UNIVERSITÉ **PARIS XIII - SORBONNE PARIS NORD**

École Doctorale Sciences, Technologies, Santé Galilée

Valeurs extrêmes de processus de branchement spatiaux

Extreme values of spatial branching processes

Thèse de doctorat présentée par

Mohamed Ali BELLOUM

UMR 7539 - LAGA - Laboratoire Analyse, Géométrie Et Applications

pour l'obtention de grade

DOCTEUR EN MATHÉMATIQUES

Soutenance le 07/12/2021 devant le jury composé de:

Pascal MAILLARD	Rapporteur
Nina GANTERT	Rapporteur
Bénédicte HAAS	Examinateur
Arvind SINGH	Examinateur
Thomas DUQUESNE	Examinateur
Bastien MALLEIN	Directeur de thèse
Yueyun HU	Directeur de thèse

Résumé

Dans cette thèse, on s'intéresse aux valeurs extrêmes de certains processus de branchement spatiaux tels que la marche aléatoire branchante et le mouvement Brownien branchant. La marche aléatoire branchante est un système de particule qui peut être modélisé de la façon suivante. On part d'une seule particule à la génération 0. Elle donne naissance à un nombre aléatoire d'enfants positionnés autour de leur parent selon un processus de points. Puis chaque enfant répète le même processus que celui de son parent indépendamment des autres particules. Le mouvement brownien branchant peut être décrit de manière similaire. On commence toujours avec une seule particule située à l'origine à l'instant $t = 0$. Elle se déplace selon un mouvement Brownien standard. Après un temps exponentiel de paramètre 1, elle meurt en donnant naissance à deux enfants. Puis chacun des deux enfants commence à son tour un mouvement Brownien branchant indépendant.

Dans la première partie de cette thèse on étudie un modèle qui interpole entre la marche aléatoire branchante et un modèle lié à la physique statistique appelé *Random Energy Model (REM)*. Le deuxième et le troisième chapitre sont consacrés à l'étude des valeurs extrêmes de certains modèles de branchement multitype. Plus précisément, on étudie le comportement asymptotique du processus extrémal d'un mouvement Brownien branchant multitype réductible.

Abstract

In this thesis, we are interested in extreme values of certain spatial branching processes such as the branching random walk and the branching Brownian motion. The branching random walk is a particle system that can be described as follows. It starts with an unique particle at generation 0. It gives birth to a random number of children positioned with respect to their parent according to a point process. Then, each child repeats the same process to that of his parent and independently of the rest of particles. The branching Brownian motion can be described similarly. It starts with an unique particle at the origin. It moves according to a standard Brownian motion. After an exponential time, it dies giving birth to two children on its current position. Then, each child starts an independent branching Brownian motion.

In the first part of this thesis, we study a model that interpolates between the branching random walk and a model linked to statistical physics which called *Random energy model (REM)*. The next part of this thesis is devoted to the study of the extreme values of certain multitype branching processes. More precisely, we study the asymptotic behaviour of the extremal process of a reducible branching Brownian motion.

Remerciements

Je tiens à remercier Monsieur Bastien MALLEIN, Maître de conférence à l'université Sorbonne Paris Nord, qui m'a encadré tout au long de cette thèse et qui m'a fait partager ses brillantes intuitions. Qu'il soit aussi remercié pour sa gentillesse, sa disponibilité permanente et pour les nombreux encouragements qu'il m'a prodigué. Je suis ravi d'avoir travaillé en sa compagnie car outre son appui scientifique, il a toujours été là pour me soutenir et me conseiller au cours de l'élaboration de cette thèse.

Je remercie Monsieur Yueyun HU, Professeur à l'université Sorbonne Paris Nord pour son encadrement, son soutien et ses recommandations pour mes candidatures aux post-doc.

J'adresse tous mes remerciements à Monsieur Pascal MAILLARD, Professeur à l'université de Toulouse III - Paul Sabatier, ainsi qu'à Madame Nina GANTERT, Professeur à l'université Technique de Munich, de l'honneur qu'ils m'ont fait en acceptant d'être rapporteurs de cette thèse.

J'exprime ma gratitude à Madame Bénédicte HAAS, Professeur à l'université sorbonne Paris Nord et à Monsieur Arvind SINGH, Chercheur au CNRS à l'université Paris Sud, qui ont bien voulu être examinateurs.

J'ai été très heureux de travailler au LAGA, je remercie les différents membres du ce laboratoire et spécialement Yolande JIMENEZ, Gilles DESERT et Jean-Philippe DOMERGUE pour leur gentillesse et leur serviabilité.

Je tiens à remercier tous les membres de LAGA, je pense particulièrement à mes collègues et amis (les thésards du LAGA et LIPN) pour les bons moments que nous avons passés ensemble.

Le dernier remerciement va à ma famille pour leur soutien inconditionnel, je pense à maman, mon frère et mon cousin Seif pour son aide.

Table des matières

Chapitre 1

Introduction

Les processus de branchement sont des modèles mathématiques qui décrivent l'évolution d'une population au fil du temps. Parmi les processus de branchement les plus étudiés dans la littérature, on trouve le processus de Bienaymé-Galton-Watson. C'est un processus qui décrit l'évolution d'une population où les individus se reproduisent selon une même loi et indépendamment les uns des autres. Ce processus a été introduit pour la première fois par Bienaymé [Bie45] en 1845. En 1874, Galton et Watson [WG75] ont étudié la probabilité du survie d'une population, et ont montré que la survie est assurée si et seulement si le nombre moyen d'enfants d'un individu est strictement supérieur à 1.

Ce processus peut être étudié avec sa généalogie, pour cela il est naturel d'introduire un arbre aléatoire associé à ce processus qu'on appelle l'arbre de Galton-Watson défini comme suit. On commence par un unique individu situé à l'origine qu'on appelle la racine. Il donne naissance à un nombre aléatoire d'enfants tiré selon une loi fixe appelée loi de reproduction. Cela constitue la première génération. Puis chaque individu à la génération $n \in \mathbb{N}$ répète le même processus indépendamment des autres individus présents dans l'arbre. Ces processus de branchements ont été l'objet d'étude de nombreux travaux (voir ceux de Kendall [Ken66, Ken75]). Une version enrichie de ces processus a été étudié par Bellman et Harris [BH48], rajoutant une durée de vie aléatoire aux individus. Puis, Crump et Mode [CM68] et Jagers [Jag69] ont continué l'étude de ces modèles en rajoutant des caractéristiques supplémentaires qui se transmettent entre les individus. Par exemple, on peut rajouter à chaque individu une valeur qui peut être sa position, sa masse, son phénotype.... À chaque reproduction, cette valeur se transmet aux nouveaux nés avec une modification aléatoire.

Parmi les processus les plus simples dans ce contexte, on trouve la **marche aléatoire branchante** (MAB) dont l'analogue en temps continu est le **mouvement Brownien branchant** (MBB). Ces deux objets sont les sujets d'étude dans cette thèse. La marche aléatoire branchante est un système de particule qui peut être modélisée de la façon suivante. On part d'une seule particule située à l'origine. Elle donne naissance à un nombre aléatoire d'enfants positionnés autour de leur parent selon un processus de points. Puis chaque enfant répète le même processus que celui de son parent indépendamment des autres particules.

Le mouvement brownien branchant peut être décrit de manière similaire. On commence toujours avec une seule particule situé à l'origine à l'instant $t = 0$. Elle se déplace selon un mouvement Brownien standard. Après un temps exponentiel de paramètre 1, elle meurt en donnant naissance à deux enfants. Puis chacun des deux enfants commence à son tour un mouvement Brownien branchant indépendant. On peut voir une illustration de ces deux processus dans la Figure 3.1. (Une définition formelle de la marche aléatoire branchante et du mouvement Brownien branchant sont données dans la section 1.1).

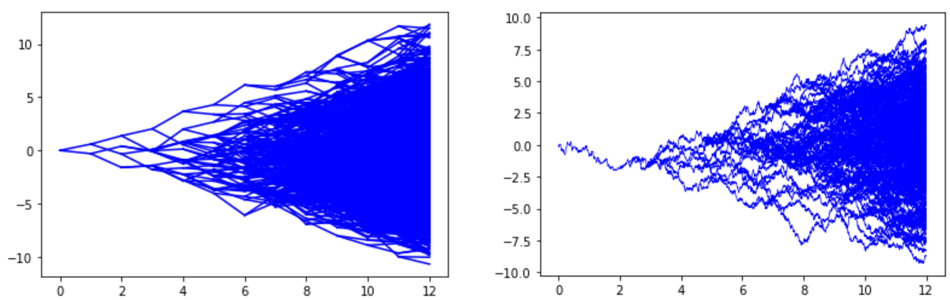


FIGURE 1.1 – Simulation de la marche aléatoire branchante à gauche et du mouvement Brownien branchant à droite.

Dans cette thèse on s'intéresse au comportement asymptotique du processus extrémal ainsi que la loi du maximum de certains processus de branchement. Le reste de l'introduction est divisé comme suit. Dans la section 1.1, on va définir de façon formelle la MAB et le MBB ainsi que les valeurs extrêmes associées à ces processus. Dans la section 1.2, on va présenter le premier modèle étudié dans cette thèse et les résultats obtenus dans l'article [Bel21] (présenté au deuxième chapitre). Il s'agit d'un modèle qui interpole entre la marche aléatoire branchante et un modèle lié à la physique statistique appelé *Random Energy Model (REM)*. La section 1.3 est consacrée à l'introduction des modèles de branchement multitype puis la discussion des résultats de l'article [BM21] présenté au chapitre 3. On étudie le comportement asymptotique du processus extrémal d'un mouvement Brownien branchant multitype réductible. On termine cette partie introductive par une présentation des résultats du chapitre 4 où on étudie les valeurs extrêmes du modèle introduit dans [BM21] dans un cas critique.

1.1 Notations et définitions

On commence par introduire la MAB et le MBB de façon formelle ainsi que les valeurs extrêmes associées à ces deux modèles. Pour définir la MAB, une première étape consiste à rappeler la notion d'arbre généalogique dans un contexte général et les notations associées.

Le système de notations qu'on va adapter a été introduit par Neveu [Nev86]. Posons

$$\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N})^n$$

l'ensemble des suites finies de $\mathbb{N}$ où $\mathbb{N}^0 = \{\varnothing\}$ par convention. L'élément $(u_1, u_2.., u_n)$ représente le $u_n^{\text{ième}}$ enfant du $u_{n-1}^{\text{ième}}$ enfant .., du $u_1^{\text{ième}}$ enfant de la racine qu'on note $\varnothing$. Si $u = (u_1, u_2.., u_n)$, on note par $u_k = (u_1, u_2.., u_k)$ la suite des k premiers valeurs de u et par $|u|$ la génération de u. On note aussi $\pi(u) = (u_1, u_2.., u_{n-1})$ le parent de u. Si $u = (u_1, u_2.., u_n)$ et $v = (v_1, v_2.., v_n)$, alors on écrit $u.v = (u_1, u_2.., u_n, v_1, v_2.., v_n)$ la concaténation de u et v. On écrit

$$|u \wedge v| := \inf\{j \leq n : u_j = v_j \text{ and } u_{j+1} \neq v_{j+1}\}.$$

Cette quantité désigne l'âge de l'ancêtre commun le plus récent entre u et v.

Un arbre $\mathcal{T}$ est un sous ensemble de $\mathcal{U}$ vérifiant les propriétés suivantes :

- $\varnothing \in \mathcal{T}$.
- Si $u \in \mathcal{T}$, alors $\pi(u) \in \mathcal{T}$.
- Si $u = (u_1, u_2, ...u_n) \in \mathcal{T}$, alors $\forall\ j \leq u_n$, $\pi(u).j \in \mathcal{T}$.

Processus et arbre de Galton Watson Soit $\mathbf{p} = (p_k)_{k\in\mathbb{N}}$ une loi de probabilité sur $\mathbb{N}$, alors le processus de Galton-Watson $(Z_n)_{n\in\mathbb{N}}$ est défini par $Z_0 = 1$ et pour tout $n \in \mathbb{N}$,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}$$

où $(\xi_{n,i})_{n\in\mathbb{N},i\in\mathbb{N}^*}$ une suite de variables aléatoires indépendantes et de même loi $\mathbf{p}$. Ici, $\xi_{n,i}$ représente le nombre d'enfants du $i^{\text{ième}}$ individu de la $n^{\text{ième}}$ génération.

On note $\mathcal{T}$ l'arbre de Galton-Watson associé qui est un sous ensemble de $\mathcal{U}$ défini de la façon suivante. Soit $(\xi_u, u \in \mathcal{U})$ une famille de variable aléatoires indépendantes et de même loi $\mathbf{p}$, alors on écrit

$$\mathcal{T} = \{u \in \mathcal{U}, \forall 1 \leq k \leq |u|, u_k \leq \xi_{u_{k-1}}\}.$$

Si on note $Z_n = \#\{u \in \mathcal{T}, |u| = n\}$ le nombre d'enfants à la génération n dans l'arbre $\mathcal{T}$, alors le processus $(Z_n, n \geq 0)$ est un processus de Galton-Watson (voir Figure 1.2).

Marche aléatoire branchante La marche aléatoire branchante est une extension de l'arbre de Galton-Watson $\mathcal{T}$ telle qu'à chaque nœud de l'arbre on associe sa position. On suppose que la moyenne de la loi de reproduction vérifie $m = \sum_{k\in\mathbb{N}} kp_k > 1$, i.e le processus de Galton-Watson survit avec une probabilité non nulle.. Soit Y une variable aléatoire. On introduit $(Y_u, u \in \mathcal{T})$ une famille de variables aléatoires i.i.d indépendante de $(\xi_u, u \in \mathcal{T})$ et de même loi que Y représentant le déplacement de l'enfant u par rapport à son parent. Alors à chaque $u \in \mathcal{T}$ on associe sa position

$$X_u = \sum_{i=1}^{|u|} Y_{u_i} \tag{1.1.1}$$

qui est la somme des déplacements le long du chemin reliant la racine à l'individu u. On s'intéresse aux valeurs extrêmes da la MAB, plus précisément à la position de l'individu le plus à droite que l'on note $M_n = \max_{|u|=n} X_u$ ainsi que le processus extrémal, i.e le processus ponctuel formé par les valeurs vues depuis la médiane du maximum (voir (1.1.3) pour une expression explicite).

On introduit la transformée log-Laplace de la MAB définie par

$$\kappa(\theta) = \log(\mathbf{E}(\sum_{|u|=1} e^{\theta X_u})) = \log(m) + \log(\mathbf{E}(e^{\theta Y})) \qquad \forall\theta > 0,$$

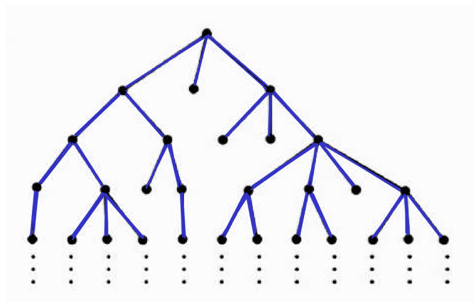


FIGURE 1.2 – Arbre de Galton Watson avec 5 générations

et la transformée de Cramér κ^* de κ, définie par

$$\begin{array}{cccc} \kappa^* & : & \mathbb{R} & \rightarrow & \mathbb{R} \\ & & x & \mapsto & \sup_{\theta>0} \theta x - \kappa(\theta) \end{array}.$$

Les deux fonctions κ et κ^* sont convexes et différentiables à l'intérieur de leurs intervalles de définition. On suppose qu'il existe $\theta^* > 0$ tel que

$$\theta^* \kappa'(\theta^*) - \kappa(\theta^*) = 0. \tag{1.1.2}$$

Les premiers travaux sur les valeurs extrêmes de la MAB remontent à Hammersley [Ham74], Kingman [Kin75] et Biggins [Big76], qui ont montré que sous certaines conditions d'intégrabilités et conditionnellement à la non-extinction de l'arbre $\mathcal{T}$,

$$\lim_{n\to\infty} \frac{M_n}{n} = \frac{\kappa(\theta^*)}{\theta^*} := v \qquad \text{p.s.}$$

Dans un second temps, Hu et Shi [HS09] d'une part ont montré que

$$\limsup_{n\to\infty} \frac{M_n - nv}{\log(n)} = \frac{-1}{2\theta^*} \text{ p.s , } \liminf_{n\to\infty} \frac{M_n - nv}{\log(n)} = \frac{-3}{2\theta^*} \text{ p.s.,}$$

et

$$\lim_{n\to\infty} \frac{M_n - nv}{\log(n)} = \frac{-3}{2\theta^*} \quad \text{en probabilité,}$$

d'autre part, Addario-Berry et Reed [AR09] ont montré la tension de la suite $(M_n - nv + \frac{3}{2\theta^*}\log(n), n \geq 1)$. La convergence en loi du maximum a été obtenu par Aidékon [Aid13].

Théorème 1.1.1. *[Aid13, Théorème 1.1] Si on pose* $m_n = nv - \frac{3}{2\theta^*}\log(n)$, *alors sous certaines hypothèses, il existe une constante* $C > 0$ *tel que*

$$\lim_{n\to\infty} \mathbf{P}(M_n \le m_n + x) = \mathbf{E}\left(e^{-CZ_\infty e^{-\theta^* x}}\right),$$

où

$$\lim_{n\to\infty} \sum_{|u|=n} (nv - X_u) e^{\theta^* X_u - n\kappa(\theta^*)} = Z_\infty \quad ,$$

est la limite de la martingale dérivée associée à la MAB.

Madaule [Mad17] a montré que le processus extrémal $\mathcal{E}_n$ défini par

$$\mathcal{E}_n = \sum_{|u|=n} \delta_{X_u - m_n}, \tag{1.1.3}$$

converge en loi vers un processus de Poisson ponctuel décoré. Plus précisément, il a montré que

$$\lim_{n\to\infty} \mathcal{E}_n = \sum_{j\in\mathbb{N}} \sum_{d\in\mathcal{D}_j} \delta_{\xi_j + d + \frac{1}{\theta^*}\log Z_\infty} := \mathcal{E}_\infty,$$

où $(\xi_i)_{i\in\mathbb{N}}$ sont les atomes d'un processus de Poisson ponctuel d'intensité $CZ_\infty e^{-\theta^* x}dx$ et $(D_j, j \in \mathbb{N})$ sont des copies i.i.d d'un processus ponctuel $\mathcal{D}$ indépendant de $(\xi_i)_{i\in\mathbb{N}}$.

Mouvement brownien branchant. Le mouvement brownien branchant est l'analogue en temps continu de la marche aléatoire branchante. C'est un système de particules qui peut être décrit de la manière suivante. On commence par une unique particule positionnée à l'origine. Elle se déplace selon un mouvement Brownien de variance σ^2. À taux β, elle meurt en donnant naissance à L enfants où L est une variable aléatoire à valeurs dans $\mathbb{N}$. Chaque enfant répète le même processus indépendamment des autres particules présentes dans le système. On supposera ici que la loi de reproduction est binaire, i.e $L = 2$ et $\beta = \sigma^2 = 1$.

On note $\mathcal{N}_t$ l'ensemble des particules en vie à l'instant t et pour $u \in \mathcal{N}_t$, on note $X_u(t)$ la position de l'individu à l'instant t. Généralement, il est plus aisé de travailler avec le mouvement Brownien branchant que la marche aléatoire branchante grâce à ses propriétés analytiques qui rendent certains calculs plus explicites. Historiquement, plusieurs résultats ont démontrés sur le MBB puis étendues au cadre plus général des MAB.

Position du maximum. De façon analogue à la MAB, si on note $M_t = \max_{u\in\mathcal{N}_t} X_u(t)$ la position du maximum, alors en se référant aux travaux de Kolmogorov [Kol37], Kingmann [Kin75], Hammersley [Ham74] et Biggins [Big76], on a

$$\lim_{t\to\infty} \frac{M_t}{t} = \sqrt{2} \qquad \text{p.s.}$$

Autrement dit, la vitesse de la particule la plus à droite dans le mouvement Brownien branchant est égale à $\sqrt{2}$. Ce résultat a été précisé par Bramson, qui a montré [Bra83] que $(M_t - \sqrt{2}t + \frac{3}{2\sqrt{2}}\log(t), t \ge 0)$ est tendue. Puis, Lalley et Selke [LS87] ont étudié la loi limite du maximum M_t centré par sa médiane m_t. Ils ont montré qu'il existe une constante $C^* > 0$ telle que

$$\lim_{t\to\infty} \mathbf{P}(M_t \le m_t + x) = \mathbf{E}\left(e^{-C^* e^{\sqrt{2}x} Z_\infty}\right) \tag{1.1.4}$$

où

$$\lim_{t\to\infty}\sum_{u\in\mathcal{N}_t}(\sqrt{2}t-X_u(t))e^{\sqrt{2}X_u(t)-2t}=Z_\infty \tag{1.1.5}$$

est la limite de la martingale dérivée associée au BBM.

Processus extrémal. Le processus extrémal du MBB standard est défini par

$$\mathcal{E}_t=\sum_{u\in\mathcal{N}_t}\delta_{X_u(t)-m_t}.$$

Le comportement asymptotique de la mesure $\mathcal{E}_t$ a été étudié par, Aidékon, Berestycki, Brunet et Shi [ABBS13] d'une part et Arguin, Bovier et Kistler [ABK11, ABK12, ABK13] d'autre part. Ils ont montré que le processus extrémal $\mathcal{E}_t$ converge en loi vers un processus de Poisson ponctuel décoré d'intensité $\sqrt{2}C^*Z_\infty e^{-\sqrt{2}x}$, avec $C^*>0$. La loi de la décoration a été décrite par [ABK13] comme la loi limite du processus extrémal vu depuis son maximum conditionnellement à l'événement $\{M_t\geq\sqrt{2}t\}$. De manière plus explicite, ils montrent l'existence d'une mesure ponctuelle $\mathcal{D}$ telle que

$$\lim_{t\to\infty}\mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{N}_t}\varphi(X_u(t)-M_t)\right)\middle|M_t\geq\sqrt{2}t\right)=\mathbf{E}\left(\exp\left(-\langle\mathcal{D},\varphi\rangle\right)\right), \tag{1.1.6}$$

pour toute fonction φ continue bornée à support borné à gauche.

On peut décrire la loi limite $\mathcal{E}_\infty$ de la façon suivante. Soit $(\xi_i)_{i\in\mathbb{N}}$ les atomes d'un processus de Poisson ponctuel d'intensité $C^*\sqrt{2}Z_\infty e^{-\sqrt{2}x}dx$ et $(\mathcal{D}_j,j\in\mathbb{N})$ des copies i.i.d du processus ponctuel $\mathcal{D}$ indépendants de $(\xi_i)_{i\in\mathbb{N}}$, alors on pose

$$\mathcal{E}_\infty=\sum_{j\in\mathbb{N}}\sum_{d\in\mathcal{D}_j}\delta_{\xi_j+d+\frac{1}{\sqrt{2}}\log Z_\infty},$$

où $\sum_{d\in\mathcal{D}_j}$ est la somme sur l'ensemble des atomes de la mesure ponctuelle $\mathcal{D}_j$.

1.2 Le *GREM* avec un grand nombre de niveaux

Dans la partie suivante on va présenter les résultats de l'article [Bel21] qui font l'objet du deuxième chapitre de la thèse.

Dans un premier temps, on va introduire le modèle d'énergie aléatoire et sa version généralisée ainsi que les valeurs extrêmes associés et les travaux récents qui ont motivé notre étude.

Random Energy Model (REM). Le modèle d'énergie aléatoire (REM) a été introduite par Bernard Derrida en 1981 [Der81] dans le but d'étudier les verres spins. C'est un modèle à champs moyen comportant N éléments, chacun associé à une valeur $+1$ où -1 nommé le spin de l'élément. On obtient alors 2^N configurations de spins. Si on note $\Sigma_N=\{+1,-1\}^N$ l'espace des configurations, alors à chaque configuration $\sigma\in\Sigma_N$, on associe une variable aléatoire gaussienne indépendante X_σ de variance N qui modélise son énergie.

En termes de MAB, le REM peut être généré comme un arbre aléatoire où on a seulement une racine à l'origine et 2^N feuilles. À chaque branche de l'arbre, on associe une variable aléatoire gaussienne indépendante X_i, $1\leq i\leq 2^N$ de variance N (voir la première figure à gauche dans 1.3 pour une illustration).

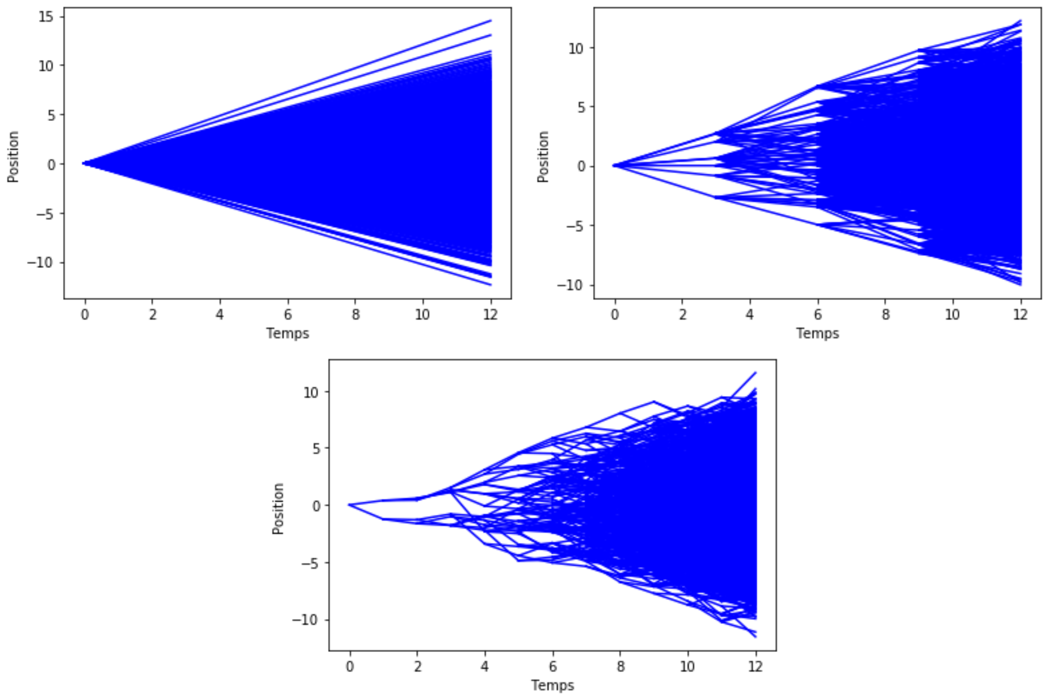


FIGURE 1.3 – $GREM_N(K)$ avec $K = 1, 4, 12$ et $N = 12$

Dans la théorie des valeurs extrêmes, le modèle d'énergie aléatoire (REM) est le plus simple à étudier. Les questions d'intérêts dans ce cas sont la loi de maximum $M_n = \max_{1 \leq i \leq 2^N} X_i$, ainsi que le processus extrémal défini par $\mathcal{E}_N = \sum_{1 \leq i \leq 2^N} \delta_{X_i - m_n}$.

Théorème 1.2.1. *[BS02, Section 8.3] Si on pose $m_N = \beta_c N - \frac{1}{2\beta_c}\log(N)$ avec $\beta_c = \sqrt{2\log(2)}$, alors le processus extrémal $\mathcal{E}_N$ converge en loi vers un processus de Poisson ponctuel d'intensité $\frac{1}{\sqrt{2\pi}} e^{-\beta_c x} dx$.*

De plus la loi du maximum $M_N = \max_{1 \leq i \leq 2^N} X_i$ centré par m_N converge vers une loi de Gumbel.

Generalized Random Energy Model(GREM) Une version généralisée de REM a été introduite par Derrida [Der85], connue sous le nom de GREM en rajoutant des corrélation aux énergies associées aux configurations. Fixons $K \leq N$ un nombre entier qui représente le nombre de niveaux dans un *GREM* comportant N sites. Alors, on décrit le $GREM_N(K)$ de la façon la suivante. On commence par une unique particule à l'origine. Elle donne naissance à $2^{\frac{N}{K}}$ avec $K \in \mathbb{N}^*$. À chaque niveau $1 \leq i \leq K$, chaque enfant donne naissance à $2^{\frac{N}{K}}$ de façon indépendante. À chaque branche de l'arbre on associe une variable aléatoire gaussienne indépendante de variance $\frac{N}{K}$. En terme de verres de spin, on obtient 2^N configurations au niveau K et l'énergie de chaque configuration est la somme des valeurs sur chaque branche du chemin reliant la configuration à la racine. On appelle ce modèle $GREM_N(K)$. Le REM dans ce cas est un GREM avec un seul niveau i.e $GREM_N(1)$ est la MAB à pas gaussien binaire est le $GREM_N(N)$.

Il est clair que les énergies des configurations dans ce cas ne sont plus des variables aléatoires indépendantes, mais présentant des corrélations générés par la structure de branchement dans l'arbre. Par contre cette corrélation n'a pas d'impacte sur le comportement asymptotique du processus extrémal ainsi que la loi du maximum et on a des résultats similaires au Théorème 2.1.1 pour ce modèle.

Kiestler et Schmidt [SK15] ont étudié les valeurs extrêmes du GREM avec un nombre de niveaux qui croit en N^α, $\alpha \in [0,1]$ qu'on note $GREM(N^\alpha)$, $\alpha \in [0,1)$. Ils ont obtenu le résultat suivant :

Théorème 1.2.2. *Si on pose*

$$m_N^{(\alpha)} = \beta_c N - \frac{2\alpha+1}{2\beta_c}\log(N),$$

alors le processus extrémale associé au $GREM_N(N^\alpha)$ *converge en loi vers un processus de Poisson ponctuel d'intensité* $\frac{1}{\sqrt{2\pi}}e^{-\beta_c x}dx$. *De plus la loi du maximum centrée par* $m_N^{(\alpha)}$ *converge vers une loi de Gumbel.*

On observe que par rapport au $GREM_N(K)$, $K > 0$, il y a un changement au niveau de la médiane. Plus précisément, on observe une décroissance en α de la correction logarithmique. On passe d'un facteur de correction logarithmique égale à $\frac{-1}{2\sqrt{2}}$ dans le cas du REM à un facteur égale $\frac{-1}{2\sqrt{2}} - \frac{\alpha}{\sqrt{2}}$ dans le cas $GREM(N^\alpha)$, $\alpha \in [0,1)$. En revanche, on obtient des résultats similaires à ceux du REM en termes de la loi limite du processus ponctuel extrémal et du maximum.

Le cas $\alpha = 1$, correspond à la marche aléatoire branchante binaire à pas gaussien. Dans ce cas , en utilisant le résultat de Madaule [Mad17] , le processus extrémal converge en loi vers un processus de Poisson décoré.

On constate une transition de phase : on passe d'un processus de Poisson ponctuel simple qui apparaît dans le cas du $GREM(N^\alpha)$, $\alpha \in [0,1)$, à un processus de Poisson ponctuel décoré dans le cas de la MAB (cas $\alpha = 1$). En partant de cette observation, et motivé par les travaux de Kiestler et Schmidt [SK15], on propose dans [Bel21] un modèle plus générale que celui du $GREM(N^\alpha), \alpha \in [0,1)$ qui interpole entre le REM et la MAB. Notre but est d'étudier cette transition de phase de façon plus précise, de déterminer lorsque la décoration disparaît.

Description du modèle Avant de décrire notre modèle, on va introduire les lois de déplacements et de reproductions associées. Soit $(Y_n)_{n\in\mathbb{N}}$ une marche aléatoire centrée telle que $\mathbf{V}\mathrm{ar}(Y_n) = 1$, qui représente la loi du déplacement dans notre modèle. Pour la loi de reproduction, on considère un processus de Galton-Watson $(Z_n, n \in \mathbb{N})$ que l'on suppose surcritique. On suppose aussi que Z_1 admet un moment d'ordre 2 i.e

$$\mathbf{E}(Z_1^2) < \infty. \tag{1.2.1}$$

Soit $k_n \leq n$ une suite d'entiers strictement positifs qui tend vers ∞ et $b_n = [\frac{n}{k_n}]$ la partie entière de $\frac{k_n}{n}$. On commence par un unique individu à l'origine à l'instant $n = 0$. Il se reproduit pendant b_n étapes selon un processus de Galton Watson. Puis, chaque descendant fait b_n étapes de déplacements de façon indépendante selon la loi Y_1. De sorte que, à la première génération, on obtient Z_{b_n} enfants positionnés selon des variables aléatoires i.i.d et qui ont la même loi que Y_{b_n}. Pour chaque $1 \leq k \leq k_n$, chaque individu répète le même processus de façon indépendante.

On note $\mathcal{T}^{(n)}$ l'arbre associé à ce modèle. On adopte les notations de l'arbre généalogique introduites dans la section 1.1. Pour chaque $1 \leq k \leq k_n$, on note

$$\mathcal{H}_k = \{u \in \mathcal{T}^{(n)}, |u| = k\}$$

l'ensemble des individus à la $k^{\text{ième}}$ génération. On définit $(X_u^{(n)}, u \in \mathcal{T}^{(n)})$ une famille de v.a i.i.d qui ont la même loi que Y_{b_n}. Pour $u \in \mathcal{T}^{(n)}$, on pose

$$S_u^{(n)} = \sum_{k=1}^{|u|} X_{u_k}^{(n)}$$

la marche aléatoire branchante associée.

Notons que les cas $k_n = n$ (respectivement $k_n = 1$) correspondent à la MAB (binaire à pas gaussien) (respectivement au REM). On s'intéresse au comportement asymptotique du processus extrémal associé à ce modèle

$$\mathcal{E}_n^{(b_n)} = \sum_{u \in \mathcal{H}_{k_n}} \delta_{S_u^{(n)} - m_n} \tag{1.2.2}$$

Dans [Bel21], on se met dans les deux situations suivantes :

$(\mathbf{H_1})$: Y_1 est une v.a gaussienne et $b_n \to \infty$ quand $n \to \infty$.

$(\mathbf{H_2})$: La fonction caractéristique $\varphi(\lambda) = \mathbb{E}\left(\exp(i\lambda Y_1)\right)$ du Y_1 satisfait la condition de Cramér, i.e

$$\limsup_{|\lambda| \to \infty} |\varphi(\lambda)| < 1,$$

and $\frac{b_n}{\log(n)^2} \to_{n\to\infty} \infty$ quand $n \to \infty$.

Théorème 1.2.3. *[Bel21, Théorème 1] Sous* $(\mathbf{H_1})$ *ou* $(\mathbf{H_2})$, *en posant*

$$m_n = k_n b_n v - \frac{3}{2\theta^*}\log(n) + \frac{\log(b_n)}{\theta^*},$$

le processus extrémal

$$\mathcal{E}_n^{(b_n)} = \sum_{u \in \mathcal{H}_{k_n}} \delta_{S_u^{(n)} - m_n}$$

converge en loi vers un processus de Poisson ponctuel d'intensité $\frac{1}{\sqrt{2\pi\sigma^2}} Z_\infty e^{-\theta^* x}$.

De plus, la loi du maximum centré par m_n *converge vers une loi de Gumbel shiftée par* $\frac{1}{\theta^*}\log(Z_\infty)$, *i.e*

$$\lim_{n\to\infty} \mathbf{P}(M_n \leq m_n + x) = \mathbf{E}\left(e^{-Z_\infty e^{-\theta^* x}}\right)$$

pour tout $x \in \mathbb{R}$.

Ce résultat montre que dès que $\frac{k_n}{n} \to_{n\to\infty} 0$, la décoration disparaît et on a convergence vers un processus de Poisson ponctuel simple. Le résultat de Kiestler et Schmidt [SK15] correspond au cas sous $(\mathbf{H_1})$ avec $k_n = N^\alpha$, $\alpha \in [0, 1)$, $Z_1 = 2$, et par suite $Z_\infty = 1$.

Ces résultats sont basés en grande partie sur l'application des deux résultats classiques suivantes : Le lemme *many-to-one* et le couplage Komlos-Major-Tusnady. Le lemme *many-to-one* est un outil fondamental dans l'étude de la MAB qui permet de simplifier les calculs des moments additifs de ce processus de branchement.

Proposition 1.2.1. *[KP76, Theorem 1.1] Pour tout $j \geq 1$ et pour toute fonction mesurable g : $\mathbb{R}^j \mapsto \mathbb{R}_+$, on a*

$$\mathbb{E}\left(\sum_{|u|=j} g((S_{u_i})_{1\leq i\leq j})\right) = \mathbb{E}\left(e^{-\theta^*\overline{T_i}} g((\overline{T_i} + ib_n v)_{1\leq i\leq j})\right).$$

avec $(\overline{T_i})_{i\geq 0}$ est une marche aléatoire centrée de variance jb_n.

Ce lemme a été introduit pour la première fois par Kahane et Peyrière [KP76] dans le but d'étudier les cascades multiplicatives.

Le résultat de Komlós-Major-Tusnàdy [KMT76] permet d'approximer des marches aléatoires à pas généraux à celle à pas gaussiens et cette approximation est de l'ordre de $\log(n)$. Dans notre contexte, on l'utilise pour lier certains estimés de marches aléatoires sous l'hypothèse (**H2**) à celles satisfaisant (**H1**).

Théorème 1.2.4 (Komlós-Major-Tusnàdy)**.** *Soit $(X_i)_{1\leq i\leq n}$ une suite i.i.d de variables aléatoires tels que $\mathbb{E}(X_i)) = 0$, $0 < \mathbb{E}(X_i^2) = \sigma^2 < \infty$ et $\mathbb{E}(\exp(\theta|X_i|)) < \infty$ pour un certain $\theta > 0$. Alors il existe une suite i.i.d de variables aléatoires gaussiennes $(Z_i)_{1\leq i\leq n}$ telles que pour tout $y \geq 0$*

$$\mathbb{P}\left(\max_{1\leq k\leq n} |\sigma^{-1}\sum_{i=1}^k X_i - \sum_{i=1}^k Z_i| > C\log(n) + y\right) \leq K\exp\left(-\lambda y\right)$$

où C, K et λ sont des constantes universelles strictement positives.

1.2.1 Questions ouvertes

Dans le modèle étudié dans [Bel21], on a montré que si on considère un *GREM* où le nombre de niveaux est une suite $k_n \leq n$ strictement positive tendant vers ∞, alors dès que $\frac{k_n}{n} \to_{n\to\infty} 0$, le processus extrémal défini dans (1.2.2) converge en loi vers un processus de Poisson ponctuel simple. Alors une question naturelle serait

- Montrer l'existence d'une famille $(\mathcal{D}^\alpha, \alpha \in (0,1])$ de processus ponctuels tels que si $\frac{k_n}{n} \to_{n\to\infty} \alpha \in (0,1]$, le processus extrémal défini dans (1.2.2) converge vers un processus de Poisson ponctuel de loi de décoration $\mathcal{D}^\alpha$. Notons que le cas $\alpha = 1$, correspondant à la marche aléatoire branchante classique a été étudié par Madaule [Mad17]. Des méthodes similaires à celles dans [Mad17] peuvent s'adapter pour répondre à cette question.

- Contrairement au MBB, on n'a pas une description explicite de la loi de décoration dans le cas de la MAB, donc une deuxième question serait de donner une description précise de la famille des processus de points $(\mathcal{D}^\alpha, \alpha \in (0,1])$ et d'étudier la continuité en le paramètre α de cette famille. On s'attend à ce que des méthodes similaires à celles utilisés dans [ABBS13, ABK13, BBCM20] pour le mouvement Brownien branchant permettent de répondre à ces questions.

1.2.2 Quelques modèles liés au GREM

Au delà de l'aspect biologique du *GREM*, plusieurs autres raisons issues de la physique statistique ou des mathématiques ont motivé l'étude du mouvement brownien branchant et de la marche aléatoire branchante.

Mécanique statistique et verres de spins. Les verres de spins sont des systèmes magnétiques désordonnés où les atomes sont en interaction aléatoire. Le premier modèle de verre de spin a été introduit par Sherrington et Kirkpatrick [She75] connu sous le nom de modèle SK à champs moyen. C'est un modèle comportant N éléments où chacun est associé à une valeur $+1$ où -1 nommé le spin de l'élément. On obtient alors 2^N configurations de spins. On note l'espace des configurations $\Sigma_N = \{+1,-1\}^N$. À chaque configuration $\sigma \in \Sigma_N$, on associe sa fonction d'énergie nommée le hamiltonien qui a l'expression suivante

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1 \le i \le j \le N} J_{ij}\sigma_i\sigma_j$$

où les J_{ij} sont des variables aléatoires i.i.d gaussiennes standards.

Les physiciens ont étudié les propriétés macroscopiques de ces modèles, comme l'énergie libre entropique et les valeurs extrêmes de ces énergies. Dans [She75], les auteurs ont obtenu une première formule de l'énergie libre par des méthodes d'approximations. Une description de ces travaux peut être trouvée dans le livre [MPV87]. Dans les années 2000, Talagrand [Tal06] a obtenu une formule de l'énergie libre avec des démonstrations rigoureuses.

Valeurs extrêmes et mécanique statistique. Parmi les quantités d'intérêts dans la mécanique statistique est la fonction de partition des énergies défini par

$$Z_N(\beta) = \sum_{\sigma \in \Sigma_N} e^{\beta H_N(\sigma)} \qquad \beta > 0, \tag{1.2.3}$$

où le paramètre $-\beta$ désigne la température inverse. À partir de cette quantité on définit l'énergie libre

$$f_N(\beta) = \frac{1}{N\beta} \log(Z_N(\beta)). \tag{1.2.4}$$

En particulier quand $N \to \infty$, la fonction de l'énergie libre donne des informations sur le premier ordre de maximum à travers les inégalités suivantes

$$\frac{\max H_N(\sigma)}{N} \le f_N(\beta) \le \frac{\log(2)}{\beta} + \frac{\max H_N(\sigma)}{N}$$

d'où on en déduit en particulier

$$\lim_{N\to\infty} \frac{\max H_N(\sigma)}{N} = \lim_{\beta\to\infty} \lim_{N\to\infty} f_N(\beta)$$

quand la limite existe. Pour déterminer cette limite, on peut considérer

$$\#\{\sigma \in \Sigma_N, H_N(\sigma) > EN\},$$

le nombre des configurations ayant des énergies dépassant le niveau EN pour une certaine valeur $E > 0$. Il est bien connu [Dav06] que l'entropie, définie par

$$S_N(E) = \frac{1}{N} \log(\#\{\sigma \in \Sigma_N, H_N(\sigma) > EN\})$$

converge en probabilité quand $N \to \infty$ et on a

$$S(E) := \lim_{N\to\infty} S_N(E) = \begin{cases} 0 & \text{si } E > E_c \\ \log(2) - \frac{E^2}{2} & \text{si } E \in [0, E_c] \end{cases}$$

où $E_c = \sqrt{2\log(2)}$. Un réarrangement des termes selon les valeurs de σ permet d'obtenir l'équivalent suivant

$$f_N(\beta) \sim_{N\to\infty} \frac{1}{N} \log\left(\sum_{0\leq E\leq E_c} \exp(\beta NE + NS(E))\right).$$

Par conséquent, une formule de l'énergie libre est donnée par

$$\lim_{N\to\infty} f_N(\beta) = \max_{0\leq E\leq E_c} \{\beta E + \log(2) - \frac{E^2}{2}\} = \begin{cases} \log(2) + \frac{\beta^2}{2} & \text{si } \beta \leq E_c \\ \beta E_c & \text{si } \beta > E_c. \end{cases} \tag{1.2.5}$$

On observe que l'énergie libre présente une transition de phase dont la valeur critique est $E = E_c = \sqrt{2\log(2)}$. Cette transition s'explique par le fait qu'au-delà de l'énergie maximal, il n'existe aucune configuration avec une énergie suffisante.

Champs libre gaussien bidimensionnel. On considère une partie finie A_n de $\mathbb{Z}^2$. Afin d'expliciter un lien avec la MAB, on suppose que $\#A_n = 2^n$. Soit $(S_k, k \geq 0)$ une marche aléatoire simple tel que $S_0 = v$. On note τ_n le premier instant où la marche aléatoire quitte A_n. On pose

$$G_n(v, v') = \mathbf{E}_v\left(\sum_{k=0}^{\tau_n} \mathbb{1}_{\{S_k = v'\}}\right) = \mathbf{E}(\varphi_v \varphi_{v'}),$$

en d'autres termes, $G_n(v, v')$ est le nombre moyen de passage en v' par la marche aléatoire en partant de v et avant de quitter A_n. Alors, le champ libre gaussien discret (2DGFF) sur A_n est le champ gaussien $(\varphi_v, v \in A_n)$ centré avec conditions de Dirichlet au bord (voir [BDG01] pour plus de détails sur ces conditions) et de fonction de covariance G_n. Il est bien connu [LL10] que

$$G_n(v, v') = \frac{2}{\pi} \log\left(\frac{2^n}{\|v - v'\|}\right) + o\left(\frac{1}{\|v - v'\|^2}\right).$$

On s'intéresse au comportement asymptotique de la quantité $\max_{v\in A_n} \varphi_v$. Parmi les outils qui facilitent l'étude de cette quantité est la MAB. En effet, on peut faire apparaître une structure de MAB dans le 2DGFF de la façon suivante. On identifie $(\varphi_v, v \in A_n) \approx (X_u, u \in \{u \in \mathcal{T}, |u| = n\})$ où X_u est la MAB binaire à pas gaussien de variance $\sigma^2 = \frac{\log(2)}{\pi} n$. En terme de verre de spins, A_n représente l'espace des configurations et $-\varphi_v$ l'énergie associé à la configuration v.

Bolthausen et al [BDG01] ont montré que

$$\frac{\max_{v\in A_n} \varphi_v}{n} \to_{n\to\infty} \sqrt{\frac{2}{\pi}}.$$

Ce résultat a été raffiné par Daviaud [Dav06] en étudiant le comportement asymptotique du logarithme normalisé du nombre de configurations ayant des énergies supérieures à $\lambda\sqrt{\frac{2}{\pi}}\log(2^n)$ pour $0 < \lambda < 1$. Il a montré que

$$\lim_{n\to\infty} \frac{1}{2^n} \log(\#\{v \in A_n, \varphi_v > \lambda\sqrt{\frac{2}{\pi}}\log(2^n)\}) = 1 - \lambda^2, \text{ en probabilité }.$$

Des résultats similaires à ceux de REM ont été obtenu [AZ15] en ce qui concerne le comportement asymptotique de l'énergie libre définie en (1.2.4) pour le DGFF. La loi du maximum a été étudiée par Ding, Roy, Zeitouni [DRZ17] pour une classe plus générale de processus log-corrélés discrets. Biskup et Louidor [MO20] ont prouvé la convergence du processus extrémal vers un processus de Poisson décoré pour des classe générales de champs log-corrélés.

Fonction zêta de Riemann et polynômes caractéristiques de matrices aléatoires. Récemment, la marche aléatoire branchante a joué un rôle important dans l'étude des valeurs extrêmes de la fonction zêta de Riemann et celles des polynômes caractéristiques de matrices aléatoires. La fonction zêta de Riemann est définie sur le demi plan $\mathrm{Re}(s) > 1$ par

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ premier}} (1-p^{-s})^{-1}.$$

Cette définition peut être prolongée sur le plan complexe avec un pole au point $s = 1$. Riemann a conjecturé en 1859 que les zéros non triviaux de la fonction zêta ont une partie réelle égale à $\frac{1}{2}$, cette conjecture est connue sous l'hypothèse de Riemann. Parmi les questions d'intérêts sont les valeurs maximales de la fonction zêta de Riemann sur la ligne critique. Fydorov et al [FK14] ont conjecturé que

$$\lim_{T\to\infty} \frac{\max_{h\in[0,1]} \log(|\zeta(1/2+i(\tau+h))|) - \log\log(T)}{\log\log\log(T)} = \frac{-3}{4}, \qquad \text{p.s.},$$

où τ est une variable aléatoire uniforme sur $[T, 2T]$. Autrement dit, le maximum sur un intervalle borné de la ligne critique vérifie

$$\max_{h\in[0,1]} \log(|\zeta(1/2+i(\tau+h))|) \approx \log\log(T) - \frac{3}{4}\log\log\log(T) + O(1).$$

Le premier ordre (respectivement le deuxième ordre) de maximum ont été obtenu par [Naj18, ABB+19] (respectivement par [ABH17]). Observons que cet ordre correspond exactement à celui de la MAB binaire à pas gaussien en prenant $\log(T) = 2^n$ et $\sigma^2 = \frac{\log(2)}{2}$. Donc il est naturel de faire apparaître une structure de MAB pour étudier ce problème. Une manière de réaliser cela est la suivante.

- Une première étape consiste à écrire

$$\log(|\zeta(1/2+it) = \mathrm{Re}\left(\sum_{p\leq T} \frac{1}{p^{1/2+it}}\right) + O(1).$$

- Remplacer t par $\tau + h$ où $h \in [0,1]$ et τ est une variable aléatoire uniforme sut $[T, 2T]$.

- En observant que la suite $(p^{-i\tau}, p$ premier) converge quand $T \to \infty$ vers une suite de v.a. indépendante uniformément distribué sur le cercle unité (voir [ABH17]), on se ramène à étudier le processus suivant

$$\sum_{p\leq T} \frac{\mathrm{Re}(U_p p^{-ih})}{p^{1/2}}, h \in [0,1]$$

  où $(U_p, p$ premier) une suite de v.a. indépendante de loi uniforme sur le cercle unité.

- On pose $\log(T) = 2^n$ et on écrit

$$\sum_{p\leq T} \frac{\mathrm{Re}(U_p p^{-ih})}{p^{1/2}} = \sum_{k=0}^{n} Y_k(h)$$

où

$$Y_k(h) = \sum_{2^{k-1}\leq \log(p)\leq 2^k} \frac{\mathrm{Re}(U_p p^{-ih})}{p^{1/2}}$$

et la suite des v.a $(Y_k(h), h \in [0,1])$ vérifie $\mathbf{E}(Y_k^2(h)) = \frac{\log(2)}{2}$ et

$$\mathbf{E}\left(Y_k(h)Y_k(h')\right) = \begin{cases} \frac{\log(2)}{2} & \text{si } |h-h'| \leq 2^{-k} \\ 0 & \text{sinon} \end{cases}. \tag{1.2.6}$$

D'après (1.2.6) on conclut que si on restreint à l'ensemble $H_n = \{0, \frac{1}{2^n}, \frac{2}{2^n}, .., \frac{2^n-1}{2^n}\}$, alors la suite $(X_n(h), h \in H_n)$ définit une MAB binaire à pas gaussien approximative (voir [ABH17]) pour plus de détails). En considèrent cette approche Arguin et al [ABH17] ont prouvé que

$$\max_{h\in[0,1]} \sum_{p\leq T} \frac{\mathrm{Re}(U_p p^{-ih})}{p^{1/2}} = (1+o(1)) \log\log(T).$$

Des résultats similaires ont été obtenus en étudiant les valeurs extrêmes du polynôme caractéristique $P_n(x)$ du $n \times n$ matrices aléatoires unitaires pour $x = e^{i\theta}$ sur le cercle unité (voir conjectures 15 et 16 de [FHK12]). On s'intéresse au comportement asymptotique de la quantité $\max_{h\in[0,1]} |P_n(e^{ih})|$. Il est connu que pour h fixé, la fonction $\log(|P_n(e^{ih})|)$ normalisée par $(\frac{1}{2}\log(n))^{1/2}$ converge vers une gaussienne standard [KS00]. Une approche qui utilise des techniques de MAB et une construction similaire à celle dans le contexte de la fonction zêta de Riemann a permis de répondre à cette conjecture (le lecteur pourra consulter [ABB17, PZ18, CZ20] par exemple).

La convergence du maximum de polynôme caractéristique des matrices aléatoires unitaires vers le chaos multiplicatif gaussien a été démontré par [Web15]. Par contre, la tension de maximum autour de sa médiane reste encore un problème ouvert.

Cascades multiplicatives. Au début des années 70, Mandelbrot [Man74] propose un modèle de mesure aléatoire basé sur une construction multiplicative élémentaire. Ce modèle (connu sous le nom du cascades multiplicative de Mandelbrot) a été introduit dans le but de simuler la dissipation d'énergie en turbulence intermittente. Une simple construction qui met en lien les cascades multiplicatives et la MAB est la suivante. Soit $\mathcal{A}$ l'ensemble des intervalles dyadiques sur $[0,1]$. On définit une mesure m comme suit : Si $\varepsilon_1, ...\varepsilon_n \in \{0,1\}$ on pose

$$I_{\varepsilon_1,..,\varepsilon_n} = [\sum_{i=1}^{n} \frac{\varepsilon_i}{2^i}, \sum_{i=1}^{n} \frac{\varepsilon_i}{2^i} + \frac{1}{2^n}] \in \mathcal{A}.$$

Soit Y une v.a. positive tel que $\mathbf{E}(Y) = 1$ et $U = \cup_{n\in\mathbb{N}}\{0,1\}^n$. Si on pose $(Y_\varepsilon)_{\varepsilon\in\mathcal{U}}$ une suite de v.a. i.i.d. de même loi que Y alors

$$m(I_{\varepsilon_1,..,\varepsilon_n}) = 2^{-n} Y_{\varepsilon_1} Y_{\varepsilon_1,\varepsilon_2} ... Y_{\varepsilon_1,..,\varepsilon_n}.$$

La mesure m est construite selon un principe multiplicatif : Si $I = I_{\varepsilon_1,..,\varepsilon_n} \in \mathcal{A}_n$ et $I^{'} = I_{\varepsilon_1,..,\varepsilon_n,0}$ et $I^{''} = I_{\varepsilon_1,..,\varepsilon_n,1}$ sont deux descendants de I à la $(n+1)^{\text{ème}}$ génération, alors

$$m(I^{'}) = m(I) \times 1/2 \times Y_{\varepsilon_1,..,\varepsilon_n,0} \qquad \text{et} \qquad m(I^{''}) = m(I) \times 1/2 \times Y_{\varepsilon_1,..,\varepsilon_n,1}.$$

Il est clair que par construction de la mesure m, une cascade multiplicative n'est autre que l'exponentielle d'une marche aléatoire branchante binaire avec loi de déplacement $\log(Y)$. Ce type de mesure aléatoire a permis l'apparition de la notion de mesure multifractale introduite par Frisch et Parisi [Fri85]. Les questions d'intérêts étaient autour le caractère multifractale de ces mesures aléatoires. Comme continuité des travaux de Mandelbrot sur l'étude d'énergie en turbulence committente, Kahane [Kah85] a introduit une nouvelle mesure aléatoire connue sous le nom de "chaos multiplicative gaussien".

Chaos multiplicatif gaussien. C'est une mesure aléatoire sur un domaine $D \subset \mathbb{R}^d, d \geq 1$ qui peut être définit comme suit. Soit $(X(x))_{x\in D}$ un champ gaussien centré de noyau de covariance

$$\mathbf{E}(X(x))\mathbf{E}(X(y)) = \log(\frac{1}{|x-y|}) + g(x,y), \qquad x, y \in D,$$

où $g : D \mapsto \mathbb{R}$ une fonction continue bornée. Vu que ce noyau de covariance présente des singularités, on considère le régularisé de champs X, qu'on note $X^{\varepsilon}, \varepsilon > 0$ et on définit le chaos multiplicatif gaussien comme la mesure

$$M_{\gamma}(dx) = \lim_{\varepsilon\to 0} e^{\gamma X^{\varepsilon}(x) - (\gamma^2/2)\mathbf{E}((X^{\varepsilon}(x))^2)}$$

pour un certain paramètre $\gamma > 0$. Il a été montré [RV10] que cette limite est dégénérée si et seulement si $\gamma < 2d$. Le chaos multiplicatif gaussien intervient de façon importante dans la théorie de la physique quantique, plus précisément dans la théorie de gravité de Liouville [Dup10].

Des résultats analogues à celle de la MAB ont été transférés au chaos multiplicatif gaussien, notamment l'introduction de la fonction suivante

$$M_t^{'}(A) = \int_A -(X_t(A) - \sqrt{2d}t)e^{\sqrt{2d}X_t(x) - dt}dx, \qquad A \subset \mathcal{B}_b(\mathbb{R}^d),$$

l'analogue de la martingale dérivée pour la MAB et MBB. Il a été montré par Duplancier, Rhode et al [DRSV14] que pour tout $A \subset \mathbb{R}^d$, la martingale dérivée $(M_t^{'}(A))_{t\geq 0}$ converge p.s. vers une variable aléatoire positive noté $M^{'}(A)$. Ce dernier objet intervient dans le comportement asymptotique du maximum $\mathbf{M}_t = \sup_{x\in[0,1]^d} X_t(x)$. Dans un premier temps, Bramson et Zeitouni [BZ12] ont montré la tension de la famille $(\mathbf{M}_t - \sqrt{2d}t + \frac{3}{2\sqrt{d}}\log(t))_{t\geq 1}$ et récemment Madaule [Mad15] a montré qu'il existe une constante $C^* > 0$ tel que

$$\lim_{t\to\infty} \mathbf{P}\left(M_t([0,1]^d) \leq \frac{-3}{2\sqrt{d}}\log(t) - z\right) = \mathbf{E}\left(e^{-C^* e^{\sqrt{2d}z} M^{'}([0,1]^d)}\right).$$

Autrement dit, le maximum $\mathbf{M}_t$ centré autour de sa médiane converge en loi vers une loi de Gumbel shifté par $\frac{1}{\sqrt{2d}}\log(M^{'}([0,1]^d))$.

Dans la partie suivante de la thèse, on s'intéresse à des modèles de branchement multitype. Plus précisément on va étudier les valeurs extrêmes du mouvement Brownien branchant réductible multitype.

1.3 Des modèles multi-types

Dans de nombreux modèles de populations, les individus peuvent être de types différents. Par exemple, en épidémiologie pour décrire la dynamique de la propagation des parasites qui changent de type à chaque mutation, les mathématiciens ont recours à des processus de branchements multi-types [AK71, DE07, YY10, VD18]. Lors de la modélisation du certaines maladies, de tels processus peuvent servir à décrire l'évolution des cellules qui ont vécu plusieurs mutations [Dur15]. En physique, les rayons cosmiques qui comportent des électrons produisant des photons et des photons produisant des électrons, peuvent être modélisés par un processus de Markov branchant à deux types [Mod71]. En outre, un grand nombre d'applications des processus de branchement multi-types en biologie peuvent être trouvés dans [HJV05, KA15].

De façon analogue aux processus de Galton-Watson avec un seul type, on a une version multitype de ce processus. Pour alléger les notations, nous allons considérer seulement deux types (type 1 et type 2), mais l'extension à un nombre fini de type est immédiate.

Défintion 1. *Si on considère $\boldsymbol{p}^{(i)}, i = 1, 2$ des lois de reproductions sur $\mathbb{N}^2$ associées à chaque type $i = 1, 2$. Alors un processus de Galton-Watson à deux types est une chaîné de Markov*

$$(\boldsymbol{Z}_n = (Z_n(1), Z_n(2)), n \geq 0)$$

à valeurs dans $\mathbb{N}^2$ vérifiant

$$\mathbf{Z}_{n+1} = \sum_{i=1}^{2} \sum_{k=0}^{Z_n(i)} \xi_{k,n}^i = \begin{pmatrix} Z_{n+1}(1) \\ Z_{n+1}(2) \end{pmatrix}$$

où $(\xi_{k,n}^i, 1 \leq i \leq 2, k \in \mathbb{N})$ sont des vecteurs aléatoires indépendants de loi $\mathbf{p}^{(i)}, i = 1, 2$. Le processus $(Z_n(i), n \geq 0)$ désigne le nombre d'individus de type i à la génération n.

Il est connu que l'extinction du processus de Galton-Watson multitype dépend da la valeur propre maximale de la matrice M dont les coefficients

$$M_{i,j} = \mathbf{E}(Z_1^{(i)}(j)), \quad 1 \leq i, j \leq 2$$

où $Z_1^{(i)}(j)$ désigne le nombre de descendant de type j à la génération 1 d'un individu de type i. Par le théorème de Perron-Frobenius, elle admet une valeur propre ρ strictement positive et maximal. De façon analogue au cas monotype, le processus multitype survit presque sûrement si et seulement si $\rho > 1$.

Marche aléatoire branchante à deux types. Une MAB à deux type est un système de particules qui peut être interprété comme suit. On commence par une unique particule de type $i \in \{1, 2\}$ située à l'origine. Elle donne naissance à un nombre aléatoire d'enfants selon le processus

$$(\mathbf{Z}_1^{(i)} = (Z_1^{(i)}(1), Z_1^{(i)}(2)), 1 \leq i \leq 2)$$

positionnés autour de leurs parents selon des processus de points indépendants. Puis chaque enfant répète le même processus indépendamment des autres particules. Dans la littérature, la MAB multitype est peu étudiée en la comparant à la version monotype qui a été l'objet d'étude d'un grand nombre de travaux, notamment en ce qui concerne les valeurs extrêmes associée. Pour des références pour la version multitype on peut consulter [BNT92, BR05, Big12].

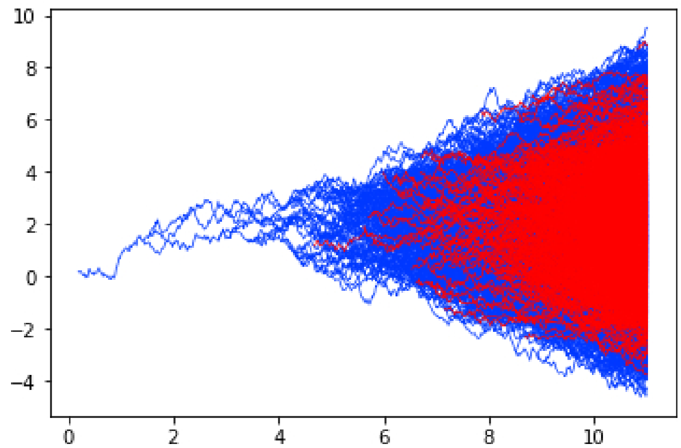


FIGURE 1.4 – Simulation du MBB à deux types réductible

Mouvement Brownien branchant à deux types réductible. Dans [BM21], on s'intéresse à la version au temps continu de la MAB à deux types. On étudie les valeurs extrêmes du mouvement Brownien branchant à deux types. C'est un système de particules qui peut être décrit de la façon suivante. Le processus commence par une unique particule de type 1 à l'origine. Elle se déplace selon un mouvement Brownien de variance σ_1^2. À taux β_1 elle donne naissance à 2 enfants de type 1 qui commencent à leurs tour deux mouvements Browniens branchants indépendants. Additionnellement, à taux α, elle donne naissance à un enfant de type 1 et un enfant de type 2. Les particules de type 2 se déplacent selon des mouvement Browniens de variance σ_2^2 et branchent à taux β_2 (Voir figure 1.4) en donnant naissance que à des enfants de type 2.

Notons que avec un changement d'échelle en espace-temps, on peut supposer, sans perte de généralité, que $\beta_2 = \sigma_2^2 = 1$.

Ce modèle a été introduit par Biggins [Big12] dans le cas de la marche aléatoire branchante à deux types réductible. Il a montré l'existence d'un phénomène d'**anomalous spreading** où les particules de type 2 vont envahir leurs environnements à une vitesse supérieur à celles des particules de type 1 ou de type 2 considérées seules.

Holzer [Hol14] a étudié le modèle introduit par Biggins en termes d'équations F-KPP. Il a conjecturé un diagramme de phase [Hol14, Figure 1] dépendant des paramètres (β, σ^2), tel que pour certain valeurs de (β, σ^2) le phénomène d'**anomalous spreading** apparaît. Notre but dans [BM21] est de confirmer ce diagramme de phase et d'étudier en détails le comportement asymptotique du processus extrémal des particules de type 2 dans le cas de MBB multitype réductible .

Le cas irréductible a été étudié dans [RY14]. C'est un système de particule où chaque type $i \in \{1, 2\}$ de particules a une probabilité non nulle d'avoir un descendant de type $j \in \{1, 2\}$ après un certain temps exponentiel. Dans ce cas, Ren et Yang ont montré des résultats similaires à ceux du MBB standard pour le comportement asymptotique du plus grand déplacement. Dans une autre direction, Blath, Jacobie et Nie [BJN21] ont étudié la vitesse d'une version modifiée du MBB standard, appelée On/Off BBM. C'est un mouvement Brownien branchant sur $\mathbb{R}$ tel que chaque particule possède un état actif ou dormant. Ce modèle peut être vu comme un MBB à deux types actif/dormant.

Dans la partie qui suit, on va discuter le modèle étudié dans [BM21]. Dans un premier temps, on va introduire le lien de mouvement brownien branchant avec les équations de réaction-diffusion. Puis on va exposer les résultats principaux du chapitre 3 et 4, et on termine par donner quelques motivations derrière notre travail.

1.3.1 Équations F-KPP

Cas monotype

Equation de réaction-diffusion F-KPP. L'équation de Fisher–Kolmogorov-Petrovskii-Piskonov (F-KPP) est une équation aux dérivées partielles semi-linéaire sous la forme

$$u_t = \frac{1}{2}u_{xx} + g(u) \tag{1.3.1}$$

où $u \mapsto g(u)$ est une de fonction $C^1([0,1])$ qui satisfait $g(0 = g(1) = 0,\ g(u) > 0,\ u \in]0,1[$ et $g'(0) = \beta > 0,\ g'(u) < \beta,\ u \in]0,1[$. Cette équation est fortement utilisée dans l'étude des phénomènes de propagation du front. Elle est présente aussi dans différents modèles liés à la biologie, l'écologie, génétique de la population et l'épidémiologie. Le terme $g(u)$ désigne la réaction et le $\frac{1}{2}u_{xx}$ représente la diffusion, d'ou le nom du réaction-diffusion.

L'étude de MBB avec un seul type est fortement liée à l'équation F-KKP. En effet, pour la fonction de réaction $g(u) = u(1-u)$ on a la correspondance suivante.

Proposition 1.3.1. *Soi $f : \mathbb{R} \mapsto [0,1]$ une fonction mesurable bornée. Alors, la fonction*

$$u(t,x) = \mathbf{E}\left(\prod_{u\in\mathcal{N}_t} f(x - X_u(t))\right)$$

est solution de l'équation (1.3.1) avec une condition initiale $u(0,x) = f(x)$.

En particulier, si on pose $M_t = \max_{u\in\mathcal{N}_t} X_u(t)$ et $f(x) = \mathbb{1}_{\{x\geq 0\}}$, alors

$$u(t,x) = \mathbf{E}\left(\prod_{u\in\mathcal{N}_t} \mathbb{1}_{\{x-X_u(t\geq 0)\}}\right) = \mathbf{E}\left(\mathbb{1}_{\{M_t\leq x\}}\right) = \mathbf{P}(M_t \leq x)$$

est solution de (1.3.1). Grâce à cette cette observation, il a été montré [McK75] que ce type d'équation admet des solutions de type onde progressive "travelling wave". En particulier, il existe une fonction $t \mapsto m_t$ tel que

$$u(t, m_t + x) \to_{t\to\infty} w(x)$$

uniformément pour $x \in \mathbb{R}$, de plus la fonction $x \mapsto w(x)$ est la solution "**travelling wave**" la plus lente de l'équation F-KPP, qui vérifie l'équation

$$1/2w_{xx} + \sqrt{2}w_x + w(w-1) = 0. \tag{1.3.2}$$

Partant de cette observation, Bramson [Bra78], a montré que

$$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t) + O(1).$$

Cas multitype

De façon analogue au MBB standard, le modèle multitype que l'on considère dans [BM21] , peut être associé à un système d'équations de réaction-diffusion. Soit $f, g : \mathbb{R} \mapsto [0,1]$ deux fonctions mesurables, et on définit pour tout $x \in \mathbb{R}$

$$
\begin{aligned}
u(t,x) &= \mathbf{E}^{(1)} \left(\prod_{u \in \mathcal{N}_t^1} f(X_u(t)+x) \prod_{u \in \mathcal{N}_t^2} g(X_u(t)+x) \right) \\
v(t,x) &= \mathbf{E}^{(2)} \left(\prod_{u \in \mathcal{N}_t^1} f(X_u(t)+x) \prod_{u \in \mathcal{N}_t^2} g(X_u(t)+x) \right) = \mathbf{E}^{(2)} \left(\prod_{u \in \mathcal{N}_t^2} g(X_u(t)+x) \right)
\end{aligned} \tag{1.3.3}
$$

où $\mathbf{P}^{(1)}$ (respectivement $\mathbf{P}^{(2)}$) désigne la loi du MBB multitype partant d'une particule de type 1 (resp. 2). Dans (1.3.3), on a utilisé le fait que notre modèle est réductible, i.e les particules de type 2 ne donnent naissance que à des particules de type 2.

Notons que sous $\mathbf{P}^{(2)}$, le processus multitype se comporte comme un MBB standard, et dans ce cas v est solution de l'équation F-KPP classique

$$
\begin{cases}
\partial_t v = \frac{1}{2} v_{xx} - v(1-v) \\
v(0,x) = g(x).
\end{cases}
$$

Pour obtenir l'équation F-KPP qui satisfait u, on observe que, sous $\mathbf{P}^{(1)}$, l'un des trois événements suivants se produit pendant les premières dt unités de temps.

- Avec probabilité $\beta dt + o(dt)$, la particule originale de type 1 se divise en 2 particules de type 1 qui commencent à leurs tours deux MBBs i.i.d de loi $\mathbf{P}^{(1)}$, issus de leurs positions de naissance.
- Avec probabilité $\alpha dt + o(dt)$, la particule originale se divise une particule de type 1 et une particule de type 2 qui commencent deux MBBs indépendants de lois $\mathbf{P}^{(1)}$ et $\mathbf{P}^{(2)}$.
- Avec probabilité $1 - (\alpha + \beta)dt$, le particule de type 1 diffuse selon un MBB de variance σ^2.

On obtient l'équation suivante

$$
\begin{aligned}
u(t+\mathrm{d}t, x) &= \beta \mathrm{d}t u(t,x)^2 + \alpha \mathrm{d}t u(t,x) v(t,x) + (1 - (\beta+\alpha)\mathrm{d}t) \mathbf{E}\left(u(t, x - \sigma B_{\mathrm{d}t})\right) + o(\mathrm{d}t) \\
&= u(t,x) + \mathrm{d}t \left(\frac{\sigma^2}{2} \Delta u(t,x) - \beta u(1-u) - \alpha u(1-v) \right).
\end{aligned}
$$

Par conséquent, le couple (u, v) est solution du couple d'équations aux dérivées partielles.

$$
\begin{cases}
\partial_t u = \frac{\sigma^2}{2} \Delta u - \beta u(1-u) - \alpha u(1-v) \\
\partial_t v = \frac{1}{2} \Delta v - v(1-v) \\
u(0,x) = f(x), \quad v(0,x) = g(x).
\end{cases} \tag{1.3.4}
$$

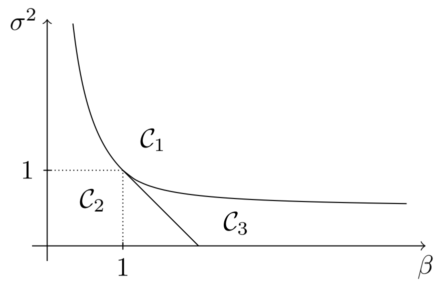


FIGURE 1.5 – Diagramme de phase de MBB réductible à deux types.

1.3.2 Résultats principaux du troisième et quatrième chapitre

Dans [BM21] on montre l'existence de trois zones $(C_i)_{1\leq i\leq 3}$ comme le montre la figure 1.5.

Chaque région dans 1.5 correspond à un comportement asymptotique différent pour le processus extrémal du MBB multitype. Avant d'étudier le processus extrémal, une première étape consiste à déterminer la vitesse du MBB multitype réductible.

Vitesse du MBB multitype

Biggins [Big10] a donné une description de la vitesse dans le cas de la MAB multitype réductible en terme d'un problème d'optimisation. Dans le contexte du mouvement brownien branchant réductible à deux types, on a

$$v = \max\left\{pa + (1-p)b : p \in [0,1],\ \ p(\frac{a^2}{2\sigma^2} - \beta) \leq 0,\ \ p(\frac{a^2}{2\sigma^2} - \beta) + (1-p)(\frac{b^2}{2} - 1) \leq 0\right\} \quad (1.3.5)$$

Ce problème d'optimisation permet de décrire la trajectoire de la particule typique menant à la position maximale. En effet, en utilisant des arguments de larges déviations, on sait que, pour $p \in [0,1]$, on a

- Si $a \leq \sqrt{2\beta\sigma^2}$, avec une grande probabilité on a environ $e^{-pt(\frac{a^2}{2\sigma^2}-\beta)(1+o(1))}$ particules de type 1 à droite de la position pta à l'instant pt.
- Si $b \geq \sqrt{2}$, la probabilité qu'une particule de type de 2 ait un descendant à droite de la position $(1-p)bt$ au bout d'un temps $(1-p)t$ est de l'ordre de $e^{(1-p)(1-b^2/2)t(1+o(1))}$.
- Pour tout triplet (p, a, b) tel que, $a \leq \sqrt{2\beta\sigma^2}, b > \sqrt{2}$ et

  $$p \in [0,1],\ p\left(\frac{a^2}{2\sigma^2} - \beta\right) \leq 0,\ p\left(\frac{a^2}{2\sigma^2} - \beta\right) + (1-p)\left(\frac{b^2}{2} - 1\right) \leq 0,$$

  on sait que avec une grande probabilité on a environ

  $$\exp\left[-t(1+o(1))\left(p(\frac{a^2}{2\sigma^2} - \beta) + (1-p)(\frac{b^2}{2} - 1)\right)\right]$$

  particules de type 2 à droite de la position $(pa + (1-p)b)t$ à l'instant t.

Si on note (p^*, a^*, b^*) le triplet qui optimise (1.3.5), alors on a les trois situations suivantes :

- Si $(\beta, \sigma^2) \in C_1$, alors $p^* = 1$ et $a^* = \sqrt{2\beta\sigma^2}$ qui est la vitesse de MBB à deux types, et correspond à la vitesse des particules de type 1 considérées seules. Dans cette situation, avec une grande probabilité, une particule de type 2 qui finit au voisinage de la position maximale est née d'une particule de type 1 à la fin du processus, i.e, à un temps $t - O(1)$.

- Si $(\beta, \sigma^2) \in C_2$, alors $p^* = 0$ et $b^* = \sqrt{2}$ qui est égale à la vitesse des particules dans un MBB de type 2. Dans ce cas, avec une grande probabilité, une particule de type 2 qui finit au voisinage du maximum est née d'une particule de type 1 au début de processus, i.e, à un temps $O(1)$.

- Si $(\beta, \sigma^2) \in C_3$, alors

$$p^* = \frac{\sigma^2 + \beta - 2}{2(1-\sigma^2)(\beta - 1)}, \quad a^* = \sigma^2 \sqrt{2\frac{\beta - 1}{1-\sigma^2}} \text{ and } b^* = \sqrt{2\frac{\beta-1}{1-\sigma^2}}.$$

  Dans cette situation la vitesse de processus multitype est égale

$$v = p^* a + (1-p^*) b = \frac{\beta - \sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}} > \max(\sqrt{2}, \sqrt{2\beta\sigma^2}).$$

  C'est la zone où on a une invasion anormale des particules de type 2. Dans ce cas, le problème d'optimisation (1.3.5) montre que, avec une grande probabilité, une particule de type 2 finissant au voisinage du maximum devrait satisfaire les propriétés suivantes.

  - Elle est née d'une particule de type 1 au voisinage de la position $p^* a^* t$ et de l'instant $p^* t$.
  - Elle a une probabilité environ $e^{(1-p^*)(1-(b^*)^2/2)t(1+o(1))}$ d'avoir un descendant à droite de la position vt au temps t.

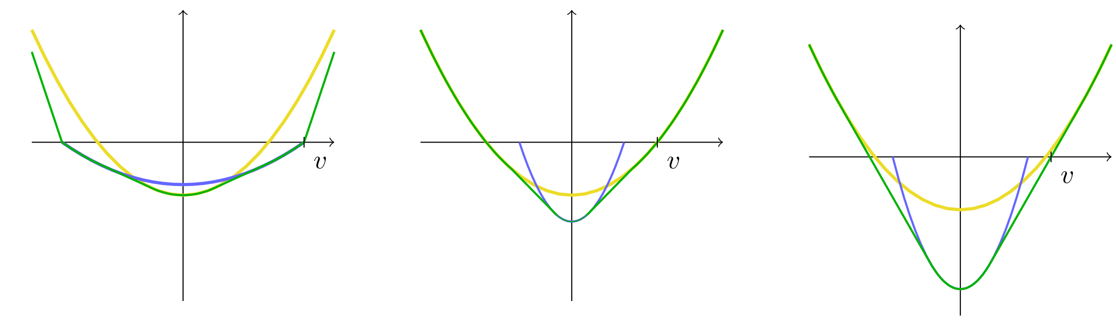
Biggins [Big12] a également montré que la vitesse v du processus multitype correspond au point d'intersection du plus grand minorant convexe des fonctions taux des particules des type 1 et 2 avec l'axe des abscisses (voir figure 1.6). On rappelle que la fonction taux des particules de type 1 (repectivement de type de 2) est égale $x \mapsto x^2/2\sigma^2 - \beta$ (respectivement $y \mapsto y^2/2 - 1$). On note g la fonction taux du processus multitype.

Dans la partie qui suit, on va discuter les résultats sur las valeurs extrêmes du MBBM réductible obtenus dans [BM21], mais avant ça, on commence par introduire quelques notations associées à notre modèle.

Notations On note $\mathcal{N}_t$ l'ensemble des particules en vie à l'instant t du processus multitype, $\mathcal{N}_t^1$ et $\mathcal{N}_t^2$ l'ensemble des particules de type 1 (respectivement de type 2). On pose $X_u(t)$ la position au temps t d'une particule $u \in \mathcal{N}_t$. Pour $u \in \mathcal{N}_t^2$, on note $T(u)$ le premier instant à lequel la première particule de type 2 est née.

Domination des particules de type 1

Le premier résultat décrit le comportement asymptotique du processus multitype dans la phase C_1. Dans ce cas, le processus extrémal du MBB à deux types est similaire à celui d'un MBB standard où on a seulement des particules de type 1.



(a) Cas I : Les particules de type 1 sont dominantes (b) Cas II :Les particules de type 2 sont dominantes (c) Cas III :Invasion anormale

FIGURE 1.6 – L'enveloppe convexe g pour (β, σ^2) dans chaque région. Les fonctions taux des particules de type 1 et 2 sont dessinées en bleu et jaune respectivement, la fonction g du processus multitype est dessinée en vert.

Théorème 1.3.1. *Si on note* $m_t^{(1)} = \sqrt{2\beta\sigma^2}t - \frac{3}{2\sqrt{2\beta/\sigma^2}}\log(t)$, *alors ils existe une constante* $c_{(1)} > 0$ *tel que*

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t^{(1)}} = \mathcal{E}_\infty^{(1)} \quad \text{en loi pour la topologie de la convergence vague,}$$

où $\mathcal{E}_\infty^{(1)}$ *est un processus de Poisson ponctuel décoré d'intensité* $\sqrt{2\beta/\sigma^2}c_{(1)}Z_\infty e^{-\sqrt{2\beta/\sigma^2}x}\mathrm{d}x$ *où* Z_∞ *est la limite de la martingale dérivée introduite dans* (1.1.5).

De plus, on a $\lim_{t\to\infty} \mathbf{P}(M_t \leq m_t^{(1)} + x) = \mathbf{E}\left(e^{-c_{(1)}Z_\infty e^{-\sqrt{2\beta/\sigma^2}x}}\right)$ *pour tout* $x \in \mathbb{R}$.

Notons que pour ce résultat, on n'a pas une description explicite de la loi décoration et qui dépend implicitement des paramètres α, β et σ^2.

Ce théorème décrit la domination des particules de type 1 dans la phase 1. Pour le prouver, on montre que :

- Avec une grande probabilité, les particules de type 2 qui contribuent au processus extrémal sont nées à un temps $t - O(1)$.
- Le processus extrémal est porté par le comportement des particules de type 1 et la loi du processus extrémal est obtenue en utilisant le principe du superposition décrit par Maillard [Mai13].
- Les descendants de type 2 qui sont nées au temps $t-O(1)$ présentent une décoration supplémentaire du processus extrémal.

Domination des particules de type 2

Si $(\beta, \sigma^2) \in C_2$, le processus extrémal du MBB à deux types est similaire à celui d'un MBB standard où on a seulement des particules de type 2.

Théorème 1.3.2. *Dans la phase* C_2*, en posant* $\mathfrak{D}$ *la loi de décoration définie dans* (4.1.5) *et si on note* $m_t^{(2)} = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t)$*, alors il existe une constante* $c_2 > 0$ *tel que*

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t^{(1)}} = \mathcal{E}_\infty^{(2)} \quad \text{en loi pour la topologie de la convergence vague,}$$

où $\mathcal{E}_\infty^{(2)}$ *est un processus de Poisson ponctuel décoré d'intensité* $\sqrt{2\beta/\sigma^2}c_2\overline{Z}_\infty e^{-\sqrt{2\beta/\sigma^2}x}\mathrm{d}x$ *et telle que* $\overline{Z}_\infty$ *est la somme de copies i.i.d. de la limite de la martingale dérivée associée aux particules de type* 2*.*

De plus, on a $\lim_{t\to\infty} \mathbf{P}(M_t \le m_t^{(2)} + x) = \mathbf{E}\left(e^{-c_{(2)}\overline{Z}_\infty e^{-\sqrt{2}x}}\right)$ *pour tout* $x \in \mathbb{R}$*.*

Contrairement à ce qui passe dans la phase 1, on montre que :

- Avec une grande probabilité les particules de type 2 qui contribuent au processus extrémal sont nées au début du processus, i.e à un temps $O(1)$.
- Le processus extrémal du MBBM est obtenu par superposition d'un nombre fini de MBBs de type 2 descendants des particules de type 1 au début du processus.

Anomalous spreading

Arrivons maintenant à la troisième situation qui est la plus importante dans notre travail où $(\beta, \sigma^2) \in C_3$. C'est le cas où on observe le phénomène d'**anomalous spreading**. Le processus extrémal dans ce cas est formé par des particules de type 2 qui vont à une vitesse supérieure à celle des particules de type 1 et 2 considérées tous seuls dans un MBB standard.

Théorème 1.3.3 (Anomalous spreading)**.** *Si* $(\beta, \sigma^2) \in \mathcal{C}_3$*, alors si on pose*

$$m_t^{(3)} = \frac{\sigma^2 - \beta}{\sqrt{2(1-\sigma^2)(\beta-1)}}t \quad \text{and} \quad \theta = \sqrt{2\frac{\beta-1}{1-\sigma^2}},$$

alors

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t^{(3)}} = \mathcal{E}_\infty^{(3)} \quad \text{en loi pour la topologie de convergence vague,}$$

où $\mathcal{E}_\infty$ *est un processus de Poisson ponctuel décoré qu'on note* $DPPP(\theta C_3 W_\infty(\theta)e^{-\theta x}\mathrm{d}x, \mathfrak{D}^{(3)})$ *avec*

- $W_\infty(\theta) = \lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^1} e^{\theta X_u(t) - t(\beta+\theta^2\sigma^2/2)}$ *est la limite de la martingale additive de MBB des particules of type* 1*.*
- $C_3 = \frac{\alpha C(\theta)}{2(\beta-1)}$ *où la constante* C *est défini dans* (3.3.16)*,*
- $\mathfrak{D}^{(3)}$ *est la loi de la mesure ponctuelle* $\mathcal{D}^\theta$ *défini dans* (3.3.19)*.*

Remarque 1.3.4. Contrairement à ce qui se passe dans les zones C_1 et C_2, on observe que dans le cas du l'invasion anormale, la médiane du maximum de plus grand déplacement des particules de type 2 ne contient pas une correction logarithmique. Une autre observation est au niveau de la loi du processus extrémal. En effet, dans le théorème 1.3.3, on observe une apparition d'une martingale additive dans la loi limite du processus multitype ce qui est différent à ce qu'on observe dans les phase C_1 et C_2) où le processus extrémal est shifté par une martingale dérivée.

Le cas $\overline{C_1} \cap \overline{C_2} \cap \overline{C_3} = \{(1,1)\}$

La quatrième partie de la thèse est en continuité avec l'article [BM21] où on étudie le comportement asymptotique du processus extrémal du mouvement brownien branchant multityoe réductible dans la phase $\overline{C_1} \cap \overline{C_2} \cap \overline{C_3}$, i.e dans le cas où $\beta = \sigma^2 = 1$.

Le résultat principal du chapitre 4 est :

Théorème 1.3.5. *Si $\beta = \sigma^2 = 1$, en posant $m_t = \sqrt{2}t - \frac{1}{2\sqrt{2}} \log(t)$, on a*

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t} = \widehat{\mathcal{E}}_\infty \quad \textit{pour la topologie de la convergence vague,}$$

où $\widehat{\mathcal{E}}_\infty$ est un processus de Poisson ponctuel décoré d'intensité $C^*\alpha\sqrt{2}Z_\infty e^{-\sqrt{2}x}dx$, ou la constante C^* est celle introduite (1.1.4) et

$$Z_\infty := \lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^1} (\sqrt{2}t - X_u(t))e^{\sqrt{2}X_u(t)-2t} \quad p.s.$$

De plus, on a $\lim_{t\to\infty} \mathbf{P}(M_t \le m_t + x) = \mathbf{E}\left(e^{-\alpha C^* Z_\infty e^{-\sqrt{2}x}}\right)$ pour tout $x \in \mathbb{R}$.

Par le théorème 4.1.1, on montre que

- Les particules de type 2 qui contribuent au processus extrémal peuvent naître à tout instant au voisinage de la position du maximum.
- Le processus extrémal est obtenu en utilisant la théorie classique des valeurs extrêmes. La loi de décoration est celle d'un MBB standard de type 1 introduite dans (4.1.5).

Remarque 1.3.6. On remarque un changement au niveau de la correction logarithmique de la médiane m_t en la comparant à celle d'un mouvement brownien branchant classique. De façon plus précise, on passe d'un facteur de correction logarithmique $\frac{-3}{2\sqrt{2}}$ dans le cas d'un MBB standard à un facteur $\frac{-3}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$ dans le cas d'un MBB à deux types réductible avec $\beta = \sigma^2 = 1$. Une simple observation de cette transition phase est que le nombre de type ainsi que le changement d'un type 1 au type 2 des particules dans le MBBM réductible ont une influence sur la correction logarithmique de la médiane.

Une extension de ce modèle consiste à considérer le MBB avec un nombre fini $k \ge 2$ de type. C'est un système de particules qui peut être interpréter comme suit. On commence par une unique particule de type 1 à l'origine. Elle se déplace selon un mouvement Brownien standard.

- À taux 1, une particule de type i, $i \ge 1$ donne naissance à 2 enfants de type i qui commencent à leurs tours deux mouvements Browniens branchants indépendants.
- À taux α, elle donne naissance à un enfant de type i et un enfant de type $i+1$.

De façon analogue au modèle à deux types, pour tout $t \ge 0$ on note $\mathcal{N}_t^{(i)}$, $i \ge 1$ l'ensemble des particules de type i envie à l'instant t. Si $u \in \mathcal{N}_t^{(i)}$, on note $X_u(t)$ la position de la particule u à l'instant t et $M_t^{(i)} = \max_{u\in\mathcal{N}_t^{(i)}} X_u(t)$ la position de la particule la plus à droite. Fixons $k \ge 2$. On s'intéresse au comportement asymptotique du processus extrémal des particules de type k définit par

$$\widehat{\mathcal{E}}_t^{(k)} = \sum_{u\in\mathcal{N}_t^{(k)}} \delta_{X_u(t)-m_t^{(k)}}.$$

On conjecture le résultat suivant.

Conjecture 1.3.2. *Posons* $m_t^{(k)} = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t) + \frac{k-1}{\sqrt{2}}\log(t)$, *alors on a*

$$\lim_{t\to\infty} \widehat{\mathcal{E}}_t^{(k)} = \widehat{\mathcal{E}}_\infty^{(k)} \quad \text{pour la topologie de la convergence vague,}$$

ou $\widehat{\mathcal{E}}_\infty$ *est un processus de Poisson ponctuel décoré process d'intensité* $C^*\frac{\alpha^{k-1}}{(k-1)!}\sqrt{2}Z_\infty e^{-\sqrt{2}x}dx$, *ou la constante* C^* *est celle introduite* (1.1.4) *et*

$$Z_\infty := \lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^{(1)}} (\sqrt{2}t - X_u(t))e^{\sqrt{2}X_u(t)-2t} \quad a.s.$$

De plus $\lim_{t\to\infty} \mathbf{P}(M_t^{(k)} \leq m_t + x) = \mathbf{E}\left(e^{-\frac{\alpha^{k-1}}{(k-1)!}C^* Z_\infty e^{-\sqrt{2}x}}\right)$ *pour tout* $x \in \mathbb{R}$.

1.3.3 Questions ouvertes

Notons que ce modèle est très riche et qu'on peut poser plusieurs questions sur ce sujet.

Par exemples, parmi les questions qu'on pourrait étudier :

- Dans [BM21], on a étudié les valeurs extrêmes du MBBM réductible dans les zones $C_1 \cup C_2 \cup C_3$, par contre, on n'a pas traité le cas où (β, σ^2) appartient aux interfaces entre $(C_i)_{1\leq i\leq 3}$. Donc une première question serait d'étudier le comportement asymptotique du processus extrémal sur chaque interface.

  Dans la zone $\overline{C_1} \cap \overline{C_3}$, on conjecture le résultat suivant pour le comportement asymptotique du processus extrémal.

  Conjecture 1.3.3. *Supposons que* $\beta > 1$ *et* $\sigma^2 = \frac{\beta}{2\beta-1}$, *alors ils existent* $c > 0$ *et* $\widetilde{\mathfrak{D}}$ *tel que*

  $$\sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-\sqrt{2\beta\sigma^2}t+\frac{1}{\sqrt{2\beta/\sigma^2}}\log t}$$

  converge en loi vers DPPP$(cZ_\infty e^{-\sqrt{2\beta/\sigma^2}x}dx, \widetilde{\mathfrak{D}})$.

  En effet, dans cette situation, les particules de type 2 contribuant au processus extrémal devraient satisfaire $t - T(u) = O(t^{1/2})$. Par conséquent, le processus extrémal est shifté par la martingale dérivée des particules de type 1, et la loi de la décoration est celle d'un MBB de type 2 conditionné à voyager à une vitesse $\sqrt{2\beta\sigma^2} > \sqrt{2}$.

  De façon similaire, dans la phase $\overline{C_2} \cap \overline{C_3}$ on conjecture le résultat suivant.

Conjecture 1.3.4. *Supposons que $\beta > 1$ et $\sigma^2 = 2-\beta$, alors ils existent $c > 0$ et une variable aléatoire $\widetilde{Z}$ tel que*

$$\sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-\sqrt{2}t+\frac{1}{\sqrt{2}}\log t}$$

converge en loi vers PPPD($c\widetilde{Z}e^{-\sqrt{2}x}dx, \mathfrak{D}$).

Dans ce cas, les particules de type 2 qui contribuent au processus extrémal doivent satisfaire $T(u) = O(t^{1/2})$.

- Étudier le processus extrémal de mouvement Brownien branchant multitype irréductible. Comme mentionné ci-dessus, Ren et Yang [RY14] ont étudié le comportement asymptotique du plus grand déplacement de MBB multitype irréductible. C'est un système de particules où pour toute paire de types i et j, les particules de type i ont une probabilité positive d'avoir au moins un descendant de type j après un certain temps. Le plus grand déplacement dans ce cas est similaire à celui d'un MBB monotype. On s'attend aussi à ce que le comportement asymptotique du processus extrémal soit similaire à celui d'un MBB standard.

- D'autres généralisations du modèle que nous considérons dans l'article [Bel21] pourraient être envisagées. On s'attend à ce qu'un mouvement brownien branchant multitype réductible avec un nombre fini de types présente un comportement similaire. Une autre extension possible consiste à considérer un mouvement brownien branchant à deux types réductible où les particules de type 1 et de type 2 donnent naissance à un nombre aléatoire d'enfants, soit L_1 pour les particules de type 1 et L_2 pour les particules de type 2. Dans ce cas, une condition naturelle à imposer aux lois de reproduction pour obtenir le comportement asymptotique observé dans le théorème 1.3.3 est
$$\mathbf{E}(L_1 \log L_1) + \mathbf{E}(L_2 \log L_2) < \infty.$$

1.3.4 Modèles liés au mouvement Brownien branchant multitype

Dans la dernière partie de cette introduction, on va donner quelques modèles et motivations derrière l'étude du mouvement brownien branchant multitype réductible. En particulier, on va introduire des modèles dont on observe le phénomène d'**anomalous spreading**.

Invasion des crapauds-Buffles en Australie Dans le début des années 1930, le crapaud buffle (Bufo-Marinus) a été introduit dans l'est du continent Australien dans le but de lutter contre les parasites et les insectes nuisibles aux cultures agricoles. Puis, ces crapauds ont commencé à envahir le sol Australien pour atteindre en 2000 la ville de Darwin au Nord de l'Australie. Les biologistes ont remarqué que la vitesse d'invasion a été multipliée par 5 par rapport le début de l'invasion. De plus les crapauds les plus avancés sont en moyenne plus gros et plus endurants.

Pour comprendre ce phénomène d'invasion, les mathématiciens ont recours à l'équation F-KPP introduite dans (1.3.1). Ce type d'équation met en évidence deux phénomènes : la diffusion des individus et leur reproduction au cours de temps. Kolmogorov [Kol37] a montré que le phénomène d'invasion se réalise à une vitesse constante. Mais ce n'est pas le cas pour les crapauds Australiens qui envahissent leur environnement avec une vitesse progressive. Une explication simple est la suivante. Les crapauds les plus endurants peuvent se disperser plus loin dans l'espace. La reproduction et la transmission génétique favorisent la montée des individus les plus dispersifs sur le front de propagation.

Une équation de type F-KPP a été introduite par [BC14] qui modélise ce phénomène et qui prend en compte que les individus ne se propagent pas à la même vitesse ainsi que l'effet de la mutation génétique. Dans [BC14], Calvez et Bouin considèrent une population structurée en trait phénotypique indexée par un ensemble $\Theta \subset \mathbb{R}$ qui représente l'espace des traits. Lorsque Θ est borné, ils ont montré que la propagation des espèces se réalise à une vitesse constante. Dans le cas où Θ est non borné, la propagation se réalise à une vitesse sur-linéaire, plus précisément on a un phénomène d'invasion anormale (voir [BCM+12, BMG15, BHR17]). Finalement, on estime que ce type de modèle peut être vu comme un MBBM et qu'on peut avoir des résultats intéressants sur le valeurs extrêmes associées à ce modèle.

Invasion anormale chez les homozygotes Ce type de modèle a été introduit par [AW75, AW78]. On considère une famille de population diploïde, i.e l'information génétique chez chaque individu est dédoublée. On suppose qu'un certain gène sur chaque paire de chromosome possède deux formes possibles d'allèles qu'on les note "a" et "A". La population se divise alors en individus "homozygotes" de type "aa" où "AA" et hétérozygotes de type "aA". On suppose aussi que les gamètes des homozygotes de type "AA" ne croisent pas avec les autres types de génotype. On note $u(x,t)$ la densité des individus hétérozygotes de type "aA" au position x à l'instant t et respectivement $v(x,t)$ la densité des individus homozygotes de type "aa" à la position x à l'instant t. Supposons que les croisements entre les hétérozygotes et les homozygotes de type "aa" sont rares. Alors ce modèle peut être modélisé par le système d'équations suivant :

$$\begin{cases} \partial_t u = u_{xx} + u(1-2u) + (uv)^2 \\ \partial_t v = \frac{1}{4} v_{xx} + u + v(18-14u) + (vu)^2. \end{cases} \tag{1.3.6}$$

Le terme $\frac{1}{4}$ vient du fait que le croisement entre les hétérozygotes de type "aA" produit une proportion d'homozygotes de type "aa" égale à $\frac{1}{4}$. Pour éviter les termes non-linéaires du système (1.3.6), Aronson et Weinberger [AW78] ont étudié le système réductible linéarisé suivant

$$\begin{cases} \partial_t u = u_{xx} + u \\ \partial_t v = \frac{1}{4} v_{xx} + u + 12v \end{cases} \tag{1.3.7}$$

Ils ont montré que pour ce type de modèle, les homozygotes se propagent à une vitesse "anormale" qui est supérieure à celle des hétérozygotes et des homozygotes considérés tous seuls. Leurs explications est la suivante. La densité des hétérozygotes $u(x,t)$ a des queues exponentiels au delà du front, alors une faible densité des hétérozygotes en avant du front produit une accélération anormale dans la propagation de l'un des homozygotes.

Revenons maintenant au système non linéarisé (1.3.6). Observons que les équations dans (1.3.6) ressemblent à les équations de type F-KPP mais ne le sont pas. Les auteurs dans [AW75] ont montré que sous certaines conditions, la densité d'un certain allèle dans la population totale vérifie approximativement une équation F-KPP.

Chapitre 2

A generalized model interpolating between the random energy model and the branching random walk

Abstract

We study a generalization of the model introduced in [SK15] that interpolates between the random energy model (REM) and the branching random walk (BRW). More precisely, we are interested in the asymptotic behaviour of the extremal process associated to this model. In [SK15], Kistler and Schmidt show that the extremal process of the $GREM(N^\alpha)$, $\alpha \in [0, 1)$ converges weakly to a simple Poisson point process. This contrasts with the extremal process of the branching random walk ($\alpha = 1$) which was shown to converge toward a *decorated* Poisson point process by Madaule [Mad17]. In this paper we propose a generalized model of the $GREM(N^\alpha)$, that has the structure of a tree with k_n levels, where ($k_n \leq n$) is a non-decreasing sequence of positive integers. We show that as long as $\frac{k_n}{n} \to_{n\to\infty} 0$, the decoration disappears and we have convergence to a simple Poisson point process. We study a generalized case, where the position of the particles are not necessarily Gaussian variables and the reproduction law is not necessarily binary.

Keywords: Extremal processes, Branching random walk, extremes of log-correlated random fields.
MSC 2020: Primary: 60G80, 60G70, 60G55. Secondary: 60G50, 60G15, 60F05.

2.1 Introduction

The random energy model (REM) was introduced by Derrida in 1981 [Der81] for the study of spin glasses. In the REM, there are 2^N spin configurations. Each configuration $\sigma \in \{-1,1\}^N$ corresponds to an independent centred Gaussian random variable X_σ with variance N, that models its energy level. It is well-known that the extremal process of the REM, which is defined as

$$\mathcal{E}_N = \sum_{\sigma \in \{-1,1\}^N} \delta_{X_\sigma - m_N}, \quad \text{where } m_N = \beta_c N - \frac{1}{2\beta_c} \log(N) \text{ and } \beta_c = \sqrt{2\log(2)}, \tag{2.1.1}$$

converges weakly in distribution to a Poisson point process with intensity $\frac{1}{\sqrt{2\pi}} e^{-\beta_c x} dx$. Additionally the law of the maximum $M_N = \max_{\sigma \in \{-1,1\}^N} X_\sigma$ centred by m_N converges weakly to a Gumbel random variable.

Derrida introduced a generalized model in 1985, called the GREM [Der85], that has the structure of a tree with K levels (K is a fixed constant in $\mathbb{N}^*$) and can be described as follows. Start by a unique individual (the root). It gives birth to $2^{\frac{N}{K}}$ (we assume that $\frac{N}{K}$ is a positive integer) children at the first level. At each level i, $1 \leq i < K$, each child gives birth independently to $2^{\frac{N}{K}}$ children. We associate each branch of this tree to an independent centred Gaussian random variable with variance $\frac{N}{K}$. In the context of spin glasses, we obtain 2^N configurations in the level K, and the level energy of each configuration is the sum of the values along the branches that forms the path from the root of the tree to the leaf corresponding to this configuration. We call this model $GREM_N(K)$. The REM in this case can be thought of as a GREM with one level, i.e. a $GREM_N(1)$. The correlation of the energy of two different configurations depends on the number of common branches shared by their paths from the root up to the node at which they split. These correlations do not have any impact on the extreme values of the energy levels, as the result described in (2.1.1) still holds even if $(X_\sigma, \sigma \in \{-1,1\}^N)$ is distributed as a $GREM_N(K)$, as $N \to \infty$.

Kistler and Schmidt [SK15] studied the asymptotic of the extremal process of a GREM with a number of levels $K_N = N^\alpha$, for $\alpha \in [0,1)$. They proved that, setting

$$m_N^{(\alpha)} = \beta_c N - \frac{2\alpha+1}{2\beta_c} \log(N),$$

the extremal process of the $GREM_N(N^\alpha)$ converges weakly to a Poisson point process with intensity $\frac{1}{\sqrt{2\pi}} e^{-\beta_c x} dx$, and the law of the maximum converges to a Gumbel distribution. In the $GREM_N(N^\alpha)$ the stronger correlations between the leaves of the tree have the effect of decreasing the median of the maximal energy level, specifically its logarithmic correction. However the limiting law of the extremal process remains unchanged. In the case of $\alpha = 1$, which corresponds to the classical binary branching random walk, the asymptotic behaviour of the extremal process is well-known. The convergence in law of the recentred maximum was proved by Aidékon [Aid13], and recently Madaule [Mad17] showed the convergence of the extremal process to a decorated Poisson point process with random intensity. Therefore a phase transition can be exhibited, from a simple Poisson point process appearing in the $GREM_N(N^\alpha)$ for $\alpha < 1$ to a decorated one for $\alpha = 1$.

The aim of this article is to have a closer look at this phase transition. We take interest in a generalized version of the $GREM_N(N^\alpha)$, that has the structure of a tree with k_n levels, where $(k_n \geq 0)$ is a non-decreasing sequence of positive integers. We study the asymptotic behaviour of the extremal point process showing that as long as $\frac{k_n}{n} \to_{n\to\infty} 0$ (in the Gaussian case), the decoration does not appear.

2.2 Notation and main result

A branching random walk on $\mathbb{R}$ is a particle system that evolves as follows. It starts with a unique individual located at the origin at time 0. At each time $n \geq 1$, each individual alive in the process dies and gives birth to a random number of children, that are positioned around their parent according to i.i.d random variables.

The process we take interest in can be described as follows. Let k_n be an integer sequence growing to ∞ such that $k_n \leq n$ for all $n \in \mathbb{N}$ and set $b_n = \lfloor \frac{n}{k_n} \rfloor$ the integer part of $\frac{n}{k_n}$. The process starts with a unique individual located at the origin at time 0. The particles reproduce for b_n consecutive steps, each particle giving birth to an i.i.d. number of children. Then each descendant of the initial ancestors moves independently, making b_n i.i.d. steps of displacements. This forms the first generation of the process. For each $1 \leq k \leq k_n$, every individual at generation k repeats independently of the others the same reproduction and displacement procedure as the original ancestor. In other words every individual creates a number of descendants given by the value at time b_n of a Galton-Watson process, whose positions are given by i.i.d. random variables with the same law as a random walk of length b_n.

To describe the model formally we introduce Ulam-Harris notation for trees. Set

$$\mathcal{U} = \bigcup_{n \geq 0} \mathbb{N}^n$$

with $\mathbb{N}^0 = \{\varnothing\}$ by convention. The element $(u_1, u_2.., u_n)$ represents the u_n^{th} child of u_{n-1}^{th} child .., of u_1 of the root particle which is noted $\varnothing$. If $u = (u_1, u_2.., u_n)$ we denote by $u_k = (u_1, u_2.., u_k)$ the sequence consisting of the k^{th} first values of u and by $|u|$ the generation of u. For $u, v \in \mathcal{U}$ we denote by $\pi(u)$ the parent of u. If $u = (u_1, u_2.., u_n)$ and $v = (v_1, v_2.., v_n)$, then we write $u.v = (u_1, u_2.., u_n, v_1, v_2.., v_n)$ for the concatenation of u and v. We write

$$|u \wedge v| := \inf\{j \leq n : u_j = v_j \text{ and } u_{j+1} \neq v_{j+1}\}.$$

This quantity is called the overlap of u and v in the context of spin glasses. A tree $\mathcal{T}$ is a subset of $\mathcal{U}$ satisfying the following assumptions:

- $\varnothing \in \mathcal{T}$.
- if $u \in \mathcal{T}$, then $\pi(u) \in \mathcal{T}$.
- if $u = (u_1, u_2, ...u_n) \in \mathcal{T}$, then $\forall\ j \leq u_n$, $\pi(u).j \in \mathcal{T}$.

We now introduce the reproduction and displacement laws associated to our process. Let $(Y_n)_{n\in\mathbb{N}}$ be a random walk such that $\mathbb{E}(Y_1) = 0$ and $\mathrm{Var}(Y_1) = 1$. We denote by $(Z_n)_{n\in\mathbb{N}}$ a Galton-Watson process such that $Z_0 = 1$ and offspring law given by the weights $(p(k))_{k\in\mathbb{N}}$ with $p_0 = 0$. Under this assumption, the Galton Watson process survives almost surely. Set $m = \sum_{k\geq 1} kp(k)$ the mean of the offspring distribution and assume that $m > 1$. Recall that the Galton-Watson process $(Z_n)_{\in\mathbb{N}}$ satisfies for all $n \in \mathbb{N}$:

$$Z_{n+1} = \sum_{j=1}^{Z_n} \xi_{n+1,j},$$

where $(\xi_{n,j})_{1\leq j\leq Z_n}$ are i.i.d random variables with law $(p(k))_{k\in\mathbb{N}}$.

Under the assumption $\mathbb{E}(Z_1 \log(Z_1)) < \infty$, Kesten and Stigum [KS66] proved that on the set of non extinction of $\mathcal{T}$ there exists a positive random variable Z_∞ such that

$$\lim_{b \to \infty} \frac{Z_b}{m^b} = Z_\infty > 0, \quad \text{a.s.} \tag{2.2.1}$$

In this article we assume that the following stronger condition holds:

$$\mathbb{E}(Z_1^2) < \infty, \tag{2.2.2}$$

which is needed for the proof of Lemma 2.4.3.

Construct a tree that we denote $\mathcal{T}^{(n)}$ as follows. Start by the ancestor $\varnothing$ located at the origin. It gives birth to Z_{b_n} children. For each $k \leq k_n$, each individual at the generation k gives birth to an independent copy of Z_{b_n}, that are positioned according to i.i.d random variables with the same law as Y_{b_n}. For $1 \leq k \leq k_n$, let

$$\mathcal{H}_k := \{u \in \mathcal{T}^{(n)} : |u| = k\},$$

the set of particles in the k^{th} generation. By construction, we have $\#\mathcal{H}_k = Z_{kb_n}$ in law for all $k \leq k_n$. We define $(X_u^{(n)}, u \in \mathcal{T}^{(n)})$ a family of i.i.d. random variables with same law as Y_{b_n}. For $u \in \mathcal{T}^{(n)}$, we write

$$S_u^{(n)} = \sum_{k=1}^{|u|} X_{u_k}^{(n)}.$$

The goal of this paper is to study the asymptotic behaviour of the extremal process associated to this model

$$\mathcal{E}_n^{(b_n)} = \sum_{u \in \mathcal{H}_{k_n}} \delta_{S_u^{(n)} - m_n},$$

where the value of the median m_n is given in Theorem 2.2.1.

Let us introduce notation associated to the displacement of the process. For all $\theta > 0$ we set

$$\Lambda(\theta) := \log\left(\mathbb{E}\left(\exp(\theta Y_1)\right)\right). \tag{2.2.3}$$

We assume that there exists $\theta > 0$ such that $\Lambda(\theta) < \infty$. We write:

$$\kappa_n(\theta) = \log \mathbb{E}\left(\sum_{|u|=1} e^{\theta X_u^{(n)}}\right).$$

Observe that $\kappa_n(\theta) = b_n(\log(m) + \Lambda(\theta))$ as

$$\mathbb{E}\left(\sum_{|u|=1} e^{\theta X_u^{(n)}}\right) = \mathbb{E}\left(\sum_{|u|=1} \mathbb{E}(e^{\theta X_u^{(n)}} | Z_{b_n})\right) = \mathbb{E}\left(Z_{b_n} \mathbb{E}(e^{\theta Y_{b_n}})\right) = e^{b_n(\log(m)+\Lambda(\theta))}.$$

The function κ_n is convex and differentiable on $\{\theta > 0, \kappa_n(\theta) < \infty\}$, its interval of definition. We assume that there exists $\theta^* > 0$ such that

$$\theta^* \Lambda'(\theta^*) - \Lambda(\theta^*) = \log(m). \tag{2.2.4}$$

We also assume that there exists $\delta > 0$ such that

$$\mathbb{E}\left(\exp((\theta^* + \delta)Y_1)\right) < \infty \tag{2.2.5}$$

Recall that the case $k_n = n$ corresponds to the classical branching random walk. Then under assumption (2.2.3) and (2.2.4), Kingman [Kin75], Hammersley [Ham74] and Biggins [?] showed that on the set of non-extinction of $\mathcal{T}$

$$\lim_{n\to\infty} \frac{M_n}{n} := \frac{\kappa(\theta^*)}{\theta^*} = v \quad \text{a.s},$$

where, $M_n = \max_{u\in\mathcal{H}_n} S_u$ and v is the speed of the right-most individual. Then, Hu and Shi [HS09] and Addario-Berry and Reed [AR09] proved that

$$M_n = nv - \frac{3}{2\theta^*}\ln(n) + O_{\mathbb{P}}(1),$$

where $O_{\mathbb{P}}(1)$ represents a tight sequence of random variables.

Throughout this paper we will assume that we are in one of the two cases:

($\mathbf{H_1}$): Y_1 is a standard Gaussian variable and $b_n \to \infty$ as $n \to \infty$.

($\mathbf{H_2}$): The characteristic function $\varphi(\lambda) = \mathbb{E}\left(\exp(i\lambda Y_1)\right)$ of Y_1 satisfies the Cramér condition, i.e

$$\limsup_{|\lambda|\to\infty} |\varphi(\lambda)| < 1,$$

and $\frac{b_n}{\log(n)^2} \to \infty$ as $n \to \infty$. The last assumption on Y_1 (under ($\mathbf{H_2}$)) comes from the fact that we used a refined version of the Stone's local limit theorem introduced in [Bor17, Theorem 2.1], more precisely in Corollary 2.3.11.

Our work is inspired by the recent works on the convergence of the extremal processes [?], [?], [SK15] and [Mad17]. The main result of this paper is the following convergence in distribution.

Theorem 2.2.1. *Assume that (2.2.2), (2.2.3), (2.2.4), (2.2.5) and either* ($\mathbf{H_1}$) *or* ($\mathbf{H_2}$) *hold, then setting*

$$m_n = k_n b_n v - \frac{3}{2\theta^*}\log(n) + \frac{\log(b_n)}{\theta^*},$$

the extremal process

$$\mathcal{E}_n^{(b_n)} = \sum_{u\in\mathcal{H}_{k_n}} \delta_{S_u^{(n)} - m_n}$$

converges in law to a Poisson point process with intensity $\frac{1}{\sqrt{2\pi\sigma^2}} Z_\infty e^{-\theta^* x}$, *where* $\sigma^2 = \kappa_n^{''}(\theta^*)$ *and* Z_∞ *is the random variable defined in equation (2.2.1). Moreover, the law of the recentered maximum converges weakly to a Gumbel distribution randomly shifted by* $\frac{1}{\theta^*}\log(Z_\infty)$.

Remark 2.2.2. Denote by $\mathcal{C}_b^{l,+}$ the set of continuous, positive and bounded functions $\varphi : \mathbb{R} \to \mathbb{R}_+$ with support bounded on the left. By [BBCM18, Lemma 4.1], it is enough to show that for all function $\varphi \in \mathcal{C}_b^{l,+}$

$$\lim_{n\to\infty} \mathbb{E}\left(e^{-\sum_{u\in\mathcal{H}_{k_n}} \varphi(S_u^{(n)} - m_n)}\right) = \mathbb{E}\left(\exp\left(-Z_\infty \frac{1}{\sqrt{2\pi\sigma^2}} \int e^{-\theta^* y}(1 - e^{-\varphi(y)})dy\right)\right).$$

The result of Kistler and Schmidt [SK15, Theorem 1.1] is covered by Theorem 1. It is the case $(\mathbf{H_1})$ with $k_n = N^\alpha$, $0 \leq \alpha < 1$ and $Z_1 = 2$ in our theorem. In that case we have $Z_\infty = 1$ and $m_n = n\beta_c - \frac{2\alpha+1}{2\beta_c}\log(n)$. Throughout this paper, we use C and c to denote generic positive constants, that may change from line to line. We say that $f_n \sim_{n\to\infty} g_n$ if $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$. For $x \in \mathbb{R}$ we write $x_+ = \max(x, 0)$.

The rest of the paper is organized as follows. In the next section, we introduce the many to one lemma, and we will give a series of useful random walk estimates. In Section 4 we introduce a modified extremal process which we show to have same asymptotic behaviour of the original extremal process defined in the principal theorem. Finally we will conclude the paper with a proof of the main result.

2.3 Many-to-one formula and random walk estimates

In this section, we introduce the many-to-one lemma, that links additive moments of branching processes to random walk estimates. We then introduce some estimates for the asymptotic behaviour of random walks conditioned to stay below a line, and prove their extension to a generalized random walk where the law of each step is given by the sum of b_n i.i.d random variables.

2.3.1 Many-to-one formula

We start by introducing the celebrated many-to-one lemma that transforms an additive function of a branching random walk into a simple function of random walk. This lemma was introduced by Kahane and Peyrière [KP76]. Before we introduce it, we need to define some change of measure and to introduce some notation.

Let $W_0 := 0$ and $(W_j - W_{j-1})_{j\geq 1}$ be a sequence of independent and identically distributed random variables such that for any measurable function $h : \mathbb{R} \mapsto \mathbb{R}$,

$$\mathbb{E}(h(W_1)) = \mathbb{E}\left(e^{\theta^* Y_1 - \Lambda(\theta^*)} h(Y_1)\right).$$

where Y_1 is the law defined in Section 2. Respectively, we introduce $(T_j^{(n)} - T_{j-1}^{(n)})_{j\geq 1}$ a sequence of i.i.d random variables such that $T_0 = 0$ and

$$\mathbb{E}(h(T_1^{(n)})) = \frac{\mathbb{E}\left(\sum_{u,|u|=1} e^{\theta^* S_u^{(n)}} h(S_u^{(n)})\right)}{\mathbb{E}(\sum_{u,|u|=1} e^{\theta^* S_u^{(n)}})} = \mathbb{E}\left(e^{\theta^* Y_{b_n} - \Lambda(\theta^*}) h(Y_{b_n})\right). \tag{2.3.1}$$

Observe that $(T_k^{(n)}, k \geq 1)$ is a sequence of random variables that have the same law as the process $(U_{kb_n} = \sum_{j=1}^{kb_n} W_j, k \geq 1)$. We now set $\overline{T}_j^{(n)} = T_j^{(n)} - jb_n v$ respectively $\overline{W}_j = W_j - jv, j \geq 1$. We have

$$\mathbb{E}(W_1) = \mathbb{E}\left(Y_1 e^{\theta^* Y_1 - \Lambda(\theta^*)}\right) = \Lambda^{'}(\theta^*),$$

and as $\Lambda^{'}(\theta^*) = \kappa_n^{'}(\theta^*) = v$, we have $\mathbb{E}(\overline{W}_1) = 0$ and similarly

$$\mathbb{E}\left(W_1^2\right) = \mathbb{E}\left(Y_1^2 e^{\theta^* Y_1 - \Lambda(\theta^*)}\right) = \Lambda^{''}(\theta^*) + (\Lambda^{'}(\theta^*))^2,$$

which gives $\text{Var}(\overline{W}_1) = \Lambda''(\theta^*) = \sigma^2$ which is finite by assumption (2.2.5). As a consequence we have $\mathbb{E}(\overline{T}_1^{(n)}) = 0$ and $\text{Var}(\overline{T}_1^{(n)})) = b_n\sigma^2 < \infty$. In the case $(\mathbf{H}_1)$, note that $\overline{W}_1$ is a standard Gaussian random variable which means that $\overline{T}_1^{(n)}$ is a centred Gaussian random variable with variance b_n.

For simplicity we write S_u in place of $S_u^{(n)}$ and T_j in place of $T_j^{(n)}$ in the rest of the article.

Proposition 2.3.1. *[Shi15, Theorem 1.1] For any $j \geq 1$ and any measurable function $g : \mathbb{R}^j \to \mathbb{R}_+$, we have*

$$\mathbb{E}\left(\sum_{|u|=j} g((S_{u_i})_{1\leq i\leq j})\right) = \mathbb{E}\left(e^{-\theta^*\overline{T_i}} g((\overline{T_i} + ib_n v)_{1\leq i\leq j})\right).$$

Proof. For $j = 1$, by (2.3.1) and using that $b_n v = \frac{\kappa_n(\theta^*)}{\theta^*}$, we have

$$\mathbb{E}\left(\sum_{|u|=1} g(S_u)\right) = \mathbb{E}(e^{-\theta^* T_1+\kappa_n(\theta^*)} g(T_1)) = \mathbb{E}\left(e^{-\theta^*\overline{T_1}} g(\overline{T_1} + b_n v)\right)$$

where $\overline{T}_1 = T_1 - b_n v$. We complete the proof by induction in the the same way as in [Shi15, Theorem 1.1]. □

2.3.2 Random walk estimates

In this section we introduce some estimates for the asymptotic behaviour of functionals of the random walks, such us the probability to stay above a boundary. We first give an estimate for the probability that a random walk stays above a boundary $(f_n)_{n\in\mathbb{N}}$, that is $O(n^{1/2-\varepsilon})$ for some $\varepsilon > 0$.

Lemma 2.3.2. *[Mal15a, Lemma 3.6]. Let $(w_n)_{n\in\mathbb{N}}$ be a centred random walk with finite variance. Fix $\varepsilon > 0$, there exists $C > 0$ such that*

$$\mathbb{P}(w_k \geq -(k^{1/2-\varepsilon} + y), k \leq n) \leq C\frac{1+y}{\sqrt{n}}$$

for any $y > 0$.

From now on we use the random walks $(T_k)_{k\geq 1}$ and $(\overline{T}_k)_{k\geq 1}$ defined in (2.3.1), unless otherwise stated. In the next lemma we will give an approximation of the probability for a random walk to end up in a finite interval using the Stone's local limit theorem [Sto67].

Lemma 2.3.3. *Let $f \in \mathcal{C}_b^{l,+}$ be a Riemann integrable function, and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, such that $\lim_{n\to\infty} \frac{r_n}{\sqrt{n}} = 0$. Set*

$$a_n = \frac{-3}{2\theta^*}\log(n) + \frac{\log(b_n)}{\theta^*}$$

then we get

$$\mathbb{E}(f(\overline{T}_{k_n} - a_n + x)e^{-\theta^*\overline{T}_{k_n}}) = \frac{e^{\theta^* x} n^{3/2}}{b_n\sqrt{2\pi\sigma^2 k_n b_n}} \int f(y)e^{-\theta^* y}dy(1 + o(1))$$

uniformly in $x \in [-r_n, r_n]$.

Proof. By setting $h(z) = e^{-\theta^* z} f(z)$, it is enough to prove that

$$\mathbb{E}(h(\overline{T}_{k_n} - a_n + x)) = \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \int h(y)dy(1 + o(1)) \tag{2.3.2}$$

uniformly in $x \in [-r_n, r_n]$. We prove this lemma by successive approximations of the function h, starting with an indicator function. Set $h(z) = \mathbf{1}_{[a,b]}(z)$ for some $a < b \in \mathbb{R}$, then we write

$$\mathbb{E}(h(\overline{T}_{k_n} - a_n + x)) = \mathbb{P}\left(\overline{T}_{k_n} - a_n + x \in [a, b]\right), \tag{2.3.3}$$

As $\overline{T}_1$ is the sum of b_n i.i.d. copies of $\overline{Z}_1$, $\overline{T}_{k_n}$ is the sum of $k_n b_n$ i.i.d. centred random variables with finite variance, therefore we can apply the Stone's local limit theorem [Sto67] to obtain

$$\mathbb{P}(\overline{T}_{k_n} - a_n + x \in [a, b]) = \frac{b-a}{\sqrt{2\pi\sigma^2 k_n b_n}} \exp\left(\frac{-(a_n - x)^2}{2k_n b_n \sigma^2}\right)(1 + o(1)) = \frac{b-a}{\sqrt{2\pi k_n b_n \sigma^2}}(1 + o(1)),$$

uniformly in $x \in [-r_n, r_n]$, which completes the proof of (2.3.2) in that case.

We now assume that h is a continuous function with compact support, we prove (2.3.2) by approximating it by simple functions. Denote by $[a, b]$ the support of h. Let $(t_i)_{0 \le i \le m}$ be an uniform subdivision of $[a, b]$ where $m \in \mathbb{N}$ is the number of the subdivisions and $t_i = a + i(b - a)/m$ for $0 \le i \le m$. Set

$$\underline{h}_m(x) = \sum_{i=0}^{m-1} m_i \mathbf{1}_{\{x \in [t_i, t_{i+1}]\}} \quad \text{and} \quad \overline{h}_m(x) = \sum_{i=0}^{m-1} M_i \mathbf{1}_{\{x \in [t_i, t_{i+1}]\}},$$

where $M_i = \sup_{z \in [t_i, t_{i+1}]} h(z)$ and $m_i = \inf_{z \in [t_i, t_{i+1}]} h(z)$. Hence using the Riemann sum approximation and the fact that f is a non-negative function, for all $\varepsilon > 0$, there exists m_0 such that for all $m \ge m_0$ we have

$$(1 - \varepsilon) \int_a^b h(y)dy \le \int_a^b \underline{h}_m(y)dy \le \int_a^b \overline{h}_m(y)dy \le (1 + \varepsilon) \int_a^b h(y)dy, \tag{2.3.4}$$

where $\int_a^b \underline{h}_m(y)dy = \sum_{i=0}^{m-1} \frac{b-a}{m} m_i$ and $\int_a^b \overline{h}_m(y)dy = \sum_{i=0}^{m-1} \frac{b-a}{m} M_i$.

Using equation (2.3.3) we have

$$\mathbb{E}\left(\overline{h}_m(\overline{T}_{k_n} - a_n + x)\right) = \sum_{i=0}^{m-1} M_i \mathbb{P}\left(\overline{T}_{k_n} - a_n + x \in [t_i, t_{i+1}[\right) = \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \sum_{i=0}^{m-1} \frac{b-a}{m} M_i (1 + o(1))$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \int_a^b \overline{h}_m(y)dy(1 + o(1)).$$

Therefore, using that $\mathbb{E}(h(\overline{T}_k - a_n + x) \le \mathbb{E}(\overline{h}_m(\overline{T_k} - a_n + x)$ and by (2.3.4) we deduce that

$$\limsup_{n\to\infty} \sup_{x \in [0, r_n]} \sqrt{k_n b_n} \mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)\right) \le (1 + \varepsilon) \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b h(y)dy.$$

Using similar arguments we have

$$\liminf_{n\to\infty} \inf_{x \in [0, r_n]} \sqrt{k_n b_n} \mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)\right) \ge (1 - \varepsilon) \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b h(y)dy.$$

Finally, letting $\varepsilon \to 0$ completes the proof of (2.3.2) when h is a compactly support function. Finally we consider the general case, and assume that f is bounded with bounded support on the left. We introduce the function

$$\chi(u) = \begin{cases} 1 & \text{if} \quad u < 0 \\ 1-u & \text{if} \quad 0 \leq u \leq 1 \\ 0 & \text{if} \quad u > 1 \end{cases}$$

then we write,

$$\mathbb{E}(h(\overline{T}_{k_n} - a_n + x)) = \mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)\chi(\overline{T}_{k_n} - a_n + x - B)\right) \\ + \mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)(1 - \chi(\overline{T}_{k_n} - a_n + x - B))\right)$$

for some $B > 0$. Observe that the function $z \mapsto h(z)\chi(z - B)$ is continuous with compact support, then using previous result, we have

$$\mathbb{E}(h(\overline{T}_{k_n} - a_n + x)) = \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \int h(y)\chi(y-B)dy(1+o(1)) \\ + \mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)(1 - \chi(\overline{T}_{k_n} - a_n + x - B))\right). \tag{2.3.5}$$

Thanks to the Stone's local limit theorem [Sto67] there exists a constant $C > 0$ such that the quantity (2.3.5) is bounded by

$$\mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)(1 - \chi(\overline{T}_{k_n} - a_n + x - B))\right) \leq \mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)\mathbf{1}_{\{\overline{T}_{k_n} - a_n + x > B\}}\right) \\ \leq ||f||_\infty \mathbb{E}\left(\sum_{j \geq B} e^{-\theta^* j}\mathbf{1}_{\{\overline{T}_{k_n} - a_n + x \in [j, j+1]\}}\right) \leq C||f||_\infty \frac{e^{-\theta^* B}}{\sqrt{k_n b_n \sigma^2}}.$$

On the other hand by the dominated convergence theorem we have

$$\lim_{B \to \infty} \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \int h(y)\chi(y-B)dy = \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \int h(y)dy,$$

as a consequence we deduce that

$$\mathbb{E}(h(\overline{T}_{k_n} - a_n + x)) = \frac{1}{\sqrt{2\pi\sigma^2 k_n b_n}} \int f(y)e^{-\theta^* y}dy(1 + o(1)),$$

which completes the proof. □

Random walk with Gaussian steps

In this section we assume that $(\mathbf{H_1})$ holds, i.e that $(\overline{T}_k)_{k \geq 0}$ is a Gaussian random walk. Let $(\beta_n(k), k \leq k_n)$ be the standard discrete Brownian bridge with k_n steps, which can be defined as,

$$\beta_n(k) = \frac{1}{\sqrt{b_n}}(\overline{T}_k - \frac{k}{k_n}\overline{T}_{k_n}).$$

In the following lemma we estimate the probability for a Brownian bridge to stay below a boundary during all his lifespan. This lemma was introduced in [?, proposition 1] for continuous time Brownian motion which also hold for the discrete time version.

Lemma 2.3.4. *Let h be the function defined by*

$$h(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k = k_n \\ a\log((k_n - k)\wedge k)b_n) + 1) & \text{otherwise.} \end{cases}$$

where a is a positive constant. There exists a constant $C > 0$ such that for all $x > 0$ and $n \geq 0$ we have

$$\mathbb{P}\left(\beta_n(k) \leq \frac{1}{\sqrt{b_n}}(h(k) + x), k \leq k_n\right) \leq C\frac{(1 + \frac{x}{\sqrt{b_n}})^2}{k_n}. \tag{2.3.6}$$

We refer to the function $k \mapsto h(k)$ as a barrier. An application of this lemma is to give an upper bound for the probability that a random walk with Gaussian steps make an excursion above a well-chosen barrier.

Lemma 2.3.5. *Let $\alpha > 0$, and for $0 \leq k \leq k_n$ we write $f_n(k) = \alpha\log(\frac{(k_n-k)b_n+1}{k_n b_n})$. There exists $C > 0$ such that for all $x \geq 0$, $a < b \in \mathbb{R}$ and $k \leq k_n$ we have*

$$\mathbb{P}\left(\overline{T_k} - f_n(k) \in [a,b], \overline{T}_j \leq f_n(j) + x, j \leq k\right) \leq C(b-a)\frac{(1 + \frac{x}{\sqrt{b_n}})^2}{\sqrt{b_n}k^{\frac{3}{2}}}.$$

Proof. For $n \in \mathbb{N}$ we have

$$\begin{aligned}
&\mathbb{P}\left(\overline{T_k} - f_n(k) \in [a,b], \overline{T}_j \leq f_n(j) + x, j \leq k\right) \\
&\leq \mathbb{P}\left(\overline{T}_k - f_n(k) \in [a,b], \overline{T}_j - \frac{j}{k}\overline{T}_k \leq f_n(j) + x - \frac{j}{k}(f_n(k) + a), j \leq k\right),
\end{aligned}$$

using independence between the discrete Brownian bridge $\overline{T}_j - \frac{j}{k}\overline{T}_k$ and $\overline{T_k}$ we obtain

$$\begin{aligned}
&\mathbb{P}\left(\overline{T_k} - f_n(k) \in [a,b], \overline{T}_j - \frac{j}{k}\overline{T}_k \leq f_n + x - \frac{j}{k}(f_n(k)), j \leq k\right) \qquad (2.3.7)\\
&\leq \mathbb{P}\left(\overline{T}_k - f_n(k) \in [a,b]\right)\mathbb{P}\left(\overline{T}_j - \frac{j}{k}\overline{T}_k \leq f_n(j) + x - \frac{j}{k}(f_n(k) + a), j \leq k\right).
\end{aligned}$$

To estimate the probability that a discrete Brownian bridge stay below a logarithmic barrier, we apply Lemma 2.3.4. First observe that the function $x \mapsto \frac{\log(x)}{x}$ is decreasing for $x \geq e$, and using that $(k_n - j)b_n + 1 \leq (k_n - k)b_n + 1 + (k-j)b_n + 1 \leq 2((k_n - k)b_n + 1)(k-j)b_n + 1))$, we have for $j \leq \frac{k}{2}$,

$$\begin{aligned}
f_n(j) + x - \frac{j}{k}(f_n(k) + a) &\leq \alpha\frac{j}{k}\left(\log(\frac{k_n b_n}{(k_n - k)b_n + 1}) - \log(\frac{k_n b_n}{(k_n - j)b_n + 1})\right) + x \\
&\leq \alpha\frac{j}{k}(\log(kb_n) + \log(2)) + x \leq \alpha(\log((jb_n \vee e)) + \log(2)) + x
\end{aligned}$$

and for $\frac{k}{2} \leq j \leq k$, we have

$$\begin{aligned}
f_n(j) + x - \frac{j}{k}(f_n(k) + a) &\leq \alpha(\log(\frac{k_n b_n}{(k_n - k)b_n + 1}) + x - \log(\frac{k_n b_n}{(k_n - j)b_n + 1})) \\
&\leq \alpha(\log(((k_n - j)b_n + 1) - \log((k_n - k)b_n + 1))) + x \\
&\leq \alpha(\log(2) + \log(1 + (k-j)b_n) + x.
\end{aligned}$$

Then by Lemma 2.3.4 we get after rescaling by $\frac{1}{\sqrt{b_n}}$ the following upper bound

$$\mathbb{P}\left(\overline{T}_j - \frac{j}{k}\overline{T}_k \le f_n(k) - \frac{j}{k}(f_n(j)-x), j \le k\right)$$
$$\le \mathbb{P}\left(\beta_n(k) \le \alpha(\log((k\wedge(k-j))+1)) + \frac{x}{\sqrt{b_n}} + 1, j\le k\right) \le C\frac{(1+\frac{x}{\sqrt{b_n}})^2}{k},$$

where C is a positive constant. To bound the first quantity in (2.3.7) we use the Gaussian estimate

$$\mathbb{P}\left(\overline{T_k} - f_n(k) \in [a,b]\right) \le \frac{b-a}{\sqrt{kb_n}}$$

which completes the proof. □

From now we denote by $B_n(k) = \frac{\overline{T}_k}{\sqrt{b_n}}$. Recall that under $(\mathbf{H}_1)$, $(B_n(k))_{k\le k_n}$ is a standard random walk with i.i.d Gaussian steps. Define the function $L:(0,\infty)\to(0,\infty)$ by $L(0)=1$ and

$$L(x) := \sum_{k\ge 0}\mathbf{P}\left(B_n(k)\ge -x, B_n(k) \le \min_{j\le k-1} B_n(j)\right) \qquad \text{for } x>0. \tag{2.3.8}$$

It is known by [Fel71, section XII.7], that the function L is the renewal function associated to the random walk $(B_n(k))_{k\ge 0}$. We will cite some properties that are mentioned in [Fel71, section XII.7]. The fundamental property of the renewal function is

$$L(x) = \mathbb{E}\left(L(x+B_n(1))\mathbf{1}_{\{x+B_n(1)\ge 0\}}\right), \tag{2.3.9}$$

and is a right-continuous and non-decreasing function. Since in case $(\mathbf{H_1})$, the initial law has no atoms, then the function L is continuous. Also, there exists a constant $c_0>0$ such that

$$\lim_{x\to\infty}\frac{L(x)}{x} = c_0. \tag{2.3.10}$$

In particular there exists a constant $C>0$ such that for all $x\in\mathbb{R}$

$$L(x) \le C(1+x_+). \tag{2.3.11}$$

Also we have by [Fel71, section XII.7] , for $x,y\ge 0$

$$L(x+y) \le 2L(x)L(y). \tag{2.3.12}$$

Similarly, we define $L_-(x)$ as the renewal function associated to $-B$.

Since $\overline{T}$ is a symmetric law we have $L_-(x) = L(x)$ for all $x\ge 0$. It is also known that there exists a positive constant C_1 such that for $y\ge 0$

$$\mathbb{P}\left(\min_{k\le k_n}(B_n(k)) \ge -y\right) \sim_{n\to\infty} C_1\frac{L(y)}{\sqrt{k_n}}. \tag{2.3.13}$$

By Theorem 3.5 in [Spi60], assuming that B is Gausian we have $C_1 = \frac{1}{\sqrt{\pi}}$. We now introduce an approximation of the probability for a random walk to stay below a line and end up in a finite interval . Set

$$\widetilde{F}_n(k) = \frac{k}{k_n}a_n = \frac{k}{k_n}(m_n - k_nb_nv), k=0...,k_n, \quad n\in\mathbb{N}.$$

Lemma 2.3.6. *Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{r_n}{\sqrt{k_n}} = 0$. Let*

$$a_n = \frac{-3}{2\theta^*}\log(n) + \frac{\log(b_n)}{\theta^*}.$$

For all $f \in \mathcal{C}_b^{l,+}$ we have

$$\mathbb{E}\left(f(\overline{T}_{k_n} - a_n + x)e^{-\theta^*\overline{T}_{k_n}}\mathbf{1}_{\{\overline{T}_k \le \overline{F}_n(k)-x, k\le k_n\}}\right) = \frac{e^{\theta^* x}}{\sqrt{2\pi}}\int_{-\infty}^{0} f(y)e^{-\theta^* y}dy\left(R(\frac{-x}{\sqrt{b_n}}) + o(1)\right).$$

uniformly in $x \in [-r_n, 0]$.

Proof. By setting $h(z) = e^{-\theta^* z}f(z)$ it is enough to prove that

$$\mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)\mathbf{1}_{\{\overline{T}_k \le \overline{F}_n(k)-x, k\le k_n\}}\right) = \frac{1}{k_n^{3/2}\sqrt{2\pi b_n}}\int_{-\infty}^{0} h(y)dy(R(\frac{-x}{\sqrt{b_n}}) + o(1)) \qquad (2.3.14)$$

uniformly in $x \in [-r_n, 0]$.

Following the same method used in Lemma 2.3.3 it is enough to prove this estimate for an indicator function. By writing $\mathbf{1}_{[-a,-b]} = \mathbf{1}_{[-a,0]} - \mathbf{1}_{[-b,0]}$ for some $a > 0, b > 0$, it is enough to prove this estimate for $h(z) = \mathbf{1}_{[-a,0]}(z)$, in that case we have

$$\mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x)\mathbf{1}_{\{\overline{T}_k \le \overline{F}_n(k)-x, k\le k_n\}}\right) = \mathbb{P}\left(\overline{T}_{k_n} - a_n + x \ge -a, \overline{T}_k \le F_n(k) - x, k \le k_n\right).$$

Define a new probability measure $\mathbb{Q}$ on $\mathbb{R}$ by

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(\overline{T}) = \exp(\frac{-a_n}{n}\overline{T} + \Lambda(\frac{a_n}{n})) \qquad (2.3.15)$$

where $\Lambda(\theta) = \frac{\theta^2}{2}$. Then we rewrite

$$\begin{aligned}&\mathbb{P}\left(\overline{T}_{k_n} - a_n + x \ge -a, \overline{T}_k \le F_n(k) - x, k \le k_n\right)\\ &= \mathbb{E}_Q\left(e^{\frac{-a_n}{n}(\sqrt{b_n}\widehat{B}_n(k_n) - \frac{a_n}{2n})}\mathbf{1}_{\{\sqrt{b_n}\widehat{B}_n(k_n)+x\ge -a, \sqrt{b_n}\widehat{B}_n(k)\le -x, k\le k_n\}}\right),\end{aligned}$$

where $\widehat{B}_n(k) = B_n(k) - \frac{k}{\sqrt{b_n}k_n}a_n$. Observe that the law of $\widehat{T}$ under $\mathbb{Q}$ is the same as the law of $\overline{T}$ under $\mathbb{P}$.

Under this change of measure, we can rewrite the probability as

$$\begin{aligned}&\mathbb{E}_Q\left(e^{-\frac{a_n}{n}\sqrt{b_n}\widehat{B}_n(k_n) + \frac{a_n^2}{2n}}\mathbf{1}_{\left\{\sqrt{b_n}\widehat{B}_n(k)\le -x, k\le k_n, \sqrt{b_n}\widehat{B}_n(k_n)+x\ge -a\right\}}\right)\\ &\le e^{\frac{a_n}{n}(x+a) + \frac{a_n^2}{2n}}\mathbb{Q}\left(\sqrt{b_n}\widehat{B}_n(k) \le -x, k \le k_n, \sqrt{b_n}\widehat{B}_n(k_n) \ge -a - x\right).\end{aligned}$$

as a consequence

$$\begin{aligned}&\limsup_{n\to\infty}\sup_{x\in[-r_n,0]}\mathbb{E}_Q\left(e^{-\frac{a_n}{n}\sqrt{b_n}\widehat{B}_n(k_n) + \frac{a_n^2}{2n}}\mathbf{1}_{\left\{\sqrt{b_n}\widehat{B}_n(k)\le -x, k\le k_n, \sqrt{b_n}\widehat{B}_n(k_n)+x\ge -a\right\}}\right)\\ &\le \limsup_{n\to\infty}\sup_{x\in[-r_n,0]}\mathbb{Q}\left(\sqrt{b_n}\widehat{B}_n(k) \le -x, k \le k_n, \sqrt{b_n}\widehat{B}_n(k) \ge -a - x\right),\end{aligned}$$

similarly we have

$$\liminf_{n\to\infty} \inf_{x\in[-r_n,0]} \mathbb{E}_Q\left(e^{-\frac{a_n}{n}\sqrt{b_n}\widehat{B}_n(k_n)+\frac{a_n^2}{2n}}\mathbf{1}_{\left\{\sqrt{b_n}\widehat{B}_n(k)\le -x, k\le k_n, \sqrt{b_n}\widehat{B}_n(k_n)+x\ge -a\right\}}\right)$$
$$\le \liminf_{n\to\infty} \inf_{x\in[-r_n,0]} \mathbb{Q}\left(\sqrt{b_n}\widehat{B}_n(k)\le -x, k\le k_n, \sqrt{b_n}\widehat{B}_n(k_n)\ge -a-x\right). \tag{2.3.16}$$

for all $a>0$. Therefore, it remains to estimate the quantity (2.3.16). Applying the Markov property at time $p=[\frac{k_n}{2}]$ we get

$$\mathbb{Q}\left(\sqrt{b_n}\widehat{B}_n(k)\le -x, k\le k_n, \sqrt{b_n}\widehat{B}_n(k_n)\ge -a-x\right) = \mathbb{E}\left(f_{x,n,a}(\sqrt{b_n}\widehat{B}_n(p))\mathbf{1}_{\left\{\sqrt{b_n}\widehat{B}_n(k)\le -x, k\le p\right\}}\right) \tag{2.3.17}$$

where for all $y\le 0$

$$f_{x,n,a}(y) = \mathbb{Q}\left(\sqrt{b_n}\widehat{B}_n(k_n-p)+y\ge -a-x, \sqrt{b_n}\widehat{B}_n(k)+y\le -x, k\le k_n-p\right).$$

Using that the process $(\sqrt{b_n}(\widehat{B}_n(k_n-p)-\widehat{B}_n(k_n-p-j)), 0\le j\le k_n-p)$ has the same law as $(\sqrt{b_n}\widehat{B}_n(j), 0\le j\le k_n-p)$ under $\mathbb{Q}$, we obtain

$$f_{x,n,a}(y) = \mathbb{Q}\left((-\sqrt{b_n}\widehat{B}_n(k))\le(-\sqrt{b_n}\widehat{B}_{k_n-p})-(x+y)\le a, k\le k_n-p\right)$$
$$= \mathbb{Q}\left(\sqrt{b_n}\widehat{B}_n(k)\le\sqrt{b_n}\widehat{B}_n(k_n-p)-(x+y)\le a, k\le k_n-p\right)$$

since $(\sqrt{b_n}\widehat{B}_n(k))_{k\ge 0}$ is a symmetric law. We write $\check{B}_n(k_n-p)=\max_{0\le j\le k_n-p}\sqrt{b_n}\widehat{B}_n(i)$, set

$$\tau_{k_n-p} = \min\left\{i:\ \ 0\le i\le k_n-p,\ \ \check{B}_n(k_n-p)=\sqrt{b_n}\widehat{B}_n(i)\right\}$$

the first time when $\sqrt{b_n}\widehat{B}_n(i)$ hits its maximum in the interval $[0,k_n-p]$. We have

$$f_{x,n,a}(y) = \sum_{i=0}^{k_n-p}\mathbb{Q}\left(\tau_{k_n-p}=i, \sqrt{b_n}\widehat{B}_n(k)\le\sqrt{b_n}\widehat{B}_n(k_n-p)-(x+y)\le a, k\le k_n-p\right).$$

Applying the Markov property at time i we get

$$f_{x,n,a}(y) = \sum_{i=0}^{k_n-p}\mathbb{E}\left(g_{x,n,y}\left(\check{B}_n(i)-a\right)\mathbf{1}_{\left\{\check{B}_n(i)=\sqrt{b_n}\widehat{B}_n(i)\le a\right\}}\right),$$

where for all $z\le 0$, $g_{x,n,y}(z)=\mathbb{Q}\left(y+x\le\sqrt{b_n}\widehat{B}_n(k_n-p-i)\le y+x-z, \check{B}_n(k_n-p-i)\le 0\right)$.

We now split the sum $\sum_{i=0}^{k_n-p}$ into $\sum_{i=0}^{i_n}+\sum_{i=i_n+1}^{k_n-p}$, where $i_n=[\sqrt{k_n}]$, then we write

$$f_{n,x,a}(y) = f^{(1)}_{n,x,a}(y)+f^{(2)}_{n,x,a}(y)$$

where

$$f^{(1)}_{n,x,a}(y) = \sum_{i=0}^{i_n}\mathbb{E}\left(g_{x,n,y}(\check{B}_n(i)-a)\mathbf{1}_{\left\{\check{B}_n(i)=\sqrt{b_n}\widehat{B}_n(i)\le a\right\}}\right),$$

and

$$f^{(2)}_{n,x,a}(y) = \sum_{i=i_n+1}^{k_n-p} \mathbb{E}\left(g_{x,n,y}(\check{B}_n(i)-a)\mathbf{1}_{\left\{\check{B}_n(i)=\sqrt{b_n}\widehat{B}_n(i)\le a\right\}}\right).$$

Set $\varphi(x) := xe^{\frac{-x^2}{2}}\mathbf{1}_{\{x\ge 0\}}$. By Theorem 1 in [Car05] of Caravenna for $n\to\infty$,

$$\begin{aligned}
&\mathbb{Q}\left(-(x+y-z)\le \sqrt{b_n}\widehat{B}_n(k_n-p-i)\le -(x+y)|\sqrt{b_n}\widehat{B}_n(j)\ge 0, j\le k_n-p-i\right)\\
&=\frac{-z}{\sqrt{(k_n-p)b_n}}\varphi\left(\frac{-y}{\sqrt{(k_n-p)b_n}}\right)+o\left(\frac{1}{\sqrt{(k_n-p)b_n}}\right),
\end{aligned}$$

uniformly in $y\le 0$, $x\in[-r_n,0]$ and z in any compact set of $\mathbb{R}_-$. As a consequence by (2.3.13) we get

$$g_{x,n,y}(z) = \frac{-z}{(k_n-p)\sqrt{b_n\pi}}\varphi\left(\frac{-y}{\sqrt{(k_n-p)b_n}}\right)+o(\frac{1}{(k_n-p)\sqrt{b_n}}),$$

uniformly in $y\le 0$, $x\in[-r_n,0]$ and $z\in[-a,0]$. For n large enough we get

$$\begin{aligned}
f^{(1)}_{n,x,a}(y) &= \frac{1}{(k_n-p)\sqrt{b_n\pi}}\varphi\left(\frac{-y}{\sqrt{(k_n-p)b_n}}\right)\sum_{i=0}^{i_n}\mathbb{E}\left(-(\widehat{B}_n(i)-\frac{a}{\sqrt{b_n}})\mathbf{1}_{\left\{\widehat{B}_n(k)\le\frac{a}{\sqrt{b_n}},k\le i\right\}}\right) \qquad (2.3.18)\\
&+o(\frac{1}{k_n\sqrt{b_n}})\sum_{i=0}^{i_n}\mathbb{Q}\left(\widehat{B}_n(k)\le\frac{a}{\sqrt{b_n}},k\le i\right).
\end{aligned}$$

We now treat the quantity

$$\mathbb{E}\left(f^{(2)}_{x,n,a}(\sqrt{b_n}\widehat{B}_n(k_n-p))\mathbf{1}_{\left\{\sqrt{b_n}\widehat{B}_n(k)\le -x,k\le k_n-p\right\}}\right).$$

Since φ is bounded, there exists a constant $C>0$ such that for all $x\in[-r_n,0], z\in\mathbb{R}$ and $0\le i\le p$

$$g_{x,n,y}(z)\le\frac{C}{\sqrt{b_n}(k_n-p-i+1)}\mathbf{1}_{\{-a\le z\le 0\}},$$

as a consequence, for all $y\le 0$ we have

$$f^{(2)}_{x,n,a}(y)\le\frac{C}{\sqrt{b_n}}\sum_{i=i_n+1}^{k_n-p}\frac{1}{k_n-p-i+1}\mathbb{P}\left(\check{B}_n(i)\le\frac{a}{\sqrt{b_n}},\widehat{B}_n(i)\ge 0\right)$$

which is bounded using Lemma 2.3.5 by

$$f^{(2)}_{x,n,a}(y)\le\frac{C}{\sqrt{b_n}}\sum_{i=i_n+1}^{k_n-p}\frac{1}{(k_n-p-i+1)i^{\frac{3}{2}}}=o(\frac{1}{k_n\sqrt{b_n}}). \qquad (2.3.19)$$

On the other hand by (2.3.13) we have $\mathbb{Q}(\widehat{B}_n(j) \leq \frac{-x}{\sqrt{b_n}}, j \leq k_n - p) \sim_{n\to\infty} \frac{\sqrt{2}}{\sqrt{\pi}} \frac{L(\frac{-x}{\sqrt{b_n}})}{\sqrt{k_n}}$ by (2.3.13), which implies together with equation (2.3.19)

$$(L(\frac{-x}{\sqrt{b_n}}))^{-1}\mathbb{E}\left(f^{(2)}_{x,n,a}(\widehat{B}_n(k_n-p))\mathbf{1}_{\left\{\sqrt{b_n}\widehat{B}_n(k)\leq -x, k\leq k_n-p\right\}}\right) = o(\frac{1}{k_n^{\frac{3}{2}}\sqrt{b_n}}). \tag{2.3.20}$$

We now return to equation (2.3.18). Letting $n\to\infty$, we have $\sum_{i=0}^{\infty}\mathbb{P}\left(\frac{\widehat{T}_i}{\sqrt{b_n}} \geq \frac{-a}{\sqrt{b_n}}\right) = R(\frac{a}{\sqrt{b_n}})$ for $a>0$ and by Fubini's theorem we have

$$\sum_{i=0}^{\infty}\mathbb{E}\left((\widehat{B}_n(i)+\frac{a}{\sqrt{b_n}})\mathbf{1}_{\left\{\widehat{B}_n(k)\geq \frac{-a}{\sqrt{b_n}}, k\leq i\right\}}\right) = \int_0^{\frac{a}{\sqrt{b_n}}} L(t)dt = \frac{1}{\sqrt{b_n}}\int_{-a}^{0} L(\frac{-t}{\sqrt{b_n}})dt.$$

By dominated convergence theorem we have

$$\frac{1}{\sqrt{b_n}}\int_{-a}^{0} L(\frac{-t}{\sqrt{b_n}})dt = \frac{a}{\sqrt{b_n}}(1+o(1)) = \frac{1}{\sqrt{b_n}}\int h(t)dt(1+o(1).$$

since $h(t) = \mathbf{1}_{[-a,0]}(t)$. This yields, for all $y\leq 0$

$$f^{(1)}_{n,x,a}(y) = \frac{1}{k_n\sqrt{b_n}\pi}\varphi(\frac{-y}{\sqrt{(k_n-p)b_n}})\int h(t)dt + o(\frac{1}{k_n\sqrt{b_n}}).$$

as a consequence, by (2.3.13)

$$\mathbb{E}\left(f^{(1)}_{x,n,a}(\sqrt{b_n}\widehat{B}_n(k_n-p))\mathbf{1}_{\left\{\widehat{B}_n(k)\leq \frac{-x}{\sqrt{b_n}}, k\leq k_n-p\right\}}\right) = \frac{2\sqrt{2}}{\pi k_n^{\frac{3}{2}}\sqrt{b_n}}\int h(t)dt$$
$$\times\, \mathbb{E}_{\frac{-x}{\sqrt{b_n}}}\left(\varphi\left(\frac{-\widehat{B}_n(k_n-p)+\frac{x}{\sqrt{b_n}}}{\sqrt{k_n-p}}\right)\big| -\widehat{B}_n(k)\geq 0, k\leq k_n-p\right)\left(L(\frac{-x}{\sqrt{b_n}})+o(1)\right).$$

On the other hand, it's known (see Lemma 2.2 in [AJ11]) that under $\mathbb{P}_y\left(.|\frac{\widehat{T}_k}{\sqrt{b_n}}\geq 0, k\leq k_n-p\right)$, $\frac{\widehat{B}_n(k_n)}{\sqrt{k_n}}$ converges weakly (as $n\to\infty$) to the Rayleigh distribution with density φ. Hence

$$\lim_{n\to\infty}\mathbb{E}_{\frac{-x}{\sqrt{b_n}}}\left(\varphi\left(\frac{\widehat{B}_n(k_n-p)+\frac{x}{\sqrt{b_n}}}{\sqrt{k_n-p}}\right)\big|\widehat{B}_n(k)\geq 0, k\leq k_n-p\right) = \int_0^{\infty}\varphi(t)^2dt = \frac{\sqrt{\pi}}{4}$$

uniformly in $x\in[-r_n,0]$. Combining this with (2.3.20) we conclude that (2.3.14) holds, which allows us to complete the proof by successive approximations. □

KMT coupling for random walk

We now introduce the well-known KMT Theorem [KMT76] which is an approximation method of a random walk satisfying (**H2**) by a Gaussian random walk. It allows us to link estimates on random walks satisfying (**H2**) to the ones previously proved under assumption (**H1**).

Theorem 2.3.7 (Komlos-Major-Tusnàdy). *Let $(X_i)_{1\leq i\leq n}$ be a sequence of i.i.d random variables such that $\mathbb{E}(X_i))=0$, $0<\mathbb{E}(X_i^2)=\sigma^2<\infty$ and $\mathbb{E}(\exp(\theta|X_i|))<\infty$ for some $\theta>0$. Then there exists a sequence of i.i.d standard normal variables $(Z_i)_{1\leq i\leq n}$ such that for all $y\geq 0$*

$$\mathbb{P}\left(\max_{1\leq k\leq n}|\sigma^{-1}\sum_{i=1}^{k}X_i-\sum_{i=1}^{k}Z_i|>C\log(n)+y\right)\leq K\exp\left(-\lambda y\right)$$

where C,K and λ are universal positive constants.

Observe that we can construct an increasing sequence of integers d_n such that $d_n/\log n\to\infty$ and $d_n^2/b_n\to 0$. Using assumption (2.2.5), we can apply Theorem 2.3.7, to note that with high probability, the random walk $(\overline{T}_k)_{k\leq k_n}$ stays within distance d_n from a random walk $(\widehat{S}_k)_{0\leq k\leq k_n}$ with Gaussian steps. More precisely, setting the event $\mathcal{W}_n=\left\{|\overline{T_k}-\sqrt{b_n}\widehat{S}_k|\leq d_n, k\leq k_n\right\}$ we have

$$\mathbb{P}(\mathcal{W}_n^c)\leq e^{-\lambda(d_n-C\log n)}=o(n^{-\gamma})$$

for all $\gamma>0$, as $d_n\gg\log n$. We start by proving a version of Lemma 2.3.2 for the random walk satisfying $(\mathbf{H_2})$.

Lemma 2.3.8. *Fix $\varepsilon>0$, there exists $C>0$ such that*

$$\mathbb{P}(\overline{T}_k\geq-(k^{1/2-\varepsilon}+y),k\leq k_n)\leq C\frac{1+\frac{y}{\sqrt{b_n}}}{\sqrt{k_n}}$$

for any $y>0$.

Proof. Let $\varepsilon>0$. The proof is an application Theorem 2.3.7. We have

$$\begin{aligned}&\mathbb{P}\left(\overline{T}_k\geq-(k^{1/2-\varepsilon}-y),k\leq k_n\right)\\&\leq\mathbb{P}\left(\overline{T}_k\geq-(k^{1/2-\varepsilon}+y),k\leq k_n,\mathcal{W}_n^c,k\leq k_n\right)+\mathbb{P}\left(\sqrt{b_n}\widehat{S}_k\geq-(k^{1/2-\varepsilon}+y)+d_n),k\leq k_n\right).\end{aligned}$$

Applying Theorem 2.3.7, there exists a constant $C>0$ such that

$$\begin{aligned}&\mathbb{P}\left(\overline{T}_k\geq-(k^{1/2-\varepsilon}+y),k\leq k_n\right)\\&\leq\frac{C}{n^\gamma}+\mathbb{P}\left(\sqrt{b_n}\widehat{S}_k\geq-(k^{1/2-\varepsilon}+y)+d_n,k\leq k_n\right),\end{aligned}$$

for all $\gamma>0$. Using the fact that $\frac{d_n}{\sqrt{b_n}}\to 0$ as $n\to\infty$, we have for n large enough

$$\mathbb{P}\left(\overline{T}_k\geq-(k^{1/2-\varepsilon}+y),k\leq k_n\right)\leq\mathbb{P}\left(\widehat{S}_k\geq-\frac{1}{\sqrt{b_n}}(k^{1/2-\varepsilon}+y)+1,k\leq k_n\right)$$

and by Lemma 2.3.2 we conclude the proof. □

In the same way we prove a similar result to Lemma 2.3.5.

Lemma 2.3.9. *Let $\alpha > 0$, and for $0 \le k \le k_n$, write $f_n(k) = \alpha \log\left(\frac{(k_n-k)b_n+1}{k_n b_n}\right)$. There exists $C > 0$ such that for all $x \ge 0$, $a \le b \in \mathbb{R}$, we have for n large enough*

$$\mathbb{P}\left(\overline{T}_{k_n} - f_n(k_n) \in [a,b], \overline{T_k} \le f_n(k) + x, k \le k_n\right) \le C(b-a+2d_n)\frac{(1+\frac{x}{\sqrt{b_n}})^2}{\sqrt{b_n \sigma^2 k_n^{\frac{3}{2}}}}.$$

Proof. Applying again Theorem 2.3.7, there exists a constant $C > 0$ such that

$$\begin{aligned}&\mathbb{P}\left(\overline{T}_{k_n} - f_n(k_n) \in [a,b], \overline{T_k} \le f_n(k) + x, k \le k_n\right)\\ &\le \frac{C}{n^\gamma} + \mathbb{P}\left(\sqrt{b_n}\widehat{S}_{k_n} - f_n(k_n) \in [a-d_n, b+d_n], \widehat{S}_k \le \frac{1}{\sqrt{b_n}}(f_n(k)+x+d_n), k \le k_n\right).\end{aligned}$$

for all $\gamma > 0$. For n large enough we obtain

$$\begin{aligned}&\mathbb{P}\left(\overline{T}_{k_n} - f_n(k_n) \in [a,b], \overline{T_k} \le f_n(k) + x, k \le k_n\right)\\ &\le \mathbb{P}\left(\sqrt{b_n}\widehat{S}_{k_n} - f_n(k_n) \in [a-d_n, b+d_n], \widehat{S}_k \le \frac{1}{\sqrt{b_n}}(f_n(k)+x)+2, k \le k_n\right),\end{aligned}$$

and we use Lemma 2.3.5 to complete the proof. □

We now prove Lemma 2.3.6 for random walk satisfying ($\mathbf{H_2}$). Set

$$\overline{F}_n(k) = \frac{k}{k_n}a_n - c_n \mathbf{1}_{k \neq 0, k_n}$$

$k = 0, ..k_n$ where $(c_n)_{n\in\mathbb{N}}$ is a sequence of integers satisfying $\lim_{n\to\infty} c_n = \infty$ and

$$\lim_{n\to\infty} \frac{c_n}{\sqrt{b_n}} = 0. \tag{2.3.21}$$

Before moving to the proof we need the following Lemma.

Lemma 2.3.10. *Uniformly in $x \in [c_n, \infty]$, we have*

$$\lim_{n\to\infty}\left|\frac{L(\frac{x-c_n}{\sqrt{b_n}})}{L(\frac{x}{\sqrt{b_n}})} - 1\right| = 0.$$

Proof. The proof follows from the properties of the renewal function introduced in (2.3.9). We first consider the ratio $\frac{L(y-\frac{c_n}{\sqrt{b_n}})}{L(y)}$ for large value of y. Let $\varepsilon > 0$, by (2.3.10), there exists a constant $A = A(\varepsilon) > 0$ sufficiently large such that for all $y \ge A$ we have

$$c_0(1-\varepsilon)y \le L(y) \le c_0(1+\varepsilon)y. \tag{2.3.22}$$

Using (2.3.22) we have

$$\sup_{\frac{x}{\sqrt{b_n}} \ge A}\left|\frac{L(\frac{x-c_n}{\sqrt{b_n}})}{L(\frac{x}{\sqrt{b_n}})} - 1\right| \le \left|\frac{1+\varepsilon}{1-\varepsilon} - 1\right| + \frac{c_0(1+\varepsilon)c_n}{A\sqrt{b_n}}. \tag{2.3.23}$$

On the other hand, recall that $y \mapsto L(y)$ is continuous. Hence it is uniformly continuous on $[0, A]$, and

$$\sup_{x\in[c_n, A\sqrt{b_n}]} \left| L(\frac{x}{\sqrt{b_n}}) - L(\frac{x-c_n}{\sqrt{b_n}}) \right| \leq \sup_{y\in[\frac{c_n}{\sqrt{b_n}}, A]} \left| L(y) - L(y - \frac{c_n}{\sqrt{b_n}}) \right| \leq w_L(\frac{c_n}{\sqrt{b_n}}),$$

where $w_L(\delta) = \sup_{\substack{s,t \\ |t-s|\leq\delta}} |L(t) - L(s)|$. Since the renewal function is increasing, we have $L(y) \geq 1$, for all $y \geq 0$, which implies that

$$\sup_{y\in[\frac{c_n}{\sqrt{b_n}}, A]} \left| \frac{L(y - \frac{c_n}{\sqrt{b_n}})}{L(y)} - 1 \right| \leq w_L(\frac{c_n}{\sqrt{b_n}}) \to 0 \text{ as } n \to \infty. \tag{2.3.24}$$

By (2.3.23) and (2.3.24) we have

$$\begin{aligned}\sup_{x\in[c_n,\infty)} \left| \frac{L(\frac{x-c_n}{\sqrt{b_n}})}{L(\frac{x}{\sqrt{b_n}})} - 1 \right| &\leq \sup_{x\in[c_n, A\sqrt{b_n}]} \left| \frac{L(\frac{x-c_n}{\sqrt{b_n}})}{L(\frac{x}{\sqrt{b_n}})} - 1 \right| + \sup_{\frac{x}{\sqrt{b_n}}\geq A} \left| \frac{L(\frac{x-c_n}{\sqrt{b_n}})}{L(\frac{x}{\sqrt{b_n}})} - 1 \right| \\ &\leq w_L(\frac{c_n}{\sqrt{b_n}}) + |\frac{1+\varepsilon}{1-\varepsilon} - 1| + \frac{c_0(1+\varepsilon)c_n}{A\sqrt{b_n}}.\end{aligned}$$

Letting $n \to \infty$ then $\varepsilon \to 0$ we conclude the proof. □

Corollary 2.3.11. *Setting*

$$a_n = \frac{-3}{2\theta^*} \ln(n) + \frac{\log(b_n)}{\theta^*},$$

then there exists $C > 0$ such that With the same notation as Lemma 2.3.6 we have

$$\mathbb{E}\left(f(\overline{T}_{k_n} - a_n + x) e^{-\theta^* \overline{T}_{k_n}} \mathbf{1}_{\{\overline{T}_k \leq \overline{F}_n(k) - x, k \leq k_n\}} \right) = \frac{e^{\theta^* x}}{\sqrt{2\pi\sigma^2 b_n}} \int_{-\infty}^{0} f(y) e^{-\theta^* y} dy \left(L(\frac{-x}{\sqrt{b_n}}) + o(1) \right), \tag{2.3.25}$$

uniformly in $x \in [-r_n, -(c_n + d_n)]$.

Proof. We will divide the proof of this corollary into two parts, proving separately an upper and a lower bound for (2.3.25). We set $h(z) = e^{-\theta^* z} f(z)$. As a consequence, it is enough to prove that

$$\mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x) \mathbf{1}_{\{\overline{T}_k \leq \overline{F}_n(k) - x, k \leq k_n\}} \right) = \frac{1}{k_n^{3/2}\sqrt{2\pi\sigma^2 b_n}} \int_{-\infty}^{0} h(y) dy (L(\frac{-x}{\sqrt{b_n}}) + o(1))$$

uniformly in $x \in [-r_n, -(c_n + d_n)]$. Using the same arguments in Lemma 2.3.6, it is enough to prove this Corollary with the function $h(z) = \mathbf{1}_{[-a,0]}(z)$ for some $a > 0$. Then we write

$$\mathbb{E}\left(h(\overline{T}_{k_n} - a_n + x) \mathbf{1}_{\{\overline{T}_k \leq \overline{F}_n(k) - x, k \leq k_n\}} \right) = \mathbb{P}(\mathcal{A}_n^{(k)}(x)) \tag{2.3.26}$$

where

$$\mathcal{A}_n^{(k)}(x) = \left\{\overline{T}_k \le \overline{F}_n(k) - x, k \le k_n, \overline{T}_{k_n} - a_n + x \ge -a\right\}$$

the event that the random walk $(\overline{T}_k)_{k \le k_n}$ stays below the barrier $k \mapsto F_n(k)$ for all $k \le k_n$ and end up in a finite interval.

- **Upper bound**

We have

$$\mathbb{P}(\mathcal{A}_n^{(k)}(x)) = \mathbb{P}\left(\mathcal{A}_n^{(k)}(x), |\overline{T}_{k_n-1} - a_n + x| \le h_n\sqrt{b_n}\right) + \mathbb{P}\left(\mathcal{A}_n^{(k)}(x), |\overline{T}_{k_n-1} - a_n + x| > h_n\sqrt{b_n}\right) \tag{2.3.27}$$

where $(h_n)_{n\in\mathbb{N}}$ is a sequence growing to ∞, that we will fix later on. We bound these two quantities separately choosing h_n such that

$$\limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2}\sqrt{b_n}\mathbb{P}\left(\mathcal{A}_n^{(k)}(x), |\overline{T}_{k_n-1} - a_n + x| > h_n\sqrt{b_n}\right) = 0.$$

We first observe that

$$\mathbb{P}\left(\mathcal{A}_n^{(k)}(x), |\overline{T}_{k_n-1} - a_n + x| > h_n\sqrt{b_n}\right) = \mathbb{P}\left(\mathcal{A}_n^{(k)}(x), \overline{T}_{k_n-1} - a_n + x < -h_n\sqrt{b_n}\right)$$

since the event $\left\{\overline{T}_{k_n-1} \le \overline{F}_n(k_n-1) - x, \overline{T}_{k_n-1} - a_n + x > h_n\sqrt{b_n}\right\}$ is impossible for n large enough. Then we have,

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2}\sqrt{b_n}\mathbb{P}\left(\mathcal{A}_n^{(k)}(x), \overline{T}_{k_n-1} - a_n + x < -h_n\sqrt{b_n}\right) \\
&\le \limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2}\sqrt{b_n}\mathbb{P}\left(\overline{T}_{k_n} - a_n + x \in [-a,0], \overline{T}_{k_n-1} - a_n + x < -h_n\sqrt{b_n}\right) \\
&\le \limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2}\sqrt{b_n}\mathbb{P}(|\overline{T}_1| \ge \sqrt{b_n}h_n).
\end{aligned}$$

On the other hand we write

$$\mathbb{P}(|\overline{T}_1| \ge \sqrt{b_n}h_n) = \mathbb{P}(\overline{T}_1 \ge \sqrt{b_n}h_n) + \mathbb{P}(-\overline{T}_1 \ge \sqrt{b_n}h_n),$$

and additionally we have

$$\mathbb{P}\left(\overline{T}_1 \ge \sqrt{b_n}h_n\right) = \mathbb{P}\left(e^{\theta\overline{T}_1} \ge e^{\theta h_n\sqrt{b_n}}\right) \le e^{b_n(\Lambda(\theta) - \theta h_n\sqrt{b_n})},$$

for all $\theta > 0$. As Λ is $\mathcal{C}^2$ on a neighbourhood of 0, we have

$$\Lambda(\theta) = \Lambda(0) + \Lambda'(0)\theta + \frac{\Lambda''(0)\theta^2}{2} + o(\theta^2)) = \Lambda''(0)\theta^2/2 + o(\theta^2).$$

Therefore, choosing $\theta = \frac{h_n}{\sqrt{b_n}}$, there exists $C > 0$ such that for all n large enough, we have

$$\mathbb{P}(\overline{T}_1 \ge \sqrt{b_n}h_n) \le e^{-Ch_n^2}.$$

Similarly, we have $\mathbb{P}(\overline{T}_1 \leq -\sqrt{b_n}h_n) \leq e^{-Ch_n^2}$ for n large enough. Finally we obtain

$$\limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2}\sqrt{b_n}\mathbb{P}\left(\mathcal{A}_n^{(k)}(x), \overline{T}_{k_n-1} - a_n + x < -h_n\sqrt{b_n}\right)$$
$$\leq \limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2}\sqrt{b_n}e^{-Ch_n^2}$$

which goes to zero as $n \to \infty$ as long as $h_n > 2\sqrt{\frac{\log(n)}{C}}$.

We now bound the first quantity in the right hand-side of (2.3.27). Applying the Markov property at time $k_n - 1$ we get

$$\mathbb{P}\left(\mathcal{A}_n^{(k)}(x), |\overline{T}_{k_n-1} - a_n + x| \leq h_n\sqrt{b_n}\right) \tag{2.3.28}$$
$$= \mathbb{E}\left(f_n(\overline{T}_j + a_n - x)\mathbf{1}_{\left\{\overline{T_k}\leq \overline{F}_n(k)-x, |\overline{T}_{k_n-1}-a_n+x|\leq h_n\sqrt{b_n}, k\leq k_n-1|\right\}}\right)$$

where $f_n(z) = \mathbb{P}_z\left(\overline{T}_1 \in [-a, 0]\right)$ for all $z \in \mathbb{R}$. We estimate the function $z \mapsto f_n(z)$, $n \in \mathbb{N}$, using the refined Stone's local limit theorem in [Bor17, Theorem 2.1]. By $(\mathbf{H_2})$ there exists a constant $c > 0$ such that for all $z \in \mathbb{R}$

$$f_n(z) \leq \frac{a}{\sqrt{2\pi b_n\sigma^2}}\exp\left(\frac{-(z - a_n + x)^2}{2b_n}\right) + \frac{c}{b_n}.$$

To approximate (2.3.28) for a random walk satisfying $(\mathbf{H}_2)$ we apply Theorem 2.3.7, there exists a constant $C > 0$ such that for all $\gamma > 0$ we have

$$\mathbb{P}\left(\mathcal{A}_n^{(k)}(x), |\overline{T}_{k_n-1} - a_n + x| \leq h_n\sqrt{b_n}\right) \tag{2.3.29}$$
$$\leq \frac{C}{n^\gamma} + \frac{a}{\sqrt{2\pi b_n\sigma^2}}\mathbb{E}\left(\sup_{|w|\leq d_n} \exp\left(\frac{-(\widehat{S}_{k_n-1} + w - a_n + x)^2}{2b_n\sigma^2}\right)\mathbf{1}_{\mathcal{B}_n^{(k)}(x)}\right) + \frac{c}{b_n}\mathbb{P}\left(\mathcal{B}_n^{(k)}(x)\right).$$

where $\mathcal{B}_n^{(k)}(x) = \left\{\widehat{S}_k \leq \widetilde{F}_n(k) + d_n - x, k \leq k_n - 1, |\widehat{S}_{k_n-1} - a_n + x| \leq h_n\sqrt{b_n} + d_n\right\}$. We write then

$$\mathbb{E}\left(\sup_{|w|\leq d_n} \exp\left(\frac{-(\widehat{S}_{k_n-1} + x - a_n + w)^2}{2b_n}\right)\mathbf{1}_{\mathcal{B}_n^{(k)}(x)}\right) \tag{2.3.30}$$
$$= \mathbb{E}_{\mathbb{Q}}\left(\sup_{|w|\leq d_n} e^{-\frac{a_n}{n}\sqrt{b_n}\widetilde{S}_{k_n-1}+n\Lambda(a_n/n)}\exp\left(\frac{-(\sqrt{b_n}\widetilde{S}_{k_n-1} + x - a_n + w)^2}{2b_n}\right)\mathbf{1}_{\mathcal{C}_n^k(x)}\right)$$

where

$$\mathcal{C}_n^{(k)}(x) = \left\{\widetilde{S}_k \leq \frac{-x + d_n - c_n}{\sqrt{b_n}}, k \leq k_n - 1, |\sqrt{b_n}\widetilde{S}_{k_n-1} + \frac{a_n}{k_n} - x| \leq h_n\sqrt{b_n} + d_n\right\}$$

and $\sqrt{b_n}\widetilde{S}_k = \sqrt{b_n}\widehat{S}_k - \frac{k}{k_n}a_n$ is a centred Gaussian random walk under the measure $\mathbb{Q}$ defined in the proof of Lemma 2.3.6. Then using the dominated convergence theorem and the fact that

$\frac{d_n^2}{b_n} \to_{n\to\infty} 0$, and rescaling by $\frac{1}{\sqrt{b_n}}$ we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}\left(\exp\left(\sup_{|w|\le d_n} \frac{-(\widehat{S}_{k_n-1}+w-a_n+x)^2}{2b_n\sigma^2}\right)\mathbf{1}_{\mathcal{C}_n^{(k)}(x)}\right)\\
&\le \limsup_{n\to\infty} \mathbb{E}_{\mathbb{Q}}\left(\exp\left(\frac{-(\sqrt{b_n}\widetilde{S}_{k_n-1}+x)^2}{2b_n\sigma^2}\right)\mathbf{1}_{\left\{\widetilde{S}_k\le \frac{1}{\sqrt{b_n}}(x+d_n-c_n),k\le k_n-1\right\}}\right)\\
&\le \limsup_{n\to\infty} \mathbb{E}_{\mathbb{Q}}\left(\exp\left(\frac{-(\widetilde{S}_{k_n-1}+\frac{-x}{\sqrt{b_n}})^2}{2\sigma^2}\right)\mathbf{1}_{\left\{\widetilde{S}_k\le \frac{-x+d_n-c_n}{\sqrt{b_n}},k\le k_n-1\right\}}\right).
\end{aligned}$$

Applying Lemma 2.3 in [Aid13] for the Gaussian random walk $(\widetilde{S}_k)_{k\le k_n}$, and using Lemma 2.3.10 we obtain for all $\frac{x}{\sqrt{b_n}} \in [-\frac{r_n}{\sqrt{b_n}}, -\frac{(c_n+d_n)}{\sqrt{b_n}}]$,

$$\begin{aligned}
&\limsup_{n\to\infty} k_n^{\frac{3}{2}} \mathbb{E}_{\mathbb{Q}}\left(\exp\left(\frac{-(\widetilde{S}_{k_n-1}+\frac{-x}{\sqrt{b_n}})^2}{2\sigma^2}\right)\mathbf{1}_{\left\{\widetilde{S}_k\le \frac{-x+d_n-c_n}{\sqrt{b_n}},k\le k_n-1\right\}}\right)\\
&\le \frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty e^{-\frac{y^2}{2\sigma^2}}L(y)dy \times L(\frac{-x+d_n-c_n}{\sqrt{b_n}}).
\end{aligned}$$

We now observe that by (2.3.9), $\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty L(y)e^{-\frac{y^2}{2\sigma^2}}dy = \mathbb{E}\left(L(\widehat{S}_1)\mathbf{1}_{\left\{\widehat{S}_1\ge 0\right\}}\right)) = L(0) = 1$, which implies

$$\begin{aligned}
&\limsup_{n\to\infty} k_n^{\frac{3}{2}} \mathbb{E}_{\mathbb{Q}}\left(\exp\left(\frac{-(\sqrt{b_n}\widetilde{S}_{k_n-1}+x)^2}{2b_n\sigma^2}\right)\mathbf{1}_{\left\{\widetilde{S}_k\le \frac{-x+d_n-c_n}{\sqrt{b_n}},k\le k_n-1\right\}}\right)\\
&\le L(\frac{-x+d_n-c_n}{\sqrt{b_n}}).
\end{aligned}$$

To complete the proof of the upper bound it remains to show that

$$\limsup_{n\to\infty} \sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1} k_n^{3/2} b_n^{-\frac{1}{2}} \mathbb{E}\left(\mathbf{1}_{\mathcal{B}_n^k(x)}\right) = 0. \qquad (2.3.31)$$

Using similar computations we have

$$\begin{aligned}
&\sup_{x\in[-r_n,-c_n]} k_n^{\frac{3}{2}} \mathbb{P}\left(\widehat{S}_k \le \widetilde{F}_n(k)+d_n-x, k\le k_n-1, |\widehat{S}_{k_n-1}-a_n+x| \le h_n\sqrt{b_n}+d_n\right)\\
&\le \sup_{x\in[-r_n,-c_n]} k_n^{\frac{3}{2}} \mathbb{P}\left(\widetilde{S}_k \le \frac{-x+d_n-c_n}{\sqrt{b_n}}, \widetilde{S}_{k_n-1} - \frac{a_n}{k_n\sqrt{b_n}} + \frac{x}{\sqrt{b_n}} \ge -h_n - \frac{d_n}{\sqrt{b_n}}, k\le k_n-1\right)\\
&\le CL(\frac{-x+d_n-c_n}{\sqrt{b_n}})\int_0^{h_n+\frac{2d_n}{\sqrt{b_n}}} L(y)dy.
\end{aligned}$$

Since the renewal function $x \mapsto R(x)$ is increasing we have by (2.3.12)

$$\sup_{x\in[-r_n,-c_n]} k_n^{\frac{3}{2}}\mathbb{P}\left(\widehat{S}_k \le \widetilde{F}_n(k)+d_n-x, k\le k_n-1, |\widehat{S}_{k_n-1}-a_n+x|\le h_n\sqrt{b_n}+d_n\right)$$
$$\le C(h_n+2\frac{d_n}{\sqrt{b_n}})L(\frac{-x+d_n-c_n}{\sqrt{b_n}})L(h_n+\frac{d_n}{\sqrt{b_n}})\le C(h_n+\frac{2d_n}{\sqrt{b_n}})^2L(\frac{-x+d_n-c_n}{\sqrt{b_n}})$$

where we used in the last inequality that $L(t_n)\le ct_n$ for some constant $c>0$ when $t_n\to_{n\to\infty}\infty$. Thanks to Lemma 2.3.10 we obtain

$$\sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1}k_n^{3/2}b_n^{-\frac{1}{2}}\mathbb{P}\left(\mathcal{B}_n^{(k)}(x)\right)\le C\frac{(h_n+\frac{2d_n}{\sqrt{b_n}})^2}{\sqrt{b_n}}$$

which goes to zero since $\frac{\log(n)+d_n}{\sqrt{b_n}}\to 0$ as $n\to\infty$. Finally by (2.3.33), (2.3.31) and Lemma 2.3.10 we deduce that

$$\limsup_{n\to\infty}\sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1}\sqrt{b_n}k_n^{\frac{3}{2}}\mathbb{E}(h(\overline{T}_{k_n}-a_n+x)\mathbf{1}_{\{\overline{T}_k\le\overline{F}_n(k)-x,k\le k_n\}}) \tag{2.3.32}$$
$$\le\limsup_{n\to\infty}\sup_{x\in[-r_n,-c_n]} L(\frac{-x}{\sqrt{b_n}})^{-1}\sqrt{b_n}k_n^{\frac{3}{2}}\mathbb{P}\left(\mathcal{A}_n^{(k)}(x),|\overline{T}_{k_n-1}-a_n+x|\le h_n\sqrt{b_n}\right)$$
$$\le\frac{1}{\sqrt{2\sigma^2\pi}}\int_{-\infty}^0 f(y)e^{-\theta^* y}dy.$$

We now treat the lower bound.

- **Lower bound**

We now compute a lower bound for $\mathbb{P}\left(\mathcal{A}_n^{(k)}(x)\right)$. Using similar arguments to these used in the upper bound we have by Theorem 2.3.7

$$\mathbb{P}\left(\overline{T}_k\le\overline{F}_n(k)-x,k\le k_n,\overline{T}_{k_n}-a_n+x\in[-a,0]\right)$$
$$\ge\frac{a}{\sqrt{2\pi b_n\sigma^2}}\mathbb{E}\left(\inf_{|w|\le d_n}\exp\left(\frac{-(\sqrt{b_n}\widehat{S}_{k_n-1}+w-a_n+x)^2}{2b_n\sigma^2}\right)\mathbf{1}_{\left\{\sqrt{b_n}\widetilde{S}_k\le\overline{F}_n(k)-d_n-x,k\le k_n-1\right\}}\right).$$

Similar computations to these used in the upper bound lead to

$$\liminf_{n\to\infty}\mathbb{E}\left(\inf_{|w|\le d_n}\exp\left(\frac{-(\sqrt{b_n}\widehat{S}_{k_n-1}+x-\frac{a_n}{n}+w)^2}{2b_n}\right)\mathbf{1}_{\left\{\widehat{S}_k\le\overline{F}_n(k)-x-d_n,k\le k_n-1\right\}}\right)$$
$$\ge\liminf_{n\to\infty}\mathbb{E}_{\mathbb{Q}}\left(\exp\left(\frac{-(\widetilde{S}_{k_n-1}+\frac{x}{\sqrt{b_n}})^2}{2\sigma^2}\right)\mathbf{1}_{\left\{\widetilde{S}_k\le\frac{-x-d_n-c_n}{\sqrt{b_n}},k\le k_n-1\right\}}\right).$$

Applying again Lemma 2.3 in [Aid13] for the Gaussian random walk $(\widetilde{S}_k)_{k\le k_n}$, we obtain for all $x\in[-r_n,-(c_n+d_n)]$,

$$\liminf_{n\to\infty}k_n^{\frac{3}{2}}\mathbb{E}_{\mathbb{Q}}\left(\exp\left(\frac{-(\widetilde{S}_{k_n-1}+\frac{x}{\sqrt{b_n}})^2}{2\sigma^2}\right)\mathbf{1}_{\left\{\widetilde{S}_k\le\frac{-x-d_n-c_n}{\sqrt{b_n}},k\le k_n-1\right\}}\right)$$
$$\ge\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty e^{-\frac{y^2}{2\sigma^2}}R(y)dy\times L(\frac{-x-(c_n+d_n)}{\sqrt{b_n}}).$$

Finally, by Lemma 2.3.10 we get

$$\liminf_{n\to\infty} \inf_{x\in[-r_n,-(c_n+d_n)]} (L(\frac{-x}{\sqrt{b_n}})^{-1}\sqrt{b_n}k_n^{\frac{3}{2}}\mathbb{E}(h(\overline{T}_{k_n}-a_n+x)\mathbf{1}_{\{\overline{T}_k\le \overline{F}_n(k)-x,k\le k_n\}}) \ge \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{0} f(y)e^{-\theta^* y}dy. \tag{2.3.33}$$

Combining equations (2.3.32) and (**??**) we deduce that

$$\mathbb{E}\left(h(\overline{T}_{k_n}-a_n+x)\mathbf{1}_{\{\overline{T}_k\le \overline{F}_n(k)-x,k\le k_n\}}\right) = \frac{e^{\theta^* x}}{\sqrt{2\pi\sigma^2}\sqrt{b_n}k_n^{\frac{3}{2}}}\int_{-\infty}^{0} f(y)e^{-\theta^* y}dy(L(\frac{-x}{\sqrt{b_n}})+o(1))$$

uniformly in $x \in [-r_n, -(c_n+d_n)]$. □

2.4 The modified extremal process

Recall that $(\overline{T_k})_{1\le k\le k_n}$ is a sequence of centred random walk with $\mathrm{Var}(\overline{T_k}) = kb_n\sigma^2$ and $a_n = m_n - k_nb_nv$. The goal of the next section is to introduce a modified extremal process and to prove that it has the same weak limit as the original extremal process $\mathcal{E}_n$.

Start by setting

$$\mathcal{E}_{n,R_n} = \sum_{u\in\mathcal{H}_{k_n}} \delta_{S_u-m_n}\mathbf{1}_{\{S_{u_k}\le R_n(k),\forall k\le k_n\}},$$

where refer to the function $R_n : \{0...,k_n\} \mapsto \mathbb{R}$ as a barrier. More precisely our objective is to prove that the weak limit of the modified extremal process $\mathcal{E}_{n,R_n}$ and the original extremal process $\mathcal{E}_n$ coincide for a well-chosen function R_n. The main steps of the proof of Theorem 2.2.1 are the following:

- We show that there exists an upper barrier such that, with high probability, all individuals stay below it all most of time.
- The second step is to locate the paths of extremal individuals. Here the method is inspired from the work of Arguin, Bovier and Kistler in [**?**] in the context of branching Brownian motion.
- We show that all individuals contributing in the extremal process split from the root.

We start by proving that there exists a barrier R_n such that, with high probability, all individuals stay below it all most of time.

Lemma 2.4.1. *Consider the barrier*

$$R_n(k) = kb_nv - \frac{3}{2\theta^*}\log(\frac{k_nb_n+1}{(k_n-k)b_n+1}) + c_n, k = 0...,k_n$$

where $(c_n)_{n\in\mathbb{N}}$ *is the sequence of integers defined in* (**??**). *It then holds:*

$$\mathbb{P}(\exists u \in \mathcal{H}_{k_n}, S_{u_k} > R_n(k),\ \textit{for some}\ k \le k_n) = o(1)\ \textit{when}\ n \uparrow \infty.$$

Proof. Using Markov inequality we get

$$\mathbb{P}(\exists |u| = k_n, S_{u_k} > R_n(k), k \leq k_n) \leq \sum_{k \leq k_n} \mathbb{E}\left(\sum_{|u|=k} \mathbf{1}_{\{S_{u_k} > R_n(k), S_{u_j} \leq R_n(j), j<k\}} \right).$$

By Proposition 2.3.1 we have,

$$\begin{aligned}
&\sum_{k \leq k_n} \mathbb{E}\left(\sum_{|u|=k} \mathbf{1}_{\{S_{u_k} > R_n(k), S_{u_j} \leq R_n(j), j<k\}} \right) \\
&\leq \sum_{k \leq k_n} \mathbb{E}\left(\exp(-\theta^* \overline{T}_k) \mathbf{1}_{\{\overline{T}_k > R_n(k) - kb_n v, \overline{T}_j \leq R_n(j) - jb_n v, j<k\}} \right) \\
&\leq e^{-\theta^* c_n} \sum_{k \leq k_n} \frac{(k_n b_n + 1)^{\frac{3}{2}}}{(k_n - k) b_n + 1)^{\frac{3}{2}}} \mathbb{P}(\overline{T}_k > \overline{R}_n(k), \overline{T}_j \leq \overline{R}_n(j), j < k),
\end{aligned} \tag{2.4.1}$$

where $\overline{R}_n(j) = R_n(j) - jb_n v$, for all $1 \leq j \leq k_n$. We compute this probability by conditioning with respect to the last step $\overline{T}_k - \overline{T}_{k-1}$ to get

$$\mathbb{P}(\overline{T}_k > \overline{R}_n(k), \overline{T}_j \leq \overline{R}_n(j), j \leq k) = \mathbb{E}(f_{k-1}(\overline{T}_k - \overline{T}_{k-1}))$$

where, $\forall y \in \mathbb{R}$

$$f_{k-1}(y) = \mathbb{P}\left(\overline{R}_n(k) - y \leq \overline{T}_{k-1} \leq \overline{R}_n(k), B_n(j) \leq \frac{1}{\sqrt{b_n}} \overline{R}_n(j)), j \leq k-1 \right).$$

Assume that $(\mathbf{H_1})$ or $(\mathbf{H_2})$ hold, by Lemma 2.3.5 or 2.3.9 and using the fact that $\frac{c_n}{\sqrt{b_n}} \to_{n\to\infty} 0$, we deduce that, for n large enough, there exists $C > 0$ such that, $\forall y \in \mathbb{R}$

$$f_{k-1}(y) \leq C \mathbf{1}_{\{y \geq 0\}} \frac{(1 + \frac{y}{\sqrt{b_n}})^3}{k^{3/2}}.$$

Now Plugging this in (2.4.1) we obtain

$$\begin{aligned}
&\mathbb{P}(\exists |u| = k_n, S_{u_k} > R_n(k), k \leq k_n) \\
&\leq C e^{-\theta^* c_n} \sum_{k \leq k_n} \frac{(k_n b_n + 1)^{\frac{3}{2}}}{((k_n - k) b_n + 1)^{\frac{3}{2}}} \frac{1}{k^{\frac{3}{2}}} (1 + \mathbb{E}(\frac{\overline{T_k} - \overline{T}_{k-1}}{\sqrt{b_n}})_+^3) \to_{n\to\infty} 0.
\end{aligned}$$

completing the proof. □

From this lemma we deduce that the extremal process $\mathcal{E}_{n,R_n}$ has the same weak limit of the one of $\mathcal{E}_n$. The second step of the proof of the main theorem is to locate the paths of extremal individuals. To do that we will consider a barrier which is lower. For the choice of such barrier we refer to the work of Arguin, Bovier and Kistler [?] in the case of branching Brownian motion.

Proposition 2.4.2. *Define the barrier*

$$F_n(k) = kb_n v + \frac{k}{k_n} a_n - c_n \mathbf{1}_{k \neq 0, k_n}, k = 0..., k_n.$$

Let $A = [a, \infty)$ where $a \in \mathbb{R}$, then we have

$$\lim_{n\to\infty} \mathbb{E}(\mathcal{E}_{n,R_n}(A) - \mathcal{E}_{n,F_n}(A)) = 0.$$

Proof. Let the following subsets of $\mathcal{H}_{k_n}$

$$\mathbf{A}_n^{(u)} = \{u \in \mathcal{H}_{k_n} : S_{k_n} - m_n \in A, S_k \le R_n(k), k \le k_n, \}$$

the set of particles at generation k_n that are close to the maximum and that stay below the barrier $k \mapsto R_n(k)$ for all $k \le k_n$. Respectively we introduce

$$\mathbf{B}_n^{(u)} = \{u \in \mathcal{H}_{k_n} : S_{k_n} - m_n \in A, S_k \le F_n(k), k \le k_n\}.$$

Set the integer-valued variable

$$\#(\mathbf{A}_n^{(u)} \cap (\mathbf{B}_n^{(u)})^c) = \#\{u \in \mathcal{H}_{k_n} : S_{k_n} - m_n \in A, S_k \le R_n(k), k \le k_n, \exists j \le k_n, S_j > F_n(j)\}.$$

Using the fact that $\mathbf{B}_n^{(u)} \subset \mathbf{A}_n^{(u)}$ we deduce that

$$\begin{aligned}&\mathbb{E}(\mathcal{E}_{n,R_n}(A) - \mathcal{E}_{n,F_n}(A)) = \mathbb{E}(\#(\mathbf{A}_n^{(u)} \cap (\mathbf{B}_n^{(u)})^c))\\ &= \mathbb{E}\left(\sum_{|u|=k_n} \mathbf{1}_{\{S_{k_n}-m_n\in A, S_k\le R_n(k), k\le k_n, \exists j\le k_n, S_j>F_n(j)\}}\right),\end{aligned}$$

then thanks to Proposition 2.3.1 we obtain

$$\begin{aligned}&\mathbb{E}\left(\sum_{|u|=k_n} \mathbf{1}_{\{S_{k_n}-m_n\in A, S_k\le R_n(k), k\le k_n, \exists j\le k_n, S_j>F_n(j)\}}\right)\\ &= \mathbb{E}\left(e^{-\theta^*\overline{T}_{k_n}} \mathbf{1}_{\{\overline{T}_{k_n}-a_n\in A, \overline{T}_k\le \overline{R}_n(k), k\le k_n, \exists j\le k_n, \overline{T}_j>\overline{F}_n(j)\}}\right),\end{aligned}$$

where $a_n = m_n - k_n b_n v$ and $\overline{F}_n(j) = \frac{j}{k_n} a_n - c_n \mathbf{1}_{j\ne 0,k_n}$ for all $1 \le j \le k_n$. Summing with respect to the value of $T_{k_n} - a_n - a$ at time k_n, we have

$$\begin{aligned}&\mathbb{E}\left(e^{-\theta^*\overline{T}_{k_n}} \mathbf{1}_{\{\overline{T}_{k_n}-a_n\in A, \overline{T}_k\le R_n(k)-kb_nv, k\le k_n, \exists j\le k_n, \overline{T}_j>\overline{F}_n(j)\}}\right)\\ &\le \frac{Cn^{\frac{3}{2}}}{b_n} \sum_{r\ge 0} e^{-\theta^* r} \sum_{0\le j\le k_n} \mathbb{P}\left(\overline{T}_{k_n} - a_n - a \in [r, r+1], \overline{T}_k \le \overline{R}_n(k), k \le k_n, \overline{T}_j > \overline{F}_n(j)\right).\end{aligned}$$

Applying the Markov property at time j we get

$$\mathbb{P}\left(\overline{T}_{k_n} - a_n - a \in [r, r+1], \overline{T}_k \le \overline{R}_n(k), k \le k_n, \overline{T}_j > \overline{F}_n(j)\right)$$

$$\le \mathbb{P}\left(\overline{T}_k \le \overline{R}_n(k), k \le j, \overline{T}_j > \overline{F}_n(j)\right) \times \tag{2.4.2}$$

$$\sup_{x\in[F_n(j), R_n(j)]} \mathbb{P}_x\left(\overline{T}_{k_n-j} - a_n - a \in [r, r+1], \overline{T}_k \le \overline{R}_n(k+j), k \le k_n - j\right). \tag{2.4.3}$$

To bound the probability (**??**), we apply the Markov property at time $l = [\frac{j}{3}]$,

$$\begin{aligned}
&\mathbb{P}(\overline{T}_j > \overline{F}_n(j), \overline{T}_k \leq \overline{R}_n(k), k \leq j) \\
&\leq \mathbb{P}(\overline{T}_k \leq \overline{R}_n(k), k \leq l) \sup_{z \leq \overline{R}_n(l)} \mathbb{P}_z\left(\overline{T}_{j-l} > \overline{F}_n(j), \overline{T}_k \leq {}^{\backprime}\overline{R}_n(k+l), k \leq j-l\right).
\end{aligned}$$

Set $\widehat{T_k} = \overline{T}_{j-l} - \overline{T}_{j-l-k}$, which is a random walk with the same law as $\overline{T_k}$. Then we obtain

$$\begin{aligned}
&\mathbb{P}_z\left(\overline{T}_{j-l} > \overline{F}_n(j), \overline{T}_k \leq \overline{R}_n(k+l), k \leq j-l\right) \\
&\leq \mathbb{P}_z\left(\overline{F}_n(j) < \overline{T}_{j-l} \leq \overline{R}_n(j), \overline{T_k} \geq F_n(j) - R_n(j-k), k \leq j-l\right).
\end{aligned} \tag{2.4.4}$$

We bound the probability in (2.4.4). We use a lower bound for the expression $(F_n(j)-R_n(j-k), k \leq j-l)$. Observe that the function $x \mapsto \frac{\log(x)}{x}$ is decreasing for $x \geq e$, and

$$(k_n - j)b_n + 1 + kb_n \leq 2((k_n-j)b_n + 1)kb_n,$$

then we have

$$\begin{aligned}
&F_n(j) - R_n(j-k) = \frac{-3}{2\theta^*}\left(\frac{j}{k_n}\log(k_n b_n) - \log(k_n b_n) + \log((k_n - j + k)b_n + 1)\right) + \frac{\ln(b_n)}{\theta^*} - 2c_n \\
&\geq \frac{-3}{2\theta^*}\left(\log((j \vee e)b_n) - \log(k_n b_n) + \log((k_n-j)b_n + 1) + \log(kb_n) + \log(2)\right) - 2c_n \\
&\geq \frac{-3}{2\theta^*}\left(\log((j \vee e) \wedge ((k_n - j) + 1)) + \log(kb_n) + \log(2)\right) - 2c_n, \forall k, j = 1....k_n.
\end{aligned}$$

Applying again the Markov property at time l we get

$$\begin{aligned}
&\mathbb{P}_z\left(\overline{F}_n(j) < \overline{T}_{j-l} \leq \overline{R}_n(j), \overline{T_k} \geq F_n(j) - R_n(j-k), k \leq j-l\right) \\
&\leq \mathbb{P}\left(\overline{T}_k \geq \frac{-3}{2\theta^*}\left(\log(kb_n) + \log(j \wedge ((k_n-j)+1)) + \log(2)\right) - 2c_n, k \leq l\right) \times \\
&\mathbb{P}_x\left(\overline{F}_n(j) - z < \overline{T}_{j-2l} \leq \overline{R}_n(j) - z\right).
\end{aligned}$$

Assume that $(\mathbf{H_1})$ or $(\mathbf{H_2})$ hold, then by Lemmas 2.3.2, 2.3.3 or 2.3.8 for n large enough we obtain

$$\begin{aligned}
&\mathbb{P}\left(\overline{T}_k \leq \overline{R}_n(k), k \leq j, \overline{T}_j > \overline{F}_n(j)\right) \\
&\underbrace{\leq \mathbb{P}\left(B_n(k) \leq \frac{1}{\sqrt{b_n}}\overline{R}_n(k), k \leq l\right)}_{\leq \frac{C}{\sqrt{j}}} \times \\
&\underbrace{\mathbb{P}\left(B_n(k) \geq \frac{-3}{2\theta^*}\left(\log(j \wedge ((k_n-j)+1)) + \log(kb_n) + \log(2)\right) - 2\frac{c_n}{\sqrt{b_n}}, k \leq l\right)}_{\leq C\frac{1+\log(j \wedge ((k_n-j)+1))}{\sqrt{j}}} \times \\
&\underbrace{\mathbb{P}_x\left(\frac{1}{\sqrt{b_n}}(\overline{F}_n(j) - z) < B_n(j-2l) \leq \frac{1}{\sqrt{b_n}}(\overline{R}_n(j) - z)\right)}_{\leq C\frac{\log(j \wedge ((k_n-j)+1))+1}{\sqrt{jb_n}}} \leq C\frac{(1 + (\log(j \wedge ((k_n-j)+1)))^2}{\sqrt{b_n} j^{\frac{3}{2}}}.
\end{aligned}$$

Using Lemma 2.3.5 or 2.3.9, for n large enough we have

$$\sup_{x\in[F_n(j),R_n(j)]} \mathbb{P}_x\left(\overline{T}_{k_n-j}-a_n-a\in[r,r+1],\overline{T}_k\leq \overline{R}_n(k+j),k\leq k_n-j\right)$$
$$\leq C\frac{(1+2d_n)}{\sqrt{b_n}}\frac{(1+\log(j\wedge((k_n-j)+1)))^2}{(k_n-j)^{\frac{3}{2}}},$$

therefore, we conclude that

$$\mathbb{E}(\mathcal{E}_{n,R_n}(A)-\mathcal{E}_{n,F_n}(A))\leq C\frac{(1+2d_n)}{\sqrt{b_n}}k_n^{\frac{3}{2}}\sum_{j\leq k_n}\frac{(1+(\log(j\wedge((k_n-j)+1))))^4}{((k_n-j)+1)^{\frac{3}{2}}j^{\frac{3}{2}}}$$
$$\leq 2C\frac{(1+2d_n)}{\sqrt{b_n}}\sum_{j\leq[k_n/2]}\frac{(1+\log(j))^4}{j^{\frac{3}{2}}}\to_{n\to\infty}0,$$

as $\sum_{j\leq[k_n/2]}\frac{(1+\log(j))^4}{j^{\frac{3}{2}}}<\infty$. □

This lemma implies that the two extremal processes $\mathcal{E}_{n,F_n}$ and $\mathcal{E}_{n,R_n}$ have the same weak limit. Consequently, the the same as the o ne of $\mathcal{E}_n^{b_n}$. For $u\in\mathcal{T}^{(n)}$, we introduce

$$H_n(u)=\{S_{u_k}\leq F_n(k),k\leq k_n\}$$

the set of individuals satisfying the F_n-barriers. The last step of the proof of our result is to show that, with high probability, the set of pairs of extremal particles that branch off at time $k\geq 1$ and stay all the time below the barrier $k\mapsto F_n(k)$ is vanishing in the large n-limit. This show that all particles contributing in the extremal process split from the root.

Lemma 2.4.3. *With the same notation used before, we have,*

$$\lim_{n\to\infty}\mathbb{E}\left(\#\{(u,v),|u\wedge v|\geq 1,H_n(u),H_n(v),S_u-m_n\in A,S_v-m_n\in A\}\right)=0.$$

Proof. By considering the positions of any pairs of individuals (u,v) at the generation k_n and at their common ancestors $u\wedge v$ we have

$$\mathbb{E}\left(\#\{(u,v),|u\wedge v|\geq 1,S_u-m_n\in A,S_v-m_n\in A,H_n(u),H_n(v)\}\right)$$
$$=\mathbb{E}\left(\sum_{j=1}^{k_n-1}\sum_{|w|=j}\mathbf{1}_{\{S_{w_i}\leq F_n(i),i\leq j\}}\sum_{(u_{j+1},v_{j+1})}\sum_{(u,v)}\mathbf{1}_{\{S_u-m_n\in A,S_v-m_n\in A,S_{u_k}\leq F_n(k),S_{v_k}\leq F_n(k),j+1\leq k\leq k_n\}}\right)$$

where the double sum $\sum_{(u_{j+1},v_{j+1})}$ is over pairs (u_{j+1},v_{j+1}) of distinct children of $w=u\wedge v$ and $\sum_{(u,v)}$ is over pairs (u,v) such that $|u|=|v|=k_n$ and u is a descendant of u_{j+1}, and v is a descendant of v_{j+1} . Applying the Markov property at time $j+1$ we get

$$\mathbb{E}\left(\#\{(u,v),|u\wedge v|\geq 1,S_u^{b_n}-m_n\in A,S_v^{b_n}-m_n\in A,H_n(u),H_n(v)\}\right)$$
$$\leq\mathbb{E}\left(\sum_{j=1}^{k_n-1}\sum_{|w|=j}\mathbf{1}_{\{S_{w_i}\leq F_n(i),i\leq j\}}\sum_{(u_{j+1},v_{j+1})}\mathbf{1}_{\{S_{u_{j+1}}\leq F_n(j+1),S_{v_{j+1}}\leq F_n(j+1)\}}\varphi_{j,n}(S_{u_{j+1}})\varphi_{j,n}(S_{v_{j+1}})\right),\quad (2.4.5)$$

where

$$\varphi_{j,n}(z) = \mathbb{E}\left(\sum_{|u|=k_n-j-1} \mathbf{1}_{\{z+S_u-m_n\in A, S_{u_k}+z\le F_n(j+k+1), k\le k_n-j-1\}}\right).$$

Now using Proposition 2.3.1 we obtain,

$$\varphi_{j,n}(z) = \mathbb{E}\left(e^{-\theta^*\overline{T}_{k_n-j-1}}\mathbf{1}_{\{z+\overline{T}_{k_n-j-1}-m_n+(k_n-j-1)b_nv\in A, \overline{T}_k+z\le F_n(j+k+1)-kb_nv, k\le k_n-j-1\}}\right).$$

Summing with respect to the value of $\overline{T}_{k_n-j-1} - m_n + (k_n - j - 1)b_nv$ we have

$$\begin{aligned}\varphi_{j,n}(z) &\le \frac{Cn^{\frac{3}{2}}}{b_n}e^{\theta^*(z-(j+1)b_nv)}\sum_{h\ge 0}e^{-\theta^*h}\\ &\times \mathbb{P}_{z-(j+1)b_nv}\left(\overline{T}_{k_n-j-1} - a_n - a \in [h, h+1], \overline{T}_k \le F_n(j+k+1) - kb_nv, k \le k_n - j - 1\right).\end{aligned} \tag{2.4.6}$$

Note that If ($\mathbf{H_2}$) holds by Theorem 2.3.7, we bound the quantity (2.4.6) by

$$\begin{aligned}&\mathbb{P}_z\left(\overline{T}_{k_n-j-1} - m_n + (k_n - j - 1)b_nv - a \in [h, h+1], \overline{T}_k \le F_n(j+k+1) - kb_nv, k \le k_n - j - 1\right)\\ &\le \mathbb{P}_{z-(j+1)b_n}\left(\mathcal{D}_n^{(k)}\right),\end{aligned}$$

where

$$\mathcal{D}_n^{(k)} = \left\{\sqrt{b_n}\widehat{S}_{k_n-j-1} - a_{k_n-j-1} - a \in [h - d_n, h+1+d_n], \sqrt{b_n}\widehat{S}_k \le \overline{F}_n(k) + d_n, , k \le k_n - j - 1\right\}.$$

Now assume that either ($\mathbf{H_1}$) or ($\mathbf{H_2}$) hold. Thanks to Lemma 2.3.5 or 2.3.9 we have,

$$\varphi_{j,n}(z) \le C(1+2d_n)k_n^{\frac{3}{2}}e^{\theta^*(z-(j+1)b_nv)}\frac{(1-\frac{z-(j+1)b_nv}{\sqrt{b_n}})^2}{(k_n-j)^{\frac{3}{2}}}.$$

By replacing this in the equation (2.4.5), we get

$$\begin{aligned}&\mathbb{E}\left(\#((u,v), |u\wedge v| \ge 1, S_u^{b_n} - m_n \in A, S_v^{b_n} - m_n \in A, H_n(u), H_n(v))\right)\\ &\le C(1+2d_n)k_n^{\frac{3}{2}}\mathbb{E}(\sum_{j=1}^{k_n-1}\frac{1}{(k_n-j)^{\frac{3}{2}}}\sum_{|w|=j}\mathbf{1}_{\{S_{w_i}\le F_n(i), i\le j\}}\times\\ &\sum_{(u_{j+1},v_{j+1})}\mathbf{1}_{\{S_{u_{j+1}}\le F_n(j+1), S_{v_{j+1}}\le F_n(j+1)\}}e^{\theta^* S_{u_{j+1}}-(j+1)b_nv+S_{v_{j+1}}-(j+1)b_nv}f_{n,j}(S_{u_{j+1}})f_{n,j}(S_{v_{j+1}}))\end{aligned}$$

where $f_{n,j}(u) = (1+\frac{z-(j+1)b_nv}{\sqrt{b_n}})^2$. In the other hand we can bound the double sum

$$\sum_{(u_{j+1},v_{j+1})}\mathbf{1}_{\{S_{u_{j+1}}\le F_n(j+1), S_{v_{j+1}}\le F_n(j+1)\}}e^{\theta^*[S_{u_{j+1}}-(j+1)b_nv)+S_{v_{j+1}}-(j+1)b_nv]}f_{n,j}(S_{u_{j+1}})f_{n,j}(S_{v_{j+1}})$$

by

$$(1+\frac{S_w-jb_nv}{\sqrt{b_n}})^4 e^{2\theta^*(S_w-jb_nv)}$$
$$\times\mathbb{E}\left(\sum_{\substack{|u|=|v|=1\\u\neq v}}(1+\frac{X_u^{(n)}-b_nv}{\sqrt{b_n}})^2(1+\frac{X_v^{(n)}-b_nv}{\sqrt{b_n}})^2 e^{\theta^*(X_u^{(n)}-b_nv+X_v^{(n)}-b_nv)}\right).$$

Using independence between $X_u^{(n)}$ and $X_v^{(n)}$ for $u\neq v$ and the fact that

$$\mathbb{E}\left(\sum_{|u|=1}(X_u^{(n)}-b_nv)e^{\theta^*(X_u^{(n)}-b_nv)}\right)=0,$$

we have

$$\mathbb{E}\left(\sum_{\substack{|u|=|v|=1\\u\neq v}}(1+\frac{X_u^{(n)}-b_nv}{\sqrt{b_n}})^2(1+\frac{X_v^{(n)}-b_nv}{\sqrt{b_n}})^2 e^{\theta^*(X_u^{(n)}-b_nv+X_v^{(n)}-b_nv)}\right)$$
$$\leq\mathbb{E}\left(\sum_{\substack{|u|=|v|=1\\u\neq v}}(1+(\frac{X_u^{(n)}-b_nv}{\sqrt{b_n}})^2)e^{\theta^*(X_u^{(n)}-b_nv)}(1+(\frac{X_v^{(n)}-b_nv}{\sqrt{b_n}})^2)e^{\theta^*(X_v^{(n)}-b_nv)}\right),$$

then conditioning on Z_{b_n} and using the following properties

$$\mathbb{E}\left((X_u^{(n)}-b_nv)^2e^{\theta^*(X_u^{(n)}-b_nv)}\right)=b_n\Lambda^{''}(\theta^*)=b_n\sigma^2 \text{ and } \mathbb{E}\left(e^{\theta^*(X_u^{(n)}-b_nv)}\right)=m^{-b_n},$$

we obtain

$$\mathbb{E}\left(\sum_{\substack{|u|=|v|=1\\u\neq v}}(1+(\frac{X_u^{(n)}-b_nv}{\sqrt{b_n}})^2)e^{\theta^*(X_u-b_nv)}(1+(\frac{X_v^{(n)}-b_nv}{\sqrt{b_n}})^2)e^{\theta^*(X_v^{(n)}-b_nv)}|Z_{b_n}\right)$$
$$\leq(1+\sigma^2)^2Z_{b_n}(Z_{b_n}-1)m^{-2b_n},$$

as a consequence, there exists a constant $C>0$ such that

$$\mathbb{E}\left(\sum_{\substack{|u|=|v|=1\\u\neq v}}(1+(\frac{X_u^{(n)}-b_nv}{\sqrt{b_n}})^2)e^{\theta^*(X_u-b_nv)}(1+(\frac{X_v^{(n)}-b_nv}{\sqrt{b_n}})^2)e^{\theta^*(X_v-b_nv)}\right)\leq C,$$

which leads to the following inequality

$$\mathbb{E}\left(\#((u,v),|u\wedge v|\geq 1,S_u^{b_n}-m_n\in A,S_v^{b_n}-m_n\in A,H_n(u),H_n(v))\right)$$
$$\leq C(1+2d_n)k_n^{\frac{3}{2}}\mathbb{E}\left(\sum_{j=1}^{k_n-1}\frac{1}{(k_n-j)^{\frac{3}{2}}}\sum_{|w|=j}(1+\frac{S_w-jb_nv}{\sqrt{b_n}})^4e^{2\theta^*(S_w-jb_nv)}\mathbf{1}_{\{S_{w_i}\leq F_n(i),i\leq j\}}\right).$$

On the other hand by Proposition 2.3.1 we obtain

$$\begin{aligned}
&\mathbb{E}\left(\#((u,v),|u\wedge v|\geq 1, S_u-m_n\in A, S_v-m_n\in A, H_n(u),H_n(v))\right)\\
&\leq C(1+2d_n)k_n^{\frac{3}{2}}\mathbb{E}\left(\sum_{j=1}^{k_n-1}\frac{1}{(k_n-j)^{\frac{3}{2}}}(1+\frac{\overline{T}_j}{\sqrt{b_n}})^4 e^{\theta^*\overline{T}_j}\mathbf{1}_{\{\overline{T}_i\leq\overline{F}_n(i),i\leq j\}}\right).
\end{aligned}$$

Summing with respect to the values of $\overline{T_j}-\overline{F}_n(j)$, by Lemma 2.3.5 or 2.3.9 and for n large enough we get

$$\begin{aligned}
&\mathbb{E}\left(\#((u,v),|u\wedge v|\geq 1, S_u-m_n\in A, S_v-m_n\in A, H_n(u),H_n(v))\right)\\
&\leq C(1+2d_n)k_n^{\frac{3}{2}}\sum_{j=1}^{k_n-1}\frac{1}{(k_n-j)^{\frac{3}{2}}}\sum_{r=0}^{\infty}\mathbb{E}\left(e^{\theta^*\overline{T}_j}(1+\frac{\overline{T_j}}{\sqrt{b_n}})^4\mathbf{1}_{\{\overline{T}_i\leq\overline{F}_n(i),\overline{T}_j-\overline{F}_n(j)\in[-r-1,-r],i\leq j\}}\right)\\
&\leq C(1+2d_n)e^{-\theta^* c_n}k_n^{\frac{3}{2}}\sum_{j=1}^{k_n-1}\frac{1}{(k_n-j)^{\frac{3}{2}}}\sum_{r=0}^{\infty}(1+r)^4 e^{-\theta^* r}\\
&\times\mathbb{P}\left(\overline{T}_i\leq\overline{F}_n(i),\overline{T}_j-\overline{F}_n(j)\in[-r-1,-r],i\leq j\right)\\
&\leq\frac{C(1+2d_n)}{\sqrt{b_n}}e^{-\theta^* c_n}\sum_{j=1}^{\lfloor\frac{k_n}{2}\rfloor}\frac{k_n^{\frac{3}{2}}}{(k_n-j)^{\frac{3}{2}}j^{\frac{3}{2}}}\to_{n\to\infty}0,
\end{aligned}$$

where we used that $\sum_{j=1}^{\lfloor\frac{k_n}{2}\rfloor}\frac{k_n^{\frac{3}{2}}}{(k_n-j)^{\frac{3}{2}}j^{\frac{3}{2}}}\leq 2^{\frac{3}{2}}\sum_{j=1}^{\lfloor\frac{k_n}{2}\rfloor}\frac{1}{i^{\frac{3}{2}}}<\infty$. □

Now we are ready to prove our main result.

Proof of Theorem 2.2.1. Let $\varphi\in\mathcal{C}_b^{l,+}$, with support $A=[a,\infty)$ where $a\in\mathbb{R}$. We have to show that

$$\lim_{n\to\infty}\mathbb{E}\left(e^{-\sum_{u\in\mathcal{H}_{k_n}}\varphi(S_u^{(n)}-m_n)}\right)=\mathbb{E}\left(\exp\left(-Z_\infty\frac{1}{\sqrt{2\pi\sigma^2}}\int e^{-\theta^* y}(1-e^{-\varphi(y)})dy\right)\right).\qquad(2.4.7)$$

First introduce

$$G_n=\{\nexists(u,v),|u\wedge v|\geq 1,S_u-m_n\geq a,S_v-m_n\geq a\}.$$

By Lemma 2.4.1 and Proposition 2.4.2, it is enough to prove (2.4.7) for the extremal process

$$\mathcal{E}_{n,F_n}=\sum_{u\in\mathcal{H}_{k_n}}\delta_{S_u-m_n}\mathbf{1}_{H_n}.$$

where $H_n=\{S_{u_k}\leq F_n(k),\forall k\leq k_n\}$. Using Lemma 2.4.3, we have $\mathbb{P}(G_n^c)\to_{n\to\infty}0$, therefore

$$\begin{aligned}
&\mathbb{E}\left(\exp -\left(\sum_{u\in\mathcal{H}_{k_n}} \varphi(S_u - m_n)\mathbf{1}_{H_n}\right)\right)\\
&= \mathbb{E}\left(\mathbf{1}_{G_n}\exp -\left(\sum_{u\in\mathcal{H}_{k_n}} \varphi(S_{u_{k_n}} - m_n)\mathbf{1}_{H_n}\right)\right) + o(1)\\
&= \mathbb{E}\left(\mathbf{1}_{G_n}\prod_{|w|=1}\left(\exp -\left(\sum_{\substack{u>w\\|u|=k_n}} \varphi(S_{u_{k_n}} - m_n)\mathbf{1}_{H_n}\right)\right)\right) + o(1).
\end{aligned}$$

Observe that $\exp\left(-\sum_{i=1}^n x_i\right) = 1 + \sum_{i=1}^n (e^{-x_i} - 1)$ if there exists at most i such that $x_i \neq 0$. Hence using that under G_n, for all w at the first generation, at most one descendant reaches level m_n, we get,

$$\begin{aligned}
&\mathbb{E}\left(\exp -\left(\sum_{u\in\mathcal{H}_{k_n}} \varphi(S_u - m_n)\mathbf{1}_{H_n}\right)\right)\\
&= \mathbb{E}\left(\mathbf{1}_{G_n}\prod_{|w|=1}\left(1 + \sum_{\substack{u>w\\|u|=k_n}} \left(\exp(-\varphi(S_u - m_n + S_w)) - 1\right)\mathbf{1}_{H_n}\right)\right) + o(1)\\
&= \mathbb{E}\left(\prod_{|w|=1}\left(1 + \sum_{\substack{u>w\\|u|=k_n}} \left(\exp(-\varphi(S_u - m_n + S_w)) - 1\right)\mathbf{1}_{H_n}\right)\right) + o(1),
\end{aligned}$$

using again that G_n^c is an event of asymptotically small probability, and that this product of random variable is bounded by 1. We now apply the Markov property at time one to obtain

$$\mathbb{E}\left(\exp -\left(\sum_{u\in\mathcal{H}_{k_n}} \varphi(S_u - m_n)\mathbf{1}_{H_n}\right)\right) = \mathbb{E}\left(\prod_{|w|=1} 1 + \psi_n(S_w)\mathbf{1}_{\left\{S_w - b_n v \leq \overline{F}_n(1)\right\}}\right) + o(1), \quad (2.4.8)$$

where

$$\psi_n : x \mapsto \mathbb{E}\left(\sum_{|u|=k_n-1}\left(e^{-\varphi(S_u - m_n + x)} - 1\right)\mathbf{1}_{\{S_{u_k} \leq F_n(k+1) - x, k<k_n\}}\right), \quad (2.4.9)$$

using Proposition 2.3.1 we have,

$$\psi_n(x) = \mathbb{E}\left(e^{-\theta^*\overline{T}_{k_n-1}} e^{-\varphi(\overline{T}_{k_n-1} - \alpha_{b_n} + x)} - 1)\mathbf{1}_{\{\overline{T}_k \leq \overline{F}_n(k+1) - (x - b_n v), k<k_n\}}\right)$$

where $\alpha_{b_n} = b_n v - \frac{3}{2\theta^*}\ln(n) + \frac{\ln(b_n)}{\theta^*}$.

Using Lemmas 2.3.6 or Corollary 2.3.11, depending on whatever we work under ($\mathbf{H}_1$) or ($\mathbf{H}_2$), we obtain the following approximation,

$$\mathbb{E}\left(e^{-\theta^*\overline{T}_{k_n-1}}e^{-\varphi(\overline{T}_{k_n-1}-\alpha_{b_n}+x)}-1)\mathbf{1}_{\{\overline{T}_k\leq\overline{F}_n(k+1)-(x-b_nv),k<k_n\}}\right)$$
$$\sim_{n\to\infty}\frac{1}{\sqrt{2\pi\sigma^2}}L(\frac{-(x-b_nv)}{\sqrt{b_n}})e^{\theta^*(x-b_nv)}\int e^{-\theta^*y}(e^{-\varphi(y)}-1)dy.$$

Plugging this in Equation (2.4.8) we get

$$\mathbb{E}\left(\exp-\left(\sum_{u\in\mathcal{H}_{k_n}}\varphi(S_u-m_n)\mathbf{1}_{H_n}\right)\right)$$
$$\sim_{n\to\infty}\mathbb{E}\left(\prod_{|w|=1}\mathbb{E}(1+L(-\frac{S_w-b_nv}{\sqrt{b_n}})e^{\theta^*(S_w-b_nv)}\mathbf{1}_{\left\{S_w-b_nv\leq\overline{F}_n(1)\right\}}\frac{1}{\sqrt{2\pi\sigma^2}}\int e^{-\theta^*y}(e^{-\varphi(y)}-1)dy)\right).$$

Recall that $v=\frac{\kappa(\theta^*)}{\theta^*}$, then by Girsanov transform we have

$$\mathbb{E}\left(L(\frac{-(S_w-b_nv)}{\sqrt{b_n}})e^{\theta^*(S_w-b_nv)}\mathbf{1}_{\left\{S_w-b_nv\leq\overline{F}_n(1)\right\}}\right)=m^{-b_n}\mathbb{E}\left(L(\frac{S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\leq\frac{\overline{F}_n(1)}{\sqrt{b_n}}\right\}}\right)$$

where $S_1^{(n)}$ is a centred random walk with variance b_n. Fix $A>0$

$$\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\leq\overline{F}_n(1)\right\}}\right)$$
$$=\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{-A\leq\frac{S_1^{(n)}}{\sqrt{b_n}}\leq\frac{\overline{F}_n(1)}{\sqrt{b_n}}\right\}}\right)+\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\leq-A\right\}}\right).$$

As the function $x\mapsto L(x)\mathbf{1}_{\left\{x\in[-A,\frac{\overline{F}_n(1)}{b_n}]\right\}}$ is bounded, by central limit theorem we have

$$\lim_{n\to\infty}\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{-A\leq\frac{-S_1^{(n)}}{\sqrt{b_n}}\leq\frac{\overline{F}_n(1)}{\sqrt{b_n}}\right\}}\right)=\frac{1}{\sqrt{2\pi}}\int_{-A}^0L(-y)e^{-\frac{y^2}{2}}dy. \tag{2.4.10}$$

Additionally, we have by Equation (2.3.11) and the Markov inequality

$$\left|\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\leq-A\right\}}\right)\right|\leq C\left|\mathbb{E}\left((1+\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\leq-A\right\}}\right)\right|$$
$$\leq C\frac{\mathbb{E}(|S_1^{(n)}|)}{\sqrt{b_n}A}+\mathbb{E}\left(\frac{|S_1^{(n)}|}{\sqrt{b_n}A}\frac{|S_1^{(n)}|}{\sqrt{b_n}}\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\leq-A\right\}}\right)\leq\frac{C}{A}.$$

Therefore, we have

$$\limsup_{n\to\infty}\left|\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\le\frac{\overline{F}_n(1)}{\sqrt{b_n}}\right\}}\right)-\frac{1}{\sqrt{2\pi}}\int_{-A}^{0}L(-y)e^{-\frac{y^2}{2}}dy\right|\le\frac{C}{A}.$$

Thus, letting $A\to\infty$ in (2.4.10), by (2.3.9) we obtain that

$$\lim_{n\to\infty}\mathbb{E}\left(L(\frac{-S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\le\frac{\overline{F}_n(1)}{\sqrt{b_n}}\right\}}\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}L(-y)e^{-\frac{y^2}{2}}dy$$

$$=\mathbb{E}\left(L(\frac{S_1^{(n)}}{\sqrt{b_n}})\mathbf{1}_{\left\{\frac{S_1^{(n)}}{\sqrt{b_n}}\ge 0\right\}}\right)=L(0)=1.$$

As a consequence we obtain

$$\mathbb{E}\left(\prod_{|w|=1}\mathbb{E}\left(1+L(-\frac{S_w-b_nv}{\sqrt{b_n}})e^{\theta^*(S_w-b_nv)}\mathbf{1}_{\left\{S_w-b_nv\le\frac{\overline{F}_n(1)}{b_n}\right\}}\frac{1}{\sqrt{2\pi\sigma^2}}\int e^{-\theta^*y}(e^{-\varphi(y)}-1)dy\right)\right)$$

$$\sim_{n\to\infty}\mathbb{E}\left(\left(1-\frac{1}{\sqrt{2\pi\sigma^2}}m^{-b_n}\int e^{-\theta^*y}(1-e^{-\varphi(y)})dy\right)^{\#\{u,|u|=1\}}\right)$$

$$=\mathbb{E}\left(\left(1-\frac{1}{\sqrt{2\pi\sigma^2}}m^{-b_n}\int e^{-\theta^*y}(1-e^{-\varphi(y)})dy\right)^{Z_{b_n}}\right).$$

Finally, applying dominated convergence theorem and by assumption (2.2.2) we deduce that (2.4.7) holds for all function $\varphi\in\mathcal{C}_b^{l,+}$, which concludes the proof using Remark 2.2.2. □

Acknowledgements. I want to thank Bastien Mallein for introducing me this subject and for the very helpful discussions.

This program has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 754362.

Chapter 3

Anomalous spreading in reducible multitype branching Brownian motion

Abstract

We consider a two-type reducible branching Brownian motion, defined as a particle system on the real line in which particles of two types move according to independent Brownian motions and create offspring at a constant rate. Particles of type 1 can give birth to particles of types 1 and 2, but particles of type 2 only give birth to descendants of type 2. Under some specific conditions, Biggins [Big12] shows that this process exhibits an anomalous spreading behaviour: the rightmost particle at time t is much further than the expected position for the rightmost particle in a branching Brownian motion consisting only of particles of type 1 or of type 2. This anomalous spreading also has been investigated from a reaction-diffusion equation standpoint by Holzer [Hol14, Hol16]. The aim of this article is to study the asymptotic behaviour of the position of the furthest particles in the two-type reducible branching Brownian motion, obtaining in particular tight estimates for the median of the maximal displacement.

3.1 Introduction

The standard *branching Brownian motion* is a particle system on the real line that can be constructed as follows. It starts with a unique particle at time 0, that moves according to a standard Brownian motion. After an exponential time of parameter 1, this particle dies and is replaced by two children. The two daughter particles then start independent copies of the branching Brownian motion from their current position. For all $t \geq 0$, we write $\mathcal{N}_t$ the set of particles alive at time t, and for $u \in \mathcal{N}_t$ we denote by $X_u(t)$ the position at time t of that particle.

The branching Brownian motion (or BBM) is strongly related to the *F-KPP reaction-diffusion equation*, defined as

$$\partial_t u = \frac{1}{2}\Delta u - u(1-u). \tag{3.1.1}$$

More precisely, given a measurable function $f : \mathbb{R} \to [0,1]$, we set for $x \in \mathbb{R}$ and $t \geq 0$

$$u_t(x) = \mathbf{E}\left(\prod_{u \in \mathcal{N}_t} f(X_u(t) + x) \right).$$

Then u is the solution of (3.1.1) with $u_0(x) = f(x)$. Hence, setting $M_t = \max_{u \in \mathcal{N}_t} X_u(t)$, we note that the tail distribution of $-M_t$ is the solution at time t of (3.1.1) with $u_0(z) = \mathbb{1}_{\{z<0\}}$.

Thanks to this observation, Bramson [Bra78] obtained an explicit formula for the asymptotic behaviour of the median of M_t. Precisely, he observed that setting

$$m_t := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t \tag{3.1.2}$$

the process $(M_t - m_t, t \geq 0)$ is tight. Lalley and Sellke [LS87] refined this result and proved that $M_t - m_t$ converges in law toward a Gumbel random variable shifted by an independent copy of $\frac{1}{\sqrt{2}} \log Z_\infty$. Here, Z_∞ is the a.s. limit as $t \to \infty$ of the derivative martingale, defined by

$$Z_t := \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}X_u(t) - 2t}.$$

The derivative martingale is called that way due to its relationship to the derivative at its critical point of the additive martingale introduced by McKean in [McK75], defined as

$$W_t(\theta) := \sum_{u \in \mathcal{N}_t} e^{\theta X_u(t) - t(1 + \frac{\theta^2}{2})}.$$

It was shown in [Nev88] that $(W_t(\theta), t \geq 0)$ is uniformly integrable if and only if $|\theta| < \sqrt{2}$ and converges to an a.s. positive limit $W_\infty(\theta)$ in that case. Otherwise, it converges to 0 a.s. This result has later been extended by Biggins [Big92a] and Lyons [Lyo97] to the branching random walk, which is a discrete-time analogous to the BBM.

The behaviour of the particles at the tip of branching Brownian motions was later investigated by Aidékon, Berestycki, Brunet and Shi [ABBS13] as well as Arguin, Bovier and Kistler [ABK11, ABK12, ABK13]. They proved that the centred *extremal process* of the standard BBM, defined by

$$\widehat{\mathcal{E}}_t^R = \sum_{u \in \mathcal{N}_t} \delta_{X_u(t) - m_t}$$

converges in distribution to a decorated Poisson point process with (random) intensity $\sqrt{2}c_\star Z_\infty e^{-\sqrt{2}x}\mathrm{d}x$. More precisely, there exists a law $\mathfrak{D}$ on point measures such that writing $(D_j, j \geq 1)$ i.i.d. point measures with law $\mathfrak{D}$ and $(\xi_j, j \geq 0)$ the atoms of an independent Poisson point process with intensity $\sqrt{2}c_\star e^{-\sqrt{2}x}\mathrm{d}x$, which are further independent of Z_∞, and defining

$$\mathcal{E}_\infty = \sum_{j\geq 1} \sum_{d \in D_j} \delta_{\xi_j + d + \frac{1}{\sqrt{2}} \log Z_\infty},$$

we have $\lim_{t\to\infty} \mathcal{E}_t = \mathcal{E}_\infty$ in law, for the topology of the vague convergence. We give more details on these results in Section 3.3.

We refer to the above limit as a DPPP$(\sqrt{2}c_\star Z_\infty e^{-\sqrt{2}x}\mathrm{d}x, \mathfrak{D})$, for decorated Poisson point process. Maillard [Mai13] obtained a characterization of this type of point processes as satisfying a stability by superposition property. This characterization was used in [Mad17] to prove a similar convergence in distribution to a DPPP for the shifted extremal process of the branching random walk. Subag and Zeitouni [SZ15] studied in more details the family of shifted randomly decorated Poisson random measures with exponential intensity.

In this article, we take interest in the *two-type reducible branching Brownian motion*. This is a particle system on the real line in which particles possess a type in addition with their position. Particles of type 1 move according to Brownian motions with diffusion coefficient σ_1^2 and branch at rate β_1 into two children of type 1. Additionally, they give birth to particles of type 2 at rate α. Particles of type 2 move according to Brownian motions with diffusion coefficient σ_2^2 and branch at rate β_2, but cannot give birth to descendants of type 1.

In [Big12], Biggins observed that in some cases multitype reducible branching random walks exhibit an *anomalous spreading* property. Precisely, the rightmost particle at time t is shown to be around position vt, with the speed v of the two-type process being larger than the speed of a branching random walk consisting only of particles of type 1 or uniquely of particles of type 2. Therefore, in that case, the multitype system invades its environment at a higher speed than the one that either particles of type 1 or particles of type 2 would be able to sustain on their own.

Holzer [Hol14, Hol16] extended the results of Biggins to this setting, by considering the associated system of F-KPP equations, describing the speed of the rightmost particle in the system in terms of $\sigma_1, \beta_1, \sigma_2$ and β_2 (the parameter α does not modify the speed of the two-type particle system). Our aim is to study in more details the position of the maximal displacement, in particular in the case when anomalous spreading occurs, for this two type BBM. We also take interest in the extremal process formed by the particles of type 2 at time t, and show it to converge towards a DPPP.

Recall that the reducible two-type BBM is defined by five parameters, the diffusion coefficient σ_1^2, σ_2^2 of particles of type 1 and 2, their branching rate β_1, β_2, and the rate α at which particles of type 1 create particles of type 2. However, up to a dilation of time and space, it is possible to modify these parameters in such a way that $\sigma_2^2 = \beta_2 = 1$. Additionally, the parameter α plays no role in the value of the speed of the multitype process. We can therefore describe the phase space of this process in terms of the two parameters $\sigma^2 := \sigma_1^2$ and $\beta = \beta_1$, and identify for which parameters does anomalous spreading occurs. This is done in Figure 3.1.

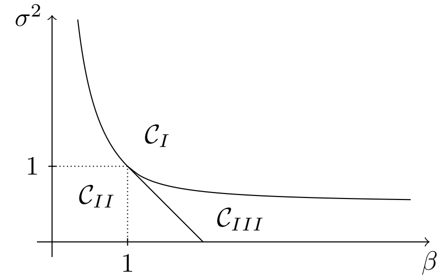


Figure 3.1 – Phase diagram of the two-type reducible BBM.

We decompose the state space $(\beta, \sigma^2) \in \mathbb{R}_+^2$ into three regions:

$$\mathcal{C}_I = \left\{ (\beta, \sigma^2) : \sigma^2 > \frac{\mathbb{1}_{\{\beta \le 1\}}}{\beta} + \mathbb{1}_{\{\beta > 1\}} \frac{\beta}{2\beta - 1} \right\}$$

$$\mathcal{C}_{II} = \left\{ (\beta, \sigma^2) : \sigma^2 < \frac{\mathbb{1}_{\{\beta \le 1\}}}{\beta} + \mathbb{1}_{\{\beta > 1\}} (2 - \beta) \right\}$$

$$\mathcal{C}_{III} = \left\{ (\beta, \sigma^2) : \sigma^2 + \beta > 2 \text{ and } \sigma^2 < \frac{\beta}{2\beta - 1} \right\}.$$

If $(\beta, \sigma^2) \in \mathcal{C}_I$, the speed of the two-type reducible BBM is $\sqrt{2\beta\sigma^2}$, which is the same as particles of type 1 alone, ignoring births of particles of type 2. Thus in this situation, the asymptotic behaviour of the extremal process is dominated by the long-time behaviour of particles of type 1. Conversely, if $(\beta, \sigma^2) \in \mathcal{C}_{II}$, then the speed of the process is $\sqrt{2}$, equal to the speed of a single BBM of particles of type 2. In that situation, the asymptotic behaviour of particles of type 2 dominates the extremal process. Finally, if $(\beta, \sigma^2) \in \mathcal{C}_{III}$, the speed of the process is larger than $\max(\sqrt{2}, \sqrt{2\beta\sigma^2})$, and we will show that in this case the extremal process will be given by a mixture of the long-time asymptotic of the processes of particles of type 1 and 2.

For all $t \ge 0$, we write $\mathcal{N}_t$ the set of all particles alive at time t, as well as $\mathcal{N}_t^1$ and $\mathcal{N}_t^2$ the set of particles of type 1 and type 2 respectively. We also write $X_u(t)$ for the position at time t of $u \in \mathcal{N}_t$, and for all $s \le t$, $X_u(s)$ for the position of the ancestor at time s of particle u. If $u \in \mathcal{N}_t^2$, we denote by $T(u)$ the time at which the oldest ancestor of type 2 of u was born. In this article, we study the asymptotic behaviour of the extremal process of particles of type 2 in this 2-type BBM for Lebesgue-almost every values of σ^2 and β (and for $\alpha \in (0, \infty)$).

We divide the main result of our article into three theorems, one for each area the pair (β, σ^2) belongs to. We begin with the asymptotic behaviour of extremal particles when $(\beta, \sigma^2) \in \mathcal{C}_I$, in which case the extremal point measure is similar to the one observed in a branching Brownian motion of particles of type 1.

Theorem 3.1.1 (Domination of particles of type 1)**.** *If $(\beta, \sigma^2) \in \mathcal{C}_I$, then there exist $c_{(I)} > 0$ and a point measure distribution $\mathfrak{D}^{(I)}$ such that if we set $m_t^{(I)} := \sqrt{2\sigma^2\beta}t - \frac{3}{2\sqrt{2\beta/\sigma^2}} \log t$ we have*

$$\lim_{t \to \infty} \sum_{u \in \mathcal{N}_t^2} \delta_{X_u(t) - m_t^{(I)}} = \mathcal{E}_\infty^{(I)} \quad \textit{in law for the topology of the vague convergence,}$$

where $\mathcal{E}^{(I)}_\infty$ *is a DPPP*$(\sqrt{2\beta/\sigma^2}c_{(I)}Z^{(1)}_\infty e^{-\sqrt{2\beta/\sigma^2}x}\mathrm{d}x, \mathfrak{D}^{(I)})$*, where*

$$Z^{(1)}_\infty := \lim_{t\to\infty} \sum_{u\in\mathcal{N}^1_t} (\sqrt{2\sigma^2\beta}t - X_u(t))e^{\sqrt{2\beta/\sigma^2}X_u(t)-2\beta t} \quad a.s.$$

Additionally, we have $\lim_{t\to\infty} \mathbf{P}(M_t \leq m^{(I)}_t + x) = \mathbf{E}\left(e^{-c_{(I)}Z^{(1)}_\infty e^{-\sqrt{2\beta/\sigma^2}x}}\right)$ *for all* $x \in \mathbb{R}$.

We underscore that in this theorem, the values of $c_{(I)}$ and $\mathfrak{D}^{(I)}$ are obtained implicitly, and depend on the parameters α, β and σ^2 of the multitype branching Brownian motion. The identification of the law of the extremal point measure $\mathcal{E}^{(I)}$ is based on the fact that it satisfies a stability by superposition property, and using Maillard's characterization [Mai13].

If $(\beta, \sigma^2) \in \mathcal{C}_{II}$, we show that the extremal process of the two-type BBM is similar to the extremal process of a single BBM of particles of type 2, up to a random shift whose law depend on the behaviour of particles of type 1.

Theorem 3.1.2 (Domination of particles of type 2)**.** *If* $(\beta, \sigma^2) \in \mathcal{C}_{II}$*, then writing* $c_\star > 0$ *the prefactor of the intensity measure in the extremal process of the standard BBM (that we recall in* (3.3.14)*) and* $\mathfrak{D}$ *the law of its decoration (defined in* (3.3.18)*), if we set* $m^{(II)}_t := m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$*, we have*

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}^2_t} \delta_{X_u(t)-m^{(II)}_t} = \mathcal{E}^{(II)}_\infty \quad \textit{in law for the topology of the vague convergence,}$$

where $\mathcal{E}^{(II)}_\infty$ *is a DPPP*$(\sqrt{2}c_\star \overline{Z}_\infty e^{-\sqrt{2}x}\mathrm{d}x, \mathfrak{D})$ *and* $\overline{Z}_\infty$ *is defined in Lemma 3.5.3. Additionally, for all* $x \in \mathbb{R}$ *we have* $\lim_{t\to\infty} \mathbf{P}(M_t \leq m^{(II)}_t + x) = \mathbf{E}\left(e^{-c_\star \overline{Z}_\infty e^{-\sqrt{2}x}}\right)$.

We finally take interest in the case $(\beta, \sigma^2) \in \mathcal{C}_{III}$, in this situation the BBM exhibits an anomalous spreading behaviour. The extremal process contains only particles of type 2, but particles travel at greater speed that would have been observed in a BBM of particles of type 1 or of type 2.

Theorem 3.1.3 (Anomalous spreading)**.** *If* $(\beta, \sigma^2) \in \mathcal{C}_{III}$*, then setting*

$$m^{(III)}_t = \frac{\sigma^2 - \beta}{\sqrt{2(1-\sigma^2)(\beta-1)}}t \quad \textit{and} \quad \theta = \sqrt{2\frac{\beta-1}{1-\sigma^2}},$$

we have

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}^2_t} \delta_{X_u(t)-m^{(III)}_t} = \mathcal{E}^{(III)}_\infty \quad \textit{in law for the topology of the vague convergence,}$$

where $\mathcal{E}_\infty$ *is a DPPP*$(\theta c_{(III)} W_\infty(\theta)e^{-\theta x}\mathrm{d}x, \mathfrak{D}^{(III)})$ *and*

- $W_\infty(\theta) = \lim_{t\to\infty} \sum_{u\in\mathcal{N}^1_t} e^{\theta X_u(t) - t(\beta+\theta^2\sigma^2/2)}$ *is the a.s. limit of an additive martingale of the BBM of particles of type* 1 *with parameter* θ*,*
- $c_{(III)} = \frac{\alpha C(\theta)}{2(\beta-1)}$ *with the function* C *being defined in* (3.3.16)*,*

- $\mathfrak{D}^{(III)}$ *is the law of the point measure* $\mathcal{D}^\theta$ *defined in* (3.3.19).

Additionally, for all $x \in \mathbb{R}$ *we have* $\lim_{t\to\infty} \mathbf{P}(M_t \leq m_t^{(III)} + x) = \mathbf{E}\left(e^{-c_{(III)} W_\infty(\theta) e^{-\theta x}}\right)$.

Remark 3.1.4. Contrarily to what happens in Theorems 3.1.1 and 3.1.2, the extremal process obtained in Theorem 3.1.3 is not shifted by a random variable associated to a derivative martingale, but by an additive martingale of the BBM. Additionally, it is worth noting that contrarily to the median of the maximal displacements in domains $\mathcal{C}_I$ and $\mathcal{C}_{II}$, when anomalous spreading occurs there is no logarithmic correction in the median of the maximal displacement.

Remark 3.1.5. Observe that in Theorems 3.1.1–3.1.3, we obtain the convergence in law for the topology of the vague convergence of extremal processes to DPPPs as well as the convergence in law of their maximum to the maximum of this DPPP. These two convergences can be synthesized into the joint convergence of the extremal process together with its maximum, which is equivalent to the convergence of $\langle \mathcal{E}_t, \varphi \rangle$ for all continuous bounded functions φ with bounded support on the left (see e.g. [BBCM20, Lemma 4.4]).

The rest of the article is organized as follows. We discuss our results in the next section, by putting them in the context of the state of the art for single type and multitype branching processes, and for coupled reaction-diffusion equations. In Section 3.3, we introduce part of the notation and results on branching Brownian motions that will be needed in our proofs, in particular for the definition of the decorations laws of the extremal process. We introduce in Section 3.4 a multitype version of the celebrated many-to-one lemma. Finally, we prove Theorem 3.1.2 in Section 3.5, Theorem 3.1.1 in Section 3.6 and Theorem 3.1.3 in Section 3.7.

3.2 Discussion of our main result

We compare our main results for the asymptotic behaviour of the two-type reducible branching Brownian motion to the pre-existing literature. We begin by introducing the optimization problem associated to the computation of the speed of the rightmost particle in this process. Loosely speaking, this optimization problem is related to the "choice" of the time between 0 and t and position at which the ancestral lineage leading to one of the rightmost positions switches from type 1 to type 2. The optimization problem was introduced by Biggins [Big12] for the computation of the speed of multitype reducible branching random walks. It allows us to describe the heuristics behind the main theorems.

We then compare Theorem 3.1.3 to the results obtained on the extremal process of time-inhomogeneous branching Brownian motions, and in particular with the results of Bovier and Hartung [BH14]. In Section 3.2.3, we apply our results to the work of Holzer [Hol14, Hol16] on coupled F-KPP equations. We end this section with the discussion of further questions of interest for multitype reducible BBMs and some conjectures and open questions.

3.2.1 Associated optimization problem and heuristic

Despite the fact that spatial multitype branching processes have a long history, the study of the asymptotic behaviour of their largest displacement has not been considered until recently. As previously mentioned, Biggins [Big12] computed the speed of multitype reducible branching random walks. This process is a discrete-time particles system in which each particle gives birth to offspring in an independent fashion around its position, with a reproduction law that depends on its type, under the assumption that the Markov chain associated to the type of a typical individual in the process is reducible. Ren and Yang [RY14] then considered the asymptotic behaviour of the maximal displacement in an *irreducible* multitype BBM.

In [Big10], Biggins gives an explicit description of the speed of a reducible two-type branching random walk as the solution of an optimization problem. In the context of the two-type BBM we consider, the optimization problem can be expressed as such:

$$v = \max \left\{ pa + (1-p)b : p \in [0,1],\ p\left(\frac{a^2}{2\sigma^2} - \beta\right) \leq 0, \right.$$
$$\left. p\left(\frac{a^2}{2\sigma^2} - \beta\right) + (1-p)\left(\frac{b^2}{2} - 1\right) \leq 0 \right\}. \quad (3.2.1)$$

This optimization problem can be interpreted as such. If $a < \sqrt{2\sigma^2\beta}$ and $b \geq \sqrt{2}$, there are with high probability around $e^{pt(\beta - a^2/2\sigma^2)+o(t)}$ particles of type 1 at time pt to the right of position pta, and a typical particle of type 2 has probability $e^{(1-p)t(1-b^2/2)+o(t)}$ of having a descendant to the right of position $(1-p)bt$ at time $(1-p)t$. Therefore, for all (p, a, b) such that

$$p \in [0,1],\ p\left(\frac{a^2}{2\sigma^2} - \beta\right) \leq 0,\ p\left(\frac{a^2}{2\sigma^2} - \beta\right) + (1-p)\left(\frac{b^2}{2} - 1\right) \leq 0,$$

by law of large numbers there should be with high probability particles of type 2 to the right of the position $t(pa + (1-p)b)$ at time t.

If we write (p^*, a^*, b^*) the triplet optimizing the problem (3.2.1), it follows from classical optimization under constraints computations that:

1. If $(\beta, \sigma^2) \in \mathcal{C}_I$, then $p^* = 1$ and $a^* = \sqrt{2\beta\sigma^2}$, which is in accordance with Theorem 3.1.1, as the extremal particle system is dominated by the behaviour of particles of type 1, and particles of type 2 contributing to the extremal process are close relatives descendants of a parent of type 1;

2. If $(\beta, \sigma^2) \in \mathcal{C}_{II}$, then $p^* = 0$ and $b^* = \sqrt{2}$, which is in accordance with Theorem 3.1.2, as the extremal particle system is dominated by the behaviour of particles of type 2, that are born at time $o(t)$ from particles of type 1;

3. If $(\beta, \sigma^2) \in \mathcal{C}_{III}$, then

   $$p^* = \frac{\sigma^2 + \beta - 2}{2(1-\sigma^2)(\beta-1)}, \quad a^* = \sigma^2\sqrt{2\frac{\beta-1}{1-\sigma^2}} \text{ and } b^* = \sqrt{2\frac{\beta-1}{1-\sigma^2}}.$$

   We then have $v = \frac{\beta - \sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}}$, which corresponds to the main result of Theorem 3.1.3. Additionally, the Lagrange multiplier associated to this optimization problem is $\theta = \sqrt{2\frac{\beta-1}{1-\sigma^2}} = b = \frac{a}{\sigma^2}$.

In particular, the optimization problem associated to the case $(\beta, \sigma^2) \in \mathcal{C}_{III}$ can be related to the following interpretation of Theorem 3.1.3. The extremal process at time t is obtained as the superposition of the extremal processes of an exponentially large number of BBMs of type 2, starting around time tp^* and position tp^*a^*. The number of these BBMs is directly related to the number of particles of type 1 that displace at speed a^*, which is known to be proportional to $W_\infty(\theta)e^{t(1-(a^*)^2/2\sigma^2)}$. It explains the apparition of this martingale in Theorem 3.1.3, whereas the decoration distribution $\mathfrak{D}^{(III)}$ is the extremal process of a BBM of type 2 conditionally on moving at the speed $b^* > v$.

For Theorem 3.1.1, a similar description can be made. We expect the asymptotic behaviour to be driven by the behaviour of particles of type 1, therefore the extremal process of particles of type 2 should be obtained as a decoration of the extremal process of particles of type 1. However, as we were not able to use result of convergence of extremal processes together with a description of the behaviour of particles at times $t - O(1)$, we do not obtain an explicit value for $c_{(I)}$ and an explicit description of the law $\mathfrak{D}^{(I)}$. However, with similar techniques as the ones used in [ABBS13] or [ABK13], such explicit constructions should be available.

Finally, in the case covered by Theorem 3.1.2, the above optimization problem indicates that the extremal process of the multitype reducible BBM should be obtained as the superposition of a finite number of BBMs of particles of type 2, descending from the first few particles of type 2 to be born. The random variable $\overline{Z}$ is then constructed as the weighted sum of i.i.d. copies of the derivative martingale of a standard BBM and the decoration is the same as the decoration of the original BBM.

To prove Theorems 3.1.1–3.1.3, we show that the above heuristic holds, i.e. that with high probability the set of particles contributing to the extremal processes are the one we identified in each case. We then use previously known results of branching Brownian motions to compute the Laplace transforms of the extremal point measures we are interested in.

The solution of the optimization problem (3.2.1) is also solution of $v = \sup\{a \in \mathbb{R} : g(a) \leq 0\}$, where g is the largest convex function such that

$$\forall |x| \leq \sqrt{2\beta\sigma^2},\ g(x) \leq \left(\frac{x^2}{2\sigma^2} - \beta\right) \quad \text{and} \quad \forall y \in \mathbb{R},\ g(y) \leq \frac{y^2}{2} - 1,$$

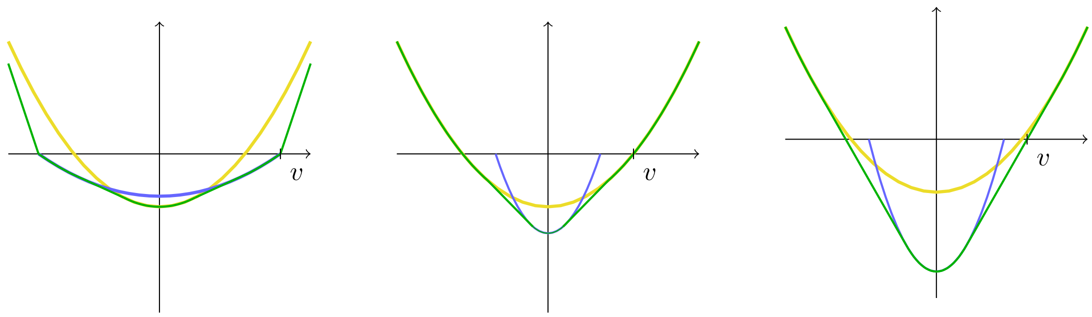
see [Big10] for precisions. The function $x \mapsto \frac{x^2}{2\sigma^2} - \beta$ is known as the rate function for particles of type 1, and $y \mapsto \frac{y^2}{2} - 1$ is the rate function for particles of type 2.

We then observe that the three cases described above are the following:

1. If $(\beta, \sigma^2) \in \mathcal{C}_I$, then $v = \sqrt{2\beta\sigma^2} = \sup\{x \in \mathbb{R} : x^2/2\sigma^2 - \beta \leq 0\}$.
2. If $(\beta, \sigma^2) \in \mathcal{C}_{II}$, then $v = \sqrt{2} = \sup\{y \in \mathbb{R} : \frac{y^2}{2} - 1 \leq 0\}$.
3. If $(\beta, \sigma^2) \in \mathcal{C}_{III}$, then $v > \max(\sqrt{2\beta\sigma^2}, \sqrt{2})$.

In other words, the anomalous spreading corresponds to the case when the convex envelope g crosses the x-axis to the right of the rate functions of particles of type 1 and 2.

As mentioned above, Ren and Yang [RY14] studied the asymptotic behaviour of irreducible multitype BBM, and computed the speed at which that process invades its environment. In that case (i.e. when for all pair of types i and j, individuals of type i have positive probability of having at least one descendant of type j after some time), this asymptotic behaviour is similar to the one obtained for a single-type BBM, with branching rate and variance obtained by considering the invariant measure of the Markov process describing the type of a typical individual. The notion of



(a) Case I: The speed of the multitype process is the same as the speed of the process of type 1 particles.

(b) Case II: The speed of the multitype process is the same as the speed of the process of type 2 particles.

(c) Case III: The anomalous spreading of the multitype process which is faster than the process consisting only of particles of type 1 or 2.

Figure 3.2 – Convex envelope g for (β, σ^2) in any of the three domains of interest. The rate function of particles of type 1 and 2 are drawn in blue and yellow respectively, the function g of the multitype branching process is drawn in green.

anomalous spreading in this case is thus very different, and the ancestral lineage of typical particles in the extremal process will present regular changes of type. As a result, we do not expect an asymptotic behaviour similar to the one observed in Theorem 3.1.3 to occur in irreducible multitype BBM.

In a different direction, Smadi and Vatutin [SV16] studied the limit in distribution of a critical reducible Galton-Watson process. It is worth noting that similarly to our results, they obtained three different behaviours for the system, with either the domination of particles of the first type, of the second type, or an interplay between the two.

3.2.2 Relation to time-inhomogeneous branching processes

The results presented here, in particular in the anomalous spreading case, are reminiscent of the known asymptotic for the extremal process of time-inhomogeneous branching Brownian motions. This model was introduced by Fang and Zeitouni [FZ12b], and is defined as follows. Given $t \geq 0$, the process is a BBM consisting only of particles of type 1 until time $t/2$, at which time they all become simultaneously particles of type 2. It has been showed [FZ12a, Mal15c] that depending on the value of (β, σ^2), the position of the maximal displacement at time t can exhibit different types of asymptotic behaviours. In particular, the logarithmic correction exhibits a strong phase transition in the phase space of (β, σ^2).

Looking more closely at the convergence of the extremes, Bovier and Hartung [BH14, BH15] obtained the convergence in distribution of the extremal process of the time-inhomogeneous BBM. In particular, for a multitype BBM with parameters $(\beta, \sigma^2) \in \mathcal{C}_{III}$ such that particles change from type 1 to type 2 at time p^*t, they showed that the extremal process converges towards $\mathcal{E}_\infty^{(III)}$, with an extra $\frac{1}{2\theta} \log t$ logarithmic correction for the centring. This is in accordance with our heuristic as we expect that the particles contributing to the extremal process at time t to have been born from particles of type 1 around time p^*t.

Generalized versions of time-inhomogeneous BBM have been studied, in which the variance of particles evolves continuously over time [MZ16, Mal15c]. In that case, the maximal displacement grows at constant speed with a negative correction of order $t^{1/3}$. It would be interesting to construct a multitype BBM, possibly with an infinite number of types, that would exhibit a similar phenomenon.

3.2.3 F-KPP type equation associated to the multitype branching Brownian motion

Observe that similarly to the standard BBM, the multitype BBM can be associated to a reaction diffusion equation in the following way. Let $f, g : \mathbb{R} \to [0, 1]$ be measurable functions, we define for all $x \in \mathbb{R}$:

$$u(t,x) = \mathbf{E}^{(1)}\left(\prod_{u\in\mathcal{N}_t^1} f(X_u(t)+x) \prod_{u\in\mathcal{N}_t^2} g(X_u(t)+x)\right)$$
$$v(t,x) = \mathbf{E}^{(2)}\left(\prod_{u\in\mathcal{N}_t^1} f(X_u(t)+x) \prod_{u\in\mathcal{N}_t^2} g(X_u(t)+x)\right) = \mathbf{E}^{(2)}\left(\prod_{u\in\mathcal{N}_t^2} g(X_u(t)+x)\right)$$

where $\mathbf{P}^{(1)}$ (respectively $\mathbf{P}^{(2)}$) is the law of the multitype BBM starting from one particle of type 1 (resp. 2), and we use the fact that particles of type 2 only produce offspring of type 2, with the usual convention $\prod_{u\in\emptyset} f(X_u(t)+x) = 1$.

As under $\mathbf{P}^{(2)}$, the process behaves as a standard BBM, the function v is a solution of the classical F-KPP reaction-diffusion equation

$$\partial_t v = \frac{1}{2}\Delta v - v(1-v) \quad \text{with } v(0,x) = g(x). \tag{3.2.2}$$

To obtain the partial differential equation satisfied by u, we observe that under law $\mathbf{P}^{(1)}$ one of the three following events might happen during the first $\mathrm{d}t$ units of time:

- with probability $\beta \mathrm{d}t + o(\mathrm{d}t)$, the original particle of type 1 branches into two offspring of type 1 that start i.i.d. processes with law $\mathbf{P}^{(1)}$;
- with probability $\alpha \mathrm{d}t + o(\mathrm{d}t)$ the particle of type 1 branches into one offspring of type 1 and one of type 2, that start independent processes with law $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ respectively;
- with probability $1-(\beta+\alpha)\mathrm{d}t$, the particle of type 2 diffuses as the Brownian motion σB with diffusion constant σ^2

As a result, we have

$$\begin{aligned} u(t+\mathrm{d}t,x) &= \beta \mathrm{d}t u(t,x)^2 + \alpha \mathrm{d}t u(t,x)v(t,x) + (1-(\beta+\alpha)\mathrm{d}t)\mathbf{E}\left(u(t,x-\sigma B_{\mathrm{d}t})\right) + o(\mathrm{d}t) \\ &= u(t,x) + \mathrm{d}t\left(\frac{\sigma^2}{2}\Delta u(t,x) - \beta u(1-u) - \alpha u(1-v)\right). \end{aligned}$$

This, together with (3.2.2) show that (u, v) is a solution of the following coupled F-KPP equation

$$\begin{cases} \partial_t u = \frac{\sigma^2}{2}\Delta - \beta u(1-u) - \alpha u(1-v) \\ \partial_t v = \frac{1}{2}\Delta v - v(1-v) \\ u(0,x) = f(x), \quad v(0,x) = g(x). \end{cases} \tag{3.2.3}$$

This non-linear coupling of F-KPP equation was introduced by Holzer [Hol14]. In that article, the author conjectured this partial differential equation to exhibit an anomalous spreading phenomenon, and conjectured a phase diagram for the model [Hol14, Figure 1]. Our main results confirm this conjecture, and the diagram we obtain in Figure 3.1 exactly matches (up to an adaptation of the notation $\sigma^2 \rightsquigarrow 2d$, $\beta \rightsquigarrow \alpha$ and $\alpha \rightsquigarrow \beta$) the one obtained by Holzer. Additionally, Theorems 3.1.1–3.1.3 give the position of the front of v_t in (3.2.3).

When starting with well chosen initial conditions f, g (for example such that there exists $A > 0$ satisfying $f(x) = g(x) = 1$ for $x < -A$ and $f(x) = g(x) = 0$ for $x > A$), we obtain the existence of a function v_t such that for all $x \in \mathbb{R}$,

$$\lim_{t\to\infty} (u(t, x - v_t), v(t, x - v_t)) = (w_1(x), w_2(x)),$$

where (w_1, w_2) is a travelling wave solution of the coupled PDE and:

1. if $(\beta, \sigma^2) \in \mathcal{C}_I$, then $v_t = \sqrt{2\beta\sigma^2}t - \frac{3}{2\sqrt{2\beta/\sigma^2}} \log t$;
2. if $(\beta, \sigma^2) \in \mathcal{C}_{II}$, then $v_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$;
3. if $(\beta, \sigma^2) \in \mathcal{C}_{III}$, then $v_t = vt$, with v defined in Theorem 3.1.3.

Holzer further studied a linearised version of (3.2.3) in [Hol16], and showed the presence of an anomalous spreading property in that context. However, the phase diagram in that case is of a different nature as the one we obtain in Figure 3.1. We believe that the phase diagram of this linearised PDE equation should be related to first moment estimates on the number of particles above a given level in the multitype BBM.

The equation (3.2.3) should also be compared to the partial differential equation studied in [BC14]. In that article, they considered a population with a family of traits indexed by a parameter $\theta \in (\theta_{\min}, \theta_{\max})$, that modifies the motility of particles. This was proposed as a model for the invasion of cane toads is Australia, as that population consists of faster individuals, that sacrifice part of their reproduction power as a trade off, and slower individuals that reproduce more easily. The multitype BBM we consider here could then be thought of as some toy-model for this partial differential equation.

3.2.4 Future developments

We recall that Theorems 3.1.1–3.1.3 cover the asymptotic behaviour of the two-type reducible BBM assuming that $(\beta, \sigma^2) \in \mathcal{C}_I \cup \mathcal{C}_{II} \cup \mathcal{C}_{III}$. However, it does not give the asymptotic behaviour of this process when (β, σ^2) belongs to the boundary of this set. Understanding the behaviour of the process at these points could help understanding the phase transitions occurring between the different areas of the state space. This would allow results similar to the ones developed in [BH20] for time-inhomogeneous BBM to be considered in reducible multitype BBM.

We conjecture the following behaviours for the branching Brownian motion at the boundary between areas $\mathcal{C}_I$ and $\mathcal{C}_{III}$.

Conjecture 3.2.1. *Assume that $\beta > 1$ and $\sigma^2 = \frac{\beta}{2\beta - 1}$, then there exist $c > 0$ and $\widetilde{\mathfrak{D}}$ such that*

$$\sum_{u \in \mathcal{N}_t^2} \delta_{X_u(t) - \sqrt{2\beta\sigma^2} t + \frac{1}{\sqrt{2\beta/\sigma^2}} \log t}$$

converges to a DPPP($cZ_\infty^{(1)} e^{-\sqrt{2\beta/\sigma^2} x} \mathrm{d}x, \widetilde{\mathfrak{D}}$).

Indeed, in this situation, particles u of type 2 contributing to the extremal process are expected to satisfy $t - T(u) = O(t^{1/2})$. Therefore, the extremal process keeps an intensity driven by the derivative martingale of particles of type 1, and the decoration point measure is given by the extremal process of a BBM of particles of type 2 conditioned to travel at speed $\sqrt{2\beta\sigma^2} > \sqrt{2}$.

Similarly, at the boundary between areas $\mathcal{C}_{II}$ and $\mathcal{C}_{III}$, the following behaviour is expected.

Conjecture 3.2.2. *Assume that $\beta > 1$ and $\sigma^2 = 2 - \beta$, then there exist $c > 0$ and a random variable $\widetilde{Z}$ such that*

$$\sum_{u \in \mathcal{N}_t^2} \delta_{X_u(t) - \sqrt{2} t + \frac{1}{\sqrt{2}} \log t}$$

converges to a DPPP($c\widetilde{Z} e^{-\sqrt{2}x} \mathrm{d}x, \mathfrak{D}$).

There, we used the fact that particles u of type 2 contributing to the extremal process are expected to satisfy $T(u) = O(t^{1/2})$.

In the case when $\beta < 1$ and $\sigma^2\beta = 1$, which corresponds to the boundary between cases $\mathcal{C}_I$ and $\mathcal{C}_{II}$, the picture is less clear as at all time s between 0 and t, particles should have the same probability to reach the maximal position, at least to the first order, as the BBM of particles of type 1 and of particles of type 2 have same speed.

Further generalisations of the model we consider in this article could be considered. A more general reducible multitype branching Brownian motion with a finite number of types would be expected to exhibit a similar behaviour. One could also allow particles to have different drift coefficients in addition to the different variance terms and branching rates. In that situation, one expects an optimization problem similar to the one studied in [Mal15a] to appear, with a similar technique of proving that the trajectory followed by particles reaching the maximal position is the same as the one inferred from the solution of the optimization problem.

Proving Theorems 3.1.1–3.1.3 for two-type reducible branching Brownian motions in which particles of type 1 and type 2 split into a random number of children at each branching event, say L_1 for particles of type 1 and L_2 for particles of type 2 would be a other natural generalisation of our results. A natural condition to put on the reproduction laws to obtain the asymptotic behaviour observed in Theorem 3.1.3 is

$$\mathbf{E}(L_1 \log L_1) + \mathbf{E}(L_2 \log L_2) < \infty.$$

It is worth noting that anomalous spreading might occur even if $\mathbf{E}(L_2) < 1$, i.e. even if the genealogical tree of a particle of type 2 is subcritical and grows extinct almost surely.

While we only take interest here in the asymptotic behaviour of the extremal particles in this article, we believe that many other features of multitype branching Brownian motions might be of interest, such as the growth rate of the number of particles of type 2 to the right of at for $a < v$, the large deviations of the maximal displacement M_t at time t, or the convergence of associated (sub)-martingales.

3.3 Preliminary results on the branching Brownian motion

We list in this section results on the standard BBM, that we use to study the two-type reducible BBM. For the rest of the section, $(X_u(t), u \in \mathcal{N}_t)_{t\geq 0}$ will denote a standard BBM, with branching rate 1 and diffusion constant 1, i.e. that has the same behaviour as particles of type 2. To translate the results of this section to the behaviour of particles of type 1 as well, it is worth noting that for all $\beta, \sigma > 0$:

$$\left(\frac{\sigma}{\sqrt{\beta}} X_u(\beta t), u \in \mathcal{N}_{\beta t}\right)_{t\geq 0} \tag{3.3.1}$$

is a branching Brownian with branching rate β and diffusion constant σ^2.

The rest of the section is organised as follows. We introduce in Section 3.3.1 the additive martingales of the BBM, and in particular the derivative martingale that plays a special role in the asymptotic behaviour of the maximal displacement of the BBM. We then provide in Section 3.3.3 a series of uniform asymptotic estimates on the maximal displacement of the BBM. Finally, in Section 3.3.4, we introduce the decoration measures and extremal processes appearing when studying particles near the rightmost one in the BBM.

3.3.1 Additive martingales of the branching Brownian motion

We begin by introducing the additive martingales of the BBM. For all $\theta \in \mathbb{R}$, the process

$$W_t(\theta) := \sum_{u\in\mathcal{N}_t} e^{\theta X_u(t) - t\left(\frac{\theta^2}{2}+1\right)}, \quad t \geq 0 \tag{3.3.2}$$

is a non-negative martingale. It is now a well-known fact that the martingale $(W_t(\theta), t \geq 0)$ is uniformly integrable if and only if $|\theta| < \sqrt{2}$, and in that case it converges towards an a.s. positive limiting random variable

$$W_\infty(\theta) := \lim_{t\to\infty} W_t(\theta). \tag{3.3.3}$$

Otherwise, we have $\lim_{t\to\infty} W_t(\theta) = 0$ a.s. This result was first shown by [Nev88]. It can also be obtained by a specific change of measure technique, called *the spinal decomposition*. This method was pioneered by Lyons, Pemantle and Peres [LPP95] for the study of the martingale of a Galton-Watson process, and extended by Lyons [Lyo97] to spatial branching processes setting.

For all $|\theta| < \sqrt{2}$ the martingale limit $W_\infty(\theta)$ is closely related to the number of particles moving at speed θ in the BBM. For example, by [Big92b, Corollary 4], for all $h > 0$ we have

$$\lim_{t\to\infty} t^{1/2} e^{t\left(\frac{\theta^2}{2}-1\right)} \sum_{u\in\mathcal{N}_t} \mathbb{1}_{\{|X_u(t)-\theta t|\leq h\}} = \frac{2\sinh(\theta h)}{\theta} W_\infty(\theta) \quad \text{a.s.} \tag{3.3.4}$$

This can be thought of as a local limit theorem result for the position of a particle sampled at random at time t, where a particle at position x is sampled with probability proportional to $e^{\theta x}$. A Donsker-type theorem was obtained in [Pai18, Section C] for this quantity, see also [GKS18]. In particular, for any continuous bounded function f, one has

$$\lim_{t\to\infty} \sum_{u\in\mathcal{N}_t} f\left(\tfrac{X_u(t)-\theta t}{t^{1/2}}\right) e^{\theta X_u(t) - t\left(\frac{\theta^2}{2}+1\right)} = \frac{W_\infty(\theta)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{z^2}{2}} f(z)\mathrm{d}z \text{ a.s.} \tag{3.3.5}$$

This justifies the fact that the variable $W_\infty(\theta)$ appears in the limiting distribution of the extremal process in the anomalous spreading case, by the heuristics described in Section 3.2.1.

To prove Theorem 3.1.3, we use the following slight generalization of the above convergence.

Lemma 3.3.1. *Let $a < b$ and $\lambda > 0$. For all continuous bounded function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, we have*

$$\lim_{t\to\infty} \frac{1}{t^{1/2}} \int_{\lambda t + a t^{1/2}}^{\lambda t + b t^{1/2}} \sum_{u \in \mathcal{N}_s} f\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) e^{\theta X_u(s) - s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s$$
$$= \frac{W_\infty(\theta)}{\sqrt{2\pi\lambda}} \int_{[a,b]\times\mathbb{R}} e^{-\frac{z^2}{2\lambda}} f(r,z) \mathrm{d}r \mathrm{d}z \text{ a.s.} \quad (3.3.6)$$

Proof. As a first step, we show that (3.3.6) holds for $f : (r,x) \mapsto \mathbb{1}_{\{r\in[a,b]\}} g(x)$, with g a continuous compactly supported function. Using that

$$\lim_{s\to\infty} \sum_{u\in\mathcal{N}_s} g\left(\lambda^{1/2} \tfrac{X_u(s)-\theta s}{s^{1/2}}\right) e^{\theta X_u(s) - s\left(\frac{\theta^2}{2}+1\right)} = \frac{W_\infty(\theta)}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{z^2}{2}} g(\lambda^{1/2} z) \mathrm{d}z \quad \text{a.s,}$$

by (3.3.5) we immediately obtain that

$$\lim_{t\to\infty} \frac{1}{t^{1/2}} \int_{\lambda t + a t^{1/2}}^{\lambda t + b t^{1/2}} \sum_{u \in \mathcal{N}_s} g\left(\lambda^{1/2} \tfrac{X_u(s)-\theta s}{s^{1/2}}\right) e^{\theta X_u(s) - s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s$$
$$= (b-a) \frac{W_\infty(\theta)}{\sqrt{2\pi\lambda}} \int_{\mathbb{R}} e^{-\frac{z^2}{2\lambda}} g(z) \mathrm{d}z \text{ a.s.}$$

We set $I_t(a,b) = [\lambda t + a t^{1/2}, \lambda t + b t^{1/2}]$. We observe that for all $s \in I_t(a,b)$, we have

$$s^{1/2} = (\lambda t + s - \lambda t)^{1/2} = (\lambda t)^{1/2} \left(1 + \frac{s - \lambda t}{\lambda t}\right)^{1/2}.$$

As $\frac{s-\lambda t}{\lambda t} \in [a t^{-1/2}/\lambda, b t^{-1/2}/\lambda]$, there exists a constant $K > 0$ such that for all $t \geq 1$, we have

$$\sup_{s\in I_t(a,b)} \left| \left(1 + \frac{s-\lambda t}{\lambda t}\right)^{1/2} - 1 \right| \leq K t^{-1/2},$$

so $\left|s^{1/2} - (\lambda t)^{1/2}\right| \leq K \lambda^{1/2}$ uniformly in $s \in I_t(a,b)$, for all t large enough.

Then, using the uniform continuity and compactness of g, for all $\varepsilon > 0$ we have

$$\sup_{s \in I_t(a,b), x \in \mathbb{R}} \left| g(\lambda^{1/2} x / s^{1/2}) - g(x/t^{1/2}) \right| \leq \varepsilon$$

for all t large enough. Therefore, for all $\varepsilon > 0$,

$$\limsup_{t\to\infty} \frac{1}{t^{1/2}} \int_{I_t(a,b)} \sum_{u\in\mathcal{N}_s} \left| g\left(\tfrac{X_u(s)-\theta s}{s^{1/2}}\right) - g\left(\tfrac{X_u(s)-\theta s}{(\lambda t)^{1/2}}\right) \right| e^{\theta X_u(s) - s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s$$
$$\leq \limsup_{t\to\infty} \frac{1}{t^{1/2}} \int_{I_t(a,b)} \sum_{u\in\mathcal{N}_s} \varepsilon e^{\theta X_u(s) - s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s = \varepsilon (b-a) W_\infty(\theta) \quad \text{a.s.}$$

Letting $\varepsilon \to 0$, we finally obtain

$$\lim_{t\to\infty} \frac{1}{t^{1/2}} \int_{I_t(a,b)} \sum_{u\in\mathcal{N}_s} f\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) e^{\theta X_u(s)-s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s$$
$$= \frac{W_\infty(\theta)}{\sqrt{2\pi\lambda}} \int_{[a,b]\times\mathbb{R}} e^{-\frac{z^2}{2\lambda}} f(r,z)\mathrm{d}r\mathrm{d}z \quad \text{a.s.} \quad (3.3.7)$$

We now assume that f is a continuous compactly supported function on $[a,b]\times\mathbb{R}$. For all $i \leq n$, we set

$$f_i(r,x) = \mathbb{1}_{\{r\in[a+i(b-a)/n, a+(i+1)(b-a)/n]\}} f(a+i(b-a)/n, x).$$

Using the uniform integrability of f, for all n large enough, we have $\left\| f - \sum_{j=1}^n f_j \right\|_\infty \leq \varepsilon$. As a result, we have

$$\limsup_{t\to\infty} \frac{1}{t^{1/2}} \int_{I_t(a,b)} \sum_{u\in\mathcal{N}_s} e^{\theta X_u(s)-s\left(\frac{\theta^2}{2}+1\right)}$$
$$\times \left| f\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) - \sum_{i=1}^n f_i\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) \right| \mathrm{d}s \leq \varepsilon W_\infty(\theta) \quad \text{a.s.}$$

Therefore, using (3.3.7), we obtain

$$\limsup_{t\to\infty} \left| \frac{1}{t^{1/2}} \int_{\lambda t+at^{1/2}}^{\lambda t+bt^{1/2}} \sum_{u\in\mathcal{N}_s} f\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) e^{\theta X_u(s)-s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s \right.$$
$$\left. - \frac{W_\infty(\theta)}{\sqrt{2\pi\lambda}} \int_{[a,b]\times\mathbb{R}} e^{-\frac{z^2}{2\lambda}} f(r,z)\mathrm{d}r\mathrm{d}z \right| \leq 2\varepsilon W_\infty(\theta) \quad \text{a.s.}$$

Letting $\varepsilon \to 0$ therefore proves that (3.3.6) holds for compactly supported continuous functions.

Finally, to complete the proof, we consider a continuous bounded function f on $[a,b]\times\mathbb{R}$. Let $R > 0$, given χ_R a continuous function on $\mathbb{R}$ such that $\mathbb{1}_{\{|x|<R\}} \leq \chi_R(x) \leq \mathbb{1}_{\{|x|\leq R+1\}}$, the previous computation shows that

$$\lim_{t\to\infty} \frac{1}{t^{1/2}} \int_{\lambda t+at^{1/2}}^{\lambda t+bt^{1/2}} \sum_{u\in\mathcal{N}_s} \chi_R\left(\tfrac{X_u(s)-\theta s}{t^{1/2}}\right) f\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) e^{\theta X_u(s)-s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s$$
$$= \frac{W_\infty(\theta)}{\sqrt{2\pi\lambda}} \int_{[a,b]\times\mathbb{R}} e^{-\frac{z^2}{2\lambda}} f(r,z)\chi_R(z)\mathrm{d}r\mathrm{d}z \text{ a.s.}$$

Additionally, setting $K = \|f\|_\infty$, for all t large enough we have

$$\left| \frac{1}{t^{1/2}} \int_{\lambda t+at^{1/2}}^{\lambda t+bt^{1/2}} \sum_{u\in\mathcal{N}_s} \left(1 - \chi_R\left(\tfrac{X_u(s)-\theta s}{t^{1/2}}\right)\right) f\left(\tfrac{s-\lambda t}{t^{1/2}}, \tfrac{X_u(s)-\theta s}{t^{1/2}}\right) e^{\theta X_u(s)-s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s \right|$$
$$\leq \frac{K}{t^{1/2}} \int_{\lambda t+at^{1/2}}^{\lambda t+bt^{1/2}} \sum_{u\in\mathcal{N}_s} \left(1 - \chi_R\left(\lambda^{1/2}\tfrac{X_u(s)-\theta s}{2s^{1/2}}\right)\right) e^{\theta X_u(s)-s\left(\frac{\theta^2}{2}+1\right)} \mathrm{d}s,$$

which converges to $\frac{(b-a)W_\infty(\theta)}{\sqrt{2\pi}} \int_{\mathbb{R}} (1 - \chi_R(\lambda^{1/2} z/2)) e^{-\frac{z^2}{2}} \mathrm{d}z$ as $t \to \infty$. Thus, letting $t \to \infty$ then $R \to \infty$ completes the proof of this lemma. □

3.3.2 The derivative martingale

The number of particles that travel at the critical speed $\sqrt{2}$ cannot be counted using the additive martingale (as it converges to 0 almost surely). In this situation, the appropriate process allowing this estimation is the derivative martingale $(Z_t, t \geq 0)$. Its name comes from the fact that Z_t can be represented as $-\frac{\partial}{\partial \theta} W_t(\theta)\big|_{\theta=\sqrt{2}}$, more precisely

$$Z_t := \sum_{u \in \mathcal{N}_t} (\sqrt{2}t - X_u(t)) e^{\sqrt{2}X_u(t) - 2t}. \tag{3.3.8}$$

Despite being a non-integrable signed martingale, it was proved by Lalley and Sellke [LS87] that it converges to an a.s. positive random variable

$$Z_\infty := \lim_{t \to \infty} Z_t \quad \text{a.s.} \tag{3.3.9}$$

In the same way that the limit of the additive martingale gives the growth rate of the number of particles moving at speed θ, the derivative martingale gives the growth rate of particles that go at speed $\sqrt{2}$. As a result, it appears in the asymptotic behaviour of the maximal displacement, and results similar to (3.3.4) and (3.3.5) can be found in [Mad17, Pai18] in the context of branching random walks.

We mention that the limit Z_∞ of the derivative martingale is non-integrable, and that its precise tail has been well-studied. In particular, Bereskycki, Berestycki and Schweinsberg [BBS13] proved that

$$\mathbf{P}(Z_\infty \geq x) \sim \frac{\sqrt{2}}{x} \text{ as } x \to \infty. \tag{3.3.10}$$

Similar results were obtained for branching random walks by Buraczewski [Bur09] and Madaule [Mad17]. They also obtained a more precise estimate on its asymptotic, that can be expressed in the two following equivalent ways

$$\mathbf{E}(Z_\infty \mathbb{1}_{\{Z_\infty \leq x\}}) = \sqrt{2} \log x + O(1) \quad \text{as } x \to \infty, \tag{3.3.11}$$

$$1 - \mathbf{E}(e^{-\lambda Z_\infty}) = \sqrt{2}\lambda \log \lambda + O(\lambda) \quad \text{as } \lambda \to 0. \tag{3.3.12}$$

Maillard and Pain [MP19] improved on these statements and gave necessary and sufficient conditions for the asymptotic developments of these quantities up to a $o(1)$. We mention that the equivalence between (3.3.11) and (3.3.12) can be found in [BIM21, Lemma 8.1], which obtain similar necessary and sufficient conditions for the asymptotic development of the Laplace transform of the derivative martingale of the branching random walk under optimal integrability conditions.

3.3.3 Maximal displacement of the branching Brownian motion

A large body of work has been dedicated to the study of the maximal displacement of the BBM, defined by $M_t = \max_{u \in \mathcal{N}_t} X_u(t)$. We recall here some estimates related to its study. We begin by observing that the BBM travels in a triangular-shaped array, and that for all $y \geq 0$

$$\mathbf{P}\left(\exists t \geq 0, u \in \mathcal{N}_t : X_u(t) \geq \sqrt{2}t + y\right) \leq e^{-\sqrt{2}y}, \tag{3.3.13}$$

which shows that with high probability, all particles at time t are smaller than $\sqrt{2}t + y$ in absolute value.

Recall that Lalley and Sellke [LS87] proved that setting $m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, the maximal displacement of the BBM centered by m_t converges in distribution to a shifted Gumbel distribution. More precisely, there exists $c_\star > 0$ such that

$$\lim_{t\to\infty} \mathbf{P}(M_t \leq m_t + z) = \mathbf{E}\left(\exp\left(-c_\star Z_\infty e^{-\sqrt{2}z}\right)\right). \tag{3.3.14}$$

An uniform upper bound is also known for the right tail of the maximal displacement. There exists $C > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}$, we have

$$\mathbf{P}(M_t \geq m_t + x) \leq C(1 + x_+)e^{-\sqrt{2}x}, \tag{3.3.15}$$

where $x_+ = \max(x, 0)$. This estimate can be obtained by first moment methods, we refer e.g. to [Hu16] for a similar estimate in the branching random walk, which immediately implies a similar bound for the BBM.

In the context of the anomalous spreading, seen from the heuristics in Section 3.2.1, it will also be necessary to use tight estimates on the large deviations of the BBM. These large deviations were first studied by Chauvin and Rouault [CR88]. Precise large deviations for the maximal displacement were recently obtained in [DMS16, GH18, BM19, BBCM20], proving that for all $\rho > \sqrt{2}$, there exists $C(\rho) \in (0, 1)$ such that

$$\mathbf{P}(M_t \geq \rho t + y) \sim_{t\to\infty} \frac{C(\rho)}{\sqrt{2\pi t}\rho} e^{-(\rho^2/2-1)t} e^{-\rho y - \frac{y^2}{2t}}, \tag{3.3.16}$$

uniformly in $|y| \leq r_t$, for all function $r_t = o(t)$.

Additionally, from a simple first moment estimate, one can obtain an uniform upper bound for this large deviations estimate on the maximal displacement.

Lemma 3.3.2. *For all $\rho > \sqrt{2}$ and $A > 0$, there exists $C > 0$ such that for all t large enough and all $y \geq -At^{1/2}$, we have*

$$\mathbf{P}(M_t \geq \rho t + y) \leq \frac{Ce^{-(\rho^2/2-1)t}}{t^{1/2}} e^{-\rho y - \frac{y^2}{2t}}.$$

This result is based on Markov inequality and classical Gaussian estimates, that appear later in our paper in more complicate settings. We thus give a short proof of this statement.

Proof. Observe that for t large enough, we have $\rho t + y \geq \delta t$ for some positive constant δ. Then, by Markov inequality, we have

$$\mathbf{P}(M_t \geq \rho t + y) = \mathbf{P}\left(\exists u \in \mathcal{N}_t : X_u(t) \geq \rho t + y\right) \leq \mathbf{E}\left(\sum_{u\in\mathcal{N}_t} \mathbb{1}_{\{X_u(t)\geq \rho t + y\}}\right).$$

Using that there are on average e^t particles alive at time t and that the displacements of particles are Brownian motions, that are independent of the total number of particles in the process, we have

$$\mathbf{E}\left(\sum_{u\in\mathcal{N}_t} \mathbb{1}_{\{X_u(t)\geq \rho t + y\}}\right) = e^t \mathbf{P}(B_t \geq \rho t + y).$$

This fact is often called the many-to-one lemma in the literature (see e.g. [Shi15, Theorem 1]). We develop in Section 3.4 a multitype versions of this result.

We now use the following well-known asymptotic estimate on the tail of the Gaussian random variable that

$$\mathbf{P}(B_1 \geq x) \leq \frac{1}{\sqrt{2\pi} x} e^{-\frac{x^2}{2}} \quad \text{for all } x \geq 0. \tag{3.3.17}$$

This yields

$$\begin{aligned}\mathbf{E}\left(\sum_{u\in\mathcal{N}_t} \mathbb{1}_{\{X_u(t)\geq \rho t + y\}}\right) &= e^t \mathbf{P}\left(B_1 \geq \rho t^{1/2} + y t^{-1/2}\right) \leq C t^{-1/2} e^{-\frac{(\rho t+y)^2}{2t}} \\ &\leq C t^{-1/2} e^{t(1-\rho^2/2)} e^{-\rho y - \frac{y^2}{2t}}\end{aligned}$$

completing the proof. □

3.3.4 Decorations of the branching Brownian motion

We now turn to results related to the extremal process of the BBM. Before stating these, we introduce a general tool that allows the obtention of the joint convergence in distribution of the maximal displacement and the extremal process of a particle system. Denote by $\mathcal{T}$ the set of continuous non-negative bounded functions, with support bounded on the left. The following result can be found in [BBCM20, Lemma 4.4].

Proposition 3.3.3. *Let $\mathcal{P}_n, \mathcal{P}$ be point measures on the real line. We denote by $\max \mathcal{P}_n$ (respectively $\max \mathcal{P}$), the position of the rightmost atom in this point measure. The following statements are equivalent*

1. $\lim_{n\to\infty} \mathcal{P}_n = \mathcal{P}$ *and* $\lim_{n\to\infty} \max \mathcal{P}_n = \max \mathcal{P}$ *in law.*
2. $\lim_{n\to\infty} (\mathcal{P}_n, \max \mathcal{P}_n) = (\mathcal{P}, \max \mathcal{P})$ *in law.*
3. *for all* $\varphi \in \mathcal{T}$, $\lim_{n\to\infty} \mathbf{E}\left(e^{-\langle \mathcal{P}_n, \varphi\rangle}\right) = \mathbf{E}\left(e^{-\langle \mathcal{P}, \varphi\rangle}\right)$.

In other words, considering continuous bounded functions with support bounded on the left instead of continuous compactly supported functions allows us to capture the joint convergence in law of the maximal displacement and the extremal process. We refer to the set $\mathcal{T}$ as the set of test functions, against which we test the convergence of our point measures of interest.

The convergence in distribution of the extremal process of a BBM has been obtained by Aïdékon, Berestycki, Brunet and Shi [ABBS13], and by Arguin, Bovier and Kistler [ABK13]. They proved that setting

$$\mathcal{E}_t = \sum_{u\in\mathcal{N}_t} \delta_{X_u(t) - m_t},$$

this extremal process converges in distribution towards a decorated Poisson point process with intensity $c_\star Z_\infty \sqrt{2} e^{-\sqrt{2}z} \mathrm{d}z$. The law of the decoration is described in [ABK13] as the limiting distribution of the maximal displacement seen from the rightmost particle, conditioned on being larger than $\sqrt{2}t$ at time t. More precisely, they proved that there exists a point measure $\mathcal{D}$ such that

$$\lim_{t\to\infty} \mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{N}_t} \varphi(X_u(t) - M_t)\right) \middle| M_t \geq \sqrt{2}t\right) = \mathbf{E}\left(\exp\left(-\langle \mathcal{D}, \varphi\rangle\right)\right) \tag{3.3.18}$$

for all function $\varphi \in \mathcal{T}$. Note that $\mathcal{D}$ is supported on $(-\infty, 0]$ and has an atom at 0.

The limiting extremal process $\mathcal{E}_\infty$ can be constructed as follows. Let $(\xi_j)_{j\in\mathbb{N}}$ be the atoms of a Poisson point process with intensity $c_\star\sqrt{2}e^{-\sqrt{2}z}\mathrm{d}z$, and $(\mathcal{D}_j, j \in \mathbb{N})$ i.i.d. point measures, then set

$$\mathcal{E}_\infty = \sum_{j\in\mathbb{N}} \sum_{d\in\mathcal{D}_j} \delta_{\xi_j + d + \frac{1}{\sqrt{2}}\log Z_\infty},$$

where $\sum_{d\in\mathcal{D}_j}$ represents a sum on the set of atoms of the point measure $\mathcal{D}_j$.

In view of Proposition 3.3.3 and (3.3.14), we can rewrite as follows the convergence in law of the extremal process of the BBM, with simple Poisson computations.

Lemma 3.3.4. *For all function $\varphi \in \mathcal{T}$, we have*

$$\lim_{t\to\infty} \mathbf{E}\left(e^{-\langle \mathcal{E}_t, \varphi\rangle}\right) = \mathbf{E}\left(\exp\left(-c_\star Z_\infty \int (1 - e^{-\Psi[\varphi](z)})\sqrt{2}e^{-\sqrt{2}z}\mathrm{d}z\right)\right),$$

where we have set $\Psi[\varphi] : z \mapsto -\log \mathbf{E}\left(e^{-\langle \mathcal{D}, \varphi(\cdot + z)\rangle}\right)$.

In the context of large deviations of BBM, a one-parameter family of point measures, similar to the one defined in (3.3.18) can be introduced. These point measures have first been studied by Bovier and Hartung [BH14] when considering the extremal process of the time-inhomogeneous BBM. More precisely, they proved that for all $\rho > \sqrt{2}$, there exists a point measure $\mathcal{D}^\rho$ such that

$$\lim_{t\to\infty} \mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{N}_t} \varphi(X_u(t) - M_t)\right)\middle| M_t \geq \rho t\right) = \mathbf{E}\left(\exp\left(-\langle \mathcal{D}^\rho, \varphi\rangle\right)\right). \tag{3.3.19}$$

In [BBCM20], an alternative construction of this one parameter family of point measures was introduced, which allows its representation as a point measure conditioned on an event of positive probability instead of a large deviation event of probability decaying exponentially fast in t. Let us begin by introducing some notation. Let $(B_t, t \geq 0)$ be a standard Brownian motion, $(\tau_k, k \geq 1)$ the atoms of an independent Poisson point process of intensity 2, and $(X_u^{(k)}(t), u \in \mathcal{N}_t^{(k)}, t \geq 0)$ i.i.d. BBMs, which are further independent of B and τ. For $\rho > \sqrt{2}$, we set

$$\widetilde{\mathcal{D}}_t^\rho = \delta_0 + \sum_{k\geq 1} \mathbb{1}_{\{\tau_k < t\}} \sum_{u\in\mathcal{N}_{\tau_k}^{(k)}} \delta_{B_{\tau_k} - \rho\tau_k + X_u(\tau_k)} \quad \text{and} \quad \widetilde{\mathcal{D}}^\rho = \lim_{t\to\infty} \widetilde{\mathcal{D}}_t^\rho. \tag{3.3.20}$$

In words, the process $\widetilde{\mathcal{D}}^\rho$ is constructed using one particle that starts from 0 and travels backwards in time according to a Brownian motion with drift ρ. This particle gives birth to offspring at rate 2, each newborn child starting an independent BBM from its current position, forward in time. The point measure $\widetilde{\mathcal{D}}^\rho$ then consists of the position of all particles alive at time 0.

As a first step, we mention the following result, which can be thought of as a spinal decomposition argument with respect to the rightmost particle. This result can be found in [BBCM20, Lemma 2.1].

Proposition 3.3.5. *For all $t \geq 0$ set*

$$\mathcal{E}_t^* = \sum_{u\in\mathcal{N}_t} \delta_{X_u(t) - M_t}$$

the extremal process seen from the rightmost position. For all measurable non-negative functions f, F, we have

$$\mathbf{E}\left(f(M_t - \rho t)F(\mathcal{E}_t^*)\right) = e^{(1-\rho^2/2)t}\mathbf{E}\left(e^{-\rho B_t} f(B_t) F(\widetilde{\mathcal{D}}_t^\rho)\mathbb{1}_{\left\{\widetilde{\mathcal{D}}_t^\rho((0,\infty))=0\right\}}\right)$$

It then follows from (3.3.16) and the above proposition that the law $\mathcal{D}^\rho$ can be represented by conditioning the point measure $\widetilde{\mathcal{D}}^\rho$, as was obtained in [BBCM20, Theorem 1.1].

Lemma 3.3.6. *For all $\rho > \sqrt{2}$,*

- *the constant $C(\rho)$ introduced in (3.3.16) is given by $C(\rho) = \mathbf{P}(\widetilde{\mathcal{D}}^\rho((0,\infty)) = 0)$.*
- *the law of the point measure $\mathcal{D}^\rho$ introduced in (3.3.19) can be constructed as*

$$\mathbf{P}(\mathcal{D}^\rho \in \cdot) = \mathbf{P}(\widetilde{\mathcal{D}}^\rho \in \cdot | \widetilde{\mathcal{D}}^\rho((0,\infty)) = 0).$$

We end this section with an uniform estimate on the Laplace transform of the extremal process of the BBM, that generalizes both (3.3.16) and (3.3.19).

Lemma 3.3.7. *Let $\rho > \sqrt{2}$, we set*

$$\mathcal{E}_t^\rho(x) = \sum_{u \in \mathcal{N}_t} \delta_{X_u(t) - \rho t + x}.$$

Let $A > 0$, for all $\varphi \in \mathcal{T}$, we have

$$\mathbf{E}\left(1 - e^{-\left\langle \mathcal{E}_t^\rho(x), \varphi\right\rangle}\right) = C(\rho)\frac{e^{\left(1-\rho^2/2\right)t}}{\sqrt{2\pi t}} e^{\rho x - \frac{x^2}{2t}} \int e^{-\rho z}\left(1 - e^{-\Psi^\rho[\varphi](z)}\right) \mathrm{d}z(1 + o(1)),$$

uniformly in $|x| \leq At^{1/2}$, as $t \to \infty$, where $\Psi^\rho[\varphi] : z \mapsto -\log \mathbf{E}\left(e^{-\langle \mathcal{D}^\rho, \varphi(\cdot + z)\rangle}\right)$.

Proof. Let $L > 0$, recall from Lemma 3.3.2 that

$$\mathbf{P}(M_t \geq \rho t - x + L) \leq Ct^{-1/2} e^{t(1-\rho^2/2)} e^{\rho x - \frac{x^2}{2t}} e^{-\rho L}.$$

Thus, as φ is non-negative, we have

$$0 \leq \mathbf{E}\left(1 - e^{-\left\langle \mathcal{E}_t^\rho(x), \varphi\right\rangle}\right) - \mathbf{E}\left(\left(1 - e^{-\left\langle \mathcal{E}_t^\rho(x), \varphi\right\rangle}\right)\mathbb{1}_{\{M_t \leq \rho t - x + L\}}\right) \leq Ct^{-1/2} e^{t(1-\rho^2/2)} e^{\rho x - \frac{x^2}{2t}} e^{-\rho L}. \quad (3.3.21)$$

We also recall that the support of φ is bounded on the left, i.e. is included on $[R, \infty)$ for some $R \in \mathbb{R}$. Observe then that $e^{-\left\langle \mathcal{E}_t^\rho(x), \varphi\right\rangle} = 1$ on the event $\{M_t \leq \rho t - x + R\}$.

We now use Proposition 3.3.5 to compute

$$\mathbf{E}\left(\left(1 - e^{-\left\langle \mathcal{E}_t^\rho(x), \varphi\right\rangle}\right)\mathbb{1}_{\{M_t - \rho t + x \in [R, L]\}}\right) = e^{t(1-\rho^2/2)}\mathbf{E}\left(e^{\rho B_t}\left(1 - e^{-\left\langle \widetilde{\mathcal{D}}_t^\rho, \tau_{B_t + x}\varphi\right\rangle}\right)\mathbb{1}_{\left\{x + B_t \in [R, L], \widetilde{\mathcal{D}}_t^\rho((0,\infty))=0\right\}}\right),$$

where $\tau_z(\varphi)(\cdot) = \varphi(z+\cdot)$. Therefore, setting

$$G_t(x,z) = \mathbf{E}\left(\left(1 - e^{-\left\langle \widetilde{\mathcal{D}}^\rho_t, \tau_z\varphi \right\rangle}\right) \mathbb{1}_{\left\{\widetilde{\mathcal{D}}^\rho_t((0,\infty))=0\right\}} \middle| B_t = z - x \right),$$

we have

$$e^{t(\rho^2/2-1)}\sqrt{2\pi t}e^{-\rho x + \frac{x^2}{2t}} \mathbf{E}\left(\left(1 - e^{-\left\langle \mathcal{E}^\rho_t(x), \varphi \right\rangle}\right) \mathbb{1}_{\{M_t - \rho t + x \in [R,L]\}} \right)$$
$$= \int_R^L e^{-\rho y + o(t^{-1/2})} G_t(x,y) \mathrm{d}y, \quad (3.3.22)$$

with the $o(t^{-1/2})$ term being uniform in $|x| \leq At^{1/2}$.

With the same computations as in the proof of [BBCM20, Lemma 3.4], we obtain

$$\lim_{t\to\infty} \sup_{|x|\leq At^{1/2}} G_t(x,y) = \lim_{t\to\infty} \inf_{|x|\leq At^{1/2}} G_t(x,y) = \mathbf{E}\left(\left(1 - e^{-\left\langle \widetilde{\mathcal{D}}^\rho, \tau_y\varphi \right\rangle}\right) \mathbb{1}_{\left\{\widetilde{\mathcal{D}}^\rho((0,\infty))=0\right\}} \right)$$
$$= C(\rho)\mathbf{E}\left(1 - e^{-\langle \mathcal{D}^\rho, \tau_y\varphi \rangle}\right),$$

using the construction of $\mathcal{D}^\rho$ given in Lemma 3.3.6. Therefore, using (3.3.21) and applying the dominated convergence theorem, equation (3.3.22) yields

$$\limsup_{t\to\infty} \sup_{|x|\leq At^{1/2}} \left| e^{t(\rho^2/2-1)}\sqrt{2\pi t}e^{-\rho x + \frac{x^2}{2t}} \mathbf{E}\left(\left(1 - e^{-\left\langle \mathcal{E}^\rho_t(x), \varphi \right\rangle}\right)\right) \right.$$
$$\left. - C(\rho) \int_{\mathbb{R}} e^{-\rho y} \mathbf{E}\left(1 - e^{-\langle \mathcal{D}^\rho, \tau_y\varphi \rangle}\right) \mathrm{d}y \right| \leq Ce^{-\rho L},$$

which, letting $L \to \infty$, completes the proof. □

Remark 3.3.8. Note that applying Lemma 3.3.7 to function $\varphi(z) = \mathbb{1}_{\{z\geq 0\}}$ yields (3.3.16), and up simple computations, this lemma can also be used to obtain (3.3.19).

3.4 Multitype many-to-one lemmas

The many-to-one lemma is an ubiquitous result in the study of branching Brownian motions. This result links additive moments of the BBM with Brownian motion estimates. We first recall the classical version of this lemma, before giving a multitype version that applies to our process.

Let $(X_u(t), u \in \mathcal{N}_t)$ be a standard BBM with branching rate 1. The classical many-to-one lemma can be tracked back at least to the work of Kahane and Peyrière [**?**, Sch76] on multiplicative cascades. It can be expressed as follows: for all $t \geq 0$ and measurable non-negative functions f, we have

$$\mathbf{E}\left(\sum_{u\in\mathcal{N}_t} f(X_u(s), s \leq t) \right) = e^t \mathbf{E}(f(B_s, s \leq t)), \quad (3.4.1)$$

where B is a standard Brownian motion.

Recall that $\mathcal{N}_t^1$ (respectively $\mathcal{N}_t^2$) is the set of particles of type 1 (resp. type 2) alive at time t. Note that the process $(X_u(t), u \in \mathcal{N}_t^1)_{t\geq 0}$ is a BBM with branching rate β and diffusion σ^2. Thus in view of (3.3.1), (3.4.1) implies that for all measurable non-negative function f

$$\mathbf{E}\left(\sum_{u\in\mathcal{N}_t^1} f(X_u(s), s \leq t)\right) = e^{\beta t}\mathbf{E}(f(\sigma B_s, s\leq t)).$$

We denote by $\mathbf{P}^{(2)}$ the law of the process starting from a single particle of type 2. As this particle behaves as in a standard BBM and only gives birth of particles of type 2, this process again is a BBM, therefore

$$\mathbf{E}^{(2)}\left(\sum_{u\in\mathcal{N}_t^2} f(X_u(s), s \leq t)\right) = e^{t}\mathbf{E}(f(B_s, s\leq t)),$$

writing $\mathbf{E}^{(2)}$ for the expectation associated to $\mathbf{P}^{(2)}$.

The main aim of this section is to prove the following result, which allows to represent an additive functional of particles of type 2 appearing in the multitype BBM by a variable speed Brownian motion.

Proposition 3.4.1. *For all measurable non-negative function f, we have*

$$\mathbf{E}\left(\sum_{u\in\mathcal{N}_t^2} f((X_u(s), s\leq t), T(u))\right) = \alpha\int_0^t e^{\beta s+(t-s)}\mathbf{E}\left(f((\sigma B_{u\wedge s} + (B_u - B_{u\wedge s}), u\leq t), s)\right)\mathrm{d}s,$$

where we recall that $T(u)$ is the birth time of the first ancestor of type 2 of u.

To prove this result, we begin by investigating the set $\mathcal{B}$ of particles of type 2 that are born from a particle of type 1, that can be defined as

$$\mathcal{B} := \left\{u \in \cup_{t\geq 0}\mathcal{N}^2(t) : T(u) = b_u\right\}.$$

We observe that $\mathcal{B}$ can be thought of as a Poisson point process with random intensity.

Lemma 3.4.2. *We set $\mathcal{F}^1 = \sigma(X_u(t), u \in \mathcal{N}_t^1, t\geq 0)$. The point measure $\sum_{u\in\mathcal{B}}\delta_{(X_u(s), s\leq T(u))}$, conditionally on $\mathcal{F}^1$ is a Poisson point process with intensity $\alpha\mathrm{d}t \otimes \sum_{u\in\mathcal{N}_t^1}\delta_{(X_u(s),s\leq t)}$.*

Proof. This is a straightforward consequence of the definition of the two-type BBM and the superposition principle for Poisson process. Over its lifetime, a particle of type 1 gives birth to particles of type 2 according to a Poisson process with intensity α, and the trajectory leading to the newborn particle at time t is exactly the same as the trajectory of its parent particle up to time t. □

A direct consequence of the above lemma is the following applications of Poisson summation formula.

Corollary 3.4.3. *For all measurable non-negative function f, we have*

$$\mathbf{E}\left(\sum_{u\in\mathcal{B}} f(X_u(s), s\le T(u))\right) = \alpha\int_0^\infty e^{\beta t}\mathbf{E}(f(\sigma B_s, s\le t))\mathrm{d}t, \tag{3.4.2}$$

$$\mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{B}} f(X_u(s), s\le T(u))\right)\right) = \mathbf{E}\left(\exp\left(-\alpha\int_0^\infty \sum_{u\in\mathcal{N}_t^1} 1-e^{-f(X_u(s),s\le t)}\mathrm{d}t\right)\right) \tag{3.4.3}$$

Proof. We denote by $\mathcal{F}^1 = \sigma(X_u(s), u\in\mathcal{N}_s^1, s\ge 0)$ the filtration generated by all particles of type 1. We can compute

$$\mathbf{E}\left(\sum_{u\in\mathcal{B}} f(X_u(s), s\le T(u))\middle|\mathcal{F}^1\right) = \alpha\int_0^\infty \sum_{u\in\mathcal{N}_t^1} f(X_u(s), s\le t)\mathrm{d}t$$

using Lemma 3.4.2. Then using Fubini's theorem and (3.4.1), we conclude that

$$\mathbf{E}\left(\sum_{u\in\mathcal{B}} f(X_u(s), s\le T(u))\right) = \alpha\int_0^\infty e^{\beta t}\mathbf{E}(f(\sigma B_s, s\le t))\mathrm{d}t.$$

Similarly, using the exponential Poisson formula, we have

$$\mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{B}} f(X_u(s), s\le T(u))\right)\middle|\mathcal{F}^1\right) = \exp\left(-\alpha\int_0^\infty \sum_{u\in\mathcal{N}_t^1} 1-e^{-f(X_u(s),s\le t)}\mathrm{d}t\right).$$

Taking the expectation of this formula completes the proof of this corollary. □

We now turn to the proof of the multitype many-to-one lemma.

Proof of Proposition 3.4.1. Let f, g be two measurable bounded functions. For any u, u' particles in the BBM, we write $u' \succcurlyeq u$ to denote that u' is a descendant of u. We compute

$$\begin{aligned}
&\mathbf{E}\left(\sum_{u\in\mathcal{N}_t^2} f(X_u(s), s\le T(u))g(X_u(s), s\in[T(u),t])\right)\\
=&\mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\le t\}} f(X_u(s), s\le T(u)) \sum_{\substack{u'\in\mathcal{N}_t^2\\ u'\succcurlyeq u}} g(X_{u'}(s), s\in[T(u),t])\right)\\
=&\mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\le t\}} f(X_u(s), s\le T(u))\varphi(T(u), X_u(T(u)))\right),
\end{aligned}$$

using the branching property for the BBM: every particle $u\in\mathcal{B}$ starts an independent BBM from time $T(u)$ and position $X_u(T(u))$. Here, we have set for $x\in\mathbb{R}$ and $s\ge 0$

$$\begin{aligned}
\varphi(s,x) &= \mathbf{E}^{(2)}\left(\sum_{u\in\mathcal{N}_{t-s}^2} g\left(x+X_u(r-s), r\in[s,t]\right)\right)\\
&= e^{t-s}\mathbf{E}\left(g\left(x+B_{r-s}, r\in[s,t]\right)\right),
\end{aligned}$$

by the standard many-to-one lemma. Additionally, by Corollary 4.2.2, we have

$$\begin{aligned}
&\mathbf{E}\left(\sum_{u\in\mathcal{N}_t^2} f(X_u(s), s\leq T(u))g(X_u(s), s\in[T(u),t])\right)\\
=&\alpha\int_0^t e^{\beta s}\mathbf{E}(f(\sigma B_r, r\leq s)\varphi(s,\sigma B_s))\mathrm{d}s\\
=&\alpha\int_0^t e^{\beta s+t-s}\mathbf{E}(f(\sigma B_r, r\leq s)g\left(\sigma B_s+(B_r-B_s), r\in[s,t]\right))\,\mathrm{d}s.
\end{aligned}$$

Using the monotone class theorem, the proof of Proposition 3.4.1 is now complete. □

3.5 Proof of Theorem 3.1.2

We assume in this section that $(\beta,\sigma^2)\in\mathcal{C}_{II}$, i.e. that either $\sigma^2>1$ and $\sigma^2<\frac{1}{\beta}$ or $\sigma^2\leq 1$ and $\sigma^2<2-\beta$. In that case, we show that the extremal process is dominated by the behaviour of particles of type 2 that are born at the beginning of the process. The main steps of the proof of Theorem 3.1.2 are the following:

1. We show that for all $A>0$, there exists $R>0$ such that with high probability, every particle u of type 2 to the right of $m_t^{(II)}-A$ satisfy $T(u)\leq R$.
2. We use the convergence in distribution of the extremal process of a single-type branching Brownian motion to demonstrate that the extremal process generated by the individuals born of type 2 before time R converges as $t\to\infty$.
3. We prove that letting $R\to\infty$, the above extremal process converges, and the limiting point measure is the point measure of the full two-type branching Brownian motion.

In this section, we write $v=\sqrt{2\beta\sigma^2}$ and $\theta=\sqrt{2\beta/\sigma^2}$, which are respectively the speed and critical parameter of the branching Brownian motion of particles of type 1. Recall that $m_t^{(II)}=\sqrt{2}t-\frac{3}{2\sqrt{2}}\log t$, we write

$$\widehat{\mathcal{E}}_t=\sum_{u\in\mathcal{N}_t^2}\delta_{X_u(t)-m_t^{(II)}},$$

the extremal process of particles of type 2 in the branching Brownian motion, centred around $m_t^{(II)}$. We begin by proving that with high probability, no particle of type 2 that was born from a particle of type 1 after time R has a descendant close to $m_t^{(II)}$.

Lemma 3.5.1. *Assuming that $(\beta,\sigma^2)\in\mathcal{C}_{II}$, for all $A>0$, we have*

$$\lim_{R\to\infty}\limsup_{t\to\infty}\mathbf{P}(\exists u\in\mathcal{N}_t^2: T(u)\geq R, X_u(t)\geq m_t^{(II)}-A)=0.$$

Proof. Let $K>0$, we first recall that by (3.3.1) and (3.3.13), we have

$$\mathbf{P}\left(\exists t\geq 0, u\in\mathcal{N}_t^1: X_u(t)\geq vt+K\right)\leq e^{-\theta K},$$

i.e. that with high probability, all particles of type 1 stay below the curve $s \mapsto vs + K$.

We now set, for $R, A, K \geq 0$ and $t \geq 0$:

$$Y_t(A, R, K) = \sum_{u \in \mathcal{B}} \mathbb{1}_{\{T(u) > R, X_u(T(u)) \leq vT(u) + K\}} \mathbb{1}_{\left\{M_t^u \geq m_t^{(II)} - A\right\}},$$

where M_t^u is the position of the rightmost descendant at time t of the individual u. In other words, $Y_t(A, R, K)$ is the number of particles of type 2 born from a particle of type 1 after time R, that were born below the curve $s \mapsto vs + K$ and have a member of their family to the right of $m_t^{(II)} - A$. Observe that by Markov inequality, we have

$$\begin{aligned}
&\mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \geq R, X_u(t) \geq m_t^{(II)} - A) \\
\leq& \, \mathbf{P}\left(\exists t \geq 0, u \in \mathcal{N}_t^1 : X_u(t) \geq vs + K\right) + \mathbf{P}(Y_t(A, R, K) \geq 1) \\
\leq& \, e^{-\theta K} + \mathbf{E}(Y_t(A, R, K)).
\end{aligned}$$

To complete the proof, it is therefore enough to bound $\limsup_{t \to \infty} \mathbf{E}(Y_t(A, R, K))$. Using the branching property and Corollary 4.2.2, we have

$$\begin{aligned}
\mathbf{E}(Y_t(A, R, K)) &= \mathbf{E}\left(\sum_{u \in \mathcal{B}} \mathbb{1}_{\{T(u) \in [R, t]\}} \mathbb{1}_{\{X_u(T(u)) \leq vT(u) + K\}} F\left(t - T(u), X_u(T(u))\right)\right) \\
&= \alpha \int_R^t e^{\beta s} \mathbf{E}\left(F\left(t - s, \sigma B_s\right) \mathbb{1}_{\{\sigma B_s \leq vs + K\}}\right) \mathrm{d}s, \qquad (3.5.1)
\end{aligned}$$

where we have set $F(r, x) = \mathbf{P}^{(2)}\left(x + M_r \geq m_t^{(II)} - A\right)$.

By (3.3.15), there exists $C > 0$ such that for all $x \in \mathbb{R}$ and $t \geq 0$, we have

$$\mathbf{P}^{(2)}\left(M_t \geq m_t^{(II)} + x\right) \leq C(1 + x_+)e^{-\sqrt{2}x},$$

so that for all $s \leq t$,

$$\begin{aligned}
F(t - s, x) &= \mathbf{P}^{(2)}\left(M_{t-s} \geq m_{t-s}^{(II)} + \sqrt{2}s + \tfrac{3}{2\sqrt{2}} \log \tfrac{t-s+1}{t+1} - A - x\right) \\
&\leq C\left(\frac{t+1}{t-s+1}\right)^{\frac{3}{2}} \left(1 + \sqrt{2}s + (-x)_+\right) e^{-\sqrt{2}(\sqrt{2}s - x - A)}. \qquad (3.5.2)
\end{aligned}$$

We bound $\mathbf{E}(Y_t(A, R, K))$ in two different ways, depending on the sign of $\sigma^2 - 1$.

First, if $\sigma^2 \leq 1$, we observe that the condition $X_u(s) \leq vs + K$ does not play a major role in the asymptotic behaviour of $\mathbf{E}(Y_t(A, R, K))$. As a result, (4.3.9) and (4.3.2) yield

$$\mathbf{E}(Y_t(A, R, K)) \leq Ce^A \int_R^t \left(\frac{t+1}{t-s+1}\right)^{3/2} e^{s(\beta-2)} \mathbf{E}\left(\left(1 + \sqrt{2}s + \sigma(-B_s)_+\right) e^{\sqrt{2}\sigma B_s}\right) \mathrm{d}s,$$

and as $\mathbf{E}\left(\left(1 + \sqrt{2}s + \sigma(-B_s)_+\right) e^{\sqrt{2}\sigma B_s}\right) \leq C(1 + s)e^{\sigma^2 s}$, we have

$$\mathbf{E}(Y_t(A, R, K)) \leq Ce^A \int_R^t \left(\frac{t+1}{t-s+1}\right)^{\frac{3}{2}} (s+1) \exp\left(s\left(\beta + \sigma^2 - 2\right)\right) \mathrm{d}s.$$

Hence, as $(\beta,\sigma^2)\in\mathcal{C}_{II}$ and $\sigma^2\leq 1$, we have $\beta+\sigma^2-2<0$. Therefore, by dominated convergence theorem,

$$\limsup_{t\to\infty}\mathbf{E}(Y_t(A,R,K))\leq Ce^A\int_R^\infty (s+1)\exp\left(s\left(\beta+\sigma^2-2\right)\right)\mathrm{d}s,$$

which goes to 0 as $R\to\infty$, completing the proof in that case.

We now assume that $\sigma^2>1$. In that case, the condition $X_u(s)\leq\sqrt{2\beta\sigma^2}s+K$ is needed to keep our upper bound small enough, as events of the form $\{X_u(s)\geq vs\}$ have small probability but $Y_t(A,R,K)$ is large on that event. Using the Girsanov transform, (4.3.9) yields

$$\begin{aligned}&\mathbf{E}(Y_t(A,R,K))\\ \leq&\alpha\int_R^t\mathbf{E}\left(e^{-\theta\sigma B_s}F(t-s,\sigma B_s+vs)\mathbb{1}_{\{\sigma B_s\leq K\}}\right)\mathrm{d}s\\ \leq&C\alpha e^{\sqrt{2}A}\int_R^t e^{-\sqrt{2}(\sqrt{2}-v)s}\left(\frac{t+1}{t-s+1}\right)^{\frac{3}{2}}\\ &\qquad\mathbf{E}\left(e^{(\sqrt{2}-\theta)\sigma B_s}\left(1+(v+\sqrt{2})s+(-B_s)_+\right)\mathbb{1}_{\{B_s\leq K\}}\right)\mathrm{d}s,\end{aligned}$$

using (4.3.2). As $(\beta,\sigma^2)\in\mathcal{C}_{II}$ and $\sigma^2>1$, we have $\beta\sigma^2<1$. This yields in particular $\beta<\sigma^2$ hence $\sqrt{2}-\theta>0$. Integrating with respect to the Brownian density, there exists $C>0$ such that

$$\mathbf{E}\left(e^{(\sqrt{2}-\theta)\sigma B_s}\left(1+\sqrt{2}(\sqrt{\beta\sigma^2}+1)s+(-B_s)_+\right)\mathbb{1}_{\{B_s\leq K\}}\right)\leq C(1+s)^{\frac{1}{2}}e^{(\sqrt{2}-\theta)\sigma K},$$

yielding

$$\mathbf{E}(Y_t(A,R,K))\leq C\alpha e^{\sqrt{2}A+(\sqrt{2}-\theta)\sigma K}\int_R^t e^{-2(1-\sqrt{\beta\sigma^2})s}\left(\frac{t+1}{t-s+1}\right)^{\frac{3}{2}}(1+s)^{\frac{1}{2}}\mathrm{d}s.$$

Then by dominated convergence, as $1-\sqrt{\beta\sigma^2}>0$, we deduce that

$$\limsup_{t\to\infty}\mathbf{E}(Y_t(A,R,K))\leq C\alpha e^{\sqrt{2}A+(\sqrt{2}-\theta)\sigma K}\int_R^\infty e^{-2(1-\sqrt{\beta\sigma^2})s}(1+s)^{\frac{1}{2}}\mathrm{d}s,$$

which decreases to 0 as $R\to\infty$, completing the proof. $\square$

We now use the known asymptotic behaviour of the extremal process of the branching Brownian motion, recalled in Section 3.3, to compute the asymptotic behaviour of the extremal process of particles satisfying $T(u)\leq R$, defined as

$$\widehat{\mathcal{E}}_t^R:=\sum_{u\in\mathcal{N}_t^2}\mathbb{1}_{\{T(u)\leq R\}}\delta_{X_u(t)-m_t^{(II)}}.$$

For any $u\in\mathcal{B}$, and $t\geq 0$, we set

$$Z_t^{(u)}:=\sum_{\substack{u'\in\mathcal{N}_t^2\\ u'\succcurlyeq u}}(\sqrt{2}t-X_{u'}(t))e^{\sqrt{2}(X_{u'}(t)-\sqrt{2}t)},$$

where we recall that $u'\succcurlyeq u$ denotes that u' is a descendant of u. Note that by (3.3.9) and the branching property, $Z_t^{(u)}$ converges a.s. to the variable $Z_\infty^{(u)}:=\liminf_{t\to\infty}Z_t^{(u)}$. Moreover, $e^{-\sqrt{2}(X_u(T(u))-\sqrt{2}T(u))}Z_\infty^{(u)}\overset{(d)}{=}Z_\infty$, where Z_∞ is the limit of the derivative martingale of a standard branching Brownian motion.

Lemma 3.5.2. *For all $\varphi \in \mathcal{T}$, we have $\lim_{t\to\infty} \left\langle \widehat{\mathcal{E}}_t^R, \varphi \right\rangle = \left\langle \widehat{\mathcal{E}}_\infty^R, \varphi \right\rangle$ in law, where $\widehat{\mathcal{E}}_\infty^R$ is a decorated Poisson point process with intensity $c_\star \overline{Z}_R \sqrt{2} e^{-\sqrt{2}x} \mathrm{d}x$ and decoration law $\mathfrak{D}$, with $\overline{Z}_R := \sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\leq R\}} Z_\infty^{(u)}$.*

Proof. Let $\varphi \in \mathcal{T}$ be a test function. Observe that using the branching property of the branching Brownian motion, we have

$$\mathbf{E}\left(\exp\left(-\left\langle \widehat{\mathcal{E}}_t^R, \varphi \right\rangle\right)\right) = \mathbf{E}\left(\prod_{u\in\mathcal{B}:T(u)\leq R} F_t(T(u), X_u(T(u)))\right),$$

where $F_t(s,x) = \mathbf{E}^{(2)}\left(\exp\left(-\sum_{u\in\mathcal{N}_t} \varphi\left(x + X_u(t-s) - m_t^{(II)}\right)\right)\right)$ for $0 \leq s \leq t$ and $x \in \mathbb{R}$. Using again that $m_t^{(II)} = m_{t-s}^{(II)} + \sqrt{2}s + o(1)$ as $t \to \infty$ and applying Lemma 3.3.4, we have for all $s \geq 0$

$$\lim_{t\to\infty} F_t(s,x) = \mathbf{E}^{(2)}\left(\exp\left(-c_\star Z_\infty e^{\sqrt{2}x - 2s} \int (1 - e^{-\Psi[\varphi](z)})\sqrt{2}e^{-\sqrt{2}z}\mathrm{d}z\right)\right),$$

where Z_∞ is the limit of the derivative martingale in a standard branching Brownian motion. Therefore, by dominated convergence theorem,

$$\lim_{t\to\infty} \mathbf{E}\left(\exp\left(-\left\langle \widehat{\mathcal{E}}_t^R, \varphi \right\rangle\right)\right) = \mathbf{E}\left(\exp\left(-c_\star \overline{Z}_R \int (1 - e^{-\Psi[\varphi](z)})\sqrt{2}e^{-\sqrt{2}z}\mathrm{d}z\right)\right),$$

with $\overline{Z}_R = \sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\leq R\}} Z_\infty^{(u)}$, completing the proof. $\square$

We then observe that $\widehat{\mathcal{E}}_\infty^R$ converges in law as $R \to \infty$ to $\mathcal{E}_\infty^{(II)}$ the point measure defined in Theorem 3.1.2.

Lemma 3.5.3. *For all $\varphi \in \mathcal{T}$, we have $\lim_{R\to\infty} \left\langle \widehat{\mathcal{E}}_\infty^R, \varphi \right\rangle = \left\langle \mathcal{E}_\infty^{(II)}, \varphi \right\rangle$ in law, where we define $\overline{Z}_\infty := \sum_{u\in\mathcal{B}} Z_\infty^{(u)}$.*

Proof. Recall that $Z_\infty \geq 0$ a.s. therefore $(\overline{Z}_R, R \geq 0)$ is increasing and $\overline{Z}_\infty = \lim_{R\to\infty} \overline{Z}_R$ exists a.s. Given that for all function $\varphi \in \mathcal{T}$,

$$\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_\infty^R, \varphi \right\rangle}\right) = \mathbf{E}\left(\exp\left(-c_\star \overline{Z}_R \int (1 - e^{-\Psi[\varphi](z)})\sqrt{2}e^{-\sqrt{2}z}\mathrm{d}z\right)\right),$$

to prove that $\mathcal{E}_\infty^R$ converges in law, it is enough to show that $\overline{Z}_\infty < \infty$ a.s.

We recall that $v = \sqrt{2\beta\sigma^2}$ the speed of the branching Brownian motion of particles of type 1. As

$$\lim_{t\to\infty} \max_{u\in\mathcal{N}_t^1} X_u(t) - vt = -\infty \quad \text{a.s.}$$

(which is a consequence of the fact that the additive martingale at the critical parameter converges to 0 a.s.), there is almost surely finitely many $u \in \mathcal{B}$ with $X_u(T(u)) \geq vT(u)$. To prove the finiteness of $\overline{Z}_\infty$, we then use the following variation on Kolmogorov's three series theorem: If we have

$$\sum_{u\in\mathcal{B}} \mathbb{1}_{\{X_u(T(u))\leq vT(u)\}} \mathbf{E}\left(Z_\infty^{(u)} \wedge 1 \middle| \mathcal{F}^1 \vee \sigma(\mathcal{B})\right) < \infty \quad \text{a.s.,} \tag{3.5.3}$$

then $\widetilde{Z}_\infty := \sum_{u\in\mathcal{B}} \mathbb{1}_{\{X_u(T(u))\leq vT(u)\}} Z_\infty^{(u)} < \infty$ a.s, where $\mathcal{F}^1 = \sigma(X_u(t), u \in \mathcal{N}_t, t \geq 0)$. Using that $\overline{Z}_\infty$ is obtained by adding a finite number of finite random variables to $\widetilde{Z}_\infty$, it implies that $\overline{Z}_\infty < \infty$ a.s

Indeed, if we assume (3.5.3), using the Markov inequality and the Borel-Catelli lemma, we deduce that almost surely there are finitely many $u \in \mathcal{B}$ whose contribution to $\widetilde{Z}_\infty$ is larger than 1. Additionally, (3.5.3) also implies that the sum of all the other contributions to $\widetilde{Z}_\infty$ has finite mean. Hence, we have $\widetilde{Z}_\infty < \infty$ a.s.

We now prove (3.5.3), using that $\mathbf{E}((Z_\infty e^x) \wedge 1) \leq C(1 + (-x)_+)e^x$ for all $x \in \mathbb{R}$, by (3.3.10) and (3.3.11). Hence, using that $Z_\infty^{(u)} \overset{(d)}{=} e^{\sqrt{2}(X_u(T(u))-\sqrt{2}T(u))} Z_\infty$ it is enough to show that

$$\sum_{u\in\mathcal{B}} \mathbb{1}_{\{X_u(T(u))\leq vT(u)\}} \left(1 + \left(2T(u) - \sqrt{2}X_u(T(u))\right)_+\right) e^{\sqrt{2}X_u(T(u))-2T(u)} < \infty \quad \text{a.s.} \tag{3.5.4}$$

This quantity being a series of positive random variables, we prove that this series has finite mean to conclude. By Corollary 4.2.2, we have

$$\mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\{X_u(T(u))\leq vT(u)\}} \left(1 + \left(2T(u) - \sqrt{2}X_u(T(u))\right)_+\right) e^{\sqrt{2}X_u(T(u))-2T(u)}\right)$$
$$= \alpha \int_0^\infty e^{\beta s} \mathbf{E}\left(\mathbb{1}_{\{\sigma B_s \leq vs\}} \left(1 + \left(2s - \sqrt{2}\sigma B_s\right)_+\right) e^{\sqrt{2}\sigma B_s - 2s}\right) \mathrm{d}s.$$

Similarly to the proof of Lemma 3.5.1, we bound the above quantity in two different ways depending on whether $\sigma^2 > 1$ or $\sigma^2 \leq 1$.

If $\sigma^2 \leq 1$, we have

$$e^{\beta s}\mathbf{E}\left(\left(1 + \left(\sqrt{2}\sigma B_s - 2s\right)_+\right) e^{\sqrt{2}\sigma B_s - 2s}\right) \leq C(1+s)\exp\left(s\left(\sigma^2 + \beta - 2\right)\right),$$

which decays exponentially fast as $(\beta, \sigma^2) \in \mathcal{C}_{II}$ and $\sigma^2 \leq 1$. Therefore, we have

$$\mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\{X_u(T(u))\leq vT(u)\}} \left(1 + \left(2T(u) - \sqrt{2}X_u(T(u))\right)_+\right) e^{\sqrt{2}X_u(T(u))-2T(u)}\right) < \infty,$$

proving (3.5.4), hence (3.5.3), therefore that $\overline{Z}_\infty < \infty$ a.s. in that case.

If $\sigma^2 > 1$, we have

$$e^{\beta s}\mathbf{E}\left(\mathbb{1}_{\{\sigma B_s \leq vs\}} \left(1 + \left(2s - \sqrt{2}\sigma B_s\right)_+\right) e^{\sqrt{2}\sigma B_s - 2s}\right)$$
$$= \mathbf{E}\left(\mathbb{1}_{\{B_s \leq 0\}} \left(1 + \left(2(1 - \sqrt{\sigma^2\beta})s - \sqrt{2}\sigma B_s\right)_+\right) e^{\sqrt{2}\sigma B_s}\right) e^{2\left(\sqrt{\sigma^2\beta}-1\right)s}$$
$$\leq C(1+s)\exp\left(s\left(\sqrt{\sigma^2\beta} - 1\right)\right).$$

As $(\beta, \sigma^2) \in \mathcal{C}_{II}$ and $\sigma^2 < 1$ we have once again

$$\mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\{X_u(T(u))\leq vT(u)\}} \left(1 + \left(2T(u) - \sqrt{2}X_u(T(u))\right)_+\right) e^{\sqrt{2}X_u(T(u))-2T(u)}\right) < \infty,$$

which proves that $\overline{Z}_\infty < \infty$ a.s. in that case as well. $\square$

Using the above results, we finally obtain the asymptotic behaviour of the extremal process in case $\mathcal{C}_{II}$.

Proof of Theorem 3.1.2. Recall that $\widehat{\mathcal{E}}_t = \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t}$. Using Proposition 3.3.3, we only need to prove that for all $\varphi \in \mathcal{T}$, we have

$$\lim_{t\to\infty} \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) = \mathbf{E}\left(e^{-\left\langle \mathcal{E}_\infty^{(II)},\varphi\right\rangle}\right).$$

Let $\varphi \in \mathcal{T}$, and set $A \in \mathbb{R}$ such that $\varphi(z) = 0$ for all $z \leq A$. By Lemma 3.5.1, for all $\varepsilon > 0$, there exists $R \geq 0$ such that $\mathbf{P}\left(\widehat{\mathcal{E}}_t^R(\varphi) \neq \widehat{\mathcal{E}}_t(\varphi)\right) \leq \varepsilon$. Then, using that φ is non-negative, so that $\left\langle \widehat{\mathcal{E}}_t^R, \varphi\right\rangle \leq \left\langle \widehat{\mathcal{E}}_t, \varphi\right\rangle$ and $e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}$ is bounded by 1, we have $\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) \leq \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) \leq \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) + \varepsilon$. Applying Lemma 3.5.2 and Lemma 3.5.3 to let t, then R, grow to ∞, we obtain

$$\mathbf{E}\left(e^{-\left\langle \mathcal{E}_\infty^{(II)},\varphi\right\rangle}\right) \leq \liminf_{t\to\infty} \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) \leq \limsup_{t\to\infty} \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) \leq \mathbf{E}\left(e^{-\left\langle \mathcal{E}_\infty^{(II)},\varphi\right\rangle}\right) + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain that $\lim_{t\to\infty} \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) = \mathbf{E}\left(e^{-\left\langle \mathcal{E}_\infty^{(II)},\varphi\right\rangle}\right)$ for all $\varphi \in \mathcal{T}$, which completes the proof of Theorem 3.1.2 by Remark 3.1.5. $\square$

We end this section by conjecturing a possible direct formula for the computation of $\overline{Z}_\infty$ as the limit of a sub-martingale of the multitype BBM.

Conjecture 3.5.4. *We have* $\lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^2} (\sqrt{2}t - X_u(t))e^{\sqrt{2}X_u(t)-2t} = \overline{Z}_\infty$ *a.s.*

3.6 Proof of Theorem 3.1.1

In this section, we assume that $(\beta, \sigma^2) \in \mathcal{C}_I$, that is either $\sigma^2 \leq 1$ and $\sigma^2 > \frac{1}{\beta}$, or $\sigma^2 > 1$ and $\sigma^2 > \frac{\beta}{2\beta-1}$. In that situation, we show that the extremal process of particles of type 2 is mainly driven by the asymptotic of particles of type 1, and that any particle of type 2 significantly contributing to the extremal process at time t satisfies $t - T(u) = O(1)$, meaning that they have a close ancestor of type 1.

For the rest of the section, we denote by $v = \sqrt{2\beta\sigma^2}$ and $\theta = \sqrt{2\beta/\sigma^2}$ the speed and critical parameter of the BBM of particles of type 1. Recall that $m_t^{(I)} = vt - \frac{3}{2\theta}\log t$, and we set

$$\widehat{\mathcal{E}}_t := \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t^{(I)}},$$

the extremal process of particles of type 2, centred around $m_t^{(I)}$.

To prove Theorem 3.1.1, we first show that for all $\varphi \in \mathcal{T}$, $\left\langle \widehat{\mathcal{E}}_t, \varphi\right\rangle$ converges, as $t \to \infty$ to a proper random variable. By [Kal21, Lemma 5.1], this is enough to conclude that $\widehat{\mathcal{E}}_t$ converges vaguely in law to a limiting point measure $\overline{\mathcal{E}}$. We then use that with high probability, no particle of type 2 born before time R contributes to the extremal process of the multitype BBM. Then, by the branching property, it shows that $\overline{\mathcal{E}}$ satisfies a stability under superposition probability which, by [Mai13, Corollary 3.2], can be identified as a decorated Poisson point process with intensity proportional to $Z_\infty^{(I)} e^{-\theta x} \mathrm{d}x$.

To prove the results of this section, we make use of the following extension of (3.3.13). For all $t \geq 0$, we write $a_t = \frac{3}{2\theta}\log(t+1)$. There exists $C > 0$ such that for all $t \geq 0$ and $K > 0$, we have

$$\mathbf{P}\left(\exists s \leq t, u \in \mathcal{N}_s^1 : X_u(s) \geq vs - a_t + a_{t-s} + K\right) \leq C(K+1)e^{-\theta K}. \tag{3.6.1}$$

This result was proved in [Mal15a] in the context of branching random walks, and has been adapted to continuous-time settings in [Mal15b, Lemma 3.1].

We first show the tightness of the law of the number of particles of type 2 born to the right of $m_t^{(I)} - A$.

Lemma 3.6.1. *We assume that $(\beta, \sigma^2) \in \mathcal{C}_I$. For all $A, K > 0$, there exists $C_{A,K} > 0$ and $\delta > 0$ such that for all $R \geq 0$, we have*

$$\limsup_{t\to\infty} \mathbf{E}\left(\sum_{u \in \mathcal{N}_t^2} \mathbb{1}_{\left\{X_u(t) \geq m_t^{(I)} - A\right\}} \mathbb{1}_{\{T(u) \leq t-R\}} \mathbb{1}_{\{X_u(s) \leq vs - a_t + a_{t-s} + K, s \leq T(u)\}}\right) \leq C_{A,K} e^{-\delta R}.$$

Proof. Let $A, K, R > 0$, we set

$$Y_t(A,K,R) = \sum_{u \in \mathcal{N}_t^2} \mathbb{1}_{\left\{X_u(t) \geq m_t^{(I)} - A\right\}} \mathbb{1}_{\{T(u) \leq t-R\}} \mathbb{1}_{\{X_u(s) \leq vs - a_t + a_{t-s} + K, s \leq T(u)\}}.$$

We use Proposition 3.4.1 to compute the mean of $Y_t(A,K,R)$ as

$$\begin{aligned} &\mathbf{E}\left(Y_t(A,K,R)\right) \\ &\leq \int_0^{t-R} e^{\beta s + t - s} \mathbf{P}\left(\sigma B_s + B_t - B_s \geq m_t^{(I)} - A, \sigma B_r \leq vr - a_t + a_{t-r} + K, r \leq s\right) \mathrm{d}s \\ &\leq \int_0^{t-R} \mathbf{E}\left(e^{-\theta\sigma B_s} F(t-s, \sigma B_s - a_t) \mathbb{1}_{\{\sigma B_r \leq a_{t-r} - a_t + K, r \leq s\}}\right) \mathrm{d}s, \end{aligned}$$

where $F(r,x) = e^r \mathbf{P}(B_r \geq vr - x)$, using the Markov property at time s and the Girsanov transform. We have $F(r,x) \leq e^{\lambda x} e^{r\left(1 - \lambda v + \frac{\lambda^2}{2}\right)}$ for all $\lambda > 0$, by the exponential Markov inequality. This implies

$$\mathbf{E}(Y_t(A,K,R)) \leq \int_0^{t-R} e^{(t-s)(1-\lambda v + \frac{\lambda^2}{2})} (t+1)^{\frac{3\lambda}{2\theta}} \mathbf{E}\left(e^{\sigma(\lambda-\theta)B_s} \mathbb{1}_{\{\sigma B_r \leq a_{t-r} - a_t + K, r \leq s\}}\right) \mathrm{d}s. \tag{3.6.2}$$

We now bound this quantity in two different ways depending on the sign of $\sigma^2 - 1$.

First, if $\sigma^2 > 1$, then $v > \theta$, in which case using (3.6.2) with $\lambda = v$, we obtain

$$\mathbf{E}(Y_t(A,K,R)) \leq \int_0^{t-R} e^{(t-s)(1-v^2/2)} (t+1)^{\frac{3v}{2\theta}} \mathbf{E}\left(e^{\sigma(v-\theta)B_s} \mathbb{1}_{\{\sigma B_r \leq a_{t-r} - a_t + K, r \leq s\}}\right) \mathrm{d}s.$$

We now use that for all $\lambda > 0$, there exists $C > 0$ such that for all $0 \leq s \leq t$, we have

$$\mathbf{E}\left(e^{\lambda\sigma B_s}\mathbb{1}_{\{\sigma B_r \leq a_{t-r}-a_t+K, r\leq s\}}\right) \leq Ce^{\lambda K}\left(\tfrac{t-s+1}{t+1}\right)^{\frac{3\lambda}{2\theta}}(s+1)^{-\frac{3}{2}}. \quad (3.6.3)$$

This bound can be obtained by classical Gaussian estimates, rewriting

$$\begin{aligned}&\mathbf{E}\left(e^{\lambda\sigma B_s}\mathbb{1}_{\{\sigma B_r \leq a_{t-r}-a_t+K, r\leq s\}}\right)\\ \leq & Ce^{\lambda K}\left(\tfrac{t-s+1}{t+1}\right)^{\frac{3\lambda}{2\theta}}\sum_{k\geq 0} e^{-\lambda k}\mathbf{P}\left(\begin{array}{l}\sigma B_s - a_s + a_t + K \in [-k-1,-k]\\ \sigma B_r \leq a_{t-r} - a_t + K, r \leq s\end{array}\right),\end{aligned}$$

and showing that the associated probability can be bounded uniformly in k, t and $s \leq t$ by $Ck(s+1)^{-\frac{3}{2}}$, with computations similar to the ones used in [Mal15a, Lemma 3.8] for random walks. Therefore, (3.6.3) implies that

$$\mathbf{E}(Y_t(A,K,R)) \leq C_{A,K}\int_0^{t-R} e^{(t-s)(1-v^2/2)}\frac{(t+1)^{\frac{3}{2}}(t-s+1)^{\frac{3v}{2\theta}}}{(s+1)^{\frac{3}{2}}}\mathrm{d}s.$$

As $(\beta,\sigma^2) \in \mathcal{C}_I$, we have $v = \sqrt{2\beta\sigma^2} > \sqrt{2}$, so $1 - \frac{v^2}{2} < 0$. As a result

$$\int_0^{\frac{t}{2}} e^{(t-s)(1-v^2/2)}\frac{(t+1)^{\frac{3}{2}}(t-s+1)^{\frac{3v}{2\theta}}}{(s+1)^{\frac{3}{2}}}\mathrm{d}s \leq Ce^{(1-v^2/2)t}(t+1)^{\frac{5\theta+3v}{2\theta}},$$

which converges to 0 as $t \to \infty$, and

$$\int_{\frac{t}{2}}^{t-R} e^{(t-s)(1-v^2/2)}\frac{(t+1)^{\frac{3}{2}}(t-s+1)^{\frac{3v}{2\theta}}}{(s+1)^{\frac{3}{2}}}\mathrm{d}s \leq C\int_R^\infty e^{(1-v^2/2)s}(s+1)^{\frac{3v}{2\theta}}\mathrm{d}s \leq Ce^{(1-v^2/2)R/2}.$$

This completes the proof of the lemma in the case $\sigma^2 > 1$.

We now assume that $\sigma^2 < 1$. We have that

$$1 - \theta v + \frac{\theta^2}{2} = 1 - 2\beta + \frac{\beta}{\sigma^2} = \beta\left(\tfrac{1}{\sigma^2} - 2\right) + 1.$$

Therefore, as long as $\sigma^2 > \frac{\beta}{2\beta-1}$, which is the case as $\sigma^2 < 1$ and $(\beta,\sigma^2) \in \mathcal{C}_I$, we have $1-\theta v+\frac{\theta^2}{2} < 0$. Therefore, for all $\delta > 0$ small enough such that

$$1 - (\theta+\delta)v + \tfrac{(\theta+\delta)^2}{2} < 0,$$

using (3.6.2) with $\lambda = \theta + \delta$, we have

$$\begin{aligned}&\mathbf{E}(Y_t(A,K,R))\\ &\qquad \leq \int_0^{t-R} e^{(t-s)(1-(\theta+\delta)v+(\theta+\delta)^2/2)}(t+1)^{\frac{3(\theta+\delta)}{2\theta}}\mathbf{E}\left(e^{\sigma\delta B_s}\mathbb{1}_{\{\sigma B_r \leq a_{t-r}-a_t+K, r\leq s\}}\right)\mathrm{d}s.\end{aligned}$$

So with the same computations as above, we obtain once again that

$$\limsup_{t\to\infty}\mathbf{E}(Y_t(A,K,R)) \leq C_{A,K}e^{-\delta R},$$

which completes the proof. □

Using the above computation, we deduce that with high probability, only particles of type 2 having an ancestor of type 1 at time $t - O(1)$ contribute substantially to the extremal process at time t.

Lemma 3.6.2. *Assuming that $(\beta, \sigma^2) \in \mathcal{C}_I$, for all $A > 0$, we have*

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \leq t - R, X_u(t) \geq m_t^{(I)} - A) = 0.$$

Proof. We observe that for all $K > 0$, we have

$$\mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \leq t - R, X_u(t) \geq m_t^{(I)} - A) \leq \mathbf{P}(\exists s \leq t, u \in \mathcal{N}_s^1 : X_u(s) \geq vs + a_t - a_{t-s})$$
$$+ \mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \leq t - R, X_u(t) \geq m_t^{(I)} - A, X_u(s) \leq vs - a_t + a_{t-s}, s \leq T(u)).$$

Then, using (3.6.1), the Markov inequality and Lemma 3.6.1, we obtain

$$\limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \leq t - R, X_u(t) \geq m_t^{(I)} - A) \leq C(K+1)e^{-\theta K} + C_{A,K}e^{-R}.$$

Letting $R \to \infty$ then $K \to \infty$, the proof is now complete. □

For all $R > 0$, we set

$$\widehat{\mathcal{E}}_t^R := \sum_{u \in \mathcal{N}_t^2} \mathbb{1}_{\{T(u) \geq t-R\}} \delta_{X_u(t) - m_t^{(I)}}.$$

We now show that $\widehat{\mathcal{E}}_t^R$ converges in law as $t \to \infty$.

Lemma 3.6.3. *Assume that $(\beta, \sigma^2) \in \mathcal{C}_I$, there exists $c_R > 0$ and a point measure distribution $\mathcal{D}^R$ such that for all $\varphi \in \mathcal{T}$, we have*

$$\lim_{t\to\infty} \left\langle \widehat{\mathcal{E}}_t^R, \varphi \right\rangle = \left\langle \widehat{\mathcal{E}}_\infty^R, \varphi \right\rangle \quad \text{in law,}$$

where $\widehat{\mathcal{E}}_\infty^R$ is a DPPP($c_R Z_\infty e^{-\theta x} \mathrm{d}x$,$\mathfrak{D}^R$).

Proof. We can rewrite

$$\widehat{\mathcal{E}}_t^R = \sum_{u \in \mathcal{N}_{t-R}^1} \sum_{\substack{u' \in \mathcal{N}_t^2 \\ u' \succcurlyeq u}} \delta_{X_{u'}(t) - X_u(t-R) + X_u(t-R) - m_t^{(I)}} = \sum_{u \in \mathcal{N}_{t-R}^1} \tau_{X_u(t-R) - m_t^{(I)}} \widehat{\mathcal{E}}_R^{(u)},$$

where τ_z is the operator of translation by z of point measures, and $\widehat{\mathcal{E}}_R^{(u)}$ is the point process of descendants of type 2 of individual $u \in \mathcal{N}_{t-R}$ at time t, centred around the position of u at time $t - R$. Note that conditionally on $\mathcal{F}_{t-R}^1$, $(\widehat{\mathcal{E}}_R^{(u)}, u \in \mathcal{N}_{t-R})$ are i.i.d. point measures with same law as $\overline{\mathcal{E}}_R := \sum_{u \in \mathcal{N}_R^2} \delta_{X_u(R)}$.

Let $\varphi \in \mathcal{T}$, we set $L \in \mathbb{R}$ such that $\varphi(x) = 0$ for all $x \leq L$. By the branching property, we have

$$\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R, \varphi \right\rangle}\right) = \mathbf{E}\left(\prod_{u \in \mathcal{N}_{t-R}^1} F_R(X_u(t-R) - m_t^{(I)})\right)$$
$$= \mathbf{E}\left(e^{-\sum_{u \in \mathcal{N}_{t-R}^1} - \log F_R(X_u(t-R) - m_t^{(I)})}\right),$$

where $F_R(x) = \mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{N}_R^2}\varphi(x+X_u(R))\right)\right)$. Observe that by Jensen transform, we have

$$-\log F_R(x) \leq \mathbf{E}\left(\sum_{u\in\mathcal{N}_R^2}\varphi(x+X_u(R))\right) \leq \int_0^R e^{\beta s+(R-s)}\mathbf{E}(\varphi(x+\sigma B_s + B_t - B_s))\mathrm{d}s$$
$$\leq ||\varphi||_\infty R e^{(\beta+1)R}\mathbf{P}\left(B_1 \geq \frac{-x}{\sqrt{R(\sigma^2+1)}}\right).$$

Therefore, by (3.3.17), we have $-\log F_R(x) \leq C_R e^{(\theta+\delta)x} \wedge 1$ for all $x \in \mathbb{R}$.

By Lemma 3.3.4, recall that $\sum_{u\in\mathcal{N}_{t-R}^1}\delta_{X_u(t-R)-m_t^{(I)}}$ converges vaguely in law to a DPPP $\mathcal{E}^1$ with intensity $c_\star\theta Z_\infty e^{-\theta(z+vR)}\mathrm{d}z$ and decoration law $\mathfrak{D}_{\beta,\sigma^2}$ the law of the decoration point measure of the BBM with branching rate β and variance σ^2. Additionally, it was proved by Madaule [Mad17] in the context of branching random walks, and extended in [CHL19] to BBM settings, that $\left\langle\mathcal{E}^1, e_{\theta+\delta}\right\rangle < \infty$ a.s. for all $\delta > 0$, where $e_{\theta+\delta}(x) = e^{(\theta+\delta)x}$. As a result, using the monotone convergence theorem, we obtain

$$\lim_{t\to\infty}\mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) = \mathbf{E}\left(e^{-\left\langle\mathcal{E}^1,-\log F_R\right\rangle}\right).$$

This proves that $\widehat{\mathcal{E}}_t^R$ converges in law, as $t\to\infty$, to a point process that can be obtained from $\mathcal{E}^1$ by replacing each atom of $\mathcal{E}^1$ by an independent copy of the point measure $\overline{\mathcal{E}}_R$. □

We now complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Let $\varphi \in \mathcal{T}$. We fix $A > 0$ such that $\varphi(x) = 0$ for all $x \leq -A$. We observe that

$$0 \leq \mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) - \mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) \leq \mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \leq t-R, X_u(t) \geq m_t^{(I)} - A),$$

which goes to 0 as t then $R\to\infty$, by Lemma 3.6.2. Additionally, by Lemma 3.6.3, we have

$$\lim_{t\to\infty}\mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) = \mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_\infty^R,\varphi\right\rangle}\right).$$

Moreover, using that $R \mapsto \mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right)$ is decreasing, we deduce that

$$\lim_{t\to\infty}\mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t,\varphi\right\rangle}\right) = \lim_{R\to\infty}\mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_\infty^R,\varphi\right\rangle}\right). \tag{3.6.4}$$

Additionally, as $R \mapsto \widehat{\mathcal{E}}_t^R$ is increasing in the space of point measures, we observe that we can construct the family of point measures $(\widehat{\mathcal{E}}_\infty^R, R \geq 0)$ on the same probability space in such a way that almost surely, $\left\langle\widehat{\mathcal{E}}_\infty^R,\varphi\right\rangle$ is increasing for all φ. We denote by $\mu(\varphi)$ its limit.

By [Kal21, Lemma 5.1], to prove that $\widehat{\mathcal{E}}_t$ admits a limit in distribution for the topology of vague convergence, it is enough to show that for all non-negative continuous functions with compact support, $\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle$ admits a limit in law which is a proper random variable. By (3.6.4), using the monotonicity of $\widehat{\mathcal{E}}^R_\infty$, we immediately obtain that $\lim_{t\to\infty} \left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle = \mu(\varphi)$ in law. Therefore, to prove that $\widehat{\mathcal{E}}_t$ converges vaguely in distribution, it is enough to show that for all $\varphi \in \mathcal{T}$, $\mu(\varphi) < \infty$ a.s. which will be a consequence of the tightness of $\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle$.

Let $\varphi \in \mathcal{T}$, we write $L \in \mathbb{R}$ such that $\varphi(x) = 0$ for all $x < L$. For all $A > 0$ and $K > 0$, we have

$$\mathbf{P}\left(\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle \geq A\right) \leq \mathbf{P}(\exists s \leq t, \exists u \in \mathcal{N}_s : X_u(s) \geq vs - a_t + a_{t-s} + K) \\ + \frac{1}{A}\mathbf{E}\left(\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle \mathbb{1}_{\left\{\max_{u\in\mathcal{N}^1_s} X_u(s) \leq vs - a_t + a_{t-s} + K, s \leq t\right\}}\right).$$

The first quantity goes to 0 as $K \to \infty$ by (3.6.1). Therefore, for all $\varepsilon > 0$, we can fix K large enough so that it remains smaller than $\varepsilon/2$. Then, using Lemma 3.6.1 with $R = 0$, we have

$$\frac{1}{A}\mathbf{E}\left(\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle \mathbb{1}_{\left\{\max_{u\in\mathcal{N}^1_s} X_u(s) \leq vs - a_t + a_{t-s} + K, s \leq t\right\}}\right) \leq \frac{C_{L,K}}{A}.$$

Therefore, we can choose A large enough such that for all $t \geq 0$, $\mathbf{P}(\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle \geq A) \leq \varepsilon$, which completes the proof of the tightness of $\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle$.

We then conclude that $\widehat{\mathcal{E}}_t$ converges vaguely in law as $t \to \infty$ to a limiting point measure that we write $\overline{\mathcal{E}}$. This point measure also is the limit as $R \to \infty$ of $\widehat{\mathcal{E}}^R_\infty$, by (3.6.4). This allows us to show that $\left\langle \widehat{\mathcal{E}}_t, \varphi \right\rangle \to \langle \overline{\mathcal{E}}, \varphi \rangle$ in law for all $\varphi \in \mathcal{T}$, so we conclude by Proposition 3.3.3 that the position of the rightmost atom in $\widehat{\mathcal{E}}_t$ also converges to the position of the rightmost atom in $\overline{\mathcal{E}}$.

To complete the proof of Theorem 3.1.1, we have to describe the law of $\overline{\mathcal{E}}$. For all $s \geq 0$, using the branching property, we have

$$\widehat{\mathcal{E}}_t = \sum_{u\in\mathcal{N}_s} \tau_{X_u(s)-vs+a_{t-s}-a_t} \widehat{\mathcal{E}}^{(u)}_{t-s}$$

where conditionally on $\mathcal{F}_s$, $(\widehat{\mathcal{E}}^{(u)}_{t-s}, u \in \mathcal{N}_s)$ is a family of independent point measures with same law as $\widehat{\mathcal{E}}_{t-s}$, under law $\mathbf{P}^{(1)}$ or $\mathbf{P}^{(2)}$ depending on the type of u. As no particle of type 2 born at an early time will have a descendant contributing in the extremal process by Lemma 3.6.2, we obtain that, letting $t \to \infty$,

$$\overline{\mathcal{E}} \overset{(d)}{=} \sum_{u\in\mathcal{N}^1_s} \tau_{X_u(s)-vs} \overline{\mathcal{E}}^{(u)}, \tag{3.6.5}$$

where $\overline{\mathcal{E}}^{(u)}$ are i.i.d. copies of $\overline{\mathcal{E}}$, that are further independent of $\mathcal{F}_s$. This superposition property characterizes the law of $\overline{\mathcal{E}}$ as a decorated Poisson point process with intensity proportional to $e^{-\theta x}\mathrm{d}x$, shifted by the logarithm of the derivative martingale of the branching Brownian motion by [Mai13, Corollary 3.2], with similar computations as in [Mad17, Section 2.2]. A general study of such point measures satisfying the branching property (3.6.5) is carried out in [**?**]. $\square$

3.7 Asymptotic behaviour in the anomalous spreading case

We assume in this section that $(\sigma^2, \beta) \in \mathcal{C}_{III}$, i.e. that $\beta + \sigma^2 > 2$ and $\sigma^2 < \frac{\beta}{2\beta-1}$. In particular, it implies that $\beta > 1$ and $\sigma^2 < 1$. Under these conditions, we set

$$\theta := \sqrt{2\frac{\beta-1}{1-\sigma^2}}, \quad a := \sigma^2\theta, \quad b := \theta \quad \text{and} \quad p := \frac{\sigma^2+\beta-2}{2(\beta-1)(1-\sigma^2)},$$

which are the values of a, b and p solutions of (3.2.1), described in terms of the parameter θ which plays the role of a Lagrange multiplier in the optimization problem. Note that $a < \sqrt{2\beta\sigma^2}$, $b > \sqrt{2}$ and $p \in (0,1)$. Recall that in this situation, the maximal displacement is expected to satisfy

$$m_t^{(III)} = vt, \quad \text{where} \ \ v = ap + b(1-p) = \frac{\beta - \sigma^2}{\sqrt{2(\beta-1)(1-\sigma^2)}}.$$

As in the previous sections, we set

$$\widehat{\mathcal{E}}_t = \sum_{u \in \mathcal{N}_t^2} \delta_{X_u(t) - m_t^{(III)}}$$

the appropriately centred extremal process of particles of type 2.

As mentioned in Section 3.2.1, under the above assumption, we are in the anomalous behaviour regime. In this regime, we have $v > \max(\sqrt{2}, \sqrt{2\beta\sigma^2})$, in other words, this furthest particle travelled at a larger speed than the ones observed in the BBM of particles of type 1, or in a BBM of particles of type 2. Moreover, given the heuristic explanation for (3.2.1), we expect the furthest particle u of type 2 at time t to satisfy $T(u) \approx pt$ and $X_u(T(u)) \approx apt$.

The idea of the proof of Theorem 3.1.3 is to show that this heuristic holds, and that all particles participating to the extremal process of the multitype BBM are of type 2, and satisfy $T(u) \approx pt$ and $X_u(T(u)) \approx apt$. We then use the asymptotic behaviour of the growth rate of the number of particles of type 1 growing at speed a to complete the proof. We begin by proving that with high probability, there is no particle of type 2 far above level $m_t^{(III)}$ at time t.

Lemma 3.7.1. *Assuming that $(\sigma^2, \beta) \in \mathcal{C}_{III}$, we have*

$$\lim_{A\to\infty} \limsup_{t\to\infty} \mathbf{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} + A\right) = 0.$$

Proof. The proof of this result is based on a first moment method. For $A > 0$, we compute, using the many-to-one lemma, the mean of $X_t(A) = \sum_{u\in\mathcal{N}_t^2} \mathbb{1}_{\left\{X_u(t) \geq m_t^{(III)}+A\right\}}$. Using (3.3.17), there exists $C > 0$ such that for all $t \geq 1$, we have

$$\begin{aligned}
&\mathbf{E}(X_t(A)) \\
&= \int_0^t e^{\beta s + t - s} \mathbf{P}\left(\sigma B_s + (B_t - B_s) \geq m_t^{(III)} + A\right) \mathrm{d}s \\
&\leq \int_0^t e^{\beta s + (t-s)} \frac{C\sqrt{\sigma^2 s + t - s}}{(vt + A)} e^{-\frac{(vt+A)^2}{2(\sigma^2 s + t - s)}} \mathrm{d}s \leq Ct^{-1/2} \int_0^t e^{\beta s + (t-s)} e^{-\frac{(vt+A)^2}{2(\sigma^2 s + t - s)}} \mathrm{d}s.
\end{aligned}$$

Therefore, setting $\varphi : u \mapsto \beta u + 1 - u - \frac{v^2}{2(\sigma^2 u+1-u)}$, by change of variable we have, for all t large enough

$$\mathbf{E}(X_t(A)) \leq Ct^{1/2} \int_0^1 \exp\left(t\varphi(u)\right) e^{-A\frac{v}{(\sigma^2 u+1-u)}} \mathrm{d}u.$$

We observe that

$$\varphi'(u) = \beta - 1 - (1-\sigma^2)\frac{v^2}{2(\sigma^2 u + 1 - u)^2} \quad \text{and} \quad \varphi''(u) = -2(1-\sigma^2)^2 \frac{v^2}{2(\sigma^2 u + 1 - u)^3},$$

hence φ is concave, and maximal at point $u = p$, with a maximum equal to 0. By Taylor expansion, there exists $\delta > 0$ such that $\varphi(u) \leq -\delta(u-p)^2$ for all $u \in [0, 1]$. Therefore, we have

$$\mathbf{E}(X_t(A)) \leq Ce^{-Av} t^{1/2} \int_0^1 e^{-\delta(u-p)^2 t} \mathrm{d}u \leq Ce^{-Av}\sqrt{\pi/\delta}.$$

As a result, applying the Markov inequality, we have

$$\mathbf{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} + A\right) = \mathbf{P}(X_t(A) \geq 1) \leq \mathbf{E}(X_t(A)),$$

thus there exists $C > 0$ such that

$$\limsup_{t\to\infty} \mathbf{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} + A\right) \leq Ce^{-Av},$$

which converges to 0 as $A \to \infty$. □

Next, we show that every particle of type 2 that contributes to the extremal process of the BBM branched from a particle of type 1 at a time and position close to (pt, apt).

Lemma 3.7.2. *Assuming that $(\sigma^2, \beta) \in \mathcal{C}_{III}$, for all $A > 0$, we have*

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |T(u) - pt| \geq Rt^{1/2}) = 0, \tag{3.7.1}$$

and

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |X_u(T(u)) - apt| \geq Rt^{1/2}) = 0. \tag{3.7.2}$$

Proof. Let $A > 0$ and $\varepsilon > 0$. By Lemma 3.7.1, there exists $K > 0$ such that with probability $(1-\varepsilon)$ no particle of type 2 is above level $m_t^{(III)} + K$ at time t for all t large enough. For $R > 0$, we now compute the mean of

$$Y_t^{(1)}(A, K, R) = \sum_{u\in\mathcal{N}_t^2} \mathbb{1}_{\left\{X_u(t) - m_t^{(III)} \in [-A,K]\right\}} \mathbb{1}_{\left\{|T(u)-pt|\geq Rt^{1/2}\right\}}.$$

By Proposition 3.4.1, setting $I_t(R) = [0, t]\backslash[pt - Rt^{1/2}, pt + Rt^{1/2}]$ we have

$$\begin{aligned}\mathbf{E}\left(Y_t^{(1)}(A, K, R)\right) &= \int_{I_t(R)} e^{\beta s + (t-s)} \mathbf{P}\left(\sigma B_s + (B_t - B_s) - m_t^{(III)} \in [-A, K]\right) \mathrm{d}s \\ &\leq Ce^{Av} t^{1/2} \left(\int_0^{p-Rt^{-1/2}} e^{t\varphi(u)} \mathrm{d}u + \int_{p+Rt^{-1/2}}^1 e^{t\varphi(u)} \mathrm{d}u\right),\end{aligned}$$

using the same notation and computation techniques as in the proof of Lemma 3.7.1. Thus, using again that there exists $\delta > 0$ such that $\varphi(u) \leq -\delta(u-p)^2$ for some $\delta > 0$, by change of variable $z = t^{1/2}(u-p)$ we obtain that

$$\mathbf{E}\left(Y_t^{(1)}(A,K,R)\right) \leq Ce^{Av} \int_{\mathbb{R}\setminus[-R,R]} e^{-\delta z^2} \mathrm{d}z.$$

Therefore, by Markov inequality, we obtain that

$$\limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |T(u) - pt| \geq Rt^{1/2}) \leq \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} + K) + Ce^{Av} \int_{\mathbb{R}\setminus[-R,R]} e^{-\delta z^2} \mathrm{d}z.$$

As a result, with the choice previously made for the constant K, we obtain that

$$\limsup_{R\to\infty} \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |T(u) - pt| \geq Rt^{1/2}) \leq \varepsilon.$$

By letting $\varepsilon \to 0$, we complete the proof of (3.7.1).

We now turn to the proof of (3.7.2). By (3.7.1), we can assume, up to enlarging the value of K that

$$\limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |T(u) - pt| \geq Kt^{1/2}) \leq \varepsilon.$$

We now compute the mean of

$$Y_t^{(2)}(A,K,R) = \sum_{u\in\mathcal{N}_t^2} \mathbb{1}_{\left\{X_u(t) - m_t^{(III)} \in [-A,K]\right\}} \mathbb{1}_{\left\{|T(u)-pt| \leq Kt^{1/2}\right\}} \mathbb{1}_{\left\{|X_u(T(u)) - apt| \geq Rt^{1/2}\right\}}.$$

Using again Proposition 3.4.1, we have

$$\begin{aligned}
&\mathbf{E}\left(Y_t^{(2)}(A,K,R)\right) \\
&= \int_{pt-Kt^{1/2}}^{pt+Kt^{1/2}} e^{\beta s + t - s} \mathbf{P}\left(\sigma B_s + (B_t - B_s) - m_t^{(III)} \in [-A,K], |\sigma B_s - apt| \geq Rt^{1/2}\right) \mathrm{d}s \\
&= \int_{pt-Kt^{1/2}}^{pt+Kt^{1/2}} e^{2(\beta-1)(s-pt)} \mathbf{E}\left(e^{\theta(\sigma B_s + B_t - B_s)} \mathbb{1}_{\left\{\substack{\sigma B_s + (B_t - B_s) + (b-a)(pt-s) \in [-A,K] \\ |\sigma B_s - a(pt-s)| \geq Rt^{1/2}}\right\}}\right) \mathrm{d}s,
\end{aligned}$$

by Girsanov transform, using that $\beta s + t - s - \frac{\theta^2}{2}(\sigma^2 s + t - s) = 2(\beta-1)(pt-s)$ and straightforward computations. Next, using that $\theta(b-a)(pt-s) = -2(\beta-1)(s-pt)$, for R large enough, we obtain

$$\mathbf{E}\left(Y_t^{(2)}(A,K,R)\right) \leq 2Kt^{1/2}e^{\theta K} \sup_{|r|\leq Kt^{1/2}} \mathbf{P}\left(\begin{array}{l} \sigma B_{pt+r} + B_t - B_{pt+r} + (b-a)r \in [-A,K] \\ |\sigma B_{pt+r} - ar| \geq Rt^{1/2} \end{array}\right).$$

We then use that $\sigma B_{pt+r} - (\sigma B_{pt+r} + B_t - B_{pt+r}) \frac{\sigma^2 pt+r}{\sigma^2(pt+r)+t(1-p)-r}$ is independent of $\sigma B_{pt+r} + B_t - B_{pt+r}$ by classical Gaussian computations. We deduce that for R large enough, we have for all t large enough

$$\sup_{|r|\leq Kt^{1/2}} \mathbf{P}\left(\sigma B_{pt+r} + B_t - B_{pt+r} + (b-a)r \in [-A,K], |\sigma B_{pt+r} - ar| \geq Rt^{1/2}\right) \leq C(A+K)t^{-1/2}\mathbf{P}\left(|B_1| \geq \frac{R}{2}\sqrt{\tfrac{(\sigma^2 p+1-p)}{4\sigma^2 p(1-p)}}\right).$$

As a result, using again the Markov inequality, we have

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |X_u(T(u)) - apt| \geq Rt^{1/2}) \\
&\leq \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |T(u) - pt| \geq Kt^{1/2}) \\
&+ \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} + K) + C(A+K)Ke^{\theta K}\mathbf{P}\left(|B_1| \geq R\sqrt{\tfrac{(\sigma^2 p+1-p)}{4\sigma^2 p(1-p)}}\right).
\end{aligned}$$

Hence, letting $R \to \infty$, with the choice made for the constant K, we obtain

$$\limsup_{R\to\infty} \limsup_{t\to\infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A, |X_u(T(u)) - apt| \geq Rt^{1/2}) \leq 2\varepsilon,$$

and letting $\varepsilon \to 0$ completes the proof of (3.7.2). □

The above lemma shows that typical particles of type 2 that contribute to the extremal process of the multitype BBM have their last ancestor of type 1 around time pt and position pat. We now prove Theorem 3.1.3, using this localization of birth times and positions of particles in $\mathcal{B}$ that have a descendant contribution to the extremal process at time t, with high probability. Then, using Lemmas 3.3.1 and 3.4.2 we compute the quantity of contributing particles and with Lemma 3.3.7 to obtain the value associated to each contribution.

Proof of Theorem 3.1.3. Let $R > 0$, we set

$$\widehat{\mathcal{E}}_t^R := \sum_{u \in \mathcal{N}_t^2} \mathbb{1}_{\left\{|T(u)-pt| \leq Rt^{1/2}, |X_u(T(u))-apt| \leq Rt^{1/2}\right\}} \delta_{X_u(t) - m_t^{(III)}}.$$

Lemma 3.7.2 states that the extremal process $\widehat{\mathcal{E}}_t^R$ is close to the extremal process of the BBM. Precisely, for all $\varphi \in \mathcal{T}$ we have

$$\left|\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R, \varphi\right\rangle}\right) - \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t, \varphi\right\rangle}\right)\right| \leq \mathbf{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t^{(III)} - A,\ (T(u) - pt, X_u(T(u)) - apt) \notin [-Rt^{1/2}, Rt^{1/2}]^2\right)$$

where A is such that the support of φ is contained in $[-A, \infty)$. As a result, by Lemma 3.7.2 we have

$$\lim_{R\to\infty} \limsup_{t\to\infty} \left|\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R, \varphi\right\rangle}\right) - \mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t, \varphi\right\rangle}\right)\right| = 0, \tag{3.7.3}$$

so to compute the asymptotic behaviour of $\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t,\varphi\right\rangle}\right)$, it is enough to study the convergence of $\widehat{\mathcal{E}}_t^R$ as t then R grow to ∞.

Let $R > 0$ and $\varphi \in \mathcal{T}$. Using the branching property and Corollary 4.2.2, we have

$$\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) = \mathbf{E}\left(\exp\left(-\alpha \int_{pt-Rt^{1/2}}^{pt+Rt^{1/2}} \sum_{u\in\mathcal{N}_s^1} \mathbb{1}_{\left\{|X_u(s)-apt|\leq Rt^{1/2}\right\}} F(t-s, X_u(s)-apt)\mathrm{d}s\right)\right),$$

with $F(r,x) = 1 - \mathbf{E}^{(2)}\left(e^{-\sum_{u\in\mathcal{N}_r^2}\varphi(X_u(r)-br+x-b((1-p)t-r))}\right)$. Additionally, by Lemma 3.3.7, we have

$$\begin{aligned}
&F((1-p)t-r,x)\\
&= C(b)\frac{e^{(1-\frac{b^2}{2})((1-p)t-r)}}{\sqrt{2\pi t(1-p)}} e^{b(x-br)-\frac{(x-br)^2}{2((1-p)t-r)}} \int e^{-\theta z}\left(1-e^{-\Psi^b[\varphi](z)}\right)\mathrm{d}z(1+o(1))\\
&= C(b)\frac{e^{(1-\frac{b^2}{2})(1-p)t}}{\sqrt{2\pi t(1-p)}} e^{\theta x-(1+\frac{\theta^2}{2})r-\frac{(x-br)^2}{2(1-p)t}} \int e^{-\theta z}\left(1-e^{-\Psi^b[\varphi](z)}\right)\mathrm{d}z(1+o(1)),
\end{aligned}$$

as $t \to \infty$, uniformly in $|r| \leq Rt^{1/2}$ and $|x| \leq Rt^{1/2}$, where we used that $\theta = b$. Thus, setting

$$\Theta(\varphi) := \alpha C(b) \int e^{-\theta z}\left(1-e^{-\Psi^b[\varphi](z)}\right)\mathrm{d}z$$

and $G_R(r,x) = \mathbb{1}_{\{|x+ar|\leq R\}} e^{-\frac{(x+(a-b)r)^2}{2(1-p)}}$ we can now rewrite the Laplace transform of $\widehat{\mathcal{E}}_t^R$ as

$$\begin{aligned}
\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) = \mathbf{E}\Bigg(\exp\Bigg(&-\Theta(\varphi)(1+o(1))\frac{e^{(1-\frac{b^2}{2})(1-p)t}}{\sqrt{2\pi t(1-p)}}\\
&\times\int_{pt-Rt^{1/2}}^{pt+Rt^{1/2}} \sum_{u\in\mathcal{N}_s^1} e^{\theta(X_u(s)-apt)-(1+\frac{\theta^2}{2})(s-pt)} G_R\left(\tfrac{s-pt}{t^{1/2}}, \tfrac{X_u(s)-as}{t^{1/2}}\right)\mathrm{d}s\Bigg)\Bigg).
\end{aligned}$$

We have $\beta + \frac{\theta^2\sigma^2}{2} = 1 + \frac{\theta^2}{2} = \frac{\beta-\sigma^2}{1-\sigma^2}$ and $\theta a = \beta + \frac{\theta^2\sigma^2}{2} - \left(\beta - \frac{a^2}{2\sigma^2}\right)$, therefore we can rewrite

$$\begin{aligned}
\mathbf{E}\left(e^{-\left\langle \widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right) = \mathbf{E}\Bigg(\exp\Bigg(&-\Theta(\varphi)\frac{1+o(1)}{\sqrt{2\pi t(1-p)}}\\
&\times\int_{pt-Rt^{1/2}}^{pt+Rt^{1/2}} \sum_{u\in\mathcal{N}_s^1} e^{\theta X_u(s)-(\beta+\frac{\theta^2\sigma^2}{2})s} G_R\left(\tfrac{s-pt}{t^{1/2}}, \tfrac{X_u(s)-as}{t^{1/2}}\right)\mathrm{d}s\Bigg)\Bigg), \quad (3.7.4)
\end{aligned}$$

where we used that $(1-p)(1-\frac{b^2}{2}) + p(1-\frac{a^2}{2\sigma^2}) = 0$.

We now observe that by Lemma 3.3.1, using (3.3.1), we have

$$\lim_{t\to\infty}\frac{1}{t^{1/2}}\int_{pt-Rt^{1/2}}^{pt+Rt^{1/2}}\sum_{u\in\mathcal{N}_s^1}e^{\theta X_u(s)-(\beta+\frac{\theta^2\sigma^2}{2})s}G_R\left(\tfrac{s-pt}{t^{1/2}},\tfrac{X_u(s)-apt}{t^{1/2}}\right)\mathrm{d}s$$
$$=\frac{W_\infty(\theta)}{\sqrt{2\pi p\sigma^2}}\int_{[-R,R]\times\mathbb{R}}e^{-\frac{z^2}{2\sigma^2 p}}e^{-\frac{(z+(a-b)r)^2}{2(1-p)}}\mathbb{1}_{\{|z+ar|\leq R\}}\mathrm{d}r\mathrm{d}z$$

where $W_\infty(\theta)=\lim_{t\to\infty}\sum_{u\in\mathcal{N}_t^1}e^{\theta X_u(t)-t(\beta+\frac{\sigma^2\theta^2}{2})}$ is the limit of the additive martingale with parameter θ for the branching Brownian motion of type 1. As a result, writing

$$c_R=\frac{1}{2\pi\sqrt{p(1-p)\sigma^2}}\int_{[-R,R]\times\mathbb{R}}e^{-\frac{z^2}{2\sigma^2 p}}e^{-\frac{(z+(a-b)r)^2}{2(1-p)}}\mathbb{1}_{\{|z+ar|\leq R\}}\mathrm{d}r\mathrm{d}z\in(0,\infty),$$

by dominated convergence theorem, (3.7.4) yields

$$\lim_{t\to\infty}\mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t^R,\varphi\right\rangle}\right)=\mathbf{E}\left(\exp\left(-c_R W_\infty(\theta)\Theta(\varphi)\right)\right).$$

This convergence holds for all $\varphi\in\mathcal{T}$. Then by [BBCM20, Lemma 4.4], the process $\widehat{\mathcal{E}}_t^R$ converges vaguely in distribution as $t\to\infty$ to a DPPP$(\theta c_R W_\infty(\theta)e^{-\theta z}\mathrm{d}z,\mathfrak{D}^b)$, as $t\to\infty$, where $\mathfrak{D}^b$ is the law of $\mathcal{D}^b$, the point measure defined in (3.3.19).

To complete the proof, we now observe that by monotone convergence theorem, we have

$$\lim_{R\to\infty}c_R=\frac{1}{2\pi\sqrt{p(1-p)\sigma^2}}\int_{\mathbb{R}\times\mathbb{R}}e^{-\frac{z^2}{2\sigma^2 p}}e^{-\frac{(z+(a-b)r)^2}{2(1-p)}}\mathrm{d}r=\frac{1}{b-a}=\frac{1}{\theta(1-\sigma^2)}.$$

Therefore, letting $t\to\infty$ then $R\to\infty$, (3.7.3) yields

$$\lim_{t\to\infty}\mathbf{E}\left(e^{-\left\langle\widehat{\mathcal{E}}_t,\varphi\right\rangle}\right)=\mathbf{E}\left(\exp\left(-\frac{\alpha C(b)W_\infty(\theta)}{2(\beta-1)}\int\theta e^{-\theta z}\left(1-e^{-\Psi^b[\varphi](z)}\right)\right)\right).$$

As a result, using [BBCM20, Lemma 4.4], the proof of Theorem 3.1.3 is now complete, with $c^{(III)}=\frac{\alpha C(b)}{2(\beta-1)}$ and $\mathfrak{D}^{(III)}$ the law of $\mathcal{D}^b$ defined in (3.3.19). □

This program has received funding from the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant agreement No 754362.

Chapter 4

The extremal process of a cascading family of branching Brownian motion

Abstract

We study the asymptotic behaviour of the extremal process of a cascading family of branching Brownian motions. This is a particle system on the real line such that each particle has a type in addition to his position. Particles of type 1 move on the real line according to Brownian motions and branch at rate 1 into two children of type 1. Furthermore, at rate α, they give birth to children too of type 2. Particles of type 2 move according to standard Brownian motion and branch at rate 1, but cannot give birth to descendants of type 1. We obtain the asymptotic behaviour of the extremal process of particles of type 2.

Keywords: Extremal process, Branching Brownian motion, multitype branching process.
MSC 2020: Primary: 60G55; 60j80.
Secondary: 60G70; 92D25.

4.1 Introduction

The branching Brownian motion is a particle system on $\mathbb{R}$ that can be described as follows. Start with on particle at the origin at time $t = 0$. After an exponential random time of mean one, this particle splits in two children. The new particles then start independent copies of the branching Brownian motion from their positions. Denote by $\mathcal{N}_t$ the set of particles alive at time t. For $u \in \mathcal{N}_t$, we denote by $X_u(t)$ the position at time t of that particle. We define

$$M_t = \max_{u\in\mathcal{N}_t} X_u(t)$$

the position of the right most particle in the Branching Brownian motion.

There is a fundamental link between the BBM and the well known Fisher-Kolmogorov-Petrovsky-Piskunov (F-KPP) reaction-diffusion equation

$$\partial_t u = \frac{1}{2}\Delta u - u(1-u). \tag{4.1.1}$$

If we denote by $u(t,x) = \mathbf{P}(M_t \le x)$, the function u is the solution of the F-KPP equation (4.1.1) with Heaviside initial condition. It is also known [Kol37] that there exists a function m_t such that

$$u(t, m_t + x) \to w(x)$$

uniformly in x, where w is called a travelling wave solution of the F-KPP equation, that satisfies

$$\frac{1}{2}w_{xx} + \sqrt{2}w_x + w(w-1) = 0. \tag{4.1.2}$$

Using these observations, Kolmogorov, Petrovskii and Piskunov [Kol37] proved that $\lim_{t\to\infty} \frac{m_t}{t} = \sqrt{2}$. Bramson [Bra78] showed, using the connection with the BBM, that

$$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t) + O(1).$$

The convergence in distribution of the centred maximal displacement was obtained by Lalley and Sellke [LS87]. They proved that the centered maximum $M_t - m_t$ converges in law to a randomly shifted Gumbel distribution. More precisely, if we denote by

$$Z_t = \sum_{u\in\mathcal{N}_t} (\sqrt{2}t - X_u(t))e^{-\sqrt{2}(\sqrt{2}t - X_u(t))}$$

the so-called derivative martingale of the BBM, they proved that there exists a constant $C^* > 0$ such that for all $x \ge 0$

$$\lim_{t\to\infty} u(t, m_t + x) = \mathbf{E}\left(\exp(-C^* Z_\infty e^{-\sqrt{2}x})\right) \tag{4.1.3}$$

where

$$Z_\infty := \lim_{t\to\infty} Z_t \text{ a.s.} \tag{4.1.4}$$

One of the most interesting questions in the context of branching Brownian motion, is the study of the asymptotic behaviour of the extremal process defined by

$$\mathcal{E}_t = \sum_{u\in\mathcal{N}_t} \delta_{X_u(t)-m_t}.$$

It was shown by Aidékon, Berestycki, Brunet and Shi [ABBS13] as well as Arguin, Bovier and Kistler [ABK13] that this point measure converges in law to a decorated Poisson point process with intensity $\sqrt{2}C^* Z_\infty e^{-\sqrt{2}x}dx$. A description of the law of decoration was given by Arguin, Bovier and Kistler [ABK13], they proved that there exists a point measure $\mathcal{D}$ such that

$$\lim_{t\to\infty} \mathbf{E}\left(\exp\left(-\sum_{u\in\mathcal{N}_t}\varphi(X_u(t)-M_t)\right)\middle| M_t \geq \sqrt{2}t\right) = \mathbf{E}\left(\exp\left(-\langle\mathcal{D},\varphi\rangle\right)\right), \tag{4.1.5}$$

for all continuous function φ with a bounded support on the left. Moreover, the point measure $\mathcal{D}$ is supported on $\mathbb{R}_-$, with an atom on 0. The limiting law of the extremal process of the branching Brownian motion, that we denote $\mathcal{E}_\infty$, can be described as follows. Let $(\xi_i)_{i\in\mathbb{N}}$ be the atoms of a Poisson point process with intensity $C^*\sqrt{2}e^{-\sqrt{2}x}dx$. For each atom ξ_j, we attached a point measure $\mathcal{D}_j$ where $(\mathcal{D}_j, j\in\mathbb{N})$ are i.i.d copies of the point measure $\mathcal{D}$ which are independents of $(\xi_i)_{i\in\mathbb{N}}$, then we set

$$\mathcal{E}_\infty = \sum_{j\in\mathbb{N}}\sum_{d\in\mathcal{D}_j} \delta_{\xi_j+d+\frac{1}{\sqrt{2}}\log Z_\infty},$$

where $\sum_{d\in\mathcal{D}_j}$ is the sum on the set of atoms of the point measure $\mathcal{D}_j$.

In this paper we study the asymptotic behaviour of the extremal process of a of a cascading family of branching Brownian motion. This is a particle system on the real line such that each particle has a type in addition to his position. Particles of type 1 move on the real line according to Brownian motion with variance 1 and branch at rate 1 into two children of type 1. Additionally , at rate α, they give birth to children of type 2. Particles of type 2 move according to standard Brownian motion and branch at rate 1, but cannot give birth to descendants of type 1.

In a recent paper [BM21], we studied the asymptotic behaviour of the extremal process of a two-type reducible branching Brownian motion where particles move and reproduce at a different rate. Note that the model we considered here can be seen as a critical case of the one studied in [BM21].

For all $t\geq 0$ we write $\mathcal{N}_t$ for the set of particle alive at time t, separated into $\mathcal{N}_t^1$ (respectively $\mathcal{N}_t^2$) the set of particles of type 1 (respectively type 2). If $u\in\mathcal{N}_t^2$, we denote by $T(u)$ the time at which the oldest ancestor of type 2 of u was born from a particle of type 1. We also write $X_u(t)$ the position at time t of $u\in\mathcal{N}_t$ and $\widehat{M}_t = \max_{u\in\mathcal{N}_t^2} X_u(t)$. We studied the asymptotic behaviour of the two-type reducible branching Brownian motion according to a phase diagram containing three regions. However, we didn't consider the boundary case.

Multitype branching processes are widely used in biology ans ecology. For example, when modeling certain diseases, such processes can be used to describe the evolution of cells that have carried out different numbers of mutations [Dur15]. In epidemiology, multi-type continuous time Markov branching process may be used to describe the dynamics of the spread of parasites of two types that can mutate into each other in a common population [BKR13]. Many applications of multi-type branching processes in biology can be found in [HHJV05, KA15].

Historically, questions about the extreme values of spatial multitype branching processes were not a main subject of interest. Biggins in [Big12] gave an explicit formula for the speed of the reducible multitype process of the branching random walk. The irreducible case (i.e. when for all pair of types i and j, particles of type i have positive probability of having at least one descendant of type j after an exponential time) has been studied by Ren and Yang [RY14]. They showed that the asymptotic behaviour of the maximal displacement is similar to that of classical branching Brownian motion. However, the extremal process has not yet been studied. We expect that in this case, results should be similar to what is observed in the standard BBM.

Blath, Jacobie and Nie [BJN21] studied the asymptotic speed in a modified version of the standard BBM, called the On/Off BBM. It is a branching Brownian motion on $\mathbb{R}$ such that each particle has an active or dormant state. They studied the asymptotic behaviour of the maximal displacement of the On/Off BBM using the *McKean representation* [McK75] of the position of the rightmost particle as a solution of F-KPP equation. We observe here that this model can be seen as a two-type active/dormant BBM.

We now introduce our main result.

Theorem 4.1.1. *Setting* $m_t = \sqrt{2}t - \frac{1}{2\sqrt{2}}\log(t)$, *and* $\widehat{\mathcal{E}}_t = \sum_{u\in\mathcal{N}_t^2} \delta_{X_u(t)-m_t}$, *we have*

$$\lim_{t\to\infty} \widehat{\mathcal{E}}_t = \widehat{\mathcal{E}}_\infty \quad \textit{for the topology of the vague convergence,}$$

where $\widehat{\mathcal{E}}_\infty$ *is a decorated Poisson point process with intensity* $C^*\alpha\sqrt{2}Z_\infty e^{-\sqrt{2}x}dx$, *the constant* C^* *is the one introduced in* (4.1.3) *and*

$$Z_\infty := \lim_{t\to\infty} \sum_{u\in\mathcal{N}_t^1} (\sqrt{2}t - X_u(t))e^{\sqrt{2}X_u(t)-2t} \quad a.s..$$

Moreover, we have $\lim_{t\to\infty} \mathbf{P}(M_t \le m_t + x) = \mathbf{E}\left(e^{-\alpha C^* Z_\infty e^{-\sqrt{2}x}}\right)$ *for all* $x \in \mathbb{R}$.

The random variable Z_∞ is the same limit as in (4.1.4) since $(X_u(t), u \in \mathcal{N}_t^1)$ is a standard branching Brownian motion.

An extension of this model is to consider the cascading BBM. It is a particle system that can be described as follows. Particles of type i, $i \ge 1$ move according standard Brownian motion and branch at rate 1 into two children of type i. Additionally, at rate α, they give birth to one particle of type i and one particle of type $i+1$.

For all $t \ge 0$, we write $\mathcal{N}_t^{(i)}$, $i \ge 1$ the set of particles of type i. Fix $k \ge 2$, we conjecture the asymptotic behaviour of the extremal process of particles of type k.

Conjecture 4.1.2. *Setting* $m_t^{(k)} = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t) + \frac{k-1}{\sqrt{2}}\log(t)$ *and* $\widehat{\mathcal{E}}_t^{(k)} = \sum_{u\in\mathcal{N}_t^{(k)}} \delta_{X_u(t)-m_t^{(k)}}$, *we have*

$$\lim_{t\to\infty} \widehat{\mathcal{E}}_t^{(k)} = \widehat{\mathcal{E}}_\infty^{(k)} \quad \textit{for the topology of the vague convergence,}$$

where $\widehat{\mathcal{E}}_\infty$ *is a decorated Poisson point process with intensity* $C^*\frac{\alpha^{k-1}}{(k-1)!}\sqrt{2}Z_\infty e^{-\sqrt{2}x}dx$, *with same notation as in Theorem* 4.1.1.

Remark 4.1.3. We observe a change in the logarithmic correction of the median m_t comparing it to that of a classical branching Brownian motion. More precisely, we pass from a multiplicative factor $\frac{-3}{2\sqrt{2}}$ in the case of a standard BBM to a factor $\frac{-3}{2\sqrt{2}} + \frac{k-1}{\sqrt{2}}$ for particles of type k.

Notation. Throughout the paper, we use C and c to denote a generic positive constant, that may change from line to line. We say that $f_n \sim g_n$ if $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$. For $x \in \mathbb{R}$, we write $x_+ = \max(x, 0)$.

Organisation of the paper. The rest of the paper is organised as follows. In the next section, we recall a version of a multitype many-to-one lemma that was introduced in [BM21]. In Section 3, we introduce some useful lemmas in the context of standard BBM. Finally, we conclude the paper with a proof of the main result.

4.2 Multitype many-to-one formula and Brownian motion estimates

The classical many to-one lemma was first introduced by Kahane and Peyrière [KP76]. This lemma links an additive functional of branching Brownian motion with a simple function of Brownian motion. Let us recall the standard version of many-to-one in the context of classical BBM.

Lemma 4.2.1 (Many to-on-lemma)**.** *For any $t \geq 0$, and measurable positive function f, we have*

$$\mathbf{E}\left(\sum_{u\in\mathcal{N}_t} f(X_u(s), s \leq t)\right) = e^t \mathbf{E}(f(B_s, s \leq t)), \tag{4.2.1}$$

where B is a standard Brownian motion.

Before we introduce a multitype version, we will set some notation. We write

$$\mathcal{B} = \{u \in \cup_{t\geq 0}\mathcal{N}_t^2, T(u) = b_u\}$$

for the set of particles of type 2 that are born from a particle of type 1. Recall that $T(u)$ is the time at which the oldest ancestor of type 2 of u was born. The following proposition was introduced in [BM21, Corollary 4.3].

Proposition 4.2.2. *For any measurable non-negative function f, we have*

$$\mathbb{E}\left(\sum_{u\in\mathcal{B}} f(X_u(s), s \leq T(u))\right) = \alpha \int_0^\infty e^{\beta t}\mathbb{E}(f(B_s, s \leq t))dt,$$

$$\mathbb{E}\left(\exp\left(-\sum_{u\in\mathcal{B}} f(X_u(s), s \leq T(u))\right)\right) = \mathbb{E}\left(\exp\left(-\alpha \int_0^\infty \sum_{u\in\mathcal{N}_t^1} 1 - e^{-f(X_u(s), s\leq t)} dt\right)\right). \tag{4.2.2}$$

4.2.1 Brownian motion estimates

We now introduce some Brownian motion estimates that will be needed in the proof of the main result. Let $(B_s)_{s\geq 0}$ be a standard Brownian motion. The quantity $\sup_{0\leq s\leq t} B_s$ has the same law as $|B_t|$. As a consequence, there exists $C > 0$ such that for all $t \geq 1$, $y \geq 1$ we have

$$\mathbf{P}(B_s \geq -y, s \leq t) = \mathbf{P}(|B_1| \leq y/\sqrt{t}) \leq C\frac{y \wedge \sqrt{t}}{\sqrt{t}}. \tag{4.2.3}$$

We need also an estimate for the Brownian motion to stay below a line and end up in a finite interval. For all $K \geq 1$ and $y \geq 1$ we have

$$\mathbf{P}\left(B_s \leq K, s \leq t, B_t > K - y\right) \leq C\frac{(1+K)(1+y)}{(t+1)^{3/2}} \tag{4.2.4}$$

This estimate can be obtained using similar computations to these used in [Mal15a, Lemma 3.8] for random walks.

We next introduce the 3-dimensional Bessel process that we denote $(R_s)_{s\geq 0}$. We have the following link between the process $(R_s)_{s\geq 0}$ and the Brownian motion: For all $t \geq 0$, $x > 0$ and any measurable positive function g, we have

$$\mathbf{E}_x\left(g(B_s, s \in [0,t])\mathbb{1}_{\{B_s>0, s\leq t\}}\right) = \mathbf{E}_x\left(\frac{x}{R_s}g(R_s, s \in [0,t])\right). \tag{4.2.5}$$

In other words, R corresponds to the law of the Brownian motion conditioned on not hitting 0 in the sense of Doobs's h-transform. Let $p_s(x,z)$ be the transition density of R_s started from x at time s. We have

$$p_s(x,z) = \sqrt{\frac{2}{\pi}}e^{-(z-x)^2/2s}\mathbb{1}_{\{z>0\}} \times \begin{cases} \frac{z}{2x\sqrt{s}}(1-e^{-2xz/s}) & \text{if } x > 0 \\ \frac{z^2}{s^{3/2}} & \text{if } x = 0 \end{cases}.$$

4.2.2 Branching Brownian motion estimates

In this section, we denote by $(X_t(u), u \in \mathcal{N}_t)$ a standard branching Brownian motion. We recall here some useful estimates on this process, that will be used to prove Theorem 4.1.1.

We know that with high probability all particles in the one-type BBM are smaller than $\sqrt{2}t+y$, for all $y \geq 0$. More precisely, we have the following upper bound.

Proposition 4.2.3. *There exists a constant $C > 0$ such that for any $t \geq 1$ and $K \geq 1$*

$$\mathbf{P}\left(\exists s \geq 0, u \in \mathcal{N}_s^1 : X_u(s) \geq \sqrt{2}s + K\right) \leq C(K+1)e^{-\sqrt{2}K}.$$

Proof. Let $l \geq 1$ be an integer. Define $\tau = \inf\{s \leq t, \exists u \in \mathcal{N}_t^1 : X_u(t) \geq \sqrt{2}s + K\}$ and Z_l to be the number of particle in $\mathcal{N}_l^1$ that stay below the barrier $s \mapsto \sqrt{2}s + K$ for all $s \leq l-1$ and such that $X_u(t) > \sqrt{2}s + K$ for some $t \in [l-1, l]$. Then, by the Markov inequality, we have

$$\mathbf{P}\left(\exists s \geq 0, u \in \mathcal{N}_s^1 : X_u(s) \geq \sqrt{2}s + K\right) \leq \sum_{l=1}^{\infty}\mathbf{P}(\tau \in [l-1,l]) = \sum_{l=1}^{\infty}\mathbf{E}(Z_l).$$

Using Lemma 4.2.1, we obtain

$$\mathbf{E}(Z_l) \leq \mathbf{P}\left(B_s \leq \sqrt{2}s + K, s \leq l-1, B_r > \sqrt{2}r + K \text{ for some } r \in [l-1,l]\right). \tag{4.2.6}$$

Applying the Markov property at time $l-1$, we get

$$\mathbf{P}\left(B_s \leq \sqrt{2}s + K, s \leq l-1, B_r > \sqrt{2}r + K \text{ for some } r \in [l-1,l]\right) \leq \mathbf{E}(g(\sup_{0\leq s\leq 1} B_s))$$

where $g(x) = \mathbf{P}\left(B_s \leq \sqrt{2}s + K, s \leq l-1, B_{l-1} > \sqrt{2}(l-1) + K - x\right)$. Moreover, using Girsanov theorem we have

$$\begin{aligned}
&\mathbf{P}\left(B_s \leq \sqrt{2}s + K, s \leq l-1, B_{l-1} > \sqrt{2}(l-1) + K - x\right) \\
&\leq \mathbf{E}\left(e^{-(\sqrt{2}B_{l-1}+l-1)}\mathbb{1}_{\{B_s \leq K, s \leq l-1, B_{l-1} > K-x\}}\right) \\
&\leq e^{-l+1}e^{\sqrt{2}(x-K)}\mathbf{P}\left(B_s \leq K, s \leq l-1, B_{l-1} > K - x\right) \\
&\leq Ce^{-l}e^{\sqrt{2}(x-K)}\frac{(1+K)(1+x)}{(l+1)^{3/2}}
\end{aligned}$$

where in the last inequality we used (4.2.4). Plugging all this in (4.2.6) and using that $\sup_{0\leq s\leq 1} B_s$ has the same law as $|B_1|$ (see Section 4.2.1), then easy computations lead to

$$\begin{aligned}
&\mathbf{P}\left(\exists s \geq 0, u \in \mathcal{N}_s : X_u(s) \geq \sqrt{2}s + K\right) \\
&\leq C(K+1)e^{-\sqrt{2}K}\sum_{l=1}^{\infty}\frac{1}{(l+1)^{3/2}} \leq C(K+1)e^{-\sqrt{2}K}.
\end{aligned}$$

which completes the proof. □

We also have an upper bound on the tail of the maximal displacement that was introduced by Bramson [Bra78] and refined by Arguin, Bovier and Kiestler [ABK12]. We write $\widetilde{m}_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t)$.

Proposition 4.2.4. *[ABK12, Corollary 10] There exists $t_0 > 0$ such that $\forall t \geq t_0$ and $y \in \mathbb{R}_+$*

$$\mathbb{P}(M_t > \widetilde{m}_t + y) \leq C(1+y_+)e^{-\sqrt{2}y - \frac{y^2}{2t}}$$

for some constant $C > 0$.

We next recall recall a link between the FKPP equation and the branching Brownian motion.

Lemma 4.2.5. *Let $f : \mathbb{R} \mapsto [0,1]$ a measurable function and*

$$u_f(t,x) = 1 - \mathbf{E}[\prod_{u\in\mathcal{N}_\sqcup}(1 - f(x - X_u(t)))]. \tag{4.2.7}$$

Then u_f solves the FKPP equation with the initial condition $u_f(0,x) = f(x)$.

In our work we need an uniform estimate of general solutions of the F-KPP equation that is useful for the computation of the asymptotics of the Laplace transform of the extremal process of the BBM. Before that, let us recall a result of Bramson [Bra78] on the convergence of the solutions of F-KPP equation to travelling wave (see also Theorem 4.2 in [ABK13].)

Theorem 4.2.6. *[Bra78, Theorems A, B] Let u_f be a solution of the F-KPP equation in the form of (4.2.7) with the initial condition $u(0,x) = f(x)$, where the function f satisfying*

$$\begin{aligned}
&(ii) \quad 0 \leq f(x) \leq 1 \\
&(iii) \quad \text{For some } y > 0, N > 0, M > 0, \int_x^{x+N} u(0,z)\mathrm{d}z > y \text{ for all } x \leq -M, \qquad (4.2.8) \\
&(iv) \quad \sup\{x \in \mathbb{R}, f(x) > 0\} < \infty,
\end{aligned}$$

then

$$u_f(t, \widetilde{m}_t + x) \to w(x), \text{ uniformly in } x \text{ as } t \to \infty,$$

where w is the unique solution (up to translation) of the equation (4.1.2).

The next proposition follows from Proposition 4.3 and Lemma 4.5 in [ABK13].

Proposition 4.2.7. *Let u_f be a solution of the F-KPP equation in the form of (4.2.7) with the initial condition $u(0,x) = f(x)$ and satisfying the assumptions of Theorem 4.2.6. Then, for any fixed $\varepsilon > 0$, uniformly in $x \in [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]$, we have the convergence*

$$\lim_{t\to\infty} \frac{e^{-\sqrt{2}x}}{(-x)} t^{3/2} e^{x^2/2t} u_f(t, \sqrt{2}t - x) = \gamma(f), \tag{4.2.9}$$

where $\gamma(f) = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int u_f(r, z + \sqrt{2}r) z e^{\sqrt{2}z} \mathrm{d}z$.

Proof. Fix $\varepsilon > 0$, using Proposition 4.3 in [ABK13] for r large enough, $t \geq 8r$ and $-x \geq 8r - \frac{3}{2\sqrt{2}} \log(t)$, we have

$$\rho^{-1}(r)\psi(r, t, -x + y + \sqrt{2})t \leq u_f(t, -x + y + \sqrt{2}t) \leq \rho(r)\psi(r, t, -x + y + \sqrt{2})$$

where $\rho(r) \to 1$ as $r \to \infty$ and

$$\psi(r, t, -x+y+\sqrt{2}) = \frac{e^{-\sqrt{2}(y-x)}}{\sqrt{2\pi(t-r)}} \int_0^\infty u_f(r, z+\sqrt{r}) e^{\sqrt{2}z} e^{-(z+x-y)^2/2(t-r)} \left(1 - e^{-2z\frac{(x+\frac{3}{2\sqrt{2}\log(t)})^2}{t-r}}\right) dz.$$

Using Lemma 4.5 in [ABK13], and since $\rho(r) \to 1$ we have

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{x\in[-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]} \frac{e^{\sqrt{2}(y-x)}}{-x} t^{3/2} e^{-a^2/2} u_f(t, -x + y + \sqrt{2}t) \\ &\leq \liminf_{r\to\infty} \limsup_{t\to\infty} \sup_{x\in[-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]} \frac{e^{\sqrt{2}(-x+y)}}{-x} t^{3/2} e^{-a^2/2} \psi(r, t, -x + y + \sqrt{2}) \leq \gamma(f) \end{aligned}$$

and similarly

$$\liminf_{t\to\infty} \inf_{x\in[-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]} \frac{e^{\sqrt{2}(-x+y)}}{-x} t^{3/2} e^{-a^2/2} u_f(t, -x + y + \sqrt{2}t) \geq \gamma(f)$$

for some constant $\gamma(\varphi)$ given in Lemma 4.5 in [ABK13], which completes the proof. □

In particular, by setting $f(x) = \mathbb{1}_{\{x\leq 0\}}$, we have $u(t, \sqrt{2}t - x + y) = \mathbb{P}(M_t > \sqrt{2}t - x + y)$, and the following uniform estimate of the tail of M_t.

Corollary 4.2.8. *For all $\varepsilon > 0$ and $y \in \mathbb{R}_+$, we have*

$$\mathbb{P}(M_t > \sqrt{2}t - x + y) \sim_{t\to\infty} \frac{C^*}{t^{3/2}} (-x) e^{-\sqrt{2}(y-x)} e^{-x^2/2t} \tag{4.2.10}$$

uniformly in $x \in [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]$, where the constant C^* is the one introduced in (4.1.3).

We end this section by an uniform estimate of the Laplace transform of the extremal process of the BBM that generalizes (4.2.10). Denote by $\mathcal{T}$ the set of non-negative, continuous, bounded functions $\varphi : \mathbb{R} \mapsto \mathbb{R}_+$ with support bounded on the left.

Corollary 4.2.9. *Fix $\varepsilon > 0$. Setting*

$$\mathcal{E}_t(x) = \sum_{u \in \mathcal{N}_t} \delta_{X_u(t) - \sqrt{2}t + x},$$

we have for all $\varphi \in \mathcal{T}$

$$\mathbf{E}\left(1 - e^{-\sum_{u \in \mathcal{N}_t} \varphi(x + X_u(t) - \sqrt{2}t)}\right) = C^* \sqrt{2} \frac{e^{\sqrt{2}x - \frac{x^2}{2t}}}{t^{3/2}} \int e^{-\sqrt{2}z} \left(1 - \mathbf{E}(e^{-\langle \mathcal{D}, \varphi(.+z) \rangle})\right) \mathrm{d}z (1 + o(1)),$$

uniformly in $x \in [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]$, as $t \to \infty$.

Proof. The proof follows from Proposition 4.2.7. By setting $f(x) = 1 - e^{-\varphi(-x)}$, we have

$$u_f(t, \sqrt{2}t - x - X_u(t)) = \mathbf{E}\left(1 - e^{-\sum_{u \in \mathcal{N}_t} \varphi(x + X_u(t) - \sqrt{2}t)}\right).$$

Now observe that for all $\varphi \in \mathcal{T}$ the function $x \mapsto f(x) = 1 - e^{-\varphi(-x)}$ satisfies assumptions of Theorem 4.2.6, then in view of Proposition 4.2.7, we obtain

$$\mathbf{E}\left(1 - e^{-\sum_{u \in \mathcal{N}_t} \varphi(x + X_u(t) - \sqrt{2}t)}\right) = \frac{e^{\sqrt{2}x - \frac{x^2}{2t}}}{t^{3/2}} \gamma(\varphi)(1 + o(1)).$$

On the other hand, it is known, using Corollary 4.12 in [ABK13] ,that the constant $\gamma(\varphi)$ can be expressed through the decoration $\mathcal{D}$ defined in (4.1.5), as follows

$$\gamma(\varphi) = C^* \sqrt{2} \int e^{-\sqrt{2}z} \left(1 - \mathbf{E}(e^{-\langle \mathcal{D}, \varphi(.+z) \rangle})\right),$$

where the constant C^* is introduced in (4.1.3), which completes the proof. □

4.3 Proof of the main result

Using [BBCM20, Lemma 4.1], it is enough to show that for all $\varphi \in \mathcal{T}$

$$\lim_{t \to \infty} E\left(e^{-\langle \widehat{\mathcal{E}}_t, \varphi \rangle}\right) = \mathbb{E}\left(\exp(-\alpha C^* \sqrt{2} Z_\infty \int e^{-\sqrt{2}z} \left(1 - \mathbf{E}(e^{-\langle \mathcal{D}, \varphi(.+z) \rangle})\right) \mathrm{d}z)\right).$$

where $\mathcal{D}$ is the law of the point measure defined in (4.1.5).

The first step of the proof of Theorem 4.1.1 is to show that for all $A \geq 0$ and $\varepsilon > 0$, every particle u of type 2 to the right of $m_t - A$ at time t satisfy $T(u) \in [\varepsilon t, (1 - \varepsilon)t]$ with high probability.

Proposition 4.3.1. *Fix $A > 0$, $m_t = \sqrt{2}t - \frac{1}{2\sqrt{2}} \log(t)$. We have*

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} \mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \notin [\varepsilon t, (1 - \varepsilon)t], X_u(t) \geq m_t - A) = 0 \qquad (4.3.1)$$

Proof. We first set, for $\varepsilon, A, K \geq 0$ and $t \geq 0$:

$$Z_t(A,\varepsilon,K) = \sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\leq \varepsilon t\}} \mathbb{1}_{\left\{X_u(r)\leq r\sqrt{2}+K, r\leq T(u)\right\}} \mathbb{1}_{\left\{M_t^u \geq m_t - A\right\}},$$

and

$$\widetilde{Z}_t(A,\varepsilon,K) = \sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\geq (1-\varepsilon) t\}} \mathbb{1}_{\left\{X_u(r)\leq r\sqrt{2}+K, r\leq T(u)\right\}} \mathbb{1}_{\left\{M_t^u \geq m_t - A\right\}},$$

where M_t^u is the position of the rightmost descendant at time t of the individual u. Observe that by Markov inequality and Proposition 4.2.3 we have

$$\begin{aligned}
&\mathbf{P}(\exists u \in \mathcal{N}_t^2 : T(u) \notin [\varepsilon t, (1-\varepsilon)t], X_u(t) \geq m_t - A)\\
&\leq \mathbf{P}\left(\exists t \geq 0, u \in \mathcal{N}_t^1 : X_u(t) \geq \sqrt{2}s + K\right) + \mathbf{P}(Z_t(A,\varepsilon,K) \geq 1) + \mathbf{P}(\widetilde{Z}_t(A,\varepsilon,K) \geq 1)\\
&\leq C(K+1)e^{-\theta K} + \mathbf{E}(Z_t(A,\varepsilon,K)) + \mathbf{E}(\widetilde{Z}_t(A,\varepsilon,K)).
\end{aligned}$$

Hence by fixing K large enough, it is enough to prove that $\limsup_{t\to\infty} \mathbf{E}(Z_t(A,\varepsilon,K))$ and $\limsup_{t\to\infty} \mathbf{E}(\widetilde{Z}_t(A,\varepsilon,K))$ are both $o_\varepsilon(1)$ to complete the proof.

Using the branching property and Corollary 4.2.2, we have

$$\begin{aligned}
\mathbf{E}(Z_t(A,\varepsilon,K)) &= \mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\{T(u)\leq\varepsilon t\}} \mathbb{1}_{\left\{X_u(r)\leq\sqrt{2}r+K, r\leq T(u)\right\}} F\left(t - T(u), X_u(T(u))\right)\right)\\
&= \alpha \int_0^{\varepsilon t} e^s \mathbf{E}\left(F\left(t-s, B_s\right) \mathbb{1}_{\left\{B_r \leq \sqrt{2}r + K, r\leq s\right\}}\right) \mathrm{d}s\\
&= \alpha \int_0^{\varepsilon t} \mathbf{E}\left(e^{-\sqrt{2}B_s} F\left(t-s, B_s + \sqrt{2}s\right) \mathbb{1}_{\{B_r\leq K, r\leq s\}}\right) \mathrm{d}s,
\end{aligned}$$

where we have set $F(r,x) = \mathbf{P}^{(2)}\left(x + M_r \geq m_t - A\right)$.

By Proposition 4.2.3, there exists $C > 0$ such that for all $x \in \mathbb{R}$ and $t \geq 0$, we have

$$\mathbf{P}^{(2)}\left(M_t \geq m_t + x\right) \leq C(1 + x_+)e^{-\sqrt{2}x},$$

so that for all $s \leq t$,

$$\begin{aligned}
F(t-s,x) &= \mathbf{P}^{(2)}\left(M_{t-s} \geq \sqrt{2}(t-s) - \tfrac{1}{2\sqrt{2}}\log(t) - A - (x - \sqrt{2}s)\right)\\
&\leq C\frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}}\left(1 + \frac{\log(t)}{\sqrt{2}} + (-x)_+\right) e^{-\sqrt{2}(\sqrt{2}s - x - A)}.
\end{aligned} \tag{4.3.2}$$

As a result using that $s \leq \varepsilon t$

$$\mathbf{E}(Z_t(A,\varepsilon,K)) \leq \alpha\frac{2C}{t}\int_0^{\varepsilon t} \mathbf{E}\left((c + \frac{\log(t)}{\sqrt{2}} + (-B_s)_+)\mathbb{1}_{\{B_r\leq K, r\leq s\}}\right) \mathrm{d}s.$$

Using (4.2.3) and the definition of Bessel process (4.2.5), we get

$$\mathbf{E}(Z_t(A,\varepsilon,K)) \leq \frac{\alpha C}{t}(c + \frac{\log(t)}{\sqrt{2}}) \int_0^{\varepsilon t} \frac{1}{\sqrt{s}} \mathrm{d}s + 2K\alpha C\varepsilon = \frac{\alpha C}{\sqrt{t}}(c + \frac{\log(t)}{\sqrt{2}})\sqrt{\varepsilon} + 2K\alpha C\varepsilon \tag{4.3.3}$$

We now estimate $\mathbf{E}(\widetilde{Z}_t(A,\varepsilon,K))$. Using similar calculation we have

$$\mathbf{E}(\widetilde{Z}_t(A,\varepsilon,K)) \leq \alpha \int_{(1-\varepsilon)t}^{1} \mathbf{E}\left(e^{-\sqrt{2}B_s} F\left(t-s, B_s+\sqrt{2}s\right) \mathbb{1}_{\{B_r\leq K, r\leq s\}}\right) \mathrm{d}s$$

where again $F(r,x) = \mathbf{P}^{(2)}(x+M_r \geq m_t - A)$. Using Proposition 4.2.8 we have the following upper bound

$$\mathbf{E}(\widetilde{Z}_t(A,\varepsilon,K)) \leq \alpha C \int_{(1-\varepsilon)t}^{t} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}} \mathbf{E}\left(e^{-\frac{B_s^2}{2(t-s)}}\left(c+\frac{\log(t)}{\sqrt{2}}+(-B_s)_+\right)\mathbb{1}_{\{B_r\leq K, r\leq s\}}\right)\mathrm{d}s.$$

By the definition of a Bessel process w obtain

$$\mathbf{E}(\widetilde{Z}t(A,\varepsilon,K)) \leq \alpha C \int_{(1-\varepsilon)t}^{t} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}} \mathbf{E}_K\left(\frac{K}{R_s} e^{-\frac{R_s^2}{2(t-s)}}\left(c+\frac{\log(t)}{\sqrt{2}}+R_s\right)\right)\mathrm{d}s$$

$$\leq 2\alpha C \int_{(1-\varepsilon)t}^{t} \frac{1}{(t-s+1)^{\frac{3}{2}}} \mathbf{E}_{K/\sqrt{s}}\left(\frac{K}{R_1} e^{-\frac{sR_1^2}{2(t-s)}}\left(c+\frac{\log(t)}{\sqrt{2}}\right)\right)\mathrm{d}s \tag{4.3.4}$$

$$+\alpha C K \int_{(1-\varepsilon)t}^{t} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}} \mathbf{E}_{K/\sqrt{s}}\left(e^{-\frac{sR_1^2}{2(t-s)}}\right) ds. \tag{4.3.5}$$

where we used Bessel scaling in (4.3.4).

On the one hand, we know that the density of R_1 under $\mathbf{P}_x$ for $x>0$ is equal to

$$y \mapsto \frac{y}{x}\frac{e^{-(y-x)^2/2}}{\sqrt{2\pi}}(1-e^{-2xy})\mathbb{1}_{\{y>0\}}.$$

Using that for $x,y>0$, $1-e^{-2xy}\leq 2xy$ we have

$$\mathbf{E}_x\left(\frac{1}{R_1}e^{-\frac{sR_1^2}{2(t-s)}}\right) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{x} e^{-\frac{sy^2}{2(t-s)}} e^{-(y-x)^2/2}(1-e^{-xy})dy$$

$$\leq \frac{1}{\sqrt{2\pi}}\int_0^\infty y e^{-\frac{sy^2}{2(t-s)}} e^{-(y-x)^2/2} dy = \frac{2}{\sqrt{2\pi}}\int_{-x(t-s/t)}^\infty (y+x(t-s/t)) e^{-\frac{ty^2}{2(t-s)}} e^{-x^2 s/2t} dy.$$

Plugging this in equation (4.3.4) and by the change of variable $u=\frac{s}{t}$, we have

$$\int_{(1-\varepsilon)t}^{t} \frac{\sqrt{t+1}}{\sqrt{s}(t-s+1)^{\frac{3}{2}}} \mathbf{E}_x\left(\frac{K}{R_1}e^{-\frac{sR_1^2}{2(t-s)}}\left(c+\frac{\log(t)}{\sqrt{2}}\right)\right)\mathrm{d}s$$

$$\leq C\frac{\left(c+\frac{\log(t)}{\sqrt{2}}\right)}{\sqrt{t}}\int_{1-\varepsilon}^1\int_{-x(1-u)}^\infty \frac{e^{-\frac{y^2}{2(1-u)}}}{(1-u)^{\frac{3}{2}}}(y+x(1-u))e^{-x^2u/2}dydu \tag{4.3.6}$$

with $x=K/\sqrt{tu}$. On the other hand, we bound

$$\int_{-x(1-u)}^\infty \frac{e^{-\frac{y^2}{2(1-u)}}}{(1-u)^{\frac{3}{2}}}(y+x(1-u))e^{-x^2u/2}dy \leq \frac{e^{-\frac{x^2(1-u)}{2}}}{(1-u)^{\frac{1}{2}}}+x.$$

Plugging this in (4.3.6), for t large enough we deduce that

$$\int_{(1-\varepsilon)t}^{t} \frac{\sqrt{t+1}}{\sqrt{s}(t-s+1)^{\frac{3}{2}}} \mathbf{E}_{K/\sqrt{s}} \left(\frac{K}{R_1} e^{-\frac{sR_1^2}{2(t-s)}} \left(c + \frac{\log(t)}{\sqrt{2}} \right) \right) \mathrm{d}s$$
$$\leq C \frac{\left(c + \frac{\log(t)}{\sqrt{2}}\right)}{\sqrt{t}} \int_0^{\varepsilon} \frac{e^{-\frac{K^2 u}{2t(1-u)}}}{\sqrt{u}} du + \frac{C\varepsilon}{\sqrt{t}}.$$

Similarly we bound equation (4.3.5)

$$\int_{(1-\varepsilon)t}^{t} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}} \mathbf{E}_x \left(e^{-\frac{sR_1^2}{2(t-s)}} \right) \mathrm{d}s$$
$$\leq \int_{1-\varepsilon}^{1} \int_{-x(1-u)}^{\infty} \frac{e^{-\frac{y^2}{2(1-u)}}}{(1-u)^{\frac{3}{2}}} y(y + x(1-u)) e^{-x^2 u/2} dy du$$
$$\leq \int_{1-\varepsilon}^{1} 1 + \sqrt{1-u} e^{-\frac{K^2(1-u)}{2tu}} du.$$

We finally obtain, for t large enough

$$\mathbf{E}(\widetilde{Z}_t(A, \varepsilon, K)) \tag{4.3.7}$$
$$\leq 2C\alpha \frac{\left(c + \frac{\log(t)}{\sqrt{2}}\right)}{\sqrt{t}} \int_0^{\varepsilon} \frac{e^{-\frac{K^2 u}{2t(1-u)}}}{\sqrt{u}} du + \frac{C\varepsilon}{\sqrt{t}} + \alpha C K \int_{1-\varepsilon}^{1} 1 + \sqrt{1-u} e^{-\frac{K^2(1-u)}{2tu}} du$$
$$\leq 2\alpha C \frac{\left(c + \frac{\log(t)}{\sqrt{2}}\right)}{\sqrt{t}} \sqrt{\varepsilon} + \frac{C\varepsilon}{\sqrt{t}} + 2\alpha C K \varepsilon,$$

letting $t \to \infty$ then $\varepsilon \to 0$ in (4.3.3) and (4.3.7) we conclude that

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} \mathbb{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t - A, T(u) \notin [\varepsilon t, (1-\varepsilon)t] \right) = 0,$$

which completes the proof. □

We now show that, with high probability, every particle of type 2 that contributes to the extremal process of the BBM satisfy $X_u(T(u)) - \sqrt{2}T(u) \notin [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]$.

Proposition 4.3.2. *Fix $A > 0$. We have*

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} \mathbb{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t - A, X_u(T(u)) - \sqrt{2}T(u) \notin [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}] \right) = 0$$

Proof. We write

$$\mathbb{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t - A, X_u(T(u)) - \sqrt{2}T(u) \notin [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}] \right)$$
$$\leq \mathbb{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t - A, T(u) \notin [\varepsilon t, (1-\varepsilon)t] \right)$$
$$+ \mathbb{P}\left(\exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t - A, T(u) \in [\varepsilon t, (1-\varepsilon)t], X_u(T(u)) - \sqrt{2}T(u) \notin [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}] \right). \tag{4.3.8}$$

Using Proposition 4.3.1 it is enough to estimate (4.3.8). We set, for $\varepsilon, A, K \geq 0$ and $t \geq 0$:

$$Y_t(A,\varepsilon,K) = \sum_{u\in\mathcal{B}} \mathbb{1}_{\left\{T(u)\in[\varepsilon t,(1-\varepsilon)t], X_u(T(u))-\sqrt{2}T(u)\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}]\right\}} \mathbb{1}_{\left\{X_u(r)\leq\sqrt{2}r+K, r\leq T(u)\right\}} \mathbb{1}_{\left\{M_t^u\geq m_t-A\right\}}.$$

By the Markov inequality, it is enough to estimate $\mathbf{E}(Y_t(A,\varepsilon,K))$. We have

$$\begin{aligned}
&\mathbf{E}(Y_t(A,B,K)) \\
&= \mathbf{E}\left(\sum_{u\in\mathcal{B}} \mathbb{1}_{\left\{T(u)\notin[\varepsilon t,(1-\varepsilon)t], X_u(T(u))-\sqrt{2}T(u)\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}]\right\}} \mathbb{1}_{\left\{X_u(r)\leq\sqrt{2}r+K, r\leq T(u)\right\}} F\left(t-T(u), X_u(T(u))\right)\right) \\
&= \alpha\int_{\varepsilon t}^{(1-\varepsilon)t} e^s \times \mathbf{E}\left(F(t-s,B_s)\mathbb{1}_{\left\{B_s-\sqrt{2}s\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}], B_r\leq\sqrt{2}r+K, r\leq s\right\}}\right)\mathrm{d}s \\
&= \alpha\int_{\varepsilon t}^{(1-\varepsilon)t} \mathbf{E}\left(e^{-\sqrt{2}B_s}F\left(t-s,B_s+\sqrt{2}s\right)\mathbb{1}_{\left\{B_s\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}], B_r\leq K, r\leq s\right\}}\right)\mathrm{d}s,
\end{aligned}$$

where we have set $F(r,x) = \mathbf{P}^{(2)}(x+M_r \geq m_t - A)$. Using Proposition 4.2.8 we have

$$\mathbf{E}(Y_t(A,\varepsilon,K)) \leq \alpha C\int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}}\mathbf{E}\left(e^{-\frac{B_s^2}{2(t-s)}}\left(c+\frac{\log(t)}{\sqrt{2}}+(-B_s)_+\right)\mathbb{1}_{\left\{B_s\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}], B_r\leq K, r\leq s\right\}}\right)\mathrm{d}s.$$

By (4.2.5), we get

$$\begin{aligned}
&\int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}}\mathbf{E}\left(e^{-\frac{B_s^2}{2(t-s)}}\left(c+\frac{\log(t)}{\sqrt{2}}+(-B_s)\right)\mathbb{1}_{\left\{B_s\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}], B_r\leq K, r\leq s\right\}}\right)\mathrm{d}s \\
&\leq (c+\frac{\log(t)}{\sqrt{2}})\int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{\frac{3}{2}}}\mathbf{E}_K\left(\frac{K}{R_s}\mathbb{1}_{\left\{R_s\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}]\right\}}\right)\mathrm{d}s + \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{\frac{3}{2}}}\mathbf{E}_K\left(\mathbb{1}_{\left\{R_s\notin[\varepsilon\sqrt{t},\frac{1}{\varepsilon}\sqrt{t}]\right\}}\right)\mathrm{d}s.
\end{aligned}$$

Using the change of variable $u = s/t$ and the Bessel scaling we have

$$\begin{aligned}
&\int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{\frac{3}{2}}}\mathbf{E}_K\left(\frac{K}{R_s}\mathbb{1}_{\left\{R_s\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}]\right\}}\right)\mathrm{d}s + \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{\frac{3}{2}}}\mathbf{E}_K\left(\mathbb{1}_{\left\{R_s\notin[\varepsilon\sqrt{t},\frac{1}{\varepsilon}\sqrt{t}]\right\}}\right)\mathrm{d}s \\
&\leq \frac{1}{\sqrt{t}}\int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{u}(1-u)^{3/2}}\mathbf{E}_{K/\sqrt{tu}}\left(\frac{K}{R_1}e^{-\frac{R_1^2 u}{2(1-u)}}\mathbb{1}_{\left\{R_1\notin[\varepsilon,\frac{1}{\varepsilon}]\right\}}\right)\mathrm{d}u \qquad (4.3.9) \\
&+ 2\int_{\varepsilon}^{1-\varepsilon} \frac{1}{(1-u)^{3/2}}\mathbf{E}_{K/\sqrt{tu}}\left(e^{-\frac{R_1^2 u}{2(1-u)}}\mathbb{1}_{\left\{R_1\notin[\varepsilon,\frac{1}{\varepsilon}]\right\}}\right)\mathrm{d}u.
\end{aligned}$$

We split the expectation into two parts

$$\mathbf{E}_x\left(\frac{1}{R_1}e^{-\frac{sR_1^2}{2(t-s)}}\mathbb{1}_{\left\{R_1\notin[\varepsilon,\frac{1}{\varepsilon}]\right\}}\right) = \mathbf{E}_x\left(\frac{1}{R_1}e^{-\frac{sR_1^2}{2(t-s)}}\mathbb{1}_{\{R_1\leq\varepsilon\}}\right) + \mathbf{E}_x\left(\frac{1}{R_1}e^{-\frac{sR_1^2}{2(t-s)}}\mathbb{1}_{\left\{R_1\geq\frac{1}{\varepsilon}\right\}}\right)$$

and we will deal with the two quantities in the same way. Using that $1 - e^{-2xy} \leq 2xy$, for $x = \frac{K}{\sqrt{tu}}$, $y > 0$ and t large enough we have

$$\mathbf{E}_x\left(\frac{1}{R_1}e^{-\frac{sR_1^2}{2(t-s)}}\mathbb{1}_{\{R_1\leq\varepsilon\}}\right) = \frac{1}{\sqrt{2\pi}}\int_0^{\varepsilon}\frac{1}{x}e^{-\frac{uy^2}{2(1-u)}}e^{-(y-x)^2/2}(1-e^{-2xy})dy \quad \leq \frac{2}{\sqrt{2\pi}}\int_0^{\varepsilon} ye^{-\frac{uy^2}{2(1-u)}}e^{-y^2/2}dy.$$

Then with the change of variable $v = y\sqrt{\frac{u}{1-u}}$, we obtain

$$\int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{u}(1-u)^{3/2}} e^{-\frac{uy^2}{2(1-u)}} e^{yx} du$$
$$\leq \int_{\sqrt{\frac{y\varepsilon}{1-\varepsilon}}}^{y\sqrt{\frac{1-\varepsilon}{\varepsilon}}} e^{-v^2/2} (\frac{v^2}{v^2+y^2})^{-1/2} (\frac{y^2}{v^2+y^2})^{-3/2} \frac{2vy^2}{(v^2+y^2)^2} dv = 2\int_{y\sqrt{\frac{\varepsilon}{1-\varepsilon}}}^{y\sqrt{\frac{1-\varepsilon}{\varepsilon}}} \frac{e^{-v^2/2}}{y} dv \leq \sqrt{2\pi}/y.$$

As a result, using Fubini's theorem in (4.3.9) we obtain the following upper bound

$$\int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{u}(1-u)^{3/2}} \mathbf{E}_{K/\sqrt{tu}} \left(\frac{K}{R_1} e^{-\frac{R_1^2 u}{2(1-u)}} \mathbb{1}_{\left\{R_1 \notin [\varepsilon, \frac{1}{\varepsilon}]\right\}} \right) \mathrm{d}u \leq C(\int_0^{\varepsilon} e^{-y^2/2} dy + \int_{1/\varepsilon}^{\infty} e^{-y^2/2} dy).$$

We similarly bound

$$\int_{\varepsilon}^{1-\varepsilon} \frac{1}{(1-u)^{3/2}} \mathbf{E}_{K/\sqrt{tu}} \left(e^{-\frac{R_1^2 u}{2(1-u)}} \mathbb{1}_{\left\{R_1 \notin [\varepsilon, \frac{1}{\varepsilon}]\right\}} \right) \mathrm{d}u \leq C(\int_0^{\varepsilon} e^{-y^2/2} dy + \int_{1/\varepsilon}^{\infty} e^{-y^2/2} dy).$$

As a result we obtain

$$\mathbf{E}(Y_t(A, \varepsilon, K)) \leq \alpha C \int_{\varepsilon}^{1-\varepsilon} \frac{\sqrt{t+1}}{(t-s+1)^{\frac{3}{2}}} \mathbf{E}\left(e^{-\frac{B_s^2}{2(t-s)}} \left(c + \frac{\log(t)}{\sqrt{2}} + (-B_s)_+ \right) \mathbb{1}_{\left\{B_s \notin [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}], B_r \leq K, r \leq s\right\}} \right) \mathrm{d}s \tag{4.3.10}$$
$$\leq C(\int_0^{\varepsilon} e^{-y^2/2} dy + \int_{1/\varepsilon}^{\infty} e^{-y^2/2} dy) \left(\frac{1}{\sqrt{t}} (c + \frac{\log(t)}{\sqrt{2}}) + 1 \right).$$

Letting $t \to \infty$ then $\varepsilon \to 0$, we conclude the proof. □

We now turn to the proof of the main theorem.

Proof of Theorem 4.1.1. Let $\varepsilon > 0$, we set

$$\mathcal{E}_t^{\varepsilon} := \sum_{u \in \mathcal{B}} \mathbb{1}_{\{T(u) \in [\varepsilon t, (1-\varepsilon)t]\}} \mathbb{1}_{\left\{X_u(T(u)) - \sqrt{2}T(u) \in [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]\right\}} \sum_{\substack{u' \in \mathcal{N}_t^2 \\ u' \succcurlyeq u}} \delta_{X_{u'}(t) - m_t}.$$

Let $\varphi \in \mathcal{T}$, we assume the support of φ is contained in $[-A, \infty)$ for some $A > 0$. We set

$$\mathcal{G}_t(\varepsilon) = \left\{ \exists u \in \mathcal{N}_t^2 : X_u(t) \geq m_t - A, T(u) \in [\varepsilon t, (1-\varepsilon)t], X_u(T(u)) - \sqrt{2}T(u) \in [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}] \right\}.$$

By Propositions 4.3.1 and 4.3.2 we have $\limsup_{t\to\infty} \mathbf{P}(\mathcal{G}_t(\varepsilon)^c) \to 0$ as $\varepsilon \to 0$, furthermore we have

$$\left| \mathbf{E}\left(e^{-\langle \mathcal{E}_t, \varphi \rangle}\right) - \mathbf{E}\left(e^{-\langle \mathcal{E}_t^{\varepsilon}, \varphi \rangle}\right) \right| \leq \mathbf{P}(\mathcal{G}_t(\varepsilon)^c),$$

therefore it is enough to compute the asymptotic behaviour of $\mathbf{E}\left(e^{-\langle \mathcal{E}_t^{\varepsilon}, \varphi \rangle}\right)$.

Using (4.2.2), for all $\varphi \in \mathcal{T}$, we have

$$\mathbf{E}\left(e^{-\langle \mathcal{E}_t^{\varepsilon}, \varphi \rangle}\right) = \mathbf{E}\left(\exp\left(-\alpha \int_{\varepsilon t}^{(1-\varepsilon)t} \sum_{u \in \mathcal{N}_s} \mathbb{1}_{\left\{|X_u(s) - s| \in [-\frac{1}{\varepsilon}\sqrt{t}, -\varepsilon\sqrt{t}]\right\}} F_{\varphi}(t-s, X_u(s) - \sqrt{2}s) \mathrm{d}s \right) \right),$$

with $F_\varphi(r,x) = 1 - \mathbf{E}^{(2)}\left(e^{-\sum_{u\in\mathcal{N}_r^2}\varphi(X_u(r)-m_r-x)}\right)$. Additionally, by Corollary 4.2.9, we have

$$F_\varphi(r,x) = \gamma(\varphi)\frac{\sqrt{t}(-x)e^{\sqrt{2}x}}{r^{\frac{3}{2}}}e^{-x^2/2r}(1+o(1))$$

as $r\to\infty$, uniformly in $t-r\in[\varepsilon t,(1-\varepsilon)t]$ and $x\in[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}]$.

As a result, recalling that $\gamma(\varphi) = \alpha C^*\sqrt{2}\int e^{-\sqrt{2}z}\left(1-\mathbf{E}(e^{-\langle\mathcal{D},\varphi(.+z)\rangle})\right)\mathrm{d}z$ using the notation of Corollary 4.2.9, we have

$$\limsup_{t\to\infty}\mathbf{E}\left(e^{-\langle\mathcal{E}_t^\varepsilon,\varphi\rangle}\right) \leq \limsup_{t\to\infty}\mathbf{E}\left(\exp\left(-\gamma(\varphi)\int_{\varepsilon t}^{(1-\varepsilon)t}\frac{\sqrt{t}}{(t-s)^{3/2}}\widetilde{Z}_s^\varepsilon\mathrm{d}s\right)\right) \tag{4.3.11}$$

and

$$\liminf_{t\to\infty}\mathbf{E}\left(e^{-\langle\mathcal{E}_t^\varepsilon,\varphi\rangle}\right) \geq \liminf_{t\to\infty}\mathbf{E}\left(\exp\left(-\gamma(\varphi)\int_{\varepsilon t}^{(1-\varepsilon)t}\frac{\sqrt{t}}{(t-s)^{3/2}}\widetilde{Z}_s^\varepsilon\mathrm{d}s\right)\right) \tag{4.3.12}$$

where

$$\widetilde{Z}_s^\varepsilon = \sum_{u\in\mathcal{N}_s}(\sqrt{2}s-X_u(s))e^{\sqrt{2}(X_u(s)-\sqrt{2}s)}e^{-\frac{(\sqrt{2}s-X_u(s))^2}{2(t-s)}}\mathbb{1}_{\left\{|X_u(s)-\sqrt{2}s|\notin[-\frac{1}{\varepsilon}\sqrt{t},-\varepsilon\sqrt{t}]\right\}}.$$

We set $\lambda t = s$, then we have

$$\mathbb{E}\left(\exp\left(-\gamma(\varphi)\int_{\varepsilon t}^{(1-\varepsilon)t}\frac{\sqrt{t}\widetilde{Z}_s^\varepsilon}{(t-s)^{\frac{3}{2}}}ds\right)\right) = \mathbb{E}\left(\exp\left(-\gamma(\varphi)\int_{\varepsilon}^{1-\varepsilon}\frac{\widetilde{Z}_{\lambda t}^\varepsilon}{(1-\lambda)^{\frac{3}{2}}}d\lambda\right)\right).$$

We now observe that by Theorem 1.2 in [Mad16], for all $\lambda\in[0,1]$ we have

$$\lim_{t\to\infty}\widetilde{Z}_{\lambda t}^\varepsilon = \mathbf{E}(h_{\lambda,\varepsilon}(R_1))Z_\infty$$

where $x\mapsto h_{\lambda,\varepsilon}(x) = e^{-\frac{\lambda}{2(1-\lambda)}x^2}\mathbb{1}_{\left\{\varepsilon/\sqrt{\lambda}<x\leq 1/(\varepsilon\sqrt{\lambda})\right\}}$, $(R_s)_{s\geq 0}$ is a 3-dimensional Bessel process and Z_∞ is the limit of the critical derivative martingale. As a result, writing

$$c(\varepsilon) = \int_\varepsilon^{1-\varepsilon}\frac{\mathbf{E}(h_{\lambda,\varepsilon}(R_1))}{(1-\lambda)^{3/2}}d\lambda,$$

by dominated convergence theorem, (4.3.11) and (4.3.12) yield

$$\lim_{t\to\infty}E\left(e^{-\langle\mathcal{E}_t^\varepsilon,\varphi\rangle}\right) = \mathbb{E}\left(\exp(-c(\varepsilon)\gamma(\varphi)Z_\infty)\right),$$

On the other hand, recall that the density of R_1 under $\mathbb{P}_0$ is

$$z\mapsto\sqrt{\frac{2}{\pi}}z^2e^{-z^2/2}\mathbb{1}_{\{z>0\}}.$$

Hence, using computations with respect to the density of R_1 and the monotone convergence theorem we obtain $\lim_{\varepsilon\to 0}\mathbf{E}(h_{\lambda,\varepsilon}(R_1)) = (1-\lambda)^{3/2}$, leading, using again dominated convergence theorem

$$\lim_{\varepsilon\to 0} c(\varepsilon) = 1.$$

Therefore, letting $t\to\infty$ we deduce

$$\lim_{t\to\infty} E\left(e^{-\langle \mathcal{E}_t^{\varepsilon},\varphi\rangle}\right) = \mathbb{E}\left(\exp(-\gamma(\varphi)Z_\infty)\right) = \mathbb{E}\left(\exp(-\alpha C^*\sqrt{2}Z_\infty \int e^{-\sqrt{2}z}\left(1-\mathbf{E}(e^{-\langle \mathcal{D},\varphi(.+z)\rangle})\right)\mathrm{d}z)\right),$$

which is the Laplace transform of a decorated PPP with intensity $\sqrt{2}\alpha C^* Z_\infty e^{-\sqrt{2}z}dz$. As a result using [BBCM20, Lemma 4.1], we complete the proof of Theorem 4.1.1.

□

Bibliographie

[ABB17] L-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Commun. Math. Phys.*, 349(2) :703–751, 2017.

[ABB$^+$19] L-P. Arguin, D. Belius, P. Bourgade, M. Radziwiłł, and K. Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Commun. Pure Appl. Math.*, 72(3) :500–535, 2019.

[ABBS13] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Relat. Fields*, 157(1-2) :405–451, 2013.

[ABH17] L-P. Arguin, D. Belius, and A-J. Harper. Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.*, 27(1) :178–215, 2017.

[ABK11] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Commun. Pure Appl. Math.*, 64(12) :1647–1676, 2011.

[ABK12] L-P. Arguin, A. Bovier, and N. Kistler. Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 22(4) :1693–1711, 2012.

[ABK13] L-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Relat. Fields*, 157(3-4) :535–574, 2013.

[Aid13] E. Aidékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, 41(3A) :1362–1426, 2013.

[AJ11] E. Aïdékon and B. Jaffuel. Survival of branching random walks with absorption. *Stochastic Processes Appl.*, 121(9) :1901–1937, 2011.

[AK71] K. Athreya and S. Karlin. On branching processes with random environments : I : Extinction probabilities. *The Annals of Mathematical Statistics*, 42(5) :1499–1520, 1971.

[AR09] L. Addario-Berry and B. Reed. Minima in branching random walks. *Ann. Probab.*, 37(3) :1044–1079, 2009.

[AW75] D. G. Aronson and H. F. Weinberger. Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation. Partial differ. Equat. relat. Top., Tulane Univ. 1974, Lect. Notes Math. 446, 5-49 (1975)., 1975.

[AW78] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. Math.*, 30 :33–76, 1978.

[AZ15] L-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. *Electron. J. Probab.*, 20 :19, 2015. Id/No 59.

[BBCM18] J. Berestycki, É. Brunet, A. Cortines, and B. Mallein. Extremes of branching ornstein-uhlenbeck processes, 2018.

[BBCM20] J. Berestycki, É. Brunet, A. Cortines, and B. Mallein. A simple backward construction of branching brownian motion with large displacement and applications. *arXiv :1810.05809*, oct 2020+.

[BBS13] J. Berestycki, N. Berestycki, and J. Schweinsberg. The genealogy of branching Brownian motion with absorption. *Ann. Probab.*, 41(2) :527–618, 2013.

[BC14] E. Bouin and V. Calvez. Travelling waves for the cane toads equation with bounded traits. *Nonlinearity*, 27(9) :2233–2253, 2014.

[BCM+12] E. Bouin, V. Calvez, N. Meunier, P. Mirrahimi, B. Perthame, G. Raoul, and R. Voituriez. Invasion fronts with variable motility : phenotype selection, spatial sorting and wave acceleration. *C. R., Math., Acad. Sci. Paris*, 350(15-16) :761–766, 2012.

[BDG01] E. Bolthausen, J-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4) :1670–1692, 2001.

[Bel21] M.A. Belloum. A generalized model interpolating between the random energy model and the branching random walk. *arXiv preprint arXiv :2101.09502*, 2021.

[BH48] R. Bellman and T.E. Harris. On the theory of age-dependent stochastic branching processes. *Proc. Natl. Acad. Sci. USA*, 34 :601–604, 1948.

[BH14] A. Bovier and L. Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19 :28, 2014. Id/No 18.

[BH15] A. Bovier and L. Hartung. Variable speed branching Brownian motion. I : Extremal processes in the weak correlation regime. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 12(1) :261–291, 2015.

[BH20] A. Bovier and L. Hartung. From 1 to 6 : a finer analysis of perturbed branching Brownian motion. *Commun. Pure Appl. Math.*, 73(7) :1490–1525, 2020.

[BHR17] E. Bouin, C. Henderson, and L. Ryzhik. Super-linear spreading in local and non-local cane toads equations. *J. Math. Pures Appl. (9)*, 108(5) :724–750, 2017.

[Bie45] I-J. Bienaymé. De la loi de multiplication et de la durée des familles. *Soc. Philomat. Paris Extraits, Sér*, 5(37-39) :4, 1845.

[Big76] J. D. Biggins. The first- and last-birth problems for a multitype age-dependent branching process. *Adv. Appl. Probab.*, 8 :446–459, 1976.

[Big92a] J. D. Biggins. Martingale convergence and large deviations in branching random walk. *Teor. Veroyatn. Primen.*, 37(2) :301–306, 1992.

[Big92b] J. D. Biggins. Uniform convergence of martingales in the branching random walk. *Ann. Probab.*, 20(1) :137–151, 1992.

[Big10] J.D. Biggins. Branching out. In *Probability and mathematical genetics. Papers in honour of Sir John Kingman*, pages 113–134. Cambridge : Cambridge University Press, 2010.

[Big12] J. D. Biggins. Spreading speeds in reducible multitype branching random walk. *Ann. Appl. Probab.*, 22(5) :1778–1821, 2012.

[BIM21] D. Buraczewski, A. Iksanov, and B. Mallein. On the derivative martingale in a branching random walk. *Ann. Probab.*, 49(3) :1164–1204, 2021.

[BJN21] J. Blath, D. Jacobi, and F. Nie. How the interplay of dormancy and selection affects the wave of advance of an advantageous gene. *arXiv preprint arXiv :2106.08655*, 2021.

[BKR13] K. Borovkov, D. Konstantin, and T. Rice. High host density favors greater virulence : a model of parasite–host dynamics based on multi-type branching processes. *Journal of mathematical biology*, 66(6) :1123–1153, 2013.

[BM19] D. Buraczewski and M. Maślanka. Large deviation estimates for branching random walks. *ESAIM : PS*, 23 :823–840, 2019.

[BM21] M.A. Belloum and B. Mallein. Anomalous spreading in reducible multitype branching brownian motion. *Electronic Journal of Probability*, 26 :1–39, 2021.

[BMG15] N. Berestycki, C. Mouhot, and R. Gaël. Existence of self-accelerating fronts for a non-local reaction-diffusion equations. *arXiv preprint arXiv :1512.00903*, 2015.

[BNT92] M. Bramson, P. Ney, and J. Tao. The population composition of a multitype branching random walk. *Ann. Appl. Probab.*, 2(3) :575–596, 1992.

[Bor17] A.A. Borovkov. Generalization and refinement of the integro-local stone theorem for sums of random vectors. *Theory of Probability & Its Applications*, 61(4) :590–612, 2017.

[BR05] J. D. Biggins and A. Rahimzadeh Sani. Convergence results on multitype, multivariate branching random walks. *Adv. Appl. Probab.*, 37(3) :681–705, 2005.

[Bra78] M. D. Bramson. Maximal displacement of branching Brownian motion. *Commun. Pure Appl. Math.*, 31 :531–581, 1978.

[Bra83] M. Bramson. *Convergence of solutions of the Kolmogorov equation to travelling waves*, volume 285. Providence, RI : American Mathematical Society (AMS), 1983.

[BS02] E. Bolthausen and A-S. Sznitman. *Ten lectures on random media*, volume 32. Basel : Birkhäuser, 2002.

[Bur09] D. Buraczewski. On tails of fixed points of the smoothing transform in the boundary case. *Stochastic Processes Appl.*, 119(11) :3955–3961, 2009.

[BZ12] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Commun. Pure Appl. Math.*, 65(1) :1–20, 2012.

[Car05] F. Caravenna. A local limit theorem for random walks conditioned to stay positive. *Probab. Theory Relat. Fields*, 133(4) :508–530, 2005.

[CHL19] A. Cortines, L. Hartung, and O. Louidor. The structure of extreme level sets in branching Brownian motion. *Ann. Probab.*, 47(4) :2257–2302, 2019.

[CM68] K. S. Crump and C. J. Mode. A general age-dependent branching process. I. *J. Math. Anal. Appl.*, 24 :494–508, 1968.

[CR88] B. Chauvin and A. Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Relat. Fields*, 80(2) :299–314, 1988.

[CZ20] N. Cook and O. Zeitouni. Maximum of the characteristic polynomial for a random permutation matrix. *Commun. Pure Appl. Math.*, 73(8) :1660–1731, 2020.

[Dav06] O. Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3) :962–986, 2006.

[DE07] E. Dyakonova and E. Evgen'evna. Critical multitype branching processes in a random environment. 2007.

[Der81] B. Derrida. Random-energy model : an exactly solvable model of disordered systems. *Phys. Rev. B (3)*, 24(5) :2613–2626, 1981.

[Der85] B. Derrida. A generalization of the random energy model which includes correlations between energies. *Journal de Physique Lettres*, 46(9) :401–407, 1985.

[DMS16] B. Derrida, B. Meerson, and P. V. Sasorov. Large-displacement statistics of the rightmost particle of the one-dimensional branching brownian motion. *Phys. Rev. E*, 93 :042139, Apr 2016.

[DRSV14] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas. Critical Gaussian multiplicative chaos : convergence of the derivative martingale. *Ann. Probab.*, 42(5) :1769–1808, 2014.

[DRZ17] J. Ding, R. Roy, and O. Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.*, 45(6A) :3886–3928, 2017.

[Dup10] B. Duplantier. Liouville quantum gravity & the KPZ relation : a rigorous perspective. In *XVIth international congress on mathematical physics, Prague, Czech Republic, August 3–8, 2009. With DVD*, pages 56–85. Hackensack, NJ : World Scientific, 2010.

[Dur15] R. Durrett. Branching process models of cancer. In *Branching process models of cancer*, pages 1–63. Springer, 2015.

[Fel71] W. Feller. An introduction to probability theory and its application vol ii, 1971.

[FHK12] Y. Fyodorov, G. Hiari, and J. Keating. Freezing transition, characteristic polynomials of random matrices, and the riemann zeta function. *Physical review letters*, 108(17) :170601, 2012.

[FK14] Y. Fyodorov and J. Keating. Freezing transitions and extreme values : random matrix theory, and disordered landscapes. *Philosophical Transactions of the Royal Society A : Mathematical, Physical and Engineering Sciences*, 372(2007) :20120503, 2014.

[Fri85] U. Frisch. On the singularity structure of fully developed turbulence. *Turbulence and predictability in geophysical fluid dynamics and climate dynamics*, 1985.

[FZ12a] M. Fang and O. Zeitouni. Branching random walks in time inhomogeneous environments. *Electron. J. Probab.*, 17 :18, 2012. Id/No 67.

[FZ12b] M. Fang and O. Zeitouni. Slowdown for time inhomogeneous branching Brownian motion. *J. Stat. Phys.*, 149(1) :1–9, 2012.

[GH18] N. Gantert and T. Höfelsauer. Large deviations for the maximum of a branching random walk. *Electron. Commun. Probab.*, 23 :12, 2018. Id/No 34.

[GKS18] C. Glenz, N. Kistler, and M.A. Schmidt. High points of branching Brownian motion and McKean's Martingale in the Bovier-Hartung extremal process. *Electron. Commun. Probab.*, 23 :12, 2018. Id/No 86.

[Ham74] J. M. Hammersley. Postulates for subadditive processes. *Ann. Probab.*, 2 :652–680, 1974.

[HHJV05] P. Haccou, P. Haccou, P. Jagers, and V.A. Vatutin. *Branching processes : variation, growth, and extinction of populations.* Number 5. Cambridge university press, 2005.

[HJV05] P. Haccou, P. Jagers, and V.A. Vatutin. *Branching processes. Variation, growth, and extinction of populations.* Cambridge : Cambridge University Press, 2005.

[Hol14] M. Holzer. Anomalous spreading in a system of coupled Fisher-KPP equations. *Physica D*, 270 :1–10, 2014.

[Hol16] M. Holzer. A proof of anomalous invasion speeds in a system of coupled Fisher-KPP equations. *Discrete Contin. Dyn. Syst.*, 36(4) :2069–2084, 2016.

[HS09] Y. Hu and Z. Shi. Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Probab.*, 37(2) :742–789, 2009.

[Hu16] Y. Hu. How big is the minimum of a branching random walk ? *Ann. Inst. Henri Poincaré, Probab. Stat.*, 52(1) :233–260, 2016.

[Jag69] P. Jagers. A general stochastic model for population development. *Skand. Aktuarietidskr.*, 1969 :84–103, 1969.

[KA15] M. Kimmel and D.E. Axelrod. Branching processes with infinitely many types. In *Branching Processes in Biology*, pages 155–205. Springer, 2015.

[Kah85] J-P. Kahane. Sur le chaos multiplicatif. (On multiplicative chaos). *Ann. Sci. Math. Qué.*, 9 :105–150, 1985.

[Kal21] O. Kallenberg. *Foundations of modern probability. Volume I and II*, volume 99. Cham : Springer, 2021.

[Ken66] D. G. Kendall. Branching processes since 1873. *J. Lond. Math. Soc.*, 41 :385–406, 1966.

[Ken75] D.G. Kendall. The genealogy of genealogy branching processes before (and after) 1873. *Bull. Lond. Math. Soc.*, 7 :225–253, 1975.

[Kin75] J. F. C. Kingman. The first birth problem for an age-dependent branching process. *Ann. Probab.*, 3 :790–801, 1975.

[KMT76] J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV's and the sample DF. II. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 34 :33–58, 1976.

[Kol37] A.N. Kolmogorov. Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Moskow, Ser. Internat., Sec. A*, 1 :1–25, 1937.

[KP76] J.-P. Kahane and J. Peyriere. Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.*, 22 :131–145, 1976.

[KS66] H. Kesten and B. P. Stigum. A limit theorem for multidimensional Galton-Watson processes. *Ann. Math. Stat.*, 37 :1211–1223, 1966.

[KS00] J.P. Keating and N.C. Snaith. Random matrix theory and $\zeta(1/2+it)$. *Commun. Math. Phys.*, 214(1) :57–89, 2000.

[LL10] G-F. Lawler and V. Limic. *Random walk : A modern introduction*, volume 123. Cambridge : Cambridge University Press, 2010.

[LPP95] R. Lyons, R. Pemantle, and Y. Peres. Conceptual proofs of $L\log L$ criteria for mean behavior of branching processes. *Ann. Probab.*, 23(3) :1125–1138, 1995.

[LS87] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15 :1052–1061, 1987.

[Lyo97] R. Lyons. A simple path to Biggins' martingale convergence for branching random walk. In *Classical and modern branching processes. Proceedings of the IMA workshop, Minneapolis, MN, USA, June 13–17, 1994*, pages 217–221. New York, NY : Springer, 1997.

[Mad15] T. Madaule. Maximum of a log-correlated Gaussian field. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 51(4) :1369–1431, 2015.

[Mad16] T. Madaule. First order transition for the branching random walk at the critical parameter. *Stochastic Processes Appl.*, 126(2) :470–502, 2016.

[Mad17] T. Madaule. Convergence in law for the branching random walk seen from its tip. *J. Theor. Probab.*, 30(1) :27–63, 2017.

[Mai13] P. Maillard. A note on stable point processes occurring in branching Brownian motion. *Electron. Commun. Probab.*, 18 :9, 2013. Id/No 5.

[Mal15a] B. Mallein. Maximal displacement in a branching random walk through interfaces. *Electron. J. Probab.*, 20 :40, 2015. Id/No 68.

[Mal15b] B. Mallein. Maximal displacement in the d-dimensional branching Brownian motion. *Electron. Commun. Probab.*, 20 :12, 2015. Id/No 76.

[Mal15c] B. Mallein. Maximal displacement of a branching random walk in time-inhomogeneous environment. *Stochastic Processes Appl.*, 125(10) :3958–4019, 2015.

[Man74] B. Mandelbrot. Multiplications aléatoires iterées et distributions invariantes par moyenne ponderee aléatoire : quelques extensions. *C. R. Acad. Sci., Paris, Sér. A*, 278 :355–358, 1974.

[McK75] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Commun. Pure Appl. Math.*, 28 :323–331, 1975.

[MO20] M.Biskup, , and L. Oren. Conformal symmetries in the extremal process of two-dimensional discrete gaussian free field. *Communications in Mathematical Physics*, pages 1–61, 2020.

[Mod71] C. J. Mode. Multitype branching processes. Theory and applications. Modern Analytic and Computational Methods in Science and Mathematics. No. 34. New York : American Elsevier Publishing Comp., Inc. XX, 330 p. (1971)., 1971.

[MP19] P. Maillard and M. Pain. 1-stable fluctuations in branching Brownian motion at critical temperature. I : The derivative martingale. *Ann. Probab.*, 47(5) :2953–3002, 2019.

[MPV87] M. Mézard, G. Parisi, and M.A. Virasoro. Spin glass theory and beyond, volume 9 of. *World Scientific Lecture Notes in Physics*, 1987.

[MZ16] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 52(3) :1144–1160, 2016.

[Naj18] J. Najnudel. On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab. Theory Relat. Fields*, 172(1-2) :387–452, 2018.

[Nev86] J. Neveu. Arbres et processus de Galton-Watson. (Trees and Galton-Watson processes). *Ann. Inst. Henri Poincaré, Probab. Stat.*, 22 :199–207, 1986.

[Nev88] J. Neveu. Multiplicative martingales for spatial branching processes. Stochastic processes, Semin., Princeton/New Jersey 1987, Prog. Probab. Stat. 15, 223-242 (1988)., 1988.

[Pai18] M. Pain. The near-critical Gibbs measure of the branching random walk. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 54(3) :1622–1666, 2018.

[PZ18] E. Paquette and O. Zeitouni. The maximum of the CUE field. *Int. Math. Res. Not.*, 2018(16) :5028–5119, 2018.

[RV10] R. Robert and V. Vargas. Gaussian multiplicative chaos revisited. *Ann. Probab.*, 38(2) :605–631, 2010.

[RY14] Y-X. Ren and T. Yang. Multitype branching Brownian motion and traveling waves. *Adv. Appl. Probab.*, 46(1) :217–240, 2014.

[Sch76] V. Scheffer. Turbulence and Hausdorff dimension. Turbul. Navier Stokes Equat., Proc. Conf. Univ. Paris-Sud, Orsay 1975, Lect. Notes Math. 565, 174-183 (1976)., 1976.

[She75] D. Sherrington, and K. Scott. Solvable model of a spin-glass. *Physical review letters*, 35(26) :1792, 1975.

[Shi15] Z. Shi. *Branching random walks. École d'Été de Probabilités de Saint-Flour XLII – 2012.*, volume 2151. Cham : Springer, 2015.

[SK15] M.A. Schmidt and N. Kistler. From Derrida's random energy model to branching random walks : from 1 to 3. *Electron. Commun. Probab.*, 20 :12, 2015. Id/No 47.

[Spi60] F. Spitzer. A tauberian theorem and its probability interpretation. *Transactions of the American Mathematical Society*, 94(1) :150–169, 1960.

[Sto67] C. Stone. On local and ratio limit theorems. Proc. 5th Berkeley Sympos. math. Statist. Probab., Univ. Calif. 1965/1966, 2, Part 2, 217-224 (1967)., 1967.

[SV16] C. Smadi and V.A. Vatutin. Reduced two-type decomposable critical branching processes with possibly infinite variance. *Markov Process. Relat. Fields*, 22(2) :311–358, 2016.

[SZ15] E. Subag and O. Zeitouni. Freezing and decorated Poisson point processes. *Commun. Math. Phys.*, 337(1) :55–92, 2015.

[Tal06] M. Talagrand. The parisi formula. *Annals of mathematics*, pages 221–263, 2006.

[VD18] V. Vatutin and E. Dyakonova. Multitype branching processes in random environment : probability of survival for the critical case. *Theory of Probability & Its Applications*, 62(4) :506–521, 2018.

[Web15] C. Webb. The characteristic polynomial of a random unitary matrix and Gaussian multiplicative chaos – the L^2-phase. *Electron. J. Probab.*, 20 :21, 2015. Id/No 104.

[WG75] H.W. Watson and F. Galton. On the probability of the extinction of families. *The Journal of the Anthropological Institute of Great Britain and Ireland*, 4 :138–144, 1875.

[YY10] A. Yakovlev and N. Yanev. Limiting distributions for multitype branching processes. *Stochastic analysis and applications*, 28(6) :1040–1060, 2010.