2.3.16), d'où le modèle suivant, qui peut être vu comme la version infinie de Aldous [8, Section 2.6], ou encore une généralisation de Le Gall et Lin [73, Section 2.2].

On définit une forêt à deux côtés comme une suite d'arbres avec étiquettes,

$$\mathcal{T} = ((0,\mathcal{T}_0),(1,\mathcal{T}_1),(1,\mathcal{T}_{-1}),(2,\mathcal{T}_2),(2,\mathcal{T}_{-2})\cdots),$$

où les racines $(\pm i,\varnothing)$ de $\mathcal{T}_i$ et $\mathcal{T}_{-i}$ $(i>0)$ sont identifiées (collées ensemble) comme un seul point. La loi de reproduction $k_{(i,u)}(\mathcal{T})$, déplacement $\mathbf{d}_{(i,u)}(\mathcal{T})$ et position spatiale d'un sommet $V_{(i,u)}(\mathcal{T})$ sont notés comme pour les arbres classiques. Les racines $(\pm i,\varnothing)$ ont les lois de reproduction notées par $k^{\pm}_{(i,\varnothing)}(\mathcal{T})$.



Sur l'ensemble des forêts à deux côtés, on note par $\mathbf{P}_{\mu,\theta}$ la mesure de probabilité suivante comme une variante de celle d'une marche aléatoire branchante $P_{\mu,\theta}$:

- Pour chaque $i \geq 0, u \neq \varnothing$,

$$k_{(i,u)}(\mathcal{T}) \overset{i.i.d.}{\sim} \mu,$$

$$k_{(0,\varnothing)}(\mathcal{T}) \sim \mu,$$

  et pour les autres nœuds $(\pm i, \varnothing)$ $(i > 0)$,

$$\mathbf{P}_{\mu,\theta}(k^+_{(i,\varnothing)}(\mathcal{T}) = i, k^-_{(i,\varnothing)}(\mathcal{T}) = j) = \mu(i + j + 1).$$

- Les déplacements $\mathbf{d}_{(i,u)}(\mathcal{T})$ sont i.i.d. et distribués comme θ sur chaque arête dirigée, issue de $V_{(0,\varnothing)}(\mathcal{T}) = 0$.

On remarque que si on restreint μ à la loi géométrique, et oublie les branches à chaque instant, ne laissant que des informations sur l'historique d'un site à l'infini, le modèle obtenu est un serpent discret. Son modèle limite est appelé un serpent brownien, qui a été introduit par Le Gall [70], avec des propriétés et applications très intéressantes en relation avec des équations aux dérivées partielles semi-linéaires (voir Le Gall [75] et [71]).

On va expliquer dans le chapitre 4 que ce modèle présente une explication combinatoire, conséquence de sa propriété d'invariance,

Proposition 2.3.7 (Bai and Wan [21, Proposition 2.2]). *Soit μ une loi critique sur $\mathbb{N}$ avec second moment fini, et soit θ une loi symétrique et irréductible sur $\mathbb{Z}^d$ avec moments exponentiels fini. Alors*

$$R[s,t] - X_s \overset{d}{=} R[0, t-s] \text{ sous } \mathbf{P}_{\mu,\theta}, \forall s \leq t.$$

De plus, en montrant que $\mathbf{P}_{\mu,\theta}$ est absolument continue par rapport aux marches aléatoires branchantes classiques, on a une croissance linéaire de la capacité en grandes dimensions.

2.3.2 Dimension critique et petits dimensions

L'étude de la dimension critique nécessite des estimations plus précises. Pour commencer, on définit la fonction de Green discrète par rapport à une loi aléatoire η

$$G_\eta(x,y) = G_\eta(x-y) := \sum_{n=0}^{\infty} \mathsf{P}_0^\eta(S_n = x-y).$$

Et on a

Lemme 2.3.8 (Lawler [68, Théorème 4.3.5]). *Soit η une loi apériodique et irréductible sur $\mathbb{Z}^d (d \geq 3)$ d'espérance nulle, matrice de covariance Γ_η et $(d+1)$-ème moment fini. On a*

$$G_\eta(x) = \frac{C_{d,\eta}}{J_\eta(x)^{d-2}} + O(|x|^{1-d}),$$

où

$$C_{d,\eta} = \frac{\mathbf{\Gamma}(\frac{d}{2})}{(d-2)\pi^{d/2}\sqrt{\det \Gamma_\eta}},$$

$\mathbf{\Gamma}(\cdot)$ *est la fonction Gamma, et*

$$J_\eta(x) = \sqrt{x \cdot \Gamma_\eta^{-1} x}.$$

De plus, inspiré par Lawler [67, Théorème 3.6.1], le lemme suivant nous permet d'établir une relation entre la capacité et les fonctions de Green,

Lemme 2.3.9 (Bai and Hu [19]). *Pour $d \geq 3$ et toute suite $(X_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^d$ vérifiant* (1.3.16)*, on a*

$$\frac{1}{n+1}\sum_{i=0}^{n} \mathbb{E}\left[\mathbf{1}_{\{X_0 \notin \{X_1,\cdots,X_{n-i}\}\}} \mathsf{P}_{X_0}^\eta(\tau^+_{\{X_{-i},\cdots,X_{n-i}\}} = \infty) \cdot \sum_{j=-i}^{n-i} G_\eta(X_0, X_j)\right] = 1.$$

En analysant les fonctions de Green, on peut montrer qu'en dimension $d = 6$, la somme $\sum_{j=-i}^{n-i} G_\eta(X_0, X_j)$ pour la forêt à deux côtés est concentrée autour de $C\log n$, avec une faible dépendance du paramètre i, donc ce lemme nous permet de déduire que

$$\begin{aligned}\mathbf{E}_{\mu,\theta}\mathsf{cap}_\eta R[0,n] &= \sum_{i=0}^n \mathbb{E}\Big[\mathbf{1}_{\{X_i\notin\{X_{i+1},\cdots,X_n\}\}}\mathsf{P}^\eta_{X_i}(\tau^+_{\{X_0,\cdots,X_n\}}=\infty)\Big]\\ &= \sum_{i=0}^n \mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{\{X_0\notin\{X_1,\cdots,X_{n-i}\}\}}\mathsf{P}^\eta_{X_0}(\tau^+_{\{X_{-i},\cdots,X_{n-i}\}}=\infty)\Big]\\ &\asymp \frac{n}{\log n}.\end{aligned}$$

Une estimation sur le second moment avec des méthodes similaires montre que la capacité est concentrée autour de sa valeur attendue, et on transfère la forêt de deux côtés aux marches aléatoires branchantes classiques par un argument de l'absolue continuité entre ces deux lois.

Pour les petites dimensions, une méthode différente est nécessaire pour montrer que le cadre discret converge vers son analogue continu comme dans les théorèmes 2.3.1 à 2.3.4. En fait, pour la marche simple, il existe un couplage avec le mouvement brownien tel que, pour chaque $\epsilon > 0$, il existe $\gamma > 0$ telle que (voir par exemple Lawler [66, Lemme 3.1])

$$\mathbb{P}\left(\max_{0\le s\le n}|X_{\lfloor ds\rfloor}-B_s|>n^{1/4+\epsilon}\right)<e^{-n^\gamma};$$

et pour l'image de marches aléatoires branchantes, la convergence est établie à travers des temps locaux.

Dans cette partie, on n'a obtenu que l'ordre de la capacité dans le théorème 2.3.5 en utilisant les estimations des fonctions de Green et les inégalités dérivées de lemmes comme le lemme 2.3.9.

2.4 Marches aléatoires branchantes conditionnellement à une survie rare

Dans la chapitre 6 on considère l'étendue de la dernière génération d'une marche aléatoire branchante conditionnellement à une survie rare. Considérons une marche aléatoire branchante $(V_u)_{u\in T}$ on $\mathbb{R}$, conditionnellement à la survie au génération n, on peut alors classer par ordre croissant les déplacements spatiaux $(V_u)_{|u|=n}$,

$$V_n^{(1)}\le\cdots\le V_n^{(Z_n)},$$

où Z_n est la population de génération n. On étudie alors son étendue

$$R_n := V_n^{(Z_n)} - V_n^{(1)}$$

et les écarts successifs

$$g_n^i := V_n^{(i+1)} - V_n^{(i)},\ 1 \leq i \leq Z_n - 1.$$

Le comportement de $(V_u)_{|u|=n}$ conditionnellement à la survie a été bien étudié, comme nous l'avons introduit dans le théorème 2.1.5. Dans notre cas, on étudie le comportement conditionnellement à "une survie rare", $\{Z_n = k\}$, pour une constante $k \geq 2$. Ce problème est principalement motivé par [85], [86], où Ramola, Majumdar et Schehr considèrent l'étendue R_n et les écarts (g_n^i) pour le mouvement brownien branchant par une méthode d'EDP. On l'étend aux marches branchantes générales,

Théorème 2.4.1 (Bai et Rousselin [20]). *Soit $k \geq 2$, et soit $1 \leq i \leq k-1$. Considérons une marche aléatoire branchante $P_{\mu,\theta}$, où μ a une variance finie, et θ a des moments exponentiels finis. Alors il existe des constantes explicites $C_1, C_2, C_3 > 0$ telles que, quand $x \to \infty$,*

$$\lim_{n\to\infty} P_{\mu,\theta}(R_n > x \,|\, Z_n = k) = \begin{cases} (C_1 + o(1))x^{-2}, & \mathbb{E}[\mu] = 1, \\ \exp(-(C_2 + o(1))x), & \mathbb{E}[\mu] \neq 1, \end{cases}$$

$$\lim_{n\to\infty} P_{\mu,\theta}\big(g_n^i > x \,\big|\, Z_n = k\big) = \begin{cases} (C_1C_3 + o(1))x^{-2}, & \mathbb{E}[\mu] = 1, \\ \exp(-(C_2 + o(1))x), & \mathbb{E}[\mu] \neq 1. \end{cases}$$

On remarque que pour l'arbre de Galton-Watson critique conditionnellement à la survie, typiquement $Z_n = \Theta(n)$, voir Curien et Le Gall [34] pour une étude détaillée. De plus, pour le cas grand, $Z_n \gg n$, voir Abraham, Bouaziz et Delmas [2]. Pour les arbres de Galton-Watson surcritiques, typiquement $Z_n = \Theta(m^n)$, voir Berestycki et al. [28] pour des discussions sur le cas $Z_n < \epsilon m^n$.

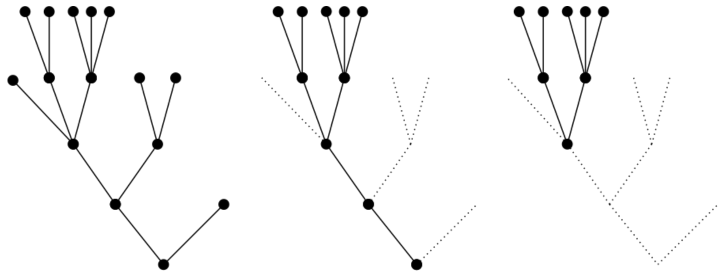
2.4.1 L'opération de tailler et couper sur l'arbre

Pour étudier ce problème, on définit d'abord les opérations de tailler et couper sur les arbres:

- Pour chaque arbre T, on construit l'arbre taillé à la hauteur n par

$$\mathtt{pru}_n(T) := \{u \in T : \exists v \in T, |v| = n, u \preceq v\}.$$

Par convention, si $Z_n(T) = 0$, on prend $\mathtt{pru}_n(T) = \{\varnothing\}$.



T $\qquad$ $\texttt{pru}(T)$ $\qquad$ $\texttt{cut}(\texttt{pru}(T))$

- De plus, on définit l'opération de couper par

$$\phi_n(T) = \bigwedge_{|u|=n,\, u\in T} u,\ h_n(T) = n - |\phi_n(T)|,\ \texttt{cut}_n(T) = T[\phi_n(T)].$$

  où par convention, $\phi_n(T) = \varnothing$ if $Z_n(T) = 0$.

Ces opérations s'étendent naturellement aux marches aléatoires branchantes. De plus, par construction, on a

$$R_n(T) = R_n(\texttt{pru}_n T) = R_{H(\texttt{cut}_n \texttt{pru}_n T)}(\texttt{cut}_n \texttt{pru}_n T),$$

et la même chose s'applique aux écarts successifs. En d'autres termes, les études de R_n et g_n^i de T sous $P_{\mu,\theta}$ se réduisent à celles de l'arbre $\texttt{cut}_n\texttt{pru}_n T$ sous $P_{\mu,\theta}(\cdot \mid Z_n = k)$.

On remarque que la structure $\texttt{cut}_n\texttt{pru}_n T$ conditionnée à la survie converge vers l'arbre de Yule, voir Curien et Le Gall [34].

2.4.2 Ratio theorem

Comme indiqué dans la sous-section précédente, notre principal objectif est d'étudier la structure généalogique de $\texttt{cut}_n\texttt{pru}_n T$. Le composant clé est alors le ratio theorem, qui fournit des probabilités de transition pour la chaîne de Markov (Z_n):

Proposition 2.4.2 (Athreya et Ney [17, Section 1.7-1.11])**.** *Soit μ une loi de reproduction telle que*

$$\mu(0), \mu(1) > 0,\ \mu(0) + \mu(1) < 1,\ \mathbb{E}Z_1 < \infty.$$

Soit $P_n(i,j)$ la probabilité de transition $P_\mu(Z_{k+n} = j \mid Z_k = i)$ pour k assez grand.

1. *Pour chaque $j \geq 1$, il existe une suite (π_j) telle que*

$$\lim_{n\to\infty} \frac{P_n(1,j)}{P_n(1,1)} \nearrow \pi_j \in (0,\infty),$$

où $\nearrow$ signifie que la limite est croissante.

2. *Pour chaque $t \in \mathbb{Z}$, $i,j,k,l \geq 1$,*

$$\lim_{n\to\infty} \frac{P_{n+t}(i,j)}{P_n(k,l)} = \gamma^t q^{i-k} \frac{i\pi_j}{k\pi_l},$$

où q est la probabilité d'extinction, f est la fonction génératrice, et $\gamma = f'(q)$.

3. *Si $\mathbb{E}Z_1 = 1, \sigma^2 := \mathtt{Var}Z_1 < \infty$, alors pour chaque $i,j \geq 1$,*

$$\lim_{n\to\infty} n^2 P_n(i,j) = \frac{2i\pi_j}{\sigma^2 \sum_{k=1}^{\infty} \pi_k(\mu(0))^k}.$$

4. *Si $\mathbb{E}Z_1 \neq 1$, $\mathbb{E}Z_1 \log Z_1 < \infty$, alors pour chaque $i,j \geq 1$,*

$$\lim_{n\to\infty} \gamma^{-n} P_n(i,j) = iq^{i-1} v_j,$$

où (v_j) est déterminé par $Q(s) = \sum_{j=0}^{\infty} v_j s^j, 0 \leq s < 1$, avec Q la solution unique de

$$Q(f(s)) = \gamma Q(s)(0 \leq s < 1),\ Q(q) = 0,\ \lim_{s\to q} Q'(s) = 1.$$

Compte tenu de ces propriétés généalogiques, on peut montrer que sous $P_\mu(\cdot \mid Z_n = k)$, l'arbre $\mathtt{cut}_n\mathtt{pru}_n T$ a sa première branche $Z_1 = 2$ avec une probabilité $1-o(1)$, et qu'il n'a pas d'autres branches jusqu'à ce qu'il s'approche de la dernière génération de l'arbre. De plus, on en déduit que la loi de $\mathtt{cut}_n\mathtt{pru}_n T$ converge quand $n \to \infty$, et on peut obtenir des propriétés spatiales en se basant sur ces observations.

Chapter 3

Cover time of λ-biased walks

This chapter is based on [18].

3.1 Introduction

3.1.1 The model

A planar tree $\mathbf{T}$ is a subset of $\cup_{n\geq 0}\mathbb{N}_+^n$ such that

- The root $\varnothing$ is in $\mathbf{T}$, where by convention, $\mathbb{N}_+^0 = \{\varnothing\}$.
- For every vertex $x = (x_1, \cdots, x_n) \in \mathbf{T}$, its parent

$$\overleftarrow{x} = (x_1, \cdots, x_{n-1}) \in \mathbf{T}.$$

- There exists an integer $\nu_x(\mathbf{T}) \geq 0$ representing the number of children of x, i.e., for every $j \in \mathbb{N}_+$,

$$(x_1, \cdots, x_n, j) \in \mathbf{T} \text{ if and only if } 1 \leq j \leq \nu_x(\mathbf{T}).$$

For $x \in \mathbf{T}$, denote by $|x| = n$ its height. For two vertices $x, y \in \mathbf{T}$, write $x \preceq y$ if x is on the simple path from $\varnothing$ to y. Denote by $z = x \wedge y$ the common ancestor, i.e. the vertex with maximum height $|z|$ such that $z \preceq x, y$. Denote the tree $\mathbf{T}$ chopped at height n by $\mathbf{T}_n = \{x \in \mathbf{T}, |x| \leq n\}$ (in $\mathbf{T}_n$, $\nu_x = 0$ if $|x| = n$), and the population in generation n by $Z_n = \sum_{x\in\mathbf{T}} \mathbf{1}_{\{|x|=n\}}$. Denote the subtree of $\mathbf{T}$ rooted at x by $\mathbf{T}^x = \{y \in \mathbf{T}, x \preceq y\}$, and the population in the n-th generation of $\mathbf{T}^x$ by $Z_n^x = \sum_{y\in\mathbf{T}^x} \mathbf{1}_{\{|y|=n\}}$. For convenience of further usage (local time related calculations), an artificial root $\overleftarrow{\varnothing}$ with height -1 is added to be the parent of $\varnothing$.

For any fixed probability distribution $\mu = (\mu_n)_{n\in\mathbb{N}}$, the Galton-Watson tree with offspring distribution μ is a measure $\mathbf{P}_{\mathrm{GW}}$ on the set of planar trees, such that all the vertices have children distributed identically and independently as μ, i.e.

$$\nu_x \overset{iid}{\sim} \mu, \forall x \neq \overleftarrow{\varnothing}.$$

The expectation under this probability measure is denoted by $\mathbf{E}_{\mathrm{GW}}$.

Let $m = \sum_{i=0}^{\infty} i\mu_i$ denote the average number of children for a vertex (except $\overleftarrow{\varnothing}$), then there is a standard result on Galton-Watson trees which states that, if $m > 1$ (so-called supercritical), then the tree extends to infinity with a positive probability. In other words, denote by $\mathcal{S}$ the event that a Galton-Watson tree survives, $\{\mathbf{T} : Z_n(\mathbf{T}) > 0, \forall n \geq 0\}$, then $\mathbf{P}_{\mathrm{GW}}(\mathcal{S}) > 0$ for $m > 1$. To study the asymptotic behavior of the cover time, we study the first n generations of a supercritical Galton-Watson tree under the conditional probability measure $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$.

By the Kesten-Stigum theorem (cf. [62]), $(\frac{Z_n}{m^n})_{n\geq 0}$ is a martingale, and that it converges to a nontrivial limit $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$-almost surely,

$$W := \lim_{n\to\infty} \frac{Z_n}{m^n} \in (0,\infty), \tag{3.1.1}$$

when $\mathbf{E}_{\mathrm{GW}}(Z_1 \log Z_1) = \sum_{k\geq 1} k \log k \mu_k < \infty$.

Given a surviving tree $\mathbf{T}$ and fix $n > 0$, let $(X_n(t))_{t\geq 0}$ be a continuous time Markov jump process on $\mathbf{T}_n$ starting at $X_n(0) = \overleftarrow{\varnothing}$, with transition probabilities $p(\overleftarrow{\varnothing}, \varnothing) = 1$,

$$p(x, \overleftarrow{x}) = \frac{\lambda}{\lambda + \nu_x},\ p(x, x^{(i)}) = \frac{1}{\lambda + \nu_x}, \forall x \in \mathbf{T}_n \backslash \{\overleftarrow{\varnothing}\}, 1 \leq i \leq \nu_x.$$

The probability measure for this random walk is denoted by $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$, and its corresponding expectation by $\mathbf{E}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$. In fact, it is more natural to call the discrete walk λ-biased random walk, and the results for both settings are the same, see Remark 3.1.2 (1).

We shall work under the following hypotheses denoted by ($\mathbf{H}$),

$$\lambda > 1,\ m > 1,\ \sum_{k\geq 0} k^2 \mu_k < \infty.$$

Conventions:

1. The probability of a generic law independent of the constructions above is denoted by $\mathbb{P}$, with its expectation $\mathbb{E}$.

2. When an integer index is needed in the presence of a real value, we mean its integral part, for instance, we write $Z_{\log n}$ for $Z_{[\log n]}$.

3. We write $f \lesssim g$, if there exists a constant $C > 0$ such that $f \leq Cg$. This constant may depend on parameters m and λ.

4. All the $O(\cdot)$, $o(\cdot)$ notations are under the limit $n \to \infty$.

5. For convenience of the readers, notations used throughout this chapter are gathered here:

$$\sigma_n = \sqrt{\frac{\lambda^{n+1} - 1}{\lambda - 1}},\ t_n^\mu = \sigma_n^2(\log Z_n + \mu),\ s_n = \sum_{i=0}^{n} \frac{Z_i}{\lambda^i}.$$

3.1.2 Main results

The goal in this chapter is to estimate the cover time defined by

$$T_n^{cov}(\mathbf{T}) = \inf\left\{t : \{X_n(s), 0 \leq s \leq t\} = \mathbf{T}_n\right\}.$$

The main result is

Theorem 3.1.1. *Under* $(\mathbf{H})$, *for* $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$*-almost surely any tree* $\mathbf{T}$, *with* $x \in \mathbb{R}$ *and* $n \to \infty$, *when* $\lambda > m$,

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(\lambda-1)T_n^{cov}}{2\lambda^{n+1}\sum_{i=0}^{\infty}\frac{Z_i}{\lambda^i}} - n\log m - \log W \leq x \,\middle|\, \mathbf{T}_n\right) \to e^{-e^{-x}};$$

when $\lambda = m$,

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(m-1)T_n^{cov}}{2m^{n+1}\sum_{i=0}^{n} Z_i} - n\log m - \log W \leq x \,\middle|\, \mathbf{T}_n\right) \to e^{-e^{-x}};$$

when $1 < \lambda < m$,

$$\mathbf{P}_{\mathrm{w}}\left(\frac{(\frac{m}{\lambda}-1)(\lambda-1)T_n^{cov}}{2Wm^{n+1}} - n\log m - \log W \leq x \,\middle|\, \mathbf{T}_n\right) \to e^{-e^{-x}}.$$

Remark 3.1.2. 1. The the same result holds for the discrete time random walk with the same transition probabilities, since $(X_n(t))_{0\leq t\leq t_0}$ takes $t_0 + O(\sqrt{t_0})$ steps, and the error is negligible for the cover time.

2. In fact, the conditions in Lemma 3.2.1 are the only requirements for $\mathbf{T}$. Therefore, the result applies to other (random) trees satisfying these conditions, not necessarily the Galton-Watson trees.

3. The λ-biased case agrees with the case of the simple random walk (cf. eg. [37]) in first order in the number of excursions (round trips from $\overleftarrow{\varnothing}$) performed, for details see Remark 3.3.5.

4. If one define the random walk on $\mathbf{T}$ instead of $\mathbf{T}_n$, the relation of λ and m discussed in Theorem 3.1.1 correspond to whether the walk is transient, positive recurrent, or null recurrent.

3.1.3 Related works

The cover time of a finite graph (by the simple random walk), $T^{cov}(G)$, is a fundamental object for a finite graph $G = (V, E)$ (Section 2, Lovász [77]). For a simple graph G with $\#V = n$, a tight bound for its cover time was given in Feige [47], [48]

$$(1 - o(1))n \log n \le T^{cov}(G) \le 4n^3/27.$$

Bounds using hitting time were given in Matthews [81],

$$\max_{S \subseteq G} \min_{u,v \in S} H(u,v)(\log(\#E) - 1) \le T^{cov}(G) \le \max_{u,v \in G} H(u,v)(1 + \log n),$$

where $H(u, v)$ is the expected time that the walk takes from u to v.

Up to the first order approximation, a general bound with Discrete Gaussian Free Field (DGFF) was given in Ding, Lee and Peres [39], then improved in Zhai [93],

$$\mathbb{P}\left(\left|T^{cov}(G) - \#EM^2\right| \ge \#E(\sqrt{sR}M + sR)\right) \le Ce^{-cs},$$

where $M = \mathbb{E}(\max_{x \in V} \eta_x)$, $R = \max_{x,y \in V} R_{\text{eff}}(x, y)$, $(\eta_x)_{x \in V}$ is a DGFF on G (centered Gaussian variables such that $\mathrm{Cov}(\eta_x, \eta_y) = R_{\text{eff}}(x, y)$), and R_{eff} is the effective resistance (cf. [11], consider each edge as a wire of electrical resistance 1, and take effective resistance in the physics sense, following Ohm's law).

Sharper results can be obtained if one restricts to trees. The first order estimate for the cover time of an m-ary trees was obtained in Aldous [10],

$$T_n^{cov} = (2 + o(1))n^2 \frac{m^{n+1}}{m-1} \log m.$$

It is showed in Andreoletti and Debs [13] that, the first $R_n = (\gamma + o(1)) \log n$ generations are covered in n steps, by a recurrent Markovian random walk on the Galton-Watson tree, where γ is an explicit constant.

The case of simple random walk on binary trees received extensive studies recently, originally as a counterexample showing that at second order, the cover time is no longer determined by the DGFF (cf. [40]). Second order asymptotics with error $O(\log\log^8 n)$ were given in Ding and Zeitouni [40], then refined to $O(1)$ in Belius, Rosen and Zeitouni [23], and a scaling limit was established independently by Cortines, Louidor and Saglietti [32] and Dembo, Rosen and Zeitouni [37],

$$\mathbb{P}\left(\frac{T_n^{cov}}{2^{n+1}n} - n\log 2 + \log n \leq s\right) = \mathbb{E}\exp\left(-CZe^{-s}\right),$$

for some explicit constant C and distribution Z.

In the studies of the cover time, the continuous counterpart for trees is the two-dimensional torus. The first order estimate of its cover time was determined by Dembo, Peres, Rosen and Zeitouni [36], then the result was improved in Ding [38], Belius and Kistler [22], Abe [1], and most recently studied by Belius, Rosen and Zeitouni [24] to the extent that

$$\lim_{K\to\infty}\limsup_{\epsilon\to 0}\mathbb{P}\left(\left|\sqrt{T^{cov}(S^2)} - 2\sqrt{2}\left(\log\epsilon^{-1} - \frac{1}{4}\log\log\epsilon^{-1}\right)\right| > K\right) = 0,$$

where $T_\epsilon^{cov}(S^2)$ is the time for the walk to intersect every ball of radius ϵ on the 2-dimensional sphere.

3.1.4 Proof outline

The cover time is determined by the excursion time and local times defined as follows.

Definition 3.1.3. *Let $\mathbf{T}$ be an infinite tree, and let $(X_n(t))_{t\geq 0}$ be a random walk on $\mathbf{T}_n$.*

1. *The excursion time is denoted by*

$$t_n^{cov} = \int_0^{T_n^{cov}} \mathbf{1}_{\{X_n(s)=\overleftarrow{\varnothing}\}} ds.$$

2. *To establish the relation between t_n^{cov} and T_n^{cov} , let*

$$\tau_n(t) = \inf\left\{r > 0 : \int_0^r \mathbf{1}_{\{X_n(s)=\overleftarrow{\varnothing}\}} ds \geq t\right\}.$$

3. *The (normalized) local time at* $x \in \mathbf{T}_n \backslash \{\overleftarrow{\varnothing}\}$ *is*

$$L_n^x(t) = \frac{1}{\pi_n(x)} \int_0^{\tau_n(t)} \mathbf{1}_{\{X_n(s)=x\}} ds,$$

where the normalization factor

$$\pi_n(x) = \frac{\lambda + \nu_x}{\lambda^{|x|+1}},\ x \in \mathbf{T}_n \backslash \{\overleftarrow{\varnothing}\},$$

is the stationary distribution of the λ*-biased walk scaled at* $\pi_n(\overleftarrow{\varnothing}) = 1$.

Intuitively, the walk $(X_n(t))$ can be seen as independent samples of random walks that starts and ends at $\overleftarrow{\varnothing}$. Each of these trials is called an excursion. Then t_n^{cov} is the number of excursions performed to cover the tree $\mathbf{T}_n$, $\tau_n(t)$ is the actual time spent in t excursions, and $L_n^x(t)$ encodes the status of x in t excursions. From these definitions, we have

$$\begin{aligned} &\tau_n(t_n^{cov}) \le T_n^{cov} \le \lim_{\epsilon \to 0+} \tau_n(t_n^{cov} + \epsilon) \\ &\tau_n(t) = \sum_{x \in \mathbf{T}_n} \pi_n(x) L_n^x(t). \end{aligned} \tag{3.1.2}$$

In fact, it is not hard to determine L_t^x (Lemma 3.2.3) and τ_n (Lemma 3.4.1), and the main part of the proof is to estimate t_n^{cov}. The key observation is that, when the tree is almost covered (at the first order estimate of the cover time), the non-visited vertices can be seen as independent. (This is inspired by the extremal landscape structure in [32], see Remark 3.3.5 (1) for details.) The scaling limit of the cover time is then established by characterizing the process afterwards.

This chapter is organized as follows. In Section 3.2, we give the regularity conditions on trees and determine the distribution of local times L_n^x. In Section 3.3, we establish the scaling limit for t_n^{cov}. And in Section 3.4, we estimate T_n^{cov} by studying τ_n, and finish the proof of Theorem 3.1.1.

3.2 Preliminaries

3.2.1 The trees

Lemma 3.2.1. *Let $c, \epsilon > 0$. Under* (**H**)*, for* $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$*-almost surely any tree* $\mathbf{T}$*, when n is large enough,*

$$\sum_{i=0}^{n} Z_i \lesssim Z_n, \tag{3.2.3}$$

$$|Z_n - m^n W| \leq m^{n/2} \log n, \tag{3.2.4}$$

$$\sum_{|x|=n-c\log n} (Z_n^x)^2 \leq Z_n^{1+\epsilon}. \tag{3.2.5}$$

Proof. By (3.1.1), $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$-almost surely, $W \in (0, \infty)$. Therefore, there exists a constant $C > 1$ (depending on $\mathbf{T}$) such that for all $n \geq 0$,

$$\frac{1}{C} W m^n < Z_n < C W m^n.$$

(There are only finitely many n violating this relation with $C = 2$, take the maximum constant among these n.) One can then deduce (3.2.3).

Moreover, by Theorem 2, [56]: $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$-almost surely,

$$\limsup_{n\to\infty} \left| \frac{Z_n - m^n W}{\sqrt{Z_n \log n}} \right| = \sqrt{\frac{2\mathrm{Var}\mu}{m^2 - m}},$$

where we recall that μ is the offspring distribution of $\mathbf{P}_{\mathrm{GW}}$.

Therefore one can take n large enough such that

$$|Z_n - m^n W| \lesssim \sqrt{Z_n \log n} \leq \sqrt{C W m^n \log n},$$

and (3.2.4) follows.

By standard calculations,

$$\mathbf{E}_{\mathrm{GW}} Z_n = m^n, \mathrm{Var}_{\mathrm{GW}}(Z_n) = m^{n-1} \frac{m^n - 1}{m - 1} \mathrm{Var}\mu \lesssim m^{2n},$$

therefore, for n large enough,

$$\begin{aligned}
\mathbf{P}_{\mathrm{GW}} \left(\sum_{|x|=n-c\log n} (Z_n^x)^2 > m^{(1+\epsilon)n} \middle| \mathcal{S} \right) &\leq \frac{\mathbf{E}_{\mathrm{GW}}(\sum_{|x|=n-c\log n} (Z_n^x)^2 |\mathcal{S})}{m^{(1+\epsilon)n}} \\
&\lesssim \frac{\mathbf{E}_{\mathrm{GW}}(Z_{n-c\log n}) m^{2c\log n}}{m^{(1+\epsilon)n}} \\
&\lesssim m^{-\frac{\epsilon n}{2}},
\end{aligned}$$

and (3.2.5) follows from the union bound,

$$\mathbf{P}_{\mathrm{GW}}\left(\exists n > N, \sum_{|x|=n-c\log n} (Z_n^x)^2 > m^{(1+\epsilon)n} \middle| \mathcal{S}\right) \lesssim \sum_{n>N} m^{-\frac{\epsilon n}{2}} \overset{N\to\infty}{\longrightarrow} 0.$$

We remark that $\sum_{|x|=n-r}(Z_n^x)^2$ is monotone increasing in r, therefore, the event (3.2.5) is decreasing in r, and once it is valid for $c\log n$, it is valid for all $0 \le r \le c\log n$ simultaneously. $\square$

3.2.2 The local times

Definition 3.2.2. *Let* $\mathbf{T}$ *be a tree.*

1. *The effective resistance (for the λ-biased walk) between $x, y \in \mathbf{T}$ is*

$$R_{\mathrm{eff}}(x,y) = \sum_{\substack{(z,\overleftarrow{z}) \\ \text{on the simple path} \\ \text{from } x \text{ to } y}} \lambda^{|z|}.$$

2. *The effective resistance between $\overleftarrow{\varnothing}$ and any vertex at generation n is abbreviated as*

$$\sigma_n^2 = \frac{\lambda^{n+1}-1}{\lambda-1}.$$

3. *For $a, b > 0$, let $\mathtt{PG}(a,b)$ be the distribution of $\sum_{i=1}^{P} E_i$, where P and E_i are independent random variables such that $P \sim \mathtt{Poisson}(a)$ has Poisson distribution of expected value a, and $E_i \sim \mathtt{Exp}(b)$ has exponential distribution of expected value $\frac{1}{b}$.*

The following lemma characterizes local times.

Lemma 3.2.3. *On any infinite tree $\mathbf{T}$, let $x, y \in \mathbf{T}_n$ such that $y \prec x$ (i.e. y is a strict ancestor of x), let $s, t > 0$. Then under $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$,*

$$L_n^x(t) \sim \mathtt{PG}\left(\frac{t}{\sigma_{|x|}^2}, \frac{1}{\sigma_{|x|}^2}\right),$$

$$(L_n^x(t)|L_n^y(t)=s) \sim \mathtt{PG}\left(\frac{s}{\sigma_{|x|}^2-\sigma_{|y|}^2}, \frac{1}{\sigma_{|x|}^2-\sigma_{|y|}^2}\right).$$

Proof. By the memoryless property of the exponential distribution (if $X \sim$ $\texttt{Exp}(1)$, then $(X-c|X>c) \stackrel{d}{=} X)$, $L_n^x(t)$ is only affected by local times on the ray from $\overleftarrow{\varnothing}$ to x, independent of movements on other branches or offspring of x.

By knowledge of reversible Markov chains (cf. eg. (3.24), p.69, [11], it can be easily checked that R_{eff} defined in Definition 3.2.2 do correspond to that of a reversible Markov chain, since there is only one simple path between two vertices),

$$\frac{1}{R_{\text{eff}}(x,y)} = \pi(x)\mathbf{P}_x(\tau_y < \tau_x^+|\mathbf{T}_n),$$

where $\mathbf{P}_x$ is the λ-biased walk starting at x, τ_y is the first hitting times of y, i.e. $\tau_y = \inf\{t > 0, X_n(t) = y\}$, and τ_x^+ is the first returning time to x, $\tau_x^+ = \inf\{t > 0, X_n(t) = x, X_n(s) \not\equiv x, 0 \le s \le t\}$. In particular,

$$\mathbf{P}_x(\tau_{\overleftarrow{\varnothing}} < \tau_x^+|\mathbf{T}_n) = \frac{1}{\sigma_{|x|}^2 \pi_n(x)}.$$

Up to excursion time t, there are $\texttt{Poisson}(t)$ departures from $\overleftarrow{\varnothing}$, and by the equation above, each trip hits x independently with probability $\frac{1}{\sigma_{|x|}^2}$, thus there are $\texttt{Poisson}\left(\frac{t}{\sigma_{|x|}^2}\right)$ arrivals on x. Upon arrival at x, the walk returns to $\overleftarrow{\varnothing}$ in exponential time, with rate $\mathbf{P}_x(\tau_{\overleftarrow{\varnothing}} < \tau_x^+|\mathbf{T}_n) = \frac{1}{\sigma_{|x|}^2\pi_n(x)}$. Therefore, the total time spent at x has distribution $\texttt{PG}\left(\frac{t}{\sigma_{|x|}^2}, \frac{1}{\sigma_{|x|}^2\pi_n(x)}\right)$. Recall that local times are normalized by $\frac{1}{\pi_n}$, and the result follows.

Conditioned at $L_n^y(t)$, the proof is similar. (Only to notice that the local times at both x, y are normalized.) □

Lemma 3.2.4. *Let $a, b > 0$, let $X \sim \texttt{PG}(a,b)$.*

1. *The* PG *distribution has basic properties*

$$bX \sim \texttt{PG}(a,1),\ \mathbb{E}(X) = \frac{a}{b},\ \text{Var}(X) = \frac{2a}{b^2},\ \mathbb{P}(X=0) = e^{-a}.$$

2. *If $a > b$, then*

$$\mathbb{P}(X \le 1) \le e^{2\sqrt{ab}-a-b}.$$

Proof. (1) is clear by definition. For (2), by Chernoff bounds, for any $\theta > 0$,

$$\begin{aligned}\mathbb{P}(X \le 1) &= \mathbb{P}\left(e^{-\theta X} \ge e^{-\theta}\right) \\ &\le e^{\theta}\sum_{k=0}^{\infty} e^{-a}\frac{a^k}{k!}\left(1+\frac{\theta}{b}\right)^{-k} = e^{\theta - \frac{\theta}{b+\theta}a},\end{aligned}$$

and the result follows by choosing $\theta = \sqrt{ab} - b$. □

3.2.3 Ray-Knight Theorem

Definition 3.2.5. *A Discrete Gaussian Free Field (DGFF) on a tree* $\mathbf{T}$ *is a family of random variables* $(\eta_x)_{x\in\mathbf{T}}$ *such that* $\eta_{\overleftarrow{\varnothing}} = 0$, $(\eta_x)_{x\neq\overleftarrow{\varnothing}}$ *are centered Gaussian variables with (both the effective resistance and the DGFF can be defined up to any scale, the factor* $\frac{1}{2}$ *is taken in accordance to [32])*

$$\mathbb{E}(\eta_x - \eta_y)^2 = \frac{1}{2}R_{\text{eff}}(x, y).$$

Remark 3.2.6. Since we have explicit relative resistances, if we attach an independent Gaussian variable $N_y \sim \mathcal{N}\left(0, \frac{\lambda^{|y|}}{2}\right)$ at each vertex $y \in \mathbf{T}\backslash\{\overleftarrow{\varnothing}\}$, and let $\eta_x = \sum_{y\preceq x} N_y$, then $(\eta_x)_{x\in\mathbf{T}}$ is a DGFF on $\mathbf{T}$.

Theorem 3.2.7. *(Second Ray-Knight theorem, [46]) For any infinite tree* $\mathbf{T}$, *let* $(\eta_x)_{x\in\mathbf{T}}$ *be a DGFF on* $\mathbf{T}$ *independent of* $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$. *For any* $t > 0$,

$$\left\{L_n^x(t) + \eta_x^2 : x \in \mathbf{T}_n\right\} \overset{d}{=} \left\{(\eta_x + \sqrt{t})^2 : x \in \mathbf{T}_n\right\}.$$

In fact, Remark 3.2.6 indicates a direct proof of this theorem by induction.

Lemma 3.2.8. *Let* $(\eta_x)_{x\in\mathbf{T}}$ *be a DGFF on a tree* $\mathbf{T}$, *let* $n \geq 0$ *and* $\mu \in \mathbb{R}$ *such that*

$$Z_n > 0,\, \log Z_n + \mu > 0,$$

we have

$$\mathbb{P}\left(\max_{|x|=n} \eta_x > \sigma_n\sqrt{\log Z_n + \mu}\right) \leq \frac{e^{-\mu}}{2\sqrt{\pi(\log Z_n + \mu)}}.$$

Proof. We first recall the Gaussian tail estimate for $x > 0$ and $X \sim \mathcal{N}(0, 1)$,

$$\mathbb{P}(X > x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(y+x)^2/2}dy \leq \frac{e^{-x^2/2}}{\sqrt{2\pi}}\int_0^\infty e^{-xy}dy = \frac{e^{-x^2/2}}{x\sqrt{2\pi}}.$$

Then by the union bound, when $\log Z_n + \mu > 0$,

$$\mathbb{P}\left(\max_{|x|=n} \eta_x > \sigma_n\sqrt{\log Z_n + \mu}\right) \leq Z_n\frac{e^{-(\log Z_n+\mu)}}{2\sqrt{\pi(\log Z_n + \mu)}} = \frac{e^{-\mu}}{2\sqrt{\pi(\log Z_n + \mu)}}.$$

□

3.3 Excursion time

As showed in the proof outline, the excursion time t_n^{cov} is compared to the quantity

$$t_n^\mu = \sigma_n^2(\log Z_n + \mu), \tag{3.3.6}$$

where we recall that

$$\sigma_n^2 = \frac{\lambda^{n+1} - 1}{\lambda - 1}.$$

To start with, we show that the first $n-c\log n$ layers have negligible influences in terms of local times.

Lemma 3.3.1. *Let $c > \frac{3}{\log\lambda}$, $\mu \in \mathbb{R}$, and $\mathbf{T}$ be an infinite tree satisfying the conditions in Lemma 3.2.1. For $n \to \infty$, with probability $1 - o(1)$ under $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$,*

$$\max_{|x|=n-c\log n} \left| \frac{L_n^x(t_n^\mu) - t_n^\mu}{\sigma_n^2} \right| \lesssim \frac{1}{\sqrt{n}}.$$

Proof. Let (η_x) be a DGFF on $\mathbf{T}$ independent of $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$. Denote the joint law of the DGFF and $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$ by $\mathbb{P}(\cdot|\mathbf{T}_n)$ in this proof.

It is guaranteed by Lemma 3.2.1 that $\log Z_n + \mu > 0$ for n large enough. Then one can apply Lemma 3.2.8,

$$\mathbb{P}\left(\max_{|x|=n-c\log n} |\eta_x| \leq \sigma_{n-c\log n}\sqrt{\log Z_{n-c\log n} + \mu} \middle| \mathbf{T}_n \right) = 1 - o(1).$$

Moreover, by Theorem 3.2.7,

$$\mathbb{P}\left(\max_{|x|=n-c\log n} \left| \sqrt{L^x(t_n^\mu) + \eta_x^2} - \sqrt{t_n^\mu} \right| \leq \sigma_{n-c\log n}\sqrt{\log Z_{n-c\log n} + \mu} \middle| \mathbf{T}_n \right) = 1 - o(1).$$

Thus with probability $1 - o(1)$, for n large enough,

$$\begin{aligned}
&\max_{|x|=n-c\log n} |L^x(t_n^\mu) - t_n^\mu| \\
&= \max_{|x|=n-c\log n} \left| \left(\sqrt{L^x(t_n^\mu) + \eta_x^2} - \sqrt{t_n^\mu} \right) \left(\sqrt{L^x(t_n^\mu) + \eta_x^2} + \sqrt{t_n^\mu} \right) - \eta_x^2 \right| \\
&\leq \max_{|x|=n-c\log n} \left| \sqrt{L^x(t_n^\mu) + \eta_x^2} - \sqrt{t_n^\mu} \right|^2 \\
&\qquad + 2\sqrt{t_n^\mu} \max_{|x|=n-c\log n} \left| \sqrt{L^x(t_n^\mu) + \eta_x^2} - \sqrt{t_n^\mu} \right| + \max_{|x|=n-c\log n} |\eta_x| \\
&\lesssim \sigma_n^2 \lambda^{-c\log n/2} (\log Z_n + \mu) \lesssim \sigma_n^2 n^{-1/2},
\end{aligned}$$

where the last line follows from Lemma 3.2.1 and the condition $c > \frac{3}{\log\lambda}$. $\square$

Now we present the key observation that, non-visited vertices up to time t_n^μ have distinct ancestors at generation $n - c\log n$.

Lemma 3.3.2. *Let $c > \frac{3}{\log\lambda}$, $\mu \in \mathbb{R}$, and $\mathbf{T}$ be an infinite tree satisfying the conditions in Lemma 3.2.1. There exists $\epsilon > 0$ such that*

$$\max_{\substack{|y\wedge z|\geq n-c\log n+1,\\ |y|=|z|=n, y\neq z}} \mathbf{P}_{\mathrm{w}}\left(L_n^y(t_n^\mu) = L_n^z(t_n^\mu) = 0 \,\middle|\, (L_n^x(t_n^\mu))_{|x|=n-c\log n}, \mathbf{T}_n\right) = o\left(Z_n^{-1-\epsilon}\right),$$

uniformly in any choices of local times at generation $n - c\log n$ satisfying the conditions in Lemma 3.3.1.

Proof. For simplicity, the conditional probability $\mathbf{P}_{\mathrm{w}}\left(\cdot|(L_n^x(t_n^\mu))_{|x|=n-c\log n}, \mathbf{T}_n\right)$ is abbreviated as $\mathbf{P}_{\mathrm{w}}(\cdot|R_n)$, and we write L^x for the local time $L_n^x(t_n^\mu)$.

Let $\delta > 0$. Fix two vertices y, z at generation n of $\mathbf{T}_n$, such that they have common ancestor $w = y \wedge z$ after generation $n - c\log n$. Let $|w| = n - s \geq n - c\log n + 1$, and denote by x the ancestor of w with $|x| = n - \log n$. Then

$$\begin{aligned}
&\mathbf{P}_{\mathrm{w}}\left(L^y = L^z = 0 \,|\, R_n\right)\\
&\leq \mathbf{P}_{\mathrm{w}}\left(L^y = L^z = 0, L^w \geq 2\delta t_n^\mu \,|\, R_n\right) + \mathbf{P}_{\mathrm{w}}\left(L^w < 2\delta t_n^\mu \,|\, R_n\right)\\
&= \mathbf{P}_{\mathrm{w}}\left(L^z = 0 \,|\, L^y = 0, L^w \geq 2\delta t_n^\mu, R_n\right)\mathbf{P}_{\mathrm{w}}\left(L^y = 0, L^w \geq 2\delta t_n^\mu \,|\, R_n\right) + \mathbf{P}_{\mathrm{w}}\left(L^w < 2\delta t_n^\mu \,|\, R_n\right)\\
&= \mathbf{P}_{\mathrm{w}}\left(L^z = 0 \,|\, L^w \geq 2\delta t_n^\mu, R_n\right)\mathbf{P}_{\mathrm{w}}\left(L^y = 0, L^w \geq 2\delta t_n^\mu \,|\, R_n\right) + \mathbf{P}_{\mathrm{w}}\left(L^w < 2\delta t_n^\mu \,|\, R_n\right)\\
&\leq \mathbf{P}_{\mathrm{w}}\left(L^z = 0 \,|\, L^w = 2\delta t_n^\mu, R_n\right)\mathbf{P}_{\mathrm{w}}\left(L^y = 0 \,|\, R_n\right) + \mathbf{P}_{\mathrm{w}}\left(L^w < 2\delta t_n^\mu \,|\, R_n\right).
\end{aligned}$$

By Lemma 3.3.1, for n large enough we have

$$L^x > (1-\delta)t_n^\mu,$$

moreover, by Lemma 3.2.3,

$$(L^w|L^x) \stackrel{d}{=} \mathtt{PG}\left(\frac{L^x}{\sigma_{n-s}^2 - \sigma_{n-c\log n}^2}, \frac{1}{\sigma_{n-s}^2 - \sigma_{n-c\log n}^2}\right).$$

where denominator above is bounded by

$$\frac{\lambda-1}{\lambda}\sigma_{n-s}^2 \leq \sigma_{n-s}^2 - \sigma_{n-s-1}^2 \leq \sigma_{n-s}^2 - \sigma_{n-c\log n}^2 \leq \sigma_{n-s}^2.$$

Therefore, for local times of the generation $n - c\log n$ satisfying Lemma 3.3.1 and for n large enough,

$$\begin{aligned}
&\mathbf{P}_{\mathrm{w}}\left(L^w < 2\delta t_n^\mu \,|\, R_n\right)\\
&= \mathbf{P}_{\mathrm{w}}\left(\mathtt{PG}\left(\frac{L^x}{\sigma_{n-s}^2 - \sigma_{n-c\log n}^2}, \frac{1}{\sigma_{n-s}^2 - \sigma_{n-c\log n}^2}\right) < 2\delta t_n^\mu \,\middle|\, R_n\right)\\
&\leq \mathbb{P}\left(\mathtt{PG}\left(\frac{(1-\delta)t_n^\mu}{\sigma_{n-s}^2}, \frac{\lambda/(\lambda-1)}{\sigma_{n-s}^2}\right) \leq 2\delta t_n^\mu\right)\\
&\leq e^{-\lambda\left(\sqrt{1-\delta}-\sqrt{2\delta\lambda/(\lambda-1)}\right)^2(\log Z_n+\mu)},
\end{aligned}$$

where the notation $\mathbb{P}(\texttt{PG}(a,b) \le x)$ stands for the probability that a random variable distribution as $\texttt{PG}(a,b)$ is smaller than x, and the last line is due to Lemma 3.2.4.

Similarly, study $(L^z|L^w)$ and $(L^y|L^x)$ by Lemma 3.2.3, for n large enough,

$$\begin{aligned}&\mathbf{P}_{\mathrm{w}}\left(L^z = 0 \,|\, L^w = 2\delta t_n^\mu, R_n\right)\mathbf{P}_{\mathrm{w}}\left(L^y = 0 \,|\, R_n\right)\\ &= e^{-\frac{2\delta t_n^\mu}{\sigma_n^2-\sigma_{n-s}^2}-\frac{L^x}{\sigma_n^2-\sigma_{n-c\log n}^2}} \le e^{-\frac{2\delta t_n^\mu}{\sigma_n^2}-\frac{(1-\delta)t_n^\mu}{\sigma_n^2}} = e^{-(1+\delta)(\log Z_n+\mu)}.\end{aligned}$$

To show that the two probabilities above are bounded by $o\left(Z_n^{-1-\epsilon}\right)$, it suffices to choose $\delta, \epsilon > 0$ such that

$$\begin{aligned}\lambda\left(\sqrt{1-\delta}-\sqrt{2\delta\lambda/(\lambda-1)}\right)^2 &> 1+\epsilon,\\ 1+\delta &> 1+\epsilon,\end{aligned} \tag{3.3.7}$$

which is always possible when $\lambda > 1$. □

The estimate above is enough to control the cover time of the n-th generation, however a Galton-Watson tree may have leaves in younger generations, in other words, the walk visits all vertices in the n-th generation does not guarantee that it covers $\mathbf{T}_n$. We treat leaves before the generation $n-1$ separately.

Lemma 3.3.3. *Let $\mu \in \mathbb{R}$, and let $\mathbf{T}$ be an infinite tree satisfying the conditions in Lemma 3.2.1, then*

$$\mathbf{P}_{\mathrm{w}}\left(\exists x \in \mathbf{T}_n, |x| \le n-1, L_n^x(t_n^\mu) = 0 |\mathbf{T}_n\right) = o(1).$$

Proof. By Lemma 3.2.3 and Lemma 3.2.4 (1), each $x \in \mathbf{T}_n$ with $|x| \le n-1$ is not visited at excursion time t_n^μ with probability

$$\mathbf{P}_{\mathrm{w}}(L_n^x(t_n^\mu) = 0|\mathbf{T}_n) = e^{-\sigma_{|x|}^{-2}t_n^\mu} \le e^{-\lambda(\log Z_n+\mu)},$$

thus by the union bound,

$$\begin{aligned}\mathbf{P}_{\mathrm{w}}\left(\exists x \in \mathbf{T}_n, |x| \le n-1, L_n^x(t_n^\mu) = 0 |\mathbf{T}_n\right) &\le e^{-\lambda(\log Z_n+\mu)} \sum_{|x|\le n-1} 1\\ &= e^{-\lambda(\log Z_n+\mu)} \sum_{i=0}^{n-1} Z_i,\end{aligned}$$

and the conclusion follows from (3.2.3). □

Returning to the n-th generation, by Lemma 3.3.2, the non-visited vertices are almost independent at excursion time t_n^μ. Intuitively, the time to cover them is a binomial random variable of $Z_{n-c\log n}$ trials, converging to a Poisson distribution. We conclude upon this intuition:

Proposition 3.3.4. *Let $\mu \in \mathbb{R}$, and let $\mathbf{T}$ be an infinite tree satisfying the conditions in Lemma 3.2.1, recall t_n^μ defined in (3.3.6), we have*

$$\# \{|x| = n, L_n^x(t_n^\mu) = 0\} \xrightarrow{d} \texttt{Poisson}(e^{-\mu}) \text{ under } \mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n), \tag{3.3.8}$$

$$\mathbf{P}_{\mathrm{w}}\left(\frac{t_n^{cov}}{\sigma_n^2} - n\log m - \log W \le \mu \middle| \mathbf{T}_n\right) \to e^{-e^{-\mu}}. \tag{3.3.9}$$

Proof. Since both the conclusions allow a $o(1)$ error, we may assume the conditions in Lemma 3.3.1 at generation $n - c\log n$ for an arbitrarily fixed $c > \frac{3}{\log\lambda}$. For simplicity, recall the abbreviation $\mathbf{P}_{\mathrm{w}}(\cdot|R_n)$ in the proof of Lemma 3.3.2, then it suffices replace $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$ by $\mathbf{P}_{\mathrm{w}}(\cdot|R_n)$. Moreover, denote by

$$F_n^\mu = \{|x| = n, L_n^x(t_n^\mu) = 0\}, E_n^\mu = \{|x| = n - c\log n + 1, \#(\mathbf{T}_n^x \cap F_n^\mu) = 1\},$$

the goal is to estimate F_n^μ, which is achieved by comparison to E_n^μ.

By Lemma 3.2.3, under $\mathbf{P}_w(\cdot|R_n)$, local times are distributed as

$$(L_n^y(t_n^\mu)|R_n) \sim \texttt{PG}\left(\frac{L_n^x(t_n^\mu)}{\sigma_n^2 - \sigma_{n-c\log n}^2}, \frac{1}{\sigma_n^2 - \sigma_{n-c\log n}^2}\right)$$

for each pair of vertices

$$x \prec y, |x| = n - c\log n, |y| = n.$$

Therefore, by Lemma 3.3.1,

$$\begin{aligned}
\mathbf{E}_{\mathrm{w}}\left(\#F_n^\mu \mid R_n\right) &= \sum_{|x|=n-c\log n} e^{-\frac{L_n^x(t_n^\mu)}{\sigma_n^2 - \sigma_{n-c\log n}^2}} Z_n^x \\
&= \sum_{|x|=n-c\log n} e^{-\log Z_n - \mu + o(1)} Z_n^x = e^{-\mu + o(1)}.
\end{aligned}$$

Furthermore, by definition one has

$$0 \le \#F_n^\mu - \#E_n^\mu \le 2 \sum_{|x|=n-c\log n+1} \sum_{y,z\in\mathbf{T}_n^x} \mathbf{1}_{\left\{y,z\in F_n^\mu\right\}},$$

where $\mathbf{E}_{\mathrm{w}}(\mathbf{1}_{\left\{y,z\in F_n^\mu\right\}}|R_n) = o(Z_n^{-(1+\epsilon)})$ by Lemma 3.3.2.

Therefore, by (3.2.5),

$$\begin{aligned}\mathbf{E}_{\mathrm{w}}\left(\#E_n^\mu \mid R_n\right) &= \mathbf{E}_{\mathrm{w}}(\#F_n^\mu|R_n) + \mathbf{E}_{\mathrm{w}}(\#E_n^\mu - \#F_n^\mu|R_n) \\ &= e^{-\mu+o(1)} + o\left(Z_n^{-(1+\epsilon)} \sum_{|x|=n-c\log n+1} (Z_n^x)^2\right) \\ &= e^{-\mu+o(1)} + o(1),\end{aligned}$$

where ϵ is the parameter in Lemma 3.3.2.

Moreover, conditioned on local times of layer $n - c\log n$, the subtrees $(\mathbf{T}^x)_{|x|=n-c\log n+1}$ are independent, thus for any $\theta > 0$, the Laplace transform of $\#E_n^\mu$ is given by

$$\begin{aligned}&\mathbf{E}_{\mathrm{w}}\left(e^{-\theta\#E_n^\mu} \,\middle|\, R_n\right) \\ =& \prod_{|x|=n-c\log n+1} \mathbf{E}_{\mathrm{w}}\left(e^{-\theta\cdot\mathbf{1}_{\{x\in E_n^\mu\}}} \,\middle|\, R_n\right) \\ =& \prod_{|x|=n-c\log n+1} \left(1 - \left(1-e^{-\theta}\right)\mathbf{P}_{\mathrm{w}}\left(x \in E_n^\mu \mid R_n\right)\right) \\ =& \prod_{|x|=n-c\log n+1} e^{-\left(1-e^{-\theta}+o(1)\right)\mathbf{P}_{\mathrm{w}}\left(x\in E_n^\mu \middle| R_n\right)} \\ =& \, e^{-\left(1-e^{-\theta}+o(1)\right)\mathbf{E}_{\mathrm{w}}\left(\#E_n^\mu \middle| R_n\right)} \to e^{-(1-e^{-\theta})e^{-\mu}},\end{aligned}$$

in other words

$$\#E_n^\mu \xrightarrow{d} \mathtt{Poisson}(e^{-\mu}).$$

Finally, by Lemma 3.3.2 and the union bound again, we have

$$\mathbf{P}_{\mathrm{w}}\left(\#E_n^\mu = \#F_n^\mu|R_n\right) = 1 - o(1),$$

thus $\#F_n^\mu$ has the same distributional limit as $\#E_n^\mu$, completing the proof of (3.3.8).

As for (3.3.9), $\{t_n^{cov} \le t_n^\mu\}$ differs from $\{\#F_n^\mu = 0\}$ by whether the leaves of $\mathbf{T}_n$ in early generations are covered, which is controlled in Lemma 3.3.3. Therefore, by (3.3.8),

$$\mathbf{P}_{\mathrm{w}}\left(t_n^{cov} \le t_n^\mu|R_n\right) = \mathbf{P}_{\mathrm{w}}(\#F_n^\mu = 0|R_n) + o(1) \to e^{-e^{-\mu}},$$

then (3.3.9) follows from the asymptotic of t_n^μ,

$$\frac{t_n^\mu}{\sigma_n^2} - n\log m - \log W \to \mu.$$

□

Remark 3.3.5. 1. For the complete binary tree, we have $m = 2$. Take $\lambda \to 1$, then $\mathbf{P}_{\mathrm{w}}(\cdot|\mathbf{T}_n)$ converges to a simple random walk, our result gives (non-rigorously)

$$t_n^{cov} \approx n^2 \log 2 + O(n),$$

whereas the cover time on the binary tree of a simple random walk is (cf. eg. [37])

$$t_n^{cov} = n^2 \log 2 - n \log n + O(n).$$

Lack of the second order term $n \log n$ is due to a difference in extremal landscapes: recall the notations in the proof of Proposition 3.3.4, in the case of a simple random walk [32], the set F_n^μ is approximately identically distributed clusters indexed by E_n^μ, whereas for the λ-biased walk, these clusters are single points instead.

2. Following exactly the same structure of the proof, one can study the maximum of a DGFF (η_x) (recall Definition 3.2.5),

$$\mathbb{P}\left(\max_{|x|=n} \eta_x \le \sigma_n \sqrt{\log Z_n - \frac{1}{2}\log\log Z_n + \mu}\right) \to \exp\left(-\frac{e^{-\mu}}{2\sqrt{\pi}}\right).$$

3. Comparing the cover time to the maximum of the corresponding DGFF, as suggested in [39], [40], one has

$$\begin{aligned} t_n^{cov} &= \frac{\lambda^{n+1}}{\lambda - 1}(n \log m + O(1)), \\ \max_{|x|=n} \eta_x^2 &= \frac{\lambda^{n+1}}{\lambda - 1}\left(n \log m - \frac{1}{2}\log n + O(1)\right). \end{aligned}$$

This difference in second order is due to different tails of Gaussian and local time distributions. (Compare Lemma 3.2.4 (2) and Lemma 3.2.8.)

3.4 From excursion time to real time

Lemma 3.4.1. *For any infinite tree* $\mathbf{T}$ *satisfying the conditions in Lemma 3.2.1, let* $s_n = \sum_{i=0}^n \frac{Z_i}{\lambda^i}$, $t > 0$, *then*

$$\mathbf{E}_{\mathrm{w}}\left(\tau_n(t)|\mathbf{T}_n\right) = \mathbf{E}_{\mathrm{w}}\left(\sum_{x \in \mathbf{T}_n} \pi_n(x) L_n^x(t) \middle| \mathbf{T}_n\right) = 2ts_n, \tag{3.4.10}$$

$$\mathrm{Var}_{\mathrm{w}}(\tau_n(t)|\mathbf{T}_n) = o\left(\frac{t\lambda^n s_n^2}{n}\right). \tag{3.4.11}$$

Proof. The expected value (3.4.10) is clear using the fundamental estimates $\mathbf{E}_{\mathrm{w}}(L^x(t)|\mathbf{T}_n) = t$ and $\sum_{|x|=k} \nu_x = Z_{k+1}$. As for (3.4.11), for any $x, y \in \mathbf{T}_n$, conditioned at $L_n^{x\wedge y}(t)$, local times $L_n^x(t)$ and $L_n^y(t)$ are independent with the same expected value $L_n^{x\wedge y}(t)$, thus by Lemma 3.2.4 (1),

$$\mathrm{Cov}_{\mathrm{w}}(L_n^x(t), L_n^y(t)|\mathbf{T}_n) = \mathrm{Var}_{\mathrm{w}}(L_n^{x\wedge y}(t)|\mathbf{T}_n) = 2t\sigma^2_{|x\wedge y|} \leq 2t\frac{\lambda}{\lambda-1}\lambda^{|x\wedge y|},$$

then by (3.1.2),

$$\begin{aligned}
&\mathrm{Var}_{\mathrm{w}}(\tau_n(t)|\mathbf{T}_n) \\
&= \mathrm{Var}_{\mathrm{w}}\left(\sum_{x\in\mathbf{T}_n} \pi(x)L_n^x(t)\middle|\mathbf{T}_n\right) \\
&= \sum_{x,y\in\mathbf{T}_n} \pi(x)\pi(y)\mathrm{Cov}_{\mathrm{w}}(L_n^x(t), L_n^y(t)|\mathbf{T}_n) \\
&\leq 2t\frac{\lambda}{\lambda-1}\sum_{x,y\in\mathbf{T}_n} \lambda^{|x\wedge y|-|x|-|y|}\left(1+\frac{\nu_x}{\lambda}+\frac{\nu_y}{\lambda}+\frac{\nu_x\nu_y}{\lambda^2}\right),
\end{aligned}$$

where ν_x is the number of children for $x \in \mathbf{T}_n$.

Moreover,

$$\lambda^{|x\wedge y|-|x|-|y|}\frac{\nu_x}{\lambda} = \sum_{\overleftarrow{z}=x} \lambda^{|z\wedge y|-|z|-|y|},$$

therefore the variance above is further bounded by

$$\mathrm{Var}_{\mathrm{w}}(\tau_n(t)|\mathbf{T}_n) \leq 8t\frac{\lambda}{\lambda-1}\sum_{x,y\in\mathbf{T}_n} \lambda^{|x\wedge y|-|x|-|y|}.$$

Now it suffices to prove that

$$\sum_{x,y\in\mathbf{T}_n} \lambda^{|x\wedge y|-|x|-|y|} = o\left(\frac{\lambda^n s_n^2}{n}\right).$$

Indeed, fix any $c > 0$,

$$\begin{aligned}
&\sum_{x,y\in\mathbf{T}_n} \lambda^{|x\wedge y|-|x|-|y|} \\
&\leq \sum_{|x\wedge y|<n-c\log n} \lambda^{n-c\log n-|x|-|y|} + \sum_{|x\wedge y|\geq n-c\log n} \lambda^{n-|x|-|y|} \\
&\leq \lambda^{n-c\log n}\left(\sum_{x\in\mathbf{T}_n} \lambda^{-|x|}\right)^2 + \sum_{|x\wedge y|\geq n-c\log n} \lambda^{n-2(n-c\log n)} \\
&\leq \frac{\lambda^n s_n^2}{n^2} + \lambda^{2c\log n-n}\sum_{|x|=n-c\log n} (Z_n^x)^2,
\end{aligned}$$

where by (3.2.5) and definition of s_n,

$$\begin{aligned}&\lambda^{2c\log n-n}\sum_{|x|=n-c\log n}(Z_n^x)^2\\ &\le \lambda^{2c\log n-n}Z_n^{1+\epsilon}\\ &\le \lambda^{2c\log n-n}(s_n\lambda^n)^{1+\epsilon}=o\left(\frac{\lambda^n s_n^2}{n}\right).\end{aligned}$$

□

Proposition 3.4.2. *Recall s_n from Lemma 3.4.1, and recall that $\sigma_n^2 = \frac{\lambda^{n+1}-1}{\lambda-1}$. Under* (**H**)*, for any $\mu\in\mathbb{R}$ and $\mathbf{P}_{\mathrm{GW}}(\cdot|\mathcal{S})$-almost surely any tree* $\mathbf{T}$*,*

$$\mathbf{P}_{\mathrm{w}}\left(\frac{T_n^{cov}}{2s_n\sigma_n^2}-n\log m-\log W\le\mu\middle|\mathbf{T}_n\right)\to e^{-e^{-\mu}}.$$

Proof. By Lemma 3.2.1,

$$\log Z_n = n\log m+\log W+o(1),$$

therefore it suffices to show that (rigorously speaking, one should prove the following convergence for $\mu\pm\epsilon$, then take $\epsilon\to 0$ to deduce the proposition),

$$\mathbf{P}_{\mathrm{w}}\left(\frac{T_n^{cov}}{2s_n\sigma_n^2}-\log Z_n\le\mu\middle|\mathbf{T}_n\right)\to e^{-e^{-\mu}}.$$

In other words (recall t_n^μ defined in (3.3.6)), it suffices to prove that

$$\mathbf{P}_{\mathrm{w}}\left(\frac{T_n^{cov}}{2s_n\sigma_n^2}\le\frac{t_n^\mu}{\sigma_n^2}\middle|\mathbf{T}_n\right)\to e^{-e^{-\mu}}.$$

By (3.1.2), for any $\alpha>0$,

$$\begin{aligned}\mathbf{P}_{\mathrm{w}}(T_n^{cov}\le 2s_nt_n^\mu|\mathbf{T}_n)&\le\mathbf{P}_{\mathrm{w}}(\tau_n(t_n^{cov})\le 2s_nt_n^\mu|\mathbf{T}_n)\\ &\le\mathbf{P}_{\mathrm{w}}(t_n^{cov}\le t_n^{\mu+\alpha}|\mathbf{T}_n)\\ &\quad+\mathbf{P}_{\mathrm{w}}(t_n^{cov}>t_n^{\mu+\alpha},|\tau_n(t_n^{\mu+\alpha})-2t_n^{\mu+\alpha}s_n|>2s_n(t_n^{\mu+\alpha}-t_n^\mu)|\mathbf{T}_n).\end{aligned}$$

For the first term, by Proposition 3.3.4,

$$\mathbf{P}_{\mathrm{w}}(t_n^{cov}\le t_n^{\mu+\alpha}|\mathbf{T}_n)\le(1+o(1))e^{-e^{-\mu-\alpha}}.$$

For the second term, by Chebyshev's inequality and Lemma 3.4.1

$$\begin{aligned}&\mathbf{P}_{\mathrm{w}}(t_n^{cov}>t_n^{\mu+\alpha},|\tau_n(t_n^{\mu+\alpha})-2t_n^{\mu+\alpha}s_n|>2s_n(t_n^{\mu+\alpha}-t_n^\mu)|\mathbf{T}_n)\\ &\le\mathbf{P}_{\mathrm{w}}(|\tau_n(t_n^{\mu+\alpha})-2t_n^{\mu+\alpha}s_n|>2s_n(t_n^{\mu+\alpha}-t_n^\mu)|\mathbf{T}_n)\\ &=o\left(\frac{\frac{1}{n}\lambda^nt_n^{\mu+\alpha}}{(t_n^{\mu+\alpha}-t_n^\mu)^2}\right)=\alpha^{-2}o\left(1\right).\end{aligned}$$

Let $\alpha \to 0+$, we have

$$\limsup_{n\to\infty} \mathbf{P}_{\mathrm{w}}(T_n^{cov} \le 2s_n t_n^{\mu} | \mathbf{T}_n) \le e^{-e^{-\mu}}.$$

Similarly, for any $\alpha > 0$, and any $\beta(\alpha) > 0$ small enough,

$$\begin{aligned}
&\mathbf{P}_{\mathrm{w}}(T_n^{cov} \le 2s_n t_n^{\mu} | \mathbf{T}_n) \\
&\ge \mathbf{P}_{\mathrm{w}}\left(\tau_n(t_n^{cov} + \beta) \le 2s_n t_n^{\mu}, t_n^{cov} \le t_n^{\mu-\alpha} | \mathbf{T}_n\right) \\
&= \mathbf{P}_{\mathrm{w}}\left(t_n^{cov} \le t_n^{\mu-\alpha} | \mathbf{T}_n\right) - \mathbf{P}_{\mathrm{w}}\left(\tau_n(t_n^{cov} + \beta) \ge 2s_n t_n^{\mu}, t_n^{cov} \le t_n^{\mu-\alpha} | \mathbf{T}_n\right) \\
&\to e^{-e^{-\mu+\alpha}}.
\end{aligned}$$

□

Proof of Theorem 3.1.1. By Proposition 3.4.2, it suffices to estimate s_n.

For $1 < \lambda < m$, one can use (3.2.4) to show that

$$s_n - \sum_{i=0}^{n} \frac{m^i}{\lambda^i} W = O\left(\frac{m^{n/2}}{\lambda^{n/2}} + \sum_{i=n/2}^{n} \frac{m^{i/2}\log n}{\lambda^i}\right) = o\left(\frac{m^n}{n\lambda^n}\right).$$

Therefore, s_n can be replaced by $(\frac{m}{\lambda}-1)^{-1}\frac{m^{n+1}}{\lambda^{n+1}}W$, and the conclusion follows.

For $\lambda > m$, similarly, the difference between $\sum_{i=0}^{\infty} \frac{Z_i}{\lambda^i}$ and s_n is negligible,

$$\sum_{i=0}^{\infty} \frac{Z_i}{\lambda^i} - s_n = \sum_{i=n+1}^{\infty} \frac{Z_i}{\lambda^i} = O\left(\frac{m^n}{\lambda^n}\right).$$

For $\lambda = m$, $s_n = \sum_{i=0}^{n} Z_i$ follows from its definition in Lemma 3.4.1. □

Chapter 4

Capacity in high dimensions

This chapter is based on [21].

4.1 Introduction

Given a probability distribution η on $\mathbb{Z}^d$ ($d \geq 3$), the *capacity* of a finite set $A \subset \mathbb{Z}^d$ (with respect to η) is defined as

$$\mathsf{cap}_\eta A := \sum_{x \in A} \mathsf{P}_x^\eta(\tau_A^+ = \infty),$$

where P_x^η refers to the law of a (discrete) random walk (S_n) started at x with transition probability η, and $\tau_A^+ := \inf\{n \geq 1 : S_n \in A\}$ is (S_n)'s first returning time to A.

Let μ be a probability distribution on $\mathbb{N}$, and θ be a probability distribution on $\mathbb{Z}^d$. Consider the process that starts with a particle at $0 \in \mathbb{Z}^d$. At each step, the particles die after generating a random number of new particles independently according to the law μ, then these new particles drift away from their precursor independently according to the law θ. This process is called *branching random walk*, whose distribution is denoted by $P_{\mu,\theta}$. The branching random walk is called *critical* if μ has mean 1, in which case, it is well-known that the process dies out in finite time almost surely (except for the trivial case that μ is the Dirac measure at $\{1\}$). The *range* R of this process, i.e. the set of points in $\mathbb{Z}^d$ visited by the branching random walk, is then almost surely finite. Moreover, we denote by $\{\#T = n\}$ the event that the branching random walk generates exactly n particles in total before dying out. The notation T actually stands for the genealogy tree of the process, see Section 4.2 for details.

In this chapter, we study the capacity of the range of critical branching random walks, denoted by $\mathtt{cap}_\eta R$, conditioned on the event $\{\#T = n\}$ as $n \to \infty$.

Throughout the chapter, we shall consider distributions μ on $\mathbb{N}$ and θ, η on $\mathbb{Z}^d$ with the assumptions

$$\left.\begin{array}{l}\mu \text{ has mean 1 and finite variance, and } \mu \not\equiv \delta_1,\\ \theta \text{ is symmetric, aperiodic and irreducible such that } \mathsf{E}_0^\theta\left[e^{\sqrt{|S_1|}}\right] < \infty,\\ \eta \text{ is aperiodic, irreducible with mean 0 and finite } (d+1)\text{-th moment },\end{array}\right\} \tag{4.1.1}$$

where E_0^θ refers to taking expectation with respect to the random walk (S_i) started at 0 with transition probability θ.

Theorem 4.1.1. *Let μ, θ, η be probability distributions with the conditions in* (4.1.1)*.*

1. *In dimension $d \geq 7$, there is a constant $C(d, \mu, \theta, \eta) > 0$ such that under $P_{\mu,\theta}(\cdot | \#T = n)$, as $n \to \infty$,*

$$\frac{\mathtt{cap}_\eta R}{n} \to C(d, \mu, \theta, \eta) \text{ in probability.}$$

2. *In dimension $d = 6$, if μ has finite 5-th moment, then under $P_{\mu,\theta}(\cdot | \#T = n)$, as $n \to \infty$,*

$$\frac{\log n}{n} \mathtt{cap}_\eta R \to 2C_G^{-1} \text{ in probability,}$$

where

$$C_G = \frac{1}{4\pi^6 \sqrt{\det \Gamma_\eta \det \Gamma_\theta}} \left(\sum_{k=0}^{\infty} (k-1) k \mu(k) \right) C_f,$$

$$C_f = \mathbb{E}\left[\int_1^e dt \int_{\mathbb{R}^6} dx \cdot J_\eta(B_t^\theta + x)^{-4} J_\theta(x)^{-4} \right],$$

$\Gamma_\eta, \Gamma_\theta$ are the covariance matrices of η, θ respectively, $J_{(\cdot)}(x) = \sqrt{x \cdot \Gamma_{(\cdot)}^{-1} x}$, and B_t^θ is the Brownian motion in $\mathbb{R}^6$ with covariance matrix Γ_θ.

Remark 4.1.2.

1. Aperiodicity and irreducibility for θ and η are assumed for convenience of the proofs. In fact the same results in Theorem 4.1.1 hold for η and θ without those assumptions.

2. For $d \geq 7$, the constant $C(d, \mu, \theta, \eta)$ is implicit. We refer the reader to Remark 4.3.4 for more details.

3. The finite variance of the offspring distribution μ is required in Lemma 4.3.2 for the high dimensions $d \geq 7$, and the finite 5-th moment of μ is required in Proposition 4.4.5 for the critical dimension $d = 6$.

4. For the displacement law θ, the moment assumption is required for the dyadic coupling in Lemma 4.2.10, and the symmetry is required for the conversion from our infinite model to finite trees, see Remark 4.2.5 for details. (We use the symmetry of θ a few times elsewhere for convenience, but they are not essential.)

5. For the random walk distribution η, the moment assumptions are required for the asymptotic estimates of Green's functions in Lemma 4.2.7.

6. If μ is the geometric distribution with parameter $\frac{1}{2}$, i.e. $\mu(k) = 2^{-k-1}$, then $P_{\mu,\theta}(\cdot|\#T = n)$ is the law of the random walk indexed by a uniformly chosen tree of n nodes considered in [73]. In this case, by Lemma 4.4.10 and the methods developed in [73, Section 3.1], the convergence in probability for dimension 6 holds in L^2-sense.

7. If μ is the geometric distribution with parameter $\frac{1}{2}$, θ and η are one-step distributions of independent simple random walks, then $C_G = 9\pi^{-3}$. We refer the reader to Proposition 4.4.7 for explicit calculations.

Historically, the study of the capacity of the range of simple random walks dates back to Jain and Orey [58], where a law of large numbers was established for $d \geq 3$. Then useful tools were developed in the book of Lawler [67]. Recently, numerous studies for the sharper estimates of the capacity appear in Chang [30] for $d = 3$ (scaled convergence in distribution), Asselah, Schapira and Sousi [15] for $d \geq 6$, [16] for $d = 4$, and Schapira [88] for $d = 5$ (central limit theorem).

If, in the definition of capacity, we simply replace the escape probability by 1, then it gives us (the size of) the range $\#R$, which is a classical object for random walks, widely studied since the work of Dvoretzky and Erdős [43], in which a law of large numbers was given for random walks in dimension $d \geq 1$. The corresponding central limit theorem was given by Jain and Orey [58] for $d \geq 5$, Jain and Pruitt [59] for $d \geq 3$, and Le Gall [69] for $d \geq 2$. See also [74] for a general study of random walks in the domain of attraction of a stable distribution (i.e. without finite variance) by Le Gall and Rosen.

For branching random walks, the law of large numbers for (the size of) its range $\#R$ was given by Le Gall and Lin in [73],[72] for every $d \geq 1$, where in the critical dimension $d = 4$ they restrict to the geometric offspring distribution case. This result (in $d = 4$) was then generalized by Zhu in [94] for general distributions. See also [64], [65] for a related topic of local times of branching random walks.

We summarize that, in view of law of large numbers, the critical dimension (the largest dimension with sublinear growth) is $d = 2$ for the range of the simple random walk (SRW) [43], $d = 4$ for the range of the branching random walk (BRW) [73], also $d = 4$ for the capacity of the SRW [58], and $d = 6$ for the capacity of the BRW.

Indeed, the SRW or the BRW can be seen as a sequence of vertices, and one can establish corresponding infinite models for them with translational invariance property, which for the SRW started at 0 is simply

$$(S_i)_{i\in\mathbb{Z}} \overset{d}{=} (S_{m+i} - S_m)_{i\in\mathbb{Z}}.$$

Intuitively, this property shows that the SRW (or the BRW) is homogeneous in time. Moreover, either the range or the capacity can be decomposed into the sum over i of the contribution of S_i, therefore, it boils down to a one-point estimate and a second moment estimate for its concentration property. One can express this one-point estimate in terms of Green's functions, and study Green's functions by moment estimates with a careful analysis of the tree (in the case of BRW) and the underlying random walk.

The rest of the chapter is organised as follows. In Section 4.2 we introduce the models and some preliminary results regarding capacities, Green's functions and the Brownian mothion. The study of capacities of BRWs in high dimension $d \geq 7$ is discussed in Section 4.3, and the case of critical dimension $d = 6$ is discussed in Section 4.4. In particular, the main model with translational invariance property is established in Section 4.2.2, the strategy with which we relate Green's functions to the capacity is showed in Section 4.2.4, the behavior of Green's functions is mainly summarized in Lemma 4.3.2 and Corollary 4.4.6, and finally, the two parts of Theorem 4.1.1 are proved in Theorem 4.3.7 and Theorem 4.4.12 respectively.

In the sequel, with a slight abuse of notations, each time we write a constant $C(*)$, where $*$ is the set of parameters that this constant will depend, it will only be used in the current paragraph.

4.2 Preliminaries

In this section, we present systematically the definitions and models in this chapter.

4.2.1 Trees and spatial trees

A tree is a set $T \subset \cup_{n\geq 0}\mathbb{N}_+^n$, such that

- The root $\varnothing \in T$, where by convention we denote $\mathbb{N}_+^0 = \{\varnothing\}$.
- If a node $u = (u_1, \cdots, u_n) \in T$, then its parent $\overleftarrow{u} := (u_1, \cdots, u_{n-1}) \in T$.
- For each node $u = (u_1, \cdots, u_n) \in T$, there exists an integer $k_u(T) \geq 0$, the number of offspring of u in T, such that for every $j \in \mathbb{N}$, $(u_1, \cdots, u_n, j) \in T$ if and only if $1 \leq j \leq k_u(T)$.

We say that $u = (u_1, \cdots, u_n) \in T$ is an ancestor of $u' = (u'_1, \cdots, u'_{n'}) \in T$ if $n < n'$ and $u_i = u'_i$, $1 \leq i \leq n$, and if this is the case, we will write $u \prec u'$. We also define the height (generation) of a node to be its length as a word, i.e. if $u = (u_1, \cdots, u_n)$, then $|u| = n$. Moreover, we denote by $\#T$ the total number of nodes. In the following, we will omit T if it is clear that to which tree the nodes belong to from the context.

Since nodes of T are sequences of natural numbers, there exists a natural lexicographical order for them. We can therefore explore T in lexicographic order

$$u_0 = \varnothing, u_1, u_2, \cdots .$$

We remark that each node appears exactly once in this sequence if the tree is finite, thus if $\#T = n$, the sequence terminates at u_{n-1}.

Consider each node as a vertex, and add an edge between a node and its parent, then one can see T as an abstract graph. If we attach a vector $\mathbf{d}_u$ in $\mathbb{Z}^d$ to each directed edge $(\overleftarrow{u}, u)$, fix the position of the root at $V_\varnothing = 0$ and let $V_u = \sum_{u' \preceq u} \mathbf{d}_{u'}$, then $(V_u)_{u\in T}$ gives a spatial tree tructure.

Given a distribution μ on $\mathbb{N}$ and a distribution θ on $\mathbb{Z}^d$, we can define a probability measure on (spatial) trees, denoted by $P_{\mu,\theta}$, under which we have that

$$k_u \overset{i.i.d.}{\sim} \mu,\ \mathbf{d}_u \overset{i.i.d.}{\sim} \theta.$$

The abstract tree T under this law is called the *Galton-Watson tree*, while the spatial tree $(V_u)_{u\in T}$ is called the *branching random walk*.

4.2.2 The infinite model

In this section, we construct an infinite model based on Galton-Watson trees that will be used throughout this chapter and may be of independent interests to other problems. Intuitively, it can be seen as the discrete limit of critical Galton-Watson trees conditioned to be large ([8, Section 2.6]), and our construction generalises the one-sided version of infinite Galton-Watson trees in [73, Section 2.2].

We define a *forest indexed by a spine* to be a sequence of trees, (here $\mathcal{T}_i$ are standard trees as in Section 4.2.1),

$$\mathcal{T} = ((0, \mathcal{T}_0), (1, \mathcal{T}_1), (1, \mathcal{T}_{-1}), (2, \mathcal{T}_2), (2, \mathcal{T}_{-2}) \cdots),$$

where the roots $(\pm i, \varnothing)$ of $\mathcal{T}_i$ and $\mathcal{T}_{-i}$ $(i > 0)$ are identified (glued together) as one single point on the spine. We write $k_{(i,u)}(\mathcal{T}) = k_u(\mathcal{T}_i)$ for the number of offspring of node $u \in \mathcal{T}_i$, and in particular, $k^+_{(i,\varnothing)}(\mathcal{T}), k^-_{(i,\varnothing)}(\mathcal{T})$ are the numbers of offspring of points $(\pm i, \varnothing)$ in the two trees $\mathcal{T}_i, \mathcal{T}_{-i}$, respectively. We call the set of points $\{(i, \varnothing), i \in \mathbb{N}\}$ the *spine* of $\mathcal{T}$, and $(0, \varnothing)$ the *base point*. Notice that by adding edges between consecutive points on the spine, the forest can also be seen as an abstract tree and the base point does not always take the role of the 'root', see Remark 4.2.5.

We embed this forest in $\mathbb{Z}^d$, by taking $\mathbf{d}_{(i,u)}(\mathcal{T}) = \mathbf{d}_u(\mathcal{T}_i)$ as the spatial displacement from its parent, and letting $V_{(i,u)}(\mathcal{T})$ be the spatial position of u by summing over all displacements along the path from the base point $(0, \varnothing)$ to (i, u).

On the set of forests, we define the following probability measure $\mathbf{P}_{\mu,\theta}$:

- Offspring distributions are independent, except for the two offspring distributions of the same node, $k^{\pm}_{(i,\varnothing)}(\mathcal{T})$. For each $i \geq 0, u \neq \varnothing$,

$$k_{(i,u)}(\mathcal{T}) \overset{i.i.d.}{\sim} \mu,$$

  moreover,

$$k_{(0,\varnothing)}(\mathcal{T}) \sim \mu,$$

  while for other nodes $(\pm i, \varnothing)$ $(i > 0)$ on the spine

$$\mathbf{P}_{\mu,\theta}(k^+_{(i,\varnothing)}(\mathcal{T}) = i, k^-_{(i,\varnothing)}(\mathcal{T}) = j) = \mu(i + j + 1).$$

- Displacements $\mathbf{d}_{(i,u)}(\mathcal{T})$ are i.i.d. distributed as θ on each directed edge including edges on the spine, with the base point fixed at the origin, $V_{(0,\varnothing)}(\mathcal{T}) = 0$.

Remark 4.2.1. The law of the spine is indeed well-defined as a probability measure, because $\sum_{i,j\geq 0} \mu(i + j + 1) = \sum_{k\geq 0} k\mu(k) = 1$ for a critical distribution μ.

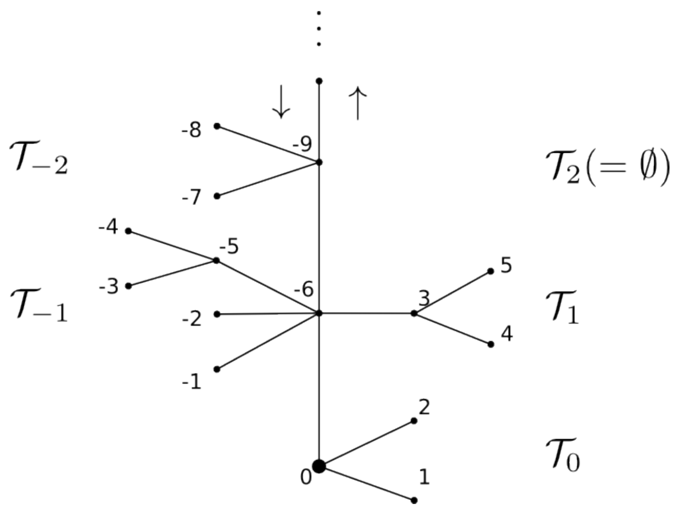


Figure 4.1: Lexicalgraphical order on the forest indexed by spine.

The lexicographical order of nodes on the forest is illustrated in Figure 4.1. We denote this sequence (seen as vertices on a graph) by

$$\cdots, u_{-1}(\mathcal{T}), u_0(\mathcal{T}) = (0, \varnothing), u_1(\mathcal{T}), \cdots, u_n(\mathcal{T}), \cdots,$$

and the corresponding spatial positions $(V_{u_i}(\mathcal{T}))$ by

$$\cdots, v_{-1}(\mathcal{T}), v_0(\mathcal{T}) = 0, v_1(\mathcal{T}), \cdots, v_n(\mathcal{T}), \cdots. \tag{4.2.2}$$

The range is defined as

$$R[i, j](\mathcal{T}) = \{v_i(\mathcal{T}), v_{i+1}(\mathcal{T}), \cdots, v_j(\mathcal{T})\}.$$

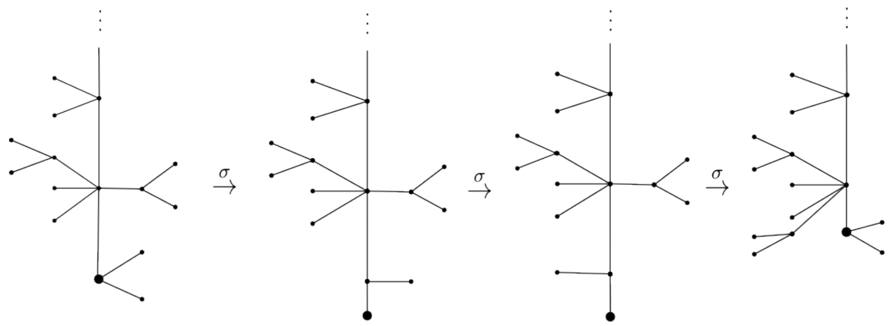


Figure 4.2: The transform σ on the tree. Base points $(0, \varnothing)$ are marked with bigger circles.

On the set of spine-indexed forests, we can then establish a shift transformation σ defined by (see Figure 4.2):

$$u_i(\sigma(\mathcal{T})) = u_{i+1}(\mathcal{T}),\ v_i(\sigma(\mathcal{T})) = v_{i+1}(\mathcal{T}) - v_1(\mathcal{T}). \tag{4.2.3}$$

One can easily check that $(u_i(\sigma(\mathcal{T})))_{i\in\mathbb{Z}}$ is the same sequence as $(u_{i+1}(\mathcal{T}))_{i\in\mathbb{Z}}$, and $(v_i(\sigma(\mathcal{T})))_{i\in\mathbb{Z}} = (v_{i+1}(\mathcal{T}) - v_1(\mathcal{T}))_{i\in\mathbb{Z}}$ is the corresponding positions of $(u_i(\sigma(\mathcal{T})))_{i\in\mathbb{Z}}$ in $\mathbb{Z}^d$, translated such that the base point $(0, \varnothing)$ stays at the origin. Moreover, the transformation is invariant under $\mathbf{P}_{\mu,\theta}$. In other words, for any measurable set A of spine-indexed forests,

$$\mathbf{P}_{\mu,\theta}(\mathcal{T} \in A) = \mathbf{P}_{\mu,\theta}(\sigma(\mathcal{T}) \in A).$$

Proposition 4.2.2. *Given the assumption* (4.1.1)*, the probability measure* $\mathbf{P}_{\mu,\theta}$ *is invariant and ergodic under* σ*. Consequently, we have that*

$$(v_i, \cdots, v_{n+i}) - v_i \stackrel{d}{=} (v_0, \cdots, v_n) \text{ under } \mathbf{P}_{\mu,\theta},\ \forall i \in \mathbb{Z}, n \in \mathbb{N}. \tag{4.2.4}$$

In other words,

$$R[i, n+i] - v_i \stackrel{d}{=} R[0, n] \text{ under } \mathbf{P}_{\mu,\theta},\ \forall i \in \mathbb{Z}, n \in \mathbb{N}.$$

Proof. Since θ is symmetric, it suffices to study the abstract tree structure (u_i).

As shown in Figure 4.3, take any node u: if it is the base point or some point not on the spine, then it has k children (thus degree $k+1$) with probability $\mu(k) = \mu(\deg(u)-1)$; otherwise, it has i children on the left and j children on the left (thus degree $i+j+2$) with probability $\mu(i+j+1) = \mu(\deg(u)-1)$.

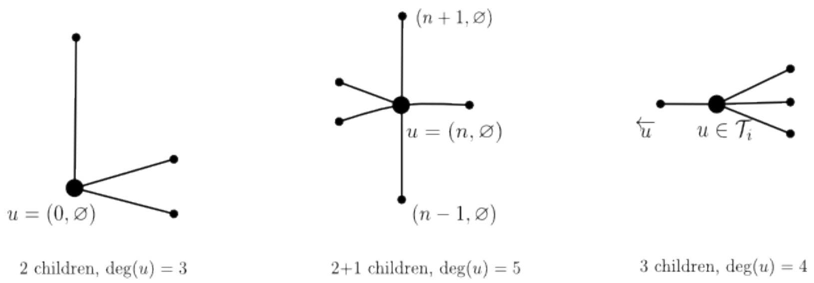


Figure 4.3: Neighborhood of a single node. Degree means the number of adjacent nodes as in an abstract graph.

Therefore, $\mathbf{P}_{\mu,\theta}$ can be seen as a probability measure on spine-indexed forests such that each node u has degree $k+1$ with probability $\mu(k)$. That is to say, $\mathbf{P}_{\mu,\theta}$ only takes into account the abstract tree structure, regardless of the base point. For example, denote by t and t' the structures depicted in Figure 4.4, and by A and A' the cylinder sets of forests whose first two or three subtrees are identical to t and t' respectively, then

$$\mathbf{P}_{\mu,\theta}(\mathcal{T} \in A) = \prod_{u \in t} \mu(\deg(u) - 1) = \prod_{u \in t'} \mu(\deg(u) - 1) = \mathbf{P}_{\mu,\theta}(\mathcal{T} \in A').$$

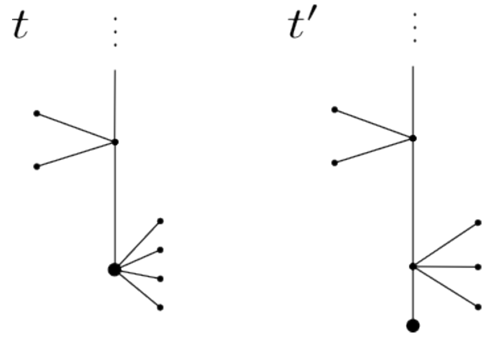


Figure 4.4: Finite trees t and t' that are only different in the position of base points.

Since σ only changes the base point, it is then invariant with respect to $\mathbf{P}_{\mu,\theta}$. Ergodicity is also clear by construction.

Then (4.2.4) follows easily by applying the invariant transform as illustrated below:

$$\begin{aligned} &\mathbf{P}_{\mu,\theta}(v_2(\mathcal{T}) - v_1(\mathcal{T}) = x) \\ =&\mathbf{P}_{\mu,\theta}(v_1(\sigma(\mathcal{T})) - v_0(\sigma(\mathcal{T})) = x) \\ =&\mathbf{P}_{\mu,\theta}(v_1(\mathcal{T}) - v_0(\mathcal{T}) = x), \end{aligned}$$

where we use the invariance property of σ with respect to $\mathbf{P}_{\mu,\theta}$ in the last line. $\square$

Remark 4.2.3. If one is only interested in the positive side,

$$((0, \mathcal{T}_0), (1, \mathcal{T}_1), (2, \mathcal{T}_2), \cdots),$$

then the spine has offspring distribution

$$\mathbf{P}_{\mu,\theta}(k_{(i,\varnothing)}(\mathcal{T}) = i) = \sum_{j=0}^{\infty} \mathbf{P}_{\mu,\theta}(k^+_{(i,\varnothing)}(\mathcal{T}) = i, k^-_{(i,\varnothing)}(\mathcal{T}) = j) = \sum_{j=0}^{\infty} \mu(i + j + 1),$$

which is consistent with the construction in [73, Section 2.2], for which the invariant transformation can be also induced by transformation σ defined in (4.2.3).

Remark 4.2.4. If we are interested in trees with n nodes instead of infinite nodes, with the same spirit as in the proof of Proposition 4.2.2, one has the equivalence between Galton-Watson trees conditioned on total population size $= n$ and *simply generated trees* in [8, Section 2.1]. For a tree with n nodes, one has to specify a root (both for the branching process and the combinatoric model), while in the infinite case, the 'root' is naturally set at infinity, and the 'base point' is actually redundant (for the combinatoric model).

Remark 4.2.5. If we replace edges in our model by directed edges of distribution θ pointing towards infinity, then Proposition 4.2.2 still holds without assuming that θ is symmetric.

In contrast, the standard branching random walk with asymmetric displacement is constructed by attaching displacements to the directed edges of the Galton-Watson tree pointing towards the root.

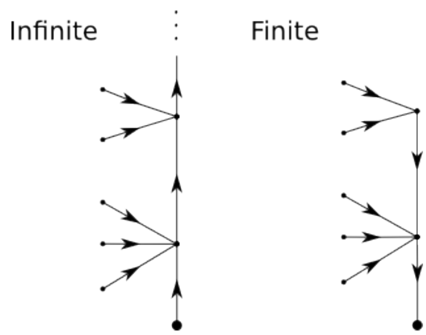


Figure 4.5: Directed edges for the infinite and finite models. Directions of edges are different on the 'spine'.

Therefore, for asymmetric θ, the role of the base point $(0, \varnothing)$ here and the role of the root in the standard branching random walk are different, and we can no longer compare them by identifying the base point of the infinite model as the root of a standard finite model, which is the method in Lemma 4.3.6. The displacement distribution θ is thus assumed symmetric.

4.2.3 Estimates on random walks and Green's functions

In this section, we present a few estimates on random walks and Green's function. We denote by P^η_x the law of the random walk started at x with transition probability η, and by (S_n) the random walk under P^η_x (or $S^{(i)}_n$ for its i.i.d. copies). Then the η-Green's function is defined as

$$G_\eta(x,y) = G_\eta(x-y) = \sum_{n=0}^{\infty} \mathsf{P}_0^\eta(S_n = x-y).$$

Lemma 4.2.6. *[68, p.24] Let η be an aperiodic and irreducible distribution on $\mathbb{Z}^d\,(d \geq 1)$ with mean 0 and finite third moment. Denote by Γ_η the covariance matrix of η. Then there exists a constant $C(d,\eta) > 0$ such that, uniformly for all $x \in \mathbb{Z}^d$,*

$$\left| \mathsf{P}_0^\eta(S_n = x) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Gamma_\eta}} e^{-\frac{x\cdot\Gamma_\eta^{-1}x}{2n}} \right| \leq C(d,\eta) n^{-\frac{d+1}{2}}$$

Lemma 4.2.7. *[68, Theorem 4.3.5] Given an aperiodic and irreducible distribution η on $\mathbb{Z}^d(d \geq 3)$ with mean 0 and covariance matrix Γ_η, if it has finite $(d+1)$-th moment $\mathsf{P}_0^\eta(|S_1|^{d+1}) < \infty$, then*

$$G_\eta(x) = \frac{C_{d,\eta}}{J_\eta(x)^{d-2}} + O(|x|^{1-d}),$$

where $C_{d,\eta} = \frac{\mathbf{\Gamma}(\frac{d}{2})}{(d-2)\pi^{d/2}\sqrt{\det \Gamma_\eta}}$, $\mathbf{\Gamma}(\cdot)$ refers to the Gamma function and $J_\eta(x) = \sqrt{x \cdot \Gamma_\eta^{-1} x}$.

Lemma 4.2.8. *Let η be an aperiodic and irreducible distribution on $\mathbb{Z}^d\,(d \geq 3)$ with mean 0 and finite third moment and $1 \leq m \leq d-1$. There exists a constant $C(d,\eta) > 0$ such that uniformly on the starting point $x_0 \in \mathbb{Z}^d$,*

$$\mathsf{E}_{x_0}^\eta(|S_n| \vee 1)^{-m} \leq C(d,\eta) n^{-\frac{m}{2}}.$$

Proof. Due to irreducibility of η, we have $J_\eta(x)^2 \geq C_1(d,\eta)|x|^2$. Then by Lemma 4.2.6, we can find $C_2(d,\eta) > 0$ such that

$$\begin{aligned}
\mathsf{E}_{x_0}^\eta(|S_n| \vee 1)^{-m} &\leq C_2(d,\eta) \sum_{x \in \mathbb{Z}^d} (|x_0 + x| \vee 1)^{-m} n^{-\frac{d}{2}} e^{-\frac{C_1(d,\eta)|x|^2}{2n}} + O(n^{-\frac{d+1}{2}}) \\
&\leq C_2(d,\eta) n^{-\frac{m}{2}} \sum_{x \in \mathbb{Z}^d/\sqrt{n}} \left(\left| \frac{x_0}{\sqrt{n}} + x \right| \vee \frac{1}{\sqrt{n}} \right)^{-m} n^{-\frac{d}{2}} e^{-\frac{C_1(d,\eta)|x|^2}{2}} + O(n^{-\frac{d+1}{2}}).
\end{aligned}$$

Moreover, denote by $B(y;r)$ the ball centered at y with radius r, then

$$
\begin{aligned}
& n^{-\frac{d}{2}} \sum_{x\in\mathbb{Z}^d/\sqrt{n}} \left(\left|\frac{x_0}{\sqrt{n}}+x\right| \vee \frac{1}{\sqrt{n}}\right)^{-m} e^{-\frac{C_1(d,\eta)|x|^2}{2}} \\
\le\, & n^{-\frac{d}{2}} \left(\sum_{x\in(\mathbb{Z}^d/\sqrt{n})\cap B(-x_0/\sqrt{n};1)} \left(\left|\frac{x_0}{\sqrt{n}}+x\right| \vee \frac{1}{\sqrt{n}}\right)^{-m} + \sum_{x\in(\mathbb{Z}^d/\sqrt{n})\setminus B(-x_0/\sqrt{n};1)} e^{-\frac{C_1(d,\eta)|x|^2}{2}} \right) \\
\underset{n\to\infty}{\longrightarrow} & \int_{B(0;1)} |x|^{-m} dx + \int_{\mathbb{R}^d} e^{-\frac{C_1(d,\eta)|x|^2}{2}} dx.
\end{aligned}
$$

which is a constant depending only on d and η. □

Corollary 4.2.9. *Let η be an aperiodic and irreducible distribution on $\mathbb{Z}^d$ ($d \ge 3$) with mean 0 and finite third moment, then for any $m \ge 1$,*

1. *there exists a constant $C(d,\eta,m) > 0$ such that uniformly for $x_0 \in \mathbb{Z}^d$,*
$$\mathsf{E}^{\eta}_{x_0}\left[\left(\sum_{i=0}^{n}(|S_i| \vee 1)^{-2}\right)^m\right] \le C(d,\eta,m)(\log n)^m;$$
2. *for any $k > 2$, there exists a constant $C'(d,\eta,m,k) > 0$ such that uniformly for $x_0 \in \mathbb{Z}^d$,*
$$\mathsf{E}^{\eta}_{x_0}\left[\left(\sum_{i=0}^{n}(|S_i| \vee 1)^{-k}\right)^m\right] \le C'(d,\eta,m,k).$$

Proof. The cases $m = 1$ for both $k = 2$ and $k > 2$ are clear by Lemma 4.2.8. For $m \ge 2$, applying Markov's property inductively gives that

$$
\begin{aligned}
& \mathsf{E}^{\eta}_{x_0}\left[\left(\sum_{i=0}^{n}(|S_i| \vee 1)^{-k}\right)^m\right] \\
\le & C'(d,\eta,m,k)\mathsf{E}^{\eta}_{x_0}\left[\sum_{i=0}^{n}\left((|S_i| \vee 1)^{-k}\right) \cdot \mathsf{E}^{\eta}_{S_i}\left[\left(\sum_{j=0}^{n-i}(|S'_j| \vee 1)^{-k}\right)^{m-1}\right]\right],
\end{aligned}
$$

where $\left(S'_j\right)$ denotes a random walk independent of (S_i). □

Lemma 4.2.10. *[45, Theorem 4] Let η be a probability distribution in $\mathbb{R}^d$ with mean 0 and covariance matrix Γ_η. If $\mathsf{E}^{\eta}_0\left[e^{\sqrt{|S_1|}}\right] < \infty$, then one can construct on the same probability space a Brownian motion (B_t) with covariance matrix Γ_η such that there exists $C, C' > 0$ depending on d, η such that*

$$\mathsf{P}^{\eta}_0\left(\max_{1\le k\le n}|S_k - B_k| \ge x\right) \le \frac{Cn}{e^{C'\sqrt{x}}}.$$

4.2.4 Capacity

Given a distribution η on $\mathbb{Z}^d$ ($d \geq 3$) and a finite set $A \subseteq \mathbb{Z}^d$, recall that the η-capacity is defined as

$$\mathtt{cap}_\eta A = \sum_{x \in A} \mathsf{P}_x^\eta(\tau_A^+ = \infty). \tag{4.2.5}$$

In this section, we give two estimates relating the η-capacity to the η-Green's function, which is defined as

$$G_\eta(x, y) = G_\eta(x - y) = \mathsf{E}_0^\eta\left[\sum_{i=0}^{\infty} \mathbf{1}_{(S_i = x - y)}\right] = \sum_{i=0}^{\infty} \mathsf{P}_0^\eta(S_i = x - y),\ x, y \in \mathbb{Z}^d.$$

Lemma 4.2.11. *Let $d \geq 3$ and η be any probability distribution on $\mathbb{Z}^d$. For any finite set $A \subset \mathbb{Z}^d$ and $k \in \mathbb{N}_+$,*

$$\mathtt{cap}_\eta A \geq \frac{\#A}{k+1} - \frac{\sum_{x,y \in A} G_\eta(x, y)}{k(k+1)}.$$

Proof. We define local times $L_A := \sum_{n=1}^{\infty} \mathbf{1}_{(S_n \in A)} \in \mathbb{N} \cup \{\infty\}$ for any finite set $A \subset \mathbb{Z}^d$, then by definition, $\mathtt{cap}_\eta A = \sum_{x \in A} \mathsf{P}_x^\eta(L_A = 0)$.

For any integers $a > 0$ and $b \geq 0$,

$$\begin{aligned}\sum_{x \in A} \mathsf{P}_x^\eta(L_A = a)\mathsf{P}_x^{-\eta}(L_A = b) &= \sum_{x,y \in A} \mathsf{P}_x^\eta(S_{\tau_A^+} = y)\mathsf{P}_y^\eta(L_A = a - 1)\mathsf{P}_x^{-\eta}(L_A = b)\\ &= \sum_{x,y \in A} \mathsf{P}_y^{-\eta}(S_{\tau_A^+} = x)\mathsf{P}_y^\eta(L_A = a - 1)\mathsf{P}_x^{-\eta}(L_A = b)\\ &= \sum_{y \in A} \mathsf{P}_y^\eta(L_A = a - 1)\mathsf{P}_y^{-\eta}(L_A = b + 1),\end{aligned}$$

where $-\eta$ refers to the distribution with $-\eta(x) := \eta(-x)$, $\forall x \in \mathbb{Z}^d$.

Thus by induction we have that

$$\sum_{x \in A} \mathsf{P}_x^\eta(L_A = a)\mathsf{P}_x^{-\eta}(L_A = b) = \sum_{x \in A} \mathsf{P}_x^\eta(L_A = 0)\mathsf{P}_x^{-\eta}(L_A = a + b).$$

By summing over $a \leq k$ and $b \geq 0$, it follows that

$$\begin{aligned}\sum_{x \in A} \mathsf{P}_x^\eta(L_A \leq k) &= \sum_{y \in A} \mathsf{P}_y^\eta(L_A = 0)\left(\sum_{a=0}^{k} \mathsf{P}_y^{-\eta}(L_A \geq a)\right)\\ &\leq (k+1)\sum_{y \in A} \mathsf{P}_y^\eta(L_A = 0).\end{aligned} \tag{4.2.6}$$

Therefore

$$\begin{aligned}\#A - \sum_{x\in A} \mathsf{P}^\eta_x(L_A > k) &= \sum_{x\in A}(1-\mathsf{P}^\eta_x(L_A>k))\\ &= \sum_{x\in A}\mathsf{P}^\eta_x(L_A\le k)\\ &\le (k+1)\sum_{y\in A}\mathsf{P}^\eta_y(L_A=0) = (k+1)\mathsf{cap}_\eta A.\end{aligned}$$

To conclude, it suffices to notice that $\mathsf{P}^\eta_x(L_A > k) \le \frac{\sum_{y\in A} G_\eta(x,y)}{k}$, which follows directly from Markov's inequality. □

Moreover, in our situation, the set $A_n = \{X_0, \cdots, X_n\}$ is the trajectory of a stationary process $(X_n)_{n\in\mathbb{Z}}$ up to translation (under some probability space $(\Omega, \mathcal{F}, \mathbb{P})$), in the sense that

$$X_0 = 0,\ \{X_0,\cdots,X_n\} \overset{d}{=} \{X_i,\cdots,X_{n+i}\} - X_i,\ \forall i\in\mathbb{Z}, \forall n\in\mathbb{N}, \tag{4.2.7}$$

where $A - x := \{a - x : a \in A\}$ for any set $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$. We can thus rewrite (4.2.5) as

$$\mathsf{cap}_\eta A_n = \sum_{i=0}^n \mathbf{1}_{\{X_i\notin\{X_{i+1},\cdots,X_n\}\}}\mathsf{P}^\eta_{X_i}(\tau^+_{A_n}=\infty). \tag{4.2.8}$$

and take expectation to get

$$\begin{aligned}\mathbb{E}\mathsf{cap}_\eta A_n &= \sum_{i=0}^n \mathbb{E}\Big[\mathbf{1}_{\{X_i\notin\{X_{i+1},\cdots,X_n\}\}}\mathsf{P}^\eta_{X_i}(\tau^+_{\{X_0,\cdots,X_n\}}=\infty)\Big]\\ &= \sum_{i=0}^n \mathbb{E}\Big[\mathbf{1}_{\{X_0\notin\{X_1,\cdots,X_{n-i}\}\}}\mathsf{P}^\eta_{X_0}(\tau^+_{\{X_{-i},\cdots,X_{n-i}\}}=\infty)\Big].\end{aligned}$$

This sum may be approximated (with a second moment method, for instance) by n times

$$\mathbb{E}\Big[\mathbf{1}_{\big\{X_0\notin\{X_1,\cdots,X_{\xi^r_n}\}\big\}}\mathsf{P}^\eta_{X_0}\Big(\tau^+_{\{X_{-\xi^l_n},\cdots,X_{\xi^r_n}\}}=\infty\Big)\Big], \tag{4.2.9}$$

where ξ^l_n and ξ^r_n are geometric killing times with parameter $\frac{1}{n}$.

The following lemma inspired by [67, Theorem 3.6.1] then allows us to establish a relation between (4.2.9) and Green's functions. Recall that ξ is a geometric variable with parameter λ if

$$\mathbb{P}(\xi = k) = \lambda(1-\lambda)^k,\ k\in\mathbb{N}.$$

Lemma 4.2.12. *Let $(X_n)_{n\in\mathbb{Z}} \in \mathbb{Z}^d$ be a stationary process up to translation in (4.2.7). Let $d \geq 3, n \geq 1$, and let ξ_n^l, ξ_n^r, ξ_n be independent geometric random variables with parameter $\frac{1}{n}$. If we set*

$$
\begin{aligned}
I_n &= \mathbf{1}_{\{X_0 \neq X_i, 0<i\leq \xi_n^r\}},\\
E_n &= \mathsf{P}^{\eta}_{X_0}\left(\tau^+_{\{X_{-\xi_n^l},\cdots,X_{\xi_n^r}\}} > \xi_n\right),\\
G_n &= \sum_{i=-\xi_n^l}^{\xi_n^r} G_\eta^{(1-\frac{1}{n})}(X_0, X_i),
\end{aligned}
$$

where $G_\eta^{(\lambda)}(x) = \sum_{k\geq 0} \lambda^k \mathsf{P}_0^\eta(S_k = x)$ denotes the Green's function with killing rate λ, then

$$\mathbb{E}[E_n G_n I_n] = 1.$$

Proof. For $m \in \mathbb{N}$ and $x_1, \cdots, x_m$ in $\mathbb{Z}^d$, we consider the event

$$B = B(m; x_1, \cdots, x_m) := \{\xi_n^l + \xi_n^r = m, X_{i-\xi_n^l} = X_{-\xi_n^l} + x_i, \forall 0 \leq i \leq m\}$$

with the convention that $x_0 = 0$. When m runs through $\mathbb{N}$ and (x_i) runs through all possible finite sequences of $\mathbb{Z}^d$, we have that

$$\sum_{m\geq 0} \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d} \mathbf{1}_{B(m;x_1,\cdots,x_m)} = 1.$$

Therefore, it suffices to prove that for any $B = B(m; x_1, \cdots, x_m)$,

$$\mathbb{E}[\mathbf{1}_B E_n G_n I_n] = \mathbb{P}(B).$$

Moreover, on a fixed B, we can define

$$B_j = \{\xi_n^l = j, \xi_n^r = m - j, X_i = X_0 + x_i, \forall 0 \leq i \leq m\},\ 0 \leq j \leq m,$$

then since E_n, I_n, G_n are all invariant under the translation $(X_i) \to (X_i - X_{-\xi_n^l})$, we have that

$$
\begin{aligned}
\mathbb{E}[\mathbf{1}_B E_n G_n I_n] &= \sum_{j=0}^{m} \mathbb{E}[\mathbf{1}_{B_j} E_n G_n I_n]\\
&= \sum_{j=0}^{m} \mathbb{P}(B_j) \mathbf{1}_{\{x_j \neq x_i, j<i\leq m\}} \mathsf{P}^\eta_{x_j}\left(\tau^+_{\{x_0,\cdots,x_m\}} > \xi_n\right) \sum_{k=0}^{m} G_\eta^{(1-\frac{1}{n})}(x_j, x_k).
\end{aligned}
$$

By the stationary property (4.2.7), we have that

$$\mathbb{P}(B_j) = \frac{\mathbb{P}(B)}{m+1},\ \forall 0 \leq j \leq m,$$

thus we can further simplify the equation above to

$$\mathbb{E}[\mathbf{1}_B E_n G_n I_n] = \frac{\mathbb{P}(B)}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{m} \mathbf{1}_{\{x_j \neq x_i, j<i\leq m\}} \mathsf{P}_{x_j}^{\eta} \left(\tau_{\{x_0,\cdots,x_m\}}^{+} > \xi_n \right) G_{\eta}^{(1-\frac{1}{n})}(x_j, x_k). \tag{4.2.10}$$

For any $A \subseteq \mathbb{Z}^d$, $z \in A$, by decomposing the random walk (S_n) started at z at the last time it hits $A \subseteq \mathbb{Z}^d$, it is not hard to see that ([67, Proposition 2.4.1 (b)])

$$\sum_{x \in A} \mathsf{P}_x^{\eta}(\tau_A^+ > \xi_n) G_{\eta}^{(1-\frac{1}{n})}(z, x) = 1.$$

Take $A = \{x_0, x_1, \cdots, x_m\}$, then we have that

$$\sum_{j=0}^{m} \mathbf{1}_{\{x_j \neq x_i, j<i\leq m\}} \mathsf{P}_{x_j}^{\eta}(\tau_A^+ > \xi_n) G_{\eta}^{(1-\frac{1}{n})}(z, x_j) = 1,\ z \in \{x_0, x_1, \cdots, x_m\}.$$

Put this into (4.2.10), then

$$\mathbb{E}[\mathbf{1}_B E_n G_n I_n] = \frac{\mathbb{P}(B)}{m+1} \sum_{k=0}^{m} 1 = \mathbb{P}(B).$$

The conclusion follows by adding up all choices of $B(m; x_1, \cdots, x_m)$. □

4.2.5 Strong mixing property for functions of the Brownian motion

The calculation for Green's functions will lead to some estimates of the following form, for which we give a concentration result in advance. This part is inspired partially by [73, Lemma 18, Lemma 19].

In this section, let $d \geq 3$, and we consider a continuous homogeneous functions of degree 2, $f : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}^+$ such that

$$f(\lambda z) = \lambda^{-2} f(z), z \in \mathbb{R}^d \backslash \{0\}, \lambda \in \mathbb{R} \backslash \{0\}. \tag{4.2.11}$$

Then in particular, f is bounded on the unit sphere, and

$$f(z) \asymp |z|^{-2},\ z \to \infty. \tag{4.2.12}$$

Consider the trajectory w of a d-dimensional Brownian motion. Let

$$F(w) = \int_1^e f(w(t))dt$$

and

$$Tw(t) = \frac{w(et)}{\sqrt{e}}.$$

Then one can easily deduce that:

1. F is almost surely finite;
2. T is invariant and ergodic in the Wierner space equipped with the probability measure of the Brownian motion;
3. $\int_1^{e^n} f(w(t)) = F(w) + F(Tw) + \cdots + F(T^{n-1}w)$.

Thus by Birkhoff's ergodic theorem, for the Brownian motion (B_t) in $\mathbb{R}^d$, the following integral converges almost surely to its expectation,

$$\frac{\int_1^{e^n} f(B_t)dt}{n} \to \mathbb{E}\left[\int_1^e f(B_t)dt\right]. \tag{4.2.13}$$

Moreover, we can improve it to a concentration property,

Proposition 4.2.13. *Let (B_t) be the Brownian motion in $\mathbb{R}^d$ $(d \geq 3)$ with non-degenerate covariance matrix Γ. Then for any $\epsilon > 0$, $m > 0$ and f satisfying (4.2.11), there exists a constant $C(d, \epsilon, m, \Gamma) > 0$ such that*

$$\mathbb{P}\left(\left|\frac{\int_1^n f(B_t)dt}{\log n} - \mathbb{E}\left[\int_1^e f(B_t)dt\right]\right| > \epsilon\right) \leq C(d, \epsilon, m, \Gamma)(\log n)^{-m}, \forall n \geq 1. \tag{4.2.14}$$

To prove this, we need the following moment estimate,

Lemma 4.2.14. *[92, Theorem 1] Let $(X_n)_{n\in\mathbb{Z}}$ be a (strictly) stationary sequence, i.e. a sequence of random variables such that for any $k \in \mathbb{N}$ and $t, t_1, \cdots, t_k \in \mathbb{Z}$*

$$(X_{t_1}, \cdots, X_{t_k}) \stackrel{d}{=} (X_{t_1+t}, \cdots, X_{t_k+t}).$$

Let $\mathcal{M}_i^j$ be the σ-field generated by $\{X_i, X_{i+1}, \cdots, X_j\}$, and let

$$\alpha(n) = \sup_{A\in\mathcal{M}_{-\infty}^0, B\in\mathcal{M}_n^\infty} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|.$$

For $r > 2, \delta > 0$, if we have $\mathbb{E}X_1 = 0$, $\mathbb{E}|X_1|^{r+\delta} < \infty$ and

$$\sum_{n=0}^{\infty}(n+1)^{\frac{r}{2}-1}\alpha(n)^{\frac{\delta}{r+\delta}} < \infty,$$

then there exists a constant $C(r, \delta) > 0$ such that

$$\mathbb{E}|X_1 + \cdots + X_n|^r \leq C(r, \delta)n^{\frac{r}{2}}, \ \forall n \geq 1.$$

Proof of Proposition 4.2.13. It suffices to prove (4.2.14) for the Brownian motion with covariance I_d. Let

$$X_n = \int_{e^{n-1}}^{e^n} f(B_t)dt - \mathbb{E}\left[\int_1^e f(B_t)dt\right] = F(T^{n-1}B_t) - \mathbb{E}\left[\int_1^e f(B_t)dt\right],$$

then by a change variable from n to e^n, it suffices to show that there exists $C(d,\epsilon,m) > 0$ with

$$\mathbb{P}(|X_1 + \cdots + X_n| > \epsilon n) \leq C(d,\epsilon,m)n^{-m}.$$

By (4.2.12), (X_n) is a stationary sequence with mean 0. Moreover, by applying the same trick as in Corollary 4.2.9, we can easily show that it also satisfies the moment requirement $\mathbb{E}|X_1|^{r+\delta} < \infty$ for all r, δ. Therefore, to apply Lemma 4.2.14, it suffices to prove that for (X_n) we have $\alpha(n) = O(e^{-cn})$ for some $c > 0$.

Since (X_n) only depends on the trajectory of the Brownian motion, which is a Markov process, we have that

$$\alpha(n) = \sup_{A,B\subseteq\mathbb{R}^6} |\mathbb{P}(B_1 \in A, B_{e^n} \in B) - \mathbb{P}(B_1 \in A)\mathbb{P}(B_{e^n} \in B)|.$$

Clearly, $\mathbb{P}(|B_1| > n) = \mathbb{P}(|B_{e^n}| > ne^{n/2}) = O(e^{-cn})$, so we may consider the supreme restricted to bounded balls in $\mathbb{R}^d$, $A \subseteq \mathrm{Ball}(0;n)$, $B \subseteq \mathrm{Ball}(0;ne^{n/2})$. Then we expand $\alpha(n)$ by definition,

$$\begin{aligned}
&\sup_{\substack{A\subseteq \mathrm{Ball}(0;n)\\ B\subseteq \mathrm{Ball}(0;ne^{n/2})}} |\mathbb{P}(B_1 \in A, B_{e^n} \in B) - \mathbb{P}(B_1 \in A)\mathbb{P}(B_{e^n} \in B)| \\
\leq &\frac{1}{(2\pi)^d} \sup_{\substack{A\subseteq \mathrm{Ball}(0;n)\\ B\subseteq \mathrm{Ball}(0;ne^{n/2})}} \int_A dx \int_B dy \cdot \left| \frac{1}{\sqrt{e^n-1}} e^{-\frac{|x|^2}{2} - \frac{|y-x|^2}{2(e^n-1)}} - \frac{1}{\sqrt{e^n}} e^{-\frac{|x|^2}{2} - \frac{|y|^2}{2e^n}} \right| \\
\leq &\frac{1}{(2\pi)^d} \sup_{\substack{A\subseteq \mathrm{Ball}(0;n)\\ B\subseteq \mathrm{Ball}(0;ne^{n/2})}} \int_A dx \int_B dy \cdot \frac{1}{\sqrt{e^n}} \left| e^{-\frac{|x|^2}{2} - \frac{|y-x|^2}{2(e^n-1)}} - e^{-\frac{|x|^2}{2} - \frac{|y|^2}{2e^n}} \right| \\
&+ \frac{1}{(2\pi)^d} \sup_{\substack{A\subseteq \mathrm{Ball}(0;n)\\ B\subseteq \mathrm{Ball}(0;ne^{n/2})}} \int_A dx \int_B dy \cdot \left| \frac{1}{\sqrt{e^n-1}} - \frac{1}{\sqrt{e^n}} \right| e^{-\frac{|x|^2}{2} - \frac{|y-x|^2}{2(e^n-1)}} \\
\leq &\frac{1}{(2\pi)^d e^{\frac{n}{2}}} \sup_{\substack{A\subseteq \mathrm{Ball}(0;n)\\ B\subseteq \mathrm{Ball}(0;ne^{n/2})}} \int_A e^{-\frac{|x|^2}{2}} dx \int_B e^{-\frac{|y|^2}{2e^n}} dy \cdot O\left(\left| \frac{|y-x|^2}{2(e^n-1)} - \frac{|y|^2}{2e^n} \right| \right) + O(e^{-n}) \\
= &O(n^2 e^{-n}),
\end{aligned}$$

where in the last line, we upper bound the integrals by 1 and use that $|x| \leq n$, $|y| \leq ne^{n/2}$.

In conclusion, we have $\alpha(n) = O(e^{-cn})$ for some $c > 0$, thus the conditions in Lemma 4.2.14 are satisfied. Therefore, let $\delta = \frac{1}{2}$, then for any $r > 2$, there exists a constant $C(r)$ such that

$$\mathbb{E}|X_1 + \cdots + X_n|^r \leq C(r)n^{\frac{r}{2}}, \ \forall n \geq 1,$$

then by a Chebyshev-type inequality,

$$\mathbb{P}\Big(|X_1 + \cdots + X_n| \geq k(C(r)n^{\frac{r}{2}})^{\frac{1}{r}}\Big) \leq k^{-r}.$$

The conclusion follows by taking $r = 2m$ and $k = \epsilon n^{\frac{1}{2}}(C(r))^{-\frac{1}{r}}$. □

Corollary 4.2.15. *Let η be a distribution satisfying the conditions in Lemma 4.2.10, and recall that f is a function satisfying* (4.2.11)*. Take an arbitrary value for $f(0)$ so that it is defined on $\mathbb{R}^d$, then for any $\epsilon > 0$ and $m > 0$, there exists a constant $C(d, \eta, \epsilon, m) > 0$*

$$\mathsf{P}_0^\eta\left(\left|\frac{\sum_{i=0}^n f(S_i)}{\log n} - \mathbb{E}\left[\int_1^e f(B_t)dt\right]\right| > \epsilon\right) \leq C(d, \eta, \epsilon, m)(\log n)^{-m}, \ \forall n \geq 0.$$

Proof. Extend the discrete process $(S_n)_{n\in\mathbb{N}}$ to a continuous-time process $(S_{\lfloor t\rfloor})_{t\geq 0}$. Using Lemma 4.2.10 and some basic estimates on the Brownian motion, we can find a Brownian motion with the same covariance matrix as S on the same probability space, a constant $C(d, \eta) > 0$, and a power index k, such that the event

$$F_n := \left\{\max_{0\leq t\leq n} |S_t - B_t| < C(d, \eta)(\log n)^2\right\} \cap \left\{\inf_{t > (\log n)^k} |B_t| > (\log n)^3\right\}$$

happens with probability $1 - O((\log n)^{-m})$.

Recall that f is continuous on $\mathbb{R}^d\backslash\{0\}$ and homogeneous of degree 2, we can easily get that for any $\delta > 0$, when n is large enough, for any $x, y \in \mathbb{R}^d$ such that $|y| > (\log n)^3$, $|x - y| < (\log n)^2$,

$$\begin{aligned} |f(x) - f(y)| &\leq \big||x|^{-2} - |y|^{-2}\big| f\left(\frac{x}{|x|}\right) + |y|^{-2}\left|f\left(\frac{x}{|x|}\right) - f\left(\frac{y}{|y|}\right)\right| \\ &= |y|^{-2}\frac{|x| + |y|}{|x|}\frac{|x| - |y|}{|x|} f\left(\frac{x}{|x|}\right) + |y|^{-2}\left|f\left(\frac{x}{|x|}\right) - f\left(\frac{y}{|y|}\right)\right| \\ &\leq \delta|y|^{-2}. \end{aligned}$$

Therefore, conditioned on the event F_n, if we write $C_f = \mathbb{E}\big[\int_1^e f(B_t)dt\big]$ for simplicity, we have that

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1} f(S_i) - C_f \log n\right| \\
\le& \sum_{i=0}^{\lceil(\log n)^k\rceil} f(S_i) + \left|\int_{(\log n)^k}^n f(B_t)dt - C_f \log n\right| + \int_{(\log n)^k}^n |f(B_t) - f(S_t)|\, dt \\
\le& \sum_{i=0}^{\lceil(\log n)^k\rceil} f(S_i) + \left|\int_{(\log n)^k}^n f(B_t)dt - C_f \log n\right| + \delta \int_{(\log n)^k}^n |B_t|^{-2} dt.
\end{aligned}$$

For the first term, by Corollary 4.2.9, we have that

$$\mathbb{P}\left(\sum_{i=0}^{\lceil(\log n)^k\rceil} f(S_i) > \epsilon \log n\right) \le C_1(d,\eta,\epsilon,m)(\log n)^{-m}.$$

Similar bounds for the second and the third term follows from Proposition 4.2.13. □

4.3 The super-critical dimensions

In this section, we prove Theorem 4.1.1 for $d \ge 7$ via the infinite model defined in Section 4.2.2. The main strategy is to establish a lower bound for the expectation of capacity using Lemma 4.2.11 and estimates on Green's functions, then deduce the desired convergence for the infinite model with the help of its ergodicity under transformation (4.2.3), and finally extend it to a similar convergence for the branching random walk indexed by the critical Galton-Watson tree conditioned to be large.

4.3.1 Estimates on Green's functions

Lemma 4.3.1. *If $d \ge 3$ and η, θ are distributions on $\mathbb{Z}^d$ satisfying* (4.1.1), *then as $n \to \infty$, there exists a constant $C(d,\eta,\theta) > 0$ such that*

$$\mathsf{E}_0^\theta[G_\eta(S_n)] \le C(d,\eta,\theta) n^{1-d/2}.$$

Proof. Recall that according to Lemma 4.2.7, there exists $C'(d,\eta,\theta)$ such that

$$(C'(d,\eta,\theta))^{-1} G_\theta(x) \le G_\eta(x) \le C'(d,\eta,\theta) G_\theta(x) \quad \text{uniformly for all } x \in \mathbb{Z}^d,$$

then it suffices to show that

$$\mathsf{E}_0^\theta[G_\theta(S_n)] \leq C(d,\theta)n^{1-d/2}.$$

In fact since θ is a symmetric distribution,

$$\begin{aligned}\mathsf{E}_0^\theta[G_\theta(S_n)] &= \sum_{x\in\mathbb{Z}^d} \mathsf{P}_0^\theta(S_n = x)G_\theta(x)\\ &= \sum_{x\in\mathbb{Z}^d}\sum_{m\geq 0} \mathsf{P}_0^\theta(S_n = x)\mathsf{P}_0^\theta(S_m = x)\\ &= \sum_{m\geq 0} \mathsf{P}_0^\theta(S_{m+n} = 0) \leq C(d,\theta)n^{1-d/2},\end{aligned}$$

where the last line follows by taking $x = 0$ in Lemma 4.2.6 and this completes the proof. □

Lemma 4.3.2. *In dimension $d \geq 7$, recall that μ, θ, η are probability distributions satisfying* (4.1.1)*, and the sequence (v_i) of the infinite model is defined in* (4.2.2)*. Then there exists a constant $C(d,\mu,\theta,\eta) > 0$ such that*

$$\mathbf{E}_{\mu,\theta}\left[\sum_{i=-\infty}^{\infty} G_\eta(v_i)\right] \leq C(d,\mu,\theta,\eta).$$

Proof. Recall that (v_i) run through all subtrees denoted by $\mathcal{T}_{\pm n} = \mathcal{T}_n \cup \mathcal{T}_{-n}$, thus

$$\begin{aligned}&\mathbf{E}_{\mu,\theta}\left[\sum_{i=-\infty}^{\infty} G_\eta(v_i)\right]\\ =&\mathbf{E}_{\mu,\theta}\otimes \mathsf{E}_0^\theta\left[\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \#\{u\in\mathcal{T}_{\pm n} : |u| = i\}G_\eta(S_{n+i})\right]\\ =&\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \mathbf{E}_{\mu,\theta}[\#\{u\in\mathcal{T}_{\pm n} : |u| = i\}]\mathsf{E}_0^\theta[G_\eta(S_{n+i})].\end{aligned}$$

If μ has finite variance, then for all n and i ,

$$\mathbf{E}_{\mu,\theta}[\#\{u\in\mathcal{T}_{\pm n} : |u| = i\}] = \mathbf{E}_{\mu,\theta}[\#\{u\in\mathcal{T}_{\pm n} : |u| = 1\}] = \sum_{i,j\geq 0}(i+j)\mu(i+j+1), \tag{4.3.15}$$

thus we have that

$$\begin{aligned}\mathbf{E}_{\mu,\theta}\left[\sum_{i=-\infty}^{\infty} G_\eta(v_i)\right] &\leq C(\mu)\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} \mathsf{E}_0^\theta[G_\eta(S_{n+i})]\\ &= C(\mu)\sum_{m=0}^{\infty}(m+1)\mathsf{E}_0^\theta[G_\eta(S_m)] \leq C(\mu,d,\eta,\theta),\end{aligned}$$

where the last line follows from Lemma 4.3.1. □

4.3.2 Limit theorem for the infinite model

Proposition 4.3.3. *In dimension $d \geq 7$, μ, θ, η are supposed to satisfy* (4.1.1) *and recall the range $R[0,n]$ defined in Section 4.2.2. Then there is a constant $C(d, \mu, \theta, \eta) > 0$ such that*

$$\frac{\mathtt{cap}_\eta(R[0,n])}{n} \to C(d,\mu,\theta,\eta) \quad \mathbf{P}_{\mu,\theta}\text{-almost surely.}$$

Proof. By definition of the capacity, for any finite sets $A, B \subset \mathbb{Z}^d$,

$$\mathtt{cap}_\eta(A \cup B) \leq \mathtt{cap}_\eta A + \mathtt{cap}_\eta B.$$

Recall the ergodic measure-preserving shift σ defined by (4.2.3). In particular we have that

$$\begin{aligned}\mathtt{cap}_\eta(R[0,n+m]) &\leq \mathtt{cap}_\eta(R[0,n]) + \mathtt{cap}_\eta(R[n,n+m]) \\ &= \mathtt{cap}_\eta(R[0,n]) + \mathtt{cap}_\eta(\sigma^n \circ R[0,m]).\end{aligned}$$

Thus Kingman's subadditive ergodic theorem suggests that there exists a constant $C(d,\mu,\theta,\eta)$ such that

$$\lim_{n\to\infty} \frac{\mathtt{cap}_\eta(R[0,n])}{n} \to C(d,\mu,\theta,\eta) \quad \mathbf{P}_{\mu,\theta}\text{-almost surely.}$$

Then it remains to prove that the constant

$$C(d,\mu,\theta,\eta) = \lim_{n\to\infty} \frac{1}{n}\mathbf{E}_{\mu,\theta}\big[\mathtt{cap}_\eta(R[0,n])\big] \tag{4.3.16}$$

is strictly positive.

In face by Lemma 4.2.11, for any $k \geq 1$,

$$\frac{1}{n}\mathbf{E}_{\mu,\theta}[\mathtt{cap}_\eta(R[0,n])] \geq \frac{\frac{1}{n}\mathbf{E}_{\mu,\theta}[\#R[0,n]]}{k+1} - \frac{\frac{1}{n}\mathbf{E}_{\mu,\theta}\left[\sum_{x,y\in R[0,n]} G_\eta(x,y)\right]}{k(k+1)}.$$

The first term $\frac{1}{n}\mathbf{E}_{\mu,\theta}[\#R[0,n]]$ converges to a strictly positive constant by [73, Proposition 5], and in the second term

$$\frac{1}{n}\mathbf{E}_{\mu,\theta}\left[\sum_{x,y\in R[0,n]} G_\eta(x,y)\right] \leq \frac{1}{n}\mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n} G_\eta(v_i,v_j)\right] \leq \mathbf{E}_{\mu,\theta}\left[\sum_{i=-\infty}^{\infty} G_\eta(v_i)\right]$$

is also finite by Lemma 4.3.2. Then (4.3.16) is strictly positive by taking k sufficiently large. □

Remark 4.3.4. The limiting constant here is implicit. In fact, in the language of Lemma 4.2.12, for high dimensions $d \geq 7$, both $E_n I_n$ and G_n will converge by monotonicity (to some random variables). Indeed, write $E_\infty I_\infty$ and G_∞ to denote their limits, then the desired constant is

$$\mathbb{E}[E_\infty I_\infty] = \mathbf{P}_{\mu,\theta} \otimes \mathsf{P}_\eta^0 \Big(v_0 \notin \{v_1, v_2, \cdots\}, \tau^+_{\{\cdots, v_{-1}, v_0, v_1, \cdots\}} = \infty \Big).$$

However, the equation $\mathbb{E}[E_\infty I_\infty \cdot G_\infty] = 1$ does not contain enough information to determine this constant, since G_∞ is a non-trivial random variable for $d \geq 7$.

4.3.3 Proof of Theorem 4.1.1 (1)

The goal of this section is to establish an intermediate structure, then compare the infinite model with large Galton-Watson trees via this new structure as in [94, p. 19].

To study $R[0, n]$, it suffices to look at (v_i) for $i \geq 0$, thus we consider the model in Remark 4.2.3, i.e. we attach one subtree $\mathcal{T}_i$ to each node $(i, \varnothing)$ on the spine and set

$$k_{(0,\varnothing)} \sim \mu,\ \mathbf{P}_{\mu,\theta}(k_{(i,\varnothing)} = n) = \mu[n+1, \infty) = \sum_{j=n+1}^{\infty} \mu(j),\ i > 0.$$

Now we construct a new probability measure $\mathbf{P}^I_{\mu,\theta}$ such that all nodes on the spine, including the base point $(0, \varnothing)$, have offspring distribution

$$\mathbf{P}^I_{\mu,\theta}(k_{(i,\varnothing)} = n) = \mu[n+1, \infty),\ i \geq 0,$$

while all other constructions (independence, offspring distribution for nodes not on the spine, and displacements) are the same as $\mathbf{P}_{\mu,\theta}$. Since $\mathbf{P}^I_{\mu,\theta}$ and $\mathbf{P}_{\mu,\theta}$ are different only in the first subtree, it follows that

Corollary 4.3.5. *In dimension $d \geq 7$, let μ, θ, η be distributions with the conditions in* (4.1.1)*. There is a constant $C(d, \mu, \theta, \eta) > 0$ such that under $\mathbf{P}^I_{\mu,\theta}$,*

$$\frac{\mathsf{cap}_\eta(R[0, n])}{n} \to C(d, \mu, \theta, \eta) \text{ in probability.}$$

Moreover, for the measure $\mathbf{P}^I_{\mu,\theta}$ we have

Lemma 4.3.6 ([94])**.** *In dimension $d \geq 3$, let μ, θ, η be distributions with the conditions in* (4.1.1)*. Recall that $P_{\mu,\theta}$ is the law of the Galton-Watson* *tree (cf. Section 4.2.1). Let $a \in (0,1)$ and let (f_n) be any uniformly bounded sequence of functions on $\mathbb{Z}^{\lfloor an \rfloor + 1}$. Then (with an abuse of the notation (v_i) for positions of nodes under both $P_{\mu,\theta}$ and $\mathbf{P}^I_{\mu,\theta}$)*

$$\lim_{n\to\infty}\left|E_{\mu,\theta}\big(f_n\big((v_i)_{0\le i\le\lfloor an\rfloor}\big)\,\big|\,\#T=n\big)-\mathbf{E}^I_{\mu,\theta}\big(f_n\big((v_i)_{0\le i\le\lfloor an\rfloor}\big)\big)g_a\Big(\frac{v_{\lfloor an\rfloor}}{\sigma n}\Big)\right|=0,$$

where $g_a(x) = (1-a)^{-\frac{3}{2}} \exp\Big(-\frac{x^2}{2(1-a)}\Big)$ and σ^2 is the variance of μ.

Proof. See (5.3), (5.4) and the disaplay that follows in [94]. □

Theorem 4.3.7. *In dimension $d \ge 7$, let μ, θ, η be distributions with the conditions in (4.1.1), and let $R[0,n]$ be the range constructed in Section 4.2.2 (abused to denote the range of other trees as well). There is a constant $C = C(d,\mu,\theta,\eta) > 0$ such that under the law of a (standard) Galton-Watson tree conditioned to have $n+1$ nodes, $P_{\mu,\theta}(\cdot|\#T = n+1)$,*

$$\frac{\mathsf{cap}_\eta(R[0,n])}{n} \to C \text{ in probability.}$$

Proof. For any $\epsilon > 0$, take

$$f_n = \mathbf{1}_{\left|\frac{1}{n}\mathsf{cap}_\eta R[0,an]-aC\right|>\epsilon}$$

in Lemma 4.3.6. Then by Corollary 4.3.5, we have that

$$\lim_{n\to\infty} P_{\mu,\theta}\left(\left|\frac{1}{n}\mathsf{cap}_\eta R[0,an]-aC\right|>\epsilon\,\middle|\,\#T=n+1\right)=0, \tag{4.3.17}$$

Moreover, since

$$\begin{aligned}
&\left|\frac{1}{n}\mathsf{cap}_\eta(R[0,n])-C\right|\\
\le&\left|\frac{1}{n}\mathsf{cap}_\eta(R[0,n])-\frac{1}{n}\mathsf{cap}_\eta(R[0,\lfloor an\rfloor])\right|+\left|\frac{1}{n}\mathsf{cap}_\eta(R[0,\lfloor an\rfloor])-aC\right|+|aC-C|\\
\le&(1-a)+\left|\frac{1}{n}\mathsf{cap}_\eta(R[0,\lfloor an\rfloor])-aC\right|+(1-a)C,
\end{aligned}$$

we have that

$$\begin{aligned}
&\lim_{n\to\infty} P_{\mu,\theta}\left(\left|\frac{1}{n}\mathsf{cap}_\eta(R[0,n])-C\right|>\epsilon\,\middle|\,\#T=n+1\right)\\
\le&\lim_{n\to\infty} P_{\mu,\theta}\left(\left|\frac{1}{n}\mathsf{cap}_\eta(R[0,\lfloor an\rfloor])-aC\right|>\epsilon-(1-a)(1+C)\,\middle|\,\#T=n+1\right)=0,
\end{aligned}$$

where the last line holds by (4.3.17) if a is taken sufficiently close to 1. □

4.4 The critical dimension

In this section, we consider the critical dimension $d = 6$. The main strategy is to estimate Green's functions for the infinite model established in Section 4.2.2, so that we can use Lemma 4.2.12 and a second moment method to get the desired convergence. Finally similar argument as in Theorem 4.3.7 allows us to prove the convergence result of capacity for large Galton-Watson trees.

4.4.1 Estimates on Green's functions

Proposition 4.4.1. *In dimension $d = 6$, let μ, θ, η be distributions with assumptions in* (4.1.1)*. Let $P_{\mu,\theta}$ be the law of a (standard) branching random walk $(V_u)_{u \in T}$ indexed by a (standard) Galton-Watson tree T (cf. Section 4.2.1). Then*

1. *As $z \to \infty$, we have that*

$$E_{\mu,\theta}\left[\sum_{u \in T} G_\eta(z + V_u)\right] = F_{\eta,\theta}(z) + O(|z|^{-3}),$$

where the function

$$F_{\eta,\theta}(z) := C_{6,\eta} C_{6,\theta} \int_{\mathbb{R}^6} J_\eta(z + x)^{-4} J_\theta(x)^{-4} dx,$$

is a continuous function defined on $\mathbb{R}^6 \backslash \{0\}$ with $F_{\eta,\theta}(\lambda z) = \lambda^{-2} F(z)$ for all $\lambda > 0$, with $C_{6,(\cdot)}$ and $J_{(\cdot)}$ defined in Lemma 4.2.7.

2. *For any $m \geq 2$, if μ has finite m-th moment, then there exists a constant $C(m, \mu, \theta, \eta) > 0$, so that for any $z \neq 0$,*

$$E_{\mu,\theta}\left[\left(\sum_{u \in T} G_\eta(z + V_u)\right)^m\right] \leq C(m, \mu, \theta, \eta) |z|^{-2}.$$

Proof. Because μ is critical, we have $E_{\mu,\theta}[\#\{u \in T : |u| = n\}] = 1$ for all

$n \geq 1$. Then

$$\begin{aligned}
E_{\mu,\theta}\left[\sum_{u\in T} G_\eta(z+V_u)\right] &= E_{\mu,\theta}\left[\sum_{n=0}^{\infty} \#\{u \in T : |u| = n\}\mathsf{E}_0^\theta[G_\eta(z+S_n)]\right] \\
&= \sum_{n=0}^{\infty} \mathsf{E}_0^\theta[G_\eta(z+S_n)] \\
&= \sum_{n=0}^{\infty}\sum_{x\in\mathbb{Z}^6} G_\eta(z+x)\mathsf{P}_0^\theta(S_n = x) \\
&= \sum_{x\in\mathbb{Z}^6} G_\eta(z+x)G_\theta(x).
\end{aligned}$$

By Lemma 4.2.7, we then have

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^6} G_\eta(z+x)G_\theta(x) \\
=&C_{6,\eta}C_{6,\theta}\sum_{x\in\mathbb{Z}^6} J_\eta(z+x)^{-4}J_\theta(x)^{-4} + O\left(\sum_{x\in\mathbb{Z}^6}|z+x|^{-5}|x|^{-4}\right),
\end{aligned}$$

and it is elementary to show that (by approximating the sum by an integral)

$$O\left(\sum_{x\in\mathbb{Z}^6}|z+x|^{-5}|x|^{-4}\right) = O(|z|^{-3}).$$

Moreover, the difference between $C_{6,\eta}C_{6,\theta}\sum_{x\in\mathbb{Z}^6} J_\eta(z+x)^{-4}J_\theta(x)^{-4}$ and $F_{\eta,\theta}(z)$ is of the same order as $O\big(\sum_{x\in\mathbb{Z}^6}|z+x|^{-5}|x|^{-4}\big)$ by mean value theorem. Therefore,

$$E_{\mu,\theta}\left[\sum_{u\in T} G_\eta(z+V_u)\right] = F_{\eta,\theta}(z) + O(|z|^{-3}).$$

The asymptotic and the scaling relation for $F_{\eta,\theta}$ are easy to check by using $J.(x) \asymp |x|, J.(\lambda x) = \lambda J.(x)$.

As for Part (2), let $(S_n^{(i)})(1 \leq i \leq k)$ be independent θ-random walks started at 0. Given any $z \in \mathbb{Z}^6, k \geq 2$, by Part (1) and Lemma 4.2.7,

$$\mathsf{E}_0^\theta\left[\prod_{i=1}^{k}\sum_{j=0}^{\infty} G_\eta(z+S_j^{(i)})\right] \leq C_1(\theta,\eta)\prod_{i=1}^{k}(|z|\vee 1)^{-2} \leq C_2(\theta,\eta)G_\eta(z)^{k/2}. \tag{4.4.18}$$

To deal with the second moment, $m = 2$, we need to study the positions of two nodes u, u'. Given that $|u \wedge u'| = k, |u| = k + i, |u'| = k + j$, where

$u \wedge u'$ denotes their youngest common ancestor), then their contribution to the second moment is

$$\mathsf{E}_0^\theta G_\eta(z + S_k + S_i^{(1)}) G_\eta(z + S_k + S_j^{(2)}).$$

Summing up all possible tree-structures, we have that

$$\begin{aligned} &E_{\mu,\theta}\left[\left(\sum_{u\in T} G_\eta(z+V_u)\right)^2\right] \\ =&\sum_{i,j,k=0}^{\infty} \mathsf{E}_0^\theta\Big[G_\eta(z + S_k + S_i^{(1)}) G_\eta(z + S_k + S_j^{(2)})\Big] E_{\mu,\theta}[N(k;i,j)], \end{aligned}$$

where

$$N(k;i,j) = \#\{u, u' \in T : |u \wedge u'| = k, |u| = k+i, |u'| = k+j\}.$$

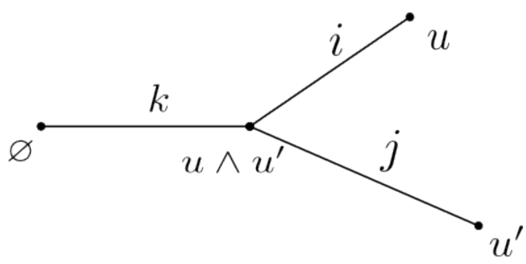


Figure 4.6: $N(k;i,j)$

We can then count $N(k;i,j)$ as illustrated in Figure 4.6 on critical Galton-Watson trees. Set $Z_n := \#\{u \in T : |u| = n\}$), then

$$E_{\mu,\theta}[N(k;i,j)] = E_{\mu,\theta}[Z_k]E_{\mu,\theta}[Z_1(Z_1-1)]E_{\mu,\theta}[Z_{i-1}]E_{\mu,\theta}[Z_{j-1}] = E_{\mu,\theta}[Z_1(Z_1-1)]$$

for $i, j \geq 1$, which is finite as long as μ has finite second moment (the case i or $j = 0$ can be easily treated alone). Then we apply (4.4.18) with $k = 2$,

$$\begin{aligned} &E_{\mu,\theta}\left[\left(\sum_{u\in T} G_\eta(z+V_u)\right)^2\right] \\ \leq& E_{\mu,\theta}[Z_1(Z_1-1)] \sum_{i,j,k=0}^{\infty} \mathsf{E}_0^\theta\Big[G_\eta(z + S_k + S_i^{(1)}) G_\eta(z + S_k + S_j^{(2)})\Big] \\ \leq& C(\theta,\eta) E_{\mu,\theta}[Z_1(Z_1-1)] \sum_{k=0}^{\infty} \mathsf{E}_0^\theta[G_\eta(z+S_k)] \leq C(\mu,\theta,\eta)|z|^{-2}, \end{aligned}$$

where the last inequality follows from Part (1).

Similar argument works for $m \geq 3$, by counting all possible hierarchy structures of m vertices as for $N(k; i, j)$, and perform (4.4.18) recursively on those structures. $\square$

Remark 4.4.2. By (4.4.18), one may expect an $O(|z|^{-m})$ result in Part (2), however, $O(|z|^{-2})$ is in fact optimal for all $m \geq 3$. Take $m = 3$ for instance. To estimate the contribution of 'binary' branching structure $u^{(i)} (i = 1, 2, 3)$ with (see Figure 4.7)

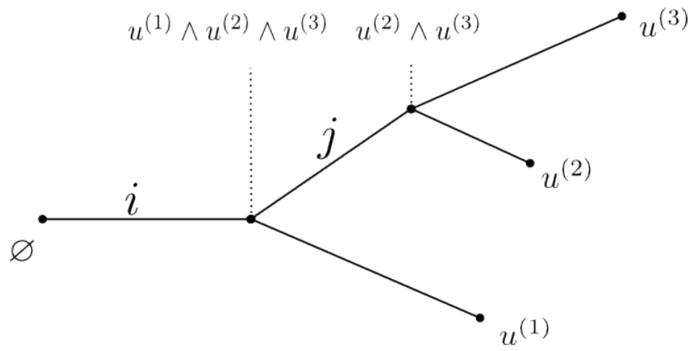


Figure 4.7: 'binary' branching structures for $k = 3$

$$|u^{(1)} \wedge u^{(2)} \wedge u^{(3)}| = i, |u^{(2)} \wedge u^{(3)}| = i + j > i,$$

we need to perform (4.4.18) with $k = 2$ twice, instead of the equation with $k = 3$:

$$\begin{aligned}
&\sum_{i,j,k,l,h=0}^{\infty} \mathsf{E}_0^{\theta}\Big[G_\eta(z + S_i + S_j^{(1)} + S_k^{(2)})G_\eta(z + S_i + S_j^{(1)} + S_l^{(3)})G_\eta(z + S_i + S_h^{(4)})\Big] \\
\leq &C_1(\mu, \theta, \eta) \sum_{i,j,h=0}^{\infty} \mathsf{E}_0^{\theta}\Big[G_\eta(z + S_i + S_j^{(1)})G_\eta(z + S_i + S_h^{(4)})\Big] \\
\leq &C_2(\mu, \theta, \eta) \sum_{i=0}^{\infty} \mathsf{E}_0^{\theta}[G_\eta(z + S_i)] \leq C_3(\mu, \theta, \eta)|z|^{-2}.
\end{aligned}$$

It is only when $u^{(1)}, u^{(2)}, u^{(3)}$ all branch at the same node (i.e. $j = 0$ in Figure 4.7) that one can apply (4.4.18) with $k = 3$. Thus our method gives the bound $O(|z|^{-2})$ for all m-th moment for $m \geq 2$.

Since the infinite model has offspring distributions different from μ on the spine, we include the following corollary, whose proof is clear by that of Proposition 4.4.1.

Corollary 4.4.3. *In the setting of Proposition 4.4.1, take an arbitrary distribution μ^* on $\mathbb{N}$, and consider a random tree whose offspring distribution of the first generation is replaced by μ^*, with other construction same as $P_{\mu,\theta}$, write its law as $P^*_{\mu,\theta}$, then*

1. *As $z \to \infty$, we have that*

$$E^*_{\mu,\theta}\left[\sum_{u\in T} G_\eta(z+V_u)\right] = \mathbb{E}[\mu^*]F_{\eta,\theta}(z) + O(|z|^{-3}).$$

2. *For any $m \geq 2$, if μ^* and μ have finite m-th moment, then there exists a constant $C(m,\mu,\mu^*,\theta,\eta) > 0$*

$$E^*_{\mu,\theta}\left[\left(\sum_{u\in T} G_\eta(z+V_u)\right)^m\right] \leq C(m,\mu,\mu^*,\theta,\eta)|z|^{-2}.$$

Before going to the main estimate, we attach here a moment estimate for independent random variables.

Lemma 4.4.4. *[51, Corollary 4.4] Let $t \geq 2$, and $(X_i), i = 1,\cdots,n$ be independent random variables such that*

$$\mathbb{E}X_i = 0, \text{ and } \mathbb{E}|X_i|^t < \infty,$$

then

$$\mathbb{P}\left(\sum_{i=1}^n X_i \geq x\right) \leq C_1 x^{-t}\sum_{i=1}^n \mathbb{E}|X_i|^t + \exp\left(-C_2x^2/\sum_{i=1}^n \mathbb{E}|X_i|^2\right),$$

where $C_1 = (1+2/t)^t, C_2 = 2(t+2)^{-1}e^{-t}$.

We are now ready to treat Green's functions for the infinite model.

Proposition 4.4.5. *In dimension $d = 6$, let μ,θ,η be distributions with assumptions in (4.1.1). Recall the infinite model in Section 4.2.2. Let ζ_{-n},ζ_n be indexes such that*

$$R[\zeta_{-n},\zeta_n] = \{v_{\zeta_{-n}},\cdots,v_{\zeta_n}\}$$

is the range formed by the displacement of all nodes in

$$\{(0,\mathcal{T}_0),(1,\mathcal{T}_{\pm 1}),\cdots,(n,\mathcal{T}_{\pm n})\}.$$

1. *If μ has finite* 5*-th moment, then for any fixed $\epsilon > 0$, as $n \to \infty$,*

$$\mathbf{P}_{\mu,\theta}\left(\left|\sum_{i=\zeta_{-n}}^{\zeta_n} G_\eta(v_i) - C_G \log n\right| > \epsilon \log n\right) = o((\log n)^{-2})$$

where $C_G = \sum_{k=1}^{\infty}(k-1)k\mu(k)\cdot\mathbb{E}\big[\int_1^e F_{\eta,\theta}(B_t^\theta)dt\big]$, B_t^θ is a Brownian motion with covariance matrix Γ_θ, and $F_{\eta,\theta}$ is the function defined in Proposition 4.4.1.

2. *For any $m \geq 2$, if μ has finite $(m+1)$-th moment, then as $n \to \infty$,*

$$\mathbf{E}_{\mu,\theta}\left[\left(\sum_{i=\zeta_{-n}}^{\zeta_n} G_\eta(v_i)\right)^m\right] = O((\log n)^m).$$

Proof. We merge the two subtrees $(n, \mathcal{T}_{\pm n})$ into a single tree, whose first generation has offspring distribution $\mu^*(k) := \sum_{\{i,j\,:\,i+j=k\}}\mu(i+j+1) = (k+1)\mu(k+1)$. Then we need μ to have finite $(m+1)$-th moment in order that μ^* has finite m-th moment. For simplicity, we denote by $G_\eta(\mathcal{T}_{\pm n})$ the sum of Green's functions over the range of $(n, \mathcal{T}_{\pm n})$, and we denote by $\mathcal{S}_0 = 0, \mathcal{S}_1, \cdots, \mathcal{S}_n$ the spatial positions of the spine $(0, \varnothing), \cdots, (n, \varnothing)$. Clearly,

$$\sum_{i=\zeta_{-n}}^{\zeta_n} G_\eta(v_i) = G_\eta(\mathcal{T}_0) + \sum_{i=1}^{n} G_\eta(\mathcal{T}_{\pm i}),$$

and $(G_\eta(\mathcal{T}_{\pm i}))$ are independent conditioned on $(\mathcal{S}_i)$.

For Part (1), we have that

$$\begin{aligned}\left|\sum_{i=\zeta_{-n}}^{\zeta_n} G_\eta(v_i) - C_G \log n\right| \leq &\left|\sum_{i=0}^{n} G_\eta(\mathcal{T}_{\pm i}) - \sum_{i=0}^{n}\mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i}) \,|\, (\mathcal{S}_i)_{0\leq i\leq n}]\right| \\ &+ \left|\sum_{i=0}^{n}\mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i}) \,|\, (\mathcal{S}_i)_{0\leq i\leq n}] - \mathbb{E}[\mu^*]\sum_{i=1}^{n} F_{\eta,\theta}(\mathcal{S}_i)\right| \\ &+ \left|\mathbb{E}[\mu^*]\sum_{i=1}^{n} F_{\eta,\theta}(\mathcal{S}_i) - C_G \log n\right|\end{aligned} \tag{4.4.19}$$

and it suffices to estimate each of the three terms here.

Indeed, for the third term in (4.4.19), by Corollary 4.2.15,

$$\mathbf{P}_{\mu,\theta}\left(\left|\mathbb{E}[\mu^*]\sum_{i=1}^n F_{\eta,\theta}(\mathcal{S}_i) - C_G\log n\right| > \epsilon\log n\right) = o((\log n)^{-2}),$$

For the second term in (4.4.19), by Corollary 4.4.3 we have that

$$\left|\sum_{i=0}^n \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i})\,|\,(\mathcal{S}_i)_{0\le i\le n}] - \mathbb{E}[\mu^*]\sum_{i=1}^n F_{\eta,\theta}(\mathcal{S}_i)\right| = O\left(\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-3}\right),$$

which is in turn deduced by Corollary 4.2.9 (2) with $k=3$, $m=1,2,3$ (and a Chebyshev-type inequality for the 3rd moment),

$$\mathbf{P}_{\mu,\theta}\left(\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-3} > \epsilon\log n\right) = o((\log n)^{-2}).$$

As for the first term in (4.4.19), by Corollary 4.4.3 with $m=2,4$ (here we need finite fourth moment for μ^*, thus finite fifth moment for μ), we have that

$$\sum_{i=0}^n \mathbf{E}_{\mu,\theta}\big[(G_\eta(\mathcal{T}_{\pm i}))^2\,\big|\,(\mathcal{S}_i)_{0\le i\le n}\big] \le C_1(\mu,\theta,\eta)\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2},$$
$$\sum_{i=0}^n \mathbf{E}_{\mu,\theta}\big[(G_\eta(\mathcal{T}_{\pm i}))^4\,\big|\,(\mathcal{S}_i)_{0\le i\le n}\big] \le C_2(\mu,\theta,\eta)\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2},$$

so

$$\sum_{i=0}^n \mathbf{E}_{\mu,\theta}\big[(G_\eta(\mathcal{T}_{\pm i}) - \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i})\,|\,(\mathcal{S}_i)_{0\le i\le n}])^2\,\big|\,(\mathcal{S}_i)_{0\le i\le n}\big] \le C_3(\mu,\theta,\eta)\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2},$$
$$\sum_{i=0}^n \mathbf{E}_{\mu,\theta}\big[(G_\eta(\mathcal{T}_{\pm i}) - \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i})\,|\,(\mathcal{S}_i)_{0\le i\le n}])^4\,\big|\,(\mathcal{S}_i)_{0\le i\le n}\big] \le C_4(\mu,\theta,\eta)\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2}.$$

Then we apply Lemma 4.4.4 with $X_i = G_\eta(\mathcal{T}_{\pm i}) - \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i})\,|\,(\mathcal{S}_i)_{0\le i\le n}]$, $t=4$, and $\mathbb{P} = \mathbf{P}_{\mu,\theta}(\cdot\,|\,(\mathcal{S}_i)_{0\le i\le n})$,

$$\begin{aligned}&\mathbf{P}_{\mu,\theta}\left(\left|\sum_{i=0}^n G_\eta(\mathcal{T}_{\pm i}) - \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i})\,|\,(\mathcal{S}_i)_{0\le i\le n}]\right| \ge \epsilon\log n\;\middle|\;(\mathcal{S}_i)_{0\le i\le n}\right)\\
\le& C_5(\mu,\theta,\eta)\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2}(\epsilon\log n)^{-4} + \exp\left(-C_6(\mu,\theta,\eta)(\epsilon\log n)^2/\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2}\right)\\
\le& C_5(\mu,\theta,\eta)\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2}(\epsilon\log n)^{-4} + e^{-C_6(\mu,\theta,\eta)\epsilon^2\log n/\log\log n} + \mathbf{1}_{\sum_{i=0}^n (|\mathcal{S}_i|\vee 1)^{-2} > \log n\log\log n},\end{aligned}$$

If we take expectation $\mathbf{E}_{\mu,\theta}$ on both sides, all these terms are $o((\log n)^{-2})$ by Corollary 4.2.9, then we have that

$$\mathbf{P}_{\mu,\theta}\left(\left|\sum_{i=0}^{n} G_\eta(\mathcal{T}_{\pm i}) - \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i}) \mid (\mathcal{S}_i)_{0\le i\le n}]\right| \ge \epsilon \log n\right) = o((\log n)^{-2}).$$

The conclusion follows by combining the estimates for the three terms on the right-hand side of (4.4.19) individually.

For the second part, we illustrate the $m = 2$ case, since the proof for m other than 2 is similar. Indeed,

$$\begin{aligned}
&\mathbf{E}_{\mu,\theta}\left[\left(\sum_{i=\zeta_{-n}}^{\zeta_n} G_\eta(v_i)\right)^2 \middle| (\mathcal{S}_i)_{0\le i\le n}\right] \\
=&\mathbf{E}_{\mu,\theta}\left[\left(\sum_{i=0}^{n} G_\eta(\mathcal{T}_{\pm i})\right)^2 \middle| (\mathcal{S}_i)_{0\le i\le n}\right] \\
=&\sum_{i=0}^{n} \mathbf{E}_{\mu,\theta}\left[G_\eta(\mathcal{T}_{\pm i})^2 \mid (\mathcal{S}_i)_{0\le i\le n}\right] + \\
&2\sum_{0\le i<j\le n} \mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm i}) \mid (\mathcal{S}_i)_{0\le i\le n}]\mathbf{E}_{\mu,\theta}[G_\eta(\mathcal{T}_{\pm j}) \mid (\mathcal{S}_i)_{0\le i\le n}].
\end{aligned}$$

By Corollary 4.4.3, if μ has finite 3rd moment, then this sum is of the order

$$\begin{aligned}
&\sum_{i=0}^{n} (|\mathcal{S}_i| \vee 1)^{-2} + 2\sum_{0\le i<j\le n} (|\mathcal{S}_i| \vee 1)^{-2}(|\mathcal{S}_j| \vee 1)^{-2} \\
\asymp&\sum_{i=0}^{n} (|\mathcal{S}_i| \vee 1)^{-2} + \left(\sum_{i=0}^{n} (|\mathcal{S}_i| \vee 1)^{-2}\right)^2,
\end{aligned}$$

Take expectation $\mathbf{E}_{\mu,\theta}$, and we can conclude by Corollary 4.2.9. □

Corollary 4.4.6. *Under the same setting of Proposition 4.4.5 (1),*

$$\mathbf{P}_{\mu,\theta}\left(\left|\sum_{i=-n}^{n} G_\eta(v_i) - \frac{1}{2}C_G \log n\right| > \epsilon \log n\right) = o((\log n)^{-2}).$$

Proof. By standard tools of Kemperman's formula (see e.g. [44, Section 3]), denote by ζ_n' the total population of n Galton-Watson trees of offspring distribution μ, and by (Y_i) an i.i.d. sequence distributed as $\mu - 1$, then

$$\mathbf{P}_{\mu,\theta}(\zeta_n' = m) = \frac{n}{m}\mathbb{P}(Y_1 + \cdots + Y_m = n).$$

Apply Lemma 4.2.6 with $d = 1$ and the random walk with displacements (Y_i) (where μ being critical implies that $\mathbb{E}Y_i = 0$, and finite fifth moment required in Proposition 4.4.5 (1) implies the finite third moment of Y_i), we have that

$$\left|\mathbf{P}_{\mu,\theta}(\zeta'_n = m) - \frac{n}{m}\frac{C_1(\mu)}{\sqrt{m}}e^{-\frac{C_2(\mu)n^2}{m}}\right| \le \frac{C_3(\mu)}{m}.$$

Sum over m, then

$$\mathbf{P}_{\mu,\theta}\left(\zeta'_n \ge n^2(\log n)^5\right) = o((\log n)^{-2}).$$

Moreover, by [68, Proposition 2.1.2 (a)] with $k = 2$ (guaranteed by the finite fifth moment in Proposition 4.4.5 (1)),

$$\begin{aligned}
&\mathbf{P}_{\mu,\theta}\left(\zeta'_n \le n^2(\log n)^{-2}\right) \\
\le& \sum_{m=1}^{n^2(\log n)^{-2}} \frac{n}{m}\mathbb{P}(Y_1 + \cdots + Y_m = n) \\
\le& \left(\sum_{m=1}^{n^2(\log n)^{-2}} \frac{n}{m}\right) \cdot \mathbb{P}\left(\max_{1\le j\le n^2(\log n)^{-2}} Y_1 + \cdots + Y_i \ge n\right) \\
=& o((\log n)^{-2}).
\end{aligned}$$

In summary,

$$\mathbf{P}_{\mu,\theta}\left(n^2(\log n)^{-2} < \zeta'_n < n^2(\log n)^5\right) = 1 - o((\log n)^{-2}).$$

Moreover, recall the probability distribution μ^* in the proof of Proposition 4.4.5, take an i.i.d. sequence (X_i) distributed as μ^*, then

$$\zeta_n \overset{d}{=} \zeta'_{1+X_1+\cdots+X_n}.$$

Apply [68, Proposition 2.1.2 (a)] again for the sequence $(X_i - \mathbb{E}X_i)$, we can show that for any constants $0 < C_4(\mu) < \mathbb{E}X_i < C_5(\mu)$,

$$\mathbb{P}(C_4(\mu)n < 1 + X_1 + \cdots + X_n < C_5(\mu)n) = 1 - o((\log n)^{-2}).$$

Thus for any $0 < C_6(\mu) < (\mathbb{E}[X_i])^2 < C_7(\mu)$,

$$\mathbf{P}_{\mu,\theta}\left(C_6(\mu)n^2(\log n)^{-2} < \zeta_n < C_7(\mu)n^2(\log n)^5\right) = 1 - o((\log n)^{-2}). \tag{4.4.20}$$

The same estimate holds for ζ_{-n}, thus we conclude by Proposition 4.4.5. □

Before ending this section, we give a brief calculation of C_G for the simplest case:

Proposition 4.4.7. *If μ is the geometric distribution with parameter $\frac{1}{2}$, i.e. $\mu(k) = 2^{-k-1}$, and θ and η are one-step distributions of independent simple random walks in $\mathbb{R}^6$, then $C_G = 9\pi^{-3}$.*

Proof. Recall from Proposition 4.4.5 that

$$C_G = \sum_{k=1}^{\infty}(k-1)k\mu(k) \cdot \mathbb{E}\left[\int_1^e F_{\eta,\theta}(B_t^\theta)dt\right].$$

The first term is just the variance of the geometric distribution,

$$\sum_{k=1}^{\infty}(k-1)k\mu(k) = 2.$$

For the second term, we first determine $F_{\eta,\theta}$. Denote by $(S_n), (\tilde{S}_n)$ two independent simple random walks in $\mathbb{R}^6$ started from 0, then by Proposition 4.4.1, for $|z| \to \infty$,

$$\begin{aligned} F_{\eta,\theta}(z) &= E_{\mu,\theta}\left[\sum_{u\in T} G_\eta(z+V_u)\right] + O(|z|^{-3}) \\ &= \mathbb{E}\left[\sum_{n=0}^{\infty} G_\eta(z+S_n)\right] + O(|z|^{-3}) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathbb{P}(\tilde{S}_m = z + S_n) + O(|z|^{-3}) \\ &= \sum_{k=0}^{\infty}(k+1)\mathbb{P}(S_k = z) + O(|z|^{-3}). \end{aligned}$$

Then simplify the sum by Lemma 4.2.6, we have

$$F_{\eta,\theta}(z) = 9\pi^{-3}|z|^{-2} + O(|z|^{-3}).$$

By definition, $F_{\eta,\theta}(\lambda z) = \lambda^{-2}F_{\eta,\theta}(z)$ for any $z \neq 0$, so

$$F_{\eta,\theta}(z) = 9\pi^{-3}|z|^{-2},\ z \neq 0.$$

We can then conclude by the fact that for a 6-dimensional Brownian motion with covariance matrix $\frac{1}{6}\mathtt{I}_6$,

$$\mathbb{E}\left[\int_1^e |B_t^\theta|^{-2}dt\right] = \frac{1}{2}.$$

$\square$

4.4.2 Limit theorem for the infinite model

In this section, we apply the estimates of Green's functions to deduce the estimates for the capacity using Lemma 4.2.12. We begin by estimating the term G_n in Lemma 4.2.12.

Lemma 4.4.8. *In dimension $d = 6$, let η be a distribution with conditions in* (4.1.1)*, and let $G_\eta^{(1-\frac{1}{n})}(x) = \sum_{i\geq 0}(1-\frac{1}{n})^i \mathsf{P}_0^\eta(S_i = x)$ as in Lemma 4.2.12. There exists $C(\eta) > 0$ such that for all $x \in \mathbb{Z}^6$ and $n \geq 1$,*

$$G_\eta(x) - G_\eta^{(1-\frac{1}{n})}(x) \leq \frac{C(\eta)}{n}.$$

Proof. Since $(1-\frac{k}{n}) \vee 0 \leq (1-\frac{1}{n})^k$, we have that

$$G_\eta^{(1-\frac{1}{n})}(x) \geq \sum_{k=0}^{n}(1-\frac{k}{n})\mathsf{P}_0^\eta(S_k = x) \geq G_\eta(x) - \sum_{k\in\mathbb{N}} \frac{k\wedge n}{n}\mathsf{P}_0^\eta(S_k = x).$$

Then the desired estimate follows because there exists $C(\eta) > 0$ such that $\mathsf{P}_0^\eta(S_k = x) \leq C(\eta)k^{-3}$ uniformly in $x \in \mathbb{Z}^6$ by Lemma 4.2.6. □

Lemma 4.4.9. *In dimension $d = 6$, let μ, θ, η be distributions with assumptions in* (4.1.1) *and recall the infinite model in Section 4.2.2. In the setting of Lemma 4.2.12, apply G_n to the sequence (v_i). If μ has finite 5-th moment, then as $n \to \infty$,*

$$\mathbf{P}_{\mu,\theta}\left(\left|G_n - \frac{1}{2}C_G \log n\right| > \epsilon \log n\right) = o((\log n)^{-2}),$$

where C_G is the constant in Proposition 4.4.5. If μ has finite $(m+1)$-th moment for $m \geq 2$, then as $n \to \infty$,

$$\mathbf{E}_{\mu,\theta}[(G_n)^m] = O((\log n)^m).$$

Proof. If ξ_n is a geometric random variable with parameter $\frac{1}{n}$, it is not hard to see that

$$\mathbb{P}(n(\log n)^{-3} \leq \xi_n < n \log n) = 1 - o((\log n)^{-2}).$$

Therefore,

$$\begin{aligned}
&\mathbf{P}_{\mu,\theta}\left(G_n > \frac{1}{2}C_G \log n + \epsilon \log n\right) \\
=&\mathbf{P}_{\mu,\theta}\left(G_n > \frac{1}{2}C_G \log n + \epsilon \log n,\ \xi_n^l, \xi_n^r < n\log n\right) + o((\log n)^{-2}) \\
\leq&\mathbf{P}_{\mu,\theta}\left(\sum_{i=-n\log n}^{n\log n} G_\eta(v_i) > \frac{1}{2}C_G \log n + \epsilon \log n\right) + o((\log n)^{-2}) = o((\log n)^{-2}),
\end{aligned}$$

where the last line follows from Corollary 4.4.6. For the other side, we have that

$$\begin{aligned}
&\mathbf{P}_{\mu,\theta}\left(G_n < \frac{1}{2}C_G\log n - \epsilon\log n\right)\\
=&\mathbf{P}_{\mu,\theta}\left(G_n < \frac{1}{2}C_G\log n - \epsilon\log n,\ \xi_n^l, \xi_n^r \geq n(\log n)^{-3}\right) + o((\log n)^{-2})\\
\leq&\mathbf{P}_{\mu,\theta}\left(\sum_{i=-n(\log n)^{-3}}^{n(\log n)^{-3}} G_\eta(v_i) < \frac{1}{2}C_G\log n - \epsilon\log n + 2C(\eta)(\log n)^{-3}\right) + o((\log n)^{-2})\\
=&o((\log n)^{-2}),
\end{aligned}$$

where $C(\eta)$ is the constant in Lemma 4.4.8.

Moreover, by Proposition 4.4.5, the m-th moment is bounded by

$$\begin{aligned}
\mathbf{E}_{\mu,\theta}[(G_n)^m] &\leq \mathbf{E}_{\mu,\theta}\left[\left(\sum_{i=-\xi_n^l}^{\xi_n^r} G_\eta(v_i)\right)^m\right]\\
&\leq C_1(\mu,\theta,\eta)\sum_{k\geq 0}\mathbb{P}(\max(\xi_n^l,\xi_n^r)=k)(\log k)^m \leq C_2(\mu,\theta,\eta)(\log n)^m.
\end{aligned}$$

□

We are now ready to go from Green's functions to (the contribution of the origin of) the capacity.

Lemma 4.4.10. *In dimension $d = 6$, let μ, θ, η be distributions with assumptions in* (4.1.1) *and that μ has finite* 5*-th moment. Recall the infinite model in Section 4.2.2,*

$$\lim_{n\to\infty}(\log n)\mathsf{P}_0^\eta \otimes \mathbf{P}_{\mu,\theta}\left(0 \notin R[1,n], \tau^+_{R[-n,n]} = \infty\right) = 2C_G^{-1},$$

where C_G is the constant in Proposition 4.4.5.

Proof. We apply Lemma 4.2.12 to (the displacements of) the infinite model (v_i). For any fixed $\epsilon > 0$ sufficiently small, let

$$A_{n,\epsilon} = \left\{\left|G_n - \frac{1}{2}C_G\log n\right| \leq \epsilon\log n\right\},$$

which, by Lemma 4.4.9, happens with probability $1 - o((\log n)^{-2})$.

By Cauchy-Schwarz, we have that

$$\mathbf{E}_{\mu,\theta}[E_n I G_n \mathbf{1}_{A^c_{n,\epsilon}}] \leq \sqrt{\mathbf{P}_{\mu,\theta}(A^c_{n,\epsilon})\mathbf{E}_{\mu,\theta}(G_n^2)} = o(1),$$

because $0 \leq E_n, I_n \leq 1$ (by definition), $\mathbf{P}_{\mu,\theta}(A^c_{n,\epsilon}) = o((\log n)^{-2})$, and $\mathbf{E}_{\mu,\theta}(G_n^2) = O((\log n)^2)$ by Lemma 4.4.9. This together with Lemma 4.2.12 then shows that

$$\mathbf{E}_{\mu,\theta}[E_n I_n G_n \mathbf{1}_{A_{n,\epsilon}}] = 1 - o(1).$$

Moreover since $0 \leq E_n, I_n \leq 1$, we have that

$$\left(\frac{1}{2}C_G - \epsilon\right)(\log n)\mathbf{E}_{\mu,\theta}[E_n I_n] \leq \mathbf{E}_{\mu,\theta}[E_n I_n G_n \mathbf{1}_{A_{n,\epsilon}}] \leq \left(\frac{1}{2}C_G + \epsilon\right)(\log n)\mathbf{E}_{\mu,\theta}[E_n I_n],$$

thus

$$\limsup_{n\to\infty}\left(\frac{1}{2}C_G - \epsilon\right)(\log n)\mathbf{E}_{\mu,\theta}[E_n I_n] \leq 1$$
$$\liminf_{n\to\infty}\left(\frac{1}{2}C_G + \epsilon\right)(\log n)\mathbf{E}_{\mu,\theta}[E_n I_n] \geq 1.$$

Since this holds for any ϵ, we have $\frac{1}{2}C_G(\log n)\mathbf{E}_{\mu,\theta}[E_n I_n] = 1 + o(1)$. That is to say

$$\mathsf{P}_0^\eta \otimes \mathbf{P}_{\mu,\theta}\left(0 \notin R[1, \xi_n^r], \tau^+_{R[-\xi_n^l, \xi_n^r]} > \xi_n\right) = \frac{2 + o(1)}{C_G \log n}.$$

Moreover, apply the standard estimate

$$\mathbb{P}(n(\log n)^{-3} \leq \xi_n < n \log n) = 1 - o((\log n)^{-2})$$

for all three random variables ξ_n, ξ_n^l, ξ_n^r, by monotonicity we have that

$$\mathsf{P}_0^\eta \otimes \mathbf{P}_{\mu,\theta}\left(0 \notin R[1, n],\, \tau^+_{R[-n,n]} \geq n\right) = \frac{2 + o(1)}{C_G \log n}.$$

The statement of Lemma 4.4.10 now follows since

$$\begin{aligned}\mathsf{P}_0^\eta \otimes \mathbf{P}_{\mu,\theta}\left(n < \tau^+_{R[-n,n]} < \infty\right) &\leq \sum_{k>n} \mathsf{P}_0^\eta \otimes \mathbf{P}_{\mu,\theta}(S_k \in R[-n, n]) \\ &\lesssim \sum_{k>n} n \sup_{z\in\mathbb{Z}^6} \mathsf{P}_0^\eta(S_k = z) \asymp n^{-1}\end{aligned}$$

is negligible, where in the last line we use Lemma 4.2.6. □

Finally, we conclude for the capacity of the infinite model by a second moment method, analogue to [73, Theorem 14].

Proposition 4.4.11. *In dimension $d = 6$, let μ, θ, η be distributions with assumptions* (4.1.1) *and that μ has finite 5-th moment. Recall the infinite model in Section 4.2.2. As $n \to \infty$, under $\mathbf{P}_{\mu,\theta}$,*

$$\frac{\log n}{n}\mathtt{cap}_\eta R[0,n] \xrightarrow{\mathbb{L}^2} 2C_G^{-1},$$

where C_G is that in Proposition 4.4.5.

Proof. Decompose the capacity as discussed in (4.2.8). By (4.2.4) and Lemma 4.4.10 we have that

$$\begin{aligned}
&\frac{\log n}{n}\mathbf{E}_{\mu,\theta}[\mathtt{cap}_\eta R[0,n]]\\
=&\frac{\log n}{n}\sum_{i=0}^{n}\mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{v_i\notin R[i+1,n]}\mathsf{P}^\eta_{v_i}\Big(\tau^+_{R[0,n]}=\infty\Big)\Big]\\
=&\frac{\log n}{n}\sum_{i=0}^{n}\mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{0\notin R[1,n-i]}\mathsf{P}^\eta_{0}\Big(\tau^+_{R[-i,n-i]}=\infty\Big)\Big]\\
\geq&(\log n)\mathsf{P}^\eta_0\otimes\mathbf{P}_{\mu,\theta}\Big(0\notin R[1,n],\tau^+_{R[-n,n]}=\infty\Big)\overset{n\to\infty}{\longrightarrow}2C_G^{-1}.
\end{aligned}$$

Then it suffices to show that

$$\limsup_{n\to\infty}\left(\frac{\log n}{n}\right)^2\mathbf{E}_{\mu,\theta}\big[(\mathtt{cap}_\eta R[0,n])^2\big]\leq\big(2C_G^{-1}\big)^2. \tag{4.4.21}$$

In fact, for any $\alpha\in(0,\frac{1}{4})$, set

$$D(\alpha)=\{(i,j)\colon 0<i<j<n \text{ and } i,j-i,n-j>n^{1-\alpha}\},$$

then

$$\begin{aligned}
&\mathbf{E}_{\mu,\theta}[(\mathtt{cap}_\eta R[0,n])^2]\\
=&\sum_{i,j=0}^{n}\mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{v_i\notin R[i+1,n]}\mathbf{1}_{v_j\notin R[j+1,n]}\mathsf{P}^\eta_{v_i}\Big(\tau^+_{R[0,n]}=\infty\Big)\mathsf{P}^\eta_{v_j}\Big(\tau^+_{R[0,n]}=\infty\Big)\Big]\\
=&2\sum_{D(\alpha)}\mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{v_i\notin R[i+1,n]}\mathbf{1}_{v_j\notin R[j+1,n]}\mathsf{P}^\eta_{v_i}\Big(\tau^+_{R[0,n]}=\infty\Big)\mathsf{P}^\eta_{v_j}\Big(\tau^+_{R[0,n]}=\infty\Big)\Big]+o\bigg(\frac{n^2}{(\log n)^2}\bigg).
\end{aligned}$$

Moreover, write $k=j-i$ for simplicity, then for $(i,j)\in D(\alpha)$, by (4.2.4),

$$\begin{aligned}
&\mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{v_i\notin R[i+1,n]}\mathbf{1}_{v_j\notin R[j+1,n]}\mathsf{P}^\eta_{v_i}\Big(\tau^+_{R[0,n]}=\infty\Big)\mathsf{P}^\eta_{v_j}\Big(\tau^+_{R[0,n]}=\infty\Big)\Big]\\
\leq&\mathbf{E}_{\mu,\theta}\Big[\mathbf{1}_{0\notin R[1,n^{1-3\alpha}]}\mathbf{1}_{v_k\notin R[k+1,k+n^{1-3\alpha}]}\times\\
&\qquad\mathsf{P}^\eta_0\Big(\tau^+_{R[-n^{1-3\alpha},n^{1-3\alpha}]}=\infty\Big)\mathsf{P}^\eta_{v_k}\Big(\tau^+_{R[k-n^{1-3\alpha},k+n^{1-3\alpha}]}=\infty\Big)\Big]
\end{aligned}$$

By (4.4.20), with probability $1-o((\log n)^{-2})$, one has $\left|\zeta_{\pm n^{\frac{1}{2}-\alpha}}\right| \in [2n^{1-3\alpha}, n^{1-\alpha}]$. And under this condition, the range $R[-n^{1-3\alpha}, n^{1-3\alpha}]$ and $R[k-n^{1-3\alpha}, k+n^{1-3\alpha}]$ correspond to disjoint subtrees in $\mathcal{T}$, thus by strong Markov property applied at the node $(n^{\frac{1}{2}-\alpha}, \varnothing)$, we can bound the probability above by

$$\begin{aligned}
&\left(\mathsf{P}_0^\eta \otimes \mathbf{P}_{\mu,\theta}(0 \notin R[1, n^{1-3\alpha}], \tau^+_{R[-n^{1-3\alpha}, n^{1-3\alpha}]} = \infty)\right)^2 + o((\log n)^{-2}) \\
&= \left(\left(2C_G^{-1}(1-3\alpha)^{-1}\right)^2 + o(1)\right)(\log n)^{-2}
\end{aligned}$$

using Lemma 4.4.10. Then (4.4.21) follows by summing over all indices in $D(\alpha)$ and let $\alpha \to 0+$. □

4.4.3 Proof of Theorem 4.1.1 (2)

We use the same treatment as for high dimensions to extend the result on the infinite model to that of a standard branching process.

Theorem 4.4.12. *In dimension $d = 6$, let μ, θ, η be distributions with the conditions in (4.1.1) and that μ has finite 5-th moment, and let $R[0, n]$ be the range constructed in Section 4.2.2 (abused to denote the range of other trees as well). Under the law of a (standard) Galton-Watson tree conditioned to have $n+1$ nodes, $P_{\mu,\theta}(\cdot | \#T = n+1)$,*

$$\frac{\log n}{n}\mathtt{cap}_\eta(R[0,n]) \to 2C_G^{-1} \text{ in probability,}$$

where C_G is the constant in Proposition 4.4.5.

Proof. As in the proof of Theorem 4.3.7, we can prove by Lemma 4.3.6 that for any $a \in (0,1), \epsilon > 0$,

$$\lim_{n\to\infty} P_{\mu,\theta}\left(\left|\frac{\log n}{n}\mathtt{cap}_\eta(R[0,an]) - 2aC_G^{-1}\right| > \epsilon \,\middle|\, \#T = n+1\right) = 0$$

Take $a \to 1-$, then we have a lower bound for $\mathtt{cap}_\eta R[0,n]$,

$$\lim_{n\to\infty} P_{\mu,\theta}\left(\frac{\log n}{n}\mathtt{cap}_\eta(R[0,n]) - 2C_G^{-1} < -\epsilon \,\middle|\, \#T = n+1\right) = 0$$

If we reverse the order for nodes on a tree T, and set the range of its last an nodes by $R[0,an]^-$, then $R[0,an]^-$ will satisfy the same estimate as

$R[0, an]$. Moreover, $R[0, n/2], R[0, n/2]^-$ will cover all the tree expect for a negligible number of nodes ([94, p. 20]), thus

$$\begin{aligned}
&\lim_{n\to\infty} P_{\mu,\theta}\left(\frac{\log n}{n}\mathtt{cap}_\eta(R[0,n]) - 2C_G^{-1} > \epsilon \,\middle|\, \#T = n+1\right) \\
&= \lim_{n\to\infty} P_{\mu,\theta}\left(\frac{\log n}{n}\mathtt{cap}_\eta(R[0,n/2] \cup R[0,n/2]^-) - 2C_G^{-1} > \epsilon \,\middle|\, \#T = n+1\right) \\
&\le \lim_{n\to\infty} P_{\mu,\theta}\left(\frac{\log n}{n}(\mathtt{cap}_\eta R[0,n/2] + \mathtt{cap}_\eta R[0,n/2]^-) - 2C_G^{-1} > \epsilon \,\middle|\, \#T = n+1\right) = 0.
\end{aligned}$$

□

Chapter 5

Capacity in low dimensions

This chapter is based on [19].

5.1 Introduction

Let $d \geq 3$ and η be a probability distribution on $\mathbb{Z}^d$. The η-capacity of a finite set $A \subset \mathbb{Z}^d$ (with respect to η) is defined as

$$\mathsf{cap}_\eta A := \sum_{x \in A} \mathsf{P}_x^\eta(\tau_A^+ = \infty),$$

where P_x^η denotes the law of a (discrete) random walk (S_n) with jump distribution η started at x, and $\tau_A^+ := \inf\{n \geq 1 : S_n \in A\}$ is (S_n)'s first returning time to A.

Let μ be a probability distribution on $\mathbb{N}$. A μ-Galton-Watson tree starts with one initial ancestor which produces a random number of children according to μ, and these children form the first generation. Then particles in the first generation produce their children independently in the same way, forming the second generation. The system goes on until infinity, or until when there is no particle in a generation. In this paper, we are interested in the critical case, i.e. the case when $\sum_{k=0}^\infty k\mu(k) = 1$. In this case, it is well-known that the Galton-Watson tree extincts (stops with no particle in finitely many generations) almost surely. To avoid extinction, we consider the Galton-Watson forest defined as follows. Let $(\mathcal{T}_n)_{n\geq 0}$ be a sequence of independent μ-Galton-Watson trees. As showed in Figure 5.1, we start with a fixed infinite ray $(w_n)_{n\geq 0}$ called spine, and attach $\mathcal{T}_n$ to each w_n. For every $n \geq 1$, w_{n-1} is considered as the parent of w_n and the whole forest is rooted at w_0 which we denote by $\varnothing$. As all $\mathcal{T}_n$ are finite, this Galton-Watson forest is in fact an infinite rooted tree, denoted by $\mathcal{T}^{IGW}$. Let $\mathbb{P}_\mu$ be the law of $\mathcal{T}^{IGW}$.

Let θ be a probability distribution on $\mathbb{Z}^d$. Given a (finite or infinite) tree $\mathcal{T}$, we can define a tree-indexed random walk $(V_u)_{u\in\mathcal{T}}$ in $\mathbb{Z}^d$ as follows: To all edges of $\mathcal{T}$ we attach i.i.d. random variables which are distributed as θ, independent of $\mathcal{T}$. Define $V_\varnothing := 0$. For each $u \in \mathcal{T}\backslash\{\varnothing\}$, let V_u be the sum of those random variables which are attached to the edges in the (unique) simple path relating u to the root $\varnothing$. Clearly $\mathcal{T}$ describes the genealogy of $(V_u)_{u\in\mathcal{T}}$. We may also call $(V_u)_{u\in\mathcal{T}}$ a branching random walk when its genealogy tree is a Galton-Watson tree (or forest).

Denote by $\mathbb{P}_{\mu,\theta}$ the law of the branching random walk $(V_u)_{u\in\mathcal{T}^{IGW}}$ when $\mathcal{T}^{IGW}$ is the Galton-Watson forest distributed as $\mathbb{P}_\mu$.

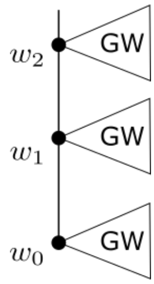


Figure 5.1: The Galton Watson forest $\mathcal{T}^{IGW}$.

Under the measure $\mathbb{P}_{\mu,\theta}$, let $R_n := \{V_u, u \in \cup_{j=0}^{n-1}\mathcal{T}_j\}$ be the set of points in $\mathbb{Z}^d$ visited by the branching random walk (V_u) when the index u explores the first n subtrees of $\mathcal{T}^{IGW}$. Our main result is:

Theorem 5.1.1. *In dimensions $d = 3, 4, 5$, let μ be a probability measure in $\mathbb{N}$, let θ, η be probability measures in $\mathbb{Z}^d$, with the conditions*

$$\left.\begin{array}{l} \mu \textit{ has mean } 1 \textit{ and finite variance, and } \mu \not\equiv \delta_1, \\ \eta \textit{ is aperiodic, irreducible, with mean 0 and finite } (d+1)\textit{-th moment,} \\ \theta \textit{ is symmetric, irreducible, with some finite exponential moments.} \end{array}\right\} \tag{5.1.1}$$

Then almost surely under $\mathbb{P}_{\mu,\theta}$, as $n \to \infty$,

$$\mathsf{cap}_\eta R_n = n^{\frac{d-2}{2}+o_{\mathsf{as}}(1)},$$

where here and in the sequel, $o_{\mathsf{as}}(1)$ denotes a quantity which converges to 0 almost surely as $n \to \infty$.

Remark 5.1.2. We need the finite second moment of μ in Lemma 5.2.4 and Lemma 5.2.6, and use the symmetry and finite exponential moments of θ in Corollary 5.2.5, Lemma 5.3.3 and Lemma 5.4.4, whereas the finite $(d+1)$-th moment of η is needed in Lemma 4.2.7. Finally, we assume the irreducibility for brevity, indeed the result remains true as long as a θ-walk generates a d-dimensional subspace of $\mathbb{Z}^d$. $\square$

A few comments are in order. First, it will be clear from our proof that Theorem 5.1.1 holds when $\mathcal{T}^{IGW}$ is replaced by a more general tree with one unique infinite ray, for example if we attach to each spine $w_i, i \geq 0$, an i.i.d. random number of independent μ-Galton-Watson trees, as long as this random number has finite second moment. In particular Theorem 5.1.1 holds for the Kesten tree which is the μ-Galton-Watson tree conditioned to survive forever if μ has finite third moment (because by the spine decomposition, the number of children of w_i in the Kesten tree has the size-biased law of μ).

Second, to avoid the extinction of a critical μ-Galton-Watson tree $\mathcal{T}$, we may condition $\mathcal{T}$ to have n vertices, thus we obtain a random tree, say $\mathcal{T}_n^{cond}$. Let $R_n^{cond} := \{V_u, u \in \mathcal{T}_n^{cond}\}$ be the range of $(V_u)_{\mathcal{T}_n^{cond}}$ when the underlying genealogy tree is $\mathcal{T}_n^{cond}$. Le Gall and Lin [72, 73] studied in detail $\#R_n^{cond}$, the cardinality of the range R_n^{cond}, and obtained various scaling limits for all dimensions. In particular, their results show that the critical dimension for the range of the tree-indexed walk is $d = 4$: for $d \geq 5$, $\#R_n^{cond}$ grows linearly whereas for $d = 4$, $\#R_n^{cond}$ is sub-linear and for $d \leq 3$, $\#R_n^{cond}$ is of order $n^{d/4}$.

The study of the capacity of the range R_n^{cond} was initiated in [21] where the authors proved that $\mathtt{cap}_\eta R_n^{cond}$ grows linearly for $d \geq 7$ and is sub-linear for $d = 6$. This suggests, also as conjectured in [21], that $d = 6$ should be the critical dimension for the capacity of the range. The main motivation of the present work is to confirm this prediction, by giving the growth order of $\mathtt{cap}_\eta R_n^{cond}$ for $d \in \{3, 4, 5\}$, this will be stated in the forthcoming Remark 5.2.2, see (5.2.2).

At last, let us mention the systematical studies on the capacity of the range for a simple random walk on $\mathbb{Z}^d$, see Asselah, Schapira and Sousi [15] and the references therein.

The rest of the paper is organized as follows: In Section 2, we order the vertices in the Galton-Watson forest $\mathcal{T}^{IGW}$ and state the corresponding result for the range of the walk indexed by the first n vertices (Proposition 5.2.1). Then Theorem 5.1.1 follows as a consequence of Proposition 5.2.1 and Lemma 5.2.3. Sections 3 and 4 are devoted to the proofs of the upper and lower bound of Proposition 5.2.1 respectively.

Notation: Under P_x^θ (resp: P_x^η), $(S_n)_{n\geq 0}$ denotes a random walk on $\mathbb{Z}^d$ starting from x with jump distribution θ (resp: η). For brevity, we call (S_n) a θ (resp: η)-random walk. Finally, $C_i, 1 \leq i \leq 12$ denote some positive constants.

5.2 On the Galton-Watson forest

It will be more convenient to study the capacity for n vertices than n subtrees, then we order the vertices in the Galton-Watson forest. On $\mathcal{T}^{IGW}$, we visit the vertices in the order illustrated in Figure 5.2: starting with the first subtree $\mathcal{T}_0$ rooted at w_0, one visits every vertex in the order of Depth-First Search (lexicographical order). Then we continue with the subtree $\mathcal{T}_1$ rooted at w_1 and iterate the process. We denote the sequence of vertices in this order by $(u_i)_{i\geq 0}$.

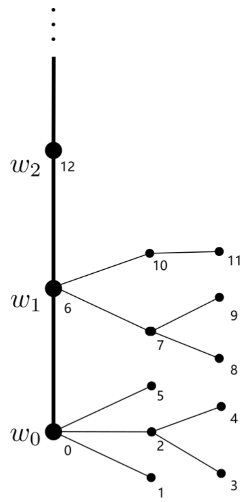


Figure 5.2: A sample of the μ-Galton Watson forest. The path in bold is the spine (w_n). Labels correspond to the sequence (u_i). For example, $u_0 = w_0 = \varnothing$ and $u_6 = w_1$.

Under the measure $\mathbb{P}_{\mu,\theta}$, the sequence (u_i) then induces a sequence of points in $\mathbb{Z}^d$, (V_{u_i}) the positions of (u_i), and we define

$$R[0,n] = \{V_{u_0}, V_{u_1}, \cdots, V_{u_n}\}.$$

The main part of this paper will be devoted to prove that

Proposition 5.2.1. *In dimensions $d = 3,4,5$, let μ, θ, η be probability distributions with the conditions* (5.1.1) *Then almost surely under $\mathbb{P}_{\mu,\theta}$,*

$$\mathtt{cap}_\eta R[0,n] = n^{\frac{d-2}{4}+o_{\mathrm{as}}(1)}.$$

Remark 5.2.2. As for Theorem 5.1.1, Proposition 5.2.1 also holds for more general trees with one unique infinite ray: if we attach to each w_i an i.i.d. random number ν_i of μ-Galton-Watson tree, then the same conclusion holds as long as $\mathbb{E}_{\mu,\theta}[\nu_i^2] < \infty$. The proof follows in the same way as that of Proposition 5.2.1 and we skip the details.

Now let $R_n^{cond} := \{V_u, u \in \mathcal{T}_n^{cond}\}$ be as before the range of $(V_u)_{\mathcal{T}_n^{cond}}$, where $\mathcal{T}_n^{cond}$ is the μ-Galton-Watson tree conditioned to have n vertices. Assume (5.1.1) and furthermore that μ has finite third moment, then in probability

$$\mathtt{cap}_\eta R_n^{cond} = n^{\frac{d-2}{4} + o_{\mathtt{P}}(1)}, \tag{5.2.2}$$

where $o_{\mathtt{P}}(1)$ denotes a quantity which converges to 0 in probability as $n \to \infty$. The conclusion (5.2.2) follows from the aforementioned generalized version of Proposition 5.2.1 with $\mathbb{P}_{\mu,\theta}(\nu_i = k) = \sum_{j=k+1}^{\infty} \mu(j), k \geq 0$, and the arguments in Zhu [94], Section 5 for the coupling between the infinite tree model and $\mathcal{T}_n^{cond}$. Indeed, we first observe that ν_i has finite second moment thanks to the assumption on μ. Fix $0 < a < 1$. By the generalized version of Proposition 5.2.1 (with ν_i) and [21, Lemma 3.6], we have

$$\mathtt{cap}_\eta R_n^{cond}[0, \lfloor an \rfloor] = n^{\frac{d-2}{4} + o_{\mathtt{P}}(1)},$$

where $R_n^{cond}[0, \lfloor an \rfloor]$ is the range of $(V_u)_{\mathcal{T}_n^{cond}}$ when u runs over the first $1 + \lfloor an \rfloor$ vertices of $\mathcal{T}_n^{cond}$ in the lexicographical order. This gives a lower bound of (5.2.2),

$$\mathtt{cap}_\eta R_n^{cond} \geq \mathtt{cap}_\eta R_n^{cond}[0, \lfloor an \rfloor] = n^{\frac{d-2}{4} + o_{\mathtt{P}}(1)}.$$

Moreover, by exploring the tree $\mathcal{T}_n^{cond}$ in the reversed order, we get that

$$\mathtt{cap}_\eta R_n^{cond}[\lfloor an \rfloor, n] = n^{\frac{d-2}{4} + o_{\mathtt{P}}(1)},$$

yielding the upper bound because

$$\mathtt{cap}_\eta R_n^{cond} \leq \mathtt{cap}_\eta R_n^{cond}[0, \lfloor an \rfloor] + \mathtt{cap}_\eta R_n^{cond}[\lfloor an \rfloor, n] = n^{\frac{d-2}{4} + o_{\mathtt{P}}(1)}.$$

□

Admitting Proposition 5.2.1, we deduce Theorem 5.1.1 from the following lemma on population of the first n subtrees.

Lemma 5.2.3. *Let $\mu \not\equiv \delta_1$ be a probability measure on $\mathbb{N}$ with mean 1 and finite variance, then $\mathbb{P}_\mu$-almost surely, there are $n^{2+o_{\mathtt{as}}(1)}$ vertices in the first n subtrees rooted at $w_0, \cdots, w_{n-1}$.*

Proof. The proof is easy. In fact, if we denote by $\#\mathcal{T}$ the total number of vertices of a finite tree $\mathcal{T}$, then $\#\mathcal{T}_0$ is in the domain of attraction of a stable law of index $\frac{1}{2}$, see the forthcoming (5.2.6). To get the Lemma, it suffices to apply the almost sure fluctuation results for the partial sum of

n independent copies of $\#\mathcal{T}_0$. However, to unify the presentation we give another proof based on the Lukasiewicz path.

It is well-known (see [42], Section 0.2) that there exists a random walk Y on $\mathbb{Z}$ with $Y_0 = 0$ and jump distribution $\mathbb{P}_\mu(Y_1 = k) = \mu(k+1)$ for $k = -1, 0, 1, 2, ...$, such that

$$\#\mathcal{T}_0 + ... + \#\mathcal{T}_{n-1} = \inf\{k \geq 1 : Y_k = -n\}, \qquad n \geq 1. \tag{5.2.3}$$

Observe that $\mathbb{E}_\mu(Y_1) = 0$ and $\mathrm{Var}(Y_1) \in (0, \infty)$. By the classical Khintchine and Hirsch laws of iterated logarithm for the random walk (Y_k) (see Csáki [33] for Hirsch's law of iterated logarithm under the second moment assumption),

$$-\min_{0 \leq k \leq n} Y_k = n^{\frac{1}{2} + o_{\mathrm{as}}(1)}, \qquad \text{a.s.} \tag{5.2.4}$$

It follows that $\#\mathcal{T}_0 + ... + \#\mathcal{T}_{n-1} = n^{2+o_{\mathrm{as}}(1)}$ a.s. □

The rest of the paper is devoted to the proof of Proposition 5.2.1. At first, we need the following estimates on the population of the Galton-Watson forest $\mathcal{T}^{IGW}$. For any $u, v \in \mathcal{T}^{IGW}$, let $\mathrm{dist}(u, v)$ be the graph distance between u and v.

Lemma 5.2.4. *Let $\mu \not\equiv \delta_1$ be a probability measure on $\mathbb{N}$ with mean 1 and finite variance, then $\mathbb{P}_\mu$-almost surely,*

$$\max_{0 \leq i \leq n} \mathrm{dist}(\varnothing, u_i) = n^{\frac{1}{2} + o_{\mathrm{as}}(1)}.$$

Proof. Recall that (w_n) are the roots of subtrees $(\mathcal{T}_n)$ in the forest. As showed in Figure 5.3, let ζ_n be the index of the subtree that u_n belongs to, i.e. $u_n \in \mathcal{T}_{\zeta_n}$. Then let $H_n = \mathrm{dist}(u_n, w_{\zeta_n})$ be the height of u_n in $\mathcal{T}_{\zeta_n}$. Therefore,

$$\mathrm{dist}(\varnothing, u_n) = \zeta_n + H_n, \qquad \forall n \geq 0.$$

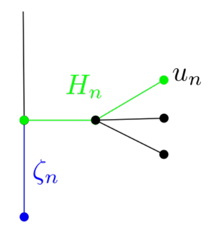


Figure 5.3: The decomposition $\mathrm{dist}(\varnothing, u_n) = \zeta_n + H_n$.

By Lemma 5.2.3, we have

$$\zeta_n = n^{\frac{1}{2}+o_{\mathrm{as}}(1)}.$$

It thus suffices to show that $\mathbb{P}_\mu$-almost surely,

$$\max_{0\le i\le n} H_i \le n^{\frac{1}{2}+o_{\mathrm{as}}(1)}. \tag{5.2.5}$$

Note that the process (H_n) is distributed as the height process in the sense of [42, Section 0.2]: Using the random walk (Y_k) introduced in the proof of Lemma 5.2.3, we have

$$H_n = \sum_{k=0}^{n-1} 1_{\{Y_k=\min_{k\le j\le n} Y_j\}}, \qquad n\ge 1.$$

For any fixed n, by considering $Y_n - Y_{n-k}, 0 \le k \le n$, we see that H_n is distributed as $\sum_{k=1}^n 1_{\{Y_k=\max_{0\le j\le k} Y_k\}}$. In other words, let $\mathtt{t}_0 := 0$ and for $j \ge 1$, $\mathtt{t}_j := \inf\{k > \mathtt{t}_{j-1} : Y_k \ge Y_{\mathtt{t}_{j-1}}\}$ be the sequence of (weak) ascending ladder epochs of Y. Then for all $n, \ell \ge 1$,

$$\mathbb{P}_\mu(H_n \ge \ell) = \mathbb{P}_\mu(\mathtt{t}_\ell \le n) \le \inf_{\lambda>0} e^{\lambda n}\big(\mathbb{E}_\mu(e^{-\lambda \mathtt{t}_1})\big)^\ell,$$

where in the above inequality we have used the fact that $\mathtt{t}_k - \mathtt{t}_{k-1}, k \ge 1$ are i.i.d and distributed as $\mathtt{t}_1$. The Laplace transform of $\mathbb{E}_\mu(e^{-\lambda\mathtt{t}_1})$ can be computed by the Sparre-Anderson identity, whose asymptotic is given by Kersting and Vatutin ([60], proof of Theorem 4.6, Page 75):

$$1 - \mathbb{E}_\mu(e^{-\lambda \mathtt{t}_1}) \sim C_1\sqrt{\lambda}, \qquad \lambda \to 0. \tag{5.2.6}$$

Take $\lambda = \frac{1}{n}$ we see that for all $n \ge 1$, $\mathbb{P}_\mu(H_n \ge n^{\frac{1}{2}}(\log n)^2) \le e^{1-C_2(\log n)^2}$. It follows that $\mathbb{P}_\mu(\max_{1\le k\le n} H_k \ge n^{\frac{1}{2}}(\log n)^2) \le n\, e^{1-C_2(\log n)^2}$ whose sum over n converges. We get (5.2.5) by the Borel-Cantelli lemma. $\square$

Corollary 5.2.5. *Let μ, θ be probability measures satisfying* (5.1.1)*, then $\mathbb{P}_{\mu,\theta}$-almost surely,*

$$\max_{0\le i\le n} |V_{u_i}| = n^{\frac{1}{4}+o_{\mathrm{as}}(1)}.$$

Proof. Conditionally on $u \in \mathcal{T}^{IGW}$ with $\mathrm{dist}(\varnothing, u) = k$, V_u is distributed as S_k, where $(S_n)_{n\ge 0}$ is a θ-random walk started at 0, i.e. a random walk in $\mathbb{Z}^d$ whose law is P_0^θ. By assumption (5.1.1), $\mathsf{E}_0^\theta(S_1) = 0$ and S_1 has some finite exponential moments.

Notice that $(V_{w_j}, 0 \le j \le \zeta_n)$ is a θ-random walk on $\mathbb{Z}^d$, and $\zeta_n = n^{\frac{1}{2}+o_{\mathrm{as}}(1)}$, we have the lower bound $\max_{0\le i\le n} |V_{u_i}| \ge \max_{0\le j\le \zeta_n} |V_{w_j}| = \max_{0\le j\le n^{\frac{1}{2}+o_{as}(1)}} |V_{w_j}| = n^{\frac{1}{4}+o_{\mathrm{as}}(1)}$ by applying (5.2.4) to the θ-random walk $(V_{w_j})_{j\ge 0}$ instead of $(Y_j)_{j\ge 0}$.

Below we show the upper bound $\max_{0\le i\le n} |V_{u_i}| \le n^{\frac{1}{4}+o_{\mathrm{as}}(1)}$. Indeed, applying Petrov ([84], Theorem 2.7 and Lemma 2.2) gives that for all $n \ge 1$ and $\lambda > 0$,

$$\mathsf{P}_0^\theta\big(|S_n| \ge \lambda\big) \le \max(e^{-C_3\frac{\lambda^2}{n}}, e^{-C_3\lambda}). \tag{5.2.7}$$

It follows that for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}_{\mu,\theta}\Big(\max_{0\le i\le n} |V_{u_i}| \ge n^{\frac{1}{4}+\varepsilon}, \max_{0\le i\le n} \operatorname{dist}(\varnothing, u_i) \le n^{\frac{1}{2}+\varepsilon}\Big) \;&\le\; n \max_{0\le k\le n^{\frac{1}{2}+\varepsilon}} \mathsf{P}_0^\theta\big(|S_k| \ge n^{\frac{1}{4}+\varepsilon}\big)\\ &\le\; n\, e^{-C_3 n^{\varepsilon}},\end{aligned}$$

whose sum over n converges. By using the Borel-Cantelli lemma and Lemma 5.2.4, we get the Corollary. $\square$

For $\varepsilon \in (0, \frac{1}{4})$, let

$$F_\varepsilon(n) := \Big\{\max_{0\le i\le n} \operatorname{dist}(\varnothing, u_i) < n^{\frac{1}{2}+\varepsilon}\Big\}. \tag{5.2.8}$$

By Lemma 5.2.4, almost surely $F_\varepsilon(n)$ holds for all large n.

Lemma 5.2.6. *Let $\mu \not\equiv \delta_1$ be a probability measure on $\mathbb{N}$ with mean 1 and finite second moment, then for any $k \ge 0$,*

$$\mathbb{E}_\mu\big[\#\{(i,j) : 0 \le i \le j \le n, \operatorname{dist}(u_i, u_j) = k\}\, 1_{F_n(\varepsilon)}\big] \le (k+1)^2 n^{\frac{1}{2}+\varepsilon} + C_4(k+1)n^{1+2\varepsilon},$$

where $C_4 := \sum_{j=0}^\infty j^2\mu(j)$.

Proof. For any $u, v \in \mathcal{T}^{IGW}$, we $u \preceq v$, if u is an ancestor of v and denote by $u \wedge v$ their most recent common ancestor. We consider the two cases: $u_i \wedge u_j \in \{w_\ell : \ell \ge 0\}$, $u_i \wedge u_j \notin \{w_\ell : \ell \ge 0\}$ separately.

First case: $u_i \wedge u_j = w_\ell$ for some $\ell \ge 0$.

Note that the subtree rooted at w_ℓ, $\mathcal{T}_\ell = \{u : w_\ell \preceq u, w_{\ell+1} \not\preceq u\}$, is a critical Galton-Watson tree,

$$\mathbb{E}_\mu[\#\{u \in \mathcal{T}_\ell : \operatorname{dist}(u, w_\ell) = k\}] = 1, \qquad \forall k \ge 1. \tag{5.2.9}$$

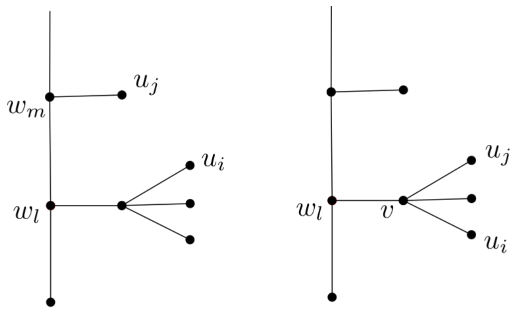


Figure 5.4: The relative position of u_i, u_j and w_ell.

As is shown in Figure 5.4(a),

$$\begin{aligned}
&\mathbb{E}_\mu\big[\#\{(i,j): 0\le i\le j\le n, \mathrm{dist}(u_i,u_j)=k, u_i\wedge u_j\in\{w_\ell:\ell\ge 0\}\}\,1_{F_n(\varepsilon)}\big]\\
\le\ & \sum_{r=0}^{k}\sum_{0\le\ell<m\le\ell+k-r} 1_{\{m\le n^{\frac12+\varepsilon}\}}\,\mathbb{E}_\mu\Big[\sum_{u\in\mathcal{T}_\ell, u'\in\mathcal{T}_m} 1_{\{\mathrm{dist}(u,w_\ell)=r,\,\mathrm{dist}(u',w_m)=k-r-(m-\ell)\}}\Big]\\
=\ & \sum_{r=0}^{k}\sum_{0\le\ell<m\le\ell+k-r} 1_{\{m<n^{\frac12+\varepsilon}\}}\\
\le\ & (k+1)^2\, n^{\frac12+\varepsilon}, \qquad (5.2.10)
\end{aligned}$$

where the above equality follows from (5.2.9) and the independence of $\mathcal{T}_\ell$ and $\mathcal{T}_m$.

Second (and last) case: $u_i\wedge u_j\notin\{w_\ell:\ell\ge 0\}$.

For this case, similarly as shown in Figure 5.4(b), let $v=u_i\wedge u_j$. On $F_n(\varepsilon)$, $\mathrm{dist}(\varnothing,v)\le n^{\frac12+\varepsilon}$. Then

$$\begin{aligned}
&\mathbb{E}_\mu\big[\#\{(i,j): 0\le i\le j\le n, \mathrm{dist}(u_i,u_j)=k, u_i\wedge u_j\notin\{w_\ell:\ell\ge 0\}\}\,1_{F_n(\varepsilon)}\big]\\
\le\ & \sum_{r=0}^{k}\sum_{0\le\ell,t<n^{\frac12+\varepsilon}}\mathbb{E}_\mu\Big[\sum_{v\in\mathcal{T}_\ell,\mathrm{dist}(v,w_\ell)=t}\ \sum_{u\wedge u'=v} 1_{\{\mathrm{dist}(u,v)=r,\,\mathrm{dist}(u',v)=k-r\}}\Big].
\end{aligned}$$

Conditionally on the number of children of v, say j, by using (5.2.9), the expectation of $\sum_{u\wedge u'=v} 1_{\{\mathrm{dist}(u,v)=r,\,\mathrm{dist}(u',v)=k-r\}}$ is dominated by j^2. Again using (5.2.9), we deduce from the branching property that

$$\mathbb{E}_\mu\big[\sum_{v\in\mathcal{T}_\ell,\mathrm{dist}(v,w_\ell)=t}\ \sum_{u\wedge u'=v} 1_{\{\mathrm{dist}(u,v)=r,\,\mathrm{dist}(u',v)=k-r\}}\big]\le\sum_{j=0}^{\infty} j^2\mu(j),$$

which together with (5.2.10) yield the Lemma. $\square$

5.3 Proof of the upper bound in Proposition 5.2.1

Before studying the capacity, we need the basic notation of Green's function:

$$G_\eta(x,y) = G_\eta(y-x) := \mathsf{E}_0^\eta\left[\sum_{i=0}^{\infty} \mathbf{1}_{\{S_i=y-x\}}\right] = \sum_{i=0}^{\infty} \mathsf{P}_0^\eta(S_i = y-x), \qquad x,y \in \mathbb{Z}^d.$$

The Green function $G_\eta(x)$ has the following asymptotic estimate:

Lemma 5.3.1 (Lawler and Limic [68, Theorem 4.3.5])**.** *Given an aperiodic and irreducible distribution η on $\mathbb{Z}^d(d \geq 3)$ with mean 0 and covariance matrix Γ_η, if it has finite $(d+1)$-th moment $\mathsf{E}_0^\eta[|S_1|^{d+1}] < \infty$, then*

$$G_\eta(x) = \frac{C_{d,\eta}}{J_\eta(x)^{d-2}} + O(|x|^{1-d}),$$

where $C_{d,\eta} = \frac{\mathbf{\Gamma}(\frac{d}{2})}{(d-2)\pi^{d/2}\sqrt{\det \Gamma_\eta}}$, $\mathbf{\Gamma}(\cdot)$ refers to the Gamma function and $J_\eta(x) = \sqrt{x \cdot \Gamma_\eta^{-1} x}$.

Below is a lemma that connects the capacity with Green's function, which is inspired from [21, Lemma 2.12].

Lemma 5.3.2. *Let η be a probability distribution in $\mathbb{Z}^d$, $d \geq 3$. For any sequence $(x_n)_{n\geq 0} \in \mathbb{Z}^d$,*

$$\frac{1}{n+1}\sum_{i=0}^{n} 1_{x_i \notin \{x_{i+1},\cdots,x_n\}} \mathsf{P}_{x_i}^\eta(\tau_{\{x_0,\cdots,x_n\}}^+ = \infty)\sum_{j=0}^{n} G_\eta(x_j,x_i) = 1,$$

where under P_x^η, (S_n) is a random walk on $\mathbb{Z}^d$ started at x and with jump distribution η, and $\tau_A^+ := \inf\{i \geq 1 : S_i \in A\}$ denotes as before the first returning time of A, for any finite $A \subset \mathbb{Z}^d$.

Proof. Since the random walk (S_n) in dimension $d \geq 3$ is transient, for any finite set $A \subset \mathbb{Z}^d$ and $z \in A$, let $\sigma_A := \sup\{i \geq 0 : S_i \in A\}$ be the last-passage time, then

$$\begin{aligned}
1 &= \mathsf{P}_z^\eta(\sigma_A < \infty) \\
&= \sum_{x\in A}\sum_{i=0}^{\infty} \mathsf{P}_z^\eta(S_i = x)\mathsf{P}_x^\eta(\tau_A^+ = \infty) \\
&= \sum_{x\in A} G_\eta(z,x)\mathsf{P}_x^\eta(\tau_A^+ = \infty).
\end{aligned}$$

Take $A = \{x_0, \cdots, x_n\}$ in this equation, then

$$\sum_{i=0}^{n} \mathbf{1}_{x_i \notin \{x_{i+1},\cdots,x_n\}} \mathsf{P}^{\eta}_{x_i}(\tau^{+}_{\{x_0,\cdots,x_n\}} = \infty) G_\eta(z, x_i) = 1,$$

and the conclusion follows by summing over $z = x_0, \cdots, x_n$. □

Then we estimate the sum of Green's functions.

Lemma 5.3.3. *In dimensions $d = 3, 4, 5$, let μ, θ, η be probability distributions with the conditions in* (5.1.1)*. Then $\mathbb{P}_{\mu,\theta}$-almost surely,*

$$\min_{0\le i\le n} \sum_{j=0}^{n} G_\eta(v_i, v_j) \ge n^{\frac{6-d}{4} + o_{\mathrm{as}}(1)}.$$

Proof. By Lemma 5.3.1, it suffices to show that $\mathbb{P}_{\mu,\theta}$-almost surely,

$$\min_{0\le i\le n} \sum_{j=0}^{n} \frac{1}{(1+|v_i - v_j|)^{d-2}} \ge n^{\frac{6-d}{4} + o_{\mathrm{as}}(1)}. \tag{5.3.11}$$

Denote as before by $\mathrm{dist}(u_i, u_j)$ the graph distance between the two vertices u_i, u_j on the tree, then

$$v_i - v_j \overset{d}{=} S_{\mathrm{dist}(u_i,u_j)},$$

where $(S_n)_{n\ge 0}$ is the θ-random walk started at 0, independent of $\mathrm{dist}(u_i, u_j)$.

For any $\varepsilon \in (0, \frac{1}{4})$, using the union bound and (5.2.7) we get that

$$\mathbb{P}_{\mu,\theta}\Big(\cup_{0\le i,j\le n} \{|v_i - v_j| \ge n^{\varepsilon}\sqrt{1 + \mathrm{dist}(u_i, u_j)}\}\Big) \le (n+1)^2 \, e^{-C_3\, n^{\varepsilon}}.$$

By the Borel-Cantelli lemma, almost surely the above event cannot happen infinitely often. Thus to prove (5.3.11), it suffices to show that $\mathbb{P}_{\mu,\theta}$-almost surely,

$$\min_{0\le i\le n} \sum_{j=0}^{n} \frac{1}{(1+\mathrm{dist}(u_i,u_j))^{\frac{d-2}{2}}} \ge n^{\frac{6-d}{4} + o_{\mathrm{as}}(1)},$$

or more generally, for any $\alpha > 0$, $\mathbb{P}_{\mu,\theta}$-almost surely

$$\min_{0\le i\le n} \sum_{j=0}^{n} \frac{1}{(1+\mathrm{dist}(u_i,u_j))^{\alpha}} \ge n^{1-\frac{\alpha}{2} + o_{\mathrm{as}}(1)}. \tag{5.3.12}$$

Observe that

$$\min_{0\le i\le n}\sum_{j=0}^{n}\frac{1}{(1+\mathrm{dist}(u_i,u_j))^{\alpha}} \ge (n+1)\Big(1+\max_{0\le i,j\le n}\mathrm{dist}(u_i,u_j)\Big)^{-\alpha}$$
$$\ge (n+1)\Big(1+2\max_{0\le i\le n}\mathrm{dist}(\varnothing,u_i)\Big)^{-\alpha},$$

then (5.3.12) follows from Lemma 5.2.4. □

Proof of the upper bound in Proposition 5.2.1: Applying Lemma 5.3.2 to $\{V_{u_0},\cdots,V_{u_n}\}$, we deduce from the definition of the η-capacity that

$$\mathtt{cap}_\eta R[0,n] = \sum_{i=0}^{n} 1_{V_{u_i}\notin\{V_{u_{i+1}},\cdots,V_{u_n}\}}\,\mathsf{P}^{\eta}_{V_{u_i}}\Big(\tau^+_{\{V_{u_0},\cdots,V_{u_n}\}}=\infty \,|\, \{V_{u_0},\cdots,V_{u_n}\}\Big)$$
$$\le \frac{n+1}{\min_{0\le i\le n}\sum_{j=0}^{n}G_\eta(V_{u_i},V_{u_j})},$$

and the conclusion follows from Lemma 5.3.3. □

5.4 Proof of the lower bound in Proposition 5.2.1

For the lower bound, our main tool is the following lemma.

Lemma 5.4.1 ([21, Lemma 2.11])**.** *Let $d\ge 3$ and η be any probability distribution on $\mathbb{Z}^d$. For any finite set $A\subset\mathbb{Z}^d$ and $k\in\mathbb{N}_+$,*

$$\mathtt{cap}_\eta A \ge \frac{\#A}{k+1}-\frac{\sum_{x,y\in A}G_\eta(x,y)}{k(k+1)}.$$

According to this lemma, the capacity $\mathtt{cap}_\eta R[0,n]$ can be bounded below by estimates of $\#R[0,n]$ and the sum of Green's functions. We start with $\#R[0,n]$. Let

$$L_n^x := \sum_{i=0}^{n}\mathbf{1}_{\{V_{u_i}=x\}},\qquad \forall x\in\mathbb{Z}^d,\ n\ge 0,$$

denote the local times, then we can write the range as

$$R[0,n]=\{x\in\mathbb{Z}^d : L_n^x\ge 1\}.$$

The following second moment estimate for local times is inspired by the proof of Le Gall and Lin [72, Lemma 3].

Lemma 5.4.2. *Let $d \geq 3$. With the conditions in* (5.1.1), $\mathbb{P}_{\mu,\theta}$*-almost surely, as* $n \to \infty$,

$$\sum_{x\in\mathbb{Z}^d} (L_n^x)^2 \leq n^{\max(\frac{8-d}{4},1)+o_{\mathrm{as}}(1)}.$$

Proof. Let $\varepsilon \in (0, \frac{1}{4})$ and recall the event $F_\varepsilon(n)$ defined in (5.2.8). We are going to prove that for all $n \geq 1$,

$$\sum_{x\in\mathbb{Z}^d,|x|\leq n} \mathbb{E}_{\mu,\theta}[(L_n^x)^2 \, 1_{F_\varepsilon(n)}] \leq C_5 \, n^{\max(\frac{8-d}{4},1)+4\varepsilon}. \tag{5.4.13}$$

Admitting for the moment (5.4.13) we can give the proof of Lemma 5.4.2. Let $\xi_n := \sum_{x\in\mathbb{Z}^d,|x|\leq n}(L_n^x)^2$, $\gamma := \max(\frac{8-d}{4}, 1) + 5\varepsilon$ and $n_j := 2^j$ for $j \geq 1$. By Markov's inequality, (5.4.13) implies that for all $j \geq 1$,

$$\mathbb{P}_{\mu,\theta}\big(\xi_{n_j} \geq n_{j-1}^\gamma, F_\varepsilon(n_j)\big) \leq C_5 \frac{n_j^{\gamma-\varepsilon}}{n_{j-1}^\gamma} \leq C_6 \, 2^{-\varepsilon j}.$$

The Borel-Cantelli lemma says that almost surely for all large j, either $\xi_{n_j} < n_{j-1}^\gamma$ or $F_\varepsilon(n_j)$ does not hold. However by Lemma 5.2.4, almost surely $F_\varepsilon(n_j)$ holds for all large j, hence we have proved that almost surely for all large j, $\xi_{n_j} < n_{j-1}^\gamma$. On the other hand, by Corollary 5.2.5, almost surely for all large j, $L_{n_j}^x = 0$ for all $|x| > n_j$, hence $\sum_{x\in\mathbb{Z}^d}(L_{n_j}^x)^2 = \xi_{n_j} < n_{j-1}^\gamma$. Then by monotonicity for all large n, $\sum_{x\in\mathbb{Z}^d}(L_n^x)^2 < n^\gamma$ a.s. Since ε can be arbitrarily small, we have proved Lemma 5.4.2.

It remains to show (5.4.13). To this end, we denote the transition probabilities for a θ-walk $(S_n)_{n\geq 0}$ by

$$\pi_m(x) := \mathsf{P}_0^\theta(S_m = x), \qquad m \geq 0, x \in \mathbb{Z}^d,$$

for simplicity. Then there exists a constant $C_7 > 0$ depending on d and θ such that for all $x \in \mathbb{Z}^d$ and $m \geq 0$,

$$\pi_m(x) \leq C_7 \, (1 + |x|)^{-d}, \tag{5.4.14}$$

$$\pi_m(x) \leq C_7 \, (1 + m)^{-\frac{d}{2}}, \tag{5.4.15}$$

$$\sum_{x\in\mathbb{Z}^d} \pi_m(x) = 1. \tag{5.4.16}$$

where (5.4.14) follows from [68, Proposition 2.4.6], and (5.4.15) follows from [68, p.24]. (In [68], θ is also required to be aperiodic, but since we only need an upper bound, these results can be easily extended to periodic cases.) Then we decompose the second moment in (5.4.13) as

$$\sum_{x\in\mathbb{Z}^d,|x|\le n} \mathbb{E}_{\mu,\theta}\left[(L_n^x)^2\, 1_{F_\varepsilon(n)}\right] = \sum_{x\in\mathbb{Z}^d,|x|\le n} \sum_{i,j=0}^{n} \mathbf{P}_{\mu,\theta}\Big(V_{u_i}=V_{u_j}=x, F_\varepsilon(n)\Big).$$

For notational brevity, we write $u=u_i\wedge u_j$ for the most recent common ancestor of u_i,u_j, and $y=V_u$ for the spatial location of u. We also write $a=\mathrm{dist}(\varnothing,u)$, $b=\mathrm{dist}(u,u_i)$ and $c=\mathrm{dist}(u,u_j)$ for the graph distances between these particles, as shown in Figure 5.5.

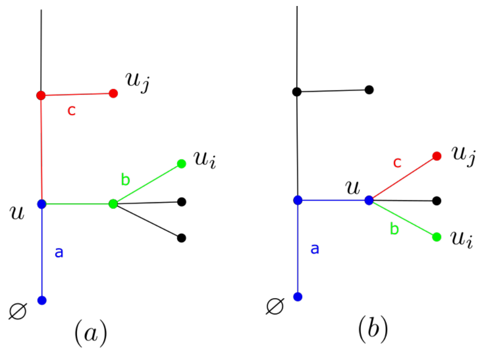


Figure 5.5: An illustration for the relative positions of u,u_i,u_j.

We assume without loss of generality that $b\ge c$, then

$$b\ge \frac{1}{2}\mathrm{dist}(u_i,u_j). \tag{5.4.17}$$

Therefore (keeping in mind that a,b,c depend on u_i,u_j),

$$\begin{aligned}
&\mathbb{E}_{\mu,\theta}\left[(L_n^x)^2\, 1_{F_\varepsilon(n)}\right]\\
=&\sum_{i,j=0}^{n}\mathbf{P}_{\mu,\theta}(V_{u_i}=V_{u_j}=x,F_\varepsilon(n))\\
=&\sum_{i,j=0}^{n}\sum_{y\in\mathbb{Z}^d}\mathbf{P}_{\mu,\theta}(V_u=y,V_{u_i}=V_{u_j}=x,F_\varepsilon(n))\\
=&\mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\sum_{y\in\mathbb{Z}^d}\pi_a(y)\pi_b(x-y)\pi_c(x-y)\,1_{F_\varepsilon(n)}\right]\\
=&A+B,
\end{aligned}$$

where

$$
\begin{aligned}
A &:= \mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\sum_{|y|\geq|x|/2}\pi_a(y)\pi_b(x-y)\pi_c(x-y)\,1_{F_\varepsilon(n)}\right],\\
B &:= \mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\sum_{|y|<|x|/2}\pi_a(y)\pi_b(x-y)\pi_c(x-y)\,1_{F_\varepsilon(n)}\right].
\end{aligned}
$$

For A, we use (5.4.14) for π_a, (5.4.15) and (5.4.17) for π_b and (5.4.16) for π_c, then

$$
\begin{aligned}
A &\leq C_7^2\,\mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\left(1+\frac{|x|}{2}\right)^{-d}\left(1+\frac{1}{2}\mathrm{dist}(u_i,u_j)\right)^{-\frac{d}{2}}\sum_{y\in\mathbb{Z}^d}\pi_c(x-y)\,1_{F_\varepsilon(n)}\right]\\
&= C_7^2\,\mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\left(1+\frac{|x|}{2}\right)^{-d}\left(1+\frac{1}{2}\mathrm{dist}(u_i,u_j)\right)^{-\frac{d}{2}}1_{F_\varepsilon(n)}\right]\\
&= C_7^2\left(1+\frac{|x|}{2}\right)^{-d}\sum_{k=0}^{\infty}\left(1+\frac{k}{2}\right)^{-\frac{d}{2}}\mathbf{E}_{\mu,\theta}\big[\#\{0\leq i,j\leq n:\mathrm{dist}(u_i,u_j)=k\}\,1_{F_\varepsilon(n)}\big].
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\left(1+\frac{k}{2}\right)^{-\frac{d}{2}}\mathbf{E}_{\mu,\theta}\big[\#\{0\leq i,j\leq n:\mathrm{dist}(u_i,u_j)=k\}\,1_{F_\varepsilon(n)}\big]\\
\leq\;&\sum_{0\leq k\leq n^{\frac{1}{2}}}\left(1+\frac{k}{2}\right)^{-\frac{d}{2}}\mathbf{E}_{\mu,\theta}\big[\#\{0\leq i,j\leq n:\mathrm{dist}(u_i,u_j)=k\}\,1_{F_\varepsilon(n)}\big]+\left(1+\frac{1}{2}n^{\frac{1}{2}}\right)^{-\frac{d}{2}}n^2,
\end{aligned}
$$

which by Lemma 5.2.6, is further bounded by

$$
\sum_{0\leq k\leq n^{\frac{1}{2}}}\left(1+\frac{k}{2}\right)^{-\frac{d}{2}}\big[(k+1)^2n^{\frac{1}{2}+\varepsilon}+C_4(k+1)n^{1+2\varepsilon}\big]+\left(1+\frac{1}{2}n^{\frac{1}{2}}\right)^{-\frac{d}{2}}n^2\leq C_8\,n^{\max(\frac{8-d}{4},1)+3\varepsilon}.
$$

Therefore we have proved that for any $x\in\mathbb{Z}^d$ and $n\geq 1$,

$$
A\leq C_7^2\,C_8\left(1+\frac{|x|}{2}\right)^{-d}n^{\max(\frac{8-d}{4},1)+3\varepsilon}.
$$

We may deal with the term B in a similar way. If $|y| < \frac{|x|}{2}$, then $|x-y| \geq \frac{|x|}{2}$, so we use (5.4.16) for π_a, (5.4.15) and (5.4.17) for π_b and (5.4.14) for π_c,

$$\begin{aligned} B &\leq C_7^2\,\mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\sum_{y\in\mathbb{Z}^d}\pi_a(y)\left(1+\frac{1}{2}\mathrm{dist}(u_i,u_j)\right)^{-\frac{d}{2}}\left(1+\frac{|x|}{2}\right)^{-d} 1_{F_\varepsilon(n)}\right] \\ &= C_7^2\,\mathbf{E}_{\mu,\theta}\left[\sum_{i,j=0}^{n}\left(1+\frac{|x|}{2}\right)^{-d}\left(1+\frac{1}{2}\mathrm{dist}(u_i,u_j)\right)^{-\frac{d}{2}} 1_{F_\varepsilon(n)}\right] \\ &\leq C_7^2\,C_8\left(1+\frac{|x|}{2}\right)^{-d} n^{\max(\frac{8-d}{4},1)+3\varepsilon}. \end{aligned}$$

Then for any $x \in \mathbb{Z}^d$, we have

$$\mathbf{E}_{\mu,\theta}\big[(L_n^x)^2\,1_{F_\varepsilon(n)}\big] = A + B \leq 2C_7^2\,C_8\left(1+\frac{|x|}{2}\right)^{-d} n^{\max(\frac{8-d}{4},1)+3\varepsilon}.$$

Taking the sum over $|x| \leq n$ gives (5.4.13). This completes the proof of Lemma 5.4.2. □

From this lemma we deduce an almost-sure lower bound for $\#R[0,n]$:

Proposition 5.4.3. *For $d \geq 3$, let μ, θ be probability distributions with the conditions in (5.1.1), then $\mathbb{P}_{\mu,\theta}$-almost surely for all large n,*

$$\#R[0,n] \geq n^{\min(\frac{d}{4},1)+o_{\mathrm{as}}(1)}.$$

Proof. By definition,

$$\sum_{x\in R[0,n]} L_n^x = n+1,$$

then by Cauchy-Schwarz' inequality,

$$\#R[0,n] \geq \frac{(n+1)^2}{\sum_{x\in\mathbb{Z}^d}(L_n^x)^2}.$$

We conclude by Lemma 5.4.2. □

Lemma 5.4.4. *For $d = 3, 4, 5$, let μ, θ, η be probability distributions with the conditions in (5.1.1), then $\mathbb{P}_{\mu,\theta}$-almost surely for all large n,*

$$\sum_{x,y\in R[0,n]} G_\eta(x,y) \leq \begin{cases} n^{\frac{5}{4}+o_{\mathrm{as}}(1)}, & d=3 \\ n^{\frac{10-d}{4}+o_{\mathrm{as}}(1)}, & d=4,5 \end{cases}.$$

Remark 5.4.5. One would expect that the sum of Green's functions is monotone decreasing in d with a unified asymptotic formula. However, in dimension $d = 3$, $R[0, n]$ contains considerably less points than that in $d \in \{4, 5\}$. Therefore, we have different results and proofs for the case $d = 3$ and the case $d \in \{4, 5\}$. □

Proof. Let $\varepsilon \in (0, \frac{1}{12})$ be small.

For $d = 3$, by Corollary 5.2.5, $\mathbb{P}_{\mu,\theta}$-almost surely for all large n,

$$\sum_{x,y \in R[0,n]} G_\eta(x, y) \le \sum_{|x|,|y| \le n^{\frac{1}{4}+\varepsilon}} G_\eta(x, y) \le C_9\, n^{\frac{3}{4}+3\varepsilon},$$

where the last inequality follows from the asymptotic behaviors of G_η given in Lemma 5.3.1. This proved the case $d = 3$.

For $d \in \{4, 5\}$, recall the event $F_\varepsilon(n)$ defined in (5.2.8). Since

$$\sum_{x,y \in R[0,n]} G_\eta(x, y) \le \sum_{i,j=0}^{n} G_\eta(V_{u_i}, V_{u_j}),$$

we have

$$\mathbb{E}_{\mu,\theta}\Bigg[\sum_{x,y \in R[0,n]} G_\eta(x, y)\, 1_{F_\varepsilon(n)}\Bigg] \le \sum_{i,j=0}^{n} \mathbb{E}_{\mu,\theta}\big[G_\eta(V_{u_i}, V_{u_j})\, 1_{F_\varepsilon(n)}\big].$$

Using Lemma 5.3.1 and the fact that $V_{u_i} - V_{u_j}$ is distributed as $S_{\mathrm{dist}(u_i,u_j)}$ with S a θ-random walk independent of $\mathrm{dist}(u_i, u_j)$, we deduce from the local limit theorem for S that

$$\begin{aligned}
\mathbb{E}_{\mu,\theta}\Bigg[\sum_{x,y \in R[0,n]} G_\eta(x, y)\, 1_{F_\varepsilon(n)}\Bigg] &\le C_{10} \sum_{i,j=0}^{n} \mathbb{E}_{\mu,\theta}\Bigg[\frac{1}{(1 + |V_{u_i} - V_{u_j}|)^{d-2}}\, 1_{F_\varepsilon(n)}\Bigg] \\
&\le C_{11} \sum_{i,j=0}^{n} \mathbb{E}_{\mu,\theta}\Bigg[\frac{1}{(1 + \mathrm{dist}(u_i, u_j))^{\frac{d-2}{2}}}\, 1_{F_\varepsilon(n)}\Bigg] \\
&= C_{11} \sum_{k=0}^{\infty} \frac{\mathbb{E}_{\mu,\theta}\big[\#\{0 \le i, j \le n : \mathrm{dist}(u_i, u_j) = k\}\, 1_{F_\varepsilon(n)}\big]}{(1 + k)^{\frac{d-2}{2}}}.
\end{aligned}$$

The above sum over k is less than

$$\sum_{0 \le k \le \sqrt{n}} \frac{\mathbb{E}_{\mu,\theta}\big[\#\{0 \le i, j \le n : \mathrm{dist}(u_i, u_j) = k\}\, 1_{F_\varepsilon(n)}\big]}{(1 + k)^{\frac{d-2}{2}}} + \frac{n^2}{(1 + \sqrt{n})^{\frac{d-2}{2}}},$$

which by Lemma 5.2.6 is further bounded by

$$\sum_{0\le k\le\sqrt{n}} \left[(k+1)^{3-\frac{d}{2}} n^{\frac{1}{2}+\varepsilon} + C_4(k+1)^{2-\frac{d}{2}} n^{1+2\varepsilon}\right] + n^{\frac{10-d}{4}} \le C_{12} n^{\frac{10-d}{4}+2\varepsilon}.$$

Then we have shown that for all $n \ge 1$,

$$\mathbb{E}_{\mu,\theta}\Big(\sum_{x,y\in R[0,n]} G_\eta(x,y)\, 1_{F_\varepsilon(n)}\Big) \le C_{11}\, C_{12}\, n^{\frac{10-d}{4}+2\varepsilon}.$$

Similarly to the proof of Lemma 5.4.2, we use the Borel-Cantelli lemma and the fact that $F_\varepsilon(n)$ holds eventually for all large n (Lemma 5.2.4), to get that a.s. for all large n, $\sum_{x,y\in R[0,n]} G_\eta(x,y) \le n^{\frac{10-d}{4}+3\varepsilon}$. Since ε can be arbitrarily small, we get the Lemma for the case $d\in\{4,5\}$. $\square$

Proof of the lower bound in Proposition 5.2.1: Let $d\in\{3,4,5\}$. Let $\varepsilon \in (0,\frac{1}{12})$ be small. By Proposition 5.4.3 and Lemma 5.4.4, we see that $\mathbb{P}_{\mu,\theta}$-almost surely for all large n, $\#R[0,n] \ge n^{\min(\frac{d}{4},1)-\varepsilon}$, and

$$\sum_{x,y\in R[0,n]} G_\eta(x,y) \le \begin{cases} n^{\frac{5}{4}+\varepsilon}, & d=3 \\ n^{\frac{10-d}{4}+\varepsilon}, & d=4,5 \end{cases}.$$

Applying Lemma 5.4.1 to $A = R[0,n]$ with $k = \lfloor 2n^{\frac{1}{2}+2\varepsilon}\rfloor$ if $d=3$ and $k=\lfloor 2n^{\frac{6-d}{4}+2\varepsilon}\rfloor$ if $d\in\{4,5\}$, we get that $\mathbb{P}_{\mu,\theta}$-almost surely for all large n, $\mathtt{cap}_\eta R[0,n] \ge \frac{1}{5}\, n^{\frac{d-2}{4}-3\varepsilon}$. Since ε can be arbitrarily small, this gives the lower bound in Proposition 5.2.1. $\square$

Chapter 6

Small trees

This chapter is based on [20].

6.1 Introduction

Consider a Galton-Watson tree T with offspring distribution μ and regularity conditions

$$\begin{gathered} \mu(0), \mu(1) > 0,\ \mu(0) + \mu(1) < 1, \\ m := \sum_{k=1}^{\infty} k\mu(k) \in (0, \infty),\ \sum_{k=1}^{\infty} k^2 \mu(k) < \infty. \end{gathered} \tag{6.1.1}$$

We denote by Z_n the population of generation n, and by $\mathtt{cut}_n(\mathtt{pru}_n(T))$ the

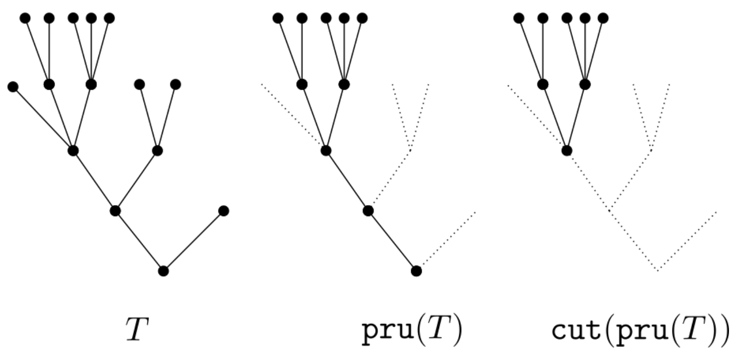


Figure 6.1: Essential structure $\mathtt{cut}_n(\mathtt{pru}_n(T))$ for $n = 4, k = 5$.

reduced tree formed by the family tree of nodes in generation n up to their youngest common ancestor, as illustrated in Figure 6.1. Then our first result is that

Theorem 6.1.1. *Fix $k \geq 1$. Under* (6.1.1)*, one can construct an explicit probability measure $\mathbf{P}_k^{st}$ (see Proposition 6.3.8) such that, as $n \to \infty$, for any set B of finite trees,*

$$P_\mu(\mathtt{cut}_n(\mathtt{pru}_n(T)) \in B \mid Z_n = k) \to \mathbf{P}_k^{st}(B).$$

Moreover, consider the branching random walk $(V_u)_{u \in T}$ indexed by a Galton-Watson tree:

$$V_u = \sum_{\varnothing < v \leq u} V_v,\ V_\varnothing = 0,$$

where $V_v \overset{\text{i.i.d.}}{\sim} \theta$ for all $v \in T \backslash \{\varnothing\}$, and θ is a distribution on $\mathbb{R}$ with regularity conditions

$$\mathbb{E}[X] = 0,\ \mathtt{Var}(X) = 1,\ \mathbb{E}[\exp(tX)] < \infty, \forall t \in \mathbb{R}, \text{ where } X \sim \theta. \tag{6.1.2}$$

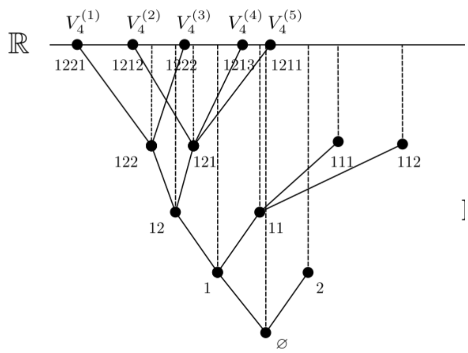


Figure 6.2: Spatial positions $(V_n^{(i)})$ illustrated with $n = 4, k = 5$,

We list the positions of nodes in generation n in increasing order,

$$V_n^{(1)} \leq \cdots \leq V_n^{(Z_n)},$$

as showed in Figure 6.2, then we study its **span**

$$R_n := V_n^{(Z_n)} - V_n^{(1)}$$

and its successive **gaps**

$$g_n^i := V_n^{(i+1)} - V_n^{(i)},\ 1 \leq i \leq Z_n - 1.$$

For simplicity, we write R and (g^i) for the span and gaps of the last generation for a finite tree. Recall that $m = \sum_{k=1}^{\infty} k\mu(k)$ is the expected number of children, and our second result is then

Theorem 6.1.2. *Let $k \geq 2$ and $1 \leq i \leq k-1$. If (6.1.1) and (6.1.2) are satisfied, then the law of R_n and (g_n^i) under $P_\mu(\cdot \mid Z_n = k)$ converges in distribution to that of R and (g^i) under $\mathbf{P}_k^{st}(\cdot)$, and there are explicit constants C_1, C_2, C_3 (see Lemma 6.4.1, Proposition 6.4.3) such that, as $x \to \infty$,*

$$\mathbf{P}_k^{st}(R > x) = \begin{cases} (C_1 + o(1))x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \neq 1, \end{cases}$$

$$\mathbf{P}_k^{st}\left(g^i > x\right) = \begin{cases} (C_1C_3 + o(1))x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \neq 1. \end{cases}$$

Remark 6.1.3. We use the unified assumptions (6.1.1) and (6.1.2) for convenience. These assumptions can be further refined for specific cases:

- In the critical case $m = 1$, θ only need to have finite $(2+\delta)$-th moment (for any $\delta > 0$) instead of exponential moments.
- In the supercritical case $m > 1$, we only need the $L \log L$ condition, $\sum_{i=1}^{\infty} i \log i \mu(i) < \infty$, instead of the finite variance condition in (6.1.1).
- In the subcritical case $m < 1$, we only need the $L \log L$ condition for Section 6.3 except in Part 2, Corollary 6.3.10.

This chapter is mainly motivated by [85] and [86], where Ramola, Majumdar and Schehr studied the span and gaps for the branching Brownian motion via a PDE method. Their result corresponds to the continuous version of Theorem 6.1.2 with geometric μ and Gaussian θ. In particular, we show that the asymptotic for gap statistics are no longer independent of k and i for the critical case with non-geometric offspring distribution, see Remark 6.4.4.

The study of the reduced Galton-Watson tree $\mathtt{pru}_n(T)$ at least dates back to Fleischmann and Prehn [49], Fleischmann and Siegmund-Schultze [50], where it is showed that the limit for the critical case $m = 1$ is the Yule tree. See also Curien and Le Gall [34] for properties and applications of the Yule tree. In particular, for the critical Galton Watson tree conditioned on non-extinction, one has $Z_n = \Theta(n)$, and the conditioned case $\{0 < Z_n \leq \phi(n)\}$, where $\phi(n) = O(n), \phi(n) \to \infty$ is recently studied in Liu and Vatutin [76].

The conditioned limiting behavior of the whole tree T (in contrast to the reduced tree) is known as the local limit. The general result (under the condition $\{Z_n > 0\}$) for the local limit is the Kesten's tree [61], see also Geiger [53]. Further, there are detailed discussions for local limits conditioned on rare events ($\{Z_n < \epsilon\mathbb{E}[Z_n \,|\, Z_n > 0]\}$ or $\{Z_n > \epsilon^{-1}\mathbb{E}[Z_n \,|\, Z_n > 0]\}$), see Abraham and Delmas [3], [4], Abraham, Bouaziz and Delmas [2].

In addition, we also establish an extension of the Ratio theorem (cf. Theorem 1.7.4, [17]) as an byproduct. Indeed, let

$$P_n(1,j) := P_\mu(Z_n = j \,|\, Z_0 = 1),$$

denote the transition probabilities for the population of n generations of the Galton-Watson tree, then we have that

Proposition 6.1.4. *Fix $k \geq 2$. Under* (6.1.1)*, we can construct constants γ (see Part 2, Proposition 6.2.1) and C_4, C_5 (see Remark 6.3.11) such that as $n \to \infty$,*

$$\frac{P_n(1,k)}{P_n(1,1)} - \frac{P_{n-1}(1,k)}{P_{n-1}(1,1)} = \begin{cases} (C_4 + o(1))n^{-2}, & m = 1, \\ (C_5 + o(1))\gamma^n, & m \neq 1, \end{cases}$$

The chapter is organized as follows. In Section 6.2 we present systematically the notations and concepts needed. In Section 6.3 we study the genealogical properties of $\mathtt{pru}_n(T)$ and $\mathtt{cut}_n(\mathtt{pru}_n(T))$, then we prove Theorem 6.1.1 and Proposition 6.1.4 in Proposition 6.3.8 and Remark 6.3.11. In Section 6.4, we study the span and gaps, and prove Theorem 6.1.2 in Proposition 6.4.2 and Proposition 6.4.3.

6.2 Preliminaries

6.2.1 Trees

A (locally finite, rooted) **planar tree** T is a subset of integer-valued words, $T \subset \cup_{n \geq 0} \mathbb{N}_+^n$, such that:

- The **root** $\varnothing \in T$, where by convention we denote $\mathbb{N}_+^0 = \{\varnothing\}$.
- If a **node** $u = (u_1, \cdots, u_n) \in T$, then its **parent** $\overleftarrow{u} := (u_1, \cdots, u_{n-1}) \in T$.
- For each node $u = (u_1, \cdots, u_n) \in T$, there exists an integer $k_u(T) \in \mathbb{N}$ called its **number of children**, such that for every $j \in \mathbb{N}, (u_1, \cdots, u_n, j) \in T$ if and only if $1 \leq j \leq k_u(T)$.

We only consider locally finite trees, i.e. $k_u(T) < \infty$, $\forall u \in T$. We give a few basic notations on trees:

- The **set of all planar trees** is denoted by $\mathcal{T}$.
- The **generation/height of a node** is its length as a word, i.e. if $u = (u_1, \cdots, u_n)$, then $|u| = n$. The **height of a tree** is then defined as $H(T) := \max\{|u| : u \in T\} \in \mathbb{N} \cup \{\infty\}$.
- The **population of generation** n is defined as $Z_n(T) := \#\{u \in T : |u| = n\}$. By construction, $Z_0(T) = 1$ for any tree T.
- A node $u = (u_1, \cdots, u_n) \in T$ is an **ancestor** of another one $v = (v_1, \cdots, v_m) \in T$, denoted by $u \prec v$, if $n < m$ and $u_i = v_i$, $1 \le i \le n$. The **(youngest) common ancestor** of two nodes $u, v \in T$, denoted by $u \wedge v$, is then the node in T with maximum height, such that $u \wedge v \preceq u, v$.
- For $u \in T$, the **subtree** rooted at $u \in u$ is defined as $T[u] = \{v \in \mathbb{N}_+^n : uv \in T\}$, where uv stands for concatenation of words. It is not hard to check that this set is a tree. In particular, given that $k_\varnothing(T) = r$, nodes in the first generations are labeled $1, 2, \cdots, r$ by construction, thus subtrees rooted at them are $T[1], \cdots, T[r]$.

When there is no confusion for the tree under consideration, we omit the reliance on T and write, for instance, Z_n for $Z_n(T)$.

Moreover, given a tree T, one can attach the (not necessarily random) spatial structure $(V_u)_{u \in T}$: We attach on each node $v \neq \varnothing$ a displacement from its parent $V_v \in \mathbb{R}$, and set the position of a node u as

$$V_u = \sum_{\varnothing \prec v \preceq u} V_v,\ V_\varnothing = 0.$$

6.2.2 The prune and cut operation

To study the relative relations of nodes in generation n while omitting irrelevant information, we define the prune and cut operations on trees (recall the illustration in Figure 6.1):

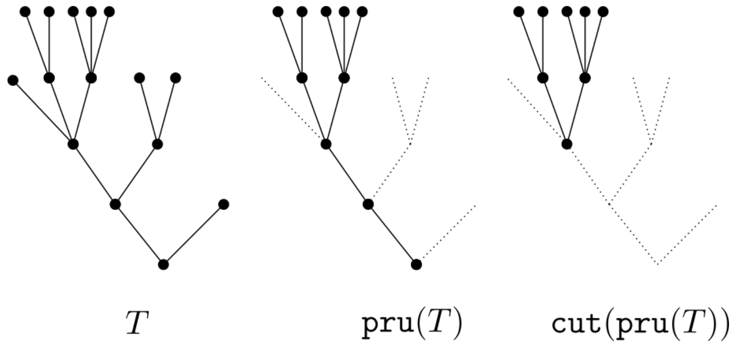
- For any $T \in \mathcal{T}$, we construct the **pruned tree** at height n by

$$\mathtt{pru}_n(T) := \{u \in T : \exists v \in T, |v| = n, u \preceq v\}.$$

  By convention, if $Z_n(T) = 0$, we take $\mathtt{pru}_n(T) = \{\varnothing\}$.
- Moreover, we define the **cut operation** by

$$\phi_n(T) = \bigwedge_{|u|=n,\, u \in T} u, \text{ and then } \mathtt{cut}_n(T) = T[\phi_n(T)].$$

  where by convention, $\phi_n(T) = \varnothing$ if $Z_n(T) = 0$.



In addition, we denote the **set of all pruned trees** at height n by $\mathcal{T}_n := \mathtt{pru}_n(\mathcal{T})$, and the set of all pruned trees at height n with $Z_n = k$ by $\mathcal{T}_{n,k}$. In particular, $\mathcal{T}_{n,0}$ contains only one element $\{\varnothing\}$. Since trees are assumed to be locally finite, we have that $\mathcal{T}_n = \cup_{k=0}^{\infty}\mathcal{T}_{n,k}$. Since all these sets are countable, the problem of measurability is trivial.

These operations naturally extend to the branching random walk indexed by these trees, by translation such that the root is always pinned at 0. Then by construction,

$$R_n(T) = R_{H(\mathtt{cut}_n(T))}(\mathtt{cut}_n(T)) = R_n(\mathtt{pru}_n(T)), \tag{6.2.3}$$

and the same thing applies to the gaps.

6.2.3 Galton-Watson tree and Ratio theorem

Let μ be a probability distribution on $\mathbb{N}$. The law of a **Galton-Watson** tree with offspring distribution μ is a probability measure P_μ on the set of planar trees $\mathcal{T}$, such that for all nodes u,

$$k_u \overset{i.i.d.}{\sim} \mu.$$

Clearly, the sequence (Z_n) is a Markov chain starting at $Z_0 = 1$ under P_μ, and one can then set its transition probabilities as

$$P_n(i,j) := P_\mu(Z_{k+n} = j \mid Z_k = i),$$

where we take k such that $P_\mu(Z_k = i) > 0$. (Under the assumption (6.1.1), this is always possible by taking k large enough.) In particular,

$$P_1(1,i) = \mu(i).$$

Moreover, we define the **generating function** of this process as

$$f(s) := E_\mu\big(s^{Z_1(T)}\big) = \sum_{i=0}^{\infty} P_1(1,i)s^i, \tag{6.2.4}$$

then its derivatives are

$$f^{(r)}(s) = r! \sum_{\ell \geq r} \binom{l}{r} P_1(1, i) s^{\ell - r}, \tag{6.2.5}$$

and its iterations are

$$f_n(s) := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}} = E_\mu\big(s^{Z_n(T)}\big) = \sum_{\ell \geq 0} P_n(1, \ell) s^\ell. \tag{6.2.6}$$

We also define the **extinction probabilities** as

$$q_n := P_n(1, 0) = f_n(0), \; q := \lim_{n \to \infty} q_n. \tag{6.2.7}$$

Clearly, (q_n) is a bounded increasing sequence, which guarantees the existence of q. Moreover, it is standard (Athreya and Ney [17, Theorem 1.5.1]) that $q = 1$ if $m \leq 1$ (except for the trivial case $\mu = \delta_1$), and $q < 1$ if $m > 1$, where $m := \sum_{i=1}^\infty i\mu(i)$ is the expected number of children.

We finish this section by citing some fundamental estimates that shall be used later,

Proposition 6.2.1 (Athreya and Ney [17, Section 1.7-1.11])**.** *Let μ be an offspring distribution such that*

$$\mu(0), \mu(1) > 0, \; \mu(0) + \mu(1) < 1, \; m < \infty.$$

1. *There exists a sequence (π_j) such that for any $j \geq 1$,*

$$\lim_{n \to \infty} \frac{P_n(1, j)}{P_n(1, 1)} \nearrow \pi_j \in (0, \infty),$$

where $\nearrow$ means non-decreasing limit.

2. *For any $t \in \mathbb{Z}$, $i, j, k, l \geq 1$,*

$$\lim_{n \to \infty} \frac{P_{n+t}(i, j)}{P_n(k, l)} = \gamma^t q^{i-k} \frac{i\pi_j}{k\pi_l},$$

where q is the extinction probability in (6.2.7), and $\gamma = f'(q)$.

3. *If $m = 1, \sigma^2 := \sum_{i=1}^\infty i^2 \mu(i) < \infty$, then for any $i, j \geq 1$,*

$$\lim_{n \to \infty} n^2 P_n(i, j) = \frac{2i\pi_j}{\sigma^2 \sum_{k=1}^\infty \pi_k (\mu(0))^k}.$$

4. *If $m \neq 1$, $\sum_{i=1}^{\infty} i \log i\mu(i) < \infty$, then for any $i, j \geq 1$,*

$$\lim_{n\to\infty} \gamma^{-n} P_n(i,j) = iq^{i-1} v_j,$$

where (v_j) is determined by $Q(s) = \sum_{j=0}^{\infty} v_j s^j, 0 \leq s < 1$, with Q the unique solution of

$$Q(f(s)) = \gamma Q(s)(0 \leq s < 1),\ Q(q) = 0,\ \lim_{s\to q} Q'(s) = 1.$$

6.2.4 Branching random walk and Cramer's theorem

On a Galton-Watson tree T, we shall consider the **branching random walk** $(V_u)_{u\in T}$ by attaching i.i.d. spatial displacements

$$V_u \overset{i.i.d.}{\sim} \theta,\ \forall u \in T\backslash\{\varnothing\},$$

whose probability measure is still denoted by P_μ for simplicity. For other probability measures on trees that we shall construct, we also abuse the same notations for the corresponding spatial process.

Moreover, we cite a fundamental estimate for random walks,

Lemma 6.2.2. *Let θ be a distribution on $\mathbb{R}$, and let (X_i) be i.i.d. random variables distributed as θ. Assume that $\mathbb{E}[X_1] = 0$, $\mathtt{Var}(X_1) = 1$.*

1. *[84, Theorem 5.7] If $\mathbb{E}\left[|X_1|^{2+\delta}\right] < \infty$, then there exists $C_1, C_2, C_3 > 0$ such that for any $n \geq 1, x > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > x\right) \leq \frac{C_1 n}{x^3} + e^{-C_2 \frac{x^2}{n}},$$

$$\left|\mathbb{P}\left(\sum_{i=1}^{n} X_i \leq x\right) - \Phi\left(\frac{x}{\sqrt{n}}\right)\right| \leq \frac{C_3}{\sqrt{n}},$$

where $\Phi(x)$ is the cumulative distribution function of the standard Gaussian distribution $\mathcal{N}(0,1)$.

2. *(Cramer's theorem) If*

$$\Lambda(t) := \log \mathbb{E}[\exp(tX_1)] < \infty,\ \forall t \in \mathbb{R},$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq nx\right)\right) = -\sup_{t\in\mathbb{R}}(tx - \Lambda(t)).$$

6.3 Galton-Watson trees conditioned on rarely survival

6.3.1 Pruned Galton-Watson trees

We first study the distribution of $\mathtt{pru}_n(T)$. Recall that $\mathcal{T}_{n,k}$ stands for the set of all pruned trees with k nodes in generation n.

Definition 6.3.1. *For any pair of integers (n, k) such that $P_n(1, k) > 0$, we denote by $\mathbf{P}^{pru}_{n,k}$ the law of $\mathtt{pru}_n(T)$, supported on $\mathcal{T}_{n,k}$, where T is sampled under the law of $P_\mu(\cdot \mid Z_n = k)$. In other words, for any $A \subset \mathcal{T}_{n,k}$,*

$$\mathbf{P}^{pru}_{n,k}(A) = P_\mu(\mathtt{pru}_n(T) \in A \mid Z_n = k) = \frac{P_\mu(\mathtt{pru}_n(T) \in A)}{P_n(1, k)}. \tag{6.3.8}$$

Recall that $T[i]$ denotes the i-th subtree of T rooted at the first generation, then

Proposition 6.3.2. *Under* (6.1.1)*, for any pair of integers (n, k) such that $P_n(1, k) > 0$, let $r \geq 1$ and let $k_1, \cdots, k_r$ be positive integers such that $\sum_{i=1}^r k_i = k$. Then for any $A_i \subseteq \mathcal{T}_{n-1,k_i}$, $1 \leq i \leq r$,*

$$\begin{aligned}&\mathbf{P}^{pru}_{n,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)\\ =&\frac{1}{P_n(1,k)} \frac{f^{(r)}(q_{n-1})}{r!} \prod_{i=1}^r P_{n-1}(1, k_i) \prod_{i=1}^r \mathbf{P}^{pru}_{n-1,k_i}(A_i),\end{aligned}$$

where f and q_{n-1} are defined in (6.2.4), (6.2.7).

Proof. By (6.3.8),

$$\begin{aligned}&P_n(1,k)\mathbf{P}^{pru}_{n,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)\\ =&P_\mu(Z_1(\mathtt{pru}_n(T)) = r, (\mathtt{pru}_n(T))[i] \in A_i, 1 \leq i \leq r)\\ =&\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_r \leq \ell} P_\mu(Z_1(T) = \ell; \mathtt{pru}_{n-1}(T[j_i]) \in A_i, i = 1, \cdots, r;\\ &\qquad\qquad Z_{n-1}(T[i]) = 0, i \notin \{j_1, \cdots, j_r\}).\end{aligned}$$

Let $B_i = \{T \in \mathcal{T} : \mathtt{pru}_{n-1}T \in A_i\}$, by the self-similarity of Galton-Watson trees, the equation above can be further simplified as

$$\begin{aligned}&P_n(1,k)\mathbf{P}^{pru}_{n,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)\\ =&\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_r \leq \ell} P_1(1,\ell) q_{n-1}^{\ell-r} \prod_{i=1}^r P_\mu(B_i).\end{aligned} \tag{6.3.9}$$

Moreover, by (6.3.8), we have that

$$P_\mu(B_i) = P_{n-1}(1, k_i)\mathbf{P}^{pru}_{n-1,k_i}(A_i),$$

put it in (6.3.9), and it suffices to show that

$$\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_r \leq \ell} P_1(1, \ell) q_{n-1}^{\ell - r} = \frac{f^{(r)}(q_{n-1})}{r!}.$$

Indeed, we have

$$\begin{aligned}&\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_r \leq \ell} P_1(1, \ell) q_{n-1}^{\ell - r} \\ =&\sum_{\ell \geq r} \binom{\ell}{r} P_1(1, \ell) q_{n-1}^{\ell - r} = \frac{f^{(r)}(q_{n-1})}{r!},\end{aligned}$$

where the second line follows from (6.2.5). $\square$

Moreover, we give two more properties of $\mathbf{P}^{pru}_{n,k}$:

Corollary 6.3.3. *Under* (6.1.1)*, for any* $1 \leq u \leq n$*, any* $A \subseteq \mathcal{T}_{u,k}$*,*

$$\begin{aligned}&\mathbf{P}^{pru}_{n,k}(Z_1(T) = \cdots = Z_{n-u}(T) = 1, T[\underbrace{11\cdots1}_{n-u \text{ times}}] \in A) \\ =&\frac{P_n(1,1)}{P_n(1,k)} \frac{P_u(1,k)}{P_u(1,1)} \mathbf{P}^{pru}_{u,k}(A),\end{aligned}$$

where $\underbrace{11\cdots1}_{n-u \text{ times}}$ *means the first node (and also the only node, under the condition* $Z_1(T) = \cdots = Z_{n-u}(T) = 1$*) in generation* $n - i$*.*

Proof. Take $r = 1$ in Proposition 6.3.2, we have that

$$\begin{aligned}\mathbf{P}^{pru}_{n,k}(Z_1(T) = 1, T[1] \in B) &= \frac{P_{n-1}(1,k)}{P_n(1,k)} f'(q_{n-1}) \mathbf{P}^{pru}_{n-1,k}(B) \\ &= \frac{P_{n-1}(1,k)}{P_n(1,k)} \frac{P_n(1,1)}{P_{n-1}(1,1)} \mathbf{P}^{pru}_{n-1,k}(B),\end{aligned}$$

for any $B \subseteq \mathcal{T}_{n-1,k}$, where we use (6.2.6) and (6.2.7) to deduce that $f'(q_{n-1}) = \frac{f_n'(0)}{f_{n-1}'(0)} = \frac{P_n(1,1)}{P_{n-1}(1,1)}$. The result follows by using this relation $n - u$ times inductively. $\square$

Corollary 6.3.4. *Under* (6.1.1), *for any* $r, k \geq 2$, *if* $f^{(r)}(q) < \infty$, *then*

$$\lim_{n\to\infty} \frac{\mathbf{P}^{pru}_{n,k}(Z_1(T) = r)}{P_n(1,1)^{r-1}} = \gamma^{-r} \frac{f^{(r)}(q)}{r!} \sum_{\substack{k_1,k_2,\cdots,k_r \geq 1 \\ k_1+\cdots+k_r = k}} \frac{\pi_{k_1} \cdots \pi_{k_r}}{\pi_k},$$

$$\lim_{n\to\infty} \frac{\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r)}{P_n(1,1)^{r-1}} = \gamma^{-r} \frac{f^{(r)}(q)}{r!} \sum_{\substack{k_1,k_2,\cdots,k_r \geq 1 \\ k_1+\cdots+k_r = k}} \frac{\pi_{k_1} \cdots \pi_{k_r}}{\pi_k},$$

where f *is defined in* (6.2.4), q, γ *and* (π_k) *are defined in Proposition 6.2.1.*

Proof. For the first equation, by Proposition 6.3.2, we take $A_i = \mathcal{T}_{n-1,k_i}$ and sum over all choices of (k_i), then

$$\mathbf{P}^{pru}_{n,k}(Z_1(T) = r) = \frac{1}{P_n(1,k)} \frac{f^{(r)}(q_{n-1})}{r!} \sum_{\substack{k_1,k_2,\cdots,k_r \geq 1 \\ k_1+\cdots+k_r = k}} \prod_{i=1}^{r} P_{n-1}(1,k_i).$$

Divide both sides by $P_n(1,1)^{r-1}$, we have that

$$\frac{\mathbf{P}^{pru}_{n,k}(Z_1(T) = r)}{P_n(1,1)^{r-1}} = \frac{P_n(1,1)}{P_n(1,k)} \frac{f^{(r)}(q_{n-1})}{r!} \sum_{\substack{k_1,k_2,\cdots,k_r \geq 1 \\ k_1+\cdots+k_r = k}} \prod_{i=1}^{r} \frac{P_{n-1}(1,k_i)}{P_n(1,1)}.$$

For fixed r and k, there are only finitely many terms on the right hand side, thus we can take the limit separately for each fraction by Proposition 6.2.1, and the result follows.

To deal with $\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r)$, we sum over

$$\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1)$$

for all $\sum_{i=1}^{r} k_i = k$. In other words, we consider the first $r-1$ subtrees to give k_i offspring each, while the rest subtrees give k_r offspring in total. By the proof of Proposition 6.3.2, we have that

$$\begin{aligned}
&P_n(1,k)\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1) \\
=&\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_{r-1} < \ell} P_\mu\Bigg(Z_1(T) = \ell; Z_{n-1}(T[j_i]) = k_i, i = 1, \cdots, r-1; \\
&\qquad \sum_{\substack{i < j_{r-1} \\ i \notin \{j_1,\cdots,j_{r-1}\}}} Z_{n-1}(T[i]) = 0; \sum_{i > j_{r-1}} Z_{n-1}(T[i]) = k_r \Bigg) \\
=&\sum_{\ell \geq r} \sum_{1 \leq j_1 < \cdots < j_{r-1} < \ell} P_1(1,\ell) \prod_{i=1}^{r-1} P_{n-1}(1,k_i) \cdot q_{n-1}^{j_{r-1}-r+1} P_{n-1}(\ell - j_{r-1}, k_r).
\end{aligned}$$

Write $j_{r-1} = j$ for short, and we can simplify this term into

$$\begin{aligned}&P_n(1,k)\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1)\\ =&P_1(1,\ell)\prod_{i=1}^{r-1} P_{n-1}(1,k_i)\sum_{\ell=r}^{\infty}\sum_{j=r-1}^{\ell-1}\binom{j-1}{r-2}q_{n-1}^{j-r+1}P_{n-1}(\ell-j,k_r).\end{aligned}$$

Divide by $(P_n(1,1))^r$, then

$$\begin{aligned}&\frac{P_n(1,k)}{(P_n(1,1))^r}\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1)\\ =&P_1(1,\ell)\prod_{i=1}^{r-1}\frac{P_{n-1}(1,k_i)}{P_n(1,1)}\sum_{\ell=r}^{\infty}\sum_{j=r-1}^{\ell-1}\binom{j-1}{r-2}q_{n-1}^{j-r+1}\frac{P_{n-1}(\ell-j,k_r)}{P_n(1,1)}.\end{aligned} \tag{6.3.10}$$

By Proposition 6.2.1 and dominated convergence, this term converges to

$$\begin{aligned}&P_1(1,\ell)\gamma^{-r}\prod_{i=1}^{r-1}\pi_{k_i}\sum_{\ell=r}^{\infty}\sum_{j=r-1}^{\ell-1}\binom{j-1}{r-2}q^{j-r+1+\ell-j-1}(\ell-j)\pi_{k_r}\\ =&P_1(1,\ell)\gamma^{-r}\prod_{i=1}^{r}\pi_{k_i}\sum_{\ell=r}^{\infty}\binom{\ell}{r}q^{\ell-r}\\ =&\gamma^{-r}\frac{f^{(r)}(q)}{r!}\prod_{i=1}^{r}\pi_{k_i},\end{aligned}$$

by the elementary identities

$$\sum_{i=r-1}^{\ell-1}(\ell-i)\binom{i-1}{r-2} = \binom{\ell}{r},\ f^{(r)}(s) = \sum_{\ell\geq r} r!\binom{\ell}{r}P_1(1,\ell)s^{\ell-r}.$$

Moreover, by Proposition 6.2.1 again, we have that

$$\frac{P_n(1,k)}{(P_n(1,1))^r} = (1+o(1))\frac{\pi_k}{(P_n(1,1))^{r-1}},$$

thus (6.3.10) gives

$$\begin{aligned}&\lim_{n\to\infty}\frac{\pi_k}{(P_n(1,1))^{r-1}}\mathbf{P}^{pru}_{n,k}(Z_1(T) \geq r, Z_{n-1}(T[i]) = k_i, 1 \leq i \leq r-1)\\ =&\gamma^{-r}\frac{f^{(r)}(q)}{r!}\pi_{k_1}\cdots\pi_{k_r},\end{aligned}$$

and the result follows by summing over all choices of (k_i). □

Remark 6.3.5. The condition $f^{(r)}(s) < \infty$ is always true if $s < 1$, while $q < 1$ if and only if $m \leq 1$. Thus the condition $f^{(r)}(q) < \infty$ in Corollary 6.3.4 is trivially satisfied if $m > 1$. If $m \leq 1$, one can verify that this condition is equivalent to the moment condition $\sum_{i=1}^{\infty} i^r\mu(i) < \infty$.

6.3.2 The small tree measure

Now we study the composition of the cut and prune operation.

Definition 6.3.6. *For any pair of integers (n,k) such that $P_n(1,k) > 0$ and $k \geq 2$, we denote by $\mathbf{P}^{st}_{n,k}$ the law of $\mathtt{cut}_n(T)$, where T is sampled under the law of $\mathbf{P}^{pru}_{n,k}$. In other words, for any $A \subseteq \mathcal{T}$,*

$$\mathbf{P}^{st}_{n,k}(A) = \mathbf{P}^{pru}_{n,k}(\mathtt{cut}_n(T) \in A).$$

We remark that k is assumed to be at least 2, since for $k = 1$, the cut operation will always give the trivial tree $\{\varnothing\}$. This measure $\mathbf{P}^{st}_{n,k}$ is developed in the following lemma:

Lemma 6.3.7. *Under* (6.1.1)*, for any pair of integers (n,k) such that $P_n(1,k) > 0$ and $k \geq 2$, $\mathbf{P}^{st}_{n,k}$ is supported on $\cup_{u=1}^{n}\mathcal{T}_{u,k} \cap \{T : Z_1 \geq 2\}$. Moreover, for any $A \subseteq \cup_{u=1}^{n}\mathcal{T}_{u,k} \cap \{T : Z_1 \geq 2\}$,*

$$\mathbf{P}^{st}_{n,k}(A) = \frac{P_n(1,1)}{P_n(1,k)} \sum_{u=1}^{n} \frac{P_u(1,k)}{P_u(1,1)} \mathbf{P}^{pru}_{u,k}(A \cap \mathcal{T}_{u,k}).$$

Proof. For the first assertion, by construction, $\mathtt{cut}_n(T)$ has at least 2 nodes in its first generation. Moreover, the trees on which we perform the cut operation are those in $\mathcal{T}_{n,k}$, so $\mathtt{cut}_n(T)$ has height at most n, with all its leaves in the last generation. Thus it is an element in $\cup_{u=1}^{n}\mathcal{T}_{u,k} \cap \{T : Z_1(T) \geq 2\}$. The converse is trivial.

Then it suffices to prove the second assertion for $A \subseteq \mathcal{T}_{u,k} \cap \{T : Z_1(T) \geq 2\}$. Indeed, by Definition 6.3.6 and Corollary 6.3.3,

$$\begin{aligned}
\mathbf{P}^{st}_{n,k}(A) &= \mathbf{P}^{pru}_{n,k}(\mathtt{cut}_n(T) \in A) \\
&= \mathbf{P}^{pru}_{n,k}(Z_1(T) = \cdots = Z_{n-u}(T) = 1, T[\underbrace{11\cdots1}_{n-u \text{ times}}] \in A) \\
&= \frac{P_n(1,1)}{P_n(1,k)} \frac{P_u(1,k)}{P_u(1,1)} \mathbf{P}^{pru}_{u,k}(A).
\end{aligned}$$

Then the conclusion follows by partitioning a general set $A \subseteq \cup_{u=1}^{n}\mathcal{T}_{u,k} \cap \{T : Z_1 \geq 2\}$ into $\cup_{u=1}^{n}(A \cap \mathcal{T}_{u,k})$, and use the above equation on each part $A \cap \mathcal{T}_{u,k}$. $\square$

In fact, we notice that n no longer plays a major role in Lemma 6.3.7, and we are motivated to take the limit $n \to \infty$. Moreover, this limit measure still has Galton-Watson-type branching properties:

Proposition 6.3.8. *Under (6.1.1), fix $k \geq 2$, let $n \to \infty$, then the measures $(\mathbf{P}^{st}_{n,k})_n$ converge to a measure $\mathbf{P}^{st}_k$ supported on $\cup_{u=1}^{\infty}\mathcal{T}_{u,k} \cap \{T : Z_1(T) \geq 2\}$, defined by*

$$\mathbf{P}^{st}_k(A) = \frac{1}{\pi_k}\sum_{u=1}^{\infty}\frac{P_u(1,k)}{P_u(1,1)}\mathbf{P}^{pru}_{u,k}(A \cap \mathcal{T}_{u,k}), \tag{6.3.11}$$

for any $A \subseteq \cup_{u=1}^{\infty}\mathcal{T}_{u,k} \cap \{T : Z_1(T) \geq 2\}$. Moreover, fix any $u > 0$ such that $P_u(1,k) > 0$, let $r \geq 2$ and $k_1, \cdots, k_r \geq 1$ such that $\sum_{i=1}^{r} k_i = k$. Then for any $A_i \subseteq \mathcal{T}_{u-1,k_i}$,

$$\begin{aligned}&\mathbf{P}^{st}_k(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)\\ =&\frac{\pi_{k_1}\cdots\pi_{k_r}}{\pi_k}\frac{P_{u-1}(1,1)^r}{P_u(1,1)}\frac{f^{(r)}(q_{u-1})}{r!}\prod_{i=1}^{r}\mathbf{P}^{st}_{k_i}(\mathtt{cut}_{u-1}(A_i)).\end{aligned} \tag{6.3.12}$$

Proof. Convergence and (6.3.11) follows directly from Lemma 6.3.7 and Part 1 of Proposition 6.2.1.

Then for (6.3.12), by (6.3.11) and Proposition 6.3.2, we have that

$$\begin{aligned}&\mathbf{P}^{st}_k(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)\\ =&\frac{1}{\pi_k}\frac{P_u(1,k)}{P_u(1,1)}\mathbf{P}^{pru}_{u,k}(Z_1(T) = r, T[i] \in A_i, 1 \leq i \leq r)\\ =&\frac{1}{\pi_k}\frac{1}{P_u(1,1)}\frac{f^{(r)}(q_{u-1})}{r!}\prod_{i=1}^{r}P_{u-1}(1,k_i)\prod_{i=1}^{r}\mathbf{P}^{pru}_{u-1,k_i}(A_i).\end{aligned} \tag{6.3.13}$$

Decompose A_i by the height of trees after the cut operation,

$$A_i^{(x)} := \{T \in A_i : H(\mathtt{cut}_{u-1}(T)) = x\},$$

then by Corollary 6.3.3,

$$\mathbf{P}^{pru}_{u-1,k_i}(A_i) = \frac{P_{u-1}(1,1)}{P_{u-1}(1,k_i)}\sum_{x=1}^{u-1}\frac{P_x(1,k_i)}{P_x(1,1)}\mathbf{P}^{pru}_{x,k_i}(\mathtt{cut}_{u-1}(A_i^{(x)})).$$

Since $\mathtt{cut}_{u-1}$ is injective on A_i, we have

$$\mathtt{cut}_{u-1}(A_i^{(x)}) = \mathtt{cut}_{u-1}(A_i) \cap \mathcal{T}_{x,k_i},$$

thus by (6.3.11) again,

$$\mathbf{P}^{pru}_{u-1,k_i}(A_i) = \frac{P_{u-1}(1,1)}{P_{u-1}(1,k_i)}\cdot\pi_{k_i}\mathbf{P}^{st}_{k_i}(\mathtt{cut}_{u-1}(A_i)).$$

Put this back into (6.3.13), and we get (6.3.12). □

This enables us to give further descriptions of $\mathbf{P}_k^{st}$. Recall that $H(T)$ is the height of a tree, then

Corollary 6.3.9. *Under* (6.1.1)*, for any* $r, k \geq 2$*, if* $f^{(r)}(q) < \infty$*, then*

$$\lim_{u\to\infty} \frac{\mathbf{P}_k^{st}(Z_1(T) = r \mid H(T) = u)}{P_u(1,1)^{r-2}}$$
$$= \lim_{u\to\infty} \frac{\mathbf{P}_k^{st}(Z_1(T) \geq r \mid H(T) = u)}{P_u(1,1)^{r-2}} = \frac{2\gamma^{2-r}}{r!}\frac{f^{(r)}(q)}{f''(q)} \frac{\sum_{\substack{k_1,k_2,\cdots,k_r\geq 1\\ k_1+\cdots+k_r=k}} \pi_{k_1}\cdots\pi_{k_r}}{\sum_{\substack{k_1,k_2\geq 1\\ k_1+k_2=k}} \pi_{k_1}\pi_{k_2}}.$$

Proof. By (6.3.11),

$$\mathbf{P}_k^{st}(Z_1(T) = r \mid H(T) = u) = \frac{\mathbf{P}_{u,k}^{pru}(Z_1(T) = r)}{\mathbf{P}_{u,k}^{pru}(Z_1(T) \geq 2)},$$

and we apply Corollary 6.3.4. Changing $Z_1(T) = r$ to $Z_1(T) \geq r$ is idem. □

Corollary 6.3.10. *Under* (6.1.1)*, fix any* $k \geq 2$*.*

1. *For any* $u \geq 1$*,*

$$\mathbf{P}_k^{st}(H(T) = u) = \frac{1}{\pi_k}\left(\frac{P_u(1,k)}{P_u(1,1)} - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)}\right).$$

2.

$$\lim_{u\to\infty} \frac{\mathbf{P}_k^{st}(H(T) = u)}{P_u(1,1)} = \gamma^{-2}\frac{f''(q)}{2}\frac{\sum_{1\leq i\leq k-1}\pi_i\pi_{k-i}}{\pi_k}.$$

Proof. 1. Take $A = \mathcal{T}_{u,k} \cap \{T : Z_1(T) \geq 2\}$ in Proposition 6.3.8, we have that

$$\begin{aligned}
&\mathbf{P}_k^{st}(H(T) = u)\\
=&\mathbf{P}_k^{st}(\mathcal{T}_{u,k} \cap \{T : Z_1(T) \geq 2\})\\
=&\frac{1}{\pi_k}\frac{P_u(1,k)}{P_u(1,1)}\mathbf{P}_{u,k}^{pru}(Z_1(T) \geq 2)\\
=&\frac{1}{\pi_k}\frac{P_u(1,k)}{P_u(1,1)}\big(1 - \mathbf{P}_{u,k}^{pru}(Z_1(T) = 1)\big),
\end{aligned}$$

then we use Corollary 6.3.3 to conclude that

$$\begin{aligned}
\frac{1}{\pi_k}\frac{P_u(1,k)}{P_u(1,1)}\big(1 - \mathbf{P}_{u,k}^{pru}(Z_1(T) = 1)\big) =&\frac{1}{\pi_k}\frac{P_u(1,k)}{P_u(1,1)}\left(1 - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)}\frac{P_u(1,1)}{P_u(1,k)}\right)\\
=&\frac{1}{\pi_k}\left(\frac{P_u(1,k)}{P_u(1,1)} - \frac{P_{u-1}(1,k)}{P_{u-1}(1,1)}\right).
\end{aligned}$$

2. In the proof of Part 1, we deduced that

$$\mathbf{P}_k^{st}(H(T)=u)=\frac{1}{\pi_k}\frac{P_u(1,k)}{P_u(1,1)}\mathbf{P}_{u,k}^{pru}(Z_1(T)\geq 2),$$

and the conclusion follows from Corollary 6.3.4 with $r=2$.

□

Remark 6.3.11. As a byproduct of Corollary 6.3.10, we have that

$$\frac{1}{\pi_k}\left(\frac{P_u(1,k)}{P_u(1,1)}-\frac{P_{u-1}(1,k)}{P_{u-1}(1,1)}\right)=(1+o(1))\gamma^{-2}\frac{f''(q)}{2}\frac{\sum_{1\leq i\leq k-1}\pi_i\pi_{k-i}}{\pi_k}P_u(1,1).$$

Together with the asymptotic of $P_u(1,1)$ in Proposition 6.2.1, we deduce that

$$\frac{P_u(1,k)}{P_u(1,1)}-\frac{P_{u-1}(1,k)}{P_{u-1}(1,1)}=\begin{cases}(C_4+o(1))u^{-2}, & m=1,\\ (C_5+o(1))\gamma^u, & m\neq 1,\end{cases}$$

where

$$C_4=\gamma^{-2}\frac{f''(q)\sum_{1\leq i\leq k-1}\pi_i\pi_{k-i}}{\sigma^2\sum_{i=1}^{\infty}\pi_i(\mu(0))^i},\ C_5=\frac{1}{2}\gamma^{-2}v_1f''(q)\sum_{1\leq i\leq k-1}\pi_i\pi_{k-i},$$

with σ^2, v_1 defined in Proposition 6.2.1.

6.4 Application to branching random walks

As we shall deal with trees without fixed heights in this section, we abbreviate $R(T)=R_{H(T)}(T)$ and $g^i(T)=g^i_{H(T)}(T)$. By (6.2.3) and Proposition 6.3.8, for the span we have that

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{P}(R_n>x\,|\,Z_n=k)\\ &=\lim_{n\to\infty}\mathbf{P}(R(\mathtt{cut}_n(\mathtt{pru}_nT))>x\,|\,Z_n=k)\\ &=\lim_{n\to\infty}\mathbf{P}_{n,k}^{st}(R(T)>x)=\mathbf{P}_k^{st}(R(T)>x),\end{aligned}\tag{6.4.14}$$

and it is idem for the gaps. In other words, R_n and (g_n^i) under $\mathbf{P}(\cdot\,|\,Z_n=k)$ converge to $R(T)$ and $(g^i(T))$ under $\mathbf{P}_k^{st}(\cdot)$ as $n\to\infty$. Thus to prove Theorem 6.1.2 it suffices to study the span and gaps under $\mathbf{P}_k^{st}$.

6.4.1 The span

Take any tree $T \in \mathcal{T}_{n,k} \cap \{Z_1(T) \geq 2\}$ under $\mathbf{P}_k^{st}$, we divide the span $R_n(T)$ into two parts: the span of the first (in lexicographical order) node in the last layer of each subtree is denoted by

$$S_n(T) := \text{the span of } \{1 \leq i \leq Z_1(T) : V_{\underbrace{i11\dots1}_{\text{length } n}}(T)\},$$

and the maximum span among each subtree is denoted by

$$G_n(T) := \max_{1 \leq i \leq Z_1(T)} \{R_{n-1}(T[i])\}.$$

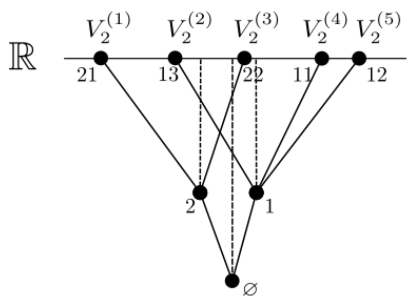


Figure 6.3: A spatial tree.

For instance, the tree in Figure 6.3 has height $n = 2$ and $k = 5$ particles in the last generation, with two subtrees at the first generation. For this tree we have $S_2(T) = V_2^{(4)}(T) - V_2^{(1)}(T)$ and $G_2(T) = \max\{V_2^{(3)}(T) - V_2^{(1)}(T), V_2^{(5)}(T) - V_2^{(2)}(T)\}$.

By the triangle inequality, we have that

$$S_n(T) \leq R_n(T) \leq S_n(T) + 2G_n(T). \tag{6.4.15}$$

For simplicity, since trees under $\mathbf{P}_k^{st}$ do not have a fixed height, we write $S(T)$, $G(T)$, $R(T)$ for $S_{H(T)}(T)$, $G_{H(T)}(T)$, $R_{H(T)}(T)$.

Lemma 6.4.1. *Fix $k \geq 2$. Under the conditions* (6.1.1) *and* (6.1.2)*, as $x \to \infty$,*

$$\mathbf{P}_k^{st}(S(T) > x) = \begin{cases} (C_1 + o(1))x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \neq 1, \end{cases}$$

where

$$C_1 = \frac{f''(q)}{\gamma^2\sigma^2} \frac{\sum_{1 \leq i \leq k-1} \pi_i \pi_{k-i}}{\pi_k \sum_{i=1}^{\infty} \pi_i (\mu(0))^i},$$

$$C_2 = \inf_{s\in(0,\infty)}\left(-s\log\gamma + \sup_{t\in\mathbb{R}}(t - s\Lambda(t))\right),$$

and all parameters appearing in the C_1, C_2 are those in Proposition 6.2.1 and Lemma 6.2.2.

Proof. Denote by $X_i^{(j)}$ independent random variables distributed as θ for all $i, j \in \mathbb{N}$. Denote

$$F(m,u,x) := \mathbb{P}\left(\max_{1\le a,b\le m}\left\{\sum_{i=1}^{u} X_i^{(a)} - \sum_{j=1}^{u} X_j^{(b)}\right\} > x\right),$$

then

$$\begin{aligned}\mathbf{P}_k^{st}(S(T) > x) &= \mathbf{E}_k^{st}(F(Z_1(T), H(T), x)) \\ &= \sum_{u\ge1}\mathbf{P}_k^{st}(H(T) = u)\mathbf{E}_k^{st}(F(Z_1(T), u, x) \mid H(T) = u).\end{aligned}$$

Moreover, by the union bound,

$$F(m,u,x) \le m^2 F(2,u,x),$$

thus

$$\begin{aligned}&\mathbf{E}_k^{st}(F(Z_1(T), H(T), x)) - \sum_{u\ge1}\mathbf{P}_k^{st}(H(T) = u)F(2,u,x) \\ \le &\sum_{u\ge1}\mathbf{P}_k^{st}(H(T) = u)\cdot k^2F(2,u,x)\mathbf{P}_k^{st}(Z_1(T)\ge 3 \mid H(T) = u).\end{aligned} \tag{6.4.16}$$

Then by Corollary 6.3.9, the error term in (6.4.16) is negligible, so

$$\mathbf{P}_k^{st}(S(T) > x) = \mathbf{E}_k^{st}(F(Z_1(T), H(T), x)) = (1+o(1))\sum_{u\ge1}\mathbf{P}_k^{st}(H(T) = u)F(2,u,x).$$

Thus it suffices to show that

$$\sum_{u\ge1}\mathbf{P}_k^{st}(H(T) = u)F(2,u,x) = \begin{cases}(C_1 + o(1))x^{-2}, & m = 1,\\ \exp(-(C_2 + o(1))x), & m \ne 1.\end{cases}$$

If $m = 1$, by Part 2 of Corollary 6.3.10, Part 3 of Proposition 6.2.1 and

Part 1 of Lemma 6.2.2, we have that

$$
\begin{aligned}
&\sum_{u\geq 1}\mathbf{P}_k^{st}(H(T)=u)F(2,u,x)\\
=&\sum_{u>x^{3/2}}\frac{C_1+o(1)}{u^2}\left[2\left(1-\Phi\left(\frac{x}{\sqrt{2u}}\right)\right)+O\left(\frac{1}{\sqrt{u}}\right)\right]+\sum_{u\leq x^{3/2}}o\Big(\frac{1}{u^2}\cdot\frac{u}{x^2\log x}\Big)\\
=&(2C_1+o(1))\int_{x^{3/2}}^{\infty}\frac{1-\Phi(x/\sqrt{2y})}{y^2}dy+o(x^{-2})\\
=&\frac{2C_1+o(1)}{x^2}\int_0^{\infty}\frac{1-\Phi(1/\sqrt{2z})}{z^2}dz=\frac{2C_1+o(1)}{x^2}.
\end{aligned}
$$

If $m\neq 1$, similarly, we can choose suitable constants C, s_1, s_2 such that

$$
\begin{aligned}
&\sum_{u\geq 1}\mathbf{P}_k^{st}(H(T)=u)F(2,u,x)\\
=&\sum_{s_1x<u<s_2x}(C+o(1))\gamma^u e^{-(1+o(1))\sup_{t\in\mathbb{R}}(tx-u\Lambda(t))}+O(R(2,s_1x,x)+\gamma^{s_2x})\\
=&e^{-(C_2+o(1))x}.
\end{aligned}
$$

We remark that since

$$
\begin{aligned}
\lim_{s\to+\infty}\left(-s\log\gamma+\sup_{t\in\mathbb{R}}(t-s\Lambda(t))\right)&\geq\lim_{s\to+\infty}(-s\log\gamma+(0-s\Lambda(0)))\\
&=\lim_{s\to+\infty}(-s\log\gamma)=+\infty,
\end{aligned}
$$

and

$$
\lim_{s\to 0+}\left(-s\log\gamma+\sup_{t\in\mathbb{R}}(t-s\Lambda(t))\right)=\lim_{s\to 0+}\sup_{t\in\mathbb{R}}(t-s\Lambda(t))=+\infty,
$$

the infimum over $(0,\infty)$ in C_2 is equivalent to the infimum among a bounded interval $[\epsilon,\epsilon^{-1}]$. □

Proposition 6.4.2. *Fix $k\geq 2$. Under the conditions* (6.1.1) *and* (6.1.2)*, as $x\to\infty$,*

$$
\mathbf{P}_k^{st}(R(T)>x)=\begin{cases}(C_1+o(1))x^{-2}, & m=1,\\ \exp(-(C_2+o(1))x), & m\neq 1,\end{cases}
$$

where C_1, C_2 are those in Lemma 6.4.1.

Proof. By (6.4.15) and Lemma 6.4.1, it suffices to show that

$$
\mathbf{P}_k^{st}(G(T)>x^{1-\epsilon})=o(\mathbf{P}_k^{st}(S(T)>x)) \tag{6.4.17}
$$

for some $\epsilon > 0$.

In fact, $G(T)$ is determined by the structures of $\texttt{cut}(T[i])$. If we denote by

$$\tilde{H}(T) := \max_{1 \le i \le Z_1(T)} H(\texttt{cut}(T[i])),$$

then by the union bound, $G(T)$ is determined by at most k^2 pairs of nodes within the same subtrees, in other words,

$$\begin{aligned}\mathbf{P}_k^{st}(G(T) > x^{1-\epsilon}) &\le k^2 \mathbf{E}_k^{st}\left[\max_{1 \le u \le \tilde{H}(T)} F(2, u, x^{1-\epsilon})\right] \\ &\asymp \sum_{u \ge 1} \mathbf{P}_k^{st}(\tilde{H}(T) = u) F(2, u, x^{1-\epsilon}).\end{aligned}$$

Sum over all possible cases by (6.3.12), notice that r and (k_i) can only take integer values at most k, we have that

$$\mathbf{P}_k^{st}(\tilde{H}(T) = u) \lesssim P_u(1,1) \mathbf{P}_k^{st}(H(T) = u).$$

To sum up,

$$\mathbf{P}_k^{st}(G(T) > x^{1-\epsilon}) \lesssim \sum_{u \ge 1} P_u(1,1) \mathbf{P}_k^{st}(H(T) = u) F(2, u, x^{1-\epsilon}),$$

while

$$\mathbf{P}_k^{st}(S(T) > x) \asymp \sum_{u \ge 1} \mathbf{P}_k^{st}(H(T) = u) F(2, u, x).$$

Therefore, by the proof of Lemma 6.4.1, we have the desired dominance in (6.4.17). □

6.4.2 The gaps

Proposition 6.4.3. *Let $k \ge 2$ and $1 \le i \le k-1$. Under* (6.1.1) *and* (6.1.2)*, as $x \to \infty$,*

$$\begin{aligned}\mathbf{P}_k^{st}(g^i(T) > x) &= (C_3 + o(1)) \mathbf{P}_k^{st}(R(T) > x) \\ &= \begin{cases} (C_1 C_3 + o(1)) x^{-2}, & m = 1, \\ \exp(-(C_2 + o(1))x), & m \ne 1, \end{cases}\end{aligned}$$

where $C_3 = \frac{\pi_i \pi_{k-i}}{\sum_{j=1}^{k-1} \pi_j \pi_{k-j}}$.

Proof. Recall that trees under $\mathbf{P}_k^{st}$ have k nodes in the last generation, and we write their positions in increasing order,

$$V^{(1)}(T) \leq \cdots \leq V^{(k)}(T).$$

In the previous section, we showed that

$$\begin{aligned}
&\mathbf{P}_k^{st}(Z_1(T) \geq 3, R(T) > x) = o(\mathbf{P}_k^{st}(R(T) > x)),\\
&\mathbf{P}_k^{st}\big(G(T) > x^{1-\epsilon}\big) = o(\mathbf{P}_k^{st}(R(T) > x)),
\end{aligned}$$

thus it suffices to consider the case where

$$\{V^{(1)}(T), \cdots, V^{(i)}(T)\} \text{ and } \{V^{(i+1)}(T), \cdots, V^{(k)}(T)\}$$

are exactly positions of nodes in the last generation of the two subtrees $T[1], T[2]$. In other words,

$$\begin{aligned}
&\mathbf{P}_k^{st}(g^i(T) > x)\\
=&\frac{1}{2}\mathbf{P}_k^{st}(S(T) > x, \#T[1] = i, \#T[2] = k-i)\\
&+\frac{1}{2}\mathbf{P}_k^{st}(S(T) > x, \#T[1] = k-i, \#T[2] = i) + o(\mathbf{P}_k^{st}(R(T) > x)),
\end{aligned}$$

where the factors $\frac{1}{2}$ are to distinguish the symmetric cases $g^i(T) > x$ and $g^{k-i}(T) > x$.

Moreover, by Proposition 6.3.8, we have that

$$\begin{aligned}
&\mathbf{P}_k^{st}(S(T) > x, \#T[1] = i, \#T[2] = k-i)\\
=&\sum_{u\geq 1} F(2,u,x)\mathbf{P}_k^{st}(H(T) = u, \#T[1] = i, \#T[2] = k-i)\\
=&(C_3 + o(1))\sum_{u\geq 1} F(2,u,x)\mathbf{P}_k^{st}(H(T) = u),
\end{aligned}$$

and the conclusion follows from the proof of Lemma 6.4.1. □

Remark 6.4.4. As an example, consider the canonical case where the offspring distribution μ is the geometric distribution with parameter $\frac{1}{2}$, i.e. $\mu(k) = 2^{-k-1}$, then one can explicitly show (cf. eg. [17, Section 1.4]) that its generating function satisfies

$$f_n(s) = \sum_{i=0}^{\infty} \mathbb{P}(Z_n = i)s^i = 1 - \frac{1}{n + \frac{1}{1-s}},$$

and its transition probabilities are

$$P_n(1,k) = \frac{1}{k!}\frac{d^k f_n(s)}{ds^k}|_{s=0} = \frac{n^{k-1}}{(n+1)^{k+1}}.$$

Therefore,

$$\pi_k = \lim_{n\to\infty} \frac{P_n(1,k)}{P_n(1,1)} = 1,\ \forall k \in \mathbb{N}_+.$$

Thus in this case, the constant

$$C_1 C_3 = \frac{f''(q)}{\gamma^2\sigma^2}\frac{\pi_i \pi_{k-i}}{\pi_k \sum_{i=1}^{\infty} \pi_i (\mu(0))^i} = \frac{f''(q)}{\gamma^2\sigma^2}\frac{1}{\sum_{i=1}^{\infty} (\mu(0))^i}$$

in Proposition 6.4.3 no longer depends on the choice of i or k, as is showed in [86].

Finally, we formally conclude that

Proof of Theorem 6.1.2. The conclusion follows from (6.4.14) (with its counterpart for the gaps), Proposition 6.4.2 and Proposition 6.4.3. □

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