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Théorèmes limites pour la méthode MLMC pour plusieurs modèles: processus exponentiel Lévy, EDS dirigée par un processus de Lévy à sauts purs et processus de diffusion avec une approximation antithétique

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"La pensée n'est qu'un éclair au milieu d'une longue nuit. Mais c'est cet éclair qui est tout."

Henri Poincaré

"Mathematics is a game played according to certain rules with meaningless marks on paper."

David Hilbert

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Résumé

Équipe Probabilité et Statistique

Laboratoire Analyse, Géométrie et Applications

Docteur en Mathématiques

Limit theorems for MLMC method for several models : exponential Lévy process, SDE driven by a pure jumps Lévy process and diffusion process with an antithetic approximation

by Thi Bao Trâm Ngô

Motivés par la méthode multilevel Monte Carlo (MLMC), introduite par Giles, 2008b permettant d'améliorer la vitesse de la méthode Monte Carlo classique, nous nous intéressons à développer des théorèmes limites autour de cette méthode dans des cadres différents. La thèse se compose de trois parties :

Dans la première partie, nous démontrons un théorème de la limite centrale sur la méthode MLMC pour le calcul des prix d'options de type vanille en finance lorsque l'actif sous-jacent est donné par un modèle exponentiel de Lévy. Pour prouver ce résultat, nous donnons un théorème limite fonctionnel sur le comportement asymptotique de la distribution de l'erreur du processus d'approximation entre deux niveaux consécutifs de la méthode MLMC. De plus, nous fournissons une analyse de la complexité de l'algorithme montrant que la méthode MLMC réduit efficacement le coût de calcul par rapport à une méthode classique de Monte Carlo et dans certains cas particuliers pour une précision ε donnée elle atteint la complexité optimale $O(\varepsilon^{-2})$ qui correspond à la méthode de Monte Carlo non biaisée. Nous illustrons la suprématie de la méthode MLMC sur les méthodes de Monte Carlo à travers des tests numériques pour un modèle exponentiel de CGMY.

Dans la deuxième partie, nous étudions le comportement asymptotique du processus d'erreur normalisé $u_{n,m}(X^n - X^{nm})$ où X^n et X^{nm} sont respectivement des approximations d'Euler avec des pas de temps $1/n$ et $1/nm$ d'une équation différentielle stochastique dirigée par un processus de Lévy à sauts purs. Dans cet article, nous prouvons que cette erreur de type multilevel converge vers un processus limite non trivial avec une vitesse de convergence $u_{n,m}$. Les résultats obtenus sont en continuité avec ceux de Jacod, 2004 établis pour l'erreur normalisée $u_n(X^n - X)$. Cependant, contrairement à Jacod, 2004, dans nos preuves, nous traitons le comportement de la loi jointe de m tableaux triangulaires dépendants. Formellement, lorsque m tend vers l'infini, nous récupérons les processus limites de Jacod, 2004.

Dans la dernière partie, nous introduisons l'estimateur MLMC antithétique pour une diffusion multi-dimensionnelle qui est une extension de la méthode MLMC antithétique originale introduite par Giles and Szpruch, 2014. Notre objectif est d'étudier le comportement asymptotique des erreurs faibles impliquées dans ce nouvel algorithme. Parmi les résultats obtenus, nous montrons que l'erreur entre d'une part la moyenne du schéma de Milstein sans l'aire de Lévy et sa version antithétique construits sur la grille fine et d'autre part l'approximation grossière converge en loi stablement avec une vitesse d'ordre 1. Nous montrons également que l'erreur entre le schéma de Milstein sans l'aire de Lévy et sa version antithétique converge en loi stablement avec une vitesse d'ordre 1/2. Plus précisément, nous avons un théorème de limite fonctionnelle sur le comportement asymptotique de la loi jointe de ces deux erreurs basé sur une approche par tableau triangulaire. Grâce à ce résultat, nous établissons un théorème central limite de type Lindeberg-Feller pour l'estimateur MLMC antithétique. Une analyse de la complexité de l'algorithme est effectuée.

Mots-clés: Schéma d'Euler, Schéma de Milstein, Méthodes Multilevel Monte Carlo (MLMC), processus de Lévy, équations différentielles stochastiques, modèle CGMY, processus exponentiel Lévy, théorèmes limites fonctionnels, théorèmes limites centraux.

ABSTRACT

Motivated by the multilevel Monte Carlo method introduced by Giles, 2008b to improve the rate of convergence by the Monte Carlo method, we are interested in developing limit theorems for different settings. The thesis consists of three parts:

For the first part, we prove a central limit theorem on the Multilevel Monte Carlo method for pricing vanilla type options when the underlying asset is given by an exponential Lévy model. To prove this result we give a functional limit theorem on the asymptotic behavior of the error distribution of the approximating process between two consecutive levels of the Multilevel Monte Carlo method. Moreover we provide an analysis of the time complexity and it turns out that the MLMC method reduces efficiently the time cost compared to a classical Monte Carlo method and in some particular cases for a given precision ε it reaches the optimal complexity $O(\varepsilon^{-2})$ so that it behaves like an unbiased Monte Carlo method. We illustrate the supremacy of the MLMC method over the Monte Carlo methods through numerical tests for pricing European call options under an exponential Lévy model where the Lévy process is given by the CGMY model that covers a general class of Lévy processes.

For the second part, we study the asymptotic behavior of the normalized error process $u_{n,m}(X^n - X^{nm})$ where X^n and X^{nm} are respectively Euler approximations with time steps $1/n$ and $1/nm$ of a given stochastic differential equation driven by a pure jump Lévy process. In this paper, we prove that this multilevel error process converges to some non-trivial limiting process with a sharp rate $u_{n,m}$. The obtained results extend those of Jacod, 2004 for the normalized error $u_n(X^n - X)$. For the multilevel error, the proofs of the current paper are challenging since unlike Jacod, 2004 we need to deal with m dependent triangular arrays instead of one. Formally, when letting m tends to infinity, we recover limit processes of Jacod, 2004.

For the last part, we introduce our antithetic MLMC estimator for a multi-dimensional diffusion which is an extended version of the original antithetic MLMC one introduced by Giles and Szpruch, 2014. Our aim is to study the asymptotic behavior of the weak errors involved in this new algorithm. Among the obtained results, we prove that the error between on the one hand the average of the Milstein scheme without Lévy area and its antithetic version build on the finer grid and on the other hand the coarse approximation stably converges in distribution with a rate of order 1. We also prove that the error between the Milstein scheme without Lévy area and its antithetic version stably converges in distribution with a rate of order 1/2. More precisely, we have a functional limit theorem on the asymptotic behavior of the joined distribution of these errors based on a triangular array approach. Thanks to this result, we establish a central limit theorem of Lindeberg-Feller type for the antithetic MLMC estimator. The time complexity of the algorithm is carried out.

Keywords: Euler scheme, Milstein scheme, Multilevel Monte Carlo methods, Lévy processes, stochastic differential equations, CGMY model, exponential Lévy, functional limit theorems, central limit theorems.

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Contents

List of Symbols

C	some generic constant that can chane from lines to lines
càdlàg	right continuous with left limits
$\mathcal{S}_m$	the set of all permutations of order m
$(\mathbb{R}^{d\times q})^{m\times n}$	the set of $d \times q$-block matrices of $m \times n$-matrices
$\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$	Skorokhod space
$\mathbb{1}_A$	the indicator function of the set A
$[x]$	the integer part of $x \in \mathbb{R}_+$
$\lvert . \rvert$	the ℓ_1-matrix norm
$\sim$	equivalent between measures
$\overset{L^p}{\to}$	the uniform convergence in L^p
$\overset{\mathbb{P}}{\to}$	the uniform convergence in $\mathbb{P}$
$\overset{\nu}{\to}$	the vague convergence
$\overset{\mathcal{L}}{\longrightarrow}$	the convergence in law
$\overset{\text{stable}}{\Rightarrow}$	the stable convergence in law
$\ll$	absolutely continuous

Dedicated to my family

Chapter 1

Introduction and main results

1.1 Introduction and motivation

In many applications, particularly for pricing of financial securities, the effective computation of the quantity $\mathbb{E}(\varphi(X_T))$, $T > 0$, where $(X_t)_{0\leq t\leq T}$ is some underlying asset price and φ is a given payoff function is of great interest within the last decades (see e.g. Kloeden and Platen, 1992 and Glasserman, 2003). For the one-dimensional setting, the computation of $\mathbb{E}(\varphi(X_T))$ can be done efficiently using Fourier transform methods or numerical methods for partial differential equations. However, for the high dimensional setting, the Monte Carlo methods remain the most competitive in practice for this aim. This method consists of two steps. First, we approximate the process $(X_t)_{0\leq t\leq T}$ by the discretization scheme $(X_t^n)_{0\leq t\leq T}$ with time step T/n. Then approximate $E(\varphi(X_T^n))$ by $\frac{1}{N}\sum_{k=1}^{n}\varphi(X_{T,k}^n)$, where $(X_{T,k}^n)_{1\leq k\leq N}$ is a sample of N independent copies of X_T^n. The Statistical Romberg (SR) method introduced by Kebaier, 2005 for the setting of discretization schemes for Brownian stochastic differential equations is a two-level Monte Carlo estimator. This method reduces efficiently the time complexity compared to the classical Monte Carlo method. It uses two Euler schemes $(X_t^n)_{0\leq t\leq T}$ and $(\hat{X}_t^{n^\beta})_{0\leq t\leq T}$ with time steps T/n and T/n^β, $\beta\in(0,1)$ and approximates $E(\varphi(X_T))$ by $\frac{1}{N_1}\sum_{k=1}^{N_1}\varphi(\hat{X}_{T,k}^{n^\beta}) + \frac{1}{N_2}\sum_{k=1}^{N_2}(\varphi(X_{T,k}^n) - \varphi(X_{T,k}^{n^\beta}))$ where the Brownian paths used for X_T^n and $X_T^{n^\beta}$ has to be independent of the Brownian paths used to simulate $\hat{X}_T^{n^\beta}$. Recently, an extension of the SR method introduced by Giles, 2008b (see also Heinrich, 2001 for an earlier variant of the computational concept) called multilevel Monte Carlo (MLMC) method reduces efficiently the time complexity in the context of discretization schemes for Brownian stochastic differential equations. Interesting numerical tests, comparing three methods (crude Monte Carlo, statistical Romberg and the multilevel Monte Carlo), were processed in Korn, Korn, and Kroisandt, 2010. Giles's approach used a root mean squared error (RMSE) for the optimization of the size of the sample paths in order to run the MLMC method. The study of the multilevel method and all related topics interest a wide international community, we refer to the webpage of Giles `https://people.maths.ox.ac.uk/gilesm/mlmc_community.html`, see also Giles, 2008a, Giles, Higham, and Mao, 2008, Creutzig, Dereich, and Müller-Gronbach, 2009, Dereich, 2011, Giles and Szpruch, 2013b, Hutzenthaler, Jentzen, and Kloeden, 2013, Lemaire and Pagès, 2017 and Giorgi, Lemaire, and Pagès, 2017 for related results. Many new schemes have been developped to improve the order of convergence using the MLMC combined with other method of variance reduction, such as improved MLMC with Milstein scheme of Giles, 2008a, the antithetic MLMC scheme of Giles and Szpruch, 2013a, the nested MLMC

Giorgi, Lemaire, and Pagès, 2020 or the coupling importance sampling and MLMC of Kebaier and Lelong, 2018 (see also the earlier results of Ben Alaya, Hajji, and Kebaier, 2016). In a general framework, whenever a discretization scheme is used, we can implement the multilevel algorithm. The MLMC method uses information from a sequence of computations with decreasing step sizes and approximates the quantity $\mathbb{E}(\varphi(X_T))$ by

$$\mathcal{Q}_n = \frac{1}{N_0}\sum_{k=1}^{N_0}\varphi(X^1_{T,k}) + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}(\varphi(X^{\ell,m^\ell}_{T,k}) - \varphi(X^{\ell,m^{\ell-1}}_{T,k})), \quad m \in \mathbb{N}\backslash\{0,1\},$$

where $\ell \in \{0,...,L\}$, with $L = \frac{\log n}{\log m}$ and $(X^{m^\ell}_t)_{0\le t\le T}$ denotes some discretization scheme with time step $m^{-\ell}T$. Concerning the first empirical mean, the processes $(X^1_{t,k})_{0\le t\le T}$, $1 \le k \le N_0$ are independent copies of $(X^1_t)_{0\le t\le T}$. Concerning the second one, for $\ell \in \{0,...,L\}$, the processes $(X^{\ell,m^\ell}_{t,k}, X^{\ell,m^{\ell-1}}_{t,k})_{0\le t\le T}$, $1 \le k \le N_\ell$ are independent copies of $(X^{m^\ell}_t, X^{m^{\ell-1}}_t)_{0\le t\le T}$. However, for fixed ℓ, the simulation of $(X^{\ell,m^\ell}_t)_{0\le t\le T}$ and $(X^{\ell,m^{\ell-1}}_t)_{0\le t\le T}$ have to be based on the same path. Therefore, it is important to study of the asymptotic behaviors of the MLMC type error $X^{\ell,m^\ell} - X^{\ell,m^{\ell-1}}$ as $\ell \to \infty$ which is also the main topic of this thesis.

The above objective can be reduced to the study of the error in general form $\boxed{X^{nm} - X^n}$ as $n \to \infty$ where X^{nm} and X^n stand for the discretization schemes with time steps T/nm and T/n built on the same path. Next, we recall some results on the convergence orders of strong and weak error of Euler and Milstein discretizations. Some results on the MLMC with theses discretization are also recalled.

1.1.1 Euler MLMC scheme

The Euler MLMC method approximates the quantity $\mathbb{E}(\varphi(X_T))$ by $\mathcal{Q}_n$ corresponding to the Euler discretization. Let us consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ and a process $(X_t)_{0\le t\le T}$ which is solution to the following classical type stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t f(X_{s-})dY_s, \quad t \in [0,T], \quad T > 0,$$

where $x_0 \in \mathbb{R}^d$, Y is a semimartingale and f is a function regular enough. First, we recall the Euler scheme (see, e.g., Kloeden and Platen, 1992 for more details on discretization schemes) and its analytical results on the order of strong and weak convergence. We divide the interval $[0,T]$ into n partitions with the same length and for $k \in \{0,\dots,n\}$, we denote $t_k = \frac{kT}{n}$. Now, for $s \in [0,T]$, we denote $\eta_n(s) = \lfloor \frac{ns}{T} \rfloor \frac{T}{n}$ and the continuous Euler scheme starting at x_0 is defined by

$$\forall 0 \le k \le n-1, \quad \forall t \in [t_k, t_{k+1}], \quad d\hat{X}^n_t = f(\hat{X}^n_{\eta_n(t)})dY_t.$$

Another version called the discrete Euler scheme is defined by the induction on the grids $(t_k)_k$: $\bar{X}^n_{t_{k+1}} = \bar{X}^n_{t_k} + f(\bar{X}^n_{t_k})(Y_{t_{k+1}} - Y_{t_k})$, $\bar{X}_0 = x_0$. The Euler scheme is a well-known method of approximation of solutions of stochastic differential equations (SDEs) which is sometimes called the Euler-Maruyama approximation (see Maruyama, 1955). Until now, there are many results concerning the precision of this approximation in case of equations driven by a drift and a Brownian motion. More recently, people are interested in the approximation of solutions of SDEs driven by a general Lévy process. To review the analytical results available in the literature until now of this scheme and the Euler MLMC scheme, we discuss on two types of model: SDE driven by multi-dimensional Brownian motion with drift and SDE driven by a Lévy process.

Stochastic differential equation driven by a multi-dimensional Brownian motion with drift: Here, Y in the equation above is continuous, more precisely, let Y be a multi-dimensional Brownian motion containing a drift. It is well-known that under the global Lipschitz condition

$$\exists C_T > 0, \quad s.t., \quad |f(x) - f(y)| \leq C_T |x - y|, \quad x, y \in \mathbb{R}^d$$

the Euler scheme satisfies the following property (see, e.g., Bouleau and Lepingle, 1995), for any $p \geq 1$,

$$\sup_{0\leq t\leq T} |X_t| \in L^p, \quad \sup_{0\leq t\leq T} |\hat{X}_t^n| \in L^p,$$
$$\mathbb{E}(\sup_{0\leq t\leq T} |\hat{X}_t^n - X_t|^p) \leq \frac{K_p(T)}{n^{p/2}}, \quad K_p(T) > 0.$$

Under the weaker condition where f is locally Lipschitz with linear growth, Jacod and Protter, 1998, Theorem 3.1 states that we have the uniform convergence in probability, namely the property

$$\sup_{0\leq t\leq T} |\hat{X}_t^n - X_t| \xrightarrow{\mathbb{P}} 0.$$

When computing an approximation of the expected value $\mathbb{E}(\varphi(X_T))$ for φ smooth enough, one problem is to evaluate the discretization error $\varepsilon_n = \mathbb{E}(\varphi(\hat{X}_T^n) - \varphi(X_T))$. For the case where Y is continuous with a coefficient function f smooth enough, Talay and Tubaro, 1990 and Bally and Talay, 1996 (see also Bally and Talay, 1995) proved that ε_n is of order $1/n$ and an expansion of ε_n as increasing powers of $1/n$ is even given. More precisely, they obtained the following results that can be found in Pagès, 2018, Theorem 7.8.

- Talay and Tubaro, 1990 proved that if $f \in \mathcal{C}^\infty$ bounded with bounded partial derivatives and the function $\varphi \in \mathcal{C}^\infty$ with partial derivatives having polynominal growth, then, there exists a sequence $(C_i)_{i\geq 1}$ of real numbers depending on T, f and φ such that for any order $R \in \mathbb{N}^*$, we have

$$\varepsilon_n = \frac{C_1}{n} + \frac{C_2}{n^2} + \ldots + \frac{C_R}{n^R} + O(\frac{1}{n^{R+1}}), \quad \text{as } n \to \infty.$$

- Bally and Talay, 1996 extended the same result for bounded Borel function φ, where the function $f \in \mathcal{C}^\infty$ with bounded partial derivatives and the diffusion coefficient is uniformly elliptic.

In the context of possibly degenerate diffusions, when φ satisfies $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^p + |y|^p)|x - y|$ for $C > 0$, $p \geq 0$, using the result of strong error, the weak error ε_n is bounded by $c/\sqrt{n}$ with c a positive constant. In a more general context, for possibly degenerate diffusion X with f globally Lipschitz and φ satisfying $\mathbb{P}(X_T \notin \mathcal{D}_\varphi) = 0$ where $\mathcal{D}_\varphi = \{x \in \mathbb{R}^d : \varphi \text{ is differentiable at } x\}$, Kebaier, 2005 proved that the rate of convergence of the discretization error ε_n can be $1/n^\alpha$ for any $\alpha \in [1/2, 1]$. All the above developments on the Euler scheme are very useful for studying the MLMC error. In fact, more recently, based on these analytical results, Ben Alaya and Kebaier, 2015 proved the following functional limit theorem for all $m \in 2, 3, ...$, as $n \to \infty$,

$$(Y, \sqrt{\frac{mn}{(m-1)T}}(X^{mn} - X^n)) \stackrel{stably}{\Rightarrow} (Y, U),$$

where

$$U_t = \frac{1}{\sqrt{2}} Z_t \int_0^t (Z_s)^{-1} f(X_s) f'(X_s) dW_s,$$

with W is a standard Brownian Motion independent of Y and $Z_t = 1 + \int_0^t f'(X_s) Z_s dY_s$. Afterward, they established a central limit theorem of Lindeberg-Feller type for the MLMC estimator.

Stochastic differential equation driven by a Lévy process To give a flavor in a very simple setting, we consider a one-dimensional Lévy process Y with characteristics (b, c, F) such that

$$\mathbb{E}(e^{\mathrm{i}uY_t}) = \exp\left\{t\left(\mathrm{i}ub - \frac{cu^2}{2} + \int_{\mathbb{R}} F(dx)(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\mathbb{1}_{|x|\le 1})\right)\right\}.$$

From the point of view of financial modelling, Lévy processes provide a class of models with jumps that is both sufficiently rich to reproduce empirical data and simple enough to do many computations analytically (see e.g. Cont and Tankov, 2004). For a concise treatment of Lévy processes, we refer to the textbooks Applebaum, 2009 and Sato, 1999. Again, one problem of computing an approximation of the expected value $\mathbb{E}(\varphi(X_T))$ for smooth enough functions φ, we need to evaluate the error $\varepsilon_n = \mathbb{E}(\varphi(\hat{X}_T^n)) - \mathbb{E}(\varphi(X_T))$. For Y a given Lévy process, Protter and Talay, 1997 and Jacod et al., 2005 proved that ε_n is of order $1/n$ and with some appropriate assumptions on the coefficient function f, they showed that the error ε_n can be expanded in successive powers of $1/n$. Concerning the asymptotic behavior of the MLMC type error for SDE driven by a Lévy process, there are some available results in litterature. When Y is Lévy process with non null continuous part that is Y has a charcteristic triplet (b, c, F) where $c \neq 0$, then for a cut-off sequence h_n such that $F(h_n, \infty) + F(-\infty, -h_n) \sim \frac{\theta}{t_n} \geq 0$ and $\lim_{n\to\infty} h_n/\sqrt{t_n} = 0$, $t_n = M^{-n}T$ with $M \in \{2, 3, ...\}$ is fixed, Dereich and Li, 2016 proved that

$$(Y, t_n^{-1/2}(X^{n+1} - X^n)) \stackrel{\mathcal{L}}{\longrightarrow} (Y, U),$$

with

$$U_t = \int_0^t f'(X_{s-})U_{s-}dY_s + c\Gamma \int_0^t f(X_{s-})f'(X_{s-})dW_s + \sum_{s\in(0,t]:\Delta Y_s \neq 0} \sqrt{c_s}\xi_s f(X_{s-})f'(X_{s-})\Delta Y_s,$$

where for any s, $\xi_s \sim \mathcal{N}(0, 1)$ independent of Y, $\Gamma^2 = \begin{cases} \frac{e^{-\theta}-1+\theta}{\theta^2}\left(1 - \frac{1}{M}\right), & \theta > 0 \\ \frac{1}{2}\left(1 - \frac{1}{M}\right), & \theta = 0 \end{cases}$, W is a standard Brownian motion independent of Y and (ξ_s), and (c_s) are independent positive marks well-defined as a function of a family of independent uniform random variable on $[0, 1]$ and exponential random variables with parameters θ and $(M-1)\theta$. Thanks to this limit theorem, Dereich and Li, 2016 obtained for this case a central limit theorem for MLMC estimator. When Y is a Lévy process without a continuous Gaussian part that is Y has a characteristic triplet $(b, 0, F)$, being inspired by Jacod, 2004, we are interested in proving limit theorems for MLMC type error. More precisely, Jacod, 2004 found normalizing sharp rates u_n such that the sequence $(u_n(X^n_{\eta_n(.)} - X_{\eta_n(.)}))_{n\geq 0}$ is tight where the rates of convergence u_n mainly depend on the behavior of the Lévy measure of Y around 0 as follows. For this aim, he introduced hypotheses with Blumenthal-Getoor index $\alpha \in (0, 2)$:

$$F(\beta, +\infty) + F(-\infty, -\beta) \leq \frac{C}{\beta^\alpha} \text{ for all } \beta \in (0, 1], \qquad (\mathbf{H}1\alpha)$$

and

$$\beta^\alpha F(\beta, +\infty) \to \theta_+ \text{ and } \beta^\alpha F(-\infty, -\beta) \to \theta_- \text{ as } \beta \to 0 \qquad (\mathbf{H}2\alpha)$$

for some constants θ_+, $\theta_- \geq 0$ satisfying $\theta_- + \theta_+ > 0$. Then, he considered five cases

- If ($\mathbf{H}2\alpha$) holds for $\alpha > 1$, F is either symmetric or non-symmetric, then $u_n = \left(\frac{n}{\log n}\right)^{1/\alpha}$.
- If ($\mathbf{H}2\alpha$) holds for $\alpha = 1$ and F is non-symmetric, then $u_n = \frac{n}{(\log n)^2}$.
- If ($\mathbf{H}2\alpha$) holds for $\alpha = 1$ and F is symmetric, then $u_n = \frac{n}{\log n}$.
- If ($\mathbf{H}1\alpha$) holds for $\alpha < 1$, F is symmetric and $b \neq 0$ or F is non-symmetric and $d := b - \int_{|x|\leq 1} xF(dx) \neq 0$, then $u_n = n$.
- If ($\mathbf{H}2\alpha$) holds for $\alpha < 1$ and F is symmetric and $b = 0$, then $u_n = \left(\frac{n}{\log n}\right)^{1/\alpha}$.

For each case, the limit process is non-trivial and well-defined. Inspired by Jacod, 2004, we succeeded to analyse the MLMC type error which is the main contribution of chapter 3 below.

1.1.2 Milstein MLMC scheme

Let $X := (X_t)_{0\leq t\leq T}$ be the process with values in $\mathbb{R}^d$, solution to

$$dX_t = f(X_t)dt + g(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d, \qquad (1.1.1)$$

where $W = (W^1, ..., W^q)$ is a q-dimensional Brownian motion on some given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with $(\mathcal{F}_t)_{t\geq 0}$ is the standard filtration, f and g are, respectively, $\mathbb{R}^d$ and $\mathbb{R}^{d\times q}$ valued functions regular enough. As seen for the Euler scheme, the convergence order of weak error can be adjusted, it seems to be interesting for many authors to improve the order of the strong error. Comparing to the Euler scheme, the Milstein scheme is built by using higher order expansion when we approximate our integrals. The speed of convergence of strong error is then improved to $1/n$ instead of $1/\sqrt{n}$ as for the Euler scheme (see, e.g. Kloeden and Platen, 1992). However, a weakness of the Milstein discretisation is that in multidimensional setting it generally requires the simulation of iterated Itô integrals known as Lévy areas, for which there is no known efficient method except in dimension 2 (see, e.g. Gaines and Lyons, 1994, Rydén and Wiktorsson, 2001, Wiktorsson, 2001). In what follows, we first recall the original scheme for this model and a modified version studied by Giles and Szpruch, 2013a to get rid of the part Lévy area.

The original Milstein scheme Milstein, 1974 has introduced a scheme that uses additionally the multiple Itô-integrals $\int_{t_k}^{t_{k+1}}(W_s^m - W_{t_k}^m)dW_s^j$ for $j, m \in \{1, \dots, q\}$. We split the interval $[0, T]$ into n partitions with the same length and for $k \in \{0, \dots, n\}$, we denote the uniform time step $\Delta t = \frac{T}{n}$ and $t_k = k\Delta t$. For $s \in [t_k, t_{k+1}]$ and $i \in \{1, \dots, d\}$, $j \in \{1, \dots, q\}$, we have

$$\begin{aligned} g_{ij}(X_s) &\simeq g_{ij}(X_{t_k}) + \sum_{\ell=1}^{d} \frac{\partial g_{ij}}{\partial x_\ell}(X_{t_k})(X_s^\ell - X_{t_k}^\ell) \\ &\simeq g_{ij}(X_{t_k}) + \sum_{\ell=1}^{d} \frac{\partial g_{ij}}{\partial x_\ell}(X_{t_k}) \sum_{m=1}^{q} g_{\ell m}(X_{t_k})(W_s^m - W_{t_k}^m). \end{aligned}$$

Then, we get

$$g(X_s)dW_s \simeq g(X_{t_k})dW_s + \sum_{j,m=1}^{q} \nabla g_{\bullet j}(X_{t_k}) g_{\bullet m}(X_{t_k})(W_s^m - W_{t_k}^m)dW_s^j$$

where for $j \in \{1, \dots, q\}$, $g_{\bullet j} = (g_{1j}, \dots, g_{dj})^\top \in \mathbb{R}^d$ and $\nabla g_{\bullet j} = (\nabla g_{1j}, \dots, \nabla g_{dj})^\top \in \mathbb{R}^{d\times d}$. The original Milstein scheme starting at x_0 can be rewritten in a compact form given by the following induction on $k \in \{0, \dots, n-1\}$

$$\begin{aligned} X_{t_{k+1}}^{\mathrm{Mil},n} = X_{t_k}^{\mathrm{Mil},n} &+ f(X_{t_k}^{\mathrm{Mil},n})\Delta t + g(X_{t_k}^{\mathrm{Mil},n})(W_{t_{k+1}} - W_{t_k}) \\ &+ \sum_{j,m=1}^{q} \nabla g_{\bullet j}(X_{t_k}^{\mathrm{Mil},n}) g_{\bullet m}(X_{t_k}^{\mathrm{Mil},n}) \int_{t_k}^{t_{k+1}} (W_s^m - W_{t_k}^m)dW_s^j. \end{aligned}$$

Milstein scheme without Lévy area (strong and weak error) By an integration by parts formula, the original scheme above can be rewritten as

$$\begin{aligned} &X_{t_{k+1}}^{\mathrm{Mil},n} = X_{t_k}^{\mathrm{Mil},n} + f(X_{t_k}^{\mathrm{Mil},n})\Delta t + g(X_{t_k}^{\mathrm{Mil},n})(W_{t_{k+1}} - W_{t_k}) \\ &+ \frac{1}{2}\sum_{j,m=1}^{q} \nabla g_{\bullet j}(X_{t_k}^{\mathrm{Mil},n}) g_{\bullet m}(X_{t_k}^{\mathrm{Mil},n})((W_{t_{k+1}}^j - W_{t_k}^j)(W_{t_{k+1}}^m - W_{t_k}^m) - \Omega_{jm}\Delta t - \mathcal{A}_{kjm}), \end{aligned}$$

where Ω is the correlation matrix for the driving Brownian paths and $\mathcal{A}_k \in \mathbb{R}^{q\times q}$ is the Lévy area defined by

$$\mathcal{A}_{kjm} = \int_{t_k}^{t_{k+1}} (W_s^j - W_{t_k}^j)dW_s^m - \int_{t_k}^{t_{k+1}} (W_s^m - W_{t_k}^m)dW_s^j, \qquad j, m \in \{1, \dots, q\}.$$

In some applications, the diffusion coefficient $g(x)$ has a commutativity property which gives $\nabla g_{ij}(x) g_{im}(x) = \nabla g_{im}(x) g_{ij}(x)$ for all $i \in \{1, \dots, d\}, j, m \in \{1, \dots, q\}$. In that case, because the Lévy areas are anti-symmetric (i.e., $A_{kjm} = -A_{kmj}$), it follows that $\nabla g_{ij}(X_{t_k}^{\mathrm{Mil},n}) g_{im}(X_{t_k}^{\mathrm{Mil},n}) A_{kjm} + \nabla g_{im}(X_{t_k}^{\mathrm{Mil},n}) g_{ij}(X_{t_k}^{\mathrm{Mil},n}) A_{kmj} = 0$ and therefore the terms involving the Lévy areas cancel and so it is not necessary to simulate them. However, this only happens in special cases. Let us introduce the so called Milstein scheme without Lévy area starting at x_0 defined by induction on the integer $k \in \{1, \dots, n-1\}$

$$X_{t_{k+1}}^n = X_{t_k}^n + f(X_{t_k}^n)\Delta t + g(X_{t_k}^n)(W_{t_{k+1}} - W_{t_k})$$

$$+\frac{1}{2}\sum_{j,m=1}^{q}\nabla g_{\bullet j}(X^n_{t_k})g_{\bullet m}(X^n_{t_k})((W^j_{t_{k+1}}-W^j_{t_k})(W^m_{t_{k+1}}-W^m_{t_k})-\Omega_{jm}\Delta t).$$

Clark and Cameron, 1980 proved for a particular SDE that it is impossible to achieve a better order of strong convergence than the Euler-Maruyama discretisation when using just the discrete increments of the underlying Brownian motion. In the two dimensional setting, Clark and Cameron also showed that the order of strong convergence of the Milstein scheme is 1 while the Milstein scheme without Lévy area is 1/2. The analysis was extended by Müller-Gronbach, 2002 to general SDEs. Suppose that $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^{d\times q})$ and there exists a constant L such that for any $x \in \mathbb{R}^d$ and for all $1 \le i \le d$ and $1 \le j, k, \ell \le q$,

$$|\frac{\partial f_i}{\partial x_\ell}(x)| \le L, \quad |\frac{\partial g_{ij}}{\partial x_\ell}(x)| \le L, \quad |\frac{\partial h_{ijk}}{\partial x_\ell}(x)| \le L,$$
$$|\frac{\partial^2 f_i}{\partial x_\ell \partial x_k}(x)| \le L, \quad |\frac{\partial^2 g_{ij}}{\partial x_\ell \partial x_k}(x)| \le L.$$

Then, for $p \ge 2$, there exists a constant K_p, independent of the time step, such that

$$\mathbb{E}(\sup_{0\le t\le T}|X^{\mathrm{Mil},n}_t|^p) \le K_p$$

and $X^{\mathrm{Mil},n}_t$ strongly converges to the solution of the SDE (1.1.1). Theses results remain correct for the Milstein scheme without Lévy area. The proof given in Müller-Gronbach, 2002, Lemma 2 page 137 follows the standard method of analysis in references such as Kloeden and Platen, 1992 and Milstein and Tretyakov, 2004, see also Giles and Szpruch, 2013a. It is proved that the strong error of the original Milstein scheme has order 1 that is $\mathbb{E}(\sup_{0\le t\le T}|X^{\mathrm{Mil},n}_t - X_t|^p) \le K_p\Delta t^p$ which is an improvement comparing to the Euler scheme. However, with the Milstein scheme without Lévy area, this order is the same as the Euler scheme $\mathbb{E}(\sup_{0\le t\le T}|X^n_t - X_t|^p) \le K_p\Delta t^{p/2}$. For similar settings as above presented in the part of Euler scheme, Talay and Tubaro, 1990 proved that the weak convergence order of the original Milstein scheme is 1. The efficiency of the MLMC method mainly depends on the strong convergence order of the discretisation. As a consequence if we use the standard MLMC method with the Milstein scheme without simulating the Lévy areas the complexity will remain the same as for Euler-Maruyama scheme. Nevertheless, Giles and Szpruch, 2013a showed that by constructing a suitable antithetic MLMC estimator one can neglect the Lévy areas and still obtain a multilevel correction estimator with a variance which decays at the same rate as the original Milstein estimator. For this Milstein MLMC method, there is not yet the analysis of the MLMC error and CLT for the MLMC estimator which is the main contribution of chapter 4.

For future research, to extend our results in chapter 4 for the setting of a SDE driven by a Lévy process, we would like to go through some interesting results on Milstein scheme for this model. For the case of jump-diffusion SDE, Xia, 2013 and Platen and Bruti-Liberati, 2010 gave some interesting analysis on the jump-adapted Milstein discrtisation. If the coefficient functions satisfy some Lipschitz conditions and linear growth bound condition, they proved that the order of strong convergence is 1 (see e.g. Xia, 2013, Theorem 3.3.1). There are also the works of Wang and Gan, 2013, Kumar and Sabanis, 2017 and Kumar, 2021. In addition, for the SDE driven by continuous semimartingales, Yan, 2005 studied the assymptotic error considering the Milstein scheme.

1.2 Outline of the thesis

The main objective of this thesis is to study the asymptotic behavior of the MLMC type error and then establish a central limit theorem (CLT). In general, CLTs illustrate how the choice of parameters affects the efficiency of the scheme and they are a central tool for tuning the parameters. Indeed, the appeal of a central limit theorem is that it provides the fair rate of convergence and gives the exact asymptotic variance. Moreover, it allows us to build an automatic algorithm where the sample size of each level is explicitly given without any precomputation procedure and yields a more accurate confidence interval. The thesis consists of four chapters. The first chapter is the introduction where we summarize at first some basic knowledges of the Euler and Milstein schemes for SDE driven by Lévy process, then some results on the MLMC method equiped with the two schemes and some useful tools to work with. The second chapter shows a central limit theorem for the MLMC method with a jump Lévy model or even with an exponential Lévy model and provides a detailed complexity analysis of the MLMC method for the case of the celebrated CGMY process. Next, in the third chapter, we consider a stochastic differential equation driven by a pure jump Lévy process and we follow the idea presented in Jacod, 2004 to get the rate of convergence of the MLMC type error. Finally, in chapter 4, we present an improvement in the speed of convergence of MLMC type error used with an antithetic Milstein scheme comparing to the one used with Euler scheme (see e.g. Jacod and Protter, 1998, Ben Alaya and Kebaier, 2015). The principal part of this thesis is deduced from the submitted papers Ben Alaya, Kebaier, and Ngô, 2021b, Ben Alaya, Kebaier, and Ngô, 2021a and Ben Alaya, Kebaier, and Ngô, 2020.

1.3 Main results

1.3.1 The multilevel Monte Carlo method for jump Lévy models: Central limit theorem

An important family of stochastic processes arising in many areas of applied probability is the class of Lévy processes. Generally such processes are not simulatable especially for those with infinite activity. In practice, it is common to approximate them by truncating the jumps at some cut-off size ε ($\varepsilon \searrow 0$) meaning an approximation obtained by neglecting jumps with absolute size smaller than ε. This procedure leads us to consider a simulatable compound Poisson process. We are interested on the effective computation of option price given by $\mathbb{E}F(L_T)$, $T > 0$, where the underlying asset price $(L_t)_{0\le t\le T}$ is a $\mathbb{R}^d$-valued pure jump Lévy process, $d \ge 1$ and $F : \mathbb{R}^d \mapsto \mathbb{R}$ is a given function. The aim of the current work is to develop a central limit theorem for the MLMC method for option pricing under exponential Lévy models. To do so, we first obtain a functional limit theorem for the error process $L^{m^{-j+1}} - L^{m^{-j}}$, $0 \le j \le J$, between two consecutive levels of the MLMC method, where m^{-j} (resp. m^{-j+1}) stands for the fine (resp. coarse) truncation size of the small jumps of L. By virtue of this latter result we prove a central limit theorem for the MLMC estimator

$$Q_{m^{-J}} = \frac{1}{N_0}\sum_{k=1}^{N_0} F(L^1_{T,k}) + \sum_{j=1}^{J}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(F(L^{j,m^{-j}}_{T,k}) - F(L^{j,m^{-j+1}}_{T,k})\right),$$

for a given $\mathcal{C}^1$ payoff function F and also for F with the following form

$$F(x) = f(e^{x_1}, \dots, e^{x_d}), \quad \text{for all } x = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

with $f : \mathbb{R}_+^d \to \mathbb{R}$ a given $\mathcal{C}^1$ Lipschitz continuous function, to cover the exponential Lévy model setting. The obtained central limit theorem provides a better confidence interval than the one provided by the RMSE approach. Unlike the RMSE approach our central limit theorem provides an explicit description of the choice of the sample sizes $(N_j)_{0\le j\le J}$ that does not need any pre-computation step. Moreover, we provide an optimization analysis of the time complexity of the MLMC method. It turns out that for Lévy processes with a Lévy measure ν having a density of the form $L(x)/|x|^{Y+1}$, where L is a positive slowly varying function, the optimal time complexity is given by $C_{\text{MLMC}} = O(\varepsilon^{-\frac{2}{(1-\frac{\eta}{2})(2-Y)}(2(Y-1)+(1-\frac{\eta}{2})(2-Y))})$ for small $\eta > 0$ and $Y \in (1,2)$ and $C_{\text{MLMC}} = O(\varepsilon^{-2})$ for $Y \in (0,1)$. This latter time complexity corresponds to the optimal one that the MLMC method can reach so that it behaves like an unbiased Monte Carlo estimator. We also illustrate the supremacy of the MLMC estimator over the classical Monte Carlo method for pricing European Call options for an exponential Lévy model driven by the CGMY process introduced by Carr, Geman, Madan and Yor Carr et al., 2002. This work in Ben Alaya, Kebaier, and Ngô, 2021b is accepted for publication as a book chapter in Application of Lévy processes (2021), Nova Science publishers.

1.3.2 The multilevel Monte Carlo Euler method for Lévy driven stochastic differential equations: Limit theorems

In this work, we study the quantity $\mathbb{E}(\varphi(X_T))$, $T > 0$, where the process $(X_t)_{0\le t\le T}$ is the solution of the Lévy driven stochastic differential equation

$$X_t = x_0 + \int_0^t f(X_{s-})dY_s, \qquad t \in [0,T],\ T > 0$$

with $f \in \mathcal{C}^3$ is regular enough and the Lévy process Y with characteristics $(b, 0, F)$ with F is an infinite measure. We recall the setting on the model and some notations of Jacod, 2004 to analyze the asymptotic behavior of the normalized error process $u_{n,m}(X^n - X^{nm})$, where X^n and X^{nm} are two consecutive Euler approximations and with $u_{n,m}$ must be a sharp rate going to infinity when $n \to \infty$. This means that this error process converges to some non-trivial limit process. Being motivated by Jacod's paper for the Euler scheme, we consider five cases corresponding to five different choices of our $u_{n,m}$ for MLMC scheme. Without loss of generality we can reduce ourselves to study the case where we have bounded jumps and coefficient f with compact support. Indeed, adapting the same arguments as in Proposition 2.4 in Jacod, 2004 to the multilevel error setting, we can easily recover our main results for non-bounded jumps and coefficient f without a compact support. By triangular array approach using the Kallenberg, 2002, Corollary 15.16, we found exactly the same limit process as Jacod's case except in our Case 1, we obtain different limit. However, in this special case, when letting m tend to infinity, we also recover Jacod's limit. Although the ideas seem natural, the proofs in our case were more challenging comparing to his case since we have to deal with triangular arrays without the i.i.d. property. This work represents the first foundation stone for proving generalized limit theorems for the MLMC method for stochastic differential equation driven by a pure jump Lévy process. The special technical tool used in this paper is a well-known trick, called the "subsequences principle" for weak convergence (see Jacod and Protter, 2012).

This work in Ben Alaya, Kebaier, and Ngô, 2021a was submitted.

1.3.3 The antithetic multilevel Monte Carlo Milstein method for SDE driven by a standard Brownian motion with drift: Central limit theorem

In this paper, we consider $(X_t)_{0\leq t\leq T}$ as a diffusion of the d-dimensional SDE driven by a q-dimensional Brownian motion $W=(W^1,\ldots,W^q)^\top$, $q\geq 1$

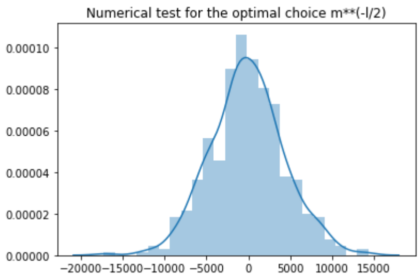
$$X_t = x_0 + \int_0^t f(X_s)ds + \int_0^t g(X_s)dW_s, \text{ for } t\in[0,T],\, T>0,$$

where $x_0\in\mathbb{R}^d$, $f\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^d)$ and $g\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R}^{d\times q})$ are regular enough. We give a natural extension of the antithetic multilevel Monte Carlo (MLMC) estimator for a multi-dimensional diffusion introduced by Giles and Szpruch, 2014 by considering the permutation between m Brownian increments, $m\geq 2$, instead of using two increments as in the original paper. Considering SDE driven by multidimensional Brownian motion with drift, Giles, 2008b showed that for the simple Euler discretisation with a Lipschitz payoff, $m=7$ gives twice the computational efficiency of $m=2$. Therefore, it is worth to extend the scheme for general m. Our aim is to establish a central limit theorem on this extended antithetic MLMC algorithm that is parametrized by a permutation $\boxed{\sigma(k)=m-k+1\in\mathcal{S}_m}$ corresponding to a reversal of time for each finer Brownian increment. To do so, based on a triangular array approach (see e.g. Jacod, 1997) and using "subsequences principle" for the stable convergence, we proved a functional limit theorem for the normalized error on two consecutive levels for the joined distribution of the couple

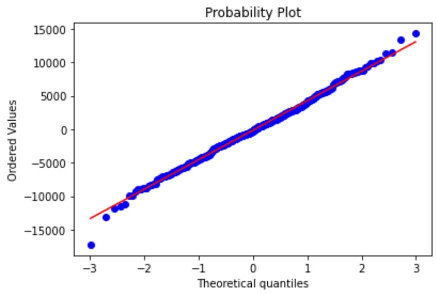
$$\left(\sqrt{n}(X^{nm}-X^{\sigma,nm}), n((X^{nm}+X^{\sigma,nm})/2-X^n)\right),$$

where X^{nm} denotes the Milstein scheme with time step T/mn without Lévy area and $X^{\sigma,nm}$ is its antithetic version. This result extends the stable convergence limit theorem obtained by Ben Alaya and Kebaier, 2015 for the normalized error on two consecutive levels $\sqrt{n}(\tilde{X}^{mn}-\tilde{X}^n)$ where $\tilde{X}^n$ denotes the Euler scheme with time step T/n. Thanks to this result, we establish a central limit theorem of Lindeberg-Feller type for the antithetic MLMC estimator. The time complexity of the algorithm is analyzed. By Cauchy-Schwarz, the minimum of the complexity C_{AMLMC} is reached for the choice of the weight $a^*_\ell = m^{-\ell/2}$, $\ell\in\{1,\ldots,L\}$. This optimal choice a^*_ℓ leads to the complexity $O(n^2)$ and the sample size $N_\ell = \frac{n^{2\alpha}}{m^{3\ell/2-2}}\left(1-\frac{1}{\sqrt{n}}\right)$. However, it does not satisfy our needed Lyapunov condition. Then, it seems natural to try to check experimentally if the central limit theorem is satisfied or not and we proceed to some numerical tests. In the setting of the Clark-Cameron model $d=2$, $q=2$, $f(x)=0_{\mathbb{R}^2}$ and $g(x)=\begin{pmatrix}1 & 0\\ 0 & x_1\end{pmatrix}$ for any $x=(x_1,x_2)\in\mathbb{R}^2$ and with $x_0=(100,100)\in\mathbb{R}^2$, we consider the European call option with payoff $\varphi(x)=(x_1+x_2-200)_+$. In Figure 1 we plot at the left the data histogram of 500 samples of $\hat{\mathcal{Q}}^2_n$ correctly renormalized and at the right we proceed to the quantile-quantile test where the horizontal axis means quantiles of a standard normal distribution and the vertical axis indicates the empirical quantiles of the same data. According to these numerical tests, the central limit theorem seems to be true despite the lack of theoretical proof.

This work in Ben Alaya, Kebaier, and Ngô, 2020 is under minor revisions for the journal Annals of Applied Probability.



(A) Histogram



(B) Q-Q plot

FIGURE 1.1: Numerical tests for the optimal choice $a_\ell^* = 4^{-\ell/2}$

1.4 Perspectives

In this thesis, we obtained several types of results motivated by the MLMC method as shown in the three last chapters $2, 3$ and 4. This PhD period gave me a lot of experiences and knowledges so that I can study further. I am willing to find a new oppotunity for my further researches on multilevel Monte Carlo method, variance reduction methods, stochastic models, stochastic analysis, Malliavin Calculus and related problems. After this thesis, I will continue with my supervisors to improve the results obtained particularly on four types of problem:

- Evaluating the weak error $\varepsilon_n = \mathbb{E}(\varphi(X_T^n)) - \mathbb{E}(\varphi(X_T))$ where X^n denotes some discretization scheme with time step T/n
- Improving the rate of weak convergence of the error process and studying the methods of variance reduction.
- Proving a CLT on the MLMC method when in the context of SDEs driven by pure jump Lévy process
- To have a hint on the rate of convergence of the MLMC method in this context, we will explore numerically the MLMC error which will be helpful to tackle the theoretical study of it.

More precisely, corresponding to the last chapter, the studies on the weak error for the case of our new method, antithetic MLMC, need to be taken into account. For instance, to compute an approximation of the expected value $\mathbb{E}(\varphi(X_T))$, we gave some hypostheses on this weak error. Corresponding to the third chapter, we save for the future research the study of the sharp rate for the rest case (C6) where $(\mathbf{H}_1^\alpha)$ satisfies with $\alpha < 1$, $d = 0$ and $(\mathbf{H}_3)$ does not hold.

Chapter 2

The Multilevel Monte Carlo Method for jump Lévy Models: Central Limit Theorem

In this chapter, we prove a central limit theorem on the Multilevel Monte Carlo method for pricing vanilla type options when the underlying asset is given by an exponential Lévy model. To prove this result we give a functional limit theorem on the asymptotic behavior of the error distribution of the approximating process between two consecutive levels of the Multilevel Monte Carlo method. Moreover we provide an analysis of the time complexity and it turns out that the MLMC method reduces efficiently the time cost compared to a classical Monte Carlo method and in some particular cases for a given precision ε it reaches the optimal complexity $O(\varepsilon^{-2})$ so that it behaves like an unbiased Monte Carlo method. We illustrate the supremacy of the MLMC method over the Monte Carlo methods through numerical tests for pricing European call options under an exponential Lévy model where the Lévy process is given by the CGMY model that covers a general class of Lévy processes.

The original paper Ben Alaya, Kebaier, and Ngô, 2021b of this work is accepted for publication as a book chapter in Application of Lévy processes (2021), Nova Science publishers.

2.1 Introduction

In recent decades, there has been a growing use of jump processes in financial applications since they are an effective excellent tool for pricing financial securities and modeling stock asset price. Indeed, it has been noted by experts in the field that asset prices do jump and that simple pure diffusion models were not able to emulate stylized facts of real financial markets such as the phenomenon of very steep implied volatility smile for short-dated option prices. In this work, we are interested on the effective computation of option price given by

$$\mathbb{E}F(L_T),\ T > 0, \tag{2.1.1}$$

where the underlying asset price $(L_t)_{0\leq t\leq T}$ is a $\mathbb{R}^d$-valued pure jump Lévy process, $d \geq 1$ and $F : \mathbb{R}^d \mapsto \mathbb{R}$ is a given function. One of the main features of such models is that they preserve the independence and stationarity properties of the log-returns of the jumping asset price. (see e. g. Cont and Tankov, 2006 and Schoutens, 2003). In the one-dimensional setting the computation of $\mathbb{E}F(L_T)$ can be done efficiently using Fourier transform methods (see e. g. Carr and Madan, 1999 and Fang and Oosterlee, 2008 or numerical methods for partial integral differential equations (see e.g. Cont and Voltchkova, 2005 and references therein). However, for the high dimensional setting, the Monte Carlo methods remain the most competitive in practice for this aim. In a recent work, Ben Alaya, Hajji, and Kebaier, 2016 used the Statistical Romberg (SR) method for pricing (2.1.1). The SR method introduced by Kebaier, 2005 for the setting of discretization schemes for Brownian stochastic differential equations is a two-level Monte Carlo estimator that reduces efficiently the time complexity compared to the classical Monte Carlo method. At a first glance, it seems quite unlikely that such a procedure with pure-jump Lévy processes would work, since the design of MLMC methods requires the use of a discretization scheme or at least an inner iterative routine that can be recycled from the finest level to crudest one. However, it is known in the literature (see e.g. Asmussen and Rosiński, 2001) that when the increments of the jump process cannot be simulated, L can be represented as a sum of a compound Poisson process and an almost sure limit of compensated compound Poisson process $L_t = \lim_{\varepsilon\to 0} L_t^\varepsilon$ $a.s.$ where for $0 < \varepsilon < 1$

$$L_t^\varepsilon = \gamma t + \sum_{0<s\le t} \Delta L_s \mathbb{1}_{|\Delta L_s|>1} + \Big(\sum_{0<s\le t} \Delta L_s \mathbb{1}_{\varepsilon\le|\Delta L_s|\le 1} - t\int_{\varepsilon\le|x|\le 1} x\nu(dx)\Big), \quad t\ge 0.$$

The error process $R^\varepsilon := L - L^\varepsilon$ is also a Lévy process independent of L^ε with characteristic function

$$\mathbb{E}e^{iu.R_t^\varepsilon} = \exp\left\{t\int_{|x|\le\varepsilon}(e^{iu.x} - 1 - iu.x)\nu(dx)\right\}.$$

This independence feature of the error process noticed by Ben Alaya, Hajji, and Kebaier, 2016 is the keystone on which we build the implementation of SR type methods, for this setting of pure jump processes (see Kebaier, 2017 for more details). The use of Multilevel Monte Carlo (MLMC) method, which is an extension of the SR method introduced by Giles, 2008b in the context of discretization schemes of Brownian stochastic differential equation that reduces efficiently the time complexity, in the setting of exponential Lévy models was also studied by Giles and Xia, 2017 using a root mean squared error (RMSE) approach for the optimization of the size of the sample paths in order to run the MLMC method. The aim of the current work is to develop a central limit theorem for the MLMC method for option pricing under exponential Lévy models. To do so, we first obtain a functional limit theorem for the error process $L^{m^{-j+1}} - L^{m^{-j}}$, $0\le j\le J$, between two consecutive levels of the MLMC method, where m^{-j} (resp. m^{-j+1}) stands for the fine (resp. coarse) truncation size of the small jumps of L (see Theorem 2.3.1). By virtue of this latter result we prove a central limit theorem for the MLMC estimator

$$Q_{m^{-J}} = \frac{1}{N_0}\sum_{k=1}^{N_0} F(L_{T,k}^1) + \sum_{j=1}^{J}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(F(L_{T,k}^{j,m^{-j}}) - F(L_{T,k}^{j,m^{-j+1}})\right),$$

for a given $\mathcal{C}^1$ payoff function F (see Theorem 2.3.7) and also for F with the following form

$$F(x) = f(e^{x_1},\dots,e^{x_d}), \quad \text{for all } x = (x_1,\dots,x_d)\in\mathbb{R}^d,$$

with $f:\mathbb{R}_+^d\to\mathbb{R}$ a given $\mathcal{C}^1$ Lipschitz continuous function, to cover the exponential Lévy model setting (see Corollary 2.3.8). The obtained central limit theorem provides a better confidence interval than the one provided by the RMSE approach. Moreover, unlike the RMSE approach our central limit theorem provides an explicit descriptions of the choice of the sample sizes $(N_j)_{0\le j\le J}$ that does not need any pre-computation step.

The rest of the paper is organized as follows. In Section 2.2, we introduce our general framework and some preliminary results. In Section 2.3, we give and prove our main results namely the functional limit theorem on the asymptotic behavior of the error distribution $L^{m^{-j+1}} - L^{m^{-j}}$, $0 \le j \le J$ and miscellaneous versions of the central limit theorem on the MLMC estimator $Q_{m^{-J}}$. Moreover, we provide an optimization analysis of the time complexity of the MLMC method. It turns out that for Lévy processes with a Lévy measure ν having a density of the form $L(x)/|x|^{Y+1}$, where L is a positive slowly varying function, the optimal time complexity is given by $C_{\text{MLMC}} = O(\varepsilon^{-\frac{2}{(1-\frac{\eta}{2})(2-Y)}(2(Y-1)+(1-\frac{\eta}{2})(2-Y))})$ for small $\eta > 0$ and $Y \in (1,2)$ and $C_{\text{MLMC}} = O(\varepsilon^{-2})$ for $Y \in (0,1)$. This latter time complexity corresponds to the optimal one that the MLMC method can reach so that it behaves like an unbiased Monte Carlo estimator. Let us recall that, to achieve the same precision ε, the optimal complexity for a classical Monte Carlo method is $C_{\text{MC}} = O(\varepsilon^{-\frac{4}{(2-\eta)(2-Y)}(Y+(1-\frac{\eta}{2})(2-Y))})$ which clearly has a larger order than the MLMC time complexities obtained in both cases $Y \in (0,1)$ and $Y \in (1,2)$. Section 2.4 is devoted to the numerical tests. More precisely, we illustrate the supremacy of the MLMC estimator over the classical Monte Carlo method for pricing European Call options for an exponential Lévy model driven by the CGMY process introduced by Carr et al., 2002. The Appendix section is devoted to recall several useful technical results.

2.2 General Framework and preliminary results

We consider a stochastic process $(L_t)_{t\ge 0}$ on a given probability space $(\Omega, \mathbb{F}, \mathbb{P})$ taking values in $\mathbb{R}^d$ such that $L_0 = 0$ and L has càdlàg sample paths. The process $(L_t)_{t\ge 0}$ is a Lévy process if it has independent and stationary increments. In what follows, we will consider the canonical filtration $(\mathcal{F}_t)_{0\le t\le T}$ where $\mathcal{F}_t = \sigma(L_s, s \le t)$. The characteristic function of a Lévy process L with generating triplet (γ, A, ν) is given by the well known Lévy Kintchine representation

$$\mathbb{E}e^{iu.L_t} = \exp\left\{t\left(i\gamma.u - \frac{1}{2}u.Au + \int_{\mathbb{R}^d}(e^{iu.x} - 1 - iu.x\mathbb{1}_{|x|\le 1})\nu(dx)\right)\right\}, \quad u \in \mathbb{R}^d,$$

where $\gamma \in \mathbb{R}^d$, A is a symmetric nonnegative-definite $d \times d$ matrix and ν is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ verifying $\int_{\mathbb{R}^d\setminus\{0\}}(|x|^2 \wedge 1)\nu(dx) < \infty$. (Given vectors x and $y \in \mathbb{R}^d$, $x.y$ denotes the inner product of x and y). From now on, as we are interested in studying pure jump Lévy processes we only consider Lévy processes $(L_t)_{t\ge 0}$ with generating triplet $(\gamma, 0, \nu)$. Let us recall that the simulation of a Lévy process with infinite Lévy measure can not generally be straightforward. From the Lévy-Itô decomposition (see for example Theorem 19.2 Sato, 1999), we know that L can be represented as a sum of a compound Poisson process and and an almost sure limit of compensated compound Poisson processes $L_t = \lim_{\varepsilon\to 0} L_t^\varepsilon$ *a.s.* where for $0 < \varepsilon < 1$

$$L_t^\varepsilon = \gamma t + \sum_{0<s\le t} \Delta L_s \mathbb{1}_{|\Delta L_s|>1} + \Big(\sum_{0<s\le t} \Delta L_s \mathbb{1}_{\varepsilon\le|\Delta L_s|\le 1} - t\int_{\varepsilon\le|x|\le 1} x\nu(dx)\Big), \quad t \ge 0. \tag{2.2.1}$$

Note that without the compensation $t\int_{\varepsilon\le|x|\le 1} x\nu(dx)$, the sum of jumps $\sum_{0<s\le t}\Delta L_s \mathbb{1}_{\varepsilon\le|\Delta L_s|\le 1}$ may not converge as ε goes to zero. We denote the approximation error by

$$R^\varepsilon = L - L^\varepsilon. \tag{2.2.2}$$

It is worth noticing that R^ε is also a Lévy process independent of L^ε with characteristic function

$$\mathbb{E}e^{iu.R^\varepsilon_t} = \exp\left\{t\int_{|x|\le\varepsilon}(e^{iu.x} - 1 - iu.x)\nu(dx)\right\}.$$

Therefore, $\mathbb{E}[R^\varepsilon_t] = 0$ and the variance-covariance matrix $\mathbb{E}[R^\varepsilon_t(R^\varepsilon_t)'] = t\Sigma_\varepsilon$, where

$$\Sigma_\varepsilon = \int_{|x|\le\varepsilon} xx'\nu(dx).$$

(Here, we denote by A' the transpose of a matrix A).

Let us recall that the asymptotic behavior of the distribution of R^ε is firstly studied by Asmussen and Rosiński, 2001 in the one dimensional case and later extended to the multidimensional case by Cohen and Rosiński, 2007 (See Theorem 2.2.1 below). In what follows, $W = (W_t)_{t\ge 0}$ denotes a standard Brownian motion in $\mathbb{R}^d$ independent of $(L_t)_{t\ge 0}$.

Theorem 2.2.1. *Under the above notation, suppose that Σ_ε is invertible for every $\varepsilon\in(0,1]$. Then as $\varepsilon\to 0$,*

$$\Sigma_\varepsilon^{-1/2}R^\varepsilon \Rightarrow W,$$

if and only if for each $k>0$

$$\lim_{\varepsilon\to 0}\int_{\langle\Sigma_\varepsilon^{-1}x,x\rangle>k}\langle\Sigma_\varepsilon^{-1}x,x\rangle\mathbb{1}_{|x|\le\varepsilon}\nu(dx) = 0. \tag{2.2.3}$$

Here "$\Rightarrow$" stands for the convergence in distribution.

Moreover, if ν is given in polar coordinates by

$$\nu(dr,du) = \mu(dr|u)\lambda(du), \quad r>0, u\in S^{d-1}, \tag{2.2.4}$$

where $\{\mu(\cdot|u) : u\in S^{d-1}\}$ is a measurable family of Lévy measures on $(0,\infty)$ and λ is a finite measure on the unit sphere S^{d-1} then

$$\Sigma_\varepsilon = \int_{S^{d-1}}\int_0^\varepsilon r^2uu'\mu(dr|u)\lambda(du).$$

If we define $\sigma^2(\varepsilon,u) := \int_0^\varepsilon r^2\mu(dr|u)$ and $\sigma^2(\varepsilon) := \int_{S^{d-1}}\sigma^2(\varepsilon,u)\lambda(du)$ then

$$\mathbb{E}|L_t - L^\varepsilon_t|^2 = t\mathrm{Tr}(\Sigma_\varepsilon) = t\sigma^2(\varepsilon). \tag{2.2.5}$$

Now, let us recall some relevant remarks and properties form Ben Alaya, Hajji, and Kebaier, 2016.

Remark 2.2.2. *In the one dimensional case Assmussen and Rosiński proved the convergence of $\sigma^{-1}(\varepsilon)R^\varepsilon$ to a standard Brownian motion if and only if for each $k>0$, $\sigma(k\sigma(\varepsilon)\wedge\varepsilon)\sim\sigma(\varepsilon)$ which is satisfied as soon as $\lim_{\varepsilon\to 0}\frac{\sigma(\varepsilon)}{\varepsilon} = \infty$ (see Theorem 2.1 and Proposition 2.1 in Asmussen and Rosiński, 2001). An extension to this sufficient condition in the multidimensional case is given by Theorem 2.5 in Cohen and Rosiński,* *2007. More precisely, if the support of the measure λ is not contained in any proper linear subspace of $\mathbb{R}^d$, they proved if*

$$\lim_{\varepsilon\to 0}\frac{\sigma(\varepsilon,u)}{\varepsilon}=\infty,\lambda-a.e. \tag{2.2.6}$$

then Σ_ε is invertible and condition (2.2.3) of Theorem 2.2.1 holds.

In addition, there is a finite L^q-upper bound of the error approximation in the one dimensional case for any real $q>0$ (see Proposition 2.1 of Dia, 2013). This latter property remains valid for the multidimensional case. Indeed, if we consider the d-dimensional error Lévy process R^ε defined by (2.2.2), then we can deduce that

$$\mathbb{E}|R_T^\varepsilon|^q\le K_{q,T}\bar{\sigma}(\varepsilon)^q, \tag{SE}$$

where $\bar{\sigma}(\varepsilon)=\sigma(\varepsilon)\vee\varepsilon$ and $K_{q,T}$ is a positive constant. For the weak error, if F denotes a real valued Lipschitz function with Lipschitz constant $C>0$, then it is easy to see that

$$|\mathbb{E}F(L_T)-\mathbb{E}F(L_T^\varepsilon)|\le C\sqrt{T}\sigma(\varepsilon) \tag{2.2.7}$$

Moreover, under some regularity conditions on the function F we can obtain an expansion of the weak error as in Proposition 2.2 and Remark 2.3 in Dia, 2013. So, it is worth introducing the following assumption: there are $C_F\in\mathbb{R}$ and $v_\varepsilon\searrow 0$ as $\varepsilon\searrow 0$ such that

$$v_\varepsilon^{-1}\left(\mathbb{E}F(L_T)-\mathbb{E}F(L_T^\varepsilon)\right)\to C_F. \tag{WE}$$

2.3 Main results

The idea of the Multilevel Monte Carlo method in the Lévy process setting is to apply the Monte Carlo method for a decreasing sequence of cut-off sizes $(m^{-j})_{1\le j\le J}$, m and $J\in\mathbb{N}\setminus\{0,1\}$, and to compute different numbers of paths on each cut-off size, from a few paths when the cut-off size is small to many paths when the cut-off size is large. More precisely, we will approximate the quantity $\mathbb{E}F(L_T)$ by

$$Q_{m^{-J}}=\frac{1}{N_0}\sum_{k=1}^{N_0}F(L_{T,k}^1)+\sum_{j=1}^{J}\frac{1}{N_j}\sum_{k=1}^{N_j}\left(F(L_{T,k}^{j,m^{-j}})-F(L_{T,k}^{j,m^{-j+1}})\right).$$

Here, for $j\in\{1,\cdots,J\}$, the processes $(L_{T,k}^{j,m^{-j}},L_{T,k}^{j,m^{-j+1}})_{k\in\{1,\cdots,N_j\}}$ are independent copies of $(L_T^{j,m^{-j}},L_T^{j,m^{-j+1}})$ whose components denote respectively the approximations with cut-off sizes m^{-j} and m^{-j+1}. However, for fixed j, the simulation of $(L_T^{j,m^{-j}})$ and $(L_T^{j,m^{-j+1}})$ has to be based on the same path. Concerning the first empirical mean, the processes $(L_{T,k}^1)_{k\in\{1,\cdots,N_0\}}$ are independent copies of L_T^1 which denotes the approximation with cut-off size equal to one. Here, it is important to point out that all these $J+1$ Monte Carlo estimators have to be based on different independent samples.

In order to study the asymptotic behavior of this estimator, we prove a convergence theorem for the cut-off approximation on two consecutive levels m^{-j} and m^{-j+1} of the type obtained by Asmussen and Rosiński, 2001 in the one dimensional case or more generally Cohen and Rosiński, 2007 in the multidimensional case. For more clarity and to make the paper self-contained, we adapt and rewrite the limit theorem of Cohen and Rosiński, 2007 to the MLMC setting.

2.3.1 A functional limit theorem

We have the following functional result.

Theorem 2.3.1. *Under the above notations, let* $\Sigma_{j,m} := Var(L_1^{m^{-j+1}} - L_1^{m^{-j}})$ *for all* $j \in \mathbb{N} \setminus \{0\}$ *and assume that* $\Sigma_{j,m}$ *is non singular for* j *sufficiently large. Then, the sequence* $\Sigma_{j,m}^{-1/2}(L^{m^{-j}} - L^{m^{-j+1}})$ *converges in distribution to a standard Brownian motion* W *if and only if for every* $\kappa > 0$

$$\lim_{j\to+\infty} \int_{\langle \Sigma_{j,m}^{-1}x,x\rangle>\kappa} \langle \Sigma_{j,m}^{-1}x, x\rangle \mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx) = 0. \tag{2.3.1}$$

Proof. According to the Lévy-Itô decomposition, the approximation error on two consecutive levels m^{-j} and m^{-j+1} is a Lévy process with generating triplet given by $(0, 0, \nu_{\{m^{-j}<|x|\le m^{-j+1}\}})$. Hence, by putting $Y_{j,m} = \Sigma_j^{-1/2}(L^{m^{-j}} - L^{m^{-j+1}})$ and using the push forward of $\nu_{\{m^{-j}<|x|\le m^{-j+1}\}}$ (the restriction of ν on the set $\{m^{-j} < |x| \le m^{-j+1}\}$) by the map $x \to \Sigma_{j,m}^{-1/2}x$, which is nothing but the measure $\nu_{j,m}$ defined by

$$\nu_{j,m}(B) := \nu(\Sigma_{j,m}^{1/2}B \cap \{m^{-j} < |x| \le m^{-j+1}\}), \quad B \in \mathcal{B}(\mathbb{R}^d),$$

it is easy to check that $Y_{j,m}$ is also a Lévy process with generating triplet $(\gamma_{j,m}, 0, \nu_{j,m})$ where

$$\gamma_{j,m} = -\int_{|x|>1} x\nu_{j,m}(dx) = -\Sigma_{j,m}^{-1/2} \int_{|\Sigma_{j,m}^{1/2}x|>1} x\mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx).$$

Since $Y_{j,m}$ is a Lévy process we have only to prove the convergence in distribution of $Y_{j,m}(1)$ to a standard normal distribution. Thanks to Theorem 15.14 in Kallenberg, 2002 (see Theorem 2.5.1 in Appendix) we have this convergence if and only if for $0 < h < 1$ fixed,

$$\begin{aligned}
&\int_{|x|\le h} xx^\top \nu_{j,m}(dx) \to I_d, \quad \gamma_{j,m} - \int_{h<|x|\le 1} x\nu_{j,m}(dx) \to 0 \\
&\text{and} \quad \nu_{j,m}(|x| > \kappa) \to 0, \quad \forall \kappa > 0
\end{aligned} \tag{2.3.2}$$

as $j \to +\infty$. We first notice that

$$\begin{aligned}
\int_{\mathbb{R}^d} xx^\top \nu_{j,m}(dx) &= \int_{\mathbb{R}^d} (\Sigma_{j,m}^{-1/2}x)(\Sigma_{j,m}^{-1/2}x)^\top \mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx) \\
&= \Sigma_{j,m}^{-1/2} \int_{\mathbb{R}^d} xx^\top \mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx)\Sigma_{j,m}^{-1/2} = I_d.
\end{aligned} \tag{2.3.3}$$

Now, concerning the first term in (2.3.2), using (2.3.3) and Cauchy-Schwarz inequality we have

$$\begin{aligned}
|I_d - \int_{|x|\le h} xx^\top \nu_{j,m}(dx)| &= |\int_{|\Sigma_{j,m}^{-1/2}x|>h} (\Sigma_{j,m}^{-1/2}x)(\Sigma_{j,m}^{-1/2}x)^\top \mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx)| \\
&\le \int_{|\Sigma_{j,m}^{-1/2}x|>h} |\Sigma_{j,m}^{-1/2}x|^2 \mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx) \\
&\le \int_{\langle\Sigma_{j,m}^{-1}x,x\rangle>h^2} \langle\Sigma_{j,m}^{-1}x, x\rangle \mathbb{1}_{\{m^{-j}<|x|\le m^{-j+1}\}}\nu(dx),
\end{aligned}$$

which converges to zero thanks to the condition (2.3.1) is satisfied. Now, it remains to prove the last two convergences in relation (2.3.2) provided that relation (2.3.1) is satisfied. At first, let $\kappa > 0$, it is easy to check that

$$\begin{aligned}\nu_{j,m}(|x| > \kappa) \leq & \frac{1}{\kappa^2}\int_{|x|>\kappa} |x|^2 \nu_{j,m}(dx) \\ \leq & \frac{1}{\kappa^2}\int_{|\Sigma_{j,m}^{-1/2}x|>\kappa} |\Sigma_{j,m}^{-1/2}x|^2 \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \\ \leq & \frac{1}{\kappa^2}\int_{\langle \Sigma_{j,m}^{-1}x,x\rangle>\kappa^2} \langle \Sigma_{j,m}^{-1}x, x\rangle \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \to 0,\end{aligned}$$

as $j \to +\infty$, since relation (2.3.1) is satisfied. For the last term, we have

$$\begin{aligned}|\gamma_{j,m} - \int_{h<|x|\leq 1} x\nu_{j,m}(dx)| = & |\int_{|x|>h} x\nu_{j,m}(dx)| \\ \leq & \int_{|\Sigma_{j,m}^{-1/2}x|>h} |\Sigma_{j,m}^{-1/2}x| \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \\ \leq & \int_{|\Sigma_{j,m}^{-1/2}x|>h} |\Sigma_{j,m}^{-1/2}x|^2 \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \\ \leq & \int_{\langle \Sigma_{j,m}^{-1}x,x\rangle>h^2} \langle \Sigma_{j,m}^{-1}x, x\rangle \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \to 0,\end{aligned}$$

as $j \to +\infty$. Therefore, we proved (2.3.1) implies the convergence in law. To prove the converse, thanks to Theorem 2.5.1, we have for any $\kappa > 0$, $\int_{|x|\leq\kappa} xx^\top \nu_{j,m}(dx) \to I_d$. Then by (2.3.3), we deduce the matrix $\int_{|x|>\kappa} xx^\top \nu_{j,m}(dx)$ vanishes as $j \to \infty$. To obtain condition (2.3.1), we write

$$\begin{aligned}& \int_{\langle \Sigma_{j,m}^{-1}x,x\rangle>\kappa} \langle \Sigma_{j,m}^{-1}x, x\rangle \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \\ = & \int_{|\Sigma_j^{-1/2}x|>\kappa^{1/2}} |\Sigma_{j,m}^{-1/2}x|^2 \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) \\ = & \int_{|x|>\kappa^{1/2}} |x|^2 \nu_{j,m}(dx) \\ = & \mathrm{Tr}(\int_{|x|>\kappa^{1/2}} xx^\top \nu_{j,m}(dx)) \to 0,\end{aligned}$$

as $j \to +\infty$. This completes the proof. □

The above result rewrites in the one-dimensional case as follows.

Corollary 2.3.2. *Let us consider the one-dimensional case corresponding to $d = 1$ and let $\sigma_{j,m}^2 := \mathrm{Tr}(\Sigma_{j,m})$ for all $j \in \mathbb{N}\setminus\{0\}$ and assume that $\sigma_{j,m} > 0$ for j sufficiently large. Then, the sequence $\sigma_{j,m}^{-1}(L^{m^{-j}} - L^{m^{-j+1}})$ converges in distribution to a standard Brownian motion W if and only if for each $\kappa > 0$*

$$\lim_{j\to+\infty} \sigma_{j,m}^{-2} \int_{|x|>\kappa^{1/2}\sigma_{j,m}} x^2 \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx) = 0. \tag{2.3.4}$$

Remark 2.3.3. *In the one dimensional case, by similar arguments of Asmussen and Rosiński, 2001, it is clear that the condition (2.3.4) is equivalent to for any* $\kappa > 0$,

$$\int_{\kappa^{1/2}\sigma_{j,m}\wedge m^{-j}}^{\kappa^{1/2}\sigma_{j,m}\wedge m^{-j+1}} x^2\nu(dx) \sim \sigma_{j,m}^2 \tag{2.3.5}$$

which is obviously satisfied as soon as

$$\lim_{j\to+\infty} \frac{\sigma_{j,m}}{m^{-j}} = +\infty. \tag{2.3.6}$$

The following theorem provides an extension to dimension $d \geq 2$ of the last sufficient condition in the above remark which ensures that the sequence $\Sigma_{j,m}^{-1/2}(L^{m^{-j}} - L^{m^{-j+1}})$ converges in distribution to a standard Brownian motion W. The proof follows the same steps as in the proof of Theorem 2.5 in Cohen and Rosiński, 2007.

Theorem 2.3.4. *Let* ν *be a Lévy measure on* $\mathbb{R}^d$ *given by (2.2.4) such that the support of* λ *is not contained in any proper linear subspace of* $\mathbb{R}^d$*. If*

$$\lim_{j\to\infty} m^{-2j} \int_{m^{-j}<r\leq m^{-j+1}} r^2\mu(dr|u) = \infty, \quad \lambda - a.e., \tag{2.3.7}$$

then $\Sigma_{j,m}$ *is non-singular and condition (2.3.1) of Theorem 2.3.1 holds.*

If we use twice the result of Dia, we get $\mathbb{E}|L_t^{m^{-j}} - L_t^{m^{-j+1}}|^q \leq K_{q,t}\sigma_0(m^{-j+1})$. Therefore, the Proposition below gives the multilevel-type bound for the moments of $L^{m^{-j}} - L^{m^{-j+1}}$. Its proof can be easily adapted from Proposition 2.1 of Dia, 2013.

Proposition 2.3.5. *Let* $\overline{\sigma}_{j,m} = \max(\sigma_{j,m}, m^{-j+1})$ *for any* $j \in \{1, \dots, J\}$*. Then for any real* $q > 0$ *and* $t \geq 0$

$$\mathbb{E}|L_t^{m^{-j}} - L_t^{m^{-j+1}}|^q \leq K_{q,t}\overline{\sigma}_{j,m}^q$$

where $K_{q,t}$ *is a positive constant depending only on* q *and* t.

Proof. Without loss of generality, we suppose $d = 1$. Let $n = \lceil q/2 \rceil$, then we have $0 < q/2n \leq 1$. From Jensen's inequality for concave function, $\mathbb{E}|L_t^{m^{-j}} - L_t^{m^{-j+1}}|^q \leq (\mathbb{E}|L_t^{m^{-j}} - L_t^{m^{-j+1}}|^{2n})^{q/2n}$ and now we prove instead that for any $n \in \mathbb{N}^*$,

$$|\mathbb{E}(L_t^{m^{-j}} - L_t^{m^{-j+1}})^n| \leq K_{n,t}\overline{\sigma}_{j,m}^n. \tag{2.3.8}$$

Let us proceed by induction, the relation is trivial when $n = 1$ or $n = 2$. We suppose that (2.3.8) holds for all $n < k$. Let $c_{t,k}^j$ denote the kth cumulant of $L_t^{m^{-j}} - L_t^{m^{-j+1}}$. We have $c_{t,1}^j = 0$ and $c_{t,k}^j = t\int_{m^{-j}\leq|x|\leq m^{-j+1}} x^k\nu(dx)$ for $k \geq 2$ (see e.g. Proposition 1.2 of Tankov, 2004). Then, by the well known relation between the moments (see e.g. Theorem 2 of Morris, 1983) we have for all $m \geq 2$,

$$|\mathbb{E}(L_t^{m^{-j}} - L_t^{m^{-j+1}})^k| \leq \sum_{n=0}^{k-2} \binom{k-1}{n} |\mathbb{E}(L_t^{m^{-j}} - L_t^{m^{-j+1}})^n||c_{t,k-n}^j|.$$

Now, for $k - n \leq 2$, we have

$$|c_{t,k-n}^j| \leq t\int_{m^{-j}\leq|x|\leq m^{-j+1}} |x|^{k-n}\nu(dx)$$

$$\leq t(m^{-j+1})^{k-n-2}\int_{m^{-j}\leq|x|\leq m^{-j+1}}|x|^2\nu(dx)\leq t\overline{\sigma}_{j,m}^{k-n}.$$

Therefore, (2.3.8) holds for $n=k$ and this completes the proof. □

2.3.2 Central limit theorem

Thanks to the Theorem 2.3.1, we are now able to prove our central limit theorem for the MLMC method. To do so, we introduce a crucial tool called Toeplitz lemma.

Lemma 2.3.6. *Let $(a_j)_{j\in\mathbb{N}}$ be a real sequence of positive terms such that*

$$\lim_{J\to\infty}\sum_{j=1}^{J}a_j=\infty \quad \text{and} \quad \lim_{J\to\infty}\frac{1}{(\sum_{j=1}^{J}a_j)^{p/2}}\sum_{j=1}^{J}a_j^{p/2}=0,\quad p>2. \tag{W$_\text{p}$}$$

The first assumption of property (W$_\text{p}$) implies that if $(x_j)_{j\in\mathbb{N}}$ is a sequence converging to $x\in\mathbb{R}$ as j tends to infinity then

$$\lim_{J\to\infty}\frac{\sum_{j=1}^{J}a_jx_j}{\sum_{j=1}^{J}a_j}=x.$$

Theorem 2.3.7. *Let $(a_j)_{j\in\mathbb{N}}$ be a real sequence of positive terms satisfying the condition (W$_\text{p}$) for some $p>2$. Let $F:\mathbb{R}^d\to\mathbb{R}$ be a $\mathcal{C}^1$ function satisfying assumption (WE) and such that $\mathbb{E}F^2(L_T^1)$ and $\sup_{j\geq 1}\mathbb{E}|\sigma_{j,m}^{-1}(F(L_T^{m^{-j}})-F(L_T^{m^{-j+1}}))|^p$ are finite. Moreover, assume that*

(H) *the condition (2.3.1) in Theorem 2.3.1 holds, and there exists a positive definite matrix Σ such that*

$$\lim_{j\to\infty}\sigma_{j,m}^{-2}\Sigma_{j,m}=\Sigma.$$

For

$$N_j=\frac{\upsilon_{m^{-J}}^{-2}(\sigma_{j,m}^2\mathbb{1}_{\{j\neq 0\}}+\sigma^2(1)\mathbb{1}_{\{j=0\}})}{a_j}\sum_{j=1}^{J}a_j,\quad j\in\{0,1,\dots,J\},$$

we have

$$\upsilon_{m^{-J}}^{-1}(Q_{m^{-J}}-\mathbb{E}F(L_T))\Rightarrow\mathcal{N}(C_F,T\mathbb{E}(\nabla F(L_T)\Sigma\nabla^\top F(L_T))),\quad \text{as}\quad J\nearrow\infty.$$

Proof. At first, we rewrite the error term as follows

$$Q_{m^{-J}}-\mathbb{E}F(L_T)=Q^1_{m^{-J}}+Q^2_{m^{-J}}+\mathbb{E}F(L_T^{m^{-J}})-\mathbb{E}F(L_T),\quad\text{where}$$

$$\begin{aligned}
Q^1_{m^{-J}}=&\frac{1}{N_0}\sum_{k=1}^{N_0}(F(L^1_{T,k})-\mathbb{E}F(L^1_T)),\\
Q^2_{m^{-J}}=&\sum_{j=1}^{J}\frac{1}{N_j}\sum_{k=1}^{N_j}F(L_{T,k}^{j,m^{-j}})-F(L_{T,k}^{j,m^{-j+1}})-\mathbb{E}\left[F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}})\right].
\end{aligned}$$

From the assumption (WE), we have the convergence of $\upsilon_{m^{-J}}^{-1}\left(\mathbb{E}F(L_T)-\mathbb{E}F(L_T^{m^{-J}})\right)$ toward C_F as J goes to infinity. Now, with $N_0=\frac{\upsilon_{m^{-J}}^{-2}\sigma^2(1)}{a_0}\sum_{j=1}^J a_j$ we can apply the classic central limit theorem to get

$$\sqrt{N_0}Q^1_{m^{-J}}\Rightarrow\mathcal{N}(0,\mathrm{Var}(F(L_T^1)))\quad as\ J\nearrow\infty.$$

As $\lim_{J\to\infty}\sum_{j=1}^J a_j=\infty$, we deduce that $\upsilon_{m^{-J}}^{-1}Q^1_{m^{-J}}\xrightarrow{\mathbb{P}}0$ when J goes to infinity. Finally, we only need to study the convergence of $\upsilon_{m^{-J}}^{-1}Q^2_{m^{-J}}$. To do so, we verify conditions (1) and (3) of Theorem 2.5.2 and set

$$X_{m^{-J},j}:=\frac{\upsilon_{m^{-J}}^{-1}}{N_j}\sum_{k=1}^{N_j}Z_{T,k}^{m^{-j},m^{-j+1}},\text{ where }(Z_{T,k}^{m^{-j},m^{-j+1}})_{1\le k\le N_j}\text{ are independent copies of}$$
$$Z_T^{m^{-j},m^{-j+1}}:=F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}})-\mathbb{E}\left[F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}})\right].$$

Step 1. We check the limit variance of $\upsilon_{m^{-J}}^{-1}Q^2_{m^{-J}}$. We have

$$\sum_{j=1}^J\mathbb{E}(X_{m^{-J},j})^2=\sum_{j=1}^J\frac{\upsilon_{m^{-J}}^{-2}}{N_j}\mathrm{Var}(Z_T^{m^{-j},m^{-j+1}})=\sum_{j=1}^J\frac{1}{\sum_{j=1}^J a_j}a_j\sigma_{j,m}^{-2}\mathrm{Var}(Z_T^{m^{-j},m^{-j+1}}).\tag{2.3.9}$$

Besides, since $F\in\mathcal{C}^1$, applying Taylor-Young's expansion we get

$$\begin{aligned}&F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}})\\&=\nabla F(L_T^{j,m^{-j+1}})(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})+(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})\epsilon(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}}),\end{aligned}$$

where $\epsilon(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})\xrightarrow{a.s.}0$ as $j\nearrow\infty$. Now, thanks to assumption (H), by applying the Theorem 2.3.1 we obtain $\sigma_{j,m}^{-1}(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})\Rightarrow\Sigma^{1/2}W_T$ as $j\nearrow\infty$. For, the second term, we use the tightness of $\sigma_{j,m}^{-1}(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})$ to deduce that $\sigma_{j,m}^{-1}(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})\epsilon(L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}})\xrightarrow{a.s.}0$ as $j\nearrow\infty$. Finally, since $L_T^{j,m^{-j+1}}$ is independent of $L_T^{j,m^{-j}}-L_T^{j,m^{-j+1}}$ and $\nabla F(L_T^{j,m^{-j+1}})\xrightarrow{a.s.}\nabla F(L_T)$ as $j\nearrow\infty$, we deduce that as $j\to\infty$

$$\sigma_{j,m}^{-1}(F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}}))\Rightarrow\nabla F(L_T).\Sigma^{1/2}W_T.$$

From the assumption $\sup_{j\ge1}\mathbb{E}|\sigma_{j,m}^{-1}(F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}}))|^p<\infty$, we have the uniform integrability and we get for $k\in\{1,2\}$

$$\mathbb{E}\left[\sigma_{j,m}^{-1}(F(L_T^{j,m^{-j}})-F(L_T^{j,m^{-j+1}}))\right]^k\underset{j\to\infty}{\longrightarrow}\mathbb{E}\left(\nabla F(L_T).\Sigma^{1/2}W_T\right)^k.$$

Consequently, $\sigma_{j,m}^{-2}\mathrm{Var}(Z_{T,1}^{m^{-j},m^{-j+1}})\underset{j\to\infty}{\longrightarrow}\mathrm{Var}\left(\nabla F(L_T).\Sigma^{1/2}W_T\right)<\infty$. Thus, from (2.3.9) and Lemma 2.3.6, $\lim_{J\uparrow\infty}\sum_{j=1}^J\mathbb{E}(X_{m^{-J},j})^2=T\mathbb{E}(\nabla F(L_T)\Sigma\nabla^\top F(L_T))$.

Step 2. We only need to check the Lyapunov condition. In what follows, let C_p be a generic positive constant depending on p that may change from line to line. By

Burkholder and Jensen inequalities, we get

$$\mathbb{E}|X_{m^{-J},j}|^p = \frac{\upsilon_{m^{-J}}^{-p}}{N_j^p}\mathbb{E}|\sum_{k=1}^{N_j} Z_{T,k}^{m^{-j},m^{-j+1}}|^p \leq C_p\frac{\upsilon_{m^{-J}}^{-p}}{N_j^{p/2}}\mathbb{E}|Z_T^{m^{-j},m^{-j+1}}|^p.$$

Besides, by Jensen inequality, we have

$$\sigma_{j,m}^{-p}\mathbb{E}|Z_T^{m^{-j},m^{-j+1}}|^p \leq C_p\mathbb{E}|\sigma_{j,m}^{-1}(F(L_T^{j,m^{-j}}) - F(L_T^{j,m^{-j+1}}))|^p < \infty,$$

thanks to our uniform integrability assumption. Therefore,

$$\sum_{j=1}^{J}\mathbb{E}|X_{m^{-J},j}|^p \leq C_p\sum_{j=1}^{J}\frac{\upsilon_{m^{-J}}^{-p}}{N_j^{p/2}}\sigma_{j,m}^p \leq C_p\frac{1}{\left(\sum_{j=1}^J a_j\right)^{p/2}}\sum_{j=1}^{J}a_j^{p/2} \overset{J\to\infty}{\to} 0.$$

We complete the proof using (W$_\mathrm{p}$).

□

In what follows, we derive a central limit theorem for the exponential Lévy model where we assume that the payoff function F has the following form

$$F(x) = f(e^{x_1},\ldots,e^{x_d}), \quad \text{for all } x = (x_1,\ldots,x_d) \in \mathbb{R}^d,$$

with $f : \mathbb{R}_+^d \to \mathbb{R}$ a given $\mathcal{C}^1$ Lipschitz continuous function.

Corollary 2.3.8. *Let $(a_j)_{j\in\mathbb{N}}$ be a real sequence of positive terms satisfying the condition (W$_\mathrm{p}$) for some $p > 2$. Assume that $\int_{|z|>1} e^{p|z|}\nu(dz)$ is finite. Then, in the setting of an exponential Lévy model there is $C > 0$ such that for all $0 \leq j \leq J$,*

$$|\mathbb{E}F(L_T^{m^{-j}}) - \mathbb{E}F(L_T)| \leq C\sigma(m^{-j}) \quad \textit{and} \quad |\mathbb{E}F(L_T^{m^{-j}}) - \mathbb{E}F(L_T^{m^{-j+1}})| \leq C\sigma_{j,m}.$$

Moreover, assume that $\sigma_{j,m} > m^{-j+1}$ for all $j > 0$ and the condition (H) of Theorem 2.3.7 is satisfied. Then, for any $0 < \eta < 2$, if we choose

$$N_j = \frac{\sigma^{-2+\eta}(m^{-J})(\sigma_{j,m}^2\mathbb{1}_{\{j\neq 0\}} + \sigma^2(1)\mathbb{1}_{\{j=0\}})}{a_j}\sum_{j=1}^{J}a_j, \quad j \in \{0,1,\ldots,J\},$$

we have

$$\sigma^{-1+\eta/2}(m^{-J})(Q_{m^{-J}} - \mathbb{E}F(L_T)) \Rightarrow \mathcal{N}(0, T\mathbb{E}(\nabla F(L_T)\Sigma\nabla^\top F(L_T))), \quad \textit{as} \quad J \nearrow \infty.$$

Proof. At first, according to Corollary 3.1 in Ben Alaya, Hajji, and Kebaier, 2016 we have the existence of a positive constant C such that for all $0 \leq j \leq J$, $|\mathbb{E}F(L_T^{m^{-j}}) - \mathbb{E}F(L_T)| \leq C\sigma(m^{-j})$. Now, by Theorem 2.3.7, it remains to prove that $\sup_{j\geq 1}\mathbb{E}|\sigma_{j,m}^{-1}(F(L_T^{m^{-j}}) - F(L_T^{m^{-j+1}}))|^p < \infty$ for $p \geq 2$ and $\mathbb{E}F^2(L_T^1) < \infty$ are satisfied. In order to prove the first assertion, it is enough to find an upper bound of $\mathbb{E}|e^{L_T^{m^{-j}}} - e^{L_T^{m^{-j+1}}}|^p$, $p \geq 2$, since f is Lipschitz. By some basic exponential inequalities and using the independence of $L_T^{m^{-j+1}}$ and $L_T^{m^{-j}} - L_T^{m^{-j+1}}$ and Cauchy-Schwarz's inequality, there is a positif constant C such that

$$\mathbb{E}|e^{L_T^{m^{-j}}} - e^{L_T^{m^{-j+1}}}|^p \leq C\mathbb{E}|e^{L_T^{m^{-j+1}}}|^p\mathbb{E}(|L_T^{m^{-j}} - L_T^{m^{-j+1}}|^p e^{p|L_T^{m^{-j}} - L_T^{m^{-j+1}}|})$$

$$\leq C\mathbb{E}|e^{L_T^{m^{-j+1}}}|^p\| |L_T^{m^{-j}} - L_T^{m^{-j+1}}|^p\|_2 \|e^{p|L_T^{m^{-j}} - L_T^{m^{-j+1}}|}\|_2.$$

By the assumption $\int_{|z|>1} e^{p|z|}\nu(dz) < \infty$ and using Theorem 2.5.5, the finiteness of $\mathbb{E}|e^{L_T^{m^{-j+1}}}|^p$ is ensured. Now, using the inequality $e^{|x|} \leq \prod_{j=1}^d (e^{x_j} + e^{-x_j})$ this last upper bound can be written as a sum of finite number of exponential functions evaluated at points which are a linear combination of the components of the vector x. Therefore there exists a family of $\mathbb{R}^d$-valued vectors $(b_k)_{1\leq k\leq 2^d}$ such that $e^{2p|L_T^{m^{-j}} - L_T^{m^{-j+1}}|} \leq \sum_{k=1}^{2^d} e^{b_k\cdot(L_T^{m^{-j}} - L_T^{m^{-j+1}})}$. Now by virtue of Lemmas 25.6 and 25.7 in Sato, 1999 we deduce the boundedness of $\|e^{p|L_T^{m^{-j}} - L_T^{m^{-j+1}}|}\|_2^2$. Indeed, we have

$$\begin{aligned}
\|e^{p|L_T^{m^{-j}} - L_T^{m^{-j+1}}|}\|_2^2 &\leq \sum_{k=1}^{2^d} \exp\left\{T\int_{m^{-j}\leq|x|\leq m^{-j+1}} (e^{b_k.x} - 1 - b_k.x)\nu(dx)\right\} \\
&\leq \sum_{k=1}^{2^d} \exp\left\{T\int_{0\leq|x|\leq 1} |e^{b_k.x} - 1 - b_k.x|\nu(dx)\right\} \\
&\leq \sum_{k=1}^{2^d} \exp\left\{Tc_k\int_{0\leq|x|\leq 1} |x|^2\nu(dx)\right\}, \quad c_k > 0.
\end{aligned}$$

Combining all these results together and applying Proposition 2.3.5, there exists a contant C such that

$$\mathbb{E}|\sigma_{j,m}^{-1}(F(L_T^{m^{-j}}) - F(L_T^{m^{-j+1}}))|^p \leq C\sigma_{j,m}^{-p}\bar{\sigma}^p(m^{-j+1}) = C.$$

Now, by the linear growth of f and the condition $\int_{|z|>1} e^{p|z|}\nu(dz) < \infty$, the second condition $\mathbb{E}F^2(L_T^1) < \infty$ holds using Theorem 2.5.5. Hence, if we choose $\upsilon_{m^{-J}} = \sigma^{1-\eta/2}(m^{-J})$ then Theorem 2.3.7 completes the proof. □

2.3.3 The time complexity

We consider the one-dimensional case for which $\upsilon_{m^{-J}} = \sigma^{1-\frac{\eta}{2}}(m^{-J})$, with $\eta \in (0,2)$. Assume that the measure ν has a density of the form $L(x)/|x|^{Y+1}$ for a small x, where $L(x)$ is a positive function that is slowly varying as $x \to 0$ and $Y \in (0,2)$. Then the positive (resp. negative) part of the approximation $(L_t^{m^{-j}})_{0\leq t\leq T}$, $0 \leq j \leq J$, is a compound Poisson process with intensity $\nu([m^{-j}, +\infty))$ (resp. $\nu((-\infty, -m^{-j}]))$. Then the cost necessary of a single simulation is random, with expectation of order $\mathcal{K}(m^{-j}) = \nu(|x| \geq m^{-j})$. Thus, by Theorem 2.3.7, the mean of time complexity of the MLMC method needed to achieve an accuracy of order $\sigma^{1-\frac{\eta}{2}}(m^{-J})$ is

$$\begin{aligned}
C_{\text{MLMC}} &= C \times \sum_{j=0}^{J} \mathcal{K}(m^{-j})N_j, \\
&= C \times \sigma^{-(2-\eta)}(m^{-J}) \sum_{j=0}^{J} \mathcal{K}(m^{-j}) \frac{\sigma_{j,m}^2 \mathbb{1}_{\{j\neq 0\}} + \sigma^2(1)\mathbb{1}_{\{j=0\}}}{a_j} \sum_{j=1}^{J} a_j,
\end{aligned}$$

where C is a positive constant that may change from line to line. By Karamata's theorem (see e.g. Bingham, Goldie, and Teugels, 1987 or Feller, 1971) we have

$$\sigma^2(m^{-j}) \underset{j\to\infty}{\sim} \frac{L(m^{-j}) + L(-m^{-j})}{2-Y} m^{-j(2-Y)} \quad \text{and} \quad \mathcal{K}(m^{-j}) \underset{j\to\infty}{\sim} \frac{L(m^{-j}) + L(-m^{-j})}{Y} m^{jY}.$$

Using the slowly property and the positivity of L we get

$$\sigma^2(m^{-j+1}) \underset{j\to\infty}{\sim} \frac{L(m^{-j}) + L(-m^{-j})}{2-Y} m^{(-j+1)(2-Y)}$$

and by the decomposition $\sigma^2_{j,m} = \sigma^2(m^{-j+1}) - \sigma^2(m^{-j})$ we deduce that

$$\sigma^2_{j,m} \underset{j\to\infty}{\sim} \frac{L(m^{-j}) + L(-m^{-j})}{2-Y} m^{-j(2-Y)}(m^{2-Y} - 1).$$

In what follows, we assume in addition that the function L is bounded, which is the case for the CGMY process (See Section 2.4).

The case $Y \in (1, 2)$. For the choice $a_j = m^{-j(1-Y)}$, as $j \to \infty$ we have $\mathcal{K}(m^{-j})\frac{\sigma^2_{j,m}}{a_j} = O(m^{-j(1-Y)})$ and then as $J \to \infty$, the time complexity needed to achieve a precision of order $\sigma^{1-\frac{\eta}{2}}(m^{-J}) = O(m^{-\frac{J(2-Y)(2-\eta)}{4}})$ is $C_{\text{MLMC}} = O(m^{J(2(Y-1)+(1-\frac{\eta}{2})(2-Y))})$. Clearly, the closer to zero η is, the smaller the MLMC time complexity is. Thus, to achieve a precision of order ε the time complexity for the MLMC method is $C_{\text{MLMC}} = O(\varepsilon^{-\frac{4}{(2-\eta)(2-Y)}(2(Y-1)+(1-\frac{\eta}{2})(2-Y))})$. Note that for the particular case where $Y \simeq 1$, the MLMC method may reach its optimal time complexity given by

$$C^*_{\text{MLMC}} = O(\varepsilon^{-2})$$

so that the MLMC estimator would behave like an unbiased Monte Carlo estimator. It is worth noticing that the weight $a_j = m^{-j(1-Y)}$ does not satisfy the second part of the technical condition ($\mathrm{W_p}$) needed to prove the central limit theorem. However, we may choose $a_j = 1$ to ensure the validity of the Central Limit Theorem 2.3.7 and in this case to achieve a precision of order ε the MLMC method reaches an optimal time complexity given by $C_{\text{MLMC}} = O(\varepsilon^{-\frac{4}{(2-\eta)(2-Y)}(2(Y-1)+(1-\frac{\eta}{2})(2-Y))}\log(\frac{1}{\varepsilon}))$.

The case $Y \in (0, 1)$. For the choice $a_j = m^{-j(1-Y)}$, as $J \to \infty$, the time complexity needed to achieve a precision of order $\sigma^{1-\frac{\eta}{2}}(m^{-J}) = O(m^{-\frac{J(2-Y)(2-\eta)}{4}})$ is $C_{\text{MLMC}} = O(m^{J(2-Y)(1-\frac{\eta}{2})})$. Thus, for a given precision ε, the time complexity of the MLMC method is the optimal one $C^*_{\text{MLMC}} = O(\varepsilon^{-2})$. Of course, as mentioned in the previous case, to ensure the validity of the Central Limit Theorem 2.3.7 we have to adapt the choice of the weights a_j. Following Ben Alaya and Kebaier, 2014, for the particular choice $a_j = \frac{1}{j}$ (resp $a_j = \frac{1}{j\log(j)}$) it is easy to check that the time complexity needed to achieve the accuracy ε is $C_{\text{MLMC}} = O(\varepsilon^{-2}\log\log(\frac{1}{\varepsilon}))$ (resp. $C_{\text{MLMC}} = O(\varepsilon^{-2}\log\log\log(\frac{1}{\varepsilon}))$). Let us recall that according to Corollary 3.1 in Ben Alaya, Hajji, and Kebaier, 2016, for a precision ε the optimal time complexity of a Monte Carlo method is given by $C_{\text{MC}} = O(\varepsilon^{-\frac{4}{(2-\eta)(2-Y)}(Y+(1-\frac{\eta}{2})(2-Y))})$ which clearly has a larger order than the above MLMC time complexities obtained in both cases $Y \in (0, 1)$ and $Y \in (1, 2)$. It is worth noticing, that the gain between both methods dramatically decreases when Y is chosen close to zero.

2.4 Numerical results

For this section we illustrate the efficiency of the MLMC method compared to the classical Monte Carlo method for pricing european calls under exponential Lévy models. More precisely, we consider an underlying asset following an exponential pure jump CGMY model. Let us recall that the CGMY process presented in Carr et al., 2002 provides a rich jump model for the equity log-returns. The CGMY process covers a general class of Lévy processes since its particular parametrization allows pure diffusion or pure jumps, infinite or finite variation, and infinite or finite arrival rates. The practical option pricing under the CGMY model using the classical Monte Carlo methods has been introduced in several works by Madan and Yor, 2008, Poirot and Tankov, 2006 and Rosiński, 2007. The use of a two-level Monte Carlo method for pricing vanilla options under the CGMY model has been tackled by Ben Alaya, Hajji, and Kebaier, 2016. For more clarity, we first give a brief summary on the main properties of such a process. The CGMY process is a pure jump process with generating triplet $(0, 0, \nu)$ where for $C > 0, G > 0, M > 0$ and $Y \in (0, 2)$

$$\nu(dx) = \frac{Ce^{-Mx}}{x^{1+Y}} \mathbb{1}_{x>0} dx + \frac{Ce^{-G|x|}}{|x|^{1+Y}} \mathbb{1}_{x<0} dx. \tag{2.4.1}$$

It has a Lévy-Khintchine representation formula with a truncation function h and a characteristic exponent given by

$$\psi(u) = i\gamma_h u + \int_{\mathbb{R}} (e^{iux} - 1 - iuh(x))\nu(dx) \text{ with } \gamma_h = \int_{\mathbb{R}} (h(x) - x\mathbb{1}_{\{|x|\leq 1\}})\nu(dx), \; u \in \mathbb{R}.$$

- For $1 < Y < 2$ and $h(x) = x$, we have $\gamma_h = \int_{|x|\geq 1} x\nu(dx)$ and

$$\psi(u) = iu\gamma_h + C\Gamma(-Y)\left[M^Y\left((1 - \frac{iu}{M})^Y - 1 + \frac{iuY}{M}\right) + G^Y\left((1 + \frac{iu}{G})^Y - 1 - \frac{iuY}{G}\right)\right].$$

- For $0 < Y < 1$ and $h(x) = 0$, we have $\gamma_h = \int_{|x|\leq 1} x\nu(dx)$ and

$$\psi(u) = iu\gamma_h + C\Gamma(-Y)\left[M^Y\left((1 - \frac{iu}{M})^Y - 1\right) + G^Y\left((1 + \frac{iu}{G})^Y - 1\right)\right].$$

In what follows, we focus on the CGMY process $(L_t)_{0\leq t\leq T}$ with generating triplet $(\gamma, 0, \nu)$, $\gamma \in \mathbb{R}$ and consider the stock price under risk neutral probability given by

$$S_t = S_0 \exp(rt + L_t), \text{ where } r > 0 \text{ is the interest rate and } S_0 > 0.$$

Moreover, we assume that

$$\int_{|x|\geq 1} e^x \nu(dx) < \infty \quad \text{and} \quad \gamma + \int_{\mathbb{R}} (e^y - 1 - y\, 1_{\{|y|\leq 1\}})\nu(dy) = 0. \tag{2.4.2}$$

This above assumption is essential to guarantee the validity of the martingale property of the discounted asset price $(e^{-rt}S_t)_{0\leq t\leq T}$. It is worth noticing that the first condition of assumption (2.4.2) is satisfied as soon as $M > 1$.

Now, for a fixed level $j \in \mathbb{N} \setminus \{0\}$ the fine (respectively coarse) approximation of the CGMY process $(L_t)_{0\leq t\leq T}$ is given by $(L_t^{m^{-j}})_{0\leq t\leq T}$ (respectively $(L_t^{m^{-j+1}})_{0\leq t\leq T}$). The fine approximation is a Lévy process with characteristic triplet given by $(\gamma, 0, \nu^f_{m^{-j}})$ where $\nu^f_{m^{-j}}(dx) := \mathbb{1}_{\{|x|\geq m^{-j}\}}\nu(dx)$ that is simulated as a compound Poisson process with drift $\gamma^f_{m^{-j}} := \gamma - \int_{m^{-j}\leq|x|\leq 1} x\nu(dx)$, see (2.2.1). More precisely, this compound process is generated as the difference between two independent compound processes, namely a positive (resp. negative) compound process with jump size

$\nu^{f,+}_{m^{-j}} = \mathbb{1}_{\{x\geq m^{-j}\}}\frac{\nu(dx)}{\nu([m^{-j},+\infty[)}$ (resp. $\nu^{f,-}_{m^{-j}} = \mathbb{1}_{\{x\leq -m^{-j}\}}\frac{\nu(dx)}{\nu(]-\infty,-m^{-j}])}$) and intensity $\nu([m^{-j},+\infty[)$ (resp. $\nu(]-\infty,-m^{-j}])$). Here we follow the sampling method proposed by Rosiński, 2001 (see Algorithm 1 below) that simulates the paths of $\nu^{f,+}_{m^{-j}}$ from those of $\nu^{f,+}_{0,m^{-j}}$ by only accepting all jumps x in the paths of $\nu^{f,+}_{0,m^{-j}}$ for which $d\nu^{f,+}_{m^{-j}}/d\nu^{f,+}_{0,m^{-j}}(x) > u$ where u is an independent random variable draw from uniform distribution.

Algorithm 1 Simulating the positive jump size Z of the CGMY process using Rosinski's rejection

Require: U_1 and U_2 are uniform random variables and $Z = m^{-j}U_1^{-\frac{1}{Y}}$
  if $U_2 > \exp -M.Z$ **then**
    $Z = 0$
  end if
  return Z

The negative jump part is sampled by replacing in the above algorithm the parameter M by G. The coarse approximation $(L_t^{m^{-j+1}})_{0\leq t\leq T}$ is generated from the paths of the fine one using that for $t\in[0,T]$

$$L_t^{m^{-j+1}} = L_t^{m^{-j}} + \tilde{L}_t^{m,j},$$

where $(\tilde{L}_t^{m,j})_{0\leq t\ T}$ is an independent Lévy process with generating triplet given by $(0,0,\nu_{\{m^{-j}<|x|\leq m^{-j+1}\}})$ with $\nu_{\{m^{-j}<|x|\leq m^{-j+1}\}} := \mathbb{1}_{\{m^{-j}<|x|\leq m^{-j+1}\}}\nu(dx)$.

The aim now is to test the performance of the MLMC method that approximates the price $\mathbb{E}F(S_T)$ with a payoff function $F(x) = e^{-rT}(x-K)_+$ by

$$Q_{m^{-J}} = \frac{1}{N_0}\sum_{k=1}^{N_0} F(L^1_{T,k}) + \sum_{j=1}^{J}\frac{1}{N_j}\sum_{k=1}^{N_j}\Big(F(L^{j,m^{-j}}_{T,k}) - F(L^{j,m^{-j+1}}_{T,k})\Big). \tag{2.4.3}$$

The CGMY parameters are chosen as follows: $S_0 = 100, K = 100, C = 0.0244, G = 0.0765, M = 7.5515, Y = 1.2945$, the free interest rate $r = \log(1.1)$, maturity time $T = 1$ and $m = 4$. For this set of parameters, the benchmark price is equal to 13.496508 and is computed using the Fourier-cosine method introduced by Fang and Oosterlee Fang and Oosterlee, 2008 that reaches an accuracy of order 10^{-10}. This method is available in the free online version of Premia platform `https://www.rocq.inria.fr/mathfi/Premia/index.html`. In this case, we have $\sigma^2(m^{-j}) \underset{\varepsilon\to 0}{\simeq} 2Cm^{-j(2-Y)}/(2-Y)$ and according to Subsection 2.3.3 we set $v_\varepsilon = \sigma^{1-\eta/2}(\varepsilon)$ with $\eta = 0.04$. For different values of ε, we give in Figure 2.1 below the log-log plot of the obtained RMSE versus the CPU time for the classical Monte Carlo and the MLMC method.

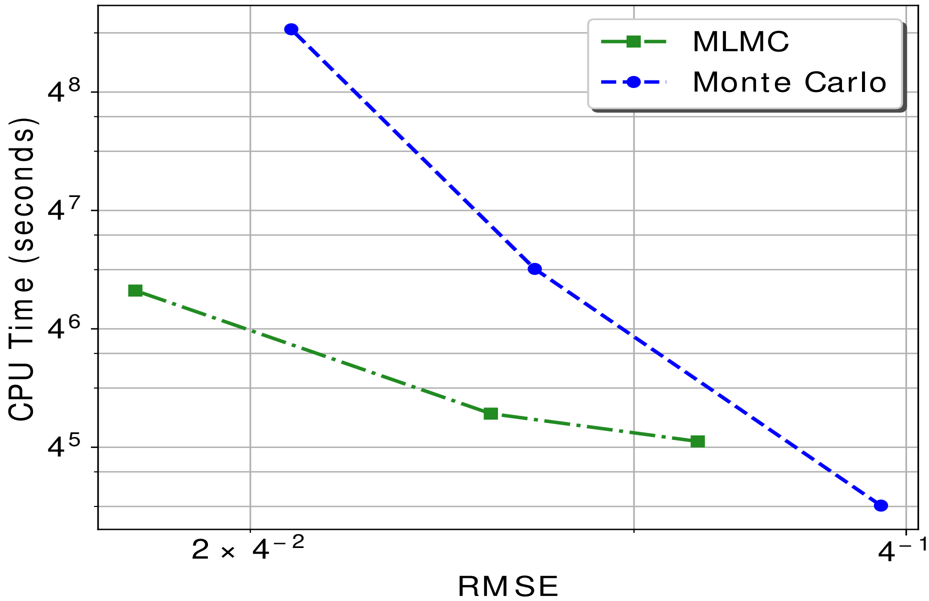


FIGURE 2.1: CPU time versus RMSE.

To do so, we compute for each method the CPU time (per second) (the computations are done on a PC with a 1.6 GHz Intel Core i5 dual core) and the RSME given by

$$\text{RMSE} = \sqrt{\frac{1}{50}\sum_{i=1}^{50}(\text{Benchmark price} - \text{Approximated value})^2}. \tag{2.4.4}$$

As expected, we see in Figure 2.1 that the MLMC method is asymptotically more efficient than the classical Monte Carlo one. Indeed, according to our numerical results for a fixed RMSE of order 10^{-1}, the MLMC method reduces the CPU time by a factor of 21.31 compared to the Monte Carlo method.

2.5 Appendix

2.5.1 Convergence of infinitely divisible distributions

The following theorem is about the convergence of infinitely divisible distributions. We recall Theorem 15.14(i) in Kallenberg, 2002. Justified by the one-to-one correspondence between infinitely divisible distributions μ and their characteristics (a, b, ν), we may write $\mu = id(a, b, \nu)$. Define for any $h > 0$,

$$a^h = a + \int_{|x|\leq h} xx^\top \nu(dx), \quad b^h = b - \int_{h<|x|\leq 1} x\nu(dx),$$

where $\int_{h<|x|\leq 1} = -\int_{1<|x|\leq h}$ when $h > 1$.
Let $\overline{\mathbb{R}^d}$ denote the one-point compactification of $\mathbb{R}^d$.

Theorem 2.5.1. *Let $\mu = id(a, b, \nu)$ and $\mu_n = id(a_n, b_n, \nu_n)$ on $\mathbb{R}^d$, and fix any $h > 0$ with $\nu(|x| = h) = 0$. Then $\mu_n \xrightarrow{w} \mu$ iff $a_n^h \to a^h$, $b_n^h \to b^h$ and $\nu_n \xrightarrow{v} \nu$ on $\overline{\mathbb{R}^d}\backslash\{0\}$.*

2.5.2 Lindeberg-Feller central limit theorem

We recall also the following central limit theorem for triangular array (see, e.g., Theorem 7.2 and 7.3 in Billingsley, 1999).

Theorem 2.5.2. *Let $(k_n)_{n\in\mathbb{N}}$ be a sequence such that $k_n \to \infty$ as $n \to \infty$. For each n, let $X_{n,1},\dots,X_{n,k_n}$ be k_n independent random variables with finite variance such that $\mathbb{E}(X_{n,k}) = 0$ for all $k \in \{1,\dots,k_n\}$. Suppose that the following conditions hold:*

1. $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k}|^2 = \vartheta$, $\vartheta > 0$.
2. *Lindeberg's condition: For all $\epsilon > 0$, $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}(|X_{n,k}|^2 \mathbb{1}_{|X_{n,k}|>\epsilon}) = 0$.* *Then*

$$\sum_{k=1}^{k_n} X_{n,k} \Rightarrow \mathcal{N}(0,\vartheta), \qquad \text{as } n \to \infty.$$

Moreover, if the $X_{n,k}$ have moments of order $p > 2$, then the Lindeberg's condition can be obtained by the following one:

3. *Lyapunov's condition: $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k}|^p = 0$.*

2.5.3 A useful lemma from the paper of Cohen and Rosiński

We recall Lemma 2.1 in Cohen and Rosiński, 2007.

Lemma 2.5.3. *Let ν is a measure such that $\int_{\mathbb{R}^d} |x|^2 \nu(dx) < \infty$ and $\Sigma = \int_{\mathbb{R}^d} xx^\top \nu(dx)$. The following conditions are equivalent*

1. *Σ is non-singular,*
2. *the smallest linear space supporting ν equals $\mathbb{R}^d$. (ν is not concentrated on any proper linear subspace of $\mathbb{R}^d$)*

2.5.4 Tools used for the exponential Lévy model seeting

In what follows, we recall the definition of a submultiplicative function (see e.g. Definition 2.1 in Ben Alaya, Hajji, and Kebaier, 2016) and then an important property of Lévy processes (see e.g. Theorem 25.3 in Sato, 1999).

Definition 2.5.4. *A function $f : \mathbb{R}^d \mapsto [0,\infty)$ is said to be submultiplicative if there exists a positive constant c such that $f(x+y) \le cf(x)f(y)$ for $x, y \in \mathbb{R}^d$. The product of two submultiplicative functions is also a submultiplicative function.*

Theorem 2.5.5. *Let f be a submultiplicative, locally bounded, measurable function on $\mathbb{R}^d$, and let $(L_t)_{t\ge 0}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure ν. Then, $\mathbb{E}f(L_t)$ is finite for every $t > 0$ iff $\int\limits_{|z|\ge 1} f(z)\nu(dz) < +\infty$.*

Chapter 3

Asymptotic behavior of the error between two different Euler schemes for the Lévy driven SDEs

In this chapter, motivated by the multilevel Monte Carlo method introduced by Giles, 2008b, we study the the asymptotic behavior of the normalized error process $u_{n,m}(X^n - X^{nm})$ where X^n and X^{nm} are respectively Euler approximations with time steps $1/n$ and $1/nm$ of a given stochastic differential equation driven by a pure jump Lévy process Y. In this paper, we prove that this multilevel error process converges to some non-trivial limiting process with a sharp rate $u_{n,m}$. The obtained results extend those of Jacod, 2004 for the normalized error $u_n(X^n - X)$. For the multilevel error, the proofs of the current paper are challenging since unlike Jacod, 2004 we need to deal with m dependent triangular arrays instead of one. Formally, when letting m tends to infinity, we recover limit processes of Jacod, 2004.

The original paper Ben Alaya, Kebaier, and Ngô, 2021a of this work is submitted.

3.1 Introduction

Suppose that we are in the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ endowed with the filtration $\mathcal{F}_s = \sigma(Y_u, u \le s)$, where Y is a Lévy process with characteristics (b, c, F) with respect to the truncation function $h(x) = x\mathbb{1}_{\{|x|\le 1\}}$, meaning

$$\mathbb{E}(e^{\mathtt{i}uY_t}) = \exp\Big\{t\big(\mathtt{i}ub - \frac{cu^2}{2} + \int F(dx)(e^{\mathtt{i}ux} - 1 - \mathtt{i}ux\mathbb{1}_{\{|x|\le 1\}})\big)\Big\}.$$

We consider the Lévy driven stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t f(X_{s-})dY_s, \qquad t \in [0, T],\ T > 0 \tag{3.1.1}$$

where $x_0 \in \mathbb{R}$, $f \in \mathcal{C}^3$ (a three-times-differentiable function). Without loss of generality, we assume that $T = 1$. In what follows, we consider the continuous Euler approximation

$$dX_t^n = f(X^n_{\eta_n(t-)})dY_t, \qquad t \in [0, 1] \tag{3.1.2}$$

with time step $1/n$, where $\eta_n(t) = \frac{[nt]}{n}$.

For the error process $X^n - X$, Jacod and Protter, 1998 proved that the sharp rate is $\sqrt{n}$ when the characteristic triplet corresponds to (b, c, F) with $c > 0$. Let us precise that a rate is called sharp if the normalized error converges to a non-trivial limiting process. Then, Jacod, 2004 established new sharp rates of convergence different from the classical $\sqrt{n}$ rate for several cases with a Lévy characteristic triplet $(b, 0, F)$ and $F(\mathbb{R}) = \infty$. Those cases correspond to different behaviors of the Lévy measure F around zero. More recently, Wang, 2015 extended the results of Jacod, 2004 for the case of general Itô semimartingales.

In the current paper, motivated by the multilevel Monte Carlo method introduced by Giles, 2008b, we study instead the error between two Euler schemes with different time steps. In particular, we are interested in determining sharp rates $u_{n,m}$ for the weak convergence of the error between two consecutive Euler approximations $X^n - X^{nm}$ and identifying the corresponding limiting processes. Here, X^n and X^{nm} stand for the Euler schemes with respectively time steps $1/n$ and $1/nm$ that are build on the same Lévy paths. In the literature, several papers studied this multilevel type error. Indeed, when the characteristic triplet of Y is $(b, 1, 0)$, Ben Alaya and Kebaier, 2015 proved that

$$(Y, \sqrt{\frac{mn}{(m-1)}}(X^{nm} - X^n)) \stackrel{stably}{\Rightarrow} (Y, U), \text{ as } n \to \infty$$

with $m \in \mathbb{N}\backslash\{0, 1\}$,

$$U_t = \int_0^t f'(X_s) U_s dY_s + \frac{1}{\sqrt{2}} \int_0^t f(X_s) f'(X_s) dW_s$$

and W is a new standard Brownian motion independent of Y. When the characteristic triplet of Y is (b, c, F) with $c \neq 0$, Dereich and Li, 2016 proved a similar result with a sharp rate $\sqrt{n}$ under some regularity condition on F around zero with an explicit limiting process.

Therefore, to fill the gap in the literature for the analysis of this type of error, we consider Y as a Lévy process with characteristics $(b, 0, F)$ where F is an infinite measure. More precisely, for the same cases studied by Jacod, 2004 we consider in the current work the multilevel type error between two consecutive Euler approximations defined by

$$U_t^{n,m} = X^n_{\eta_{nm}(t)} - X^{nm}_{\eta_{nm}(t)}, \quad t \in [0, 1]. \tag{3.1.3}$$

For this multilevel type error, we use triangular arrays technics to find the sharp rate of convergence that turns out to be faster than $\sqrt{n}$ which is the usual rate when Y incorporates a continuous Gaussian part. It is worth noticing that the work of Jacod, 2004 when studying the error $X^n_{\eta_n} - X_{\eta_n}$ needs only to treat one main contributing triangular array. However, in the current work, the technical challenge we faced when proving the convergence of the correctly normalized multilevel error (3.1.3) consists in studying the asymptotic behavior of the joint probability distribution of m triangular arrays. These dependent triangular arrays appear naturally when studying the multilevel error obtained between the finer discretization with time step $1/nm$ and the coarse one with time step $1/n$. To overcome this problem, we develop new treatments and proofs to handle these m dependent terms that contribute in the limit in different ways depending on the assumptions taken on the original Lévy measure F around zero. In more details, besides using the "subsequences principle" trick (see e.g. Jacod and Protter, 2012), we use arguments of Sato, 1999, Ex.12.8-12.10 that let us avoid complicated calculations of multi-dimensional integrals and rather focus on the pairwise asymptotic behavior of the m marginals and we conclude using technical criteria of Kallenberg, 2002, Theorem 15.14 and Corollary 15.16 to prove the weak convergence to the limiting process.

The rest of the paper is organized as follows. In Section 3.2, using similar notations, we recall from Jacod, 2004 some assumptions and estimates on the Lévy measure and also the semimartingale decomposition. Here, in the spirit of Jacod, 2004, we precise our consideration for five specific cases. In Section 3.3, we introduce and prove our main results namely Theorem 3.3.1 for the tightness, Theorem 3.3.2 and Theorem 3.3.4 the functional limit theorems for the couple of normalized errors. Section 3.4 gives the details of the error analysis to prove our main results with specifying the main and rest terms for each cases and the study of the asymptotic behaviors of the joint distribution of the main terms. The rest terms are treated in appendix 3.5. Appendices 3.6 and 3.7 are dedicated to recall some technical tools that we use throughout the paper.

3.2 General settings and notations

Let f denotes a real-valued function f satisfying

$$f \in \mathcal{C}^3 \text{ and globally Lipschitz.} \qquad (\mathbf{H}_f)$$

It is well known that assumption $(\mathbf{H}_f)$ guarantees that (3.1.1) has an unique non-exploding solution. The crucial factor to find the sharp rates of the multilevel type error (3.1.3) is the behavior of the Lévy measure F near 0, which will be expressed through the following functions on $\mathbb{R}_+$:

$$\begin{aligned} \theta_+(\beta) = F(\beta, +\infty), \qquad \theta_-(\beta) = F(-\infty, -\beta), \\ \theta(\beta) = \theta_+(\beta) + \theta_-(\beta). \end{aligned}$$

Note that from now on, we denote C as a generic constant which may change from line to line and α denotes our basic index. We keep the same framework as in Jacod, 2004 and we introduce the four main classes of Lévy measures F that we are interested in:

$$\text{We have } \theta(\beta) \leq \frac{C}{\beta^\alpha} \text{ for all } \beta \in (0,1]. \qquad (\mathbf{H}_1^\alpha)$$

$$\begin{aligned} &\text{We have } \beta^\alpha\theta_+(\beta) \to \theta_+ \text{ and } \beta^\alpha\theta_-(\beta) \to \theta_- \text{ as } \beta \to 0 \text{ for some constants } \theta_+, \theta_- \geq 0. \\ &\text{We set } \theta = \theta_+ + \theta_- > 0 \text{ and } \theta' = \theta_+ - \theta_-, \text{ as } \beta \to 0, \ \theta(\beta) \sim \tfrac{\theta}{\beta^\alpha}. \end{aligned} \qquad (\mathbf{H}_2^\alpha)$$

$$\text{The measure } F \text{ is symmetrical about } 0. \qquad (\mathbf{H}_3)$$

$$\text{We have } b = 0. \qquad (\mathbf{H}_4)$$

We note that $(\mathbf{H}_1^\alpha)$ is weaker than $(\mathbf{H}_2^\alpha)$. Here, the Hypothesis $(\mathbf{H_1^2})$ always holds because the Lévy measure F integrates $x \mapsto |x|^2 \wedge 1$. That is

$$|\beta|^2\theta(\beta) = \int_{|x|>\beta} (|\beta|^2 \wedge 1)F(dx) \leq \int_{|x|>\beta} (|x|^2 \wedge 1)F(dx) < +\infty.$$

Now, let us give an example of a process in finance which satisfies the first two hypotheses.

Example 1. *We consider the CGMY process with Lévy density*

$$F(dx) = \begin{cases} Ce^{-Mx}/x^{1+Y} & x > 0 \\ Ce^{Gx}/|x|^{1+Y} & x < 0 \end{cases}$$

where $C, G, M > 0$ and $0 < Y < 2$. This process always satisfies Hypothesis $(\mathbf{H}_2^\alpha)$ and therefore it satisfies Hypothesis $(\mathbf{H}_1^\alpha)$. More precisely, we have

$$\beta_n^Y\theta(\beta_n) = \frac{2C}{Y} + C\beta_n^Y \int_{x>\beta_n} \frac{e^{-Mx} + e^{-Gx} - 2}{x^{1+Y}} dx.$$

So, noticing that

$$C\beta_n^Y \int_{x>\beta_n} \frac{e^{-Mx} + e^{-Gx} - 2}{x^{1+Y}} dx \leq \frac{C}{Y}(e^{-M\beta_n} + e^{-G\beta_n} - 2) \underset{n\to\infty}{\to} 0.$$

we deduce that $\beta_n^Y\theta(\beta_n) \to \theta$ as $n \to \infty$, where $\theta = \frac{2C}{Y}$.

In the same spirit as in Jacod, 2004, we prove sharp rate $u_{n,m}$ for our multilevel error $U^{n,m}$ (3.1.3) with pointing out the technical choice for the sequence β_n tending to 0 that truncates the small jumps. To do so, we consider five different cases depending on some reasonably general circumstances.

(C1) If $(\mathbf{H}_1^\alpha)$ is valid for some $\alpha < 1$ and $d := b - \int_{|x|\leq 1} xF(dx) \neq 0$, then we choose $u_{n,m} = \frac{nm}{m-1}$ and $\beta_n = \frac{(\log n)^2}{n}$. (See the Remark 3.2.3 for the finiteness of d)

(C2) If $(\mathbf{H}_2^\alpha)$ is valid for some $\alpha < 1$ and hypotheses $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ are also valid then we choose $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \left(\frac{\log n}{n}\right)^{1/\alpha}$.

(C3) If $(\mathbf{H}_2^\alpha)$ is valid for $\alpha = 1$ and F is non-symmetric then we choose $u_{n,m} = \frac{mn}{(m-1)(\log n)^2}$ and $\beta_n = \frac{\log n}{n}$.

(C4) If $(\mathbf{H}_2^\alpha)$ is valid for $\alpha = 1$ and hypothesis $(\mathbf{H}_3)$ is also valid then we choose $u_{n,m} = \frac{mn}{(m-1)\log n}$ and $\beta_n = \frac{\log n}{n}$.

(C5) If $(\mathbf{H}_2^\alpha)$ is valid for some $\alpha > 1$ then we choose $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$.

3.2.1 Some estimates on Lévy measure

In what follows we consider the same notations as in Jacod, 2004 and for $\beta > 0$, we denote

$$c(\beta) = \int_{|x|\leq\beta} |x|^2 F(dx),$$

$$
\begin{aligned}
d_+(\beta) &= \int_{x>\beta} |x|F(dx), & d_-(\beta) &= \int_{x<-\beta} |x|F(dx),\\
\rho_+(\beta) &= \int_{x>\beta} |x|^\alpha F(dx), & \rho_-(\beta) &= \int_{x<-\beta} |x|^\alpha F(dx),\\
\delta(\beta) &= d_+(\beta) + d_-(\beta), & \rho(\beta) &= \rho_+(\beta) + \rho_-(\beta),\\
d'(\beta) &= d_+(\beta) - d_-(\beta), & b' &= b + \int_{|x|>1} xF(dx),\\
d(\beta) &= b' - d'(\beta).
\end{aligned} \tag{3.2.1}
$$

Note that $d(\beta) = b - \int_{\beta<|x|\leq 1} xF(dx)$ if $\beta < 1$ and $d(\beta) = b$ if $\beta = 1$.

• Without loss of generality we can reduce ourselves to study the case where we have bounded jumps and coefficient f with compact support. Thus, from now on we assume that

(A) $f \in \mathcal{C}^3$ with compact support and $|\Delta Y| \leq p$ for some integer $p \geq 1$, which amounts to say that $\theta(p) = 0$.

Indeed, adapting the same arguments as in Proposition 2.4 in Jacod, 2004 to the multilevel error setting, we can easily recover our main results namely Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.3.4 for non-bounded jumps and coefficient f without a compact support.

Remark 3.2.1. *Note that under (A) the quantity $\int_{\mathbb{R}} |x|^a F(dx)$, $a \geq 2$ is finite.*

In this part, we recall from Jacod, 2004 under assumption (A) some useful estimations on the above quantities introduced in (3.2.1). We provide some details for the proofs of the following lemmas in appendix 3.6.

Lemma 3.2.2. *Since $\theta(p) = 0$, under $(\mathbf{H}_1^\alpha)$, we have for any $\beta \in (0,1]$*

$$
\begin{cases}
c(\beta) \leq C\beta^{2-\alpha}, \qquad \rho(\beta) \leq C \log\left(\frac{1}{\beta}\right)\\
\int_{|x|>\beta} |x|^{\alpha/2} F(dx) \leq \frac{C}{\beta^{\alpha/2}}\\
\delta(\beta) + |d(\beta)| + d_+(\beta) + d_-(\beta) + |d'(\beta)| \leq Cs(\beta)
\end{cases}
\quad \text{where } s(\beta) = \begin{cases} 1, & \alpha < 1\\ \log(\frac{1}{\beta}), & \alpha = 1\\ \beta^{1-\alpha}, & \alpha > 1 \end{cases} \tag{3.2.2}
$$

Remark 3.2.3. *Note that under $(\mathbf{H}_1^\alpha)$ with $\alpha < 1$, we have $|\delta(\beta)| < C$ for all $\beta \in (0,1]$, then by the monotone convergence theorem $\int |x|F(dx) < \infty$.*

Lemma 3.2.4. *If further $(\mathbf{H}_2^\alpha)$ holds, then we obtain the following equivalences or convergences as β goes to 0,*

$$
\begin{cases}
c(\beta) \sim \frac{\alpha\theta}{2-\alpha}\beta^{2-\alpha}, & & \\
\frac{\rho_+(\beta)}{\log 1/\beta} \to \alpha\theta_+ & \frac{\rho_-(\beta)}{\log 1/\beta} \to \alpha\theta_-, & \\
\beta^{\alpha-1} d_+(\beta) \to \frac{\alpha\theta_+}{\alpha-1} & \beta^{\alpha-1} d_-(\beta) \to \frac{\alpha\theta_-}{\alpha-1} & \text{if } \alpha > 1\\
\frac{d_+(\beta)}{\log(1/\beta)} \to \theta_+ & \frac{d_-(\beta)}{\log(1/\beta)} \to \theta_- & \text{if } \alpha = 1\\
d_+(\beta) \to d_+ & d_-(\beta) \to d_- & \text{if } \alpha < 1
\end{cases} \tag{3.2.3}
$$

with some positive constants d_+ and d_-.

Lemma 3.2.5. *When $\alpha = 1$, under assumption ($\mathbf{H}_2^\alpha$), we have for every $b > 0$ and as $\beta \to 0$*

$$\frac{1}{(\log(1/\beta))^2} \int_{\beta<|x|\le b} (x \log|x|) F(dx) \to -\frac{\theta'}{2}. \tag{3.2.4}$$

Besides, for a given truncating sequence $(\beta_n)_{n\ge 0}$ that tends to zero as n tends to infinity, we introduce

$$\boxed{c_n := c(\beta_n),\ \ d_n := d(\beta_n),\ \ d'_n := d'(\beta_n),\ \ \rho_n := \rho(\beta_n) \ \ \text{and } \delta_n := \delta(\beta_n),}$$

to make the notations less cluttered. We deduce easily from (3.2.2) that under ($\mathbf{H}_1^\alpha$), we have

$$\begin{cases} c_n \le C\beta_n^{2-\alpha}, \\ d'_n + |d_n| + \delta_n \le C s(\beta_n), \\ \rho_n \le C \log\left(\frac{1}{\beta_n}\right). \end{cases} \tag{3.2.5}$$

3.2.2 Semimartingale decomposition

Now, we give a decomposition of the process Y.

• For a predictable real function δ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ and a real measure m, we denote $\delta * m$ the stochastic integral process given by $\delta * m_t = \int_0^t \int_{\mathbb{R}^d} \delta(s,x) m(ds,dx)$ for $t \ge 0$. Let μ denotes the jump measure of our driving Lévy process Y and $\nu(ds,dx) = ds \otimes F(dx)$ is its predictable compensator. For $\beta > 0$, we can write

$$\begin{aligned} &Y = Y^\beta + N^\beta, \ \text{ where } Y^\beta = A^\beta + M^\beta \ \text{ with} \\ &A_t^\beta = d(\beta)t, \quad M_t^\beta = x\mathbb{1}_{\{|x|\le\beta\}} * (\mu - \nu)_t \quad \text{and} \quad N_t^\beta = x\mathbb{1}_{\{|x|>\beta\}} * \mu_t. \end{aligned} \tag{3.2.6}$$

Then M^β is a square-integrable martingale with predictable bracket $< M^\beta, M^\beta >_t = c(\beta)t$. Moreover under assumption (A), for $\beta \ge p$ we have $N^\beta = 0$ and then $Y = A^\beta + M^\beta$ with $A_t^\beta = b't$, whereas for $\beta = 1$ we have $A_t^1 = bt$.

• In the context of the multilevel type error (3.1.3), we consider two time discretization grids. The coarse grid with time step $1/n$ and with associated times $t_i := \frac{i-1}{n}$ for all $i \in \{1,\dots,n+1\}$. The finer grid with time step $1/nm$ and with associated times

$$t_i^k := \frac{m(i-1)+k-1}{nm},$$

with $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m+1\}$ and $m \in \mathbb{N}\setminus\{0,1\}$. Note that $t_1^1 = t_0$ and $t_i^1 = t_i$ corresponding to the coarser grid. We note also that the point of the coarse grid can be written either as a final point t_{i-1}^{m+1} or as the next point t_i^1 on the same grid. Now, for a given truncating sequence $(\beta_n)_{n\ge0}$ that tends to zero as $n \to \infty$, we denote

$$\boxed{M_{t_i^k,t}^{\beta_n} = M_t^{\beta_n} - M_{t_i^k}^{\beta_n}} \quad \text{for } t \in I(nm,i,k) := (t_i^k, t_i^{k+1}],$$

with $(i,k) \in \{1,\dots,n\} \times \{1,\dots,m\}$. Further, let $\left(T_p^{\beta_n}(t_i^k)\right)_{p\ge1}$ denotes the sequence of the successive jump times of Y after time t_i^k and of size bigger than or equal to β_n. Let also $K(t_i^k)$ denotes the random number of jumps occurring in the time interval

$I(nm, i, k)$ that satisfies $T^{\beta_n}_{K(t^k_i)}(t^k_i) \le t^{k+1}_i < T^{\beta_n}_{K(t^k_i)+1}(t^k_i)$. Note that the random number $K(t^k_i)$ is well-defined as we use the cut-off of size β_n. Then, the following two main properties hold:

(P1) Conditionally on $\mathcal{F}_{t^k_i}$, the random variables $(\Delta Y_{T^{\beta_n}_p(t^k_i)})_{p\ge 1}$, $K(t^k_i)$ and $\{M^{\beta_n}_{t^k_i,t}, t \in I(nm, i, k)\}$ are independent. Each $\Delta Y_{T^{\beta_n}_p(t^k_i)}$ has the density $\frac{1}{\theta(\beta_n)} 1_{|x|>\beta_n} F(dx)$ and $K(t^k_i)$ has Poisson law with parameter

$$\boxed{\lambda_{n,m} = \frac{\theta(\beta_n)}{nm}.}$$

(P2) The process $\{M^{\beta_n}_{t^k_i,t}, t \in I(nm, i, k)\}$ is a Lévy process, independent of $\mathcal{F}_{t^k_i}$ and satisfying for $t \in I(nm, i, k)$, for $v \in \mathbb{R}$

$$\mathbb{E}(e^{\mathtt{i}vM^{\beta_n}_{t^k_i,t}}) = \exp\left\{\left(t - t^k_i\right)\int_{|y|\le\beta_n} (e^{\mathtt{i}vy} - 1 - \mathtt{i}vy)F(dy)\right\}.$$

We also note that under Hypothesis $(\mathbf{H}^\alpha_1)$, with some choice β_n going to 0, we have $\lambda_{n,m} \to 0$ as $n \to \infty$.

3.3 Main results

Our main results are to prove the convergence in law of $u_{n,m}U^{n,m}_{\eta_n(.)}$ to the limit process with the above choices of the rate $u_{n,m}$ corresponding to those cases. First of all, we assume that function f always satisfies assumption $(\mathbf{H}_f)$. The theorem below is about the tightness which can be easily deduced by Lemma 3.4.5, Lemma 3.4.4, Lemma 3.4.8, Lemma 3.4.7, Lemma 3.4.11, Lemma 3.4.10, Lemma 3.4.14, Lemma 3.4.13, Corollary 3.4.3 and Lemma 3.7.2 in appendix 3.7.

Theorem 3.3.1. *Assume that hypothesis $(\mathbf{H}^\alpha_1)$ holds for some $\alpha \in (0, 2)$. Then, with the above choice of $u_{n,m}$ in the previous section, the sequence $(u_{n,m}U^{n,m}_{\eta_n(.)})$ is tight.*

Let $\overline{Y}^n$ be the discretized process associated with Y, that is $\overline{Y}^n_t = Y_{\eta_n(t)}$. We observe that the sequence $\overline{Y}^n$ converges pointwise to the process Y for the Skorohod topology. The following limit theorem considering the error between two consecutive levels Euler approximations is covered by (C1).

Theorem 3.3.2. *For case (C1), the sequence*

$$(\overline{Y}^n, u_{n,m}U^{n,m}_{\eta_n(.)}) \xrightarrow{\mathcal{L}} (Y, U), \ \textit{when } n \to \infty$$

where U is the unique solution of the linear equation

$$U_t = \int_0^t f'(X_{s-})U_{s-}dY_s - Z_t, \quad t \in [0, 1] \tag{3.3.1}$$

and when letting $n \to \infty$,

$$Z_t = d\sum_{k:R_k\le t}\left(ff'(X_{R_k-})\Delta Y_{R_k}\frac{\lfloor m\Upsilon_k\rfloor}{m-1} + (f(X_{R_k}) - f(X_{R_k-}))(1 - \frac{\lfloor m\Upsilon_k\rfloor}{m-1})\right)$$

$$+\frac{d^2}{2}\int_0^t f(X_{s-})f'(X_{s-})ds \quad (3.3.2)$$

where $d = b - \int_{|x|\leq 1} xF(dx)$, $(R_k)_{k\geq 1}$ denotes an enumeration of the jump times of Y (or of X) and $(\Upsilon_k)_{k\geq 1}$ is a sequence of i.i.d. variables, uniform on $[0,1]$ and independent of Y.

Proof. This proof uses some results in Section 3.4. In particular, we begin with the decomposition of $u_{n,m}U^{n,m}_{\eta_n(.)}$ from (3.4.2). From Theorem 3.4.6 and Lemma 3.4.4, we have $(\overline{Y}^n, u_{n,m}Z^{n,m}) \xrightarrow{\mathcal{L}} (Y, Z)$. Then the result is straightforward according to Theorem 3.4.2. □

Remark 3.3.3. *From Theorem 3.3.2, if we let $m \to \infty$, we will recover the same limit as the case of Euler (Case 3a in Jacod, 2004), that is*

$$Z_t = d\sum_{k:R_k\leq t}([f(X_{R_k}) - f(X_{R_k-})]\Upsilon_k + f'(X_{R_k-})\Delta X_{R_k}(1-\Upsilon_k)) + \frac{d^2}{2}\int_0^t f(X_{s-})f'(X_{s-})ds,$$

where $d = b - \int_{|x|\leq 1} xF(dx)$ and $(\Upsilon_k)_{k\geq 1}$ is a sequence of i.i.d. variables, uniform on $[0,1]$ and independent of Y, and $(R_k)_{k\geq 1}$ is an enumeration of the jump times of Y (or of X).

The following limit theorem is of the rest cases and with stronger assumption $(\mathbf{H}_2^\alpha)$ for some $\alpha \in (0,2)$.

Theorem 3.3.4. *We have that the sequence*

$$(\overline{Y}^n, u_{n,m}U^{n,m}_{\eta_n(.)}) \xrightarrow{\mathcal{L}} (Y, U), \text{ when } n\to\infty$$

where U is the unique solution of the linear equation (3.3.1) and the process Z is described as follows:

(a) In (C2) and (C4),

$$Z_t = \int_0^t f(X_{s-})f'(X_{s-})dV_s \quad (3.3.3)$$

where V is another Lévy process, independent of Y and characterized by

$$\mathbb{E}(e^{iuV_t}) = \exp\left(\frac{t\theta^2\alpha}{4}\int\frac{1}{|x|^{1+\alpha}}(e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}})dx\right). \quad (3.3.4)$$

Hence, V is a symmetric stable process with index α.

(b) In (C3),

$$Z_t = -\frac{\theta'^2}{4}\int_0^t f(X_{s-})f'(X_{s-})ds, \quad (3.3.5)$$

and we even have that $u_{n,m}U^{n,m}_{\eta_n(.)}$ converges to U in probability (locally uniformly in time).

(c) In (C5), we have (3.3.3), where V is a Lévy process, independent of Y and characterized by

$$\mathbb{E}(e^{iuV_t}) = \exp\left(\frac{t\alpha}{2}\int\left\{\left[(\theta_+^2+\theta_-^2)\mathbb{1}_{\{x>0\}}+2\theta_+\theta_-\mathbb{1}_{\{x<0\}}\right]\frac{1}{|x|^{1+\alpha}}(e^{iux}-1-iux)\right\}dx\right). \tag{3.3.6}$$

Hence, V is stable with index α.

Proof. This proof also uses some of the results in Section 3.4. In particular, we begin with the decomposition of $u_{n,m}U^{n,m}_{\eta_n(\cdot)}$ from (3.4.2). From Theorem 3.4.9, Theorem 3.4.12, Theorem 3.4.15, Lemma 3.4.7, Lemma 3.4.10 and Lemma 3.4.13, we have $(\overline{Y}^n, u_{n,m}Z^{n,m}) \xrightarrow{\mathcal{L}} (Y, Z)$. Then the result is straightforward according to Theorem 3.4.2. □

3.4 Error analysis

As mentioned in subsection 3.2.1, with no loss of generality, we develop our error analysis under assumption (A). For $t \in [0,1]$, we first recall that $\eta_n(t) = \frac{[nt]}{n}$. The error between two consecutive levels Euler approximations $U_t^{n,m} := X^n_{\eta_{nm}(t)} - X^{nm}_{\eta_{nm}(t)}$ is given by

$$\begin{aligned}U^{n,m}_{\eta_n(t)} &= \int_0^{\eta_n(t)}\Big(f(X^n_{\eta_{nm}(s-)}) - f(X^{nm}_{\eta_{nm}(s-)})\Big)dY_s - \int_0^{\eta_n(t)}\Big(f(X^n_{\eta_{nm}(s-)}) - f(X^n_{\eta_n(s-)})\Big)dY_s\\ &= \int_0^{\eta_n(t)}\Big(f(X^{nm}_{\eta_{nm}(s-)} + U^{n,m}_{s-}) - f(X^{nm}_{\eta_{nm}(s-)})\Big)dY_s\\ &\qquad - \int_0^{\eta_n(t)}\Big(f(X^n_{\eta_{nm}(s-)}) - f(X^n_{\eta_n(s-)})\Big)dY_s.\end{aligned}$$

Let $G(x,y) := f(x+yf(x)) - f(x)$, using the interpolation Euler scheme $X^n_{\eta_{nm}(s-)} = X^n_{\eta_n(s-)} + f(X^n_{\eta_n(s-)})(Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})$, we deduce that

$$\begin{aligned}U^{n,m}_{\eta_n(t)} = &\int_0^{\eta_n(t)}\Big(f(X^{nm}_{\eta_{nm}(s-)} + U^{n,m}_{s-}) - f(X^{nm}_{\eta_{nm}(s-)})\Big)dY_s\\ &- \int_0^{\eta_n(t)} G(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})dY_s.\end{aligned} \tag{3.4.1}$$

Remark 3.4.1. *Under assumption (A), using Taylor's expansion we write*

$$G(x,y) = yff'(x) + y^2k(x,y),$$

where k is a $\mathcal{C}^1$ function that vanishes outside $K\times\mathbb{R}$ for some compact subset $K\subset\mathbb{R}$. Also, we note that ff' has compact support.

Consequently, given a deterministic rate of convergence $u_{n,m}$, we write

$$u_{n,m}U^{n,m}_{\eta_n(t)} = u_{n,m}\int_0^{\eta_n(t)}\Big(f(X^{nm}_{\eta_{nm}(s-)} + U^{n,m}_{s-}) - f(X^{nm}_{\eta_{nm}(s-)})\Big)dY_s - u_{n,m}Z^{n,m}_t, \tag{3.4.2}$$

where

$$Z_t^{n,m} := \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})dY_s + \int_0^{\eta_n(t)} k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})(Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})^2 dY_s. \quad (3.4.3)$$

Now, recalling the notation $\overline{Y}^n_t = Y_{\eta_n(t)}$, for $t \in [0,1]$, we follow the same arguments as in Graham et al., 1995, Theorem 9.3 page 40 to prove the following result.

Proposition 3.4.2. *For $n \to \infty$, if $(\overline{Y}^n, u_{n,m}Z^{n,m}) \xrightarrow{\mathcal{L}} (Y, Z)$ then $(\overline{Y}^n, u_{n,m}U^{n,m}_{\eta_n(.)}) \xrightarrow{\mathcal{L}} (Y, U)$, where the limiting process U is solution to (3.3.1).*

Proof. By (3.4.2), we have

$$u_{n,m}U^{n,m}_{\eta_n(t)} = \int_0^{\eta_n(t)} \frac{f(X^{nm}_{\eta_{nm}(s-)} + U^{n,m}_{s-}) - f(X^{nm}_{\eta_{nm}(s-)})}{U^{n,m}_{s-}} (u_{n,m}U^{n,m}_{s-}) \mathbb{1}_{\{U^{n,m}_{s-} \neq 0\}} dY_s - u_{n,m}Z^{n,m}_t.$$

Let $T^{n,a} = \inf\{t > 0 : |u_{n,m}U^{n,m}_t| > a\}$. Then the sequence $(u_{n,m}U^{n,m}_{t \wedge T^{n,a}})$ is relatively compact, and any limit point will satisfy (3.3.1) on $[0, T^a]$, where $T^a = \inf\{t > 0 : |U_t| > a\}$. But $\lim_{a \uparrow \infty} T^a = \infty$ a.s., so $u_{n,m}U^{n,m}_{\eta_n(.)} \xrightarrow{\mathcal{L}} U$. □

Corollary 3.4.3. *The tightness of the sequence $(\overline{Y}^n, u_{n,m}U^{n,m}_{\eta_n(.)})$ is a straightforward consequence of the tightness of the sequence $(\overline{Y}^n, u_{n,m}Z^{n,m})$.*

Thus, the aim now is to study the asymptotic behavior of the couple $(\overline{Y}^n, u_{n,m}Z^{n,m})$. To do so, on the one hand, we set

$$u_{n,m}Z^{n,m}_t =: \mathcal{M}^{n,m}_t + \mathcal{R}^{n,m}_t, \quad (3.4.4)$$

where $\mathcal{M}^{n,m}$ stands for the main term contributing in the limit behavior and $\mathcal{R}^{n,m}_t$ stands for the rest term that will tend to zero. In the sequel, the expression of the above decomposition has to be specified for each case (C1)-(C5). It is worth noticing that the second term in (3.4.3) will not contribute in the limit and will be considered as a part of $\mathcal{R}^{n,m}$ except for (C1) where it will be considered as a part of $\mathcal{M}^{n,m}$. On the other hand, we also need to rewrite the process $\overline{Y}^n$ in a triangular array form. To do so, recalling our notations given in subsection 3.2.2, with the formula (3.2.6) and taking into account the number of jumps $K(t^k_i)$ occuring in the time interval $I(nm, i, k)$, with a truncating sequence $(\beta_n)_{n \geq 0}$, we write

$$\begin{aligned} &\overline{Y}^n := \overline{Y}^n(1) + \overline{Y}^n(2), \quad \text{where} \\ &\overline{Y}^n_t(1) = \textstyle\sum_{i=1}^{[nt]} y^{n,m}_i(1), \quad y^{n,m}_i(1) := \sum_{k=1}^m (M^{\beta_n}_{t^k_i, t^{k+1}_i} + \sum_{j \geq 2} \Delta Y_{T^{\beta_n}_j(t^k_i)} \mathbb{1}_{\{K(t^k_i) \geq j\}}), \\ &\overline{Y}^n_t(2) = \textstyle\sum_{i=1}^{[nt]} y^{n,m}_i(2), \quad y^{n,m}_i(2) := \sum_{k=1}^m \left(\frac{d_n}{nm} + \Delta Y_{T^{\beta_n}_1(t^k_i)} \mathbb{1}_{\{K(t^k_i) \geq 1\}} \right). \end{aligned} \quad (3.4.5)$$

In particular, $(\overline{Y}^n(1))_{n \geq 0}$ converges uniformly in probability to zero for all cases except case (C3) and $(\overline{Y}^n(2))_{n \geq 0}$ is tight for all cases. Each subsection below is dedicated to study separately each case. Note that from now on, we denote C as some positive generic constant that can be changed from line to line. Moreover, by the notation

$\Gamma^n \xrightarrow{\mathbb{P}} 0$, we mean that $\sup_{s\le t}|\Gamma^n_s|$ goes to 0 in probability for all t as n tends to infinity.

3.4.1 Asymptotic behavior of the couple $(\overline{Y}^n, u_{n,m}Z^{n,m})$ for case (C1).

For this subsection, we first need to introduce some complementary notations. Following subsection 3.2.1, let $(T_p^{\beta_n}(t_i^1,t_i^k))_{p\ge1}$ denotes the sequence of jump times and $K(t_i^1,t_i^k)$ the random number of jumps that occur on the interval $(t_i^1,t_i^k]$, where we recall that $t_i^1 = \frac{i-1}{n}$ and $t_i^k = \frac{m(i-1)+k-1}{nm}$. Then, each $\Delta Y_{T_p^{\beta_n}(t_i^1,t_i^k)}$ has the density $\frac{1}{\theta(\beta_n)}\mathbb{1}_{\{|x|>\beta_n\}}F(dx)$ and $K(t_i^1,t_i^k)$ has a Poisson distribution with parameter $(k-1)\lambda_{n,m}$. By the representation formula (3.2.6), we obtain a decomposition for the normalized error term $u_{n,m}Z_t^{n,m}$. More precisely, from (3.4.3), we write

$$u_{n,m}Z_t^{n,m} = \sum_{i=1}^{5}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(i)$$
$$+\overline{\Gamma}^n_t(1) + \int_0^{\eta_n(t)} k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})d\overline{\Gamma}^n_s(3) + \overline{\Gamma}^n_t(4) + \overline{\Gamma}^n_t(5),$$

where

$$\begin{cases}
\Gamma^n_t(1) = & u_{n,m}\int_0^t (A^{\beta_n}_{\eta_{nm}(s-)} - A^{\beta_n}_{\eta_n(s-)})dA^{\beta_n}_s,\\
\Gamma^n_t(2) = & u_{n,m}\int_0^t (A^{\beta_n}_{\eta_{nm}(s-)} - A^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s + u_{n,m}\int_0^t (N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})dA^{\beta_n}_s,\\
\Gamma^n_t(3) = & u_{n,m}\int_0^t (M^{\beta_n}_{\eta_{nm}(s-)} - M^{\beta_n}_{\eta_n(s-)})dY_s,\\
\Gamma^n_t(4) = & u_{n,m}\int_0^t (N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s,\\
\Gamma^n_t(5) = & u_{n,m}\int_0^t [(A^{\beta_n}_{\eta_{nm}(s-)} - A^{\beta_n}_{\eta_n(s-)}) + (N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})]dM^{\beta_n}_s,\\
\overline{\Gamma}^n_t(1) = & \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^m k(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})^2,\\
\overline{\Gamma}^n_t(2) = & u_{n,m}\int_0^t (N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})^2 dA^{\beta_n}_s,\\
\overline{\Gamma}^n_t(3) = & u_{n,m}\int_0^t (Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})^2 dY_s - \overline{\Gamma}^n_t(2),\\
\overline{\Gamma}^n_t(4) = & \int_0^{\eta_n(t)} (k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)}) - k(X^n_{\eta_n(s-)}, N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)}))d\overline{\Gamma}^n_s(2),\\
\overline{\Gamma}^n_t(5) = & \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^m (k(X^n_{t_i^1}, N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1}) - k(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)}))(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})^2.
\end{cases}$$

Now, we rewrite $\overline{\Gamma}^n_t(1) = \overline{\Gamma}^n_t(1,1) + \overline{\Gamma}^n_t(1,2)$, where

$$\overline{\Gamma}^n_t(1,1) = \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^m k(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})(\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})^2\mathbb{1}_{\{K(t_i^1,t_i^k)\ge1\}},$$

$$\overline{\Gamma}^n_t(1,2) = \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^m k(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})(\sum_{h=2}^{K(t_i^1,t_i^k)}(\Delta Y_{T_h^{\beta_n}(t_i^1,t_i^k)})^2$$
$$+\sum_{\substack{h,h'=2\\h\ne h'}}^{K(t_i^1,t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^1,t_i^k)}\Delta Y_{T_{h'}^{\beta_n}(t_i^1,t_i^k)}),$$

with the convention $\sum_i^j$ for $j < i$ equals to zero. For the term driven by $\Gamma^n(2)$, we rewrite

$$\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(2) = \Gamma^n_t(2,1) + \Gamma^n_t(2,2) + \Gamma^n_t(2,3),$$

where

$$\Gamma^n_t(2,1) = \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\big[(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1,t_i^k)=0\}}$$
$$+\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)}\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}\big],$$
$$\Gamma^n_t(2,2) = \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\Big[(k-1)\sum_{h=2}^{K(t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^k)} + \sum_{h=2}^{K(t_i^1,t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^1,t_i^k)}\Big],$$
$$\Gamma^n_t(2,3) = \frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}.$$

Then we rewrite $\Gamma^n_t(2,1) + \overline{\Gamma}^n_t(1,1) := \sum_{i=1}^{[nt]}(\tilde{z}_i^n(1) + \tilde{z}_i^n(2))$ where

$$\tilde{z}_i^n(1) =\frac{u_{n,m}d_n}{nm}\sum_{k=2}^{m} ff'(X^n_{t_i^1})(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1,t_i^k)=0\}},$$
$$\tilde{z}_i^n(2) =\frac{u_{n,m}d_n}{nm}\sum_{k=2}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}.$$

Now, concerning $\tilde{z}_i^n(2)$, we rewrite the sum $\sum_{k=2}^m G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}$ as follows

$$\sum_{k=2}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1)\geq 1\}} + \sum_{k=2}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}1_{K(t_i^1)=0}$$
$$=(m-1)G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1)})\mathbb{1}_{\{K(t_i^1)\geq 1\}} + \sum_{k=2}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1)=0\}}\mathbb{1}_{\{K(t_i^2)\geq 1\}}$$
$$+\sum_{k=2}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1)=0\}}\mathbb{1}_{\{K(t_i^2)=0\}}$$

As $K(t_i^k) = K(t_i^k, t_i^{k+1})$ for $k \in \{1, \ldots, m\}$, the first term in the first for $k = 2$ vanishes and the 2 first terms in the second sum vanishes also. Then, we have

$$\sum_{k=2}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}$$
$$=(m-1)G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1)})\mathbb{1}_{\{K(t_i^1)\geq 1\}} + (m-2)G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^2)})\mathbb{1}_{\{K(t_i^1,t_i^2)=0\}}\mathbb{1}_{\{K(t_i^2)\geq 1\}}$$
$$+\sum_{k=4}^{m} G(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1,t_i^2)=0\}}.$$

Then, by induction, we deduce

$$\tilde{z}_i^n(2) = \frac{u_{n,m}d_n}{nm}\sum_{k=1}^{m-1}(m-k)G(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^k)})\mathbb{1}_{\{K(t_i^1,t_i^k)=0\}}\mathbb{1}_{\{K(t_i^k)\geq 1\}}.$$

Therefore, we can rewrite $\Gamma_t^n(2,1)+\overline{\Gamma}_t^n(1,1) = \frac{u_{n,m}d_n}{nm}\sum_{k=1}^m\sum_{i=1}^{[nt]}\tilde{z}^n_{i,k}$, where

$$\tilde{z}^n_{i,k} = [ff'(X^n_{t_i^1})(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)}+(m-k)G(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^k)})]\mathbb{1}_{\{K(t_i^1,t_i^k)=0\}}\mathbb{1}_{\{K(t_i^k)\geq 1\}}. \tag{3.4.6}$$

In this case, we have $u_{n,m}Z_t^{n,m} = \mathcal{M}_t^{n,m}+\mathcal{R}_t^{n,m}$, with

$$\begin{aligned}
\mathcal{M}_t^{n,m} &= \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma_s^n(1) + \Gamma_t^n(2,1)+\overline{\Gamma}_t^n(1,1) \text{ and}\\
\mathcal{R}_t^{n,m} &= \Gamma_t^n(2,2)+\Gamma_t^n(2,3)+\sum_{i=3}^{5}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma_s^n(i) \qquad (3.4.7)\\
&\quad + \overline{\Gamma}_t^n(1,2) + \int_0^{\eta_n(t)} k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})d\overline{\Gamma}_s^n(3) + \overline{\Gamma}_t^n(4)+\overline{\Gamma}_t^n(5).
\end{aligned}$$

The proof of the following lemma is postponed in Appendix 3.5 below.

Lemma 3.4.4. *For case (C1), we have as $n\to\infty$ the sequences of processes $(\overline{Y}^n(1))_{n\geq 0}$ and $(\mathcal{R}^{n,m})_{n\geq 0}$ converge uniformly in probability to 0.*

Lemma 3.4.5. *For case (C1), we have the sequences $(\overline{Y}^n(2))_{n\geq 0}$ and $(\mathcal{M}^{n,m})_{n\geq 0}$ are tight.*

Proof. First, we consider $\overline{Y}_t^n(2)$ given by (3.4.5). By the Property (P1), the relation (3.2.1) in particular $d_n = b' - d_n'$, the fact that $1-\lambda_{n,m}-e^{-\lambda_{n,m}} \leq \frac{\lambda^2_{n,m}}{2}$, we have

$$|\mathbb{E}(y_i^n(2)|\mathcal{F}_{t_i^1})| = \left|\frac{d_n}{n} + (1-e^{-\lambda_{n,m}})\frac{d'_n}{n\lambda_{n,m}}\right| \leq C\frac{1+\lambda_{n,m}|d'_n|}{n}. \tag{3.4.8}$$

Further, by the Jensen's inequality, $\int_{\mathbb{R}}|x|^2F(dx) < +\infty$ (see Remark 3.2.1) and the inequality $1-e^{-\lambda_{n,m}} \leq \lambda_{n,m}$, we have

$$\mathbb{E}(|y_i^n(2)|^2|\mathcal{F}_{t_i^1}) \leq C(\frac{d_n^2}{n^2} + (1-e^{-\lambda_{n,m}})\frac{1}{n\lambda_{n,m}}) \leq C\frac{1}{n}(\frac{d_n^2}{n}+1). \tag{3.4.9}$$

Then, in case (C1), using the boundedness of $|d_n'|$ and $|d_n|$ (see (3.2.2)), $\lambda_{n,m}\to 0$ as $n\to\infty$, $y_i^n(2)$ satisfies (3.7.3) and (3.7.6) ensuring the tightness from the second part of Lemma 3.7.2.
Now, we consider the tightness of the sequence $(\mathcal{M}^{n,m})_{n\geq 0}$. In this case, we recall that $\mathcal{M}_t^{n,m} = \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma_s^n(1)+\Gamma_t^n(2,1)+\overline{\Gamma}_t^n(1,1)$. We rewrite $\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma_s^n(1) = \sum_{i=1}^{[nt]}\zeta_i^n(1)$, $\zeta_i^n(1) = ff'(X^n_{t_i^k})\frac{m(m-1)u_{n,m}d_n^2}{2n^2m^2}$,

$$\begin{aligned}
\overline{\Gamma}_t^n(1,1) &= \textstyle\sum_{i=1}^{[nt]} z_i^n(1), \quad z_i^n(1) = \frac{u_{n,m}d_n}{nm}\sum_{k=2}^m k(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})(\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})^2\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}},\\
\Gamma_t^n(2,1) &= \textstyle\sum_{i=1}^{[nt]} z_i^n(2), \quad z_i^n(2) = \frac{u_{n,m}d_n}{nm}\sum_{k=2}^m ff'(X^n_{t_i^1})\Big[(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^1,t_i^k)=0\}}\\
&\qquad\qquad + \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)}\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 1\}}\Big].
\end{aligned}$$

By similar arguments and the fact that the functions ff' and k are bounded, we have $\mathbb{E}(|\zeta_i^n(1)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}d_n^2}{n^2}$, $\mathbb{E}(|z_i^n(1)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|}{n^2}$ and $\mathbb{E}(|z_i^n(2)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|\delta_n}{n^2}$. Then, we are in case (C1), using the boundedness $|d_n|$ and δ_n (see (3.2.2)) and $u_{n,m} = \frac{nm}{m-1}$, we obtain that $\zeta_i^n(1)$, $z_i^n(1)$ and $z_i^n(2)$ satisfy (3.7.2) ensuring the tightness from Lemma 3.7.2. □

Theorem 3.4.6. *For case (C1), we have*

$$(\overline{Y}^n(2), \mathcal{M}^{n,m}) \xrightarrow{\mathcal{L}} (Y, Z),$$

where Z is the limit process given in (3.3.2).

Proof. Since ff' is Lipschitz-continuous, by virtue of Lemma 3.7.10, in order to prove the convergence in law of $(\overline{Y}^n, \mathcal{M}^{n,m})$ it suffices to prove the convergence of $(\overline{Y}_1^n(2), \Gamma_1^n(1), \Gamma_1^n(2,1) + \overline{\Gamma}_1^n(1,1))$.
First as $u_{n,m} = \frac{nm}{m-1}$, we use (3.2.2) to get $d_n \to d = b - \int_{|x|<1} xF(dx)$ and then we have

$$\Gamma_1^n(1) = u_{n,m} \sum_{i=1}^{n} \sum_{k=2}^{m} (A^{\beta_n}_{t_i^k} - A^{\beta_n}_{t_i^1})(A^{\beta_n}_{t_i^{k+1}} - A^{\beta_n}_{t_i^k}) = \frac{(m-1)u_{n,m}d_n^2}{2mn} \underset{n\to\infty}{\longrightarrow} \frac{d^2}{2}.$$

Thus, it is enough to prove the convergence in law of the pair $(\overline{Y}_1^n(2), \Gamma_1^n(2,1) + \overline{\Gamma}_1^n(1,1))$. To do so, we recall that $\Gamma_1^n(2,1) + \overline{\Gamma}_1^n(1,1) = \frac{u_{n,m}d_n}{nm} \sum_{k=1}^m \sum_{i=1}^n \tilde{z}_{i,k}^n$, where $\tilde{z}_{i,k}^n$ is given by (3.4.6). Now, following Jacod, 2004, we first study the above triangular array with freezing the component $X^n_{t_i^1}$. Thus, we treat the triangular array with

$$\tilde{z}_{i,k}^n(z) = [ff'(z)(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)} + (m-k)G(z, \Delta Y_{T_1^{\beta_n}(t_i^k)})]\mathbb{1}_{\{K(t_i^1, t_i^k)=0\}}\mathbb{1}_{\{K(t_i^k)\geq 1\}},$$

$i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$ with an arbitrary value $z \in \mathbb{R}$. Next, based on the denotations in (3.4.5), we introduce $\overline{Y'}_t^n(2) = \sum_{i=1}^{[nt]} \sum_{k=1}^m y'^n_{i,k}(2)$, where $y'^n_{i,k}(2) = \Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\geq 1\}}$. In this case, we observe that $\overline{Y}_1^n(2) - \overline{Y'}_1^n(2) = \sum_{i=1}^n \sum_{k=1}^m \frac{d_n}{nm} \underset{n\to\infty}{\longrightarrow} d$ and $\frac{u_{n,m}d_n}{nm} \underset{n\to\infty}{\longrightarrow} \frac{d}{(m-1)}$. Then instead of working with $(\overline{Y}_1^n(2), \Gamma_1^n(2,1) + \overline{\Gamma}_1^n(1,1))$, it is enough to prove the convergence in law of $((\sum_{i=1}^n y'^n_{i,k}(2), \sum_{i=1}^n \tilde{z}_{i,k}^n(z)), k \in \{1, \dots, m\}) \in (\mathbb{R}^2)^m$ to m independent $\mathbb{R}^2$-Lévy processes. First, since this $(\mathbb{R}^2)^m$-vector is tight, it is enough to prove that every weakly convergent subsequence has the same limit. In what follows, we omit the notation for the subsequence for more readability. For the independence of the components of the limit vector, by using Ex.12.8-12.10 in Sato, 1999, we only need to prove the independence between the limit marginals of $(\sum_{i=1}^n y'^n_{i,k}(2), \sum_{i=1}^n y'^n_{i,k'}(2))$, $(\sum_{i=1}^n y'^n_{i,k}(2), \sum_{i=1}^n \tilde{z}^n_{i,k'}(z))$ and $(\sum_{i=1}^n \tilde{z}^n_{i,k}(z), \sum_{i=1}^n \tilde{z}^n_{i,k'}(z))$, for any $k, k' \in \{1, \dots, m\}, k \neq k'$.

- First, by the independent structure of the subsequence marginals $\sum_{i=1}^n (y'^n_{i,k}(2), y'^n_{i,k'}(2))$ for any $k, k' \in \{1, \dots, m\}, k \neq k'$, it is obvious that the limit marginals are independent.

- Second, for fixed $k, k' \in \{1, \dots, m\}, k \neq k'$, we consider the sequence $\sum_{i=1}^n (y'^n_{i,k}(2), \tilde{z}^n_{i,k'}(z))$ whose variables $(y'^n_{i,k}(2), \tilde{z}^n_{i,k'}(z))$, $1 \leq i \leq n$ are i.i.d. For $k > k'$, $y'^n_{i,k}(2)$ and $\tilde{z}^n_{i,k'}(z)$ are obviously independent and the independence of the limit marginals is straightforward. For $k < k'$, we use Lemma 3.7.7 to identify the limit characteristics. Let us denote the law of $(y'^n_{i,k}(2), \tilde{z}^n_{i,k'}(z))$ by $K^{k,k'}_{n,m}$. We will study the convergence of $nK^{k,k'}_{n,m}(h)$ to $K^{k,k'}(h)$ with some function h to be precised later where

$$K^{k,k'}(h) := \frac{1}{m}\int h(x,0)F(dx) + \frac{1}{m}\int h(0, ff'(z)(k'-1)y + (m-k')G(z,y))F(dy).$$

Therefore, we have $nK^{k,k'}_{n,m}(h) = n\mathbb{E}(h(y'^{n}_{i,k}(2), \tilde{z}^n_{i,k'}(z))|\mathcal{F}_{t^1_i})$

$$\begin{aligned}
=& n\mathbb{E}(h(y'^{n}_{i,k}(2), \tilde{z}^n_{i,k'}(z)); K(t^k_i) = 0, K(t^{k'}_i) = 0|\mathcal{F}_{t^1_i}) \\
&+ n\mathbb{E}(h(y'^{n}_{i,k}(2), \tilde{z}^n_{i,k'}(z)); K(t^{k'}_i) \geq 1, K(t^1_i, t^{k'}_i) = 0|\mathcal{F}_{t^1_i}) \\
&+ n\mathbb{E}(h(y'^{n}_{i,k}(2), \tilde{z}^n_{i,k'}(z)); K(t^k_i) = 0, K(t^{k'}_i) \geq 1, K(t^1_i, t^{k'}_i) \geq 1|\mathcal{F}_{t^1_i}) \\
&+ n\mathbb{E}(h(y'^{n}_{i,k}(2), \tilde{z}^n_{i,k'}(z)); K(t^k_i) \geq 1, K(t^{k'}_i) = 0|\mathcal{F}_{t^1_i}) \\
&+ n\mathbb{E}(h(y'^{n}_{i,k}(2), \tilde{z}^n_{i,k'}(z)); K(t^k_i) \geq 1, K(t^{k'}_i) \geq 1|\mathcal{F}_{t^1_i}) \\
=& ne^{-2\lambda_{n,m}}h(0,0) + e^{-\lambda_{n,m}(k'-1)}\frac{1-e^{-\lambda_{n,m}}}{m\lambda_{n,m}}\int_{|y|>\beta_n} h(0, ff'(z)(k'-1)y + (m-k')G(z,y))F(dy) \\
&+ n\mathbb{P}(K(t^k_i) = 0, K(t^1_i, t^{k'}_i) \geq 1|\mathcal{F}_{t^1_i})(1-e^{-\lambda_{n,m}})h(0,0) \\
&+ e^{-\lambda_{n,m}}\frac{(1-e^{-\lambda_{n,m}})}{m\lambda_{n,m}}\int_{|x|>\beta_n} h(x,0)F(dx) + \frac{(1-e^{-\lambda_{n,m}})^2}{m\lambda_{n,m}}\int_{|x|>\beta_n} h(x,0)F(dx).
\end{aligned}$$

Three following headings demonstrate three elements in Lemma 3.7.7, each corresponding to some specific choices of function h.

Concerning assertion (i): Since we work with bounded jumps, we observe that $|ff'(z)(k-1)x + (m-k)G(z,x)| \leq C|x|$ and $|ff'(z)(k-1)x| \leq C|x|$ on $x \in [-p,p]$. We shall choose $h = h_{u,v}$ where $h_{u,v}(x,y) = \mathbb{1}_{\{|x|\geq u, |y|\geq v\}}$ bounded and vanishing on a neighborhood of 0, with $(u,v) \neq (0,0)$. For any function P satisfying $|P(y)| \leq C|y|$, if we have $C|y| \leq v$ then $|P(y)| \leq v$, which yields $h_{u,v}(x, P(y)) = h_{u,v}(x, P(y))\mathbb{1}_{\{C|y|>v\}}$ for any $x, y \in [-p,p]$. Then, we have $nK^{k,k'}_{n,m}(h_{u,v}) \underset{n\to\infty}{\longrightarrow} K^{k,k'}(h_{u,v})$.

Concerning assertion (ii): For this case, we choose $h = h', h''$ where $h'(x,y) = x\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and $h''(x,y) = y\mathbb{1}_{\{x^2+y^2\leq 1\}}$. As $|ff'(z)(k-1)x + (m-k)G(z,x)| \leq C|x|$ and $|ff'(z)(k-1)x| \leq C|x|$ on $x \in [-p,p]$, using the dominated convergence theorem, $\int |x|F(dx) < C$ and $\frac{\theta(\beta_n)}{n} = m\lambda_{n,m}$ converges to 0 as n tends to infinity, we obtain that $nK^{k,k'}_{n,m}(h')$ and $nK^{k,k'}_{n,m}(h'')$ converge respectively to $K^{k,k'}(h')$ and $K^{k,k'}(h'')$ when n goes to infinity.

Concerning assertion (iii): Here we take $h = h_1, h_2, h_3$ where $h_1(x,y) = x^2\mathbb{1}_{\{x^2+y^2\leq 1\}}$, $h_2(x,y) = xy\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and $h_3(x,y) = y^2\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and apply similar arguments as in (ii), we get $nK^{k,k'}_{n,m}(h_1)$, $nK^{k,k'}_{n,m}(h_2)$ and $nK^{k,k'}_{n,m}(h_3)$ converge respectively to $K^{k,k'}(h_1)$, $K^{k,k'}(h_2)$ and $K^{k,k'}(h_3)$ when n goes to infinity. In conclusion, for any fixed $k < k'$ the obtained limit pair has independent marginals, since it has no Gaussian part and its Lévy measure $K^{k,k'}$ is supported on the union of the coordinate axes (see e.g. Ex.12.8 in Sato, 1999).

- Third, we consider the pair $\sum_{i=1}^n(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z))$ for fixed $k, k' \in \{1,\ldots,m\}$, $k \neq k'$ and by symmetry it is enough to study only the case $k < k'$. Let us denote the law of $(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z))$ by $L^{k,k'}_{n,m}$. We will prove that $nL^{k,k'}_{n,m}(h)$ converges to $L^{k,k'}(h)$

with some function h to be precised later and where

$$L^{k,k'}(h) := \frac{1}{m}\int h(ff'(z)(k-1)x + (m-k)G(z,x), 0)F(dx) + \frac{1}{m}\int h(0, ff'(z)(k'-1)y + (m-k')G(z,y))F(dy).$$

Therefore, we have $nL^{k,k'}_{n,m}(h) = n\mathbb{E}(h(\zeta^n_{i,k}(z), \zeta^n_{i,k'}(z))|\mathcal{F}_{t^1_i})$

$$\begin{aligned}
=&n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^k_i) = 0, K(t^{k'}_i) = 0|\mathcal{F}_{t^1_i})\\
&+ n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^{k'}_i) \geq 1, K(t^1_i, t^{k'}_i) = 0|\mathcal{F}_{t^1_i})\\
&+ n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^k_i) = 0, K(t^{k'}_i) \geq 1, K(t^1_i, t^{k'}_i) \geq 1|\mathcal{F}_{t^1_i})\\
&+ n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^k_i) \geq 1, K(t^{k'}_i) = 0, K(t^1_i, t^k_i) = 0|\mathcal{F}_{t^1_i})\\
&+ n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^k_i) \geq 1, K(t^{k'}_i) = 0, K(t^1_i, t^k_i) \geq 1|\mathcal{F}_{t^1_i})\\
&+ n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^k_i) \geq 1, K(t^{k'}_i) \geq 1, K(t^1_i, t^k_i) = 0|\mathcal{F}_{t^1_i})\\
&+ n\mathbb{E}(h(\tilde{z}^n_{i,k}(z), \tilde{z}^n_{i,k'}(z)); K(t^k_i) \geq 1, K(t^{k'}_i) \geq 1, K(t^1_i, t^k_i) \geq 1|\mathcal{F}_{t^1_i})\\
=&ne^{-2\lambda_{n,m}}h(0,0)\\
&+ e^{-\lambda_{n,m}(k'-1)}\frac{1-e^{-\lambda_{n,m}}}{m\lambda_{n,m}}\int_{|y|>\beta_n} h(0, ff'(z)(k'-1)y + (m-k')G(z,y))F(dy)\\
&+ n\mathbb{P}(K(t^k_i) = 0, K(t^1_i, t^{k'}_i) \geq 1|\mathcal{F}_{t^1_i})(1-e^{-\lambda_{n,m}})h(0,0)\\
&+ e^{-\lambda_{n,m}-\lambda_{n,m}(k-1)}\frac{1-e^{-\lambda_{n,m}}}{m\lambda_{n,m}}\int_{|x|>\beta_n} h(ff'(z)(k-1)x + (m-k)G(z,x), 0)F(dx)\\
&+ ne^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})(1-e^{-\lambda_{n,m}(k-1)})h(0,0)\\
&+ \frac{(1-e^{-\lambda_{n,m}})^2 e^{-\lambda_{n,m}(k-1)}}{m\lambda_{n,m}}\int_{|x|>\beta_n} h(ff'(z)(k-1)x + (m-k)G(z,x), 0)F(dx)\\
&+ n(1-e^{-\lambda_{n,m}})^2(1-e^{-\lambda_{n,m}(k-1)})h(0,0).
\end{aligned}$$

Now, to check the three conditions of Lemma 3.7.7, we use similar arguments as in the second point above .

Concerning assertion (i): Since $|ff'(z)(k-1)x + (m-k)G(z,x)| \leq C|x|$ and $|ff'(z)(k-1)x| \leq C|x|$ on $x \in [-p,p]$, we choose $h = h_{u,v}$ where $h_{u,v}(x,y) = \mathbb{1}_{\{|x|\geq u, |y|\geq v\}}$ bounded and vanishing on a neighborhood of 0, with $(u,v) \neq (0,0)$. For any functions P_1 and P_2 satisfying $|P_1(x)| \leq C|x|$ and $|P_2(y)| \leq C|y|$, we have $h_{u,v}(P_1(x), P_2(y)) = h_{u,v}(P_1(x), P_2(y))\mathbb{1}_{\{C|x|\geq u, C|y|\geq v\}}$ for any $x, y \in [-p,p]$. By similar arguments, we have $nL^{k,k'}_{n,m}(h_{u,v}) \underset{n\to\infty}{\longrightarrow} L^{k,k'}(h_{u,v})$.

Concerning assertion (ii): For this case, we choose $h = h', h''$ where $h'(x,y) = x\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and $h''(x,y) = y\mathbb{1}_{\{x^2+y^2\leq 1\}}$. As $|ff'(z)(k-1)x + (m-k)G(z,x)| \leq C|x|$ and $|ff'(z)(k-1)x| \leq C|x|$ on $x \in [-p,p]$, and applying similar arguments used in (ii) of the second point, we obtain that $nL^{k,k'}_{n,m}(h')$ and $nL^{k,k'}_{n,m}(h'')$ converge respectively to $L^{k,k'}(h')$ and $L^{k,k'}(h'')$ when n goes to infinity.

Concerning assertion (iii): Here we take $h_1(x,y) = x^2\mathbb{1}_{\{x^2+y^2\leq 1\}}$, $h_2(x,y) = xy\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and $h_3(x,y) = y^2\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and apply similar arguments as in (iii) of the second point, we get $nL^{k,k'}_{n,m}(h_1)$, $nL^{k,k'}_{n,m}(h_2)$ and $nL^{k,k'}_{n,m}(h_3)$ converge respectively to $L^{k,k'}(h_1)$, $L^{k,k'}(h_2)$ and $L^{k,k'}(h_3)$ when n goes to infinity. In conclusion, for any fixed $k < k'$, the obtained limit pair has independent marginals, no Gaussian part and its Lévy measure $L^{k,k'}$ is supported on the union of the coordinate axes (see e.g. Ex.12.8 in Sato, 1999).
Hence, combining the three above points, the independence of the m $\mathbb{R}^2$-Lévy limits of $\{(\sum_{i=1}^n y'^n_{i,k}(2), \sum_{i=1}^n \tilde{z}^n_{i,k}(z)), k \in \{1,\dots,m\}\} \in (\mathbb{R}^2)^m$ is shown (see e.g. Ex.12.9-Ex.12.10 in Sato, 1999). Now, we turn to identify the limit marginals of our vector. Let us denote the law of $(y'^n_{i,k}(2), \tilde{z}^n_{i,k}(z))$ by $K^k_{n,m}$. We will prove that $nK^k_{n,m}(h)$ converges to $K^k(h)$ where

$$K^k(h) = \frac{1}{m}\int h(x, ff'(z)(k-1)x + (m-k)G(z,x))F(dx).$$

Therefore, we have $nK^k_{n,m}(h) = n\mathbb{E}(h(y'^n_{i,k}(2), \tilde{z}^n_{i,k}(z))|\mathcal{F}_{t^1_i})$

$$\begin{aligned}
=&n\mathbb{E}(h(y'^n_{i,k}(2), \tilde{z}^n_{i,k}(z)); K(t^k_i) = 0|\mathcal{F}_{t^1_i}) + n\mathbb{E}(h(y'^n_{i,k}(2), \tilde{z}^n_{i,k}(z)); K(t^k_i) \geq 1, K(t^1_i, t^k_i) \geq 1|\mathcal{F}_{t^1_i}) \\
&+ n\mathbb{E}(h(y'^n_{i,k}(2), \tilde{z}^n_{i,k}(z)); K(t^k_i) \geq 1, K(t^1_i, t^k_i) = 0|\mathcal{F}_{t^1_i}) \\
=&ne^{-\lambda_{n,m}}h(0,0) + \frac{(1-e^{-\lambda_{n,m}})(1-e^{-\lambda_{n,m}(k-1)})}{m\lambda_{n,m}}\int_{|x|>\beta_n} h(x,0)F(dx) + \\
&e^{-\lambda_{n,m}(k-1)}\frac{1-e^{-\lambda_{n,m}}}{m\lambda_{n,m}}\int_{|x|>\beta_n} h(x, ff'(z)(k-1)x + (m-k)G(z,x))F(dx).
\end{aligned}$$

By the same arguments and specific choices of function h as above, we easily verify the three elements in Lemma 3.7.7. In this case, for any fixed $k \in \{1,\dots,m\}$, the obtained limit pair does not have independent marginals since its Lévy measure K^k is not supported on the union of the coordinate axes. Finally, the vector $((\sum_{i=1}^n y'^n_{i,k}(2), \sum_{i=1}^n \tilde{z}^n_{i,k}(z)), k \in \{1,\dots,m\})$ is convergent in law to $((Y'^k_1, V^k_1(z)), k \in \{1,\dots,m\})$ and the sequence $(\overline{Y'}^n_1(2), \sum_{i=1}^n\sum_{k=1}^m \tilde{z}^n_{i,k}(z))$ weakly converges to $(Y_1 - d, V_1(z))$ where $d = b - \int_{|x|\leq 1} xF(dx)$ and by independence $V_1(z) = \sum_{k=1}^m V^k_1(z)$ with Lévy measure $K(h) = \sum_{k=1}^m \frac{1}{m}\int h(ff'(z)(k-1)x + (m-k)G(z,x))F(dx)$, the drift part equal to $K(x\mathbb{1}_{|x|\leq 1})$ and no Gaussian part (see (ii) and (iii) right above). Since its Lévy measure can also be rewritten as $K(h) = \int_{\mathbb{R}}\int_0^1 h(ff'(z)\lfloor mu\rfloor x + (m-1-\lfloor mu\rfloor)G(z,x))F(dx)du$, similarly to Jacod, 2004, (5.23), a possible representation of the limit process is given by

$$V_1(z) = \sum_{k:R_k\leq 1}\big(ff'(z)\lfloor m\Upsilon_k\rfloor\Delta Y_{R_k} + (m-1-\lfloor m\Upsilon_k\rfloor)(f(z+\Delta Y_{R_k}f(z)) - f(z))\big)$$

where $(R_k)_{k\geq 1}$ denotes an enumeration of the jump times of Y (or of X) and $(\Upsilon_k)_{k\geq 1}$ is a sequence of i.i.d. variables, uniform on $[0,1]$ and independent of Y. It is worth to note that this sum is of finite variation. Now, as said at the beginning, we go back to consider the convergence related to our original term $\tilde{z}^n_{i,k}$ defined in (3.4.6) where $z = X^n_{t^1_i}$ is no longer fixed. As $z \mapsto \tilde{z}^n_{i,k'}(z)$ is continous, by following step by step the proof's arguments of Jacod, 2004, Theorem 1.2(d), we obtain $(\overline{Y'}^n(2), \sum_{i=1}^{[n.]}\sum_{k=1}^m \tilde{z}^n_{i,k'})$ converges in law to $(Y_1 - d, V_1)$ where

$$V_1 = \sum_{k:R_k\le 1}[ff'(X_{R_k-})\lfloor m\Upsilon_k\rfloor\Delta Y_{R_k}+(m-1-\lfloor m\Upsilon_k\rfloor)(f(X_{R_k-}+\Delta Y_{R_k}f(X_{R_k-}))-f(X_{R_k-}))].$$

This completes the proof. □

3.4.2 Asymptotic behavior of the couple $(\overline{Y}^n, u_{n,m}Z^{n,m})$ for case (C2) and (C4).

For the first component $\overline{Y}^n$, we use the same decomposition given by the relation (3.4.5). For the second one, we consider the formula of $u_{n,m}Z^{n,m}$ given in (3.4.3). In these cases, the second term from this formula does not contribute to the limit. Therefore, we need only to give the analysis for its first term and by using the classical decomposition (3.2.6) of Y, we have

$$u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(Y_{\eta_{nm}(s-)}-Y_{\eta_n(s-)})dY_s=\sum_{i=1}^{4}\Gamma_t^n(i), \tag{3.4.10}$$

where

$$\begin{cases}\Gamma_t^n(1)= & u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(N^{\beta_n}_{\eta_{nm}(s-)}-N^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s,\\ \Gamma_t^n(2)= & u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(A^{\beta_n}_{\eta_{nm}(s-)}-A^{\beta_n}_{\eta_n(s-)})dA^{\beta_n}_s\\ & +u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(N^{\beta_n}_{\eta_{nm}(s-)}-N^{\beta_n}_{\eta_n(s-)})dA^{\beta_n}_s\\ & +u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(A^{\beta_n}_{\eta_{nm}(s-)}-A^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s,\\ \Gamma_t^n(3)= & u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(M^{\beta_n}_{\eta_{nm}(s-)}-M^{\beta_n}_{\eta_n(s-)})dY_s,\\ \Gamma_t^n(4)= & u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(A^{\beta_n}_{\eta_{nm}(s-)}-A^{\beta_n}_{\eta_n(s-)})dM^{\beta_n}_s\\ & +u_{n,m}\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(N^{\beta_n}_{\eta_{nm}(s-)}-N^{\beta_n}_{\eta_n(s-)})dM^{\beta_n}_s.\end{cases}$$

The three last terms in the above decomposition do not contribute on the limit. Then we only have to study $\Gamma_t^n(1)$ and we seperate it into 2 terms: the first term that will be the essential term of the limit contains only the first jumps and the drifts and the second term that will be sort in the rest terms contains all the other jumps. More precisely, we rewrite

$$\begin{aligned}\Gamma_t^n(1) &= u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m}\int_{I(nm,i,k)} ff'(X^n_{t_i^1})(N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})dN^{\beta_n}_s\\ &= u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\sum_{j=1}^{k-1}(N^{\beta_n}_{t_i^{j+1}}-N^{\beta_n}_{t_i^j})(N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k})=\Gamma_t^n(1,1)+\Gamma_t^n(1,2),\end{aligned}$$

where

$$\begin{aligned}\Gamma_t^n(1,1) =& u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\ge 1\}}\sum_{j=1}^{k-1}\Delta Y_{T_1^{\beta_n}(t_i^j)}\mathbb{1}_{\{K(t_i^j)\ge 1\}},\\ \Gamma_t^n(1,2) =& u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\ge 1\}}\sum_{j=1}^{k-1}\sum_{h=2}^{K(t_i^j)}\Delta Y_{T_h^{\beta_n}(t_i^j)}\\ &+ u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\sum_{h=2}^{K(t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^k)}\sum_{j=1}^{k-1}\Delta Y_{T_1^{\beta_n}(t_i^j)}\mathbb{1}_{\{K(t_i^j)\ge 1\}}\end{aligned} \tag{3.4.11}$$

$$+ u_{n,m} \sum_{i=1}^{[nt]} \sum_{k=2}^{m} ff'(X^n_{t^1_i}) \sum_{h=2}^{K(t^k_i)} \Delta Y_{T^{\beta_n}_h(t^k_i)} \sum_{j=1}^{k-1} \sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)}.$$

In this case, $u_{n,m} Z^{n,m}_t = \mathcal{M}^{n,m}_t + \mathcal{R}^{n,m}_t$, with

$$\mathcal{M}^{n,m}_t = \Gamma^n_t(1,1) \quad \text{and} \quad \mathcal{R}^{n,m}_t = \Gamma^n_t(1,2) + \sum_{i=2}^{5} \Gamma^n_t(i), \tag{3.4.12}$$

where $\Gamma^n_t(5) = u_{n,m} \int_0^{\eta_n(t)} k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})(Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})^2 dY_s$. The proof of the following lemma is postponed to Appendix 3.5.

Lemma 3.4.7. *For the cases (C2) and (C4), we have as $n \to \infty$ the sequences $(\overline{Y}^n(1))_{n\geq 0}$ and $(\mathcal{R}^{n,m})_{n\geq 0}$ converge uniformly in probability to 0.*

Lemma 3.4.8. *For the cases (C2) and (C4), the sequences $(\overline{Y}^n(2))_{n\geq 0}$ and $(\mathcal{M}^{n,m})_{n\geq 0}$ are tight.*

Proof. First, we consider $\overline{Y}^n_t(2)$ given by (3.4.5). In these cases (C2) and (C4), thanks to assumption $(\mathbf{H}_3)$, we have $d'_n = 0$ and $|d_n| \leq C$. Then from (3.4.8) and (3.4.9), $y^n_i(2)$ satisfies (3.7.3) ensuring the tightness of $(\overline{Y}^n(2))_{n\geq 0}$ from the second part of Lemma 3.7.2. Now, we rewrite that $\Gamma^n_t(1,1) = \sum_{k=1}^m \sum_{j=1}^{k-1} \sum_{i=1}^{[nt]} \zeta^n_{i,k,j}$ with

$$\zeta^n_{i,k,j} = u_{n,m} ff'(X^n_{t^1_i}) \Delta Y_{T^{\beta_n}_1(t^k_i)} \mathbb{1}_{\{K(t^k_i)\geq 1\}} \Delta Y_{T^{\beta_n}_1(t^j_i)} \mathbb{1}_{\{K(t^j_i)\geq 1\}}.$$

For $k \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k-1\}$ fixed, by using $(\mathbf{H}_3)$, Property (P1), ff' is bounded, the inequality $1 - e^{-\lambda_{n,m}} \leq \lambda_{n,m}$, $c(\beta) = \int_{|x|\leq\beta} x^2 F(dx)$ where $c(\beta) \leq C\beta^{2-\alpha}$ (see (3.2.2)) and $\theta(\beta) \leq C\beta^{-\alpha}$ (see $(\mathbf{H}^\alpha_1)$), we get

$$\begin{cases} \mathbb{E}(\zeta^n_{i,k,j} \mathbb{1}_{|\zeta^n_{i,k,j}|\leq 1} | \mathcal{F}_{t^1_i}) = ff'(X^n_{t^1_i}) u_{n,m} \frac{(1-e^{-\lambda_{n,m}})^2}{n^2 m^2 \lambda^2_{n,m}} \int_{|x|>\beta_n} \int_{\frac{1}{|ff'(X^n_{t^1_i})| u_{n,m}|x|} \geq |y| > \beta_n} xy F(dx) F(dy) = 0, \\ \mathbb{E}(|\zeta^n_{i,k,j}|^2 \mathbb{1}_{|\zeta^n_{i,k,j}|\leq 1} | \mathcal{F}_{t^1_i}) \leq C u^2_{n,m} \frac{(1-e^{-\lambda_{n,m}})^2}{n^2 \lambda^2_{n,m}} \int_{|x|>\beta_n} x^2 c(\frac{1}{|ff'(X^n_{t^1_i})| u_{n,m}|x|}) F(dx) \leq C \frac{u^\alpha_{n,m} \rho_n}{n^2}, \\ \mathbb{P}(|\zeta^n_{i,k,j}| > y | \mathcal{F}_{t^1_i}) \leq C \frac{(1-e^{-\lambda_{n,m}})^2}{n^2 \lambda^2_{n,m}} \int_{|x|>\beta_n} \theta(\frac{y}{|ff'(X^n_{t^1_i})| u_{n,m}|x|}) F(dx) \leq C \frac{u^\alpha_{n,m} \rho_n}{n^2 y^\alpha}, \quad \forall y > 1 \end{cases}.$$

Then, we conclude the tightness of $(\mathcal{M}^{n,m})_{n\geq 0}$ by $\rho_n \leq C \log 1/\beta_n$ (see (3.2.5)), the choice $u_{n,m} = \left[\frac{nm}{(m-1)\log n}\right]^{1/\alpha}$ with $\alpha \leq 1$, criteria (3.7.4) and Lemma 3.7.2. ☐

Theorem 3.4.9. *For cases (C2) and (C4), we have*

$$(\overline{Y}^n(2), \mathcal{M}^{n,m}) \xrightarrow{\mathcal{L}} (Y, Z),$$

where Z is the limit process given in (3.3.4).

Proof. Let us first introduce $\Gamma'^n_t(1,1) = \sum_{k=1}^m \sum_{j=1}^{k-1} \sum_{i=1}^{[nt]} \zeta^n_{i,j,k}(1)$, where for any $1 \leq k \leq m$ and $1 \leq j \leq k-1$,

$$\zeta^n_{i,j,k}(1) = u_{n,m} \Delta Y_{T^{\beta_n}_1(t^k_i)} \mathbb{1}_{\{K(t^k_i)\geq 1\}} \Delta Y_{T^{\beta_n}_1(t^j_i)} \mathbb{1}_{\{K(t^j_i)\geq 1\}}.$$

Since ff' is Lipschitz-continuous, by virtue of Lemma 3.7.10, in order to prove the convergence in law of $(\overline{Y}^n, \mathcal{M}^{n,m})$ it suffices to prove that $(\overline{Y}^n_1(2), \Gamma'^n_1(1,1))$ converges in law to (Y_1, V_1) where V is a Lévy process independent of Y and characterized by (3.3.4). Now, let us denote

$$\overline{Y'}^n_t(2) = \sum_{i=1}^{[nt]} \sum_{k=1}^{m} {y'}^n_{i,k}(2), \quad \text{where} \quad {y'}^n_{i,k}(2) = \Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k) \geq 1\}}.$$

From the hypothesis ($\mathbf{H}_3$), $d_n = b$ in this case, then $\overline{Y}^n_1(2) - \overline{Y'}^n_1(2) = \sum_{i=1}^{n} \sum_{k=1}^{m} \frac{d_n}{nm} = b$ which allows us to prove instead the couple $(\overline{Y'}^n_1(2), \Gamma'^n_1(1,1))$ converges to the limit process $(Y_1 - b, V_1)$ with no drift and no continuous martingale part. To do so, we choose the strategy of proving the convergence of the $\mathbb{R}^{\frac{m(m-1)}{2}+m}$-vector

$$\left(\left(\sum_{i=1}^{n} {y'}^n_{i,k}(2) \right)_{k \in \{1,\dots,m\}}, \left(\sum_{i=1}^{n} \zeta^n_{i,j',k'}(1) \right)_{\substack{k',j' \in \{1,\dots,m\} \\ j' < k'}} \right)$$

to the limit vector whose components are pairwise independent Lévy processes. First, since this $\mathbb{R}^{\frac{m(m-1)}{2}+m}$-vector is tight thanks to Lemma 3.4.8, it is enough to prove that every weakly convergent subsequence has the same limit. In what follows, we omit the notation for the subsequence for more readability. For the independence of the components of the limit vector, by using Ex.12.8-12.10 in Sato, 1999, we only need to prove the independence between the limit marginals of the pairs $((\sum_{i=1}^{n} {y'}^n_{i,k}(2), \sum_{i=1}^{n} {y'}^n_{i,k'}(2)); k \neq k')$, $(\sum_{i=1}^{n} {y'}^n_{i,k}(2), \sum_{i=1}^{n} \zeta^n_{i,j',k'}(1))$ and $((\sum_{i=1}^{n} \zeta^n_{i,j,k}(1), \sum_{i=1}^{n} \zeta^n_{i,j',k'}(1)); (k,j) \neq (k',j'))$, for any $k, k', j, j' \in \{1,\dots,m\}$, and $j < k$, $j' < k'$, then we obtain the Fourrier transform of the limit vector.

- First, for any $k, k' \in \{1,\dots,m\}, k \neq k'$, by the independent structure of the subsequence marginals $\sum_{i=1}^{n} ({y'}^n_{i,k}(2), {y'}^n_{i,k'}(2))$, it is obvious that the limit marginals are i.i.d.

- Second, for fixed $k, k', j' \in \{1,\dots,m\}$ and such that $1 \leq j' \leq k'-1$ we consider the convergence of the triangular array $\sum_{i=1}^{n} ({y'}^n_{i,k}(2), \zeta^n_{i,j',k'}(1))$ whose generic terms $(({y'}^n_{i,k}(2), \zeta^n_{i,j',k'}(1)))_{1 \leq i \leq n}$ are i.i.d. When k, k' and j' are different, the independence between the marginals is obvious. By symmetry of the roles played by j' and k' in $\zeta^n_{i,j',k'}(1)$, it is sufficient to consider the case when $k = k'$ and $j' \neq k'$. Note that the law of $({y'}^n_{i,k}(2), \zeta^n_{i,j',k}(1))$ does not depend on parameters k and j', then we denote it by $K^1_{n,m}$. We will prove that $nK^1_{n,m}(h)$ converges to $K^1(h)$ as n tends to infinity for some suitable function h where

$$K^1(dx,dy) = \frac{1}{m}\delta_0(dy)F(dx) + \frac{\theta^2\alpha}{2m(m-1)}\delta_0(dx)\frac{1}{|y|^{1+\alpha}}dy$$

and δ_0 is Dirac measure sitting at point 0. Therefore, we have

$$\begin{aligned} nK^1_{n,m}(h) &= n\mathbb{E}(h(\Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\geq 1\}}, u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\geq 1\}} \Delta Y_{T_1^{\beta_n}(t_i^{j'})} \mathbb{1}_{\{K(t_i^{j'})\geq 1\}})|\mathcal{F}_{t_i^1}) \\ &= n\mathbb{E}(h(0,0); K(t_i^k) = 0|\mathcal{F}_{t_i^1}) + n\mathbb{E}(h(\Delta Y_{T_1^{\beta_n}(t_i^k)}, 0); K(t_i^k) \geq 1, K(t_i^{j'}) = 0|\mathcal{F}_{t_i^1}) \\ &+ n\mathbb{E}(h(\Delta Y_{T_1^{\beta_n}(t_i^k)}, u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)} \Delta Y_{T_1^{\beta_n}(t_i^{j'})}); K(t_i^k) \geq 1, K(t_i^{j'}) \geq 1|\mathcal{F}_{t_i^1}) \\ &= ne^{-\lambda_{n,m}}h(0,0) + \frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})}{m\lambda_{n,m}} \int_{|x|>\beta_n} h(x,0)F(dx) \end{aligned}$$

$$+\frac{(1-e^{-\lambda_{n,m}})^2}{n(m\lambda_{n,m})^2}\int_{|x|>\beta_n}F(dx)\int_{|y|>\beta_n}h(x,u_{n,m}xy)F(dy).$$

Three following headings demonstrate three elements in Lemma 3.7.7, each corresponding to some specific choices of function h.

Concerning assertion (i): We choose $h=h_{u,v}$ where $h_{u,v}(x,y)=\mathbb{1}_{\{|x|\geq u,|y|\geq v\}}$ for all $u,v\in\mathbb{R}_+$ such that $(u,v)\neq(0,0)$.

a) For $u>0$ and $v=0$,

$$nK^1_{n,m}(h_{u,0})=\frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})}{m\lambda_{n,m}}\int_{|x|>\beta_n}\mathbb{1}_{\{|x|\geq u\}}F(dx)+\frac{(1-e^{-\lambda_{n,m}})^2}{m\lambda_{n,m}}\int_{|x|>\beta_n}\mathbb{1}_{\{|x|\geq u\}}F(dx).$$

As soon as $u>\beta_n$, we have $\int_{|x|>\beta_n}\mathbb{1}_{\{|x|\geq u\}}F(dx)=\theta(u-)$, where $\theta(u-)$ denotes the left limit at point u of the decreasing and right-continuous function $\theta(.)$. Then we get $nK^1_{n,m}(h_{u,0})\underset{n\to\infty}{\longrightarrow}K^1(h_{u,0})=\frac{\theta(u-)}{m}$.

b) For $u=0$ and $v>0$, we have

$$nK^1_{n,m}(h_{0,v})=\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{|x|>\beta_n}\int_{|y|>\beta_n}\mathbb{1}_{\{|u_{n,m}xy|\geq v\}}F(dy)F(dx).$$

Now, we denote the constant $v_m=v(\frac{m-1}{m})^{1/\alpha}$, using $u_{n,m}=\left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$, $\beta_n=\left(\frac{\log n}{n}\right)^{1/\alpha}$ and $u_{n,m}\beta_n=(\frac{m-1}{m})^{1/\alpha}$, then as soon as $\beta_n<v_m$, we have

$$\begin{aligned}&\frac{1}{nm^2}\int_{|x|>\beta_n}\int_{|y|>\beta_n}\mathbb{1}_{\{|u_{n,m}xy|\geq v\}}F(dx)F(dy)\\&=\frac{1}{nm^2}\int_{|x|>\beta_n}\theta(\beta_n\vee\frac{v_m\beta_n}{|x|})F(dx)=\frac{1}{nm^2}(\theta(\beta_n)\theta(v_m-)+\int_{\beta_n<|x|\leq v_m}\theta(\frac{v_m\beta_n}{|x|})F(dx)).\end{aligned}$$

By $(\mathbf{H}_2^\alpha)$, the first term is equivalent to $\frac{\theta}{nm^2\beta_n^\alpha}\theta(v_m-)$ which converges to 0. Considering the second term, let us denote $y_n=\frac{1}{nm^2}\int_{\beta_n<|x|\leq v_m}\theta(\frac{v_m\beta_n}{|x|})F(dx)$. Let $\varepsilon>0$, by $(\mathbf{H}_2^\alpha)$, there exists $\varepsilon'\in(0,v_m)$ such that for $\beta\in(0,\varepsilon')$ we have $|\frac{\beta^\alpha\theta(\beta)}{\theta}-1|\leq\varepsilon$. Then, we denote $y_n=y_n^{1,\varepsilon'}+y_n^{2,\varepsilon'}$, where $y_n^{1,\varepsilon'}=\frac{1}{nm^2}\int_{\frac{v_m\beta_n}{\varepsilon'}<|x|\leq v_m}\theta(\frac{v_m\beta_n}{|x|})F(dx)$ and $y_n^{2,\varepsilon'}=\frac{1}{nm^2}\int_{\beta_n<|x|\leq\frac{v_m\beta_n}{\varepsilon'}}\theta(\frac{v_m\beta_n}{|x|})F(dx)$. On the one hand, by the fact that $\theta(.)$ is decreasing and $(\mathbf{H}_1^\alpha)$, we have $y_n^{2,\varepsilon'}=\frac{1}{nm^2}\int_{\beta_n<|x|\leq\frac{v_m\beta_n}{\varepsilon'}}\theta(\frac{v_m\beta_n}{|x|})F(dx)\leq\frac{\theta(\beta_n)\theta(\varepsilon')}{nm^2}\underset{n\to\infty}{\longrightarrow}0$. On the other hand, if we denote $y'^{1,\varepsilon'}_n=\frac{\theta}{nm^2v_m^\alpha\beta_n^\alpha}\int_{\frac{v_m\beta_n}{\varepsilon'}<|x|\leq v_m}|x|^\alpha F(dx)$, thanks to $\rho_n\sim\alpha\theta\log 1/\beta_n$ (see (3.2.3)) we have $y'^{1,\varepsilon'}_n=\frac{\theta}{nm^2v_m^\alpha\beta_n^\alpha}(\rho(\frac{v_m\beta_n}{\varepsilon'})-\rho(v_m-))\sim\frac{\alpha\theta^2}{nm^2v_m^\alpha\beta_n^\alpha}\log 1/\beta_n$. From $(\mathbf{H}_2^\alpha)$, we have $y'^{1,\varepsilon'}_n(1-\varepsilon)\leq y_n^{1,\varepsilon'}\leq y'^{1,\varepsilon'}_n(1+\varepsilon)$ and since ε is arbitrarily small, $y_n^{1,\varepsilon'}\sim\frac{\alpha\theta^2}{nm^2v_m^\alpha\beta_n^\alpha}\log 1/\beta_n$. Then we get $nK^1_{n,m}(h_{0,v})\underset{n\to\infty}{\longrightarrow}K^1(h_{0,v})=\frac{\theta^2}{m(m-1)v^\alpha}$. In what follows, we will reused the obtained result

$$\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{|x|>\beta_n}\int_{|y|>\beta_n}\mathbb{1}_{\{|u_{n,m}xy|\geq v\}}F(dy)F(dx)\underset{n\to\infty}{\longrightarrow}\frac{\theta^2}{m(m-1)v^\alpha}. \tag{3.4.13}$$

c) For $u>0$ and $v>0$, as soon as $u>\beta_n$, by the inequality $1-e^{-\lambda_{n,m}}\leq\lambda_{n,m}$, we have

$$nK^1_{n,m}(h_{u,v})=\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{|x|\geq u}\int_{|y|>\beta_n}\mathbb{1}_{\{|u_{n,m}xy|\geq v\}}F(dy)F(dx)$$
$$\leq\frac{\theta(u-)\theta(\beta_n)}{nm^2}\underset{n\to\infty}{\longrightarrow}0.$$

Then we get $nK^1_{n,m}(h_{u,v})\underset{n\to\infty}{\longrightarrow}K^1(h_{u,v})=0$.

Concerning assertion (ii): We choose $h=h',h''$ where $h'(x,y)=x\mathbb{1}_{\{x^2+y^2\leq1\}}$ and $h''(x,y)=y\mathbb{1}_{\{x^2+y^2\leq1\}}$. Since $(\mathbf{H}_3)$ holds, the laws $K^1_{n,m}$ and K^1 are invariant under the map $(x,y)\mapsto(-x,y)$ and $(x,y)\mapsto(x,-y)$. Then in this case, we get immediately $nK^1_{n,m}(h)=K^1(h)=0$.

Concerning assertion (iii): Here we take $h=h_1,h_2,h_3$ where $h_1(x,y)=x^2\mathbb{1}_{\{x^2+y^2\leq1\}}$, $h_2(x,y)=xy\mathbb{1}_{\{x^2+y^2\leq1\}}$ and $h_3(x,y)=y^2\mathbb{1}_{\{x^2+y^2\leq1\}}$. First of all, as above in (ii), by hypothesis $(\mathbf{H}_3)$, we have $nK^1_{n,m}(h_2)=K^1(h_2)=0$. Now, we consider

$$nK^1_{n,m}(h_1)=\frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})}{m\lambda_{n,m}}\int_{\beta_n<|x|\leq1}x^2F(dx)$$
$$+\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{1\geq|x|>\beta_n}x^2F(dx)\int_{|y|>\beta_n}\mathbb{1}_{\{|y|\leq\frac{\sqrt{x^{-2}-1}}{u_{n,m}}\}}F(dy)$$

and

$$nK^1_{n,m}(h_3)=\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{1\geq|x|>\beta_n}\int_{|y|>\beta_n}u^2_{n,m}x^2y^2\mathbb{1}_{\{|y|\leq\frac{\sqrt{x^{-2}-1}}{u_{n,m}}\}}F(dx)F(dy).$$

Concerning the term $nK^1_{n,m}(h_1)$, it is clear that as $n\to\infty$, the first term converges to $\frac{1}{m}\int_{|x|\leq1}x^2F(dx)$ and as $\int_{\mathbb{R}}x^2F(dx)<\infty$, its second term is bounded by $\frac{C\theta(\beta_n)}{n}$ which goes to 0 as $n\to\infty$. Therefore, we get $nK^1_{n,m}(h_1)\underset{n\to\infty}{\longrightarrow}nK^1(h_1)$. Concerning the term $nK^1_{n,m}(h_3)$, let $a_m=(\frac{m-1}{m})^{1/\alpha}$ and $a'_m=\frac{a_m}{\sqrt{a_m^2+1}}$, we have for n large enough $\beta_n\leq a'_m$ and

$$nK^1_{n,m}(h_3)=\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{1\geq|x|>a'_m}\int_{|y|>\beta_n}u^2_{n,m}x^2y^2\mathbb{1}_{\{|y|\leq\frac{\sqrt{1-x^2}a_m\beta_n}{|x|}\}}F(dx)F(dy)$$
$$+\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{a'_m\geq|x|>\beta_n}u^2_{n,m}x^2\Big(c\Big(\frac{\sqrt{1-x^2}a_m\beta_n}{|x|}\Big)-c_n\Big)F(dx).$$

Using $\lim_{n\to\infty}\frac{1-e^{\lambda_{n,m}}}{\lambda_{n,m}}=1$ and $c_n\leq C\beta_n^{2-\alpha}$, it is easy to check that the first term in the right-hand side is bounded by $C\frac{u^2_{n,m}c_n}{n}$ which converges to 0 and the term $\frac{u^2_{n,m}}{nm^2}\int_{a'_m\geq|x|>\beta_n}x^2c_nF(dx)$ converges also to 0. Hence, we have $nK^1_{n,m}(h_3)\sim\frac{u^2_{n,m}}{nm^2}\int_{a'_m\geq|x|>\beta_n}x^2c(\frac{\sqrt{1-x^2}a_m\beta_n}{|x|})F(dx)$. Now, using $c(\beta)\sim\frac{\alpha\theta}{2-\alpha}\beta^{2-\alpha}$ for $\beta\to0$, then for $\varepsilon>0$, there exists $\varepsilon'\in(0,1)$ such that for $\beta\in(0,a_m\varepsilon')$ we have $|\frac{(2-\alpha)\beta^{\alpha-2}c(\beta)}{\alpha\theta}-1|\leq\varepsilon$ and for n large enough such that we have $\beta_n<a'_m\varepsilon'$, we can rewrite as follows

$\frac{u_{n,m}^2}{nm^2}\int_{a'_m\geq|x|>\beta_n} x^2 c(\frac{\sqrt{1-x^2}a_m\beta_n}{|x|})F(dx) = x_n + y_n$ where

$$x_n = \frac{u_{n,m}^2}{nm^2}\int_{\beta_n<|x|\leq\beta_n/\varepsilon'} x^2 c(\frac{\sqrt{1-x^2}a_m\beta_n}{|x|})F(dx)$$
$$y_n = \frac{u_{n,m}^2}{nm^2}\int_{a'_m\geq|x|>\beta_n/\varepsilon'} x^2 c(\frac{\sqrt{1-x^2}a_m\beta_n}{|x|})F(dx).$$

On the one hand, for x_n, as $c(.)$ is increasing, we use that $c(\frac{\sqrt{1-x^2}a_m\beta_n}{|x|})$ is bounded to deduce an upper bound equal to $\frac{Cu_{n,m}^2 c(\beta_n/\varepsilon')}{n}$ which converges to 0. On the other hand, for y_n, if we denote $y'_n = \frac{u_{n,m}^2 a_m^{2-\alpha}\beta_n^{2-\alpha}}{nm^2}\int_{a'_m\geq|x|>\beta_n/\varepsilon'}|x|^\alpha(1-x^2)^{\frac{2-\alpha}{2}}F(dx)$, we have $(1-\varepsilon)y'_n \leq y_n \leq (1+\varepsilon)y'_n$ which gives $y_n \sim y'_n$ since ε is arbitrarily small. In what follows, we rewrite $y'_n = y'^1_n + y'^2_n$ where

$$y'^1_n = \frac{\alpha\theta u_{n,m}^2 a_m^{2-\alpha}\beta_n^{2-\alpha}}{(2-\alpha)nm^2}\int_{a'_m\geq|x|>\beta_n/\varepsilon'}|x|^\alpha F(dx),$$
$$y'^2_n = \frac{\alpha\theta u_{n,m}^2 a_m^{2-\alpha}\beta_n^{2-\alpha}}{(2-\alpha)nm^2}\int_{a'_m\geq|x|>\beta_n/\varepsilon'}|x|^\alpha[(1-x^2)^{\frac{2-\alpha}{2}}-1]F(dx).$$

Then by $\rho_n\sim\alpha\theta\log 1/\beta_n$ (see (3.2.3)), $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \left(\frac{\log n}{n}\right)^{1/\alpha}$, we get that

$$y'^1_n \sim \frac{\alpha\theta u_{n,m}^2 a_m^{2-\alpha}\beta_n^{2-\alpha}}{(2-\alpha)nm^2}(\rho(\beta_n/\varepsilon')-\rho(a'_m)) \sim \frac{\alpha^2\theta^2 u_{n,m}^2 a_m^{2-\alpha}\beta_n^{2-\alpha}}{(2-\alpha)nm^2}\log(\varepsilon'/\beta_n)$$
$$\underset{n\to\infty}{\longrightarrow} \frac{\alpha\theta^2}{m(m-1)(2-\alpha)}$$

and that $y'^2_n \leq [(1-\beta_n{}^2)^{\frac{2-\alpha}{2}}-1]y'^1_n$ which converges to 0. Therefore, clearly, $nK^1_{n,m}(h_3) \underset{n\to\infty}{\longrightarrow} K^1(h_3) = \frac{\alpha\theta^2}{m(m-1)(2-\alpha)}$. Thanks to this proof, in particular we have proved that

$$\frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda_{n,m}^2}\int_{1\geq|x|>\beta_n}\int_{|y|>\beta_n} u_{n,m}^2 x^2 y^2 \mathbb{1}_{\{|y|\leq\frac{\sqrt{x^{-2}-1}}{u_{n,m}}\}}F(dx)F(dy) \underset{n\to\infty}{\longrightarrow} \frac{\alpha\theta^2}{m(m-1)(2-\alpha)}. \tag{3.4.14}$$

In conclusion, the obtained limit pair has independent marginals since it has no Gaussian part and its Lévy measure K^1 is supported on the union of the coordinate axes (see e.g. Sato, 1999, Ex.12.8).

- Third, for fixed $k, k', j, j' \in \{1,\dots,m\}$, $(j,k)\neq(j',k')$ and $1\leq j\leq k-1$, $1\leq j'\leq k'-1$ we consider the convergence of $\sum_{i=1}^n(\zeta^n_{i,j,k}(1),\zeta^n_{i,j',k'}(1))$ whose variables $(\zeta^n_{i,j,k}(1),\zeta^n_{i,j',k'}(1))$, $1\leq i\leq n$ are i.i.d. When k, k', j and j' are different, we have straightforward the independence between the marginals of the limit pair. Otherwise, by symmetry of the roles played by j and k and the roles played by j' and k', it is enough to consider the particular case where $k=k'$ and $j\neq j'$. Note that the law of $(\zeta^n_{i,j,k}(1),\zeta^n_{i,j',k}(1))$ does not depend on parameters k and j', we denote its law by

$K^2_{n,m}$. We will prove that $nK^2_{n,m} \underset{n\to\infty}{\longrightarrow} K^2$, where

$$K^2(dx,dy) = \frac{\theta^2\alpha}{2m(m-1)}\delta_0(dy)\frac{1}{|x|^{1+\alpha}}dx + \frac{\theta^2\alpha}{2m(m-1)}\delta_0(dx)\frac{1}{|y|^{1+\alpha}}dy$$

and δ_0 is Dirac measure sitting at point 0. Therefore, we have

$$\begin{aligned}
&nK^2_{n,m}(h) = n\mathbb{E}((\zeta^n_{i,j,k}(1), \zeta^n_{i,j',k}(1))|\mathcal{F}_{t_i^1}) \\
&= n\mathbb{E}(h(0,0); K(t_i^k) = 0|\mathcal{F}_{t_i^1}) + n\mathbb{E}(h(0,0); K(t_i^k) \geq 1, K(t_i^j) = 0, K(t_i^{j'}) = 0|\mathcal{F}_{t_i^1}) \\
&+ n\mathbb{E}(h(0, u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\Delta Y_{T_1^{\beta_n}(t_i^{j'})}); K(t_i^k) \geq 1, K(t_i^j) = 0, K(t_i^{j'}) \geq 1|\mathcal{F}_{t_i^1}) \\
&+ n\mathbb{E}(h(u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\Delta Y_{T_1^{\beta_n}(t_i^j)}, 0); K(t_i^k) \geq 1, K(t_i^j) \geq 1, K(t_i^{j'}) = 0|\mathcal{F}_{t_i^1}) \\
&+ n\mathbb{E}(h(u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\Delta Y_{T_1^{\beta_n}(t_i^j)}, u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\Delta Y_{T_1^{\beta_n}(t_i^{j'})}); K(t_i^k) \geq 1, K(t_i^j) \geq 1, K(t_i^{j'}) \geq 1|\mathcal{F}_{t_i^1}) \\
&= ne^{-\lambda_{n,m}}(1 + e^{-\lambda_{n,m}}(1 - e^{-\lambda_{n,m}}))h(0,0) \\
&+ \frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}} \int_{|x|>\beta_n}\int_{|y|>\beta_n} h(0, u_{n,m}xy)F(dx)F(dy) \\
&+ \frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}} \int_{|x|>\beta_n}\int_{|y|>\beta_n} h(u_{n,m}xy, 0)F(dx)F(dy) \\
&+ \frac{(1-e^{-\lambda_{n,m}})^3}{n^2m^3\lambda^3_{n,m}} \int_{|x|>\beta_n}\int_{|y|>\beta_n}\int_{|z|>\beta_n} h(u_{n,m}xy, u_{n,m}xz)F(dx)F(dy)F(dz).
\end{aligned}$$

Now, we verify the three elements in Lemma 3.7.7 with suitable choices of function h.

Concerning assertion (i): We choose $h = h_{u,v}$ where $h_{u,v}(x,y) = \mathbb{1}_{\{|x|\geq u, |y|\geq v\}}$ for all $u, v \in \mathbb{R}_+$ such that $(u,v) \neq (0,0)$.

a) For $u > 0$ and $v = 0$,

$$\begin{aligned}
nK^2_{n,m}(h_{u,0}) = \frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}} &\int_{|x|>\beta_n}\int_{|y|>\beta_n} \mathbb{1}_{\{|u_{n,m}xy|\geq u\}}F(dx)F(dy) \\
+ \frac{(1-e^{-\lambda_{n,m}})^3}{nm^2\lambda^2_{n,m}} &\int_{|x|>\beta_n}\int_{|y|>\beta_n} \mathbb{1}_{\{|u_{n,m}xy|\geq u\}}F(dx)F(dy).
\end{aligned}$$

By (3.4.13), the first term contributes at the limit and the second term vanishes when $n \to \infty$. Then we get $nK^2_{n,m}(h_{u,0}) \underset{n\to\infty}{\longrightarrow} K^2(h_{u,0}) = \frac{\theta^2}{m(m-1)u^\alpha}$.

b) For $u = 0$ and $v > 0$, we have $nK^2_{n,m}(h_{0,v}) = nK^2_{n,m}(h_{v,0})$. Then, by a) we get that $nK^2_{n,m}(h_{0,v}) \underset{n\to\infty}{\longrightarrow} K^2(h_{0,v}) = \frac{\theta^2}{m(m-1)v^\alpha}$.

c) For $u > 0$ and $v > 0$,

$$\begin{aligned}
nK^2_{n,m}(h_{u,v}) &= \frac{(1-e^{-\lambda_{n,m}})^3}{nm^2\lambda^2_{n,m}} \int_{|x|>\beta_n}\int_{|y|>\beta_n}\int_{|z|>\beta_n} \mathbb{1}_{\{|u_{n,m}xy|\geq u, |u_{n,m}xz|\geq v\}}F(dx)F(dy)F(dz) \\
&\leq \frac{(1-e^{-\lambda_{n,m}})^3}{nm^2\lambda^2_{n,m}} \int_{|x|>\beta_n}\int_{|y|>\beta_n} \mathbb{1}_{\{|u_{n,m}xy|\geq u\}}F(dx)F(dy).
\end{aligned}$$

Again, by (3.4.13), we have $nK^2_{n,m}(h_{u,v}) \underset{n\to\infty}{\longrightarrow} K^2(h_{u,v}) = 0$.

Concerning assertion (ii): We choose $h = h', h''$ where $h'(x,y) = x\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and $h''(x,y) = y\mathbb{1}_{\{x^2+y^2\leq 1\}}$. Similarly as above, by (**H**$_3$), $nK^2_{n,m}(h) = K^2(h) = 0$.

Concerning assertion (iii): We choose $h = h_1, h_2, h_3$ where $h_1(x,y) = x^2\mathbb{1}_{\{x^2+y^2\leq 1\}}$, $h_2(x,y) = xy\mathbb{1}_{\{x^2+y^2\leq 1\}}$ and $h_3(x,y) = y^2\mathbb{1}_{\{x^2+y^2\leq 1\}}$. First of all, as above in (ii), by hypothesis (**H**$_3$), we have $nK^2_{n,m}(h_2) = K^2(h_2) = 0$. Now, by symmetry, we have

$$nK^1_{n,m}(h_1) = nK^1_{n,m}(h_3) = \frac{e^{-\lambda_{n,m}}(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{|x|>\beta_n}\int_{|y|>\beta_n} u^2_{n,m}x^2y^2\mathbb{1}_{\{|u_{n,m}xy|\leq 1\}}F(dx)F(dy)$$

$$+\frac{(1-e^{-\lambda_{n,m}})^3}{n^2m^3\lambda^3_{n,m}}\int_{|x|>\beta_n}\int_{|y|>\beta_n}\int_{|z|>\beta_n} u^2_{n,m}x^2y^2\mathbb{1}_{\{|xy|^2\leq 1/u^2_{n,m}-|xz|^2\}}F(dx)F(dy)F(dz).$$

By similar estimations as in (3.4.14), we can easily deduce

$$\frac{1}{nm^2}\int_{|x|>\beta_n}\int_{|y|>\beta_n} u^2_{n,m}x^2y^2\mathbb{1}_{\{u_{n,m}|xy|\leq 1\}}F(dx)F(dy) \underset{n\to\infty}{\longrightarrow} \frac{\alpha\theta^2}{m(m-1)(2-\alpha)}$$

which gives the limit of the first term and that the second term vanishes as it is bounded by $\frac{(1-e^{-\lambda_{n,m}})^3}{nm^2\lambda^2_{n,m}}\int_{|x|>\beta_n}\int_{|y|>\beta_n} u^2_{n,m}x^2y^2\mathbb{1}_{\{u_{n,m}|xy|\leq 1\}}F(dx)F(dy)$ converging to 0 as $n\to\infty$. Therefore, $nK^2_{n,m}(h_1) \underset{n\to\infty}{\longrightarrow} K^2(h_1)$ and $nK^2_{n,m}(h_3) \underset{n\to\infty}{\longrightarrow} K^2(h_3)$. In conclusion, the obtained limit pair has i.i.d. marginals since it has no Gaussian part and its Lévy measure K^2 is supported on the union of the coordinate axes (see e.g. Ex.12.8 in Sato, 1999).

Overall, by the pairwise independence proven above, we can realize the limit of the vector $(\overline{Y'}^n_1(2), \Gamma'^n_1(1,1))$ as some vector $(\sum_{k=1}^m \overline{Y'}^k_1, \sum_{k=2}^m\sum_{j=1}^{k-1} V_1^{j,k})$ Lévy process with no drift, no Gaussian part and where

$$\mathbb{E}(e^{i(u\sum_{k=1}^m \overline{Y'}^k_1 + v\sum_{k=2}^m\sum_{j=1}^{k-1}V_1^{j,k})}) = [\mathbb{E}(e^{iu\overline{Y'}^1_1})]^m \times [\mathbb{E}(e^{ivV_1^{1,2}})]^{\frac{m(m-1)}{2}}$$

which is equal to

$$\exp\Big(\int F(dx)(e^{iux}-1-iux\mathbb{1}_{\{|x|\leq 1\}}) + \frac{\alpha\theta^2}{2m(m-1)}\int\frac{1}{|x|^{1+\alpha}}(e^{ivx}-1-ivx\mathbb{1}_{\{|x|\leq 1\}})dx\Big).$$

This completes the proof. □

3.4.3 Asymptotic behavior of the couple $(\overline{Y}^n, u_{n,m}Z^{n,m})$ for case (C3).

For the first component $\overline{Y}^n$, we use the same decomposition given by the relation (3.4.5). For the second one, we consider the formula of $u_{n,m}Z^{n,m}$ given in (3.4.3). In this case, the second term from this formula does not contribute to the limit. Therefore, we need only to give the analysis for its first term. To do so, we consider the same decomposition given in (3.4.10). The two last terms in this decomposition do not contribute on the limit. Then we only have to study $\Gamma^n_t(1)$ and $\Gamma^n_t(2)$. For the first one, we use the same decomposition as cases (C2) and (C4), namely we have $\Gamma^n_t(1) = \Gamma^n_t(1,1) + \Gamma^n_t(1,2)$ as given in (3.4.11). Now, similarly, we separate $\Gamma^n(2)$ into two terms: the first term that will be the essential term of the limit contains only the first jumps and the drifts and the second term that will be sort in the rest terms contains all the other jumps. Then, we have

$$\Gamma_t^n(2) = u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m}\Bigg[\int_{I(nm,i,k)} ff'(X^n_{t_i^1})(A^{\beta_n}_{t_i^k}-A^{\beta_n}_{t_i^1})dA^{\beta_n}_s + \int_{I(nm,i,k)} ff'(X^n_{t_i^1})(N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})dA^{\beta_n}_s$$
$$+\int_{I(nm,i,k)} ff'(X^n_{t_i^1})(A^{\beta_n}_{t_i^k}-A^{\beta_n}_{t_i^1})dN^{\beta_n}_s\Bigg]$$
$$=u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\left[\frac{d_n^2(k-1)}{n^2m^2}+\frac{d_n}{nm}\sum_{j=1}^{k-1}(N^{\beta_n}_{t_i^{j+1}}-N^{\beta_n}_{t_i^j})+\frac{d_n(k-1)}{nm}(N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k})\right]$$
$$=\Gamma_t^n(2,1)+\Gamma_t^n(2,2),$$

where

$$\Gamma_t^n(2,1) = u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\Bigg[\frac{d_n^2(k-1)}{n^2m^2}+\frac{d_n}{nm}\sum_{j=1}^{k-1}\Delta Y_{T_1^{\beta_n}(t_i^j)}\mathbb{1}_{\{K(t_i^j)\geq 1\}}$$
$$+\frac{d_n(k-1)}{nm}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\Bigg],$$
$$\Gamma_t^n(2,2) = u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=2}^{m} ff'(X^n_{t_i^1})\left[\frac{d_n}{nm}\sum_{j=1}^{k-1}\sum_{h=2}^{K(t_j^i)}\Delta Y_{T_h^{\beta_n}(t_i^j)}+\frac{d_n(k-1)}{nm}\sum_{h=2}^{K(t_i^j)}\Delta Y_{T_h^{\beta_n}(t_i^j)}\right].$$

In this case, $u_{n,m}Z_t^{n,m}=\mathcal{M}_t^{n,m}+\mathcal{R}_t^{n,m}$, where

$$\mathcal{M}_t^{n,m}=\Gamma_t^n(1,1)+\Gamma_t^n(2,1) \quad\text{and}\quad \mathcal{R}_t^{n,m}=\Gamma_t^n(1,2)+\Gamma_t^n(2,2)+\sum_{i=3}^{5}\Gamma_t^n(i) \quad (3.4.15)$$

where $\Gamma_t^n(5)=u_{n,m}\int_0^{\eta_n(t)}k(X^n_{\eta_n(s-)},Y_{\eta_{nm}(s-)}-Y_{\eta_n(s-)})(Y_{\eta_{nm}(s-)}-Y_{\eta_n(s-)})^2dY_s$. The proof of the following lemma is postponed to Appendix 3.5.

Lemma 3.4.10. *For the case (C3), we have as $n\to\infty$, the sequence $(\mathcal{R}^{n,m})_{n\geq 0}$ converges uniformly to 0 in probability.*

Lemma 3.4.11. *For the case (C3), the sequences $(\overline{Y}^n(1))_{n\geq 0}$, $(\overline{Y}^n(2))_{n\geq 0}$ and $(\mathcal{M}^{n,m})_{n\geq 0}$ are tight.*

Proof. First, instead of working with $\overline{Y}_t^n(1)=\sum_{i=1}^{[nt]}\sum_{k=1}^{m}(M^{\beta_n}_{t_i^k,t_i^{k+1}}+\sum_{j\geq 2}\Delta Y_{T_j^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq j\}})$, it is enough to prove that for each $k\in\{1,\dots,m\}$ the triangular arrays with generic terms $y_{i,k}^{n,m}(1,1)=M^{\beta_n}_{t_i^k,t_i^{k+1}}$ and $y_{i,k}^{n,m}(1,2)=\sum_{j\geq 2}\Delta Y_{T_j^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq j\}}$ are tight. By property (P1), (3.2.5) and Lemma 3.7.5, for the first one, we have

$$\mathbb{E}(y_{i,k}^{n,m}(1,1)|\mathcal{F}_{t_i^1})=0,\quad \mathbb{E}((y_{i,k}^{n,m}(1,1))^2|\mathcal{F}_{t_i^1})=\frac{c_n}{nm}.$$

and we conclude by using (3.2.5), $c_n\leq C\beta_n^{2-\alpha}$, the criteria (3.7.3) and Lemma 3.7.2. For the second one, we have

$$\mathbb{E}(|y_{i,k}^{n,m}(1,2)||\mathcal{F}_{t_i^1})\leq\frac{1}{\theta(\beta_n)}\int_{|x|>\beta_n}|x|F(dx)\sum_{j\geq 2}\mathbb{P}(K(t_i^k)\geq j|\mathcal{F}_{t_i^1})$$

$$= \frac{\delta_n}{\theta(\beta_n)}(\mathbb{E}(K(t_i^k)|\mathcal{F}_{t_i^1}) - \mathbb{P}(K(t_i^k) \geq 1|\mathcal{F}_{t_i^1})) = \frac{\delta_n}{\theta(\beta_n)}(\lambda_{n,m} + e^{-\lambda_{n,m}} - 1) \leq \frac{\delta_n \lambda_{n,m}^2}{\theta(\beta_n)} = \frac{\delta_n \lambda_{n,m}}{nm} \tag{3.4.16}$$

and we conclude by using (3.2.5), $\delta_n \leq C \log 1/\beta_n$, $\lambda_{n,m} \leq \frac{C}{n\beta_n}$ (see ($\mathbf{H}_1^\alpha$)), $\beta_n = \frac{\log n}{n}$, the criteria (3.7.2) and from the second part of Lemma 3.7.2. Therefore, it is clear that for case (C3), $(\overline{Y}^n(1))_{n\geq 0}$ is tight. Next, considering $\overline{Y}^n(2)$, from (3.4.8) and (3.4.9), as d_n and d'_n are bounded by $C \log 1/\beta_n$ from (3.2.5), $\lambda_{n,m} \leq \frac{C}{n\beta_n}$ from ($\mathbf{H}_1^\alpha$), $\beta_n = \frac{\log n}{n}$, $y_i^n(2)$ satisfies (3.7.3) ensuring the tightness of $(\overline{Y}^n(2))_{n\geq 0}$ from Lemma 3.7.2. Finally, we consider $(\mathcal{M}^{n,m})_{n\geq 0}$, equivalently, we prove that $(\Gamma^n(1,1))_{n\geq 0}$ and $(\Gamma^n(2,1))_{n\geq 0}$ are tight. Because ff' is bounded, for $k \in \{2, \dots, m\}$ and $j < k$ fixed, it is enough to prove that the triangular arrays corresponding to generic terms $\zeta_{i,k,j}^n(1)$ and $\zeta_{i,k}^n(2)$ are tight where

$$\zeta_{i,j,k}^n(1) = u_{n,m} \Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{K(t_i^k)\geq 1} \Delta Y_{T_1^{\beta_n}(t_i^j)} \mathbb{1}_{\{K(t_i^j)\geq 1\}}, \tag{3.4.17}$$
$$\zeta_{i,k}^n(2) = u_{n,m} \left[\frac{d_n^2(k-1)}{n^2 m^2} + \frac{d_n}{nm} \sum_{j=1}^{k-1} \Delta Y_{T_1^{\beta_n}(t_i^j)} \mathbb{1}_{\{K(t_i^j)\geq 1\}} + \frac{d_n(k-1)}{nm} \Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\geq 1\}} \right].$$

For the first term, by similar arguments, we have

$$\mathbb{E}(|\zeta_{i,k,j}^n(1)||\mathcal{F}_{t_i^1}) = u_{n,m} \frac{(1 - e^{-\lambda_{n,m}})^2}{(\theta(\beta_n))^2} \delta_n^2 \leq C \frac{u_{n,m} \delta_n^2}{n^2}.$$

Then, the tightness is obtained by (3.2.5) that δ_n are bounded by $C \log 1/\beta_n$, $\beta_n = \frac{\log n}{n}$, $u_{n,m} = \frac{mn}{(m-1)(\log n)^2}$, (3.7.2) and Lemma 3.7.2. For the second term, by using property (P1) and $1 - e^{\lambda_{n,m}} \leq \lambda_{n,m}$, we have

$$\mathbb{E}(|\zeta_{i,k}^n(2)||\mathcal{F}_{t_i^1}) \leq \frac{C u_{n,m} d_n}{n^2}(d_n + 2\delta_n),$$

and we conclude by d_n and δ_n are bounded by $C \log 1/\beta_n$ (see (3.2.5)), $\beta_n = \frac{\log n}{n}$, $u_{n,m} = \frac{mn}{(m-1)(\log n)^2}$, criteria (3.7.2) and Lemma 3.7.2. Therefore, we get $(\mathcal{M}^{n,m})_{n\geq 0}$ is tight. □

Theorem 3.4.12. *For the case (C3), we have*

$$(\overline{Y}^n, \mathcal{M}^{n,m}) \xrightarrow{\mathbb{P}} (Y, Z),$$

where Z is defined as (3.3.5).

Proof. Since $\overline{Y}^n$ converges pointwise to Y when $n \to \infty$ for the Skorokhod topology, then we only need to prove $\mathcal{M}^{n,m} \xrightarrow{\mathbb{P}} Z$. Since ff' is Lipschitz-continuous, by virtue of Lemma 3.7.10 or Lemma 3.7.9, it is enough to prove $\Gamma'^n_1(1,1) + \Gamma'^n_1(2,1) \xrightarrow{\mathbb{P}} -\frac{\theta'^2}{4}$ where $\Gamma'^n_1(1,1) = \sum_{i=1}^n \sum_{k=1}^m \sum_{j=1}^{k-1} \zeta_{i,j,k}^n(1)$ and $\Gamma'^n_1(2,1) = \sum_{k=2}^m \sum_{i=1}^n \zeta_{i,k}^n(2)$ with $\zeta_{i,j,k}^n(1)$ and $\zeta_{i,k}^n(2)$ given by (3.4.17). First, concerning $\Gamma'^n_1(2,1)$, for $k \in \{2, \dots, m\}$ fixed, on the one hand, by using property (P1), we have

$$\mathbb{E}(\zeta_{i,k}^n(2)|\mathcal{F}_{t_i^1}) = u_{n,m} \left[\frac{d_n^2(k-1)}{n^2 m^2} + \frac{2(k-1)d_n}{nm} \frac{1 - e^{-\lambda_{n,m}}}{\theta(\beta_n)} \int_{|x|>\beta_n} xF(dx) \right]$$

$$\sim \frac{(k-1)u_{n,m}}{n^2m^2}(d_n^2+2d_n{d'}_n).$$

From (3.2.1) and (3.2.3), we have ${d'}_n \sim \theta' \log\frac{1}{\beta_n}$ and $d_n = b' - {d'}_n$. Therefore, using $u_{n,m} = \frac{nm}{(m-1)(\log n)^2}$, $\beta_n = \frac{\log n}{n}$ and $\mathbb{E}(\zeta^n_{i,k}(2)|\mathcal{F}_{t^1_i})$ is non random and independent of i, we get $n\mathbb{E}(\zeta^n_{1,k}(2)) \underset{n\to\infty}{\longrightarrow} -\frac{(k-1)\theta'^2}{m(m-1)}$. On the other hand, using property (P1), the inequality $(a+b)^2 \le 2(a^2+b^2)$ and $\int_{\mathbb{R}} x^2F(dx) < \infty$ (see Remark 3.2.1), we have $\mathbb{E}(|\zeta^n_{i,k}(2)|^2|\mathcal{F}_{t^1_i}) \le \frac{Cu^2_{n,m}}{n^3}(\frac{d_n^4}{n}+d_n^2)$. Since $d_n \le C\log 1/\beta_n$ (see (3.2.5)), we proceed similarly as above to get $n\mathbb{E}(|\zeta^n_{1,k}(2)|^2) \underset{n\to\infty}{\longrightarrow} 0$. Then, since $\mathbb{V}\left(\sum_{i=1}^n \zeta^n_{i,k}(2)\right) \le n\mathbb{E}((\zeta^n_{1,k}(2))^2)$, we get $\sum_{i=1}^n \zeta^n_{i,k}(2) \overset{\mathbb{P}}{\to} -\frac{(k-1)\theta'^2}{m(m-1)}$ and we deduce that ${\Gamma'}^n_1(2,1) \overset{\mathbb{P}}{\to} -\frac{\theta'^2}{2}$. Secondly, concerning ${\Gamma'}^n_1(1,1)$, we prove its uniform convergence in probability by considering for $k\in\{2,\dots,m\}$ and $j<k$ fixed, the generic term $\zeta^n_{i,j,k}(1)$. To do so, we apply Lemma 3.7.7 to this sequence in which $(\zeta^n_{i,j,k}(1))_{1\le i\le n}$ are i.i.d. Note that the law of $\zeta^n_{i,j,k}(1)$ does not depend on parameters j and k, then we denote it by $K_{n,m}$. We will prove that $nK_{n,m}(h)$ converges as $n\to\infty$ for some suitable function h corresponding to each assertions of Lemma 3.7.7. Therefore, we have

$$nK_{n,m}(h) = \frac{(1-e^{-\lambda_{n,m}})^2}{nm^2\lambda^2_{n,m}}\int_{|x|>\beta_n}\int_{\{|y|>\beta_n\}} h(u_{n,m}xy)F(dy)F(dx).$$

Concerning assertion (i): Here, we choose $h=h_\omega$ where $h_\omega(x) = \mathbb{1}_{\{|x|>\omega\}}$ with some $\omega>0$. Using $1-e^{-\lambda_{n,m}} \le \lambda_{n,m}$, it is easy to check that $nK_{n,m}(h_\omega) \le \frac{1}{nm^2}\int_{|x|>\beta_n}\theta\left(\frac{\omega}{u_{n,m}|x|}\right)F(dx)$. By hypothesis $(\mathbf{H}^\alpha_1)$ for $\alpha=1$, $\theta(\beta)\le\frac{C}{\beta}$ and $\delta_n\equiv\rho_n$, we have $nK_{n,m}(h_\omega) \le \frac{Cu_{n,m}\rho_n}{n\omega}$. Then, by our choices of $u_{n,m}$, β_n and $\rho(\beta)\le C\log 1/\beta$ (see (3.2.2)), we get $nK_{n,m}(h_\omega) \underset{n\to\infty}{\longrightarrow} 0$.

Concerning assertion (ii): We choose $h=h'$ where $h'(x) = x\mathbb{1}_{\{|x|\le 1\}}$. Using $1-e^{-\lambda_{n,m}} \sim \lambda_{n,m}$, $u_{n,m}\beta_n \underset{n\to\infty}{\longrightarrow} 0$ and assumption (A), namely F vanishes outside $[-p,p]$, we have for n large enough $\beta_n \le \frac{1}{u_{n,m}p} \le \frac{1}{u_{n,m}|x|}$ and

$$\begin{aligned} nK_{n,m}(h') \sim& \frac{u_{n,m}}{nm^2}\int_{|x|>\beta_n} x\int_{\beta_n<|y|\le\frac{1}{u_{n,m}|x|}} yF(dy)F(dx)\\ =& \frac{u_{n,m}}{nm^2}\left({d'}^2_n - \int_{|x|>\beta_n} x\int_{|y|>\frac{1}{u_{n,m}|x|}} yF(dy)F(dx)\right). \end{aligned}$$

Since $u_{n,m} = \frac{nm}{(m-1)(\log n)^2}$ and using ${d'}_n \sim \theta'\log 1/\beta_n$ (see (3.2.3)), the first term in the r.h.s. is equivalent to $\frac{\theta'^2}{m(m-1)}$. Now, taking $\varepsilon>0$, there exists $\varepsilon'\in(0,1)$ such that for $\beta\in(0,\varepsilon')$ we have $|\frac{d'(\beta)}{\log(1/\beta)\theta'}-1|\le\varepsilon$. Considering the second term in the r.h.s., as for n large enough $\frac{1}{u_{n,m}\varepsilon'}\ge\beta_n$, we rewrite it as the sum of x_n and y_n with

$$\begin{cases} x_n = \frac{u_{n,m}}{nm^2}\int_{1/(u_{n,m}\varepsilon')\ge|x|>\beta_n} x\int_{|y|>\frac{1}{u_{n,m}|x|}} yF(dy)F(dx)\\ y_n = \frac{u_{n,m}}{nm^2}\int_{|x|>1/(u_{n,m}\varepsilon')} x\int_{|y|>\frac{1}{u_{n,m}|x|}} yF(dy)F(dx). \end{cases}$$

First, using $\frac{1}{u_{n,m}|x|} \geq \varepsilon'$, $\delta(.)$ is decreasing and $\delta_n \leq C \log n$ from (3.2.5) we derive that

$$|x_n| \leq \frac{u_{n,m}}{nm^2} \int_{1/(u_{n,m}\varepsilon') \geq |x| > \beta_n} |x| F(dx) \delta(\varepsilon') \leq \frac{C u_{n,m} \delta_n}{n} \leq \frac{C}{\log n},$$

which converges to 0. Second, on the one hand, if we denote

$$y'_n = \frac{u_{n,m}\theta'}{nm^2} \int_{|x|>1/(u_{n,m}\varepsilon')} x \log(u_{n,m} \mid x \mid) F(dx),$$

then we have $(1-\varepsilon)y'_n \leq y_n \leq (1+\varepsilon)y'_n$ which gives $y_n \sim y'_n$ since ε is arbitrarily small. On the other hand, using $d'(\beta) \sim \theta' \log 1/\beta$ (see (3.2.3)) we have that $\frac{u_{n,m}\theta'}{nm^2} \log(u_{n,m}) d'(\frac{1}{u_{n,m}\varepsilon'})$ converges to $\frac{\theta'^2}{m(m-1)}$ and thanks to (3.2.4) we have that $\frac{u_{n,m}\theta'}{nm^2} \int_{|x|>1/(u_{n,m}\varepsilon')} x \log(\mid x \mid) F(dx)$ converges to $-\frac{\theta'^2}{2m(m-1)}$. Then, it is clear that $nK_{n,m}(h') \underset{n\to\infty}{\longrightarrow} \frac{\theta'^2}{2m(m-1)}$.

Concerning assertion (iii): Choosing $h = h_1$ where $h_1(x) = x^2 \mathbb{1}_{\{|x|\leq 1\}}$, using the inequality $1 - e^{-\lambda_{n,m}} \leq \lambda_{n,m}$ and $c(\beta) \leq C\beta$ (see (3.2.2)), we have

$$\begin{aligned} nK_{n,m}(h_1) \leq & \frac{u_{n,m}^2}{nm^2} \int_{|x|>\beta_n} \int_{|y|>\beta_n} x^2 y^2 \mathbb{1}_{\{u_{n,m}|xy|\leq 1\}} F(dx) F(dy) \\ \leq & \frac{u_{n,m}^2}{nm^2} \int_{|x|>\beta_n} x^2 c\left(\frac{1}{u_{n,m}|x|}\right) F(dx) \leq \frac{C u_{n,m} \rho_n}{n}. \end{aligned}$$

Therefore, thanks to our choices of $u_{n,m}$, β_n and using $\rho(\beta) \leq C \log 1/\beta$ (see (3.2.2)), we get that $nK_{n,m}(h_1)$ converges to 0 as $n \to \infty$. In conclusion, the limit processes have no Gaussian part, a Lévy measure equal to 0 and a drift part equal to $\frac{\theta'^2}{2m(m-1)}$. Finally, we get $\sum_{i=1}^{n} \zeta_{i,j,k}^n(1) \xrightarrow{\mathbb{P}} \frac{\theta'^2}{2m(m-1)}$ and $\Gamma'^n_1(1,1) \xrightarrow{\mathbb{P}} \sum_{k=2}^{m} \sum_{j=1}^{k-1} \frac{\theta'^2}{2m(m-1)} = \frac{\theta'^2}{4}$. This completes the proof. □

3.4.4 Asymptotic behavior of the couple $(\overline{Y}^n, u_{n,m} Z^{n,m})$ for case (C5).

For the first component $\overline{Y}^n$, we use the same decomposition given by the relation (3.4.5). For the second one, we consider the formula of $u_{n,m} Z^{n,m}$ given in (3.4.3). In this case, the second term from this formula does not contribute to the limit. Therefore, we need only to give the analysis for its first term. We have

$$u_{n,m} \int_0^{\eta_n(t)} f f'(X_{\eta_n(s-)}^n)(Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)}) dY_s = \sum_{i=1}^{4} \Gamma_t^n(i),$$

where

$$\begin{cases} \Gamma_t^n(1) = u_{n,m}(\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(M^{\beta_n}_{\eta_{nm}(s-)} - M^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s \\ \qquad\qquad + \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})dM^{\beta_n}_s), \\ \Gamma_t^n(2) = u_{n,m}(\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(Y^{\beta_n}_{\eta_{nm}(s-)} - Y^{\beta_n}_{\eta_n(s-)})dM^{\beta_n}_s \\ \qquad\qquad + \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(Y^{\beta_n}_{\eta_{nm}(s-)} - Y^{\beta_n}_{\eta_n(s-)})dA^{\beta_n}_s), \\ \Gamma_t^n(3) = u_{n,m}(\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(A^{\beta_n}_{\eta_{nm}(s-)} - A^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s \\ \qquad\qquad + \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})dA^{\beta_n}_s), \\ \Gamma_t^n(4) = u_{n,m} \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})(N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})dN^{\beta_n}_s. \end{cases}$$

In this case, the three last terms do not contribute to the limit and we only have to study the first term $\Gamma^n(1)$. Let us first rewrite $\Gamma_t^n(1) = \sum_{i=1}^{[nt]} \zeta_i^n(1)$, with row-wise i.i.d. random variables $\zeta_i^n, i = 1, 2, \dots$ defined by

$$\zeta_i^n(1) = u_{n,m} ff'(X^n_{t_i^1}) \sum_{k=2}^m \left[(M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1})(N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}) + (N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})(M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}) \right].$$

Now, using Fubini for the second term, we have that

$$\sum_{k=2}^m [\sum_{j=1}^{k-1} (N^{\beta_n}_{t_i^{j+1}} - N^{\beta_n}_{t_i^j})](M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}) = \sum_{k=1}^{m-1} (N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k})(M^{\beta_n}_{t_i^{m+1}} - M^{\beta_n}_{t_i^{k+1}}).$$

Then we can rewrite our triangular array as follows

$$\begin{aligned} \zeta_i^n(1) =& u_{n,m} ff'(X^n_{t_i^1}) \left[\sum_{k=2}^m (M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1})(N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}) + \sum_{k=1}^{m-1} (N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k})(M^{\beta_n}_{t_i^{m+1}} - M^{\beta_n}_{t_i^{k+1}}) \right] \\ =& u_{n,m} ff'(X^n_{t_i^1}) \sum_{k=1}^m (N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k})[(M^{\beta_n}_{t_i^{m+1}} - M^{\beta_n}_{t_i^1}) - (M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k})] \\ =& u_{n,m} ff'(X^n_{t_i^1}) \sum_{k=1}^m \left(\Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k) \geq 1\}} + \sum_{j=2}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)} \right) \tilde{M}^{n,m}_{i,k}, \end{aligned}$$

where $\tilde{M}^{n,m}_{i,k} = (M^{\beta_n}_{t_i^{m+1}} - M^{\beta_n}_{t_i^1}) - (M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}) = \sum\limits_{j=1, j\neq k}^m (M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j})$. Now, we separate $\Gamma^n(1)$ into two terms: the first term which is the essential term of the limit corresponds to the part with the first jumps and the second term which will be sort in the rest terms corresponds to the part of all the other jumps. In particular, we have $\Gamma^n(1) = \Gamma^n(1,1) + \Gamma^n(1,2)$ where

$$\begin{cases} \Gamma_t^n(1,1) = u_{n,m} \sum_{i=1}^{[nt]} \sum_{k=1}^m ff'(X^n_{t_i^1}) \Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k) \geq 1\}} \tilde{M}^{n,m}_{i,k}, \\ \Gamma_t^n(1,2) = u_{n,m} \sum_{i=1}^{[nt]} \sum_{k=1}^m ff'(X^n_{t_i^1}) \sum_{j=2}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)} \tilde{M}^{n,m}_{i,k}. \end{cases} \tag{3.4.18}$$

In this case, we have

$$\mathcal{M}_t^{n,m} = \Gamma_t^n(1,1) \quad \text{and} \quad \mathcal{R}_t^{n,m} = \Gamma_t^n(1,2) + \sum_{i=2}^5 \Gamma_t^n(i), \tag{3.4.19}$$

with $\Gamma_t^n(5) = u_{n,m}\int_0^{\eta_n(t)} k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})(Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)})^2 dY_s$. The proof of the following lemma is postponed to Appendix 3.5.

Lemma 3.4.13. *For case (C5), we have as $n \to \infty$, the sequences $(\overline{Y}^n(1))_{n\geq 0}$ and $(\mathcal{R}^{n,m})_{n\geq 0}$ converge uniformly to 0 in probability.*

Lemma 3.4.14. *For case (C5), the sequences $(\overline{Y}^n(2))_{n\geq 0}$ and $(\mathcal{M}^{n,m})_{n\geq 0}$ are tight.*

Proof. First, we consider the sequence $(\overline{Y}^n(2))_{n\geq 0}$ given by (3.4.5). From (3.4.8) and (3.4.9), using hypothesis $(\mathbf{H}_1^\alpha)$, $\lambda_{n,m} \leq \frac{C}{n\beta_n}$, d_n and d'_n are bounded by $C\beta_n^{1-\alpha}$ from (3.2.2) with the choice $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$, then $y_i^n(2)$ satisfies (3.7.3) ensuring the tightness from the second part of Lemma 3.7.2. Now, we recall that $\mathcal{M}_t^{n,m} = \Gamma_t^n(1,1)$ given by (3.4.18) and as ff' is bounded, for $k,j \in \{1,\dots,m\}$ fixed and $j \neq k$, it is enough to prove that the triangular array with the generic term $\zeta^n_{i,k,j}(1,1) = u_{n,m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j})$ is tight. By using Property (P2) and the independence of the increments, for $|u| \leq 1$ we have

$$\begin{aligned}
\mathbb{E}(e^{\mathtt{i}u\zeta^n_{i,k,j}(1,1)}|\mathcal{F}_{t_i^1}) &= e^{-\lambda_{n,m}} + \frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n} F(dx)\mathbb{E}\left(e^{\mathtt{i}uu_{n,m}x(M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j})}\right)\\
&= e^{-\lambda_{n,m}} + \frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n} F(dx)e^{z_{n,m}(x,u)}\\
&= 1 + \frac{1-e^{-\lambda_{n,m}}}{nm\lambda_{n,m}}\int_{|x|>\beta_n} F(dx)(e^{z_{n,m}(x,u)}-1),
\end{aligned}$$

where $z_{n,m}(x,u) = \frac{1}{nm}\int_{|y|\leq\beta_n}(e^{\mathtt{i}uu_{n,m}xy} - 1 - \mathtt{i}uu_{n,m}xy)F(dy)$. By applying Sato, 1999, Lemma 8.6 for first and second orders, we get $|e^{\mathtt{i}uu_{n,m}xy} - 1 - \mathtt{i}uu_{n,m}xy| \leq C|uu_{n,m}xy| \wedge |uu_{n,m}xy|^2$. As $u_{n,m}\beta_n \underset{n\to\infty}{\longrightarrow} \infty$, for n large enough, combining these results with $\delta(\beta) \leq C\beta^{1-\alpha}$ and $c(\beta) \leq C\beta^{2-\alpha}$ (see (3.2.2)), we have

$$\begin{aligned}
&|z_{n,m}(x,u)| \leq \frac{C}{n}\int_{|y|\leq\beta_n}(|uu_{n,m}xy| \wedge |uu_{n,m}xy|^2)F(dy)\\
&= \frac{C}{n}|uu_{n,m}x|\int_{\beta_n\geq|y|>1/uu_{n,m}|x|)}|y|F(dy) + \frac{C}{n}|uu_{n,m}x|^2\int_{|y|\leq 1/(uu_{n,m}|x|)} y^2F(dy) \leq \frac{C}{n}|uu_{n,m}x|^\alpha.
\end{aligned} \tag{3.4.20}$$

Then, for $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$, the suprema of $|z_{n,m}(x,u)|$ over all $|x| \leq p$ and $|u| \leq 1$ goes to 0 as n tends to infinity. Therefore, using $|e^{z_{n,m}(x,u)} - 1| \leq C|z_{n,m}(x,u)|$ for n large enough by (3.7.3), $\rho_n \leq C\log\frac{1}{\beta_n}$ (see (3.2.2)) and $1 - e^{-\lambda_{n,m}} \leq \lambda_{n,m}$, we deduce that $|\mathbb{E}(e^{\mathtt{i}u\zeta^n_{i,k,j}(1,1)}|\mathcal{F}_{t_i^1}) - 1| \leq \frac{C|u|^\alpha u^\alpha_{n,m}\log\frac{1}{\beta_n}}{n^2}$. Then, $\zeta^n_{i,k,j}(1,1)$ satisfies (3.7.7) with $\xi'''_{n,u} = \frac{C|u|^\alpha u^\alpha_{n,m}}{n}\log\frac{1}{\beta_n}$ which is bounded by C for all $|u| \leq 1$. Thus, combining Lemma 3.7.3 and the second part of Lemma 3.7.2, we get the tightness of $(\mathcal{M}^{n,m})_{n\geq 0}$. □

Theorem 3.4.15. *For case (C5), we have*

$$(\overline{Y}^n(2), \mathcal{M}^{n,m}) \xrightarrow{\mathcal{L}} (Y, Z), \tag{3.4.21}$$

where Z is defined as (3.3.6).

Proof. First, we denote $\Gamma'^{n}_{t}(1,1) = u_{n,m}\sum_{i=1}^{[nt]}\sum_{k=1}^{m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\tilde{M}^{n,m}_{i,k}$. Then, as ff' is Lipschitz-continuous, by virtue of Lemma 3.7.10, in order to prove the convergence in law of the pair $(\overline{Y}^n(2), \mathcal{M}^{n,m})$, it is enough to consider the convergence of the pair $(\overline{Y}^n_1(2), \Gamma'^{n}_1(1,1))$ with $\overline{Y}^n_1(2)$ given in (3.4.5). By the independence structure, for u and v in $\mathbb{R}$, we have

$$\mathbb{E}(e^{\mathtt{i}(u\overline{Y}^n_1(2)+v\Gamma'^{n}_1(1,1))}) = e^{\mathtt{i}ud_n}\mathbb{E}(e^{\mathtt{i}\sum_{i=1}^{n}\sum_{k=1}^{m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})})$$
$$=e^{\mathtt{i}ud_n}\prod_{i=1}^{n}\mathbb{E}(e^{\mathtt{i}\sum_{k=1}^{m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})}).$$

For $i\in\{1,\dots,n\}$ fixed, again by tower property, $\mathbb{E}(e^{\mathtt{i}\sum_{k=1}^{m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})})$ equals to

$$\mathbb{E}(\mathbb{E}(e^{\mathtt{i}\sum_{k=1}^{m}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})}|\sigma(M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j}, j\in\{1,\dots,m\})))$$
$$=\mathbb{E}(\prod_{k=1}^{m}\mathbb{E}(e^{\mathtt{i}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})}|\sigma(M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j}, j\in\{1,\dots,m\}))).$$

For $k\in\{1,\dots,m\}$ fixed, note that $\tilde{M}^{n,m}_{i,k}$ is the martingale part of the small jumps which is $\sigma(M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j}, j\in\{1,\dots,m\})$-measurable, independent of $K(t_i^k)$ and $\Delta Y_{T_1^{\beta_n}(t_i^k)}$ then by (P1) we get that equals to

$$\mathbb{E}(e^{\mathtt{i}\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})}|\sigma(M^{\beta_n}_{t_i^{j+1}} - M^{\beta_n}_{t_i^j}, j\in\{1,\dots,m\})\vee\mathcal{F}_{t_i^k})$$
$$= e^{-\lambda_{n,m}} + \frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n} e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{i,k})}F(dx).$$

Therefore, using the independence structure of $(\tilde{M}^{n,m}_{i,k})_{i\in\{1,\dots,n\}}$ we can easily see by (P2) that for all $i\in\{1,\dots,n\}$ $\tilde{M}^{n,m}_{i,k}$ has the same distribution as $\tilde{M}^{n,m}_{1,k}$. Thus, we get

$$\mathbb{E}(e^{\mathtt{i}(u\overline{Y}^n_1(2)+v\Gamma'^{n}_1(1,1))}) = e^{\mathtt{i}ud_n}\left[\mathbb{E}(\prod_{k=1}^{m}(1+\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1))F(dx))\right]^n$$
$$=e^{\mathtt{i}ud_n}\left[\mathbb{E}\left(\prod_{k=1}^{m}\exp\left(\log(1+\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)F(dx)\right)\right)\right]^n$$
$$=e^{\mathtt{i}ud_n}\left[\mathbb{E}\left(\exp\left(\sum_{k=1}^{m}\log(1+\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)F(dx)\right)\right)\right]$$
$$=e^{\mathtt{i}ud_n}\left[\mathbb{E}\left(\exp\left(\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}F(dx)(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)+\sum_{k=1}^{m}\mathcal{R}^{n,m}_{1,k}\right)\right)\right]^n$$
$$=e^{\mathtt{i}ud_n}\left[\mathbb{E}\left(1+\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}F(dx)(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)+\sum_{k=1}^{m}\mathcal{R}^{n,m}_{1,k}+\mathcal{R}^{n}_{1}\right)\right]^n,$$

where

$$\mathcal{R}^{n,m}_{1,k} = \log\left(1+\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}F(dx)(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)\right)$$
$$-\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}F(dx)(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1),$$
$$\mathcal{R}^{n}_{1} = \exp\left(\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}F(dx)(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)+\sum_{k=1}^{m}\mathcal{R}^{n,m}_{1,k}\right)-1$$
$$-\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}F(dx)(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)-\sum_{k=1}^{m}\mathcal{R}^{n,m}_{1,k}.$$

Now, thanks to property (P2) we have that $\mathbb{E}(e^{\mathtt{i}(u\overline{Y}^n_1(2)+v\Gamma'^n_1(1,1))})$ equals to

$$e^{\mathtt{i}ud_n}\left(1+\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(\mathbb{E}(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})})-1)F(dx)+\sum_{k=1}^{m}\mathbb{E}(\mathcal{R}^{n,m}_{1,k})+\mathbb{E}(\mathcal{R}^{n}_{1})\right)^n$$
$$=e^{\mathtt{i}ud_n}\left(1+\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(e^{\mathtt{i}ux}\prod_{j=1,j\neq k}^{m}\mathbb{E}(e^{\mathtt{i}vu_{n,m}x(M^{\beta_n}_{t^1_{j+1}}-M^{\beta_n}_{t^1_j})})-1)F(dx)\right.$$
$$\left.+\sum_{k=1}^{m}\mathbb{E}(\mathcal{R}^{n,m}_{1,k})+\mathbb{E}(\mathcal{R}^{n}_{1})\right)^n$$
$$=e^{\mathtt{i}ud_n}\left(1+\frac{1-e^{-\lambda_{n,m}}}{n\lambda_{n,m}}\int_{|x|>\beta_n}(e^{\mathtt{i}ux+(m-1)z_{n,m}(x,v)}-1)F(dx)+\sum_{k=1}^{m}\mathbb{E}(\mathcal{R}^{n,m}_{1,k})+\mathbb{E}(\mathcal{R}^{n}_{1})\right)^n, \tag{3.4.22}$$

where $z_{n,m}(x,v)=\frac{1}{nm}\int_{|y|\le\beta_n}(e^{\mathtt{i}vu_{n,m}xy}-1-\mathtt{i}vu_{n,m}xy)F(dy)$. Now, using $|e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1|\le 2$ and $1-e^{-\lambda_{n,m}}\le\lambda_{n,m}$, then, it is easy to check that for n large enough, we have

$$\left|\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1)F(dx)\right|\le C\lambda_{n,m}\le\frac{1}{2}.$$

From this, for any $k\in\{1,\dots,m\}$, using the first evaluation in 3.7.3 and $(\mathbf{H}^{\alpha}_1)$, we have

$$n|\mathbb{E}(\mathcal{R}^{n,m}_{1,k})|\le n\mathbb{E}\left[\frac{1-e^{-\lambda_{n,m}}}{nm\lambda_{n,m}}\int_{|x|>\beta_n}|e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1|F(dx)\right]^2\le C\frac{(\theta(\beta_n))^2}{n}\le\frac{C}{n\beta_n^{2\alpha}}$$

which converges to 0 as $n\to\infty$ by the choice $\beta_n=\frac{\log n}{n^{1/(2\alpha)}}$. Similarly, by the second evaluation in 3.7.3, we have

$$n|\mathbb{E}(\mathcal{R}^{n}_{1})|\le n\mathbb{E}\left(\sum_{k=1}^{m}\frac{1-e^{-\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}|e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1|F(dx)+\sum_{k=1}^{m}|\mathcal{R}^{n,m}_{1,k}|\right)^2$$
$$\le C\left(n\mathbb{E}(\frac{1-e^{-\lambda_{n,m}}}{nm\lambda_{n,m}}\int_{|x|>\beta_n}|e^{\mathtt{i}x(u+vu_{n,m}\tilde{M}^{n,m}_{1,k})}-1|F(dx))^2+n\mathbb{E}(|\mathcal{R}^{n,m}_{1,k}|)^2\right)\le\frac{C}{n\beta_n^{2\alpha}}+\frac{C}{n^3\beta_n^{4\alpha}}$$

which converges to 0 as $n \to \infty$ by the choice $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$. Now, concerning the main term inside the bracket of (3.4.22), we have $\int_{|x|>\beta_n} (e^{\mathrm{i}ux+(m-1)z_{n,m}(x,v)} - 1)F(dx) = A_{n,m}(u) + B_{n,m}(v) + C_{n,m}(u,v)$ where

$$\begin{cases} A_{n,m}(u) = \int_{|x|>\beta_n} (e^{\mathrm{i}ux} - 1)F(dx), \quad B_{n,m}(v) = \int_{|x|>\beta_n} (e^{(m-1)z_{n,m}(v,x)} - 1)F(dx), \\ C_{n,m}(u,v) = \int_{|x|>\beta_n} (e^{\mathrm{i}ux} - 1)(e^{(m-1)z_{n,m}(x,v)} - 1)F(dx). \end{cases}$$

Since $1 - e^{-\lambda_{n,m}} \sim \lambda_{n,m}$, it is enough to prove that the three terms $A_{n,m}(u) + iud_n$, $C_{n,m}(u,v)$ and $B_{n,m}(v)$ converge.

Concerning $A_{n,m}(u) + \mathrm{i}ud_n$. As $d_n = b - \int_{\beta_n<|x|\le 1} xF(dx)$, this term is equal to

$$\mathrm{i}ub + \int_{|x|>\beta_n} (e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\mathbb{1}_{|x|\le 1})F(dx) \underset{n\to\infty}{\longrightarrow} \mathrm{i}ub + \int (e^{\mathrm{i}ux} - 1 - \mathrm{i}ux1_{|x|\le 1})F(dx).$$

Concerning $C_{n,m}(u,v)$. By (3.4.20), $|z_{n,m}(x,v)| \le \frac{C}{n}|vu_{n,m}x|^\alpha$ and the suprema of $|z_{n,m}(x,v)|$ over all $|x| \le p$ and $|v| \le 1$ go to 0 as n tends to 0. Now, using 3.7.3, we have $|e^{(m-1)z_{n,m}(x,v)} - 1| \le C|z_{n,m}(x,v)|$ and $|e^{iux} - 1| \le C|ux|$. Then, since $x \mapsto |x|^{\alpha+1}$ is F-integrable as $\alpha > 1$ (see Remark 3.2.1), we get $|C_{n,m}(u,v)| \le \frac{C}{nm}|u||v|^\alpha u_{n,m}^\alpha \underset{n\to\infty}{\longrightarrow} 0$.

Concerning $B_{n,m}(v)$ We rewrite $B_{n,m}(v) = B'_{n,m}(v) + B''_{n,m}(v)$, with

$$\begin{cases} B'_{n,m}(v) = \int_{|x|>\beta_n} (m-1)z_{n,m}(x,v)F(dx), \\ B''_{n,m}(v) = \int_{|x|>\beta_n} (e^{(m-1)z_{n,m}(x,v)} - 1 - (m-1)z_{n,m}(x,v))F(dx). \end{cases}$$

First, by same arguments as above, we get $|B''_{n,m}(v)| \le \frac{C}{n^2}|v|^{2\alpha}u_{n,m}^{2\alpha}$, hence, $B''_{n,m}(v) \underset{n\to\infty}{\longrightarrow} 0$. Second, note that $B'_{n,m}(v) = \int (e^{ivx} - 1 - ivx)K_{n,m}(dx)$, where

$$K_{n,m}(h) = \frac{m-1}{nm} \int_{|x|>\beta_n} \int_{|y|\le\beta_n} h(u_{n,m}xy)F(dy)F(dx)$$

with some function h. We will prove that $\int (e^{\mathrm{i}vx} - 1 - \mathrm{i}vx)K_{n,m}(dx) \underset{n\to\infty}{\longrightarrow} \int (e^{\mathrm{i}vx} - 1 - \mathrm{i}vx)K(dx)$, with

$$K(dx) = \frac{\alpha}{2}((\theta_+^2 + \theta_-^2)\mathbb{1}_{\{x>0\}} + 2\theta_+\theta_-\mathbb{1}_{\{x<0\}})\frac{1}{|x|^{1+\alpha}}dx.$$

To do so, we use Theorem 3.7.6, then it is reduced to prove that $K_{n,m}(h) \underset{n\to\infty}{\longrightarrow} K(h)$ for h equal either to $h_\omega = \mathbb{1}_{(\omega,\infty)}$ for $\omega > 0$, or $h'_\omega = \mathbb{1}_{(-\infty,-\omega)}$ for $\omega > 0$, or $h'(x) = x^2\mathbb{1}_{\{|x|\le 1\}}$, or $h''(x) = x\mathbb{1}_{\{|x|>1\}}$.

- ***First case*** $h = h_\omega$ Since $u_{n,m}\beta_n^2 \underset{n\to\infty}{\longrightarrow} \infty$, then for n large enough such that $\frac{\omega}{\beta_n u_{n,m}} < \beta_n$. Then, we rewrite $K_{n,m}(h_\omega) = y_{n,m}^1 + y_{n,m}^2$, where

$$\begin{cases} y_{n,m}^1 = \frac{m-1}{nm} \int_{x>\beta_n} (\theta_+(\frac{\omega}{u_{n,m}x}) - \theta_+(\beta_n))F(dx) \\ y_{n,m}^2 = \frac{m-1}{nm} \int_{x<-\beta_n} (\theta_-(\frac{-\omega}{u_{n,m}x}) - \theta_-(\beta_n))F(dx). \end{cases}$$

Note that by $(\mathbf{H}_1^\alpha)$ we have $\frac{(\theta_\pm(\beta_n))^2}{n} \le \frac{C}{n\beta_n^{2\alpha}} \underset{n\to\infty}{\longrightarrow} 0$. Further, as $\frac{\omega}{\beta_n u_{n,m} x} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $\{x > \beta_n\}$, using $(\mathbf{H}_2^\alpha)$, $\frac{\rho_+(\beta_n)}{\log(1/\beta_n)} \underset{n\to\infty}{\longrightarrow} \alpha\theta_+$ (see (3.2.3)), we get $y_{n,m}^1 \sim \frac{m-1}{nm}\int_{x>\beta_n} \frac{\theta_+ u_{n,m}^\alpha x^\alpha}{\omega^\alpha} F(dx) \sim \frac{\alpha\theta_+^2}{\omega^\alpha}\frac{(m-1)u_{n,m}^\alpha}{nm}\log\frac{1}{\beta_n} \underset{n\to\infty}{\longrightarrow} \frac{\theta_+^2}{2\omega^\alpha}$ by the choices $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$. Similarly, we have $y_{n,m}^2 \underset{n\to\infty}{\longrightarrow} \frac{\theta_-^2}{2\omega^\alpha}$, so $K_{n,m}(h_\omega) \underset{n\to\infty}{\longrightarrow} K(h_\omega)$.

- ***Second case*** $h = h'_\omega$ At first, similarly, we rewrite $K_{n,m}(h_\omega) = y'^1_{n,m} + y'^2_{n,m}$, where

$$\begin{cases} y'^1_{n,m} = \frac{m-1}{nm}\int_{x>\beta_n}(\theta_-(\frac{\omega}{u_{n,m}x}) - \theta_-(\beta_n))F(dx) \\ y'^2_{n,m} = \frac{m-1}{nm}\int_{x<-\beta_n}(\theta_+(\frac{-\omega}{u_{n,m}x}) - \theta_+(\beta_n))F(dx). \end{cases}$$

Using the same arguments as above, we easily get $y'^1_{n,m} \sim \frac{m-1}{nm}\int_{x>\beta_n}\frac{\theta_- u_{n,m}^\alpha x^\alpha}{(-\omega)^\alpha}F(dx) \sim \frac{\alpha\theta_-\theta_+ u_{n,m}^\alpha(m-1)}{(-\omega)^\alpha nm}\log 1/\beta_n \underset{n\to\infty}{\longrightarrow} \frac{\theta_+\theta_-}{2(-\omega)^\alpha}$ and also similarly $y'^2_{n,m} \underset{n\to\infty}{\longrightarrow} \frac{\theta_+\theta_-}{2(-\omega)^\alpha}$. Therefore, we have that $K_{n,m}(h'_\omega) \underset{n\to\infty}{\longrightarrow} K(h'_\omega)$.

- ***Third case*** $h = h'$ For n large enough such that $\frac{1}{\beta_n u_{n,m}} < \beta_n$, as $\frac{1}{u_{n,m}|x|} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $\{|x| > \beta_n\}$, by $c(\beta) \sim \frac{\alpha\theta}{2-\alpha}\beta^{2-\alpha}$ and $\rho(\beta) \sim \alpha\theta\log(1/\beta)$ for $\beta \to 0$ (see (3.2.3)), we have

$$K_{n,m}(h') = \frac{m-1}{nm}\int_{|x|>\beta_n}\int_{|y|\le\frac{1}{u_{n,m}|x|}} u_{n,m}^2 x^2 y^2 F(dy)F(dx)$$
$$\sim \frac{m-1}{nm}\int_{|x|>\beta_n}\frac{\alpha\theta}{2-\alpha}u_{n,m}^\alpha|x|^\alpha F(dx) \sim \frac{\alpha^2\theta^2}{2-\alpha}\frac{(m-1)u_{n,m}^\alpha}{nm}\log\frac{1}{\beta_n} \underset{n\to\infty}{\longrightarrow} \frac{\alpha\theta^2}{2(2-\alpha)} = K(h').$$

- ***Last case*** $h = h''$ For n large enough such that $\frac{1}{\beta_n u_{n,m}} < \beta_n$, we have

$$\begin{aligned} K_{n,m}(h'') &= \frac{m-1}{nm}\int_{|x|>\beta_n}\int_{\frac{1}{u_{n,m}|x|}<|y|\le\beta_n} u_{n,m} xy F(dy)F(dx) \\ &= \frac{(m-1)u_{n,m}}{nm}\int_{|x|>\beta_n} x\left(\int_{|y|>\frac{1}{u_{n,m}x}} yF(dy) - d'_n\right)F(dx). \end{aligned}$$

At first, note that by $d'_n \le C\beta_n^{1-\alpha}$ (see (3.2.2)), we have $\frac{(m-1)u_{n,m}d'^2_n}{mn} \underset{n\to\infty}{\longrightarrow} 0$ and $K_{n,m}(h'') \sim \frac{(m-1)u_{n,m}}{nm}\int_{|x|>\beta_n}\int_{|y|>\frac{1}{u_{n,m}x}} xyF(dy)F(dx)$ for $n \to \infty$. Then, as $\frac{1}{u_{n,m}|x|} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $\{|x| > \beta_n\}$, using $d'(\beta) \sim \frac{\alpha}{\alpha-1}\theta'\beta^{1-\alpha}$, $\rho_+(\beta) \sim \alpha\theta_+\log(1/\beta)$ and $\rho_-(\beta) \sim \alpha\theta_-\log(1/\beta)$ for $\beta \to 0$ (see (3.2.3)), we get for $n \to \infty$

$$\begin{aligned} K_{n,m}(h'') &\sim \frac{(m-1)\alpha\theta' u_{n,m}^\alpha}{nm(\alpha-1)}\Big(\int_{x>\beta_n}|x|^\alpha F(dx) - \int_{x<-\beta_n}|x|^\alpha F(dx)\Big) \\ &\sim \frac{m-1}{nm}\frac{\alpha^2\theta'^2}{\alpha-1}u_{n,m}^\alpha\log\frac{1}{\beta_n} \underset{n\to\infty}{\longrightarrow} \frac{\alpha\theta'^2}{2(\alpha-1)} = K(h''). \end{aligned}$$

Therefore, we get

$$B'_{n,m}(v) \underset{n\to\infty}{\longrightarrow} \int \frac{\alpha}{2}[(\theta_+^2+\theta_-^2)\mathbb{1}_{\{x>0\}} + 2\theta_+\theta_-\mathbb{1}_{\{x<0\}}]\frac{1}{|x|^{1+\alpha}}(e^{\mathtt{i}vx}-1-\mathtt{i}vx)dx.$$

Finally, we have

$$\mathbb{E}(\exp(\mathtt{i}(u\overline{Y}_1^n(2)+v\Gamma'^n_1(1,1)))) \underset{n\to\infty}{\longrightarrow} \exp\left\{\mathtt{i}ub + \int F(dx)(e^{\mathtt{i}ux}-1-\mathtt{i}ux\mathbb{1}_{\{|x|\le 1\}}) + \frac{\alpha}{2}[(\theta_+^2+\theta_-^2)\mathbb{1}_{\{x>0\}} + 2\theta_+\theta_-\mathbb{1}_{\{x<0\}}]\frac{1}{|x|^{1+\alpha}}(e^{\mathtt{i}vx}-1-\mathtt{i}vx)dx\right\},$$

which completes the proof. □

3.4.5 Conclusion

The challenge ahead is to apply these approximations to obtain a central limit theorem type for the multilevel Monte Carlo method with the stochastic differential equation (3.1.1) driven by a pure jump Lévy process, in the spirit of the ones established by Ben Alaya and Kebaier, 2015 and Ben Alaya, Kebaier, and Ngô, 2020 for the case of a diffusion process, Dereich and Li, 2016 for the case of jump-diffusion process and Giorgi, Lemaire, and Pagès, 2017 for the case of nested Multilevel Monte Carlo. We keep this work for a future research.

3.5 Appendix A: Proof of lemmas 3.4.4, 3.4.7, 3.4.10, 3.4.13 concerning the rest terms

Note that throughout this section, C is a generic constant (may depending on m) which can be changed from line to line.

3.5.1 Proof of Lemma 3.4.4

Here, we prove that the sequences of processes $(\overline{Y}^n(1))_{n\ge 0}$ and $(\mathcal{R}^{n,m})_{n\ge 0}$ converge uniformly in probability to 0 as $n\to\infty$. First, instead of considering the form $\overline{Y}_t^n(1) = \sum_{i=1}^{[nt]}\sum_{k=1}^{m}(M^{\beta_n}_{t_i^k,t_i^{k+1}} + \sum_{j\ge 2}\Delta Y_{T_j^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\ge j\}})$, it is enough to prove that for each $k\in\{1,\dots,m\}$ the triangular arrays with generic terms $y_{i,k}^{n,m}(1,1) = M^{\beta_n}_{t_i^k,t_i^{k+1}}$ and $y_{i,k}^{n,m}(1,2) = \sum_{j\ge 2}\Delta Y_{T_j^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\ge j\}}$ converge in probability to 0 as $n\to\infty$. By Property (P1), (3.2.5) and Lemma 3.7.5, for the first one, we have

$$\mathbb{E}(y_{i,k}^{n,m}(1,1)|\mathcal{F}_{t_i^1}) = 0, \quad \mathbb{E}((y_{i,k}^{n,m}(1,1))^2|\mathcal{F}_{t_i^1}) = \frac{c_n}{nm} \tag{3.5.1}$$

and therefore we conclude using (3.2.5), $c_n \le C\beta_n^{2-\alpha}$, the criteria (3.7.3) and Lemma 3.7.2. For the second one, we have

$$\begin{aligned}\mathbb{E}(|y_{i,k}^{n,m}(1,2)||\mathcal{F}_{t_i^1}) &\le \frac{1}{\theta(\beta_n)}\int_{|x|>\beta_n}|x|F(dx)\sum_{j\ge 2}\mathbb{P}(K(t_i^k)\ge j|\mathcal{F}_{t_i^1})\\ &= \frac{\delta_n}{\theta(\beta_n)}(\mathbb{E}(K(t_i^k)|\mathcal{F}_{t_i^1})-\mathbb{P}(K(t_i^k)\ge 1|\mathcal{F}_{t_i^1})) = \frac{\delta_n}{\theta(\beta_n)}(\lambda_{n,m}+e^{-\lambda_{n,m}}-1) \le \frac{\delta_n\lambda_{n,m}^2}{\theta(\beta_n)} = \frac{\delta_n\lambda_{n,m}}{nm}\end{aligned} \tag{3.5.2}$$

and therefore we conclude by using (3.2.5), the boundedness of δ_n in case (C1), the criteria (3.7.2) and Lemma 3.7.2. Thus, it is clear that for case (C1), $\overline{Y}^n(1) \xrightarrow{\mathbb{P}} 0$. Now, by using the formula of the rest term given by (3.4.7), we have

$$\begin{aligned}\mathcal{R}_t^{n,m} &= \Gamma_t^n(2,2) + \Gamma_t^n(2,3) + \sum_{i=3}^{5} \int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)}) d\Gamma_s^n(i) \\ &\quad + \overline{\Gamma}_t^n(1,2) + \int_0^{\eta_n(t)} k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)}) d\overline{\Gamma}_s^n(3) + \overline{\Gamma}_t^n(4) + \overline{\Gamma}_t^n(5).\end{aligned}$$

According to Theorem 3.7.1 (iii), in order to prove the convergence of the third and the fifth terms in the r.h.s. of the above relation, we only need to prove the convergence of each $\Gamma^n(i)$, $i \in \{3,4,5\}$ and $\overline{\Gamma}^n(3)$ to 0 as $n \to \infty$. Now, we prove that each term converges uniformly in probability to 0 when $n \to \infty$.

The term $\Gamma_t^n(2,2)$: Let us rewrite $\Gamma_t^n(2,2) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta_{i,k}^n(2,2)$ with

$$\zeta_{i,k}^n(2,2) = \frac{u_{n,m} d_n}{nm} ff'(X^n_{t_i^1})\Big((k-1) \sum_{h=2}^{K(t_i^k)} \Delta Y_{T_h^{\beta_n}(t_i^k)} + \sum_{h=2}^{K(t_i^1,t_i^k)} \Delta Y_{T_h^{\beta_n}(t_i^1,t_i^k)}\Big).$$

For each $k \in \{2,\dots,m\}$, by property (P1), (3.2.5) and the boundedness of ff', similarly to the calculations in (3.5.2), we have

$$\begin{aligned}\mathbb{E}(|\zeta_{i,k}^n(2,2)||\mathcal{F}_{t_i^1}) &\le C\frac{u_{n,m}|d_n|\delta_n}{n\theta(\beta_n)}[(k-1)(\lambda_{n,m} + e^{-\lambda_{n,m}} - 1) + (k-1)\lambda_{n,m} + e^{-(k-1)\lambda_{n,m}} - 1] \\ &\le C\frac{u_{n,m}|d_n|\delta_n\lambda_{n,m}}{n^2}.\end{aligned}$$

Then we conclude by using (3.2.5), the boundedness of $|d_n|$ and δ_n in case (C1), $u_{n,m} = \frac{nm}{m-1}$, the criteria (3.7.2) and Lemma 3.7.2.

The term $\Gamma_t^n(2,3)$: Let us rewrite $\Gamma_t^n(2,3) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta_{i,k}^n(2,3)$ with

$$\zeta_{i,k}^n(2,3) = \frac{u_{n,m} d_n}{nm} ff'(X^n_{t_i^1})(k-1)\Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\ge 1\}} \mathbb{1}_{\{K(t_i^1,t_i^k)\ge 1\}}.$$

For each $k \in \{2,\dots,m\}$, by boundedness of ff', similar as above, we have

$$\mathbb{E}(|\zeta_{i,k}^n(2,3)||\mathcal{F}_{t_i^1}) \le C\frac{u_{n,m}|d_n|\delta_n}{n\theta(\beta_n)}(1-e^{-\lambda_{n,m}})(1-e^{-(k-1)\lambda_{n,m}}) \le C\frac{u_{n,m}|d_n|\delta_n\lambda_{n,m}}{n^2}.$$

Then, we conclude similarly that $\Gamma^n(2,3) \xrightarrow{\mathbb{P}} 0$ by the criteria (3.7.2) and Lemma 3.7.2.

The term $\overline{\Gamma}_t^n(1,2)$: Let us rewrite $\overline{\Gamma}_t^n(1,2) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \overline{\zeta}_{i,k}^n(1,2)$ with

$$\overline{\zeta}_{i,k}^n(1,2) = \frac{u_{n,m} d_n}{nm} k(X^n_{t_i^1}, \Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)})\Big(\sum_{h=2}^{K(t_i^1,t_i^k)} (\Delta Y_{T_h^{\beta_n}(t_i^1,t_i^k)})^2 + \sum_{\substack{h,h'=2\\h\neq h'}}^{K(t_i^1,t_i^k)} \Delta Y_{T_h^{\beta_n}(t_i^1,t_i^k)} \Delta Y_{T_{h'}^{\beta_n}(t_i^1,t_i^k)}\Big).$$

For any fixed $k \in \{2,\dots,m\}$, by similar calculations as above, using the boundedness of the function k and $\int_{\mathbb{R}} x^2 F(dx) < \infty$ (see Remark 3.2.1), we have

$$\mathbb{E}(|\overline{\zeta}_{i,k}^n(1,2)||\mathcal{F}_{t_i^1})$$

$$\leq C\frac{u_{n,m}|d_n|}{n\theta(\beta_n)}\mathbb{E}((K(t_i^1,t_i^k)-1)\mathbb{1}_{\{K(t_i^1,t_i^k)\geq 2\}}+\frac{\delta_n^2}{\theta(\beta_n)}K(t_i^1,t_i^k)(K(t_i^1,t_i^k)-1)|\mathcal{F}_{t_i^1})$$
$$\leq C\frac{u_{n,m}|d_n|}{n^2}(\lambda_{n,m}+\frac{\delta_n^2}{n}).$$

Then, we conclude similarly that $\overline{\Gamma}^n(1,2)\xrightarrow{\mathbb{P}}0$ by the criteria (3.7.2) and Lemma 3.7.2.

The term $\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(3)$: This is equal to $\sum_{k=2}^m\sum_{i=1}^{[nt]}\zeta^n_{i,k}(3)$ with

$$\zeta^n_{i,k}(3)=u_{n,m}ff'(X^n_{t_i^1})(M^{\beta_n}_{t_i^k}-M^{\beta_n}_{t_i^1})(Y_{t_i^{k+1}}-Y_{t_i^k}).$$

For a fixed $k\in\{2,\ldots,m\}$, by the boundedness of ff', Property (P1), (3.2.5), the inequality $(a^2+b^2)\leq 2(a^2+b^2)$, $\int_{\mathbb{R}}x^2F(dx)<\infty$, lemmas 3.7.5 and 3.7.4, we have $\mathbb{E}(\zeta^n_{i,k}(3)|\mathcal{F}_{t_i^1})=0$,

$$\mathbb{E}(|\zeta^n_{i,k}(3)|^2|\mathcal{F}_{t_i^1})\leq Cu^2_{n,m}\mathbb{E}((M^{\beta_n}_{t_i^k}-M^{\beta_n}_{t_i^1})^2|\mathcal{F}_{t_i^1})\mathbb{E}((Y^{\beta_n}_{t_i^{k+1}}-Y^{\beta_n}_{t_i^k})^2+(N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k})^2|\mathcal{F}_{t_i^1})$$
$$\leq C\frac{u^2_{n,m}c_n}{n}(\frac{c_n}{nm}+\frac{d_n^2}{n^2m^2}+\frac{1}{nm})\leq C\frac{u^2_{n,m}c_n}{n^2}(c_n+\frac{d_n^2}{n}+1).$$

Then as $\alpha<1$, we conclude using the boundedness of $|d_n|$, $c_n\leq C\beta_n^{2-\alpha}$, $\beta_n=\frac{(\log n)^2}{n}$, $u_{n,m}=\frac{nm}{m-1}$, the criteria (3.7.3) and Lemma 3.7.2. Therefore, we have $\int_0^{\eta_n(.)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(3)\xrightarrow{\mathbb{P}}0$.

The term $\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(4)$: This is bounded by $C\sum_{k=2}^m\sum_{i=1}^{[nt]}\zeta^n_{i,k}(4)$ with

$$\zeta^n_{i,k}(4)=u_{n,m}|N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1}||N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k}|.$$

Let $k\in\{1,\ldots,m\}$ be fixed. Using the independence of the increments $N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1}$ and $N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k}$ and Lemma 3.7.4, we have

$$\mathbb{E}(|\zeta^n_{i,k}(4)|\mathbb{1}_{\{|\zeta^n_{i,k}(4)|\leq 1\}}|\mathcal{F}_{t_i^1})=\frac{u_{n,m}}{nm}\int_{|x|>\beta_n}|x|\mathbb{E}(|N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1}|\mathbb{1}_{\left\{|N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1}|\leq\frac{1}{u_{n,m}|x|}\right\}})F(dx)$$
$$\leq\frac{Cu_{n,m}\delta_n}{n^2}\int_{\frac{1}{u_{n,m}\beta_n}>|y|>\beta_n}|y|F(dy).$$

Then using the boundedness of δ_n, $\beta_n=\frac{(\log n)^2}{n}$, $u_{n,m}=\frac{nm}{m-1}$ and Lebesgue's theorem, we easily check that the two first points of the criteria (3.7.4) are satisfied. In order to prove the third point of this criteria, noticing that $u_{n,m}\beta_n^2\to 0$ when $n\to\infty$, then for $y>1$ we have $u_{n,m}\beta_n^2<y\Leftrightarrow\beta_n<\frac{y}{u_{n,m}\beta_n}$. Similarly, we have

$$\mathbb{P}(\zeta^n_{i,k}(4)>y|\mathcal{F}_{t_i^1})\leq\frac{C}{n^2}\int_{|x|>\beta_n}F(dx)\int_{|z|>\beta_n\vee\frac{y}{u_{n,m}|x|}}F(dz)=\frac{C}{n^2}\int_{|x|>\beta_n}F(dx)\theta\left(\frac{y}{u_{n,m}|x|}\vee\beta_n\right)$$
$$\leq\frac{C}{n^2}\left(\int_{|x|>\frac{y}{u_{n,m}\beta_n}}F(dx)\theta(\beta_n)+\int_{\frac{y}{u_{n,m}\beta_n}>|x|>\beta_n}F(dx)\theta\left(\frac{y}{u_{n,m}|x|}\right)\right).$$

Now, using $(\mathbf{H}_1^\alpha)$ and as $\frac{y}{u_{n,m}\beta_n} < 1$ for n large enough, we get

$$\mathbb{P}(\zeta^n_{i,k}(4) > y|\mathcal{F}_{t_i^1}) \leq \frac{C}{n^2}\left(\theta\left(\frac{y}{u_{n,m}\beta_n}\right)\theta\left(\beta_n\right) + \frac{u^\alpha_{n,m}}{y^\alpha}\int_{\frac{y}{u_{n,m}\beta_n}>|x|>\beta_n}|x|^\alpha F(dx)\right) \leq \frac{Cu^\alpha_{n,m}}{n^2y^\alpha}\left(1+\rho_n\right).$$

Then as $\alpha<1$, using (3.2.5), $\rho_n \leq C\log\frac{1}{\beta_n}$, $\beta_n = \frac{(\log n)^2}{n}$, $u_{n,m} = \frac{nm}{m-1}$, the third point of criteria (3.7.4) is satisfied. Therefore by Lemma 3.7.2, we have $\int_0^{\eta_n(.)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(4) \overset{\mathbb{P}}{\to} 0$.

The term $\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(5)$: This is equal to $\sum_{k=2}^m\sum_{i=1}^{[nt]}\zeta^n_{i,k}(5)$ where

$$\zeta^n_{i,k}(5) = u_{n,m}ff'(X^n_{t_i^1})\left[(A^{\beta_n}_{t_i^k} - A^{\beta_n}_{t_i^1}) + (N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})\right](M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}).$$

For any fixed $k\in\{1,\dots,m\}$, by the boundedness of ff', the independence structure, property (P1), (3.2.5), the inequality $(a+b)^2 \leq 2(a^2+b^2)$, $\int_{\mathbb{R}}x^2F(dx) < \infty$ (see Remark 3.2.1), lemmas 3.7.5 and 3.7.4, we have $\mathbb{E}(\zeta^n_{i,k}(5)|\mathcal{F}_{t_i^1}) = 0$,

$$\begin{aligned}\mathbb{E}(|\zeta^n_{i,k}(5)|^2|\mathcal{F}_{t_i^1}) \leq& Cu^2_{n,m}\mathbb{E}((M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k})^2)\left[\frac{d_n^2(k-1)^2}{n^2m^2} + \mathbb{E}((N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})^2)\right]\\ \leq& \frac{Cu^2_{n,m}c_n}{n}(\frac{d_n^2}{n^2}+\frac{1}{n}).\end{aligned}$$

Then as $\alpha<1$, we conclude using the boundedness of $|d_n|$, $c_n \leq C\beta_n^{2-\alpha}$, $u_{n,m} = \frac{nm}{m-1}$, the criteria (3.7.3) and Lemma 3.7.2. Therefore, we have $\int_0^{\eta_n(.)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(5) \overset{\mathbb{P}}{\to} 0$.

The term $\overline{\Gamma}^n_t(4)$: We recall that

$$\begin{aligned}\overline{\Gamma}^n_t(4) = u_{n,m}\int_0^{\eta_n(t)} (k(X^n_{\eta_n(s-)}, Y_{\eta_{nm}(s-)} - Y_{\eta_n(s-)}) - k(X^n_{\eta_n(s-)}, N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)}))\\ (N^{\beta_n}_{\eta_{nm}(s-)} - N^{\beta_n}_{\eta_n(s-)})^2 dA^{\beta_n}_s.\end{aligned}$$

Since $\frac{\partial k}{\partial y}(x,y)$ is bounded on $\mathbb{R}^2$, it is enough to prove that for each $k\in\{1,\dots,m\}$, the triangular array with generic term $\overline{\zeta}^n_{i,k}(4)$ converges to 0 when $n\to\infty$ where

$$\overline{\zeta}^n_{i,k}(4) = \frac{u_{n,m}|d_n|}{nm}|Y^{\beta_n}_{t_i^{k+1}} - Y^{\beta_n}_{t_i^1}|(N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^1})^2.$$

By property (P1), the independence between $Y^{\beta_n}_{t_i^{k+1}} - Y^{\beta_n}_{t_i^1}$ and $N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^1}$, Cauchy-Schwarz's inequality, $\int_{\mathbb{R}}x^2F(dx) < \infty$, Lemma 3.7.4 and Lemma 3.7.5, we have

$$\mathbb{E}(|\overline{\zeta}^n_{i,k}(4)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|}{nm}(\frac{|d_n|}{nm}+\frac{\sqrt{c_n}}{\sqrt{nm}})\frac{1}{nm}\int_{|x|>\beta_n}x^2F(dx) \leq C\frac{u_{n,m}|d_n|}{n^2}(\frac{|d_n|}{n}+\frac{\sqrt{c_n}}{\sqrt{n}}).$$

Then we conclude using the boundedness of $|d_n|$, c_n, $u_{n,m} = \frac{nm}{m-1}$, the criteria (3.7.2) and Lemma 3.7.2. Therefore, we have $\overline{\Gamma}^n(4) \overset{\mathbb{P}}{\to} 0$.

The term $\overline{\Gamma}_t^n(5)$: We recall that

$$\overline{\Gamma}_t^n(5)=\frac{u_{n,m}d_n}{nm}\sum_{i=1}^{[nt]}\sum_{k=2}^{m}(k(X^n_{t_i^1},N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})-k(X^n_{t_i^1},\Delta Y_{T_1^{\beta_n}(t_i^1,t_i^k)}))(N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})^2.$$

Similarly as for the term $\overline{\Gamma}^n(4)$, it is enough to prove that for each $k\in\{1,\dots,m\}$, the triangular array with generic term $\overline{\zeta}^n_{i,k}(5)$ converges to 0 when $n\to\infty$ with

$$\overline{\zeta}^n_{i,k}(5)=\frac{u_{n,m}|d_n|}{nm}\left|\sum_{j=2}^{K(t_i^1,t_i^k)}\Delta Y_{T_j^{\beta_n}(t_i^1,t_i^k)}\right|(N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})^2.$$

By using Cauchy-Schwarz's inequality, property (P1), Lemmas 3.7.4 and 3.7.5, the fact that $\int_{\mathbb{R}}x^4F(dx)<\infty$ (see Remark 3.2.1) and the calculations developped for the term $\overline{\Gamma}^n(1,2)$, we have

$$\begin{aligned}
&\mathbb{E}(|\overline{\zeta}^n_{i,k}(5)||\mathcal{F}_{t_i^1})\\
&\le C\frac{u_{n,m}|d_n|}{n}\left[\mathbb{E}((\sum_{j=2}^{K(t_i^1,t_i^k)}\Delta Y_{T_j^{\beta_n}(t_i^1,t_i^k)})^2|\mathcal{F}_{t_i^1})\times\mathbb{E}((N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})^4)\right]^{1/2}\\
&\le C\frac{u_{n,m}|d_n|}{n^{3/2}}\left[\mathbb{E}(\sum_{j=2}^{K(t_i^1,t_i^k)}(\Delta Y_{T_j^{\beta_n}(t_i^1,t_i^k)})^2+\sum_{\substack{j,j'=2\\ j\neq j'}}^{K(t_i^1,t_i^k)}\Delta Y_{T_j^{\beta_n}(t_i^1,t_i^k)}\Delta Y_{T_{j'}^{\beta_n}(t_i^1,t_i^k)}|\mathcal{F}_{t_i^1})\right]^{1/2}\\
&\le C\frac{u_{n,m}|d_n|}{n^{3/2}}\left[\frac{1}{n}(\lambda_{n,m}+\frac{\delta_n^2}{n})\right]^{1/2}.
\end{aligned}$$

Then we conclude using the boundedness of $|d_n|$, δ_n, $u_{n,m}=\frac{nm}{m-1}$, $\lambda_{n,m}\to 0$ when $n\to\infty$, the criteria (3.7.2) and Lemma 3.7.2. Therefore we get $\overline{\Gamma}^n(5)\xrightarrow{\mathbb{P}}0$.

The term $\int_0^{\eta_n(t)}k(X^n_{\eta_n(s-)},Y_{\eta_{nm}(s-)}-Y_{\eta_n(s-)})d\overline{\Gamma}^n_s(3)$: Since $k(x,y)$ is bounded on $\mathbb{R}^2$, using the inequality $(a+b+c)^2\le 4(a^2+b^2+c^2)$, it is enough to prove that for $k\in\{1,\dots,m\}$, the following eight triangular arrays with generic terms $\{\overline{\zeta}^n_{i,k}(3,j),j\in\{1,\dots,8\}$ converge to 0 as $n\to\infty$ with

$$\begin{aligned}
&\overline{\zeta}^n_{i,k}(3,1)=u_{n,m}(A^{\beta_n}_{t_i^k}-A^{\beta_n}_{t_i^1})^2|A^{\beta_n}_{t_i^{k+1}}-A^{\beta_n}_{t_i^k}|, && \overline{\zeta}^n_{i,k}(3,2)=u_{n,m}(A^{\beta_n}_{t_i^k}-A^{\beta_n}_{t_i^1})^2|M^{\beta_n}_{t_i^{k+1}}-M^{\beta_n}_{t_i^k}|\\
&\overline{\zeta}^n_{i,k}(3,3)=u_{n,m}(A^{\beta_n}_{t_i^k}-A^{\beta_n}_{t_i^1})^2|N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k}|, && \overline{\zeta}^n_{i,k}(3,4)=u_{n,m}(M^{\beta_n}_{t_i^k}-M^{\beta_n}_{t_i^1})^2|A^{\beta_n}_{t_i^{k+1}}-A^{\beta_n}_{t_i^k}|\\
&\overline{\zeta}^n_{i,k}(3,5)=u_{n,m}(N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})^2|M^{\beta_n}_{t_i^{k+1}}-M^{\beta_n}_{t_i^k}|, && \overline{\zeta}^n_{i,k}(3,6)=u_{n,m}(M^{\beta_n}_{t_i^k}-M^{\beta_n}_{t_i^1})^2|N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k}|\\
&\overline{\zeta}^n_{i,k}(3,7)=u_{n,m}(M^{\beta_n}_{t_i^k}-M^{\beta_n}_{t_i^1})^2|M^{\beta_n}_{t_i^{k+1}}-M^{\beta_n}_{t_i^k}|, && \overline{\zeta}^n_{i,k}(3,8)=u_{n,m}(N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1})^2|N^{\beta_n}_{t_i^{k+1}}-N^{\beta_n}_{t_i^k}|.
\end{aligned}\tag{3.5.3}$$

For the first four triangular arrays, as $A^{\beta_n}_t=d_nt$ is deterministic, applying Cauchy-Schwarz's inequality and Lemma 3.7.5 for the increment $|M^{\beta_n}_{t_i^{k+1}}-M^{\beta_n}_{t_i^k}|$ and Lemma

3.7.4 for the increment $N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}$, it is easy to check

$$\begin{cases} \mathbb{E}(|\overline{\zeta}^n_{i,k}(3,1)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|^3}{n^3}, & \mathbb{E}(|\overline{\zeta}^n_{i,k}(3,2)||\mathcal{F}_{t_i^1})\leq C\frac{u_{n,m}|d_n|^2\sqrt{c_n}}{n^2\sqrt{n}} \\ \mathbb{E}(|\overline{\zeta}^n_{i,k}(3,3)||\mathcal{F}_{t_i^1})\leq C\frac{u_{n,m}|d_n|^2\delta_n}{n^3}, & \mathbb{E}(|\overline{\zeta}^n_{i,k}(3,4)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|c_n}{n^2}. \end{cases} \tag{3.5.4}$$

Then, for our choices of $u_{n,m}$ and β_n, by the boundedness of d_n and δ_n, $c_n \leq C\beta_n^{2-\alpha}$ with $\beta_n = \frac{(\log n)^2}{n}$, the application of the criteria (3.7.2) and Lemma 3.7.2 is straightforward for these first four triangular arrays. Now, by using the independence between the increments of M^{β_n} and N^{β_n}, applying Cauchy-Schwarz's inequality and Lemma 3.7.5 for the estimation of the increment $|M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}|$, Lemma 3.7.4 and $\int_{\mathbb{R}} x^2F(dx) < \infty$, we have

$$\mathbb{E}(|\overline{\zeta}^n_{i,k}(3,5)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}\sqrt{c_n}}{n\sqrt{n}}, \quad \mathbb{E}(|\overline{\zeta}^n_{i,k}(3,6)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}c_n\delta_n}{n^2}. \tag{3.5.5}$$

By the same arguments as above, we get nc_n converges to 0 as $n \to \infty$ and then we apply the criteria (3.7.2) and Lemma 3.7.2 to get the convergence of these two triangular arrays to 0. For the seventh triangular array, we use the independence between the two increments $M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}$ and $M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1}$ and Cauchy-Schwarz's inequality to obtain

$$\mathbb{E}(|\overline{\zeta}^n_{i,k}(3,7)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}c_n^{3/2}}{n\sqrt{n}}. \tag{3.5.6}$$

Then we conclude similarly the convergence of this triangular arrays to 0. Finally, concerning the last triangular array, we use similar calculations and arguments as for the term $\int_0^{\eta_n(t)} ff'(X^n_{\eta_n(s-)})d\Gamma^n_s(4)$ and we get

$$\begin{aligned}\mathbb{E}(|\overline{\zeta}^n_{i,k}(3,8)|\mathbb{1}_{\{|\overline{\zeta}^n_{i,k}(3,8)|\leq 1\}}|\mathcal{F}_{t_i^1}) =&\frac{u_{n,m}}{nm}\int_{|x|>\beta_n}|x|\mathbb{E}(|N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1}|^2\mathbb{1}_{\left\{|N^{\beta_n}_{t_i^k}-N^{\beta_n}_{t_i^1}|\leq\frac{1}{\sqrt{u_{n,m}|x|}}\right\}})F(dx)\\ \leq&\frac{Cu_{n,m}\delta_n}{n^2}\int_{\frac{1}{\sqrt{u_{n,m}\beta_n}}>|y|>\beta_n}|y|^2F(dy),\end{aligned}$$

and using $(\mathbf{H}_1^\alpha)$ and (3.2.2), for $y > 1$ and n large enough, we have

$$\begin{aligned}&\mathbb{P}(\overline{\zeta}^n_{i,k}(3,8) > y|\mathcal{F}_{t_i^1})\\ \leq&\frac{C}{n^2}\int_{|x|>\beta_n}F(dx)\theta\left(\sqrt{\frac{y}{u_{n,m}|x|}}\vee\beta_n\right)\\ \leq&\frac{C}{n^2}\left(\int_{|x|>\sqrt{\frac{y}{u_{n,m}\beta_n}}}F(dx)\theta\left(\beta_n\right)+\int_{\sqrt{\frac{y}{u_{n,m}\beta_n}}>|x|>\beta_n}F(dx)\theta\left(\sqrt{\frac{y}{u_{n,m}|x|}}\right)\right)\\ \leq&\frac{C}{n^2}\left(\theta\left(\sqrt{\frac{y}{u_{n,m}\beta_n}}\right)\theta\left(\beta_n\right)+\frac{u_{n,m}^{\alpha/2}}{y^{\alpha/2}}\int_{\sqrt{\frac{y}{u_{n,m}\beta_n}}>|x|>\beta_n}|x|^{\alpha/2}F(dx)\right)\\ \leq&\frac{Cu_{n,m}^{\alpha/2}}{n^2y^{\alpha/2}}\left(1+\int_{|x|>\beta_n}|x|^{\alpha/2}F(dx)\right)\leq\frac{Cu_{n,m}^{\alpha/2}}{n^2y^{\alpha/2}}\left(1+\beta_n^{-\alpha/2}\right).\end{aligned}$$

Note that, as $\alpha < 1$, $\beta_n = \frac{(\log n)^2}{n}$ and $u_{n,m} = \frac{nm}{m-1}$, we have that $\frac{u_{n,m}^{\alpha/2}\beta_n^{-\alpha/2}}{n}$ converges to 0 as $n \to \infty$. Therefore, we conclude the convergence of our last triangular array by criteria (3.7.4) and Lemma 3.7.2. This completes the proof of Lemma 3.4.4.

3.5.2 Proof of Lemma 3.4.7

Here, we prove that the sequences of processes $(\overline{Y}^n(1))_{n\geq 0}$ and $(\mathcal{R}^{n,m})_{n\geq 0}$ converge uniformly in probability to 0 as $n \to \infty$. First, instead of considering the form $\overline{Y}_t^n(1)$ given in (3.4.5), it is enough to prove that for each $k \in \{1, \dots, m\}$ the triangular arrays with generic terms $y_{i,k}^{n,m}(1,1) = M^{\beta_n}_{t_i^k, t_i^{k+1}}$ and $y_{i,k}^{n,m}(1,2) = \sum_{j\geq 2} \Delta Y_{T_j^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\geq j\}}$ converge uniformly in probability to 0 as $n \to \infty$. On the one hand, from the above estimates (3.5.1), we have $\mathbb{E}(y_{i,k}^{n,m}(1,1)|\mathcal{F}_{t_i^1}) = 0, \quad \mathbb{E}(|y_{i,k}^{n,m}(1,1)|^2|\mathcal{F}_{t_i^1}) = \frac{c_n}{nm}$. Then, $y_{i,k}^{n,m}(1,1)$ satisfies (3.7.3) and we conclude using $c_n \leq C\beta_n^{2-\alpha}$ from (3.2.5) and Lemma 3.7.2. On the other hand, using similar calculations as in (3.5.2), property (P1) and $\int_{\mathbb{R}} x^2 F(dx) < \infty$, we have

$$
\begin{aligned}
&\mathbb{E}(y_{i,k}^{n,m}(1,2)|\mathcal{F}_{t_i^1}) = \frac{d'_n}{nm\lambda_{n,m}}(\lambda_{n,m} + e^{-\lambda_{n,m}} - 1),\\
&\mathbb{E}(|y_{i,k}^{n,m}(1,2)|^2|\mathcal{F}_{t_i^1}) = \mathbb{E}\Big(\sum_{j=2}^{K(t_i^k)} (\Delta Y_{T_j^{\beta_n}(t_i^k)})^2 + \sum_{\substack{j,j'=2\\ j\neq j'}}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)} \Delta Y_{T_{j'}^{\beta_n}(t_i^k)} |\mathcal{F}_{t_i^1}\Big)\\
&\leq C\mathbb{E}\Big(\frac{1}{\theta(\beta_n)} K(t_i^k)\mathbb{1}_{\{K(t_i^k)\geq 2\}} + \frac{\delta_n^2}{(\theta(\beta_n))^2} K(t_i^k)(K(t_i^k)-1)\mathbb{1}_{\{K(t_i^k)\geq 2\}}|\mathcal{F}_{t_i^1}\Big) \leq \frac{C}{n}\Big(\lambda_{n,m} + \frac{\delta_n^2}{n}\Big).
\end{aligned}
\tag{3.5.7}
$$

Then we conclude using $d'_n = 0$ by the hypothesis $(\mathbf{H}_3)$, $\delta_n \leq C \log 1/\beta_n$ (see (3.2.5)), $\beta_n = \frac{\log n}{n}$, $\lambda_{n,m} \to 0$ as $n \to \infty$, criteria (3.7.3) and Lemma 3.7.2. Therefore, we have $\overline{Y}^n(1) \xrightarrow{\mathbb{P}} 0$. Now, from the formula of the rest term given by (3.4.12), we have

$$
\mathcal{R}_t^{n,m} = \Gamma_t^n(1,2) + \sum_{i=2}^{5} \Gamma_t^n(i).
$$

In what follows, we prove that each term converges uniformly in probability to 0.
<u>**The term $\Gamma_t^n(1,2)$**</u>: We recall that

$$
\Gamma_t^n(1,2) = \sum_{i=1}^{[nt]} \sum_{k=2}^{m} \sum_{j=1}^{k-1} (\zeta_{i,k,j}^n(1,2) + \zeta'^n_{i,k,j}(1,2) + \zeta''^n_{i,k,j}(1,2)),
$$

with

$$
\begin{cases}
\zeta_{i,k,j}^n(1,2) = u_{n,m} ff'(X_{t_i^1}^n)\Delta Y_{T_1^{\beta_n}(t_i^k)} \mathbb{1}_{\{K(t_i^k)\geq 1\}} \sum_{h=2}^{K(t_i^j)} \Delta Y_{T_h^{\beta_n}(t_i^j)},\\
\zeta'^n_{i,k,j}(1,2) = u_{n,m} ff'(X_{t_i^1}^n) \sum_{h=2}^{K(t_i^k)} \Delta Y_{T_h^{\beta_n}(t_i^k)} \Delta Y_{T_1^{\beta_n}(t_i^j)} \mathbb{1}_{\{K(t_i^j)\geq 1\}},\\
\zeta''^n_{i,k,j}(1,2) = u_{n,m} ff'(X_{t_i^1}^n) \sum_{h=2}^{K(t_i^k)} \Delta Y_{T_h^{\beta_n}(t_i^k)} \sum_{h=2}^{K(t_i^j)} \Delta Y_{T_h^{\beta_n}(t_i^j)}.
\end{cases}
$$

Instead of working with $\Gamma_t^n(1,2)$, it is enough to prove that for each $k \in \{2, \dots, m\}$ and $j \in \{1, \dots, k-1\}$, the three triangular arrays with generic terms $\zeta_{i,k,j}^n(1,2)$, $\zeta'^n_{i,k,j}(1,2)$ and $\zeta''^n_{i,k,j}(1,2)$ converge uniformly in probability to 0 as $n \to \infty$. Concerning $\zeta_{i,k,j}^n(1,2)$, on the one hand, by property (P1) and hypothesis $(\mathbf{H}_3)$, we have $\mathbb{E}(\zeta_{i,k,j}^n(1,2)\mathbb{1}_{\{|\zeta_{i,k,j}^n(1,2)|\leq 1\}}|\mathcal{F}_{t_i^1}) = 0$. On the other hand, as ff' is bounded, using property (P1), the inequalities $1 - e^{\lambda_{n,m}} \leq \lambda_{n,m}$ and $(\sum_{i=1}^n |x_i|)^\alpha \leq \sum_{i=1}^n |x_i|^\alpha$ for

$x_i \in \mathbb{R}$ and $\alpha \leq 1$ and $c(\beta) \leq C\beta^{2-\alpha}$, we have

$$\begin{aligned}
&\mathbb{E}(|\zeta^n_{i,k,j}(1,2)|^2 \mathbb{1}_{\{|\zeta^n_{i,k,j}(1,2)|\leq 1\}}|\mathcal{F}_{t^1_i}) \\
&= \frac{u^2_{n,m}(1-e^{-\lambda_{n,m}})}{\theta(\beta_n)}\mathbb{E}((\sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)})^2 \int_{|x|>\beta_n} x^2 \mathbb{1}_{\{|u_{n,m} ff'(X_{t^1_i}) x \sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)}|\leq 1\}} F(dx)|\mathcal{F}_{t^1_i}) \\
&\leq \frac{Cu^2_{n,m}}{n}\mathbb{E}((\sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)})^2 c(\frac{1}{u_{n,m}|ff'(X_{t^1_i})||\sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)}|})|\mathcal{F}_{t^1_i}) \\
&\leq \frac{Cu^\alpha_{n,m}}{n}\mathbb{E}((\sum_{h=2}^{K(t^j_i)} |\Delta Y_{T^{\beta_n}_h(t^j_i)}|)^\alpha|\mathcal{F}_{t^1_i}) \leq \frac{Cu^\alpha_{n,m}}{n}\mathbb{E}(\sum_{h=2}^{K(t^j_i)} |\Delta Y_{T^{\beta_n}_h(t^j_i)}|^\alpha|\mathcal{F}_{t^1_i}) \\
&\leq \frac{Cu^\alpha_{n,m}}{n}\frac{\mathbb{E}(K(t^j_i)\mathbb{1}_{\{K(t^j_i)\geq 2\}}|\mathcal{F}_{t^1_i})\rho_n}{\theta(\beta_n)} \leq \frac{Cu^\alpha_{n,m}\rho_n\lambda_{n,m}}{n^2}.
\end{aligned}$$

Now, using similar arguments as above, the inequality $\mathbb{1}_{\{|\sum_{j=1}^{\ell} a_j|>1\}} \leq \sum_{j=1}^{\ell} \mathbb{1}_{\{|a_j|>1/\ell\}}$, $\theta(.)$ is decreasing and hypothesis $(\mathbf{H}^\alpha_1)$, we obtain for all $y > 1$

$$\begin{aligned}
&\mathbb{P}(|\zeta^n_{i,k,j}(1,2)| > y|\mathcal{F}_{t^1_i}) \leq \frac{C}{n}\int_{|x|>\beta_n} \mathbb{P}(|ff'(X_{t^1_i})u_{n,m}x \sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)}| > y|\mathcal{F}_{t^1_i})F(dx) \\
&\leq \frac{C}{n}\int_{|x|>\beta_n} \mathbb{E}(K(t^j_i)\mathbb{1}_{\{K(t^j_i)\geq 2\}}\mathbb{1}_{\{|ff'(X_{t^1_i})u_{n,m}x\Delta Y_{T^{\beta_n}_2(t^j_i)}|>y/(K(t^j_i)-1)\}}|\mathcal{F}_{t^1_i})F(dx) \\
&\leq \frac{C}{n}\int_{|x|>\beta_n} \frac{1}{\theta(\beta_n)}\mathbb{E}(K(t^j_i)\mathbb{1}_{\{K(t^j_i)\geq 2\}}\int_{|z|>\beta_n} \mathbb{1}_{\{|z|>\frac{y}{|ff'(X_{t^1_i})|(K(t^j_i)-1)u_{n,m}|x|}\}}F(dz)|\mathcal{F}_{t^1_i})F(dx) \\
&\leq \frac{C}{n}\int_{|x|>\beta_n} \frac{1}{\theta(\beta_n)}\mathbb{E}(K(t^j_i)\mathbb{1}_{\{K(t^j_i)\geq 2\}}\theta(\frac{y}{|ff'(X_{t^1_i})|(K(t^j_i)-1)u_{n,m}|x|})|\mathcal{F}_{t^1_i})F(dx) \\
&\leq \frac{C}{n}\int_{|x|>\beta_n} \frac{1}{\theta(\beta_n)y^\alpha}\mathbb{E}(K(t^j_i)^2\mathbb{1}_{\{K(t^j_i)\geq 2\}}|\mathcal{F}_{t^1_i})u^\alpha_{n,m}|x|^\alpha F(dx) \leq \frac{Cu^\alpha_{n,m}\lambda_{n,m}\rho_n}{n^2y^\alpha}.
\end{aligned}$$

Then, we conclude the convergence of the triangular array with generic term $\zeta^n_{i,k,j}(1,2)$ using $u_{n,m} = (\frac{nm}{(m-1)\log n})^{1/\alpha}$, $\beta_n = (\frac{m}{m-1})^{1/\alpha}u^{-1}_{n,m}$, $\rho_n \leq C\log 1/\beta_n$ (see (3.2.5)), $\lambda_{n,m} \to 0$ as $n \to \infty$, criteria (3.7.4) and Lemma 3.7.2. Now, concerning the triangular array with the generic term $\zeta'^n_{i,k,j}(1,2)$, noticing that j and k play a symmetric role in $\zeta^n_{i,k,j}(1,2)$ and $\zeta'^n_{i,k,j}(1,2)$, the same calculations yield the same bounds for the three conditions of criteria (3.7.4) and therefore we obtain the convergence of the triangular array with the generic term $\zeta'^n_{i,k,j}(1,2)$ in the same way. Finally, concerning $\zeta''^n_{i,k,j}(1,2)$, by similar arguments, we have $\mathbb{E}(\zeta''^n_{i,k,j}(1,2)\mathbb{1}_{\{|\zeta''^n_{i,k,j}(1,2)|\leq 1\}}|\mathcal{F}_{t^1_i}) = 0$ and

$$\begin{aligned}
&\mathbb{E}(\zeta''^n_{i,k,j}(1,2)^2\mathbb{1}_{\{|\zeta''^n_{i,k,j}(1,2)|\leq 1\}}|\mathcal{F}_{t^1_i}) \leq Cu^2_{n,m}\mathbb{E}((\sum_{h=2}^{K(t^k_i)} \Delta Y_{T^{\beta_n}_h(t^k_i)})^2(\sum_{h=2}^{K(t^j_i)} |\Delta Y_{T^{\beta_n}_h(t^j_i)}|)^{\alpha+(2-\alpha)} \\
&\times \mathbb{1}_{\{|\sum_{h=2}^{K(t^j_i)} \Delta Y_{T^{\beta_n}_h(t^j_i)}|\leq \frac{1}{u_{n,m}|ff'(X_{t^1_i})\sum_{h=2}^{K(t^k_i)} \Delta Y_{T^{\beta_n}_h(t^k_i)}|}\}}|\mathcal{F}_{t^1_i}), \\
&\leq Cu^\alpha_{n,m}\mathbb{E}((\sum_{h=2}^{K(t^k_i)} |\Delta Y_{T^{\beta_n}_h(t^k_i)}|)^\alpha(\sum_{h=2}^{K(t^j_i)} |\Delta Y_{T^{\beta_n}_h(t^j_i)}|)^\alpha|\mathcal{F}_{t^1_i})
\end{aligned}$$

$$\leq \frac{Cu_{n,m}^{\alpha}\mathbb{E}(K(t_i^k)\mathbb{1}_{\{K(t_i^k)\geq 2\}}|\mathcal{F}_{t_i^1})\mathbb{E}(K(t_i^j)\mathbb{1}_{\{K(t_i^j)\geq 2\}}|\mathcal{F}_{t_i^1})\rho_n^2}{(\theta(\beta_n))^2} \leq \frac{Cu_{n,m}^{\alpha}\lambda_{n,m}^2\rho_n^2}{n^2}.$$

By the same calculations as for $\zeta_{i,k,j}^n(1,2)$, where we use the inequality $\mathbb{1}_{\{|\sum_{j'=1}^{\ell'}\sum_{j=1}^{\ell}a_{jj'}|>1\}} \leq \sum_{j'=1}^{\ell'}\sum_{j=1}^{\ell}\mathbb{1}_{\{|a_{jj'}|>1/(\ell+\ell')\}}$, we have for $y>1$

$$\mathbb{P}(|\zeta''^n_{i,k,j}(1,2)|>y|\mathcal{F}_{t_i^1}) = \mathbb{E}(\mathbb{1}_{\{|ff'(X_{t_i^1})u_{n,m}\sum_{h=2}^{K(t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^k)}\sum_{h=2}^{K(t_i^j)}\Delta Y_{T_h^{\beta_n}(t_i^j)}|>y\}}|\mathcal{F}_{t_i^1})$$

$$\leq \mathbb{E}(K(t_i^k)K(t_i^j)\mathbb{1}_{\{K(t_i^k)\geq 2,K(t_i^j)\geq 2\}}\mathbb{1}_{\{|ff'(X_{t_i^1})u_{n,m}\Delta Y_{T_2^{\beta_n}(t_i^k)}\Delta Y_{T_2^{\beta_n}(t_i^j)}|>\frac{y}{(K(t_i^k)-1)(K(t_i^j)-1)}\}}|\mathcal{F}_{t_i^1})$$

$$\leq \mathbb{E}(\frac{K(t_i^k)K(t_i^j)\mathbb{1}_{\{K(t_i^k)\geq 2,K(t_i^j)\geq 2\}}}{(\theta(\beta_n))^2}\int_{|x|>\beta_n}\theta(\frac{y}{ff'(X_{t_i^1})u_{n,m}|x|(K(t_i^k)-1)(K(t_i^j)-1)})F(dx)|\mathcal{F}_{t_i^1})$$

$$\leq \frac{\mathbb{E}(K(t_i^k)^2K(t_i^j)^2\mathbb{1}_{\{K(t_i^k)\geq 2,K(t_i^j)\geq 2\}}|\mathcal{F}_{t_i^1})}{(\theta(\beta_n))^2}\frac{u_{n,m}^{\alpha}\rho_n}{y^{\alpha}} \leq \frac{Cu_{n,m}^{\alpha}\rho_n\lambda_{n,m}^2}{n^2y^{\alpha}}.$$

Then, we conclude using $u_{n,m}=(\frac{nm}{(m-1)\log n})^{1/\alpha}$, $\beta_n=(\frac{m}{m-1})^{1/\alpha}u_{n,m}^{-1}$, $\rho_n\leq C\log 1/\beta_n$ (see (3.2.5)), $\lambda_{n,m}\to 0$ as $n\to\infty$, criteria (3.7.4) and Lemma 3.7.2. Therefore, we get $\Gamma^n(1,2)\overset{\mathbb{P}}{\to}0$.

The term $\Gamma_t^n(2)$: Let us recall that $\Gamma_t^n(2)=\sum_{k=2}^m\sum_{i=1}^{[nt]}\zeta_{i,k}^n(2)$ where

$$\zeta_{i,k}^n(2)=u_{n,m}ff'(X_{t_i^1}^n)\frac{d_n}{nm}[\frac{k-1}{nm}d_n+(N_{t_i^k}^{\beta_n}-N_{t_i^1}^{\beta_n})+(k-1)(N_{t_i^{k+1}}^{\beta_n}-N_{t_i^k}^{\beta_n})].$$

In case (C2), from hypotheses $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ $d_n=b+\int_{|x|>1}xF(dx)-d'_n=0$ then this $\Gamma_t^n(2)$ vanishes. In case (C4), hypothesis $(\mathbf{H}_3)$ yields $d'_n=0$ and then $d_n=b$. Now, for k fixed, as ff' is bounded, using Lemma 3.7.4 and $\int_{\mathbb{R}}x^2F(dx)<\infty$, we get $|\mathbb{E}(\zeta_{i,k}^n(2)|\mathcal{F}_{t_i^1})|\leq\frac{Cu_{n,m}}{n^2}$ and

$$\begin{aligned}\mathbb{E}(|\zeta_{i,k}^n(2)|^2|\mathcal{F}_{t_i^1}) &\leq \frac{Cu_{n,m}^2}{n^2}(\frac{1}{n^2}+\mathbb{E}((N_{t_i^k}^{\beta_n}-N_{t_i^1}^{\beta_n})^2|\mathcal{F}_{t_i^1})+\mathbb{E}((N_{t_i^{k+1}}^{\beta_n}-N_{t_i^k}^{\beta_n})^2|\mathcal{F}_{t_i^1}))\\ &\leq \frac{Cu_{n,m}^2}{n^2}(\frac{1}{n^2}+\frac{1}{n})\leq\frac{Cu_{n,m}^2}{n^3}.\end{aligned}$$

Then, for case (C4), we conclude using $u_{n,m}=\frac{nm}{(m-1)\log n}$, criteria (3.7.3) and Lemma 3.7.2. Therefore, we get $\Gamma^n(2)\overset{\mathbb{P}}{\to}0$.

The term $\Gamma_t^n(3)$: Let us recall that $\Gamma_t^n(3)=\sum_{k=2}^m\sum_{i=1}^{[nt]}\zeta_{i,k}^n(3)$ where

$$\zeta_{i,k}^n(3)=u_{n,m}ff'(X_{t_i^1}^n)(M_{t_i^k}^{\beta_n}-M_{t_i^1}^{\beta_n})(Y_{t_i^{k+1}}-Y_{t_i^k}).$$

For k fixed, as ff' is bounded, by using property (P1), lemmas 3.7.4 and 3.7.5 and $\int_{\mathbb{R}}x^2F(dx)<\infty$ (see Remark 3.2.1), we have $\mathbb{E}(\zeta_{i,k}^n(3)|\mathcal{F}_{t_i^1})=0$ and

$$\begin{aligned}&\mathbb{E}(|\zeta_{i,k}^n(3)|^2|\mathcal{F}_{t_i^1})\\ &\leq Cu_{n,m}^2\mathbb{E}((M_{t_i^k}^{\beta_n}-M_{t_i^1}^{\beta_n})^2|\mathcal{F}_{t_i^1})\left[\mathbb{E}((Y_{t_i^{k+1}}^{\beta_n}-Y_{t_i^k}^{\beta_n})^2|\mathcal{F}_{t_i^1})+\mathbb{E}((N_{t_i^{k+1}}^{\beta_n}-N_{t_i^k}^{\beta_n})^2|\mathcal{F}_{t_i^1})\right]\\ &\leq \frac{Cu_{n,m}^2c_n}{n}\left(\frac{c_n}{n}+\frac{d_n^2}{n^2}+\frac{1}{n}\right).\end{aligned}$$

Then, we conclude using $(\mathbf{H}_3)$ that $d_n = b$, $u_{n,m} = (\frac{nm}{(m-1)\log n})^{1/\alpha}$, $\beta_n = (\frac{m}{m-1})^{1/\alpha} u_{n,m}^{-1}$, $c_n \leq C\beta_n^{2-\alpha}$ (see (3.2.5)), criteria (3.7.3) and Lemma 3.7.2. Therefore, we get $\Gamma^n(3) \xrightarrow{\mathbb{P}} 0$.

<u>**The term** $\Gamma_t^n(4)$:</u> Let us recall that $\Gamma_t^n(4) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta_{i,k}^n(4)$ where

$$\zeta_{i,k}^n(4) = u_{n,m} f f'(X_{t_i^1}^n)[(A_{t_i^k}^{\beta_n} - A_{t_i^1}^{\beta_n}) + (N_{t_i^k}^{\beta_n} - N_{t_i^1}^{\beta_n})](M_{t_i^{k+1}}^{\beta_n} - M_{t_i^k}^{\beta_n}).$$

For k fixed, by similar arguments as for the term $\Gamma_t^n(3)$, we have $\mathbb{E}(\zeta_{i,k}^n(4)|\mathcal{F}_{t_i^1}) = 0$ and $\mathbb{E}(|\zeta_{i,k}^n(4)|^2|\mathcal{F}_{t_i^1}) \leq \frac{Cu_{n,m}^2 c_n}{n}\left(\frac{d_n^2}{n^2} + \frac{1}{n}\right)$. Then, we conclude similarly that $\Gamma^n(4) \xrightarrow{\mathbb{P}} 0$.

<u>**The term** $\Gamma_t^n(5)$:</u> Since $k(x,y)$ is bounded on $\mathbb{R}^2$, it is enough to prove that for $k \in \{1,\ldots,m\}$, the following nine triangular arrays with generic terms $\{\zeta_{i,k}^n(5,j), j \in \{1,\ldots,9\}$ converge to 0 as $n \to \infty$ with

$$\begin{aligned}
&\zeta_{i,k}^n(5,1) = u_{n,m}(A_{t_i^k}^{\beta_n} - A_{t_i^1}^{\beta_n})^2|A_{t_i^{k+1}}^{\beta_n} - A_{t_i^k}^{\beta_n}|, &&\zeta_{i,k}^n(5,2) = u_{n,m}(A_{t_i^k}^{\beta_n} - A_{t_i^1}^{\beta_n})^2|M_{t_i^{k+1}}^{\beta_n} - M_{t_i^k}^{\beta_n}|\\
&\zeta_{i,k}^n(5,3) = u_{n,m}(A_{t_i^k}^{\beta_n} - A_{t_i^1}^{\beta_n})^2|N_{t_i^{k+1}}^{\beta_n} - N_{t_i^k}^{\beta_n}|, &&\zeta_{i,k}^n(5,4) = u_{n,m}(M_{t_i^k}^{\beta_n} - M_{t_i^1}^{\beta_n})^2|A_{t_i^{k+1}}^{\beta_n} - A_{t_i^k}^{\beta_n}|\\
&\zeta_{i,k}^n(5,5) = u_{n,m}(N_{t_i^k}^{\beta_n} - N_{t_i^1}^{\beta_n})^2|M_{t_i^{k+1}}^{\beta_n} - M_{t_i^k}^{\beta_n}|, &&\zeta_{i,k}^n(5,6) = u_{n,m}(M_{t_i^k}^{\beta_n} - M_{t_i^1}^{\beta_n})^2|N_{t_i^{k+1}}^{\beta_n} - N_{t_i^k}^{\beta_n}|\\
&\zeta_{i,k}^n(5,7) = u_{n,m}(M_{t_i^k}^{\beta_n} - M_{t_i^1}^{\beta_n})^2|M_{t_i^{k+1}}^{\beta_n} - M_{t_i^k}^{\beta_n}|, &&\zeta_{i,k}^n(5,8) = u_{n,m}(N_{t_i^k}^{\beta_n} - N_{t_i^1}^{\beta_n})^2|N_{t_i^{k+1}}^{\beta_n} - N_{t_i^k}^{\beta_n}|\\
&\zeta_{i,k}^n(5,9) = u_{n,m}(N_{t_i^k}^{\beta_n} - N_{t_i^1}^{\beta_n})^2|A_{t_i^{k+1}}^{\beta_n} - A_{t_i^k}^{\beta_n}|.
\end{aligned}$$

From (3.5.3), (3.5.4) and (3.5.5) and by same calculations, the triangular arrays with generic terms $\zeta_{i,k}^n(5,i)$, $i \in \{1,\ldots,7\}$ and $\zeta_{i,k}^n(5,9)$ are bounded as follows

$$\begin{cases}
\mathbb{E}(|\zeta_{i,k}^n(5,1)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|^3}{n^3}, & \mathbb{E}(|\zeta_{i,k}^n(5,2)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|^2\sqrt{c_n}}{n^2\sqrt{n}}\\
\mathbb{E}(|\zeta_{i,k}^n(5,3)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|^2\delta_n}{n^3}, & \mathbb{E}(|\zeta_{i,k}^n(5,4)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|c_n}{n^2}\\
\mathbb{E}(|\zeta_{i,k}^n(5,5)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}\sqrt{c_n}}{n\sqrt{n}}, & \mathbb{E}(|\zeta_{i,k}^n(5,6)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}c_n\delta_n}{n^2}\\
\mathbb{E}(|\zeta_{i,k}^n(5,7)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}c_n^{3/2}}{n\sqrt{n}}, & \mathbb{E}(|\zeta_{i,k}^n(5,9)||\mathcal{F}_{t_i^1}) \leq C\frac{u_{n,m}|d_n|}{n^2}.
\end{cases} \tag{3.5.8}$$

Then, we conclude using $u_{n,m} = (\frac{nm}{(m-1)\log n})^{1/\alpha}$, $\beta_n = (\frac{m}{m-1})^{1/\alpha} u_{n,m}^{-1}$, $c_n \leq C\beta_n^{2-\alpha}$ and $\delta_n \leq C\log 1/\beta_n$ (see (3.2.5)), $d_n = 0$ for case (C2), $d_n = b$ for case (C4), criteria (3.7.2) and Lemma 3.7.2. Therefore, we have the convergence to 0 of the eight triangular arrays corresponding to these eight generic terms. Now, concerning $\zeta_{i,k}^n(5,8)$, we rewrite as $u_{n,m}(\sum_{h=1}^{k-1}(N_{t_i^{h+1}}^{\beta_n} - N_{t_i^h}^{\beta_n}))^2|N_{t_i^{k+1}}^{\beta_n} - N_{t_i^k}^{\beta_n}|$, then by Jensen's inequality we have

$$\zeta_{i,k}^n(5,8) \leq Cu_{n,m}\sum_{h=1}^{k-1}\Big(\sum_{j=1}^{K(t_i^h)} \Delta Y_{T_j^{\beta_n}(t_i^h)}\Big)^2 \sum_{j'=1}^{K(t_i^k)} |\Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|.$$

Thus, for $h < k$ fixed, thanks to the inequality $(a+b)^2 \leq 2(a^2+b^2)$, to prove the convergence of the last triangular array, it is enough to consider the two following generic terms

$$\begin{cases}
\zeta_{i,k,h}^n(5,8) = u_{n,m}(\sum_{j=2}^{K(t_i^h)} \Delta Y_{T_j^{\beta_n}(t_i^h)})^2 \sum_{j'=1}^{K(t_i^k)} |\Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|,\\
\zeta'^n_{i,k,h}(5,8) = u_{n,m}(\Delta Y_{T_1^{\beta_n}(t_i^h)} \mathbb{1}_{\{K(t_i^h)\geq 1\}})^2 \sum_{j=1}^{K(t_i^k)} |\Delta Y_{T_j^{\beta_n}(t_i^k)}|.
\end{cases}$$

First, we consider $\zeta^n_{i,k,h}(5,8)$. By property (P1), the inequality for $\alpha \leq 1$, $(\sum_{i=1}^n |x_i|)^\alpha \leq \sum_{i=1}^n |x_i|^\alpha$ and $\int_{|x|>\beta_n} |x|^{\alpha/2} F(dx) \leq \frac{C}{\beta_n^{\alpha/2}}$ (see (3.2.2)), we have $\mathbb{E}(|\zeta^n_{i,k,h}(5,8)|\mathbb{1}_{\{|\zeta^n_{i,k,h}(5,8)|\leq 1\}}|\mathcal{F}_{t_i^1})$

$$
\begin{aligned}
=& u_{n,m}\mathbb{E}(|\sum_{j=2}^{K(t_i^h)} \Delta Y_{T_j^{\beta_n}(t_i^h)}|^{\alpha+(2-\alpha)}|\sum_{j'=1}^{K(t_i^k)} \Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|\mathbb{1}_{\{|\sum_{j=2}^{K(t_i^h)} \Delta Y_{T_j^{\beta_n}(t_i^h)}|\leq \frac{1}{\sqrt{u_{n,m}\sum_{j'=1}^{K(t_i^k)} |\Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|}}\}}|\mathcal{F}_{t_i^1})\\
\leq& u_{n,m}^{\alpha/2}\mathbb{E}((\sum_{j=2}^{K(t_i^h)} |\Delta Y_{T_j^{\beta_n}(t_i^h)}|)^\alpha(\sum_{j'=1}^{K(t_i^k)} |\Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|)^{\alpha/2}|\mathcal{F}_{t_i^1})\\
\leq& u_{n,m}^{\alpha/2}\mathbb{E}(\sum_{j=2}^{K(t_i^h)} |\Delta Y_{T_j^{\beta_n}(t_i^h)}|^\alpha \sum_{j'=1}^{K(t_i^k)} |\Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|^{\alpha/2}|\mathcal{F}_{t_i^1})\\
\leq& \frac{Cu_{n,m}^{\alpha/2}\rho_n}{(\theta(\beta_n))^2}\int_{|x|>\beta_n} |x|^{\alpha/2}F(dx)\mathbb{E}(K(t_i^h)K(t_i^k)\mathbb{1}_{\{K(t_i^h)\geq 2, K(t_i^k)\geq 1\}}|\mathcal{F}_{t_i^1}) \leq \frac{Cu_{n,m}^{\alpha/2}\lambda_{n,m}\rho_n}{n^2\beta_n^{\alpha/2}}.
\end{aligned}
$$

Now, using similar arguments as above, Jensen's inequality, the inequality $\mathbb{1}_{\{|\sum_{j'=1}^{\ell'}\sum_{j=1}^{\ell} a_{jj'}|>1\}} \leq \sum_{j'=1}^{\ell'}\sum_{j=1}^{\ell} \mathbb{1}_{\{|a_{jj'}|>1/(\ell+\ell')\}}$, $\theta(.)$ is decreasing and $(\mathbf{H}_1^\alpha)$, for $y>1$, we have

$$
\begin{aligned}
&\mathbb{P}(|\zeta^n_{i,k,h}(5,8)|>y|\mathcal{F}_{t_i^1}) \leq \mathbb{E}(\mathbb{1}_{\{u_{n,m}K(t_i^h)\sum_{j=2}^{K(t_i^h)} \Delta Y^2_{T_j^{\beta_n}(t_i^h)}\sum_{j'=1}^{K(t_i^k)} |\Delta Y_{T_{j'}^{\beta_n}(t_i^k)}|>y\}}|\mathcal{F}_{t_i^1})\\
\leq& \mathbb{E}(K(t_i^h)K(t_i^k)\mathbb{1}_{\{K(t_i^h)\geq 2, K(t_i^k)\geq 1\}}\mathbb{1}_{\{K(t_i^h)\Delta Y^2_{T_2^{\beta_n}(t_i^h)}|\Delta Y_{T_1^{\beta_n}(t_i^k)}|>\frac{y}{u_{n,m}K(t_i^h)K(t_i^k)}\}}|\mathcal{F}_{t_i^1})\\
\leq& \frac{1}{(\theta(\beta_n))^2}\mathbb{E}(K(t_i^h)K(t_i^k)\mathbb{1}_{\{K(t_i^h)\geq 2, K(t_i^k)\geq 1\}}\int_{|z|>\beta_n}\theta(\sqrt{\frac{y}{u_{n,m}K(t_i^h)^2K(t_i^k)|z|}})F(dz)|\mathcal{F}_{t_i^1})\\
\leq& \frac{1}{(\theta(\beta_n))^2}\int_{|z|>\beta_n}\frac{u_{n,m}^{\alpha/2}|z|^{\alpha/2}}{y^{\alpha/2}}F(dz)\mathbb{E}(K(t_i^h)^2K(t_i^k)^2\mathbb{1}_{\{K(t_i^h)\geq 2, K(t_i^k)\geq 1\}}|\mathcal{F}_{t_i^1}) \leq \frac{Cu_{n,m}^{\alpha/2}\lambda_{n,m}}{n^2y^{\alpha/2}\beta_n^{\alpha/2}}.
\end{aligned}
$$

We conclude using $u_{n,m} = (\frac{nm}{(m-1)\log n})^{1/\alpha}$, $\beta_n = (\frac{m}{m-1})^{1/\alpha}u_{n,m}^{-1}$, $\rho_n \leq C\log 1/\beta_n$ (see (3.2.5)), $\lambda_{n,m} \to 0$ as $n \to \infty$, criteria (3.7.4) and Lemma 3.7.2. Similarly for ${\zeta'}^n_{i,k,h}(5,8)$, using $1-e^{-\lambda_{n,m}} \leq \lambda_{n,m}$ and that $c(\beta) \leq C\beta^{2-\alpha}$, $\mathbb{E}(|{\zeta'}^n_{i,k,h}(5,8)|\mathbb{1}_{\{|{\zeta'}^n_{i,k,h}(5,8)|\leq 1\}}|\mathcal{F}_{t_i^1})$ is equal to

$$
\begin{aligned}
&u_{n,m}(1-e^{-\lambda_{n,m}})\mathbb{E}(\Delta Y^2_{T_1^{\beta_n}(t_i^h)}|\sum_{j=1}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)}|\mathbb{1}_{\{|\Delta Y_{T_1^{\beta_n}(t_i^h)}|\leq \frac{1}{\sqrt{u_{n,m}|\sum_{j=1}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)}|}}\}}|\mathcal{F}_{t_i^1})\\
\leq& \frac{u_{n,m}}{n}\mathbb{E}(c(\frac{1}{\sqrt{u_{n,m}|\sum_{j=1}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)}|}})|\sum_{j=1}^{K(t_i^k)} \Delta Y_{T_j^{\beta_n}(t_i^k)}||\mathcal{F}_{t_i^1})\\
\leq& \frac{Cu_{n,m}^{\alpha/2}}{n}\mathbb{E}(\sum_{j=1}^{K(t_i^k)} |\Delta Y_{T_j^{\beta_n}(t_i^k)}|^{\alpha/2}|\mathcal{F}_{t_i^1})\\
\leq& \frac{Cu_{n,m}^{\alpha/2}}{n\theta(\beta_n)}\int_{|x|>\beta_n}|x|^{\alpha/2}F(dx)\mathbb{E}(K(t_i^k)\mathbb{1}_{\{K(t_i^k)\geq 1\}}|\mathcal{F}_{t_i^1}) \leq \frac{Cu_{n,m}^{\alpha/2}}{n^2\beta_n^{\alpha/2}},
\end{aligned}
$$

and using the inequality $\mathbb{1}_{\{|\sum_{j=1}^{\ell} a_j|>1\}} \leq \sum_{j=1}^{\ell} \mathbb{1}_{\{|a_j|>1/\ell\}}$, $\theta(.)$ is decreasing and $(\mathbf{H}_1^{\alpha})$, we get for $y>1$,

$$\begin{aligned}\mathbb{P}(|\zeta'^n_{i,k,h}(5,8)|>y|\mathcal{F}_{t_i^1}) \leq & \lambda_{n,m}\mathbb{E}(K(t_i^k)\mathbb{1}_{\{K(t_i^k)\geq 1\}}\mathbb{1}_{\{\Delta Y^2_{T_1^{\beta_n}(t_i^h)}|\Delta Y_{T_1^{\beta_n}(t_i^k)}|>\frac{y}{u_{n,m}K(t_i^k)}\}}|\mathcal{F}_{t_i^1})\\ \leq & \frac{C\lambda_{n,m}}{(\theta(\beta_n))^2}\mathbb{E}(K(t_i^k)\mathbb{1}_{\{K(t_i^k)\geq 1\}}\int_{|z|>\beta_n}\theta(\sqrt{\frac{y}{u_{n,m}K(t_i^k)|z|}})F(dz)|\mathcal{F}_{t_i^1})\\ \leq & \frac{Cu_{n,m}^{\alpha/2}}{n\theta(\beta_n)y^{\alpha/2}}\int_{|z|>\beta_n}|z|^{\alpha/2}F(dz)\mathbb{E}(K(t_i^k)^2\mathbb{1}_{\{K(t_i^k)\geq 1\}}|\mathcal{F}_{t_i^1}) \leq \frac{Cu_{n,m}^{\alpha/2}}{n^2y^{\alpha/2}\beta_n^{\alpha/2}}.\end{aligned}$$

We conclude using $u_{n,m}=(\frac{nm}{(m-1)\log n})^{1/\alpha}$, $\beta_n=(\frac{m}{m-1})^{1/\alpha}u_{n,m}^{-1}$, criteria (3.7.4) and Lemma 3.7.2. Therefore, we get $\Gamma^n(5)\xrightarrow{\mathbb{P}}0$.

3.5.3 Proof of Lemma 3.4.10

From the formula of the rest term given by (3.4.15), we have

$$\mathcal{R}_t^{n,m}=\Gamma_t^n(1,2)+\Gamma_t^n(2,2)+\sum_{i=3}^{5}\Gamma_t^n(i).$$

The aim is to prove that each term converges uniformly in probability to 0 when $n\to\infty$.

<u>**The term** $\Gamma_t^n(1,2)$:</u> Let us recall that $\Gamma_t^n(1,2)=\sum_{k=2}^{m}\sum_{i=1}^{[nt]}\zeta_{i,k}^n(1,2)$ where

$$\zeta_{i,k}^n(1,2)=\frac{u_{n,m}d_n}{nm}ff'(X^n_{t_i^1})\left[\sum_{j=1}^{k-1}\sum_{h=2}^{K(t_j^i)}\Delta Y_{T_h^{\beta_n}(t_j^i)}+(k-1)\sum_{h=2}^{K(t_k^i)}\Delta Y_{T_h^{\beta_n}(t_k^i)}\right].$$

For $k\in\{2,\dots,m\}$, since ff' is bounded, by property (P1), Jensen's inequality and similar calculations as in (3.4.16), we have

$$\begin{aligned}\mathbb{E}(|\zeta_{i,k}^n(1,2)||\mathcal{F}_{t_i^1}) \leq & \frac{Cu_{n,m}d_n}{n}\left[\sum_{j=1}^{k-1}\mathbb{E}(\sum_{h=2}^{K(t_i^j)}|\Delta Y_{T_h^{\beta_n}(t_i^j)}||\mathcal{F}_{t_i^1})+(k-1)\mathbb{E}(\sum_{h=2}^{K(t_i^k)}|\Delta Y_{T_h^{\beta_n}(t_i^k)}||\mathcal{F}_{t_i^1})\right]\\ \leq & \frac{Cu_{n,m}d_n\delta_n\lambda_{n,m}}{n^2}.\end{aligned}$$

As we choose $u_{n,m}=\frac{nm}{(m-1)(\log n)^2}$, using $d_n\leq C\log n$, $\delta_n\leq C\log n$ (see (3.2.5)) and $\lambda_{n,m}\to 0$ as $n\to\infty$, we conclude by criteria (3.7.2) and Lemma 3.7.2 that $\Gamma^n(1,2)\xrightarrow{\mathbb{P}}0$.

<u>**The term** $\Gamma_t^n(2,2)$:</u> Let us recall that $\Gamma_t^n(2,2)=\sum_{k=2}^{m}\sum_{i=1}^{[nt]}\zeta_{i,k}^n(2,2)$ where

$$\begin{aligned}\zeta_{i,k}^n(2,2)=&u_{n,m}ff'(X^n_{t_i^1})\Bigg(\Delta Y_{T_1^{\beta_n}(t_i^k)}\mathbb{1}_{\{K(t_i^k)\geq 1\}}\sum_{j=1}^{k-1}\sum_{h=2}^{K(t_i^j)}\Delta Y_{T_h^{\beta_n}(t_i^j)}\\ &+\sum_{h=2}^{K(t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^k)}\sum_{j=1}^{k-1}\Delta Y_{T_1^{\beta_n}(t_i^j)}\mathbb{1}_{\{K(t_i^j)\geq 1\}}+\sum_{h=2}^{K(t_i^k)}\Delta Y_{T_h^{\beta_n}(t_i^k)}\sum_{j=1}^{k-1}\sum_{h=2}^{K(t_i^j)}\Delta Y_{T_h^{\beta_n}(t_i^j)}\Bigg).\end{aligned}$$

For any fixed $k \in \{2, \dots, m\}$, as ff' is bounded, by (P1), $1 - e^{-\lambda_{n,m}} \le \lambda_{n,m}$ and using the same calculations in (3.4.16), we have

$$\mathbb{E}(|\zeta^n_{i,k}(2,2)||\mathcal{F}_{t^1_i}) \le C u_{n,m} \left(2 \times \frac{\delta_n}{n} \frac{\delta_n \lambda_{n,m}}{n} + \frac{\delta_n^2 \lambda_{n,m}^2}{n^2} \right) \le C \frac{u_{n,m} \delta_n^2 \lambda_{n,m}}{n^2}.$$

Then we conclude using the same arguments as above to get $\Gamma^n(2,2) \xrightarrow{\mathbb{P}} 0$.
The term $\Gamma^n_t(3)$: Let us recall that $\Gamma^n_t(3) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta^n_{i,k}(3)$ with

$$\zeta^n_{i,k}(3) = u_{n,m} ff'(X^n_{t^1_i})(M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i})(Y_{t^{k+1}_i} - Y_{t^k_i}).$$

For $k \in \{2, \dots, m\}$, since ff' is bounded, by property (P1), the independence between $M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i}$ and $Y_{t^{k+1}_i} - Y_{t^k_i}$, decomposition (3.2.6), Lemma 3.7.4 and Lemma 3.7.5, $\int_{\mathbb{R}} x^2 F(dx) < \infty$ (see Remark 3.2.1),we get $\mathbb{E}(\zeta^n_{i,k}(3)|\mathcal{F}_{t^1_i}) = 0$ and

$$\mathbb{E}(|\zeta^n_{i,k}(3)|^2|\mathcal{F}_{t^1_i}) \le C u^2_{n,m} \mathbb{E}(|M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i}|^2|\mathcal{F}_{t^1_i}) \mathbb{E}(|Y_{t^{k+1}_i} - Y_{t^k_i}|^2|\mathcal{F}_{t^1_i}) \le \frac{C u^2_{n,m} c_n}{n}\left(\frac{d_n^2}{n^2} + \frac{c_n}{n} + \frac{1}{n}\right).$$

Then we conclude using $u_{n,m} = \frac{nm}{(m-1)(\log n)^2}$ and $\beta_n = \frac{\log n}{n}$, $d_n \le C \log 1/\beta_n$ and $c_n \le C\beta_n$ from (3.2.5), criteria (3.7.3) and Lemma 3.7.2. Therefore, we get $\Gamma^n(3) \xrightarrow{\mathbb{P}} 0$.
The term $\Gamma^n_t(4)$: Let us recall that $\Gamma^n_t(4) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta^n_{i,k}(4)$ with

$$\zeta^n_{i,k}(4) = u_{n,m} ff'(X^n_{t^1_i})(A^{\beta_n}_{t^k_i} - A^{\beta_n}_{t^1_i} + N^{\beta_n}_{t^k_i} - N^{\beta_n}_{t^1_i})(M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}).$$

For k fixed, using the same arguments as for $\Gamma^n(3)$, we get $\mathbb{E}(\zeta^n_{i,k}(4)|\mathcal{F}_{t^1_i}) = 0$, $\mathbb{E}((\zeta^n_{i,k}(4))^2|\mathcal{F}_{t^1_i}) \le \frac{C u^2_{n,m} c_n}{n}(\frac{d_n^2}{n^2} + \frac{1}{n})$ and therefore, we get $\Gamma^n(4) \xrightarrow{\mathbb{P}} 0$.
The term $\Gamma^n_t(5)$: Since $k(x,y)$ is bounded on $\mathbb{R}^2$, it is enough to prove that for $k \in \{1, \dots, m\}$, the following nine triangular arrays with generic terms $\{\zeta^n_{i,k}(5,j), j \in \{1, \dots, 9\}$ converge to 0 as $n \to \infty$ with

$$\begin{aligned}
&\zeta^n_{i,k}(5,1) = u_{n,m}(A^{\beta_n}_{t^k_i} - A^{\beta_n}_{t^1_i})^2 |A^{\beta_n}_{t^{k+1}_i} - A^{\beta_n}_{t^k_i}|, && \zeta^n_{i,k}(5,2) = u_{n,m}(A^{\beta_n}_{t^k_i} - A^{\beta_n}_{t^1_i})^2 |M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}| \\
&\zeta^n_{i,k}(5,3) = u_{n,m}(A^{\beta_n}_{t^k_i} - A^{\beta_n}_{t^1_i})^2 |N^{\beta_n}_{t^{k+1}_i} - N^{\beta_n}_{t^k_i}|, && \zeta^n_{i,k}(5,4) = u_{n,m}(M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i})^2 |A^{\beta_n}_{t^{k+1}_i} - A^{\beta_n}_{t^k_i}| \\
&\zeta^n_{i,k}(5,5) = u_{n,m}(N^{\beta_n}_{t^k_i} - N^{\beta_n}_{t^1_i})^2 |M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|, && \zeta^n_{i,k}(5,6) = u_{n,m}(M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i})^2 |N^{\beta_n}_{t^{k+1}_i} - N^{\beta_n}_{t^k_i}| \\
&\zeta^n_{i,k}(5,7) = u_{n,m}(M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i})^2 |M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|, && \zeta^n_{i,k}(5,8) = u_{n,m}(N^{\beta_n}_{t^k_i} - N^{\beta_n}_{t^1_i})^2 |N^{\beta_n}_{t^{k+1}_i} - N^{\beta_n}_{t^k_i}| \\
&\zeta^n_{i,k}(5,9) = u_{n,m}(N^{\beta_n}_{t^k_i} - N^{\beta_n}_{t^1_i})^2 |A^{\beta_n}_{t^{k+1}_i} - A^{\beta_n}_{t^k_i}|.
\end{aligned}$$

From (3.5.3), (3.5.4), (3.5.5) and (3.5.8), the triangular arrays with generic terms $\zeta^n_{i,k}(5,i)$, $i \in \{1, \dots, 9\}$ are bounded as follows

$$\begin{cases}
\mathbb{E}(|\zeta^n_{i,k}(5,1)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}|d_n|^3}{n^3}, & \mathbb{E}(|\zeta^n_{i,k}(5,2)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}|d_n|^2\sqrt{c_n}}{n^2\sqrt{n}} \\
\mathbb{E}(|\zeta^n_{i,k}(5,3)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}|d_n|^2\delta_n}{n^3}, & \mathbb{E}(|\zeta^n_{i,k}(5,4)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}|d_n|c_n}{n^2} \\
\mathbb{E}(|\zeta^n_{i,k}(5,5)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}\sqrt{c_n}}{n\sqrt{n}}, & \mathbb{E}(|\zeta^n_{i,k}(5,6)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}c_n\delta_n}{n^2} \\
\mathbb{E}(|\zeta^n_{i,k}(5,7)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}c_n^{3/2}}{n\sqrt{n}}, & \mathbb{E}(|\zeta^n_{i,k}(5,8)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}\delta_n}{n^2}, \\
\mathbb{E}(|\zeta^n_{i,k}(5,9)||\mathcal{F}_{t^1_i}) \le C\frac{u_{n,m}|d_n|}{n^2}.
\end{cases} \tag{3.5.9}$$

Then, by $u_{n,m} = \frac{nm}{(m-1)(\log n)^2}$, $d_n \le C \log 1/\beta_n$, $\delta_n \le C \log 1/\beta_n$ and $c_n \le C\beta_n$ (see (3.2.5)) with $\beta_n = \frac{\log n}{n}$, criteria (3.7.2) and Lemma 3.7.2, we conclude the convergence of these triangular arrays. Therefore, we get $\Gamma^n(5) \xrightarrow{\mathbb{P}} 0$.

3.5.4 Proof of Lemma 3.4.13

Here, we prove that the sequences of processes $(\overline{Y}^n(1))_{n\ge0}$ and $(\mathcal{R}^{n,m})_{n\ge0}$ converge uniformly in probability to 0 as $n \to \infty$. First, instead of considering the form $\overline{Y}^n_t(1)$ given in (3.4.5), it is enough to prove that for each $k \in \{1, \dots, m\}$ the triangular arrays with generic terms $y^{n,m}_{i,k}(1,1) = M^{\beta_n}_{t^k_i, t^{k+1}_i}$ and $y^{n,m}_{i,k}(1,2) = \sum_{j\ge2} \Delta Y_{T^{\beta_n}_j(t^k_i)} \mathbb{1}_{\{K(t^k_i)\ge j\}}$ converge uniformly in probability to 0 as $n \to \infty$. On the one hand, from the above (3.5.1), we have $\mathbb{E}(y^{n,m}_{i,k}(1,1)|\mathcal{F}_{t^1_i}) = 0$, $\quad \mathbb{E}((y^{n,m}_{i,k}(1,1))^2|\mathcal{F}_{t^1_i}) = \frac{c_n}{nm}$. Then, $y^{n,m}_{i,k}(1,1)$ satisfies (3.7.3) and we conclude using $c_n \le C\beta_n^{2-\alpha}$ (see (3.2.5)) and Lemma 3.7.2. On the other hand, by similar calculations as in (3.5.2), property (P1) and $\int_{\mathbb{R}} x^2 F(dx) < \infty$, we have $\mathbb{E}(|y^{n,m}_{i,k}(1,2)||\mathcal{F}_{t^1_i}) \le \frac{\delta_n \lambda_{n,m}}{nm}$. Then we conclude using $\delta_n \le C \log 1/\beta_n$ (see (3.2.5)), $\lambda_{n,m} \le \frac{C}{n\beta_n^\alpha}$ (see ($\mathbf{H}^\alpha_1$)), with the choice $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$, criteria (3.7.2) and Lemma 3.7.2. Therefore, we have $\overline{Y}^n(1) \xrightarrow{\mathbb{P}} 0$. Now, from the formula of the rest term given by (3.4.19), we have

$$\mathcal{R}^{n,m}_t = \Gamma^n_t(1,2) + \sum_{i=2}^{5} \Gamma^n_t(i).$$

Now, we will prove that each term converges uniformly in probability to 0 when $n \to \infty$.

<u>**The term** $\Gamma^n_t(1,2)$:</u> Let us recall that $\Gamma^n_t(1,2) = \sum_{k=1}^m \sum_{i=1}^{[nt]} \zeta^n_{i,k}(1,2)$, where

$$\zeta^n_{i,k}(1,2) = ff'(X_{t^1_i}) \sum_{j=2}^{K(t^i_k)} \Delta Y_{T_j(t^i_k)} \tilde{M}^{n,m}_{i,k} \quad \text{with} \quad \tilde{M}^{n,m}_{i,k} = (M^{\beta_n}_{t^{m+1}_i} - M^{\beta_n}_{t^1_i}) - (M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}).$$

For $k \in \{1, \dots, m\}$ fixed, first, by Lemma 3.7.5 we have $\mathbb{E}((\tilde{M}^{n,m}_{i,k})^2|\mathcal{F}_{t^1_i}) \le \frac{Cc_n}{n}$. As ff' is bounded, using Jensen's inequality, the independence of $\{\Delta Y_{T_j(t^k_i)}, j \ge 1, K(t^k_i)\}$ and $\tilde{M}^{n,m}_{i,k}$, property (P1) and $\int_{\mathbb{R}} x^2 F(dx) < \infty$, we get that $\mathbb{E}(\zeta^n_{i,k}(1,2)|\mathcal{F}_{t^1_i}) = 0$ and

$$\begin{aligned}\mathbb{E}((\zeta^n_{i,k}(1,2))^2|\mathcal{F}_{t^1_i}) &\le C u^2_{n,m} \mathbb{E}(K(t^k_i) \sum_{j=2}^{K(t^k_i)} (\Delta Y_{T^{\beta_n}_j(t^k_i)})^2 |\mathcal{F}_{t^1_i}) \mathbb{E}((\tilde{M}^{n,m}_{i,k})^2|\mathcal{F}_{t^1_i}) \\ &\le \frac{C u^2_{n,m} c_n}{n\theta(\beta_n)} \mathbb{E}(K(t^k_i)(K(t^k_i)-1)\mathbb{1}_{\{K(t^k_i)\ge2\}}|\mathcal{F}_{t^1_i}) \int_{|x|>\beta_n} x^2 F(dx) \le \frac{C u^2_{n,m} c_n \lambda_{n,m}}{n^2}.\end{aligned}$$

Then, we conclude by $c_n \le C\beta_n^{2-\alpha}$ (see (3.2.5)), $\lambda_{n,m} \le \frac{C}{n\beta_n^\alpha}$ (see ($\mathbf{H}^\alpha_1$)), the choices $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$ with $\alpha > 1$, criteria (3.7.3) and Lemma 3.7.2. Therefore, we get $\Gamma^n(1,2) \xrightarrow{\mathbb{P}} 0$.

<u>**The term** $\Gamma^n_t(2)$:</u> Let us recall that $\Gamma^n_t(2) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta^n_{i,k}(2)$, with

$$\zeta^n_{i,k}(2) = u_{n,m}[\frac{(k-1)d_n}{nm} + M^{\beta_n}_{t^k_i} - M^{\beta_n}_{t^1_i}](\frac{d_n}{nm} + M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}).$$

For $k \in \{2,\dots,m\}$ fixed, by the independence of the increments and Lemma 3.7.5, we get $\mathbb{E}(\zeta^n_{i,k}(2)|\mathcal{F}_{t_i^1}) = \frac{(k-1)u_{n,m}d_n^2}{n^2m^2}$ and

$$\begin{aligned}\mathbb{E}(|\zeta^n_{i,k}(2)|^2|\mathcal{F}_{t_i^1}) &\le Cu^2_{n,m}(\mathbb{E}(M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1})^2 + \frac{d_n^2}{n^2})(\mathbb{E}(M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k})^2 + \frac{d_n^2}{n^2})\\ &\le Cu^2_{n,m}\left(\frac{c_n^2}{n^2} + \frac{d_n^4}{n^4}\right).\end{aligned}$$

Then, we conclude by $c_n \le C\beta_n^{2-\alpha}$, $d_n \le C\beta_n^{1-\alpha}$ (see (3.2.5)), the choices $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$ with $\alpha > 1$, criteria (3.7.3) and Lemma 3.7.2. Therefore, we get $\Gamma^n(2) \xrightarrow{\mathbb{P}} 0$.

The term $\Gamma^n_t(3)$: Let us recall that $\Gamma^n_t(3) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta^n_{i,k}(3)$ with

$$\zeta^n_{i,k}(3) = \frac{u_{n,m}d_n(k-1)}{nm}(N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}) + \frac{u_{n,m}d_n}{nm}(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1}).$$

For $k \in \{2,\dots,m\}$ fixed, by the independence of the increments and Lemma 3.7.4, we get

$$\mathbb{E}(|\zeta^n_{i,k}(3)||\mathcal{F}_{t_i^1}) \le \frac{Cu_{n,m}|d_n|}{n}(\mathbb{E}|N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}| + \mathbb{E}|N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1}|) \le \frac{Cu_{n,m}|d_n|\delta_n}{n^2}.$$

Then, we conclude using δ_n and d_n are bounded by $C\beta_n^{1-\alpha}$ (see (3.2.5)), the choices $u_{n,m} = \left[\frac{mn}{(m-1)\log n}\right]^{1/\alpha}$ and $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$, criteria (3.7.2) and Lemma 3.7.2. Therefore, $\Gamma^n(3) \xrightarrow{\mathbb{P}} 0$.

The term $\Gamma^n_t(4)$: Let us recall that $\Gamma^n_t(4) = \sum_{k=2}^m \sum_{i=1}^{[nt]} \zeta^n_{i,k}(4)$ where

$$\zeta^n_{i,k}(4) = u_{n,m}(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})(N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}).$$

For $k \in \{2,\dots,m\}$ fixed, by the independence of the increments and Lemma 3.7.4, we have

$$\mathbb{E}(|\zeta^n_{i,k}(4)||\mathcal{F}_{t_i^1}) \le Cu_{n,m}\mathbb{E}(|N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1}|)\mathbb{E}(|N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}|) \le C\frac{u_{n,m}\delta_n^2}{n^2}.$$

Then, we conclude using $\delta_n \le C\beta_n^{1-\alpha}$ (see (3.2.5)), with our choice of $u_{n,m}$ and β_n, criteria (3.7.2) and Lemma 3.7.2. Therefore, we get $\Gamma^n(4) \xrightarrow{\mathbb{P}} 0$.

The term $\Gamma^n_t(5)$: Since $k(x,y)$ is bounded on $\mathbb{R}^2$, it is enough to prove that for $k \in \{1,\dots,m\}$, the following nine triangular arrays with generic terms $\{\zeta^n_{i,k}(5,j), j \in \{1,\dots,9\}$ converge to 0 as $n \to \infty$ with

$$\begin{aligned}
\zeta^n_{i,k}(5,1) &= u_{n,m}(A^{\beta_n}_{t_i^k} - A^{\beta_n}_{t_i^1})^2|A^{\beta_n}_{t_i^{k+1}} - A^{\beta_n}_{t_i^k}|, & \zeta^n_{i,k}(5,2) &= u_{n,m}(A^{\beta_n}_{t_i^k} - A^{\beta_n}_{t_i^1})^2|M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}|\\
\zeta^n_{i,k}(5,3) &= u_{n,m}(A^{\beta_n}_{t_i^k} - A^{\beta_n}_{t_i^1})^2|N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}|, & \zeta^n_{i,k}(5,4) &= u_{n,m}(M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1})^2|A^{\beta_n}_{t_i^{k+1}} - A^{\beta_n}_{t_i^k}|\\
\zeta^n_{i,k}(5,5) &= u_{n,m}(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})^2|M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}|, & \zeta^n_{i,k}(5,6) &= u_{n,m}(M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1})^2|N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}|\\
\zeta^n_{i,k}(5,7) &= u_{n,m}(M^{\beta_n}_{t_i^k} - M^{\beta_n}_{t_i^1})^2|M^{\beta_n}_{t_i^{k+1}} - M^{\beta_n}_{t_i^k}|, & \zeta^n_{i,k}(5,8) &= u_{n,m}(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})^2|N^{\beta_n}_{t_i^{k+1}} - N^{\beta_n}_{t_i^k}|\\
\zeta^n_{i,k}(5,9) &= u_{n,m}(N^{\beta_n}_{t_i^k} - N^{\beta_n}_{t_i^1})^2|A^{\beta_n}_{t_i^{k+1}} - A^{\beta_n}_{t_i^k}|.
\end{aligned}$$

From (3.5.9), the triangular arrays with generic terms $\zeta^n_{i,k}(5,i)$, $i \in \{1,\dots,9\}\backslash\{5\}$ are bounded as follows

$$\begin{cases} \mathbb{E}(|\zeta^n_{i,k}(5,1)||\mathcal{F}_{t^1_i}) \leq C\frac{u_{n,m}|d_n|^3}{n^3}, & \mathbb{E}(|\zeta^n_{i,k}(5,2)||\mathcal{F}_{t^1_i})\leq C\frac{u_{n,m}|d_n|^2\sqrt{c_n}}{n^2\sqrt{n}} \\ \mathbb{E}(|\zeta^n_{i,k}(5,3)||\mathcal{F}_{t^1_i})\leq C\frac{u_{n,m}|d_n|^2\delta_n}{n^3}, & \mathbb{E}(|\zeta^n_{i,k}(5,4)||\mathcal{F}_{t^1_i}) \leq C\frac{u_{n,m}|d_n|c_n}{n^2} \\ \mathbb{E}(|\zeta^n_{i,k}(5,6)||\mathcal{F}_{t^1_i}) \leq C\frac{u_{n,m}c_n\delta_n}{n^2}, & \mathbb{E}(|\zeta^n_{i,k}(5,7)||\mathcal{F}_{t^1_i}) \leq C\frac{u_{n,m}c_n^{3/2}}{n\sqrt{n}} \\ \mathbb{E}(|\zeta^n_{i,k}(5,8)||\mathcal{F}_{t^1_i}) \leq C\frac{u_{n,m}\delta_n}{n^2}, & \mathbb{E}(|\zeta^n_{i,k}(5,9)||\mathcal{F}_{t^1_i}) \leq C\frac{u_{n,m}|d_n|}{n^2}. \end{cases}$$

Then, by $u_{n,m} = \left[\frac{nm}{(m-1)\log n}\right]^{1/\alpha}$, $d_n \leq C\beta_n^{1-\alpha}$, $\delta_n \leq C\beta_n^{1-\alpha}$ and $c_n \leq C\beta_n^{2-\alpha}$ (see (3.2.5)) with $\beta_n = \frac{\log n}{n^{1/(2\alpha)}}$, criteria (3.7.2) and Lemma 3.7.2, we conclude that these triangular arrays with generic terms $\zeta^n_{i,k}(5,i)$, $i \in \{1,\dots,9\}\backslash\{5\}$ converge to 0. Next, concerning $\zeta^n_{i,k}(5,5)$, we reuse the notations in section 3.4.1 and rewrite $\zeta^n_{i,k}(5,5) = \zeta'^n_{i,k}(5,5) + \zeta''^n_{i,k}(5,5) + \zeta'''^n_{i,k}(5,5)$ where

$$\begin{cases} \zeta'^n_{i,k}(5,5) = u_{n,m}\Delta Y^2_{T_1^{\beta_n}(t^1_i,t^k_i)}\mathbb{1}_{\{K(t^1_i,t^k_i)\geq 1\}}|M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|, \\ \zeta''^n_{i,k}(5,5) = u_{n,m}\sum_{j=2}^{K(t^1_i,t^k_i)}\Delta Y^2_{T_j^{\beta_n}(t^1_i,t^k_i)}|M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|, \\ \zeta'''^n_{i,k}(5,5) = u_{n,m}\sum_{\substack{j,j'=1 \\ j\neq j'}}^{K(t^1_i,t^k_i)}|\Delta Y_{T_j^{\beta_n}(t^1_i,t^k_i)}||\Delta Y_{T_{j'}^{\beta_n}(t^1_i,t^k_i)}||M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|. \end{cases}$$

Concerning $\zeta''^n_{i,k}(5,5)$ and $\zeta'''^n_{i,k}(5,5)$, by the independence between the martingale increment and the jumps with size bigger than β_n, Cauchy-Schwarz's inequality and Lemma 3.7.5 for term $|M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|$ and the same estimates as done for the treatment of $\overline{\Gamma}^n(1,2)$ in the proof of Lemma 3.4.4, we have

$$\begin{cases} \mathbb{E}(|\zeta''^n_{i,k}(5,5)||\mathcal{F}_{t^1_i}) \leq Cu_{n,m}\sqrt{\frac{c_n}{n}}\mathbb{E}\left(\sum_{h=2}^{K(t^1_i,t^k_i)}\Delta Y^2_{T_h^{\beta_n}(t^1_i,t^k_i)}|\mathcal{F}_{t^1_i}\right) \leq \frac{Cu_{n,m}\sqrt{c_n}\lambda_{n,m}}{n\sqrt{n}} \\ \mathbb{E}(|\zeta'''^n_{i,k}(5,5)||\mathcal{F}_{t^1_i}) \leq \frac{Cu_{n,m}\sqrt{c_n}}{\sqrt{n}}\mathbb{E}\left(\sum_{\substack{j,j'=1 \\ j\neq j'}}^{K(t^1_i,t^k_i)}|\Delta Y_{T_j^{\beta_n}(t^1_i,t^k_i)}\Delta Y_{T_{j'}^{\beta_n}(t^1_i,t^k_i)}||\mathcal{F}_{t^1_i}\right) \leq \frac{Cu_{n,m}\sqrt{c_n}\delta_n^2}{n^2\sqrt{n}} \end{cases}$$

Then, for our choice of $u_{n,m}$ and β_n, we use $\delta_n \leq C\beta_n^{1-\alpha}$ and $c_n \leq C\beta_n^{2-\alpha}$ (see (3.2.5)), $\lambda_{n,m} \leq \frac{C}{n\beta_n^\alpha}$ (see ($\mathbf{H}_1^\alpha$)), criteria (3.7.2) and Lemma 3.7.2 to get the convergence to 0 of the triangular arrays with generic terms $\zeta''^n_{i,k}(5,5)$, $\zeta'''^n_{i,k}(5,5)$. Concerning the term $\zeta'^n_{i,k}(5,5)$, using the independence structure, properties (P1) and (P2), we see that

$$\mathbb{E}(e^{\mathtt{i}v\zeta'^n_{i,k}(5,5)}|\mathcal{F}_{t^1_i}) = e^{-(k-1)\lambda_{n,m}} + \frac{1-e^{-(k-1)\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}\mathbb{E}(e^{\mathtt{i}vu_{n,m}x^2|M^{\beta_n}_{t^{k+1}_i} - M^{\beta_n}_{t^k_i}|})F(dx)$$

Let us denote $z'_{n,m}(x,v) = \frac{1}{nm}\int_{|y|\leq\beta_n}(e^{\mathtt{i}vu_{n,m}x^2|y|} - 1 - \mathtt{i}vu_{n,m}x^2|y|)F(dy)$. Then we have

$$\begin{aligned} \mathbb{E}(e^{\mathtt{i}v\zeta'^n_{i,k}(5,5)}|\mathcal{F}_{t^1_i}) &= e^{-(k-1)\lambda_{n,m}} + \frac{1-e^{-(k-1)\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}e^{z'_{n,m}(x,v)}F(dx) \\ &= 1 + \frac{1-e^{-(k-1)\lambda_{n,m}}}{\theta(\beta_n)}\int_{|x|>\beta_n}(e^{z'_{n,m}(x,v)} - 1)F(dx). \end{aligned}$$

By similar calculations as in (3.4.20), we easily get $|z'_{n,m}(x,v)| \leq \frac{C}{n}|vu_{n,m}x^2|^\alpha$ and the suprema of $|z'_{n,m}(x,v)|$ over all $|x| \leq p$ and $|v| \leq 1$ goes to 0 as n tends to 0.

Now, since $|e^{z'_{n,m}(x,v)}-1|\leq C|z'_{n,m}(x,v)|$ and $x\mapsto|x|^{2\alpha}$ is F-integrable (see Remark 3.2.1), we have

$$|\mathbb{E}(e^{\mathrm{i}v\zeta'^{n}_{i,k}(5,5)}|\mathcal{F}_{t^1_i})-1|\leq\frac{C|v|^\alpha u^\alpha_{n,m}}{n^2}.$$

Then, for our choice of $u_{n,m}$, $\zeta'^{n}_{i,k}(5,5)$ satisfies (3.7.7) with $\xi'''_{n,v}=\frac{C|v|^\alpha u^\alpha_{n,m}}{n}$ which converges to 0 as $n\to\infty$ for all $v\leq 1$. Then, by Lemma 3.7.3 and Lemma 3.7.2, we get the convergence for the last triangular array. Therefore, we get $\Gamma^n(5)\stackrel{\mathbb{P}}{\to}0$.

3.6 Appendix B: Proof of some equalities and inequalities

First, we observe from the Lévy measure two relations followed:

- ***On the positive side*** For all $0\leq a<b\leq 1$ and $\gamma>0$, we have

$$\int_{a<x\leq b}|x|^\gamma F(dx)=\gamma\int_0^b y^{\gamma-1}(\theta_+(y\vee a)-\theta_+(b))dy \tag{3.6.1}$$

$$\text{and}\quad\int_{a<x\leq b}x\log xF(dx)=\int_0^b(1+\log y)(\theta_+(y\vee a)-\theta_+(b))dy. \tag{3.6.2}$$

Proof. The proofs are simply by Fubini. Concerning (3.6.1), it r.h.s. is equal to

$$\begin{aligned}
&\gamma\int_0^b y^{\gamma-1}\left(\int_{x>y\vee a}F(dx)-\int_{x>b}F(dx)\right)dy=\gamma\int_0^b y^{\gamma-1}\int_{y\vee a<x\leq b}F(dx)dy\\
&=\gamma\left(\int_0^a y^{\gamma-1}\int_{a<x\leq b}F(dx)dy+\int_a^b y^{\gamma-1}\int_{y<x\leq b}F(dx)dy\right)\\
&=\gamma\left(\int_{a<x\leq b}F(dx)\int_0^a y^{\gamma-1}dy+\int_{a<x\leq b}F(dx)\int_a^x y^{\gamma-1}dy\right)\\
&=\int_{a<x\leq b}a^\gamma F(dx)+\int_a^b(x^\gamma-a^\gamma)F(dx)=\int_{a<x\leq b}x^\gamma F(dx).
\end{aligned}$$

Concerning (3.6.2), its r.h.s is equal to

$$\begin{aligned}
&\int_0^a(1+\log y)(\theta_+(a)-\theta_+(b))dy+\int_a^b(1+\log y)(\theta_+(y)-\theta_+(b))dy\\
&=(\theta_+(a)-\theta_+(b))a\log a+\int_a^b\int_{x>y}(1+\log y)F(dx)dy-\theta_+(b)(b\log b-a\log a)\\
&=\theta_+(a)a\log a-\theta_+(b)b\log b+\int_{x>a}\int_{a<y\leq x\wedge b}(1+\log y)dyF(dx)\\
&=-\theta_+(b)b\log b+\int_{x>a}(x\wedge b)\log(x\wedge b)F(dx)=\int_{a<x\leq b}x\log xF(dx).
\end{aligned}$$

□

- ***On the negative side*** For all $-1 \le a < b \le 0$ and $\gamma > 0$, we have

$$\int_{a\le x<b} |x|^\gamma F(dx) = \gamma \int_a^0 (-y)^{\gamma-1}(\theta_-((-y)\vee(-b)) - \theta_-(-a))dy. \tag{3.6.3}$$

and $$\int_{a\le x<b} |x|\log|x| F(dx) = \int_a^0 (1+\log|y|)(\theta_-((-y)\vee(-b)) - \theta_-(-a))dy. \tag{3.6.4}$$

Proof. Here, also, the proofs are simply by Fubini. Concerning (3.6.3), it r.h.s. is equal to

$$\begin{aligned}
&\gamma \int_a^0 (-y)^{\gamma-1}\left(\int_{x<y\wedge b} F(dx) - \int_{x<a} F(dx)\right) dy = \gamma \int_a^0 (-y)^{\gamma-1} \int_{a\le|x|<y\wedge b} F(dx)dy \\
&= \gamma\left(\int_a^b (-y)^{\gamma-1}\int_{a\le|x|<y} F(dx)dy + \int_b^0 (-y)^{\gamma-1}\int_{a\le|x|<b} F(dx)dy\right) \\
&= \gamma\left(\int_{a\le x<b} F(dx)\int_x^b (-y)^{\gamma-1}dy + \int_{a\le x<b} F(dx)\int_b^0 (-y)^{\gamma-1}dy\right) \\
&= \int_{a\le x<b} ((-x)^\gamma - (-b)^\gamma)F(dx) + \int_{a\le x<b} F(dx)(-b)^\gamma = \int_{a\le x<b} (-x)^\gamma F(dx).
\end{aligned}$$

Concerning (3.6.4), its r.h.s is equal to

$$\begin{aligned}
&\int_a^b (1+\log y)(\theta_-(-y) - \theta_-(-a))dy + \int_b^0 (1+\log y)(\theta_-(-b) - \theta_-(-a))dy \\
&= \int_a^b\int_{x<y} (1+\log y)F(dx)dy - \theta_-(-a)(b\log(-b) - a\log(-a)) - (\theta_-(-b) - \theta_-(-a))b\log(-b) \\
&= \int_{x<b}\int_{x\vee a<y\le b} (1+\log y)dyF(dx) + \theta_-(-a)a\log(-a) - \theta_-(-b)b\log(-b) \\
&= \int_{x<b} (x\vee a)\log(-(x\vee a))F(dx) + \theta_-(-a)a\log(-a) = \int_{a\le x<b} |x|\log|x|F(dx).
\end{aligned}$$

□

Proof of Lemma 3.2.2. Taking advantage of (3.6.1) and (3.6.3), we have

$$\begin{aligned}
c(\beta) &= \int_{0<x\le\beta} x^2F(dx) + \int_{-\beta<x\le 0} x^2F(dx) \\
&= 2\int_0^\beta y(\theta_+(y) - \theta_+(\beta))dy + 2\int_{-\beta}^0 (-y)(\theta_-(-y) - \theta_-(\beta))dy \\
&= 2\int_0^\beta y(\theta_+(y) - \theta_+(\beta))dy + 2\int_0^\beta y(\theta_-(y) - \theta_-(\beta))dy = 2\int_0^\beta y(\theta(y) - \theta(\beta))dy.
\end{aligned} \tag{*}$$

In other hand, by $(\mathbf{H}_1^\alpha)$ $|\theta(y) - \theta(\beta)| \le |\theta(y)| + |\theta(\beta)| \le Cy^{-\alpha} + C\beta^{-\alpha}$. By $(*)$, $c(\beta) \le \left(\frac{2C}{2-\alpha} + C\right)\beta^{2-\alpha}$. Similar proofs are easily deduced from (3.6.1) and (3.6.3) for the other formulas. □

Proof of Lemma 3.2.4. By $(*)$ $c(\beta) \sim 2\int_0^\beta y\left(\frac{\theta}{y^\alpha} - \frac{\theta}{\beta^\alpha}\right)dy = \frac{\alpha\theta}{2-\alpha}\beta^{2-\alpha}$. Here, we can also obtain similar proofs for the other formulas. □

Proof of Lemma 3.2.5. Applying (3.6.2) we have

$$\frac{1}{(\log(1/\beta))^2}\int_{\beta<x\le b}(x\log x)F(dx) = \frac{1}{(\log(1/\beta))^2}\int_0^b(1+\log y)(\theta_+(y\vee\beta)-\theta_+(b))dy$$
$$\sim \frac{1}{(\log(1/\beta))^2}\theta_+(\beta)\int_0^\beta(1+\log|y|)dy + \frac{1}{(\log(1/\beta))^2}\int_\beta^b(1+\log y)\theta_+(y)dy.$$

Considering the first term, for $\beta \to 0$ we deduce from $(\mathbf{H}_2^\alpha)$ with $\alpha = 1$ that $\frac{1}{(\log(1/\beta))^2}\theta_+(\beta)\int_0^\beta(1+\log|y|)dy \sim \frac{\log\beta}{(\log(1/\beta))^2}\theta_+ \underset{n\to\infty}{\longrightarrow} 0$. Now, let $\varepsilon > 0$, there exists a $\varepsilon' \in (0,1)$ such that $\beta \in (0,\varepsilon')$ and we have $|\frac{\beta\theta_+(\beta)}{\theta_+} - 1| \le \varepsilon$. Considering the second term, we rewrite $\frac{1}{(\log(1/\beta))^2}\int_\beta^b(1+\log y)\theta_+(y)dy = x_n + y_n$ where

$$x_n = \frac{1}{(\log(1/\beta))^2}\int_{\varepsilon'}^b(1+\log y)\theta_+(y)dy, \quad y_n = \frac{1}{(\log(1/\beta))^2}\int_\beta^{\varepsilon'}(1+\log y)\theta_+(y)dy.$$

On the one hand, x_n is bounded by $\frac{\theta_+(\varepsilon')}{(\log(1/\beta))^2}\int_{\varepsilon'}^b(1+\log y)dy$ which converges to 0 as $n\to\infty$. On the other hand, if we denote $y'_n = \frac{\theta_+}{(\log(1/\beta))^2}\int_\beta^{\varepsilon'}(1+\log y)y^{-1}dy$, we have $y'_n(1-\varepsilon) \le y_n \le y'_n(1+\varepsilon)$ for any ε arbitrarily small, then $y_n \sim y'_n \sim \frac{-(\log\beta)^2}{2(\log(1/\beta))^2}\theta_+ \underset{n\to\infty}{\longrightarrow} -\frac{\theta_+}{2}$. Therefore, it is clear that $\frac{1}{(\log(1/\beta))^2}\int_{\beta<x\le\varepsilon'}(x\log x)F(dx) \underset{n\to\infty}{\longrightarrow} -\frac{\theta_+}{2}$. Similarly, applying (3.6.4), we get $\frac{1}{(\log(1/\beta))^2}\int_{-b\le x<\beta}((-x)\log(-x))F(dx) \underset{n\to\infty}{\longrightarrow} -\frac{\theta_-}{2}$ which completes the proof. □

3.7 Appendix C: Some general tools

3.7.1 Uniformly tight processes

We recall the definition of uniformly tight property (UT) defined in Jakubowski, Mémin, and Pagès, 1989. Let Z^n be a sequence of semimartingale, with the canonical decompositions

$$Z_t^n = A_t^{n,a} + M_t^{n,a} + \sum_{s\le t}\Delta Z_s^n 1_{\{|\Delta Z_s^n|>a\}}, \tag{3.7.1}$$

where $a > 0$ and $A^{n,a}$ is a predictable process with locally bounded variation and $M^{n,a}$ is a (locally bounded) local martingale. Then we say that the sequence (Z^n) satisfies (UT) if for any $t < \infty$, the sequence of real-valued random variables

$$Var(A^{n,a})_t + \langle M^{n,a}, M^{n,a}\rangle_t + \sum_{s\le t}|\Delta Z_t^n|1_{\{|\Delta Z_s^n|>a\}}$$

is tight. This property does not depend on the choice of $a \in (0,\infty)$.
If a sequence is (UT) then it has some other important properties as in the theorem below, which can also be found in Jacod and Shiryaev, 2003 or in Theorem 2.3 of Jacod and Protter, 1998.

Theorem 3.7.1. *Let X^n and Y^n be two sequences of semi-martingales,*

(i) If both sequences X^n and Y^n are (UT), then so has the sequence $X^n + Y^n$.

(ii) *Let H^n be a sequence of predictable processes such that the sequence $\sup_{s\leq t}|H^n_s|$ is tight. If the sequence X^n is (UT), so is the sequence $\int_0^\cdot H^n_s dX^n_s$.*

(iii) *Suppose that X^n weakly converges. Then (UT) is necessary and sufficient for the following property:*
For any sequence of adapted càdlàg H^n processes such that the sequence (H^n, X^n) weakly converges to (H, X), then X is a semi-martingale with respect to the filtration generated by the process (H, X), and we have $(H^n, X^n, \int_0^\cdot H^n_{s-}dX^n_s) \xrightarrow{\mathcal{L}} (H, X, \int_0^\cdot H_{s-}dX_s)$.

3.7.2 Triangular arrays

Now concerning sums of triangular arrays of the form

$$\Gamma^n_t = \sum_{i=1}^{[nt]} \zeta^n_i,$$

where for each n we have $\mathbb{R}^d$-valued random variables $(\zeta^n_i)_{i\geq 1}$ such that each ζ^n_i is $\mathcal{F}_{i/n}$-measurable. Below we give various conditions recalled in Jacod, 2004 insuring tightness or convergence of the sequence (Γ^n).

$$\mathbb{E}(|\zeta^n_i||\mathcal{F}_{(i-1)/n}) \leq \frac{\xi_n}{n}, \tag{3.7.2}$$

$$\begin{cases} |\mathbb{E}(\zeta^n_i|\mathcal{F}_{(i-1)/n})| \leq \frac{\xi_n}{n}, \\ \mathbb{E}(|\zeta^n_i|^2|\mathcal{F}_{(i-1)/n}) \leq \frac{\xi'_n}{n}, \end{cases} \tag{3.7.3}$$

$$\begin{cases} |\mathbb{E}(\zeta^n_i \mathbb{1}_{\{|\zeta^n_i|\leq 1\}}|\mathcal{F}_{(i-1)/n})| \leq \frac{\xi_n}{n}, \\ \mathbb{E}(|\zeta^n_i|^2 \mathbb{1}_{\{|\zeta^n_i|\leq 1\}}|\mathcal{F}_{(i-1)/n}) \leq \frac{\xi'_n}{n}, \\ \mathbb{P}(|\zeta^n_i| > y|\mathcal{F}_{(i-1)/n}) \leq \frac{\xi''_{n,y}}{y}, \qquad \forall y > 1. \end{cases} \tag{3.7.4}$$

Note that (3.7.3) with $\hat{\xi}_n$ and $\hat{\xi}'_n$ implies (3.7.4) with $\xi_n = \hat{\xi}_n + \hat{\xi}'_n$ and $\xi'_n = \hat{\xi}'_n$ and $\xi''_{n,y} = \hat{\xi}'_n/y^2$ (the last is from extended version of Markov inequality for monotonically increasing functions). Also, (3.7.2) with $\hat{\xi}_n$ implies (3.7.4) with $\xi_n = \xi'_n = \hat{\xi}_n$ and $\xi''_{n,y} = \hat{\xi}_n/y$.
By $\Gamma^n \xrightarrow{\mathbb{P}} 0$, we mean that $\sup_{s\leq t}|\Gamma^n_s|$ goes to 0 in probability for all t.

Lemma 3.7.2. *(Lemma 2.5 in Jacod, 2004)*

(a) *For $\Gamma^n \xrightarrow{\mathbb{P}} 0$, it is enough that either (3.7.2) or (3.7.3) or (3.7.4) hold with*

$$\lim_n \xi_n = 0, \qquad \lim_n \xi'_n = 0, \qquad \lim_n \xi''_{n,y} = 0 \qquad \forall y > 1. \tag{3.7.5}$$

(b) *For the sequence (Γ^n) to be tight for the Skorokhod topology, it is enough that the sequence of each of the d components of ζ^n_i satisfies either (3.7.2) or (3.7.3) or (3.7.4) with*

$$\limsup_n \xi_n < \infty, \quad \limsup_n \xi'_n < \infty, \quad \limsup_n \xi''_{n,y} < \infty, \quad \lim_{y\uparrow\infty} \limsup_n \xi''_{n,y} = 0. \tag{3.7.6}$$

Lemma 3.7.3. *(Lemma 2.6 in Jacod, 2004)*
Suppose that one can find constants $\xi'''_{n,v}$ such that

$$\sup_{u:|u|\leq v} |1 - \mathbb{E}(e^{iu.\zeta_i^n} | \mathcal{F}_{(i-1)/n})| \leq \frac{\xi'''_{n,v}}{n}, \quad |v| \leq 1 \tag{3.7.7}$$

then (3.7.4) holds with $\xi_n = \xi'_n = C\xi'''_{n,1}$ and $\xi''_{n,y} = C\xi'''_{n,1/y}$.

We recall some important lemmas. The following theorems are from Theorem 2.3.7. and Theorem 4.2.3. in Applebaum, 2009.

Lemma 3.7.4. *Let N be a Poisson process with intensity function μ and A be bounded below. Then if $f \in L^1(A, \mu(A))$, we have $\mathbb{E}(\int_A f(x)N(t, dx)) = t\int_A f(x)\mu(dx)$.*

Let us denote

$$\mathcal{H}_2(T, E) = \left\{ F : [0, T] \times E \times \Omega \to \mathbb{R} \mid F \text{ is predictable and } \int_0^T \int_E \mathbb{E}(|F(t, x)|^2)\rho(dt, dx) \right\}$$

and $I_T(F) = \int_0^T \int_E F(t, x)M(dt, dx)$ with M is a martingale satisfying $M(\{0\}, A) = 0$ a.s. and there exists a σ-finite measure ρ on $\mathbb{R}^+ \times E$ for which $\mathbb{E}(M(t, A)^2) = \rho(t, A)$ for any $A \in \mathcal{B}(E)$.

Lemma 3.7.5. *If $F \in \mathcal{H}_2(T, E)$ then $\mathbb{E}(I_T(F)^2) = \int_0^T \int_E \mathbb{E}(|F(t, x)|^2)\rho(dt, dx)$.*

The following theorem is about the convergence of infinitely divisible distributions. Justified by the one-to-one correspondence between infinitely divisible distributions μ and their characteristics (a, b, ν), we may write $\mu = id(a, b, \nu)$. This can be found from Theorem 15.14 (i) of Kallenberg, 2002 or equivalently Theorem VII.3.4 of Jacod and Shiryaev, 2003.

Theorem 3.7.6. *Let $\mu = id(a, b, \nu)$ and $\mu_n = id(a_n, b_n, \nu_n)$ on $\mathbb{R}^d$, and fix any $h > 0$ with $\nu\{|x| = h\} = 0$. Define*

$$a^h = a + \int_{|x|\leq h} xx^\top \nu(dx), \quad b^h = b - \int_{h<|x|\leq 1} x\nu(dx),$$

where $\int_{h<|x|\leq 1} x\nu(dx) = -\int_{1<|x|\leq h} x\nu(dx)$ when $h > 1$.
Then $\mu_n \xrightarrow{w} \mu$ iff $a_n^h \to a^h$, $b_n^h \to b^h$, and $\nu_n \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d \backslash \{0\}$.

The following Lemma shows a way to prove the convergence in distribution of a triangular array. This can be found as Corollary 15.16 of Kallenberg, 2002 or equivalently Theorem VIII.2.29 of Jacod and Shiryaev, 2003 which

Lemma 3.7.7. *Consider in $\mathbb{R}^d$ an i.i.d. array (ζ_i^n) and let Γ be $id(a, b, \nu)$. For any $h > 0$ with $\nu\{|x| = h\} = 0$, $\sum_{i=1}^{[n.]} \zeta_i^n \xrightarrow{\mathcal{L}} \Gamma$ iff*

(i) $n\mathcal{L}(\zeta_1^n) \xrightarrow{v} \nu$ on $\bar{\mathbb{R}}^d \backslash \{0\}$

(ii) $n\mathbb{E}(\zeta_1^n; |\zeta_1^n| \leq h) \to b^h$

(iii) $n\mathbb{E}(\zeta_1^n \zeta_1^{n\top}; |\zeta_1^n| \leq h) \to a^h$.

Remark 3.7.8. *To check (i), we prove that $n\mathbb{E}(1_{|\zeta_1^n|>\omega}) \to \nu\{|x| > \omega\}$, for any $\omega > 0$.*

Lemma 3.7.9. *(Lemma 2.1 in Jacod, 2004) If for each n ζ_i^n, $i = 1, 2, \ldots$ are i.i.d. random variables and Γ_1^n converges in law to a limit U, then there is a Lévy process Γ*

such that $\Gamma_1 = U$. This process Γ is unique in law and Γ^n converges in law to Γ (for the Skorokhod topology). Further, the sequence (Γ^n) has (UT).

Let $Z_t^n = \sum_{i=1}^{[nt]} \eta_i^n$, $\Gamma_t^n = \sum_{i=1}^{[nt]} \zeta_i^n$ and $\Gamma'^n_t = \sum_{i=1}^{[nt]} \zeta'^n_i$ with $\zeta_i^n = g(X^n_{\frac{i-1}{n}})\zeta'^n_i$. For each n, if the sequence (η_i^n, ζ'^n_i), $i = 1, 2, \ldots$, is i.i.d., combining Lemma 3.7.9 with Theorem 3.7.1 (iii), we get the following lemma which is very similar to Lemma 2.8 in Jacod, 2004.

Lemma 3.7.10. *We suppose that the sequence (Z^n, Γ^n) is tight. If the pair (Z_1^n, Γ'^n_1) of random variables converges in law to (Z_1, γ') with γ' a random variable independent of Z_1 and that g is a Lipschitz-continuous function, then there is a Lévy process Γ', independent of Y and unique in law, such that the processes $(Z^n, \Gamma'^n, \Gamma^n)$ converge in law to (Z, Γ', Γ), where $\Gamma_t = \int_0^t g(X_{s-})d\Gamma'_s$. If further γ' is a constant, then we get $\Gamma_t = \int_0^t g(X_{s-})\gamma' ds$, and the convergence of $(Z^n, \Gamma'^n, \Gamma^n)$ takes place in probability.*

Proof. We rewrite $\Gamma^n = \Gamma^{n,1} + \Gamma^{n,2}$ where $\Gamma_t^{n,1} = \sum_{i=1}^{[nt]}(g(X^n_{\frac{i-1}{n}}) - g(X_{\frac{i-1}{n}}))\zeta'^n_i$ and $\Gamma_t^{n,2} = \sum_{i=1}^{[nt]} g(X_{\frac{i-1}{n}}))\zeta'^n_i$. First, since (Z_1^n, Γ'^n_1) converges in law to (Z_1, γ'), then by Lemma 2.8 in Jacod, 2004, we have $(Z^n, \Gamma'^n, \Gamma^{n,2}) \xrightarrow{\mathcal{L}} (Z, \Gamma', \Gamma)$. Second, by Jacod, 2004, Theorem 1.2 and using that g is Lipschitz, we easily deduce that $g(X^n_{\eta_n(.)}) - g(X_{\eta_n(.)}) \xrightarrow{\mathbb{P}} 0$. Then, since $\Gamma^{n,2} \xrightarrow{\mathcal{L}} \Gamma$, we apply Lemma 3.7.9 to get the (UT) property of $\Gamma^{n,2}$ and Theorem 3.7.1 (iii) to obtain $\Gamma^{n,1} \xrightarrow{\mathbb{P}} 0$. Therefore, we get $(Z^n, \Gamma'^n, \Gamma^n) = (Z^n, \Gamma'^n, \Gamma^{n,1} + \Gamma^{n,2}) \xrightarrow{\mathcal{L}} (Z, \Gamma', \Gamma)$. □

3.7.3 Evaluation of logarithm and exponential functions

Here, we use the power series expansions for both functions $\log(1+z)$ and e^z for $z \in \mathbb{C}$ (see e.g. Gronwall, 1916). We know that if $z \in \mathbb{C}$ and $|z| < \frac{1}{2}$, we have $\log(1+z) - z = -\sum_{n\geq 2}(-1)^n \frac{z^n}{n}$. Then $|\log(1+z) - z| \leq \frac{1}{2}\sum_{n\geq 2}|z|^n$. By applying the formula for convergent geometric sum, we have $|\log(1+z) - z| \leq \frac{1}{2}|z|^2 \frac{1}{1-|z|}$. Now, since $|z| < \frac{1}{2}$, then $\frac{1}{1-|z|} < 2$ then $\forall z \in \mathbb{C}$ such that $|z| < \frac{1}{2}$ we have $\boxed{|\log(1+z) - z| \leq |z|^2}$. We can proceed in the same way to prove that $\forall z \in \mathbb{C}$ such that $|z| < \frac{1}{2}$ we also have $\boxed{|e^z - 1 - z| \leq |z|^2}$ and by consequence, we get $\boxed{|e^z - 1| \leq \frac{3}{2}|z|}$.

Chapter 4

Central Limit Theorem for the antithetic multilevel Monte Carlo method

In this chapter, we introduce our antithetic MLMC estimator for a multi-dimensional diffusion which is an extended version of the original antithetic MLMC one introduced by Giles and Szpruch, 2014. Our aim is to study the asymptotic behavior of the weak errors involved in this new algorithm. Among the obtained results, we prove that the error between on the one hand the average of the Milstein scheme without Lévy area and its antithetic version build on the finer grid and on the other hand the coarse approximation stably converges in distribution with a rate of order 1. We also prove that the error between the Milstein scheme without Lévy area and its antithetic version stably converges in distribution with a rate of order 1/2. More precisely, we have a functional limit theorem on the asymptotic behavior of the joined distribution of these errors based on a triangular array approach (see e.g. Jacod, 1997). Thanks to this result, we establish a central limit theorem of Lindeberg-Feller type for the antithetic MLMC estimator. The time complexity of the algorithm is carried out.

The original paper Ben Alaya, Kebaier, and Ngô, 2020 of this work is under minor revisions for the journal Annals of Applied Probability.

4.1 Introduction

In recent years, the multilevel Monte Carlo (MLMC) algorithm, used to approximate $\mathbb{E}[\varphi(X_t, 0 \leq t \leq T)]$ for a given functional φ and a stochastic process $(X_t)_{0 \leq t \leq T}$, has become a hot topic. This method introduced by Giles, 2008b, that may be seen as an extension of the works of Heinrich, 2001 and Kebaier, 2005, is well known for reducing significantly the approximation time complexity compared to a classical Monte Carlo method. Many authors have since been interested in the study of a central limit theorem associated to the MLMC estimator that can be found in the recent works by Ben Alaya and Kebaier, 2014; Ben Alaya and Kebaier, 2015, Dereich and Li, 2016, Giorgi, Lemaire, and Pagès, 2017, Hoel and Krumscheid, 2019 and Kebaier and Lelong, 2018. Like for the classical Monte Carlo method, obtaining a central limit theorem is important for the practical implementation of the MLMC method (see e.g. Hoel et al., 2014). More recently, Giles and Szpruch, 2014 introduced an antithetic version of the Milstein MLMC estimator without Lévy area that achieves the optimal complexity $O(\Delta_n^{-2})$ for a given precision Δ_n as for an unbiased Monte Carlo estimator. The efficiency of the antithetic MLMC estimator was validated through a broad array of applications that can be found in Giles and Szpruch, 2013b; Giles and Szpruch, 2013a. Since then, many new studies were interested on several types of use of the antithetic MLMC estimator (see e.g. Debrabant and Rößler, 2015, Debrabant, Ghasemifard, and Mattsson, 2019, Al Gerbi, Jourdain, and Clément, 2016; Al Gerbi, Jourdain, and Clément, 2018). However, the problem of studying the validity of the central limit theorem for the antithetic MLMC algorithm has not been addressed in previous research. In the present paper, we first introduce an extended version of this antithetic MLMC method which allows permutations between the finer m Brownian increments associated to each corse increment with $m \geq 2$. Let us emphasize that the original antithetic MLMC method introduced in Giles and Szpruch, 2014 corresponds to $m = 2$. Then, we establish a central limit theorem on this extended antithetic MLMC algorithm that is parametrized by a permutation $\sigma \in \mathcal{S}_m$. This new result fills the gap in the literature for MLMC methods and yields new insights on the practical implementation of the antithetic MLMC algorithm. Indeed, the appeal of a central limit theorem is that it provides the fair rate of convergence and gives the exact asymptotic variance. Moreover, it allows us to build an automatic algorithm where the sample size of each level is explicitly given without any precomputation procedure and yields a more accurate confidence interval. Further, the knowledge of the asymptotic variance allows for the design of efficient variance reduction techniques for the MLMC (see e.g., Ben Alaya, Hajji, and Kebaier, 2015 and Ben Alaya, Hajji, and Kebaier, 2016). In order to establish this result, we prove a functional limit theorem for the normalized error on two consecutive levels for the joined distribution of the couple

$$\left(\sqrt{n}(X^{nm} - X^{\sigma,nm}), n((X^{nm} + X^{\sigma,nm})/2 - X^{n})\right), \tag{4.1.1}$$

where X^{nm} denotes the Milstein scheme with time step T/mn without Lévy area and $X^{\sigma,nm}$ is its antithetic version. This result extends the stable convergence limit theorem obtained by Ben Alaya and Kebaier, 2015 for the normalized error on two consecutive levels $\sqrt{n}(\tilde{X}^{mn} - \tilde{X}^{n})$ where $\tilde{X}^{n}$ denotes the Euler scheme with time step T/n. The proof of this result, written in a multidimensional setting, relies on combining the limit theorems on martingale triangular arrays in Jacod, 1997 with technics used in Jacod, 2004 and Jacod and Protter, 1998.

The rest of this paper is organized as follows. In Section 2, we recall from Giles and Szpruch, 2014 the Milstein scheme without Lévy area using our own notations and we introduce our assumptions. In Section 3, we introduce the extended antithetic scheme (4.3.2) as well as the antithetic MLMC estimator (4.3.4) and prove our main results namely Theorem 4.3.2, a functional limit theorem for the couple of normalized errors (4.1.1) and Theorem 4.3.3, the central limit theorem for the associated antithetic MLMC estimator. Section 4 gives the details of the error expansion needed to prove Theorem 4.3.2 with specifying the main and rest terms. Based on this expansion, we study in Section 5 the asymptotic behaviors of the joined distribution of the main terms. The rest terms are treated in appendices 4.6 and 4.7. Appendix 4.8 is dedicated to recall some theoretical tools that we use throughout the paper.

4.2 General framework

4.2.1 Milstein scheme without Lévy area

We consider the d-dimensional SDE driven by a q-dimensional Brownian motion $W = (W^1, \dots, W^q)^\top$, $q \geq 1$, solution to

$$X_t = x_0 + \int_0^t f(X_s)ds + \int_0^t g(X_s)dW_s, \text{ for } t \in [0,T],\ T > 0, \tag{4.2.1}$$

where $x_0 \in \mathbb{R}^d$, $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^{d\times q})$. In what follows, we assume that g does not have a commutativity property (see assumption ($\mathbf{H}_{f,g}$) below). Without loss of generality we will take the solution of (4.2.1) on the interval $[0,1]$ rather than $[0,T]$, $T > 0$. We will consider a time grid on $[0,1]$ with a uniform time step $\Delta_n = \frac{1}{n}$, $n \in \mathbb{N}$.

Notations Throughout this paper, we will use the following notations:

- For $g \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}^{d\times q})$, we introduce the tensor function $\{h_{\ell jj'}, 1 \leq \ell \leq d, 1 \leq j, j' \leq q\}$ defined by

$$h_{\ell jj'}(x) = \frac{1}{2}\nabla g_{\ell j}^\top(x) g_{\bullet j'}(x) = \frac{1}{2}\sum_{\ell'=1}^{d} \frac{\partial g_{\ell j}}{\partial x_{\ell'}}(x) g_{\ell' j'}(x), \quad x \in \mathbb{R}^d$$

  with $\nabla g_{\ell j} = (\frac{\partial g_{\ell j}}{\partial x_1}, \cdots, \frac{\partial g_{\ell j}}{\partial x_d})^\top \in \mathbb{R}^d$ and $g_{\bullet j'} = (g_{1j'}, \dots, g_{dj'})^\top \in \mathbb{R}^d$ is the j'^{th}-column of g and analogously we also introduce the ℓ^{th}-row of g given by $g_{\ell\bullet} = (g_{\ell 1}, \dots, g_{\ell q})$. The notation $A^\top$ stands for the transpose of the given matrix A.

- For $\ell \in \{1, \dots, d\}$, we denote the $q \times q$-matrix $h_{\ell\bullet\bullet} = \begin{pmatrix} h_{\ell 11} & \dots & h_{\ell 1q} \\ \vdots & \ddots & \vdots \\ h_{\ell q1} & \dots & h_{\ell qq} \end{pmatrix} \in \mathbb{R}^{q\times q}$.

- For more convenience, we set $\mathbb{H} = (h_{1\bullet\bullet}, \dots, h_{d\bullet\bullet})^\top$.

- For any function $\psi : \mathbb{R}^d \to \mathbb{R}$, we denote $\nabla^2\psi = \begin{pmatrix} \frac{\partial^2\psi}{\partial x_1\partial x_1} & \cdots & \frac{\partial^2\psi}{\partial x_1\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2\psi}{\partial x_d\partial x_1} & \cdots & \frac{\partial^2\psi}{\partial x_d\partial x_d} \end{pmatrix}$ the Hessian $d \times d$ matrix of ψ.

- For any d-dimensional function f, we denote its Jacobian matrix as $\nabla f = (\nabla f_1, \dots, \nabla f_d)^\top$.

- Let $\diamond$ denote the Frobenius inner product that is for any A and $B \in M_{p\times q}(\mathbb{R})$ with $\mathcal{M}_{p\times q}(\mathbb{R})$ is the set of $\mathbb{R}$-valued $p \times q$-matrices,

$$A \diamond B = \sum_{j=1}^{p}\sum_{j'=1}^{q} A_{jj'}B_{jj'} \in \mathbb{R}.$$

Moreover, we introduce the operator $\blacklozenge$ defined by: for any $A_{\ell\ell'} \in M_{p\times q}(\mathbb{R})$, $\ell \in \{1,\dots,r\}$ and $\ell' \in \{1,\dots,s\}$ with $r,s \in \mathbb{N}\setminus\{0\}$

$$\begin{pmatrix} A_{11} & \dots & A_{1s} \\ \vdots & \ddots & \vdots \\ A_{r1} & \dots & A_{rs} \end{pmatrix} \blacklozenge B = \begin{pmatrix} A_{11} \lozenge B & \dots & A_{1s} \lozenge B \\ \vdots & \ddots & \vdots \\ A_{r1} \lozenge B & \dots & A_{rs} \lozenge B \end{pmatrix} \in \mathbb{R}^{r\times s}.$$

- We have the following property for any matrices U and A respectively in $M_{p\times 1}(\mathbb{R})$ and $M_{p\times p}(\mathbb{R})$
$$U^\top AU = A \lozenge (UU^\top). \tag{4.2.2}$$
- We denote $\eta_n(t) = \frac{[nt]}{n}$ for $t \in [0,1]$, where $[x]$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$. For $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$, $n,m \in \mathbb{N}\setminus\{0,1\}$, we denote $\Delta W_i = W_{\frac{i}{n}} - W_{\frac{i-1}{n}}$ and $\delta W_{ik} = W_{\frac{m(i-1)+k}{nm}} - W_{\frac{m(i-1)+k-1}{nm}}$.
- $\mathcal{S}_m$ stands for the set of all permutations of order m.
- For $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$, $m \in \mathbb{N}\setminus\{0,1\}$, and $\tilde{\sigma} \in \mathcal{S}_m$ we denote the σ-algebra $\mathcal{F}^{k,\tilde{\sigma}}_{\frac{i-1}{n}} = \mathcal{F}_{\frac{i-1}{n}} \bigvee \sigma(\delta W_{i\tilde{\sigma}(k')} : 1 \le k' \le k)$, where $(\mathcal{F}_t)_{t\in[0,1]}$ denotes the natural filtration of the Brownian motion W and $\bigvee$ denotes the σ-algebra generated by the union.
- For $p > 0$, let $(\Gamma^n)_{n\in\mathbb{N}}$ be a sequence of processes in L^p. By $\Gamma^n \xrightarrow{L^p} 0$ (resp. $\Gamma^n \xrightarrow{\mathbb{P}} 0$) as n tends to infinity, we mean that $\sup_{s\le 1}|\Gamma^n_s| \xrightarrow{L^p} 0$ (resp. $\sup_{s\le 1}|\Gamma^n_s| \xrightarrow{\mathbb{P}} 0$) as n tends to infinity.
- For any block matrix $A = (A_{ij})$, the notation $|A|$ stands for the L^1-matrix norm, that satisfies $|A| = \sum_{ij}|A_{ij}|$.
- The set of $p\times q$-block matrices of $m\times n$-matrices is denoted by $(\mathbb{R}^{m\times n})^{p\times q}$.

Thanks to the above notations, the original Milstein scheme introduced in Protter and Talay, 1997 starting at x_0 can be rewritten in a compact form given by the following induction on the integer $i \in \{1,\dots,n\}$

$$X^{\mathrm{Mil},n}_{\frac{i}{n}} = X^{\mathrm{Mil},n}_{\frac{i-1}{n}} + f(X^{\mathrm{Mil},n}_{\frac{i-1}{n}})\Delta_n + g(X^{\mathrm{Mil},n}_{\frac{i-1}{n}})\Delta W_i + \mathbb{H}(X^{\mathrm{Mil},n}_{\frac{i-1}{n}}) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n - \mathcal{A}_i),$$

where $\Delta W_i = W_{\frac{i}{n}} - W_{\frac{i-1}{n}}$ is the increment on the partition, $I_q = (\delta_{jj'})_{1\le j,j'\le q}$ is the correlation matrix for the driving Brownian paths and $\mathcal{A}_i \in \mathbb{R}^{q\times q}$ is the Lévy area defined by

$$\mathcal{A}_{ijj'} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} (W^j_s - W^j_{\frac{i-1}{n}})dW^{j'}_s - \int_{\frac{i-1}{n}}^{\frac{i}{n}} (W^{j'}_s - W^{j'}_{\frac{i-1}{n}})dW^j_s, \qquad j,j' \in \{1,\dots,q\}.$$

In many applications, the simulation of Lévy areas is very complicated. Recently, Giles and Szpruch, 2014 proposed to build a suitable antithetic MLMC estimator based on the Milstein scheme without the Lévy area that achieves the optimal complexity $O(\Delta_n^{-2})$ for a given precision Δ_n as for an unbiased Monte Carlo estimator. Therefore, let us introduce the so called truncated Milstein scheme starting at x_0 defined by induction on the integer $i \in \{1,\dots,n\}$

$$X^n_{\frac{i}{n}} = X^n_{\frac{i-1}{n}} + f(X^n_{\frac{i-1}{n}})\Delta_n + g(X^n_{\frac{i-1}{n}})\Delta W_i + \mathbb{H}(X^n_{\frac{i-1}{n}}) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n). \tag{4.2.3}$$

In addition, in a more general setting where SDEs are driven by continuous semi-martingales, Yan, 2005 studied the asymptotic behavior of the normalized error processes for the original Milstein scheme.

4.2.2 Settings and some standard results

In what follows we introduce our assumption ($\mathbf{H}_{f,g}$) on coefficients f and g in the spirit of Giles and Szpruch, 2014. Our condition is stricter than the one in Giles and Szpruch, 2014 as we aim to prove functional limit theorems for this method. We also recall some standard results on the moment properties of (4.2.3) (see Lemma 4.2, Corollary 4.3 and Lemma 4.4 of Giles and Szpruch, 2014).

Assumption ($\mathbf{H}_{f,g}$). *Let $f \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^d)$ and $g \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^{d\times q})$. We assume that*

- *there exists a positive constant L such that*

$$\left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}\right| \leq L, \qquad \left|\frac{\partial^{|\alpha|} g}{\partial x^\alpha}\right| \leq L, \qquad \left|\frac{\partial^{|\beta|} h}{\partial x^\beta}\right| \leq L$$

  where $\alpha, \beta \in \mathbb{N}^d$, $\alpha = (\alpha_1, \ldots, \alpha_d)^\top$, $\beta = (\beta_1, \ldots, \beta_d)^\top$ are two multi-indices such that $|\alpha| = \sum_{i=1}^d \alpha_i \leq 3$, $|\beta| = \sum_{i=1}^d \beta_i \leq 2$.

- *the diffusion coefficient g does not have a commutativity property which gives $h_{\ell jj'} = h_{\ell j'j}$ for all $\ell \in \{1, \ldots, d\}$ and $j, j' \in \{1, \ldots, q\}$.*

Lemma 4.2.1. *Under ($\mathbf{H}_{f,g}$), for $p \geq 2$ there exists a constant C_p, independent of n, such that*

$$\mathbb{E}\left(\max_{0\leq i\leq n} |X^n_{\frac{i}{n}}|^p\right) \leq C_p, \quad \text{and} \quad \mathbb{E}\left(\max_{0\leq i\leq n} |X^n_{\frac{i}{n}} - X_{\frac{i}{n}}|^p\right) \leq C_p \Delta_n^{p/2}.$$

Corollary 4.2.2. *Under ($\mathbf{H}_{f,g}$), for $p \geq 2$ there exists a constant C_p, independent of n, such that*

$$\mathbb{E}\left(\max_{0\leq i\leq n} |f_\ell(X^n_{\frac{i}{n}})|^p\right) \leq C_p, \qquad \mathbb{E}\left(\max_{0\leq i\leq n} |g_{\ell j}(X^n_{\frac{i}{n}})|^p\right) \leq C_p,$$

and

$$\mathbb{E}\left(\max_{0\leq i\leq n} |h_{\ell jj'}(X^n_{\frac{i}{n}})|^p\right) \leq C_p$$

for all $1 \leq \ell \leq d$ and $1 \leq j, j' \leq q$.

Lemma 4.2.3. *Under ($\mathbf{H}_{f,g}$), for $p \geq 2$, there exists a constant C_p, independent of n, such that*

$$\max_{1\leq i\leq n} \mathbb{E}\left(|X^n_{\frac{i}{n}} - X^n_{\frac{i-1}{n}}|^p\right) \leq C_p \Delta_n^{p/2}.$$

4.3 Main results

The extended antithetic scheme In view of running a MLMC method, we consider two types of schemes, a coarser one and a finer one. The antithetic MLMC estimator was introduced in Giles and Szpruch, 2014 for $m = 2$. For each level, the main idea consists in switching the two finer Brownian increments to obtain an antithetic version of the approximation scheme. In order to extend this idea for a general $m \in \mathbb{N}^* \setminus \{1\}$, we consider $\sigma \in \mathcal{S}_m \setminus \{\mathrm{Id}\}$ and for each level $\ell \in \{1, \ldots, L\}$, we introduce the antithetic scheme $X^{m^\ell,\sigma}$ obtained by permuting the m finer Brownian increments lying in each of the coarse intervals with length $1/m^{\ell-1}$. Based on this, we will also introduce the associated antithetic MLMC estimator. To do so, we set the scheme given by the equation (4.2.3) as the coarser approximation with time step $1/m^{\ell-1}$. The finer scheme with time step $1/m^\ell$ can be rewritten as follows : for $i \in \{1, \ldots, m^{\ell-1}\}$ and $k \in \{1, \ldots, m\}$,

$$X^{m^\ell}_{\frac{m(i-1)+k}{m^\ell}} = X^{m^\ell}_{\frac{m(i-1)+k-1}{m^\ell}} + f(X^{m^\ell}_{\frac{m(i-1)+k-1}{m^\ell}})\frac{\Delta_{m^{\ell-1}}}{m} + g(X^{m^\ell}_{\frac{m(i-1)+k-1}{m^\ell}})\delta W_{ik} + \mathbb{H}(X^{m^\ell}_{\frac{m(i-1)+k-1}{m^\ell}})^\blacklozenge(\delta W_{ik}\delta W_{ik}^\top - I_q\frac{\Delta_{m^{\ell-1}}}{m}), \quad (4.3.1)$$

where $\delta W_{ik} = W_{\frac{m(i-1)+k}{m^\ell}} - W_{\frac{m(i-1)+k-1}{m^\ell}} \in \mathbb{R}^q$. Now, for a given $\sigma \in \mathcal{S}_m \setminus \{\mathrm{Id}\}$ our σ-antithetic scheme is defined by

$$X^{m^\ell,\sigma}_{\frac{m(i-1)+k}{m^\ell}} = X^{m^\ell,\sigma}_{\frac{m(i-1)+k-1}{m^\ell}} + f(X^{m^\ell,\sigma}_{\frac{m(i-1)+k-1}{m^\ell}})\frac{\Delta_{m^{\ell-1}}}{m} + g(X^{m^\ell,\sigma}_{\frac{m(i-1)+k-1}{m^\ell}})\delta W_{i\sigma(k)} + \mathbb{H}(X^{m^\ell,\sigma}_{\frac{m(i-1)+k-1}{m^\ell}})^\blacklozenge(\delta W_{i\sigma(k)}\delta W_{i\sigma(k)}^\top - I_q\frac{\Delta_{m^{\ell-1}}}{m}). \quad (4.3.2)$$

When $\sigma = \mathrm{Id}$, we clearly have $X^{m^\ell,\mathrm{Id}} = X^{m^\ell}$. Throughout the paper we take $\boxed{\sigma(k) = m - k + 1}$ which corresponds to a reversal of time for each coarse increment. The reason for fixing σ in this way is explained in Remark 4.4.10. Since the increments $(\delta W_{ik})_{1\le i\le m^{\ell-1},1\le k\le m}$ are independent and identically distributed, it is obvious that $X^{m^\ell,\sigma} \overset{\text{Law}}{=} X^{m^\ell}$ and for any $i \in \{1, \ldots, m^{\ell-1}\}$ and $k \in \{1, \ldots, m\}$, $X^{m^\ell,\sigma}_{\frac{m(i-1)+k}{m^\ell}}$ is $\mathcal{F}^{k,\sigma}_{\frac{i-1}{m^{\ell-1}}}$-measurable.

The associated antithetic MLMC method Recall that the idea of the original multilevel Monte Carlo method (MLMC) is based on writing $\mathbb{E}(\varphi(X_1^{m^L}))$ using the following telescoping summation

$$\mathbb{E}(\varphi(X_1^{m^L})) = \mathbb{E}(\varphi(X_1^1)) + \sum_{\ell=1}^{L} \mathbb{E}(\varphi(X_1^{m^\ell}) - \varphi(X_1^{m^{\ell-1}})). \quad (4.3.3)$$

As $X^{m^\ell,\sigma} \overset{law}{=} X^{m^\ell}$, we rewrite the above telescoping sum as follows

$$\mathbb{E}(\varphi(X_1^{m^L})) = \mathbb{E}(\varphi(X_1^1)) + \sum_{\ell=1}^{L} \mathbb{E}\left(\frac{\varphi(X_1^{m^\ell}) + \varphi(X_1^{m^\ell,\sigma})}{2} - \varphi(X_1^{m^{\ell-1}})\right).$$

Then we estimate independently each expectation using an empirical mean. Thus, the σ-antithetic MLMC estimator $\hat{Q}$ approximates $\mathbb{E}(\varphi(X_1^{m^L}))$ by

$$\begin{cases} \hat{Q} = \hat{Q}_0 + \sum_{\ell=1}^{L} \hat{Q}_\ell, \text{ with} \\ \hat{Q}_0 = \frac{1}{N_0}\sum_{k=1}^{N_0} \varphi(X_{1,k}^1) \text{ and } \hat{Q}_\ell = \frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\frac{\varphi(X_{1,k}^{m^\ell})+\varphi(X_{1,k}^{m^\ell,\sigma})}{2} - \varphi(X_{1,k}^{m^{\ell-1}})\right) \end{cases} \quad (4.3.4)$$

where for each level $\ell \in \{1,\dots,L\}$, $(X_{1,k}^{m^\ell}, X_{1,k}^{m^\ell,\sigma}, X_{1,k}^{m^{\ell-1}})_{1\le k\le N_\ell}$ are independent copies of $(X_1^{m^\ell}, X_1^{m^\ell,\sigma}, X_1^{m^{\ell-1}})$ whose components are simulated using the same Brownian path and $(X_{1,k}^1)_{1\le k\le N_0}$ are independent copies of X_1^1. In order to study the error of the σ-antithetic MLMC method, we assume that $\varphi \in \mathcal{C}^2(\mathbb{R}^d,\mathbb{R})$ and introduce $\bar{X}_1^{m^\ell,\sigma} = \frac{1}{2}(X_1^{m^\ell,\sigma} + X_1^{m^\ell})$, for $\ell \in \{1,\dots,L\}$ and use a Taylor expansion to write

$$\begin{aligned}\frac{1}{2}(\varphi(X_1^{m^\ell}) + \varphi(X_1^{m^\ell,\sigma})) - \varphi(X_1^{m^{\ell-1}}) =& \nabla\varphi^\top(\xi_1)(\bar{X}_1^{m^\ell,\sigma} - X_1^{m^{\ell-1}})\\ &+ \frac{1}{8}(X_1^{m^\ell} - X_1^{m^\ell,\sigma})^\top \nabla^2\varphi(\xi_2)(X_1^{m^\ell} - X_1^{m^\ell,\sigma}),\end{aligned} \tag{4.3.5}$$

where ξ_1 is a point lying between $\bar{X}_1^{m^\ell,\sigma}$ and $X_1^{m^{\ell-1}}$, ξ_2 is a point lying between $X_1^{m^\ell,\sigma}$ and $X_1^{m^\ell}$ and $\nabla^2\varphi$ denotes the Hessian matrix of φ. More generally, if we consider the σ-antithetic MLMC method on the coarse time grid we have to introduce the error process

$$\mathcal{E}_t^{m^{\ell-1},m^\ell} = \frac{1}{2}(\varphi(X_{\eta_{m^{\ell-1}}(t)}^{m^\ell}) + \varphi(X_{\eta_{m^{\ell-1}}(t)}^{m^\ell,\sigma})) - \varphi(X_{\eta_{m^{\ell-1}}(t)}^{m^{\ell-1}}),\quad t\in[0,1].$$

The work of Giles and Szpruch, 2014 corresponds to $m = 2$ and in this case they proved the L^p boundedness of the process $m^\ell\mathcal{E}^{m^{\ell-1},m^\ell}$. In this paper, we establish this result for a general setting with $m \in \mathbb{N}\backslash\{0,1\}$ and we further study its asymptotic distribution behavior. To do so and in view of the decomposition (4.3.5), we study the couple of two errors $\bar{X}_{\eta_{m^{\ell-1}}(t)}^{m^\ell,\sigma} - X_{\eta_{m^{\ell-1}}(t)}^{m^{\ell-1}}$ and $X_{\eta_{m^{\ell-1}}(t)}^{m^\ell} - X_{\eta_{m^{\ell-1}}(t)}^{m^\ell,\sigma}$, where $\bar{X}_{\eta_{m^{\ell-1}}(t)}^{m^\ell,\sigma} = \frac{1}{2}(X_{\eta_{m^{\ell-1}}(t)}^{m^\ell,\sigma} + X_{\eta_{m^{\ell-1}}(t)}^{m^\ell})$, $t\in[0,1]$.

At first we reduce the problem to the study of the error given by the process $(\bar{X}_{\eta_n(t)}^{nm,\sigma} - X_{\eta_n(t)}^n, X_{\eta_n(t)}^{nm} - X_{\eta_n(t)}^{mn,\sigma})_{0\le t\le 1}$, where $\bar{X}_{\eta_n(t)}^{nm,\sigma} = \frac{1}{2}(X_{\eta_n(t)}^{nm,\sigma} + X_{\eta_n(t)}^{nm})$ and X^{nm}, $X^{nm,\sigma}$ and X^n respectively stand for the finer approximation scheme with time step $1/nm$, its antithetic version and the coarser approximation scheme with time step $1/n$, with $n \in \mathbb{N}\backslash\{0\}$ and $m\in\mathbb{N}\backslash\{0,1\}$. All these approximation schemes are constructed using the same Brownian path. Second, we extend Theorem 4.10. and Lemma 4.6. in Giles and Szpruch, 2014 to get the following result.

Lemma 4.3.1. *Under ($\boldsymbol{H}_{f,g}$), for $p\ge 2$, $\tilde\sigma\in\mathcal{S}_m$, there exists a constant $C_p>0$, independent of the time step, such that*

$$\mathbb{E}(\max_{0\le i\le n}|X_{\frac{i}{n}}^{nm,\tilde\sigma}|^p)\le C_p,\quad \mathbb{E}(\max_{0\le i\le n}|X_{\frac{i}{n}}^{nm} - X_{\frac{i}{n}}^{nm,\tilde\sigma}|^p)\le C_p\Delta_n^{p/2} \text{ and}$$
$$\max_{1\le i\le n}\max_{1\le k\le m}\mathbb{E}(|X_{\frac{m(i-1)+k-1}{nm}}^{nm,\tilde\sigma} - X_{\frac{i-1}{n}}^{nm,\tilde\sigma}|^p)\le C_p{\Delta_n}^{p/2}.$$

Proof. The first and the third inequalities are straightforward consequences of Lemma 4.2.1 and Lemma 4.2.3. Next, we prove the second inequality following similar arguments as in Lemma 4.6 and Theorem 4.10 in Giles and Szpruch, 2014. As $X_{\frac{i}{n}}^{nm} - X_{\frac{i}{n}}^n \overset{\text{law}}{=} X_{\frac{i}{n}}^{nm,\tilde\sigma} - X_{\frac{i}{n}}^n$, by Jensen inequality and Lemma 2.1, we have

$$\begin{aligned}\mathbb{E}(\max_{0\le i\le n}|X_{\frac{i}{n}}^{nm} - X_{\frac{i}{n}}^{nm,\tilde\sigma}|^p) \le& C_p(\mathbb{E}(\max_{0\le i\le n}|X_{\frac{i}{n}}^{nm} - X_{\frac{i}{n}}^n|^p) + \mathbb{E}(\max_{0\le i\le n}|X_{\frac{i}{n}}^{nm,\tilde\sigma} - X_{\frac{i}{n}}^n|^p))\\ \le& C_p\mathbb{E}(\max_{0\le i\le n}|X_{\frac{i}{n}}^{nm} - X_{\frac{i}{n}}^n|^p)\end{aligned}$$

$$\leq C_p(\mathbb{E}(\max_{0\leq i\leq n}|X^{nm}_{\frac{i}{n}} - X_{\frac{i}{n}}|^p) + \mathbb{E}(\max_{0\leq i\leq n}|X^{n}_{\frac{i}{n}} - X_{\frac{i}{n}}|^p)) \leq C_p\Delta_n^{p/2},$$

where C_p is a generic positive constant. □

4.3.1 Functional limit theorem for the errors

As we have the uniform L^p-boundedness of $(\sqrt{n}(X^{nm}_{\eta_n(t)} - X^{nm,\sigma}_{\eta_n(t)}))_{t\in[0,1]}$ (see Lemma 4.3.1) and $(n(\bar{X}^{nm,\sigma}_{\eta_n(t)} - X^{n}_{\eta_n(t)}))_{t\in[0,1]}$ (see Corollary 4.4.9), we get the tightness of these quantities (see e.g. Leskelä and Vihola, 2013). Then, it is natural to study the weak convergence of the couple $(\sqrt{n}(X^{nm}_{\eta_n(t)}) - X^{nm,\sigma}_{\eta_n(t)})), n(\bar{X}^{nm,\sigma}_{\eta_n(t)} - X^{n}_{\eta_n(t)}))_{t\in[0,1]}$. The following theorem is our main result.

Theorem 4.3.2. *Under the assumption ($\boldsymbol{H}_{f,g}$), let us denote $U^n_t = X^{nm}_{\eta_n(t)} - X^{nm,\sigma}_{\eta_n(t)}$ and $V^n_t = \bar{X}^{nm,\sigma}_{\eta_n(t)} - X^{n}_{\eta_n(t)}$, $t \in [0,1]$. Then we have*

$$(\sqrt{n}U^n, nV^n) \overset{\text{stably}}{\Rightarrow} (U,V), \qquad \text{as } n\to\infty, \tag{4.3.6}$$

with U and V are solutions to

$$U_t = \sum_{j=0}^{q}\int_0^t \dot{F}^j_s U_s dY^j_s + \mathcal{M}_{1,t}, \tag{4.3.7}$$

$$V_t = \sum_{j=0}^{q}\int_0^t \dot{F}^j_s V_s dY^j_s + \mathcal{M}_{2,t}, \tag{4.3.8}$$

where for $\ell \in \{1,\dots,d\}$, the ℓ-th component of $\mathcal{M}_{1,t}$ and $\mathcal{M}_{2,t}$ are given by

$$\mathcal{M}^{\ell}_{1,t} = -2\int_0^t h_{\ell\bullet\bullet}(X_s)\blacklozenge dZ_{2,s}$$

$$\mathcal{M}^{\ell}_{2,t} =$$

$$\sum_{j=0}^{q}\int_0^t\Big[\frac{m-1}{2m}\Big(\nabla f_\ell(X_s)^\top g_{\bullet j}(X_s)\mathbf{1}_{j\neq 0} + \nabla g_{\ell j}(X_s)^\top f(X_s) + \frac{1}{2}g(X_s)^\top\nabla^2 g_{\ell j}(X_s)g(X_s)\blacklozenge I^j_q\Big)$$

$$+\frac{1}{8}U_s^\top\nabla^2 g_{\ell j}(X_s)U_s\Big]dY^j_s + \frac{1}{2}\sum_{j=1}^{q}\int_0^t\Big[\nabla g_{\ell j}(X_s)^\top\mathbb{H}(X_s) + \frac{1}{2}g(X_s)^\top\nabla^2 g_{\ell j}(X_s)g(X_s)$$

$$+\dot{h}^s_{\ell\bullet\bullet}\blacklozenge g_{\bullet j}(X_s)\Big]\blacklozenge dZ^{\bullet\bullet j}_{1,s} - \frac{1}{2}\int_0^t(\dot{h}^s_{\ell\bullet\bullet}\blacklozenge U_s)\blacklozenge dZ_{2,s} + \frac{1}{2}\sum_{j,j'=1}^{q}\int_0^t\nabla g_{\ell j}(X_s)^\top[\dot{g}^s\blacklozenge g_{\bullet j''}(X_s)]dZ^{j\bullet j'}_{3,s}$$

$$+\frac{1}{2}\sum_{j=1}^{q}\int_0^t[g(X_s)^\top\nabla^2 g_{\ell j}(X_s)g(X_s)]\blacklozenge dZ^{j\bullet\bullet}_{3,s},$$

with $Y_t := (t, W^1_t,\dots,W^q_t)^\top$, $I^0_q = \mathbf{1}_{q\times q}$ is the $\mathbb{R}^{q\times q}$ matrix with all its elements equal to 1, $\dot{F}^0 = \nabla f$ and for $j\neq 0$, $I^j_q = I_q$ and $\dot{F}^j = \nabla g_{\bullet j}$, $\dot{g}^s \in (\mathbb{R}^{d\times 1})^{d\times q}$ is a block matrix such that for $\ell\in\{1,\dots,d\}$, $j\in\{1,\dots,q\}$, the ℓj-th block is given by $(\dot{g}^s)_{\ell j} = \nabla g_{\ell j}(X_s)$, $s\in[0,t]$ and the $\dot{h}^s_{\ell\bullet\bullet}\in(\mathbb{R}^{d\times 1})^{q\times q}$ is a block matrix such that for j and $j'\in\{1,\dots,q\}$, the jj'-th block is given by $(\dot{h}^s_{\ell\bullet\bullet})_{jj'} = \nabla h_{\ell jj'}(X_s)\in\mathbb{R}^{d\times 1}$, $s\in[0,t]$. Here, Z_1, Z_3 are $\mathbb{R}^{q^3}$-dimensional processes and Z_2 is a $\mathbb{R}^{q\times q}$-dimensional *process given by: for $j, j', j'' \in \{1, \ldots, q\}$,*

$$Z_{1,t}^{jj'j''} = \begin{cases} \frac{\sqrt{m-1}}{m} B_{1,t}^{jj'j''} & , \ j > j' \\ \frac{\sqrt{2(m-1)}}{m} B_{1,t}^{jjj''} & , \ j = j' \\ \frac{\sqrt{m-1}}{m} B_{1,t}^{j'jj''} & , \ j < j' \end{cases}, \quad Z_{2,t}^{jj'} = \begin{cases} \sqrt{\frac{m-1}{m}} B_{2,t}^{jj'} & , \ j > j' \\ 0 & , \ j = j' \\ -\sqrt{\frac{m-1}{m}} B_{2,t}^{j'j} & , \ j < j' \end{cases}$$

and

$$Z_{3,t}^{jj'j''} = \begin{cases} \sqrt{\frac{(m-1)(m-2)}{3m^2}} B_{3,t}^{jj'j''} & , \ j > j'' \\ \sqrt{\frac{2(m-1)(m-2)}{3m^2}} B_{3,t}^{jj'j''} & , \ j = j'' \\ \sqrt{\frac{(m-1)(m-2)}{3m^2}} B_{3,t}^{j''j'j} & , \ j < j'' \end{cases}$$

with $(B_1^{jj'j''})_{\substack{1 \le j,j',j'' \le q \\ j \ge j'}}$ and $(B_3^{jj'j''})_{\substack{1 \le j,j',j'' \le q \\ j \ge j''}}$ are two standard $q^2(q+1)/2$-dimensional Brownian motions and $(B_2^{jj'})_{1 \le j' < j \le q}$ is a standard $q(q-1)/2$-dimensional Brownian motion. Moreover, we have B_1, B_2 and B_3 are independent of the original q-dimensional Brownian motion W and also independent of each other. These processes are defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \ge 0}, \tilde{\mathbb{P}})$ of the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$. Here $Z_3^{j\bullet\bullet} = Z_3^{\bullet\bullet j}$ and we use that for $r \in \{1, 3\}$ we have

$$Z_{r,s}^{\bullet\bullet j} = \begin{pmatrix} Z_{r,s}^{11j} & \ldots & Z_{r,s}^{1qj} \\ \vdots & \ddots & \vdots \\ Z_{r,s}^{q1j} & \ldots & Z_{r,s}^{qqj} \end{pmatrix} \in \mathbb{R}^{q \times q} \quad \text{and} \quad Z_{r,t}^{j\bullet j''} = (Z_{r,t}^{j1j''}, \ldots, Z_{r,t}^{jqj''})^\top \in \mathbb{R}^{q \times 1}.$$

Proof. From section 4, equations (4.4.37) and (4.4.38), we can rewrite U^n and V^n as follows

$$U_t^n = \sum_{j=0}^{q} \int_0^t (\dot{F}_{\eta_n(s)}^{n,j} \blacklozenge U_{\eta_n(s)}^n) \mathbb{1}_{s \le \eta_n(t)} dY_s^j + J_t^{n,1},$$

$$V_t^n = \sum_{j=0}^{q} \int_0^t (\bar{\dot{F}}_{\eta_n(s)}^{n,j} \blacklozenge V_{\eta_n(s)}^n) \mathbb{1}_{s \le \eta_n(t)} dY_s^j + J_t^{n,2},$$

where $Y_t := (t, W_t^1, \ldots, W_t^q)^\top$, $J_t^{n,1} = \mathcal{M}_t^{n,1} + \mathcal{R}_t^{n,1}$, $J_t^{n,2} = \mathcal{M}_t^{n,2} + \mathcal{R}_t^{n,2}$ and for $i \in \{1, \ldots, n\}$ we denote

$$\dot{F}_{\frac{i-1}{n}}^{n,j} = \begin{cases} \dot{f}_i^n, & j = 0 \\ (\dot{g}_i^n)_{\bullet j}, & j \in \{1, \ldots, q\} \end{cases}, \quad \text{where } (\dot{g}_i^n)_{\bullet j} = ((\dot{g}_i^n)_{1j}, \ldots, (\dot{g}_i^n)_{dj})^\top, \tag{4.3.9}$$

and

$$\bar{\dot{F}}_{\frac{i-1}{n}}^{n,j} = \begin{cases} \bar{\dot{f}}_i^n, & j = 0 \\ (\bar{\dot{g}}_i^n)_{\bullet j}, & j \in \{1, \ldots, q\} \end{cases}, \quad \text{where } (\bar{\dot{g}}_i^n)_{\bullet j} = ((\bar{\dot{g}}_i^n)_{1j}, \ldots, (\bar{\dot{g}}_i^n)_{dj})^\top, \tag{4.3.10}$$

with $\dot{f}_i^n \in (\mathbb{R}^{d \times 1})^{d \times 1}$ and $\dot{g}_i^n \in (\mathbb{R}^{d \times 1})^{d \times q}$ are block matrices such that for $\ell \in \{1, \ldots, d\}$ the ℓ-th block of $\dot{f}_i^n$ is given by $(\dot{f}_i^n)_\ell = \nabla f_\ell(\xi_{\frac{i-1}{n}}^{1,n})$ and for $\ell \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, q\}$ the ℓj-th block of $\dot{g}_i^n$ is given by $(\dot{g}_i^n)_{\ell j} = \nabla g_{\ell j}(\xi_{\frac{i-1}{n}}^{2,n})$ with $\xi_{\frac{i-1}{n}}^{1,n}$ and $\xi_{\frac{i-1}{n}}^{2,n}$ are some vector points lying between $X_{\frac{i-1}{n}}^{nm}$ and $X_{\frac{i-1}{n}}^{nm,\sigma}$. In the same way, $\bar{\dot{f}}_i^n \in (\mathbb{R}^{d \times 1})^{d \times 1}$ and $\bar{\dot{g}}_i^n \in (\mathbb{R}^{d \times 1})^{d \times q}$ are block matrices such that for $\ell \in \{1, \ldots, d\}$

the ℓ-th block of $\bar{\dot{f}}_i^n$ is given by $(\bar{\dot{f}}_i^n)_\ell = \nabla f_\ell(\bar{\xi}^{1,n}_{\frac{i-1}{n}})$ and $\ell, j \in \{1,\dots,d\}$ the ℓj-th block of $\bar{\dot{g}}_i^n$ is given by $(\bar{\dot{g}}_i^n)_{\ell j} = \nabla g_{\ell j}(\bar{\xi}^{2,n}_{\frac{i-1}{n}})$ with $\bar{\xi}^{1,n}_{\frac{i-1}{n}}$ and $\bar{\xi}^{2,n}_{\frac{i-1}{n}}$ are some vector points lying between $X^n_{\frac{i-1}{n}}$ and $\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}$. The aim now is to use Theorem 4.8.6 to get the joined convergence of our couple of errors. To do so, let us introduce the processes $Z_t^n = \sum_{j=0}^q \int_0^t \dot{F}^{n,j}_{\eta_n(s)} \blacklozenge \mathbb{1}_d \mathbb{1}_{s\le\eta_n(t)}(s) dY_s^j$, $\bar{Z}_t^n = \sum_{j=0}^q \int_0^t \bar{\dot{F}}^{n,j}_{\eta_n(s)} \blacklozenge \mathbb{1}_d \mathbb{1}_{s\le\eta_n(t)}(s) dY_s^j$ and $Z_t = \sum_{j=0}^q \int_0^t \dot{F}_s^j \blacklozenge \mathbb{1}_d dY_s^j$, where $\mathbb{1}_d = (1,\dots,1)^\top \in \mathbb{R}^{d\times 1}$. Thanks to Lemma 4.2.1 and assumption ($\mathbf{H}_{f,g}$), using the Burkholder-Davis-Gundy (BDG) inequality with $p \ge 2$, there is a generic constant $C_p > 0$ such that

$$\begin{aligned}
\mathbb{E}(\sup_{0\le t\le 1} |Z_t^n - Z_t|^p) &\le C_p \mathbb{E}(|\sum_{j=1}^q \int_0^1 (\dot{F}^{n,j}_{\eta_n(s)} - \dot{F}_s^j)^2 \blacklozenge \mathbb{1}_d ds|^{p/2}) \\
&\le C_p \sum_{\ell=1}^d \sum_{j=1}^q \mathbb{E}(|\int_0^1 (\nabla g_{\ell j}(\xi^{2,n}_{\frac{i-1}{n}}) - \nabla g_{\ell j}(X_{\frac{i-1}{n}}))^2 \blacklozenge \mathbb{1}_d ds|^{p/2}) \\
&\le C_p \Delta_n^{p/2}.
\end{aligned}$$

Similarly, $\mathbb{E}(\sup_{0\le t\le 1} |\bar{Z}_t^n - Z_t|^p)$ is also bounded by $C_p\Delta_n^{p/2}$. Therefore, we have $Z^n - Z \stackrel{L^p}{\to} 0$ and $\bar{Z}^n - Z \stackrel{L^p}{\to} 0$ as $n \to \infty$. By Lemma 4.4.11 and Lemma 4.4.12 and Proposition 4.5.5, we deduce that $(\sqrt{n}J^{n,1}, nJ^{n,2}) \stackrel{\text{stably}}{\Rightarrow} (\mathcal{M}_1, \mathcal{M}_2)$ as $n \to \infty$, where the limit processes are defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\ge 0}, \tilde{\mathbb{P}})$ of the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$. By Lemma 4.8.4, we get that $(Y, \sqrt{n}J^{n,1}, nJ^{n,2}, Z^n)$ stably converges to the limit $(Y, \mathcal{M}_1, \mathcal{M}_2, Z)$ as $n \to \infty$ Finally, by Theorem 4.8.6 , we have $(Y, \sqrt{n}J^{n,1}, nJ^{n,2}, Z^n, \sqrt{n}U^n, nV^n)$ stably converges to the limit $(Y, \mathcal{M}_1, \mathcal{M}_2, Z, U, V)$ as $n \to \infty$, where U and V respectively satisfy (4.3.7) and (4.3.8). □

4.3.2 Central limit theorem

The antithetic Multilevel Monte Carlo method uses information from a sequence of computations with increasing step sizes and approximates the quantity of interest $\mathbb{E}(\varphi(X_1))$ by

$$\hat{Q}_n = \frac{1}{N_0}\sum_{k=1}^{N_0} \varphi(X^1_{1,k}) + \sum_{\ell=1}^{L} \frac{1}{N_\ell}\sum_{k=1}^{N_\ell} [\frac{1}{2}(\varphi(X^{\ell,m^\ell}_{1,k}) + \varphi(X^{\ell,m^\ell,\sigma}_{1,k})) - \varphi(X^{\ell,m^{\ell-1}}_{1,k})], \tag{4.3.11}$$

$m \in \mathbb{N}\backslash\{0,1\}$, and $L = \frac{\log n}{\log m}$. We denote the weak error $\epsilon_n = \mathbb{E}(\varphi(X_1^n)) - \varphi(X_1))$. In the spirit of Kebaier, 2005, we assume that ϵ_n is of order $1/n^\alpha$, for some $\alpha \in [1/2, 1]$. Taking advantage from Theorem 4.8.7, we are now able to establish a central limit theorem of Lindeberg Feller type on the error $\hat{Q}_n - \mathbb{E}(\varphi(X_1))$. To do so, we introduce a real sequence $(a_\ell)_{\ell\in\mathbb{N}}$ of positive weights such that

$$\lim_{L\uparrow\infty} \sum_{\ell=1}^{L} a_\ell = \infty, \text{ for } p>2, \text{ and } \lim_{L\uparrow\infty} \frac{1}{\left(\sum_{\ell=1}^L a_\ell\right)^{p/2}} \sum_{\ell=1}^{L} a_\ell^{p/2} = 0 \tag{$\mathbf{W}$}$$

and we choose the same form of N_ℓ as in Ben Alaya and Kebaier, 2015, namely

$$N_\ell = \frac{n^{2\alpha}}{m^{2(\ell-1)}a_\ell}\sum_{\ell=1}^{L} a_\ell, \qquad \ell \in \{0,\dots,L\} \text{ and } L = \frac{\log n}{\log m}. \tag{4.3.12}$$

This generic form for the sample size allows us a straightforward use of Theorem 4.3.2 to prove a central limit theorem for the antithetic MLMC estimator. In the sequel, we denote by $\tilde{E}$ and $\tilde{\mathrm{Var}}$ the expectation and the variance respectively defined on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ introduced in Theorem 4.3.2.

Theorem 4.3.3. *Assume that f and g satisfy assumption ($\boldsymbol{H}_{f,g}$). Let φ be a real-valued function satisfying:*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^p + |y|^p)|x-y|, \text{ for some constant } C \text{ and } p > 0 \text{ and } \varphi \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R}) \text{ with bounded second derivatives.} \tag{$\mathbf{H}_\varphi$}$$

Assume that for some $\alpha \in [1/2, 1]$ we have

$$\lim_{n\uparrow\infty} n^\alpha \epsilon_n = C_\varphi(\alpha). \tag{$\mathbf{H}_{\epsilon_n}$}$$

Then, for the choice of N_ℓ, $\ell \in \{0,\dots,L\}$ given by the equation (4.3.12), we have

$$n^\alpha(\hat{\mathcal{Q}}_n - \mathbb{E}(\varphi(X_1))) \Rightarrow \mathcal{N}(C_\varphi(\alpha), \mathcal{V}), \qquad \text{as } n \to \infty$$

with $\mathcal{V} = \tilde{\mathrm{Var}}(\nabla\varphi^\top(X_1)V_1 + \frac{1}{8}U_1^\top \nabla^2\varphi(X_1)U_1)$, where the limit processes U and V are explicitly given in Theorem 4.3.2.

Remark 4.3.4. *Note that our above assumption ($\mathbf{H}_\varphi$) on the payoff function φ is weaker than the one in Giles and Szpruch, 2013a where they supposed that $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$.*

Proof. To simplify our notation, we give the proof for $\alpha = 1$, the case $\alpha \in [1/2, 1)$ is straightforward by similar arguments. At first, we rewrite the error term as follows

$$\hat{\mathcal{Q}}_n - \mathbb{E}(\varphi(X_1)) = \hat{\mathcal{Q}}_n^1 + \hat{\mathcal{Q}}_n^2 + \epsilon_n, \quad \text{where}$$

$$\hat{\mathcal{Q}}_n^1 = \frac{1}{N_0}\sum_{k=1}^{N_0}(\varphi(X_{1,k}^1) - \mathbb{E}(\varphi(X_1^1))),$$

$$\hat{\mathcal{Q}}_n^2 = \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}[\frac{1}{2}(\varphi(X_{1,k}^{\ell,m^\ell}) + \varphi(X_{1,k}^{\ell,m^\ell,\sigma})) - \varphi(X_{1,k}^{\ell,m^{\ell-1}}) - \mathbb{E}(\varphi(X_1^{\ell,m^\ell}) - \varphi(X_1^{\ell,m^{\ell-1}}))].$$

For $N_0 = \frac{n^2m^2}{a_0}\sum_{\ell=1}^L a_\ell$ we simply apply the classical central limit theorem to get

$$n\hat{\mathcal{Q}}_n^1 = \sqrt{\frac{a_0}{m^2\sum_{\ell=1}^L a_\ell}}\sqrt{N_0}\hat{\mathcal{Q}}_n^1 \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty.$$

Finally, we only need to study the convergence of $n\hat{\mathcal{Q}}_n^2$ and the proof is completed by assumption ($\mathbf{H}_{\epsilon_n}$). To do so, we use Theorem 4.8.7 and set

$$X_{n,\ell} := \frac{n}{N_\ell}\sum_{k=1}^{N_\ell} Z_{1,k}^{m^\ell,m^{\ell-1}}, \text{ where } (Z_{1,k}^{m^\ell,m^{\ell-1}})_{1\le k\le N_\ell} \text{ are independent copies of}$$
$$Z_1^{m^\ell,m^{\ell-1}} := \frac{1}{2}(\varphi(X_1^{\ell,m^\ell}) + \varphi(X_1^{\ell,m^\ell,\sigma})) - \varphi(X_1^{\ell,m^{\ell-1}}) - \mathbb{E}(\varphi(X_1^{\ell,m^\ell}) - \varphi(X_1^{\ell,m^{\ell-1}})).$$

First, we check the limit variance of $n\hat{\mathcal{Q}}_n^2$. We have

$$\sum_{\ell=1}^{L}\mathbb{E}(X_{n,\ell})^2 = \sum_{\ell=1}^{L}\frac{n^2}{N_\ell}\mathrm{Var}(Z_1^{m^\ell,m^{\ell-1}}) = \sum_{\ell=1}^{L}\frac{1}{\sum_{\ell=1}^{L}a_\ell}a_\ell m^{2(\ell-1)}\mathrm{Var}(Z_1^{m^\ell,m^{\ell-1}}). \tag{4.3.13}$$

Besides, since $\varphi\in\mathcal{C}^2(\mathbb{R}^d,\mathbb{R})$, applying Taylor expansion twice we get

$$\begin{aligned}&\frac{1}{2}(\varphi(X_1^{\ell,m^\ell}) + \varphi(X_1^{\ell,m^\ell,\sigma})) - \varphi(X_1^{\ell,m^{\ell-1}})\\ &\quad= \nabla\varphi^\top(\xi_1)(\bar{X}_1^{\ell,m^\ell,\sigma} - X_1^{\ell,m^{\ell-1}}) + \frac{1}{8}(X_1^{\ell,m^\ell} - X_1^{\ell,m^\ell,\sigma})^\top\nabla^2\varphi(\xi_2)(X_1^{\ell,m^\ell} - X_1^{\ell,m^\ell,\sigma}),\end{aligned}$$

for some ξ_1 a vector point lying between X_1^{ℓ,m^ℓ} and $X_1^{\ell,m^\ell,\sigma}$ and ξ_2 a vector point lying between $\bar{X}_1^{\ell,m^\ell,\sigma}$ and $X_1^{\ell,m^{\ell-1}}$. Thus, under assumption ($\mathbf{H}_\varphi$), thanks to Theorem 4.3.2 we get as $\ell\to\infty$

$$m^{\ell-1}\left[\frac{1}{2}(\varphi(X_1^{\ell,m^\ell}) + \varphi(X_1^{\ell,m^\ell,\sigma})) - \varphi(X_1^{\ell,m^{\ell-1}})\right] \overset{\text{stably}}{\Rightarrow} \nabla\varphi^\top(X_1)V_1 + \frac{1}{8}U_1^\top\nabla^2\varphi(X_1)U_1.$$

From the uniform integrability obtained by combining ($\mathbf{H}_\varphi$) and Lemma 4.3.1, we get for $k\in\{1,2\}$

$$\begin{aligned}&\mathbb{E}\left(m^{\ell-1}\left[\frac{1}{2}(\varphi(X_1^{\ell,m^\ell}) + \varphi(X_1^{\ell,m^\ell,\sigma})) - \varphi(X_1^{\ell,m^{\ell-1}})\right]\right)^k\\ &\qquad\qquad\underset{\ell\to\infty}{\longrightarrow}\tilde{\mathbb{E}}\left(\nabla\varphi^\top(X_1)V_1 + \frac{1}{8}U_1^\top\nabla^2\varphi(X_1)U_1\right)^k.\end{aligned}$$

Consequently, $m^{2(\ell-1)}\mathrm{Var}(Z_{1,1}^{m^\ell,m^{\ell-1}})\longrightarrow\mathcal{V}$, as $\ell\to\infty$. Thus, by (4.3.13) and Toeplitz lemma we get $\lim_{L\uparrow\infty}\sum_{\ell=1}^{L}\mathbb{E}(X_{n,\ell})^2 = \mathcal{V}$. Finally, we only need to check the Lyapunov condition. By Burkholder's inequality and Jensen's inequality, we get for $p>2$,

$$\mathbb{E}|X_{n,\ell}|^p = \frac{n^p}{N_\ell^p}\mathbb{E}|\sum_{k=1}^{N_\ell} Z_1^{m^\ell,m^{\ell-1}}|^p \le C_p\frac{n^p}{N_\ell^{p/2}}\mathbb{E}|Z_1^{m^\ell,m^{\ell-1}}|^p,$$

where C_p is a generic positive constant depending on p. Besides, Lemma 4.3.1 ensures that there is a constant $K_p>0$ such that $\mathbb{E}|Z_1^{m^\ell,m^{\ell-1}}|^p\le\frac{K_p}{m^{p(\ell-1)}}$. Therefore,

$$\sum_{\ell=1}^{L}\mathbb{E}|X_{n,\ell}|^p \le C_p\sum_{\ell=1}^{L}\frac{n^p}{N_\ell^{p/2}}\frac{1}{m^{p(\ell-1)}} \le C_p\frac{1}{\left(\sum_{\ell=1}^{L}a_\ell\right)^{p/2}}\sum_{\ell=1}^{L}a_\ell^{p/2} \overset{n\to\infty}{\to} 0$$

which completes the proof. □

4.3.3 Time complexity analysis and design of the algorithm

The time complexity in the antithetic MLMC method is given by

$$\begin{aligned} C_{\text{AMLMC}} &= C \times \sum_{\ell=1}^{L} N_\ell (2m^\ell + m^{\ell-1}) \text{ with } C > 0 \\ &= C \times \sum_{\ell=1}^{L} \frac{n^2}{m^{2(\ell-1)} a_\ell} (2m^\ell + m^{\ell-1}) \sum_{\ell=1}^{L} a_\ell \\ &= C \times n^2 (2m^2 + m) \sum_{\ell=1}^{L} \frac{1}{m^\ell a_\ell} \sum_{\ell=1}^{L} a_\ell . \end{aligned}$$

This analysis is online with the one obtained by Ben Alaya and Kebaier, 2014 in the context of pricing Asian options using numerical schemes with a strong convergence order equal to 1. The optimal choice corresponding to $a_\ell^* = m^{-\ell/2}$, $\ell \in \{1, \dots, L\}$ leads to the optimal time complexity $C^*_{\text{AMLMC}} = O(n^2)$ the same one as for an unbiased Monte Carlo method having the same precision. However, this optimal weight a_ℓ^* does not satisfy (**W**) which ensures Theorem 4.3.3. In what follows, we recall from Ben Alaya and Kebaier, 2014, three examples of weights $(a_\ell)_{1\le\ell\le L}$ satisfying (**W**) and for which the time complexity gets closer and closer to C^*_{AMLMC}:

i) The choice $a_\ell = 1$, corresponding to $N_\ell = \frac{n^2}{m^{2(\ell-1)}} L$, $\ell \in \{1, \dots, L\}$ leads to the complexity $C_{\text{AMLMC}} = O(n^2 \log n)$.

ii) The choice $a_\ell = \frac{1}{\ell}$, corresponding to $N_\ell = \frac{n^2 \ell}{m^{2(\ell-1)}} \sum_{\ell=1}^{L} \frac{1}{\ell}$, $\ell \in \{1, \dots, L\}$ leads to the complexity $C_{\text{AMLMC}} = O(n^2 \log\log n)$.

iii) The choice $a_\ell = \frac{1}{\ell \log \ell}$, corresponding to $N_\ell = \frac{n^2 \ell \log \ell}{m^{2(\ell-1)}} \sum_{\ell=1}^{L} \frac{1}{\ell \log \ell}$, $\ell \in \{1, \dots, L\}$ leads to the complexity $C_{\text{AMLMC}} = O(n^2 \log\log\log n)$.

From a practical point of view, the sample sizes N_ℓ, $\ell \in \{1, \dots, L\}$ are inputs for the algorithm and are completely explicit by the simple choice of the optimal weights $(a_\ell)_{1\le\ell\le L}$. So, we do not need to add any precomputation step as like for the RMSE approach. Then, we can compute independently each empirical mean in the antithetic MLMC estimator $\hat{\mathcal{Q}}_n$ given by (4.3.11). For each level, we have a simple Monte Carlo with i.i.d. terms, their computations need only the simulation of the finer increments associated to the modified Milstein scheme X^{ℓ,m^ℓ} (given by (4.3.1)) using the permutation σ we obtain $X^{\ell,m^\ell,\sigma}$ (given by (4.3.2)). The coarser increments are deduced from the finer ones to keep the same Brownian path to obtain $X^{\ell,m^{\ell-1}}$.

4.4 Expanding analysis of the antithetic scheme

In this section, we have two main purposes. Firstly, for $\tilde{\sigma} \in \{\text{Id}, \sigma\}$, we give the expansion of two error terms $X^{nm,\tilde{\sigma}}_{\frac{i}{n}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$ and $\bar{X}^{nm,\sigma}_{\frac{i}{n}} - \bar{X}^{nm,\sigma}_{\frac{i-1}{n}}$ together with some related L^p estimates. Secondly, we give the expansions of the errors U^n and V^n with specifying the main and the rest terms. From now on we assume that assumption $(\mathbf{H}_{f,g})$ is satisfied.

4.4.1 Expansion of the error $X^{nm,\tilde\sigma}_{\frac{i}{n}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}}$, with $\tilde\sigma \in \{\mathrm{Id}, \sigma\}$

By (4.3.1) and (4.3.2), we have for all $k \in \{1,\dots,m\}$

$$X^{nm,\tilde\sigma}_{\frac{m(i-1)+k}{nm}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}} = \sum_{k'=1}^{k} f(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} + \sum_{k'=1}^{k} g(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}})\delta W_{i\tilde\sigma(k')}$$
$$+ \sum_{k'=1}^{k} \mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}) ^{\blacklozenge} (\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')} - I_q\frac{\Delta_n}{m}). \quad (4.4.1)$$

In particular, we have

$$X^{nm,\tilde\sigma}_{\frac{i}{n}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}} = \sum_{k=1}^{m} f(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})\frac{\Delta_n}{m} + \sum_{k=1}^{m} g(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})\delta W_{i\tilde\sigma(k)}$$
$$+ \sum_{k=1}^{m} \mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}) ^{\blacklozenge} (\delta W_{i\tilde\sigma(k)}\delta W^{\top}_{i\tilde\sigma(k)} - I_q\frac{\Delta_n}{m}). \quad (4.4.2)$$

This last equation can be rewritten as follows

$$X^{nm,\tilde\sigma}_{\frac{i}{n}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}} = f(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Delta_n + g(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Delta W_i + \mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}}) ^{\blacklozenge} (\Delta W_i \Delta W_i^{\top} - I_q\Delta_n)$$
$$+ \sum_{k=1}^{m}\left[f(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}) - f(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\right]\frac{\Delta_n}{m} + \sum_{k=1}^{m}\left[g(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}) - g(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\right]\delta W_{i\tilde\sigma(k)}$$
$$+ \sum_{k=1}^{m} \mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}) ^{\blacklozenge} (\delta W_{i\tilde\sigma(k)}\delta W^{\top}_{i\tilde\sigma(k)} - I_q\frac{\Delta_n}{m}) - \mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}}) ^{\blacklozenge} (\Delta W_i \Delta W_i^{\top} - I_q\Delta_n).$$
$$(4.4.3)$$

Let us start dealing with the last four terms in the right-hand side (r.h.s.) of the above equality. By a Taylor expansion, we have for any fixed index component $\ell \in \{1,\dots,d\}$,

$$f_\ell(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}) - f_\ell(X^{nm,\tilde\sigma}_{\frac{i-1}{n}}) = \nabla f_\ell^{\top}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}})$$
$$+ \frac{1}{2}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}})^{\top}\nabla^2 f_\ell(\xi^{1,n}_{ik})(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde\sigma}_{\frac{i-1}{n}}), \quad (4.4.4)$$

for some vector point $\xi^{1,n}_{ik}$ lying between $X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}$ and $X^{nm,\tilde\sigma}_{\frac{i-1}{n}}$. Then, using (4.4.1)

$$\sum_{k=1}^{m}\left[f(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}) - f(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\right]\frac{\Delta_n}{m} =: M^{nm,\tilde\sigma,1}_{\frac{i-1}{n}} + N^{nm,\tilde\sigma}_{\frac{i-1}{n}}, \quad (4.4.5)$$

where for $\ell \in \{1,\dots,d\}$ the ℓ^{th}-component of $N^{nm,\tilde\sigma}_{\frac{i-1}{n}}$ and $M^{nm,\tilde\sigma,1}_{\frac{i-1}{n}}$ are given by

$$N^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}} = \sum_{k=2}^{m}\nabla f_\ell^{\top}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}}) \quad (4.4.6)$$
$$\times \sum_{k'=1}^{k-1}\left[f(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} + \mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}) ^{\blacklozenge} (\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')} - I_q\frac{\Delta_n}{m})\right]\frac{\Delta_n}{m}$$

$$+\frac{1}{2}\sum_{k=2}^{m}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}})^\top\nabla^2 f_\ell(\xi^{1,n}_{ik})(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\frac{\Delta_n}{m}$$

$$M^{nm,\tilde\sigma,1}_{\ell,\frac{i-1}{n}}=\sum_{k=2}^{m}\nabla f_\ell{}^\top(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\sum_{k'=1}^{k-1}g(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}})\delta W_{i\tilde\sigma(k')}\frac{\Delta_n}{m}. \tag{4.4.7}$$

For the last two terms in the r.h.s. of (4.4.3) by $\Delta W_i\Delta W_i^\top=\sum_{k,k'=1}^m\delta W_{i,\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k')}$, we get

$$\begin{aligned}&\sum_{k=1}^{m}\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k)}-I_q\frac{\Delta_n}{m})-\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n)\\&=\sum_{k=2}^{m}\Big[\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})-\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Big]\blacklozenge(\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k)}-I_q\frac{\Delta_n}{m}) \qquad (4.4.8)\\&-2\sum_{1\le k'<k\le m}\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\blacklozenge\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k')}-\sum_{1\le k<k'\le m}\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\blacklozenge(\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k')}-\delta W_{i\tilde\sigma(k')}\delta W^\top_{i\tilde\sigma(k)}).\end{aligned}$$

From (4.4.3) and (4.4.8), if we denote

$$M^{nm,\tilde\sigma,2}_{\frac{i-1}{n}}=\sum_{k=2}^{m}\Big[g(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})-g(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Big]\delta W_{i\tilde\sigma(k)}-2\sum_{1\le k<k'\le m}\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\blacklozenge\delta W_{i\tilde\sigma(k')}\delta W^\top_{i\tilde\sigma(k)},$$

$$M^{nm,\tilde\sigma,3}_{\frac{i-1}{n}}=\sum_{k=2}^{m}\Big[\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})-\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Big]\blacklozenge(\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k)}-I_q\frac{\Delta_n}{m}).$$

Let us set

$$M^{nm,\tilde\sigma}=M^{nm,\tilde\sigma,1}+M^{nm,\tilde\sigma,2}+M^{nm,\tilde\sigma,3}. \tag{4.4.9}$$

Then combining (4.4.3), (4.4.5) and (4.4.9) we obtain the first assertion of the following lemma. The proof of the remaining results are postponed to appendix 4.7

Lemma 4.4.1. *The difference equation for $X^{nm,\tilde\sigma}_{\frac{i}{n}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}$, $i\in\{1,\dots,n\}$ is given by*

$$\begin{aligned}X^{nm,\tilde\sigma}_{\frac{i}{n}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}&=f(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Delta_n+g(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\Delta W_i+\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n)\\&-\mathbb{H}(X^{nm,\tilde\sigma}_{\frac{i-1}{n}})\blacklozenge\sum_{1\le k<k'\le m}(\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k')}-\delta W_{i\tilde\sigma(k')}\delta W^\top_{i\tilde\sigma(k)})+M^{nm,\tilde\sigma}_{\frac{i-1}{n}}+N^{nm,\tilde\sigma}_{\frac{i-1}{n}}, \qquad (4.4.10)\end{aligned}$$

where $\mathbb{E}(M^{nm,\tilde\sigma}_{\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=0$, and for any integer $p\ge2$ there exists a constant K_p such that

$$\max_{0\le i\le n}\mathbb{E}(|M^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^p)\le K_p\Delta_n^{3p/2}, \tag{4.4.11}$$

$$\max_{0\le i\le n}\mathbb{E}(|N^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^p)\le K_p\Delta_n^{2p}. \tag{4.4.12}$$

In what follows, we give further expansion studies for the terms $N^{nm,\tilde\sigma}_{\frac{i-1}{n}}$ and $M^{nm,\tilde\sigma,1}_{\frac{i-1}{n}}$, $M^{nm,\tilde\sigma,2}$ and $M^{nm,\tilde\sigma,3}$ defined above.

• **The term $N^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$:** Starting from relation (4.4.6) we replace the increment $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$ using (4.4.1) and we only freeze the coefficients of the contributing terms in the asymptotic behavior of the error at the limit point $X_{\frac{i-1}{n}}$. Then thanks to (4.2.2) and using that

$$\sum_{k=2}^{m}\sum_{k'=1}^{k-1}\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')} = \sum_{k=1}^{m-1}(m-k)\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)},$$

we get the following result.

Lemma 4.4.2. *For $\ell \in \{1,\dots,d\}$ the ℓ^{th}-component of $N^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$ has the following expansion*

$$N^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}} = \frac{(m-1)}{2m}\nabla f_\ell^{\top}(X_{\frac{i-1}{n}})f(X_{\frac{i-1}{n}})\Delta_n^2 +$$
$$\frac{1}{2}\Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 f_\ell(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big] \blacklozenge \sum_{k=1}^{m-1}(m-k)\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)}\frac{\Delta_n}{m} + R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0) + \tilde{R}^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0), \tag{4.4.13}$$

where

$$R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0) = \frac{\Delta_n}{m}\sum_{1\le k'<k\le m}\nabla f_\ell^{\top}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge (\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})$$
$$+ \Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 f_\ell(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big] \blacklozenge \sum_{1\le k''<k'<k\le m}\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k'')}\frac{\Delta_n}{m}$$

satisfies $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|\mathcal{F}_{\frac{i-1}{n}}) = 0$. Moreover, for any integer $p \ge 2$ there exists

$$\max_{0\le i\le n}\mathbb{E}(|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|^p) = o\Big(\Delta_n^{3p/2}\Big), \tag{4.4.14}$$
$$\max_{0\le i\le n}\mathbb{E}(|\tilde{R}^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|^p) = o\Big(\Delta_n^{2p}\Big). \tag{4.4.15}$$

The proof of the above lemma is postponed to appendix 4.7.

• **The term $M^{nm,\tilde{\sigma},1}_{\frac{i-1}{n}}$:** For this term we only need to freeze the coefficients in relation (4.4.7) at the limit point $X_{\frac{i-1}{n}}$. Then using

$$\sum_{k=2}^{m}\sum_{k'=1}^{k-1}\delta W^j_{i\tilde{\sigma}(k')} = \sum_{k=1}^{m-1}(m-k)\delta W^j_{i\tilde{\sigma}(k)},$$

we get the following result.

Lemma 4.4.3. *For $\ell \in \{1,\dots,d\}$ the ℓ^{th}-component of the term $M^{nm,\tilde{\sigma},1}_{\frac{i-1}{n}}$ has the following expansion*

$$M^{nm,\tilde{\sigma},1}_{\ell,\frac{i-1}{n}} = \Big[\nabla f_\ell{}^{\top}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\sum_{k=1}^{m-1}(m-k)\delta W_{i\tilde{\sigma}(k)}\frac{\Delta_n}{m} + R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1), \tag{4.4.16}$$

with $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|\mathcal{F}_{\frac{i-1}{n}}) = 0$. *Moreover, for any integer* $p \geq 2$ *there exists*

$$\max_{0\leq i\leq n} \mathbb{E}(|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|^p) = o\Big(\Delta_n^{3p/2}\Big). \tag{4.4.17}$$

The proof of the above lemma is postponed to the appendix 4.7.

• **The term** $M^{nm,\tilde{\sigma},2}_{\frac{i-1}{n}}$**:** For this term we first proceed similarly as in (4.4.5) and we use a Taylor expansion to write for $\ell, j \in \{1,\dots,q\}$

$$\begin{aligned}
g_{\ell j}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}) - g_{\ell j}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}) &= \nabla g_{\ell j}^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}) \\
&+ \frac{1}{2}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top \nabla^2 g_{\ell j}(\xi^{2,n}_{ik})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}),
\end{aligned} \tag{4.4.18}$$

for some vector point $\xi^{2,n}_{ik}$ lying between $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}$ and $X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$. Once again by (4.4.1) we get

$$\begin{aligned}
g_{\ell j}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}) - g_{\ell j}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}) &= \nabla g_{\ell j}^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Bigg[\sum_{k'=1}^{k-1}\Bigg(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} \\
&+ g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\delta W_{i\tilde{\sigma}(k')} + \mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge (\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})\Bigg)\Bigg] \\
&+ \frac{1}{2}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top \nabla^2 g_{\ell j}(\xi^{2,n}_{ik})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}).
\end{aligned}$$

Then we have

$$\begin{aligned}
M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}} &= \sum_{k=2}^{m}\sum_{j=1}^{q}\Bigg[\nabla g_{\ell j}^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\sum_{k'=1}^{k-1}\Bigg(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} + g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\delta W_{i\tilde{\sigma}(k')} \\
&+ \mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge (\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})\Bigg) - 2\sum_{k'=1}^{k-1}\sum_{j'=1}^{q} h_{\ell j j'}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\delta W^{j'}_{i\tilde{\sigma}(k')} \\
&+ \frac{1}{2}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top \nabla^2 g_{\ell j}(\xi^{2,n}_{ik})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Bigg]\delta W^j_{i\tilde{\sigma}(k)}.
\end{aligned}$$

Recalling that $h_{\ell j j'} = \frac{1}{2}\nabla g_{\ell j}^\top g_{\bullet j'}$ we obtain

$$\begin{aligned}
M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}} &= \sum_{k=2}^{m}\sum_{j=1}^{q}\Bigg[\nabla g_{\ell j}^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\sum_{k'=1}^{k-1}\Bigg(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} + \\
&\Big[g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) - g(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big]\delta W_{i\tilde{\sigma}(k')} + \mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge (\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})\Bigg) \\
&+ \frac{1}{2}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top \nabla^2 g_{\ell j}(\xi^{2,n}_{ik})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Bigg]\delta W^j_{i\tilde{\sigma}(k)}.
\end{aligned}$$

Again by applying Taylor expansion for each component of the matrix function g, we get $g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) - g(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}) = \dot{g}^n_{ik'} \blacklozenge (X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}) \in \mathbb{R}^{d\times q}$, where $\dot{g}^n_{ik'} \in (\mathbb{R}^{d\times 1})^{d\times q}$ is a block matrix such that for $\ell' \in \{1,\dots,d\}, j' \in \{1,\dots,q\}$, the

$\ell' j'$-th block is given by $(\dot{g}^n_{ik'})_{\ell' j'} = \nabla g_{\ell' j'}(\xi'^{2,n}_{ik'}) \in \mathbb{R}^{d\times 1}$ where $\xi'^{2,n}_{ik'}$ is a vector point lying between $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}$ and $X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$. Then, we have

$$
\begin{aligned}
M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}} = &\sum_{k=2}^{m}\sum_{j=1}^{q}\Bigg[\nabla g^\top_{\ell j}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\sum_{k'=1}^{k-1}\Bigg(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m}+\\
&\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m})\Bigg)\Bigg]\delta W^j_{i\tilde{\sigma}(k)}\\
&+\sum_{k=3}^{m}\sum_{j=1}^{q}\nabla g^\top_{\ell j}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\sum_{k'=2}^{k-1}\Big[\dot{g}^n_{ik'}\blacklozenge(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big]\delta W_{i\tilde{\sigma}(k')}\delta W^j_{i\tilde{\sigma}(k)}\\
&+\sum_{k=2}^{m}\sum_{j=1}^{q}\frac{1}{2}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\delta W^j_{i\tilde{\sigma}(k)}.
\end{aligned}
\tag{4.4.19}
$$

Now, we replace the increment $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$ using (4.4.1) and we only freeze the coefficients of the contributing terms in the asymptotic behavior of the error at the limit point $X_{\frac{i-1}{n}}$.

Lemma 4.4.4. *For $\ell \in \{1,\dots,d\}$ the ℓ^{th}-component of the term $M^{nm,\tilde{\sigma},2}_{\frac{i-1}{n}}$ has the following expansion*

$$
\begin{aligned}
&M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}} =\\
&\sum_{j=1}^{q}\nabla g^\top_{\ell j}(X_{\frac{i-1}{n}})\sum_{1\le k'<k\le m}\Bigg(f(X_{\frac{i-1}{n}})\frac{\Delta_n}{m}+\mathbb{H}(X_{\frac{i-1}{n}})\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m})\Bigg)\delta W^j_{i\tilde{\sigma}(k)}\\
&+\sum_{j=1}^{q}\nabla g^\top_{\ell j}(X_{\frac{i-1}{n}})\sum_{1\le k''<k'<k\le m}\Bigg[\dot{g}^n_i\blacklozenge\Bigg(g(X_{\frac{i-1}{n}})\delta W_{i\tilde{\sigma}(k'')}\Bigg)\Bigg]\delta W_{i\tilde{\sigma}(k')}\delta W^j_{i\tilde{\sigma}(k)}\\
&+\sum_{j=1}^{q}\frac{1}{2}\Big[g^\top(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{1\le k''<k'<k\le m}\delta W_{i\tilde{\sigma}(k'')}\delta W^\top_{i\tilde{\sigma}(k')}\delta W^j_{i\tilde{\sigma}(k)}+R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(2)
\end{aligned}
\tag{4.4.20}
$$

with $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(2)|\mathcal{F}_{\frac{i-1}{n}}) = 0$ and $\dot{g}^n_i \in (\mathbb{R}^{d\times 1})^{d\times q}$ is a block matrix such that for $\ell \in \{1,\dots,d\}$, $j \in \{1,\dots,q\}$, the ℓj-th block is given by $(\dot{g}^n_i)_{\ell j} = \nabla g_{\ell j}(X_{\frac{i-1}{n}})$. Moreover, for any integer $p \ge 2$

$$
\max_{0\le i\le n}\mathbb{E}(|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(2)|^p) = o\Big(\Delta_n^{3p/2}\Big). \tag{4.4.21}
$$

The proof of the above lemma is postponed to the appendix 4.7.

Remark 4.4.5. *The ℓ-th component of $M^{nm,\tilde{\sigma},2}$ can be rewritten as follows:*

$$
\begin{aligned}
&M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}} =\\
&\sum_{j=1}^{q}\nabla g^\top_{\ell j}(X_{\frac{i-1}{n}})\sum_{1\le k'<k\le m}\Big[f(X_{\frac{i-1}{n}})\frac{\Delta_n}{m}+\mathbb{H}(X_{\frac{i-1}{n}})\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m})\Big]\delta W^j_{i\tilde{\sigma}(k)}
\end{aligned}
$$

$$+\sum_{j=1}^{q}\nabla g_{\ell j}^{\top}(X_{\frac{i-1}{n}})\sum_{1\le k''<k'<k\le m}\Big[\dot{g}_i^n\blacklozenge\Big(g(X_{\frac{i-1}{n}})\delta W_{i\tilde{\sigma}(k'')}\Big)\Big]\delta W_{i\tilde{\sigma}(k')}\delta W^j_{i\tilde{\sigma}(k)}$$

$$+\sum_{j=1}^{q}\frac{1}{2}\Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{1\le k'<k\le m}(\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m})\delta W^j_{i\tilde{\sigma}(k)}$$

$$+\sum_{j=1}^{q}\frac{1}{2}\Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{k=2}^{m}(k-1)I_q\frac{\Delta_n}{m}\delta W^j_{i\tilde{\sigma}(k)}$$

$$+\sum_{j=1}^{q}\Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{1\le k''<k'<k\le m}\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k'')}\delta W^j_{i\tilde{\sigma}(k)}+R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(2).$$

(4.4.22)

• **The term $M^{nm,\tilde{\sigma},3}_{\frac{i-1}{n}}$:** Considering each component of $M^{nm,\tilde{\sigma},3}$, for $\ell\in\{1,\dots,d\}$ we can also consider a Taylor expansion for the components of the matrix $h_{\ell\bullet\bullet}\in\mathbb{R}^{q\times q}$ to get

$$M^{nm,\tilde{\sigma},3}_{\ell,\frac{i-1}{n}}=\sum_{k=2}^{m}\Big[\dot{h}^{n,ik}_{\ell\bullet\bullet}\blacklozenge(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big]\blacklozenge(\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)}-I_q\frac{\Delta_n}{m}),\quad(4.4.23)$$

where the $\dot{h}^{n,ik}_{\ell\bullet\bullet}\in(\mathbb{R}^{d\times1})^{q\times q}$ is a random block matrix such that for j and $j'\in\{1,\dots,q\}$, the jj'-th block is given by $(\dot{h}^{n,ik}_{\ell\bullet\bullet})_{jj'}=\nabla h_{\ell jj'}(\xi^{3,n}_{ik})\in\mathbb{R}^{d\times1}$ and $\xi^{3,n}_{ik}$ is a vector point lying between $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}$ and $X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$.

Remark 4.4.6. *Concerning $M^{nm,\tilde{\sigma},3}$, the last formula can be written differently. In fact, as $\mathbb{H}\in(\mathbb{R}^{q\times q})^{d\times1}=\mathbb{R}^{dq\times q}$, we proceed similarly as above using a Taylor expansion to get the existence of a random block matrix $\dot{\mathbb{H}}^n_{ik}$ such that*

$$M^{nm,\tilde{\sigma},3}_{\frac{i-1}{n}}=\sum_{k=2}^{m}\Big[\dot{\mathbb{H}}^n_{ik}\blacklozenge(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big]\blacklozenge(\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)}-I_q\frac{\Delta_n}{m}).$$

More precisely, we have $\dot{\mathbb{H}}^n_{ik}\in(\mathbb{R}^{d\times1})^{dq\times q}$ where for $\ell'\in\{1,\dots,dq\}$ and $j'\in\{1,\dots,q\}$, the $\ell'j'$-th block is given by $(\dot{\mathbb{H}}^n_{ik})_{\ell'j'}=\nabla h_{\ell jj'}(\xi^{3,n}_{ik})\in\mathbb{R}^{d\times1}$ where $\ell'=q(\ell-1)+j$ and $\xi^{3,n}_{ik}$ is a vector point lying between $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}$ and $X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$.

Now, we replace the increment $X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}$ using (4.4.1) and we only freeze the coefficients of the contributing terms in the asymptotic behavior of the error at the limit point $X_{\frac{i-1}{n}}$.

Lemma 4.4.7. *For $\ell\in\{1,\dots,d\}$ the ℓ^{th} component of the term $M^{nm,\tilde{\sigma},3}_{\frac{i-1}{n}}$ has the following expansion*

$$M^{nm,\tilde{\sigma},3}_{\ell,\frac{i-1}{n}}=\sum_{k=2}^{m}\sum_{k'=1}^{k-1}\Big[\dot{h}^{n,i}_{\ell\bullet\bullet}\blacklozenge(g(X_{\frac{i-1}{n}})\delta W_{i\tilde{\sigma}(k')})\Big]\blacklozenge(\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)}-I_q\frac{\Delta_n}{m})+R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)$$

(4.4.24)

with $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)|\mathcal{F}_{\frac{i-1}{n}})=0$ and $\dot{h}^{n,i}_{\ell\bullet\bullet}\in(\mathbb{R}^{d\times1})^{q\times q}$ is a block matrix such that for j and $j'\in\{1,\dots,q\}$, the jj'-th block is given by $(\dot{h}^{n,i}_{\ell\bullet\bullet})_{jj'}=\nabla h_{\ell jj'}(X_{\frac{i-1}{n}})$. Moreover, *for any integer $p \geq 2$ there exists*

$$\max_{0\leq i\leq n} \mathbb{E}(|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)|^p) = o\Big(\Delta_n^{3p/2}\Big). \tag{4.4.25}$$

The proof of the above lemma is postponed to the appendix 4.7 .

4.4.2 Expansion of the error $\bar{X}^{nm,\sigma}_{\frac{i}{n}} - \bar{X}^{nm,\sigma}_{\frac{i-1}{n}}$

We remind that $\bar{X}^{nm,\sigma} = \frac{1}{2}(X^{nm} + X^{nm,\sigma})$. By (4.4.2), we have

$$\begin{aligned}
&\bar{X}^{nm,\sigma}_{\frac{i}{n}} - \bar{X}^{nm,\sigma}_{\frac{i-1}{n}} = \frac{1}{2}\sum_{k=1}^{m}\left[f(X^{nm}_{\frac{m(i-1)+k-1}{nm}}) + f(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\right]\frac{\Delta_n}{m}\\
&+\frac{1}{2}\sum_{k=1}^{m}\left[\mathbb{H}(X^{nm}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{ik}\delta W_{ik}^{\top} - I_q\frac{\Delta_n}{m}) + \mathbb{H}(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{i\sigma(k)}\delta W_{i\sigma(k)}^{\top} - I_q\frac{\Delta_n}{m})\right]\\
&+\frac{1}{2}\sum_{k=1}^{m}\left[g(X^{nm}_{\frac{m(i-1)+k-1}{nm}})\delta W_{ik} + g(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\delta W_{i\sigma(k)}\right].
\end{aligned}$$

Then we rewrite it as follows

$$\begin{aligned}
\bar{X}^{nm,\sigma}_{\frac{i}{n}} - \bar{X}^{nm,\sigma}_{\frac{i-1}{n}} =& f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta_n + g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta W_i + \mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^{\top} - I_q\Delta_n)\\
&+ A_{\frac{i-1}{n}} + B_{\frac{i-1}{n}} + C_{\frac{i-1}{n}},
\end{aligned}$$

where

$$\begin{aligned}
A_{\frac{i-1}{n}} =& \frac{1}{2}\sum_{k=1}^{m}\left[f(X^{nm}_{\frac{m(i-1)+k-1}{nm}}) + f(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\right]\frac{\Delta_n}{m} - f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta_n,\\
B_{\frac{i-1}{n}} =& \frac{1}{2}\sum_{k=1}^{m}\left[g(X^{nm}_{\frac{m(i-1)+k-1}{nm}})\delta W_{i,k} + g(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\delta W_{i\sigma(k)}\right] - g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta W_i,\\
C_{\frac{i-1}{n}} =& \frac{1}{2}\sum_{k=1}^{m}\left[\mathbb{H}(X^{nm}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{ik}\delta W_{ik}^{\top} - I_q\frac{\Delta_n}{m}) + \mathbb{H}(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{i\sigma(k)}\delta W_{i\sigma(k)}^{\top} - I_q\frac{\Delta_n}{m})\right]\\
&- \mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^{\top} - I_q\Delta_n).
\end{aligned}$$

Now considering $A_{\frac{i-1}{n}}$, we use (4.4.5) to get

$$\begin{aligned}
A_{\frac{i-1}{n}} =& \frac{1}{2}\sum_{k=1}^{m}\left[f(X^{nm}_{\frac{m(i-1)+k-1}{nm}}) - f(X^{nm}_{\frac{i-1}{n}})\right]\frac{\Delta_n}{m} + \frac{1}{2}\sum_{k=1}^{m}\left[f(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}}) - f(X^{nm,\sigma}_{\frac{i-1}{n}})\right]\frac{\Delta_n}{m}\\
&+\frac{1}{2}\left(f(X^{nm}_{\frac{i-1}{n}}) + f(X^{nm,\sigma}_{\frac{i-1}{n}})\right)\Delta_n - f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta_n\\
=& \frac{1}{2}(M^{nm,\mathrm{Id},1}_{\frac{i-1}{n}} + M^{nm,\sigma,1}_{\frac{i-1}{n}} + N^{nm,\mathrm{Id}}_{\frac{i-1}{n}} + N^{nm,\sigma}_{\frac{i-1}{n}}) + \tilde{N}^{nm}_{\frac{i-1}{n}},
\end{aligned}$$

where $\tilde{N}^{nm}_{\frac{i-1}{n}} = \dfrac{1}{2}\big(f(X^{nm}_{\frac{i-1}{n}}) + f(X^{nm,\sigma}_{\frac{i-1}{n}})\big)\Delta_n - f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta_n$. Similarly, we have

$$B_{\frac{i-1}{n}} + C_{\frac{i-1}{n}} = \left[\frac{1}{2}\left(g(X^{nm}_{\frac{i-1}{n}}) + g(X^{nm,\sigma}_{\frac{i-1}{n}})\right) - g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\right]\Delta W_i$$

$$+\frac{1}{2}\sum_{k=1}^{m}\left[g(X^{nm}_{\frac{m(i-1)+k-1}{nm}})-g(X^{nm}_{\frac{i-1}{n}})\right]\delta W_{ik}+\frac{1}{2}\sum_{k=1}^{m}\left[g(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})-g(X^{nm,\sigma}_{\frac{i-1}{n}})\right]\delta W_{i\sigma(k)}$$
$$+\frac{1}{2}\sum_{k=1}^{m}\left[\mathbb{H}(X^{nm}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{ik}\delta W_{ik}^{\top}-I_q\frac{\Delta_n}{m})-\mathbb{H}(X^{nm}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^{\top}-I_q\Delta_n)\right]$$
$$+\frac{1}{2}\sum_{k=1}^{m}\left[\mathbb{H}(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\blacklozenge(\delta W_{i\sigma(k)}\delta W_{i\sigma(k)}^{\top}-I_q\frac{\Delta_n}{m})-\mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^{\top}-I_q\Delta_n)\right]$$
$$+\left[\frac{1}{2}\left(\mathbb{H}(X^{nm}_{\frac{i-1}{n}})+\mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})\right)-\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\right]\blacklozenge(\Delta W_i\Delta W_i^{\top}-I_q\Delta_n).$$

Now, by (4.4.8) and the expressions of $M^{nm,\tilde{\sigma},2}$ and $M^{nm,\tilde{\sigma},3}$ given above relation (4.4.9) we rearrange our terms to get

$$B_{\frac{i-1}{n}}+C_{\frac{i-1}{n}}=\frac{1}{2}(M^{nm,\mathrm{Id},2}_{\frac{i-1}{n}}+M^{nm,\sigma,2}_{\frac{i-1}{n}}+M^{nm,\mathrm{Id},3}_{\frac{i-1}{n}}+M^{nm,\sigma,3}_{\frac{i-1}{n}})+\tilde{M}^{nm,1}_{\frac{i-1}{n}}+\tilde{M}^{nm,2}_{\frac{i-1}{n}}-\frac{1}{2}\tilde{M}^{nm,3}_{\frac{i-1}{n}},$$

where

$$\tilde{M}^{nm,1}_{\frac{i-1}{n}}=\left[\frac{1}{2}\left(g(X^{nm}_{\frac{i-1}{n}})+g(X^{nm,\sigma}_{\frac{i-1}{n}})\right)-g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\right]\Delta W_i,$$
$$\tilde{M}^{nm,2}_{\frac{i-1}{n}}=\left[\frac{1}{2}\left(\mathbb{H}(X^{nm}_{\frac{i-1}{n}})+\mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})\right)-\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\right]\blacklozenge(\Delta W_i\Delta W_i^{\top}-I_q\Delta_n),$$
$$\tilde{M}^{nm,3}_{\frac{i-1}{n}}=\sum_{\tilde{\sigma}\in\{\mathrm{Id},\sigma\}}\sum_{1\le k<k'\le m}\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\blacklozenge(\delta W_{i\tilde{\sigma}(k)}\delta W_{i\tilde{\sigma}(k')}^{\top}-\delta W_{i\tilde{\sigma}(k')}\delta W_{i\tilde{\sigma}(k)}^{\top})$$

Now recalling that $\sigma(k)=m-k+1$, for all $k\in\{1,\dots,m\}$, we get

$$\tilde{M}^{nm,3}_{\frac{i-1}{n}}=\sum_{1\le k<k'\le m}\left[\mathbb{H}(X^{nm}_{\frac{i-1}{n}})-\mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})\right]\blacklozenge(\delta W_{ik}\delta W_{ik'}^{\top}-\delta W_{ik'}\delta W_{ik}^{\top}).$$

In what follows, by (4.4.9) we introduce for $i\in\{1,\dots,n\}$

$$\bar{N}^{nm}_{\frac{i-1}{n}}=\frac{1}{2}(N^{nm,\mathrm{Id}}_{\frac{i-1}{n}}+N^{nm,\sigma}_{\frac{i-1}{n}})+\tilde{N}^{nm}_{\frac{i-1}{n}},\tag{4.4.26}$$
$$\bar{M}^{nm}_{\frac{i-1}{n}}=\frac{1}{2}(M^{nm,\mathrm{Id}}_{\frac{i-1}{n}}+M^{nm,\sigma}_{\frac{i-1}{n}})+\tilde{M}^{nm,1}_{\frac{i-1}{n}}+\tilde{M}^{nm,2}_{\frac{i-1}{n}}-\frac{1}{2}\tilde{M}^{nm,3}_{\frac{i-1}{n}}.\tag{4.4.27}$$

The proof of the following lemma is postponed to the appendix 4.7.

Lemma 4.4.8. *The error $\bar{X}^{nm,\sigma}_{\frac{i}{n}}-\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}$, $i\in\{1,\dots,n\}$ can be expressed as follows*

$$\bar{X}^{nm,\sigma}_{\frac{i}{n}}-\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}=f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta_n+g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\Delta W_i$$
$$+\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})\blacklozenge(\Delta W_i\Delta W_i^{\top}-I_q\Delta_n)+\bar{M}^{mn}_{\frac{i-1}{n}}+\bar{N}^{nm}_{\frac{i-1}{n}},\tag{4.4.28}$$

where $\mathbb{E}(\bar{M}^{nm}_{\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=0$ and for any integer $p\ge 2$ there exists a constant K_p such that

$$\max_{0\le i\le n}\mathbb{E}(|\bar{M}^{nm}_{\frac{i-1}{n}}|^p)\le K_p\Delta_n^{3p/2},\tag{4.4.29}$$
$$\max_{0\le i\le n}\mathbb{E}(|\bar{N}^{nm}_{\frac{i-1}{n}}|^p)\le K_p\Delta_n^{2p}.\tag{4.4.30}$$

Corollary 4.4.9. *We have*

$$\mathbb{E}(\max_{0\le i\le n}|\bar{X}^{nm,\sigma}_{\frac{i}{n}}-X^n_{\frac{i}{n}}|^p)\le C_p\Delta_n^p.$$

Proof. Let us define $S_k=\mathbb{E}(\max_{0\le k'\le k}|\bar{X}^{nm,\sigma}_{\frac{k'}{n}}-X^n_{\frac{k'}{n}}|^p)$, for any $0\le k\le n$. For a fixed k, by summing (4.4.28) over the first k' timesteps, we obtain

$$\begin{aligned}\bar{X}^{nm,\sigma}_{\frac{k'}{n}}-X^n_{\frac{k'}{n}}=&\sum_{i=1}^{k'}(f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-f(X^n_{\frac{i-1}{n}}))\Delta_n+\sum_{i=1}^{k'}(g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-g(X^n_{\frac{i-1}{n}}))\Delta W_i\\&+\sum_{i=1}^{k'}(\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-\mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n)+\sum_{i=1}^{k'}\bar{M}^{nm}_{\frac{i-1}{n}}+\sum_{i=1}^{k'}\bar{N}^{nm}_{\frac{i-1}{n}},\end{aligned}$$

Then there is a generic constant $C_p>0$ such that

$$\begin{aligned}\mathbb{E}(\max_{0\le k'\le k}|\bar{X}^{nm,\sigma}_{\frac{k}{n}}-X^n_{\frac{k}{n}}|^p)\le\;&C_p\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}(f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-f(X^n_{\frac{i-1}{n}}))\Delta_n|^p)\\&+C_p\mathbb{E}(\max_{0\le k\le n}|\sum_{i=1}^{k'}(g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-g(X^n_{\frac{i-1}{n}}))\Delta W_i|^p)\\&+C_p\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}(\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-\mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n)|^p)\\&+C_p\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}\bar{M}^{nm}_{\frac{i-1}{n}}|^p)+C_p\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}\bar{N}^{nm}_{\frac{i-1}{n}}|^p),\end{aligned}$$

By Jensen's inequality and ($\mathbf{H}_{f,g}$), we have

$$\begin{aligned}&\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}(f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-f(X^n_{\frac{i-1}{n}}))\Delta_n|^p)\le C_p\mathbb{E}(\max_{0\le k\le n}k^{p-1}\sum_{i=1}^{k}|(f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-f(X^n_{\frac{i-1}{n}}))\Delta_n|^p)\\&\le C_pn^{p-1}\sum_{i=1}^{k}\mathbb{E}((f(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-f(X^n_{\frac{i-1}{n}})|^p)\Delta_n^p\le C_p\sum_{i=1}^{k}\mathbb{E}(\max_{0\le k\le i-1}|\bar{X}^{nm,\sigma}_{\frac{k}{n}}-X^n_{\frac{k}{n}}|^p)\Delta_n.\end{aligned}$$

Similarly, by Jensen's inequality, the independence between ΔW_i and $\mathcal{F}_{\frac{i-1}{n}}$ and the assumption ($\mathbf{H}_{f,g}$), $\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}(\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-\mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n)|^p)$ has an upper bound $C_p\sum_{i=0}^{k-1}\mathbb{E}(\max_{0\le j\le i-1}|\bar{X}^{nm,\sigma}_{\frac{j}{n}}-X^n_{\frac{j}{n}}|^p)\Delta_n$. Now, by Jensen's inequality and Lemma 4.4.8, $\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}\bar{N}^{nm}_{\frac{i-1}{n}}|^p)$ has an upper bound $C_p\Delta_n^p$. Finally, by the discrete BDG inequality in Jacod et al., 2005 combined with Jensen's inequality, we have

$$\begin{aligned}&\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}(g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-g(X^n_{\frac{i-1}{n}}))\Delta W_i|^p)\le C_p\mathbb{E}(\sum_{i=1}^{k}|g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-g(X^n_{\frac{i-1}{n}}))\Delta W_i|^2)^{p/2}\\&\le C_pn^{p/2-1}\sum_{i=1}^{k}\mathbb{E}(|g(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-g(X^n_{\frac{i-1}{n}}))|^p)\mathbb{E}(|\Delta W_i|^p)\le C_p\sum_{i=1}^{k}\mathbb{E}(\max_{0\le k\le i-1}|\bar{X}^{nm,\sigma}_{\frac{k}{n}}-X^n_{\frac{k}{n}}|^p)\Delta_n.\end{aligned}$$

Similarly, thanks to Lemma 4.4.8, $\mathbb{E}(\max_{0\le k'\le k}|\sum_{i=1}^{k'}\bar{M}^{nm}_{\frac{i-1}{n}}|^p)$ has an upper bound $C_p\Delta_n^p$. Thus, it follows that

$$S_k \le C_p(\Delta_n^p + \sum_{i=0}^{k-1} S_i\Delta_n), \quad \text{for any } 0\le k\le n.$$

By the discrete Grönwal inequality, we have

$$S_n \le C_p\Delta_n^p + C\Delta_n^{p+1}\sum_{i=0}^{n-1}\exp\{(n-1-i)\Delta_n\} \le C_p\Delta_n^p + C_p\Delta_n^{p+1}\sum_{i=0}^{n-1} e \le C_p\Delta_n^p.$$

□

In what follows we give further expansions for the terms $\tilde{N}^{nm}_{\frac{i-1}{n}}$, $\tilde{M}^{nm,1}_{\frac{i-1}{n}}$, $\tilde{M}^{nm,2}_{\frac{i-1}{n}}$ and $\tilde{M}^{nm,3}_{\frac{i-1}{n}}$ defined above. These expansions will be useful later on. To do so, we apply twice the Taylor expansion until the second order, for each $\ell\in\{1,\dots,d\}$, we get

$$\tilde{N}^{nm}_{\ell,\frac{i-1}{n}} = \frac{1}{16}(X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}})^\top\left(\nabla^2 f_\ell(\zeta^{n,1}_{\frac{i-1}{n}}) + \nabla^2 f_\ell(\zeta^{n,2}_{\frac{i-1}{n}})\right)(X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}})\Delta_n, \tag{4.4.31}$$

$$\tilde{M}^{nm,1}_{\ell,\frac{i-1}{n}} = \frac{1}{16}\sum_{j'=1}^{q}(X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}})^\top\left(\nabla^2 g_{\ell j'}(\zeta^{n,3}_{\frac{i-1}{n}}) + \nabla^2 g_{\ell j'}(\zeta^{n,4}_{\frac{i-1}{n}})\right)(X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}})\Delta W_i^{j'}. \tag{4.4.32}$$

Then using twice the Taylor expansion until the first order we get

$$\tilde{M}^{nm,2}_{\ell,\frac{i-1}{n}} = \frac{1}{4}\left[\dot{h}^{n,i,1}_{\ell\bullet\bullet} \blacklozenge (X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}})\right] \blacklozenge (\Delta W_i \Delta W_i - I_q\Delta_n) \tag{4.4.33}$$

and similarly

$$\tilde{M}^{nm,3}_{\ell,\frac{i-1}{n}} = \sum_{1\le k<k'\le m}\left[\dot{h}^{n,i,2}_{\ell\bullet\bullet} \blacklozenge (X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}})\right] \blacklozenge (\delta W_{ik}\delta W_{ik'}^\top - \delta W_{ik'}\delta W_{ik}^\top), \tag{4.4.34}$$

where for j and $j'\in\{1,\dots,q\}$, the jj'-th elements of the block matrices $\dot{h}^{n,i,1}_{\ell\bullet\bullet}$ and $\dot{h}^{n,i,2}_{\ell\bullet\bullet}$ are respectively given by $(\dot{h}^{n,i,1}_{\ell\bullet\bullet})_{jj'} = \nabla h_{\ell jj'}(\zeta^{n,5}_{\frac{i-1}{n}}) - \nabla h_{\ell jj'}(\zeta^{n,6}_{\frac{i-1}{n}}) \in \mathbb{R}^{d\times 1}$ and $(\dot{h}^{n,i,2}_{\ell\bullet\bullet})_{jj'} = \nabla h_{\ell jj'}(\zeta^{n,7}_{\frac{i-1}{n}}) \in \mathbb{R}^{d\times 1}$; for some vector point $\xi_i^{7,n}$ lying between $X^{nm,\sigma}_{\frac{i-1}{n}}$ and $X^{nm}_{\frac{i-1}{n}}$, some vector points $\zeta^{n,1}_{\frac{i-1}{n}}, \zeta^{n,3}_{\frac{i-1}{n}}, \zeta^{n,5}_{\frac{i-1}{n}}$ lying between $\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}$ and $X^{nm}_{\frac{i-1}{n}}$ and some vector points $\zeta^{n,2}_{\frac{i-1}{n}}, \zeta^{n,4}_{\frac{i-1}{n}}, \zeta^{n,6}_{\frac{i-1}{n}}$ lying between $\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}$ and $X^{nm,\sigma}_{\frac{i-1}{n}}$.

Remark 4.4.10. *In order to get the good rate of convergence, we need to assume that our σ is strictly decreasing which leads us to take the unique choice defined by $\sigma(k) = m-k+1$. Otherwise, it is easy to check that the term $n\sum_{i=1}^{[nt]}\tilde{M}^{nm,3}_{\frac{i-1}{n}}$ appearing in the decomposition of the normalized error $n(\bar{X}^{nm,\sigma}_{\eta_n(t)} - X^n_{\eta_n(t)})$ is not tight.*

4.4.3 Error analysis of U^n and V^n

For $t \in [0,1]$ we have

$$X^{nm,\sigma}_{\eta_n(t)} = x_0 + \sum_{i=1}^{[nt]}\sum_{k=1}^{m} f(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\frac{\Delta_n}{m} + \sum_{i=1}^{[nt]}\sum_{k=1}^{m} g(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}})\delta W_{i\sigma(k)}$$
$$+ \sum_{i=1}^{[nt]}\sum_{k=1}^{m} \mathbb{H}(X^{nm,\sigma}_{\frac{m(i-1)+k-1}{nm}}) \blacklozenge (\delta W_{i\sigma(k)}\delta W_{i\sigma(k)}^\top - I_q\frac{\Delta_n}{m}). \tag{4.4.35}$$

Error analysis of U^n At first, we consider the error $U^n_t = (U^{n,1}_t, \dots, U^{n,d}_t)^\top \in \mathbb{R}^d$ between the finer and the antithetic Milstein approximations given by $U^n_t = X^{nm}_{\eta_n(t)} - X^{nm,\sigma}_{\eta_n(t)}$. Then by (4.4.10), the expansion of U^n takes the following form

$$U^n_t = \sum_{i=1}^{[nt]}(f(X^{nm}_{\frac{i-1}{n}}) - f(X^{nm,\sigma}_{\frac{i-1}{n}}))\Delta_n + \sum_{i=1}^{[nt]}(g(X^{nm}_{\frac{i-1}{n}}) - g(X^{nm,\sigma}_{\frac{i-1}{n}}))\Delta W_i$$
$$+ \sum_{i=1}^{[nt]}(\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n)$$
$$- \sum_{i=1}^{[nt]}\sum_{\substack{k,k'=1\\k<k'}}^{m} (\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) + \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge \left(\delta W_{ik}\delta W_{ik'}^\top - \delta W_{ik'}\delta W_{ik}^\top\right)$$
$$+ \sum_{i=1}^{[nt]}(M^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - M^{nm,\sigma}_{\frac{i-1}{n}}) + \sum_{i=1}^{[nt]}(N^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - N^{nm,\sigma}_{\frac{i-1}{n}}).$$

By Taylor's expansion, we rewrite U^n as follows

$$U^n_t = \sum_{i=1}^{[nt]} \dot{f}^n_i \blacklozenge U^n_{\frac{i-1}{n}}\Delta_n + \sum_{i=1}^{[nt]} \left(\dot{g}^n_i \blacklozenge U^n_{\frac{i-1}{n}}\right)\Delta W_i$$
$$+ \sum_{i=1}^{[nt]}(\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n)$$
$$- \sum_{i=1}^{[nt]}\sum_{1\le k<k'\le m} (\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) + \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge \left(\delta W_{ik}\delta W_{ik'}^\top - \delta W_{ik'}\delta W_{ik}^\top\right)$$
$$+ \sum_{i=1}^{[nt]}(M^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - M^{nm,\sigma}_{\frac{i-1}{n}}) + \sum_{i=1}^{[nt]}(N^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - N^{nm,\sigma}_{\frac{i-1}{n}}), \tag{4.4.36}$$

where $\dot{f}^n_i \in (\mathbb{R}^{d\times 1})^{d\times 1}$ and $\dot{g}^n_i \in (\mathbb{R}^{d\times 1})^{d\times q}$ are block matrices such that for $\ell \in \{1,\dots,d\}$ the ℓ-th block of $\dot{f}^n_i$ is given by $(\dot{f}^n_i)_\ell = \nabla f_\ell(\xi^{1,n}_{\frac{i-1}{n}})$, for $\ell \in \{1,\dots,d\}$ and $j \in \{1,\dots,q\}$ the ℓj-th block of $\dot{g}^n_i$ is given by $(\dot{g}^n_i)_{\ell j} = \nabla g_{\ell j}(\xi^{2,n}_{\frac{i-1}{n}})$ with $\xi^{1,n}_{\frac{i-1}{n}}$ and $\xi^{2,n}_{\frac{i-1}{n}}$ are some vector points lying between $X^{nm}_{\frac{i-1}{n}}$ and $X^{nm,\sigma}_{\frac{i-1}{n}}$. Now, the equation (4.4.36) can be rewritten as

$$U^n_t = \sum_{i=1}^{[nt]} \dot{f}^n_i \blacklozenge U^n_{\frac{i-1}{n}}\Delta_n + \sum_{i=1}^{[nt]} \left(\dot{g}^n_i \blacklozenge U^n_{\frac{i-1}{n}}\right)\Delta W_i + \mathcal{M}^{n,1}_t + \mathcal{R}^{n,1}_t$$

$$= \sum_{i=1}^{[nt]} \dot{f}_i^n \blacklozenge U_{\frac{i-1}{n}}^n \Delta_n + \sum_{j=1}^{q} \sum_{i=1}^{[nt]} \left((\dot{g}_i^n)_{\bullet j} \blacklozenge U_{\frac{i-1}{n}}^n \right) \Delta W_i^j + \mathcal{M}_t^{n,1} + \mathcal{R}_t^{n,1}, \qquad (4.4.37)$$

with $(\dot{g}_i^n)_{\bullet j} = ((\dot{g}_i^n)_{1j}, \ldots, (\dot{g}_i^n)_{dj})^\top$, $\mathcal{M}^{n,1}$ is the main term and $\mathcal{R}^{n,1}$ is the rest term given by

$$\mathcal{M}_t^{n,1} = - \sum_{i=1}^{[nt]} \sum_{\substack{k,k'=1 \\ k<k'}}^{m} (\mathbb{H}(X_{\frac{i-1}{n}}^{nm}) + \mathbb{H}(X_{\frac{i-1}{n}}^{nm,\sigma})) \blacklozenge \left(\delta W_{ik} \delta W_{ik'}^\top - \delta W_{ik'} \delta W_{ik}^\top \right),$$

$$\mathcal{R}_t^{n,1} = \sum_{i=1}^{[nt]} (\mathbb{H}(X_{\frac{i-1}{n}}^{nm}) - \mathbb{H}(X_{\frac{i-1}{n}}^{nm,\sigma})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q \Delta_n) + \sum_{i=1}^{[nt]} (M_{\frac{i-1}{n}}^{nm,\mathrm{Id}} - M_{\frac{i-1}{n}}^{nm,\sigma})$$
$$+ \sum_{i=1}^{[nt]} (N_{\frac{i-1}{n}}^{nm,\mathrm{Id}} - N_{\frac{i-1}{n}}^{nm,\sigma}).$$

The proof of the following lemma is postponed to appendix 4.6.

Lemma 4.4.11. *Under the assumption ($\boldsymbol{H}_{f,g}$), we have $\sqrt{n}\mathcal{R}^{n,1} \overset{L^p}{\to} 0$ as $n \to \infty$.*

Error analysis of V^n Now, we consider the error $V_t^n = (V_t^{n,1}, \ldots, V_t^{n,d})^\top \in \mathbb{R}^d$ between the average of the finer and the coarser antithetic Milstein approximations given by $V_t^n = \bar{X}_{\eta_n(t)}^{nm,\sigma} - X_{\eta_n(t)}^n$. Similarly to the analysis of U^n, by (4.4.28) and (4.2.3), we rewrite V^n as follows

$$V_t^n = \sum_{i=1}^{[nt]} (f(\bar{X}_{\frac{i-1}{n}}^{nm,\sigma}) - f(X_{\frac{i-1}{n}}^n)) \Delta_n + \sum_{i=1}^{[nt]} (g(\bar{X}_{\frac{i-1}{n}}^{nm,\sigma}) - g(X_{\frac{i-1}{n}}^n)) \Delta W_i$$
$$+ \sum_{i=1}^{[nt]} (\mathbb{H}(\bar{X}_{\frac{i-1}{n}}^{nm,\sigma}) - \mathbb{H}(X_{\frac{i-1}{n}}^n)) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q \Delta_n) + \sum_{i=1}^{[nt]} \bar{M}_{\frac{i-1}{n}}^{nm} + \sum_{i=1}^{[nt]} \bar{N}_{\frac{i-1}{n}}^{nm},$$

where $\bar{N}_{\frac{i-1}{n}}^{nm}$ and $\bar{M}_{\frac{i-1}{n}}^{nm}$ are respectively given by (4.4.26) and (4.4.27). By the Taylor expansion, we have

$$V_t^n = \sum_{i=1}^{[nt]} \bar{\dot{f}}_i^n \blacklozenge V_{\frac{i-1}{n}}^n \Delta_n + \sum_{i=1}^{[nt]} \left(\bar{\dot{g}}_i^n \blacklozenge V_{\frac{i-1}{n}}^n \right) \Delta W_i$$
$$+ \sum_{i=1}^{[nt]} (\mathbb{H}(\bar{X}_{\frac{i-1}{n}}^{nm,\sigma}) - \mathbb{H}(X_{\frac{i-1}{n}}^n)) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q \Delta_n) + \sum_{i=1}^{[nt]} \bar{M}_{\frac{i-1}{n}} + \sum_{i=1}^{[nt]} \bar{N}_{\frac{i-1}{n}},$$

where $\bar{\dot{f}}_i^n \in (\mathbb{R}^{d\times 1})^{d\times 1}$ and $\bar{\dot{g}}_i^n \in (\mathbb{R}^{d\times 1})^{d\times q}$ are block matrices such that for $\ell \in \{1, \ldots, d\}$ the ℓ-th block of $\bar{\dot{f}}_i^n$ is given by $(\bar{\dot{f}}_i^n)_\ell = \nabla f_\ell(\bar{\xi}_{\frac{i-1}{n}}^{1,n})$, for $\ell \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, q\}$ the ℓj-th block of $\bar{\dot{g}}_i^n$ is given by $(\bar{\dot{g}}_i^n)_{\ell j} = \nabla g_{\ell j}(\bar{\xi}_{\frac{i-1}{n}}^{2,n})$ with $\bar{\xi}_{\frac{i-1}{n}}^{1,n}$ and $\bar{\xi}_{\frac{i-1}{n}}^{2,n}$ are some vector points lying between $X_{\frac{i-1}{n}}^n$ and $\bar{X}_{\frac{i-1}{n}}^{nm,\sigma}$. Thanks to Lemma 4.4.2, Lemma 4.4.3, Lemma 4.4.4 and Lemma 4.4.7, the above equation rewrites as follows

$$V_t^n = \sum_{i=1}^{[nt]} \bar{f}_i^n \blacklozenge V_{\frac{i-1}{n}}^n \Delta_n + \sum_{i=1}^{[nt]} \left(\bar{g}_i^n \blacklozenge V_{\frac{i-1}{n}}^n \right) \Delta W_i + \mathcal{M}_t^{n,2} + \mathcal{R}_t^{n,2}, \qquad (4.4.38)$$

where $\mathcal{M}^{n,2}$ stands for the main contributing term of the above error expansion and $\mathcal{R}^{n,2}$ is the rest term, for $t \in [0,1]$ they are given by

$$\mathcal{M}_t^{n,2} = \frac{1}{2} \sum_{\tilde{\sigma} \in \{\mathrm{Id}, \sigma\}} \sum_{r=1}^{4} \Gamma_t^{n,\tilde{\sigma}}(r) + \tilde{N}_t^{nm} + \tilde{M}_t^{nm,1} - \frac{1}{2} \tilde{M}_t^{nm,3}, \tag{4.4.39}$$

$$\begin{aligned}\mathcal{R}_t^{n,2} =& \frac{1}{2} \sum_{\tilde{\sigma} \in \{\mathrm{Id}, \sigma\}} \sum_{i=1}^{[nt]} \left(\tilde{R}^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}(0) + \sum_{r=0}^{3} R^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}(r) \right) + \tilde{M}_t^{nm,2} \\ &+ \sum_{i=1}^{[nt]} (\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}) - \mathbb{H}(X^n_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q \Delta_n),\end{aligned} \tag{4.4.40}$$

where for $r \in \{1,2,3\}$, $\tilde{M}_t^{nm,r} = \sum_{i=1}^{[nt]} \tilde{M}^{nm,r}_{\frac{i-1}{n}}$, $\tilde{N}_t^{nm} = \sum_{i=1}^{[nt]} \tilde{N}^{nm}_{\frac{i-1}{n}}$ with $(\tilde{M}^{nm,r}_{\frac{i-1}{n}})_{1 \le r \le 3}$, $\tilde{N}^{nm}_{\frac{i-1}{n}}$ are respectively given by (4.4.31),(4.4.32), (4.4.33) and (4.4.34) and the rest terms $R^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}(0)$ and $\tilde{R}^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}(0)$ are implicitly defined in (4.4.13) and $(R^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}(r))_{1 \le r \le 3}$ are respectively implicitly defined in (4.4.16), (4.4.20) and (4.4.24). Now, we introduce the d-dimensional processes $(\Gamma_t^{n,\tilde{\sigma}}(i), 1 \le i \le 4, t \in [0,1])$ whose ℓ^{th} components are given by

$$\begin{aligned}\Gamma_{\ell,t}^{n,\tilde{\sigma}}(1) =& \sum_{i=1}^{[nt]} \Bigg[\frac{(m-1)}{2m} \nabla f_\ell^\top (X_{\frac{i-1}{n}}) f(X_{\frac{i-1}{n}}) \Delta_n^2 \\ &+ \frac{1}{2} \Big[g^\top (X_{\frac{i-1}{n}}) \nabla^2 f_\ell (X_{\frac{i-1}{n}}) g(X_{\frac{i-1}{n}}) \Big] \blacklozenge \sum_{k=1}^{m-1} (m-k) \delta W_{i\tilde{\sigma}(k)} \delta W_{i\tilde{\sigma}(k)}^\top \frac{\Delta_n}{m} \Bigg],\end{aligned} \tag{4.4.41}$$

$$\begin{aligned}\Gamma_{\ell,t}^{n,\tilde{\sigma}}(2) =& \sum_{i=1}^{[nt]} \frac{\Delta_n}{m} \Bigg[\Big[\nabla f_\ell{}^\top (X_{\frac{i-1}{n}}) g(X_{\frac{i-1}{n}}) \Big] \sum_{k=1}^{m-1} (m-k) \delta W_{i\tilde{\sigma}(k)} \\ &+ \sum_{j=1}^{q} \nabla g_{\ell j}^\top (X_{\frac{i-1}{n}}) f(X_{\frac{i-1}{n}}) \sum_{k=1}^{m-1} (m-k) \delta W^j_{i\tilde{\sigma}(k)} \\ &+ \sum_{j=1}^{q} \frac{1}{2} g^\top (X_{\frac{i-1}{n}}) \nabla^2 g_{\ell j} (X_{\frac{i-1}{n}}) g(X_{\frac{i-1}{n}}) \blacklozenge I_q \sum_{k=2}^{m} (k-1) \delta W^j_{i\tilde{\sigma}(k)} \Bigg],\end{aligned} \tag{4.4.42}$$

$$\begin{aligned}\Gamma_{\ell,t}^{n,\tilde{\sigma}}(3) =& \sum_{i=1}^{[nt]} \sum_{j=1}^{q} \Bigg[\nabla g_{\ell j}^\top (X_{\frac{i-1}{n}}) \mathbb{H}(X_{\frac{i-1}{n}}) \blacklozenge \sum_{1 \le k' < k \le m} (\delta W_{i\tilde{\sigma}(k')} \delta W_{i\tilde{\sigma}(k')}^\top - I_q \frac{\Delta_n}{m}) \delta W^j_{i\tilde{\sigma}(k)} \\ &+ \frac{1}{2} \Big[g^\top (X_{\frac{i-1}{n}}) \nabla^2 g_{\ell j} (X_{\frac{i-1}{n}}) g(X_{\frac{i-1}{n}}) \Big] \blacklozenge \sum_{1 \le k' < k \le m} (\delta W_{i\tilde{\sigma}(k')} \delta W_{i\tilde{\sigma}(k')}^\top - I_q \frac{\Delta_n}{m}) \delta W^j_{i\tilde{\sigma}(k)} \\ &+ \Big[\dot{h}^{n,i}_{\ell \bullet \bullet} \blacklozenge g_{\bullet j} (X_{\frac{i-1}{n}}) \Big] \blacklozenge \sum_{1 \le k' < k \le m} (\delta W_{i\tilde{\sigma}(k)} \delta W_{i\tilde{\sigma}(k)}^\top - I_q \frac{\Delta_n}{m}) \delta W^j_{i\tilde{\sigma}(k')} \Bigg],\end{aligned} \tag{4.4.43}$$

$$\Gamma_{\ell,t}^{n,\tilde{\sigma}}(4) = \sum_{i=1}^{[nt]} \Bigg[\sum_{j,j'=1}^{q} \nabla g_{\ell j}^\top (X_{\frac{i-1}{n}}) \Big[\dot{g}_i^n \blacklozenge g_{\bullet j'} (X_{\frac{i-1}{n}}) \Big] \sum_{1 \le k'' < k' < k \le m} \delta W_{i\tilde{\sigma}(k')} \delta W^{j'}_{i\tilde{\sigma}(k'')} \delta W^j_{i\tilde{\sigma}(k)}$$

$$+\sum_{j=1}^{q}\Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{1\le k''<k'<k\le m}\delta W_{i\tilde\sigma(k')}\delta W_{i\tilde\sigma(k'')}^{\top}\delta W_{i\tilde\sigma(k)}^{j}\Bigg].\tag{4.4.44}$$

The proof of the following lemma is also postponed to appendix 4.6.

Lemma 4.4.12. *We have $n\mathcal{R}^{n,2}\overset{L^p}{\to}0$ as $n\to\infty$.*

Remark 4.4.13. *These processes $(\Gamma_t^{n,\tilde\sigma}(r),1\le r\le 4,t\in[0,1])$ are obtained by gathering together the main terms in (4.4.13), (4.4.16), (4.4.20) and (4.4.24), taking into account their noise types and neglecting the rest terms $\big(R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(r)\big)_{0\le r\le 3}$ and $\tilde R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(0)$, for $\ell\in\{1,\dots,d\}$.*

4.5 Asymptotic Behavior of the main terms

According to expansion (4.4.37) and (4.4.38) appearing in the decompositions of U^n and V^n we need to focus on the main terms $(\mathcal{M}^{n,1},\mathcal{M}^{n,2})$, where we recall that

$$\mathcal{M}_t^{n,1}=-\sum_{i=1}^{[nt]}\sum_{\substack{1\le k,k'\le m\\ k<k'}}(\mathbb{H}(X^{nm}_{\frac{i-1}{n}})+\mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}}))\blacklozenge\Big(\delta W_{ik}\delta W_{ik'}^{\top}-\delta W_{ik'}\delta W_{ik}^{\top}\Big),\tag{4.5.1}$$

$$\mathcal{M}_t^{n,2}=\frac{1}{2}\sum_{\tilde\sigma\in\{\mathrm{Id},\sigma\}}\sum_{r=1}^{4}\Gamma_t^{n,\tilde\sigma}(r)+\tilde N_t^{nm}+\tilde M_t^{nm,1}-\frac{1}{2}\tilde M_t^{nm,3}$$

with $\tilde N_t^{nm}$ respectively $\tilde M_t^{nm,1}$ and $\tilde M_t^{nm,3}$ are given by relation (4.4.31) respectively (4.4.32) and (4.4.34), $(\Gamma_t^{n,\tilde\sigma}(r),1\le r\le 4,t\in[0,1])$ are defined as above in (4.4.41), (4.4.42), (4.4.43) and (4.4.44).
Unlike the first main term $\mathcal{M}^{n,1}$, that has explicit form of the noise, the second main term $\mathcal{M}^{n,2}$ needs further development in order to identify its noise parts. To do so, we need the following lemma that will be proven in appendix 4.6.

Lemma 4.5.1. *Let $\bar\Gamma_t^n(r)=\frac{\Gamma_t^{n,\mathrm{Id}}(r)+\Gamma_t^{n,\sigma}(r)}{2}\in\mathbb{R}^d$, for $r\in\{1,2,3,4\}$. Then we rewrite $\bar\Gamma^n(r)$ as follows, for $\ell\in\{1,\dots,d\}$,*

$$\bar\Gamma^n_{\ell,t}(1)=\sum_{i=1}^{[nt]}\frac{(m-1)\Delta_n}{2m}\Bigg[\nabla f_\ell^{\top}(X_{\frac{i-1}{n}})f(X_{\frac{i-1}{n}})\Delta_n+\frac{1}{2}\Big[g^{\top}(X_{\frac{i-1}{n}})\nabla^2 f_\ell(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{k=1}^{m}\delta W_{ik}\delta W_{ik}^{\top}\Bigg],$$

$$\bar\Gamma^n_{\ell,t}(2)=\sum_{i=1}^{[nt]}\sum_{j=1}^{q}\frac{(m-1)\Delta_n}{2m}\Bigg[\nabla f_\ell^{\top}(X_{\frac{i-1}{n}})g_{\bullet j}(X_{\frac{i-1}{n}})\Delta W_i^j+\nabla g_{\ell j}^{\top}(X_{\frac{i-1}{n}})f(X_{\frac{i-1}{n}})\Delta W_i^j+\frac{1}{2}g^{\top}(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\blacklozenge I_q\Delta W_i^j\Bigg],$$

$$\begin{aligned}\bar{\Gamma}^n_{\ell,t}(3) =&\frac{1}{2}\sum_{i=1}^{[nt]}\sum_{j=1}^{q}\Bigg[\nabla g_{\ell j}^\top(X_{\frac{i-1}{n}})\mathbb{H}(X_{\frac{i-1}{n}})\blacklozenge\sum_{\substack{1\le k,k'\le m\\ k'\neq k}}\left(\delta W_{ik'}\delta W_{ik'}^\top - I_q\frac{\Delta_n}{m}\right)\delta W_{ik}^j\\
&+\frac{1}{2}\Big[g^\top(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{\substack{1\le k,k'\le m\\ k'\neq k}}\left(\delta W_{ik'}\delta W_{ik'}^\top - I_q\frac{\Delta_n}{m}\right)\delta W_{ik}^j\\
&+\Big[\dot{h}^{n,i}_{\ell\bullet\bullet}\blacklozenge g_{\bullet j}(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{\substack{1\le k,k'\le m\\ k'\neq k}}(\delta W_{ik'}\delta W_{ik'}^\top - I_q\frac{\Delta_n}{m})\delta W_{ik}^j\Bigg],\end{aligned}$$

$$\begin{aligned}\bar{\Gamma}^n_{\ell,t}(4) = \frac{1}{2}\sum_{i=1}^{[nt]}\Bigg[&\sum_{j,j'=1}^{q}\nabla g_{\ell j}^\top(X_{\frac{i-1}{n}})\Big[\dot{g}_i^n\blacklozenge g_{\bullet j'}(X_{\frac{i-1}{n}})\Big]\sum_{1\le k''<k'<k\le m}\delta W_{ik'}(\delta W_{ik''}^{j'}\delta W_{ik}^j+\delta W_{ik''}^j\delta W_{ik}^{j'})\\
&+\sum_{j=1}^{q}\Big[g^\top(X_{\frac{i-1}{n}})\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge\sum_{1\le k''<k'<k\le m}\delta W_{ik'}(\delta W_{ik''}^\top\delta W_{ik}^j+\delta W_{ik''}^j\delta W_{ik}^\top)\Bigg].\end{aligned}$$

Proof. Concerning $\bar{\Gamma}^n(1)$, according to the definition of σ, we only use

$$\sum_{k=1}^{m}(m-k)\delta W_{i\sigma(k)}\delta W_{i\sigma(k)}^\top = \sum_{k=1}^{m}(k-1)\delta W_{ik}\delta W_{ik}^\top.$$

Similarly, we obtain $\bar{\Gamma}^n(2)$ using

$$\sum_{k=1}^{m}(m-k)\delta W_{i\sigma(k)} = \sum_{k=1}^{m}(k-1)\delta W_{ik} \quad\text{and}\quad \sum_{k=1}^{m}(k-1)\delta W_{i\sigma(k)} = \sum_{k=1}^{m-1}(m-k)\delta W_{ik}.$$

To get $\bar{\Gamma}^n(3)$, we use

$$\sum_{\substack{k,k'=1\\ k'<k}}^{m}\left(\delta W_{i\sigma(k')}\delta W_{i\sigma(k')}^\top - \Omega\frac{\Delta_n}{m}\right)\delta W_{i\sigma(k)}^j = \sum_{\substack{k,k'=1\\ k<k'}}^{m}\left(\delta W_{ik'}\delta W_{ik'}^\top - \Omega\frac{\Delta_n}{m}\right)\delta W_{ik}^j.$$

Finally, we obtain $\bar{\Gamma}^n(4)$ using

$$\sum_{k=3}^{m}\sum_{k'=2}^{k-1}\sum_{k''=1}^{k'-1}\delta W_{i\sigma(k'')}^j\delta W_{i\sigma(k')}^{j'}\delta W_{i\sigma(k)}^{j''} = \sum_{k=1}^{m-2}\sum_{k'=k+1}^{m-1}\sum_{k''=k'+1}^{m}\delta W_{ik''}^j\delta W_{ik'}^{j'}\delta W_{ik}^{j''}.$$

□

Now, thanks to the above lemma, (4.4.38) can be rewritten in a better way as follows

$$V_t^n = \sum_{i=1}^{[nt]}\bar{f}_i^n\blacklozenge V^n_{\frac{i-1}{n}}\Delta_n + \sum_{i=1}^{[nt]}\left(\bar{g}_i^n\blacklozenge V^n_{\frac{i-1}{n}}\right)\Delta W_i + \mathcal{M}_t^{n,2} + \mathcal{R}_t^{n,2},$$

where for $t \in [0,1]$,

$$\mathcal{M}_t^{n,2} = \sum_{r=1}^{4} \bar{\Gamma}_t^n(r) + \tilde{N}_t^{nm} + \tilde{M}_t^{nm,1} - \frac{1}{2}\tilde{M}_t^{nm,3}, \tag{4.5.2}$$

here we recall that for $r \in \{1,3\}$, $\tilde{M}_t^{nm,r} = \sum_{i=1}^{[nt]} \tilde{M}_{\frac{i-1}{n}}^{nm,r}$, $\tilde{N}_t^{nm} = \sum_{i=1}^{[nt]} \tilde{N}_{\frac{i-1}{n}}^{nm}$ with $\tilde{N}_{\frac{i-1}{n}}^{nm}$ and $(\tilde{M}_{\frac{i-1}{n}}^{nm,r})_{r\in\{1,3\}}$ are respectively given by (4.4.31),(4.4.32) and (4.4.34).
Now, in order to prove the convergence in law of the couple $(\mathcal{M}^{n,1}, \mathcal{M}^{n,2})$, we first need to study the asymptotic behavior of the distribution of the noises vector $(Z_0^n, Z_1^n, Z_2^n, Z_3^n)$, where $Z_0^n = (Z_0^{n,jj'})_{j,j'\in\{1,\dots,q\}}$, $Z_2^n = (Z_2^{n,jj'})_{j,j'\in\{1,\dots,q\}}$ are q^2-matrices

$$\begin{aligned} Z_{0,t}^{n,jj'} &= \sum_{i=1}^{[nt]}\sum_{k=1}^{m} \delta W_{ik}^j \delta W_{ik}^{j'}, \\ Z_{2,t}^{n,jj'} &= \sqrt{n}\sum_{i=1}^{[nt]} \sum_{1\le k<k'\le m} \left(\delta W_{ik}^j \delta W_{ik'}^{j'} - \delta W_{ik'}^j \delta W_{ik}^{j'}\right), \end{aligned}$$

and $Z_1^n = (Z_1^{n,jj'j''})_{j,j',j''\in\{1,\dots,q\}}$ and $Z_3^n = (Z_3^{n,jj'j''})_{j,j',j''\in\{1,\dots,q\}}$ are q^3-matrices

$$\begin{aligned} Z_{1,t}^{n,jj'j''} &= n\sum_{i=1}^{[nt]} \sum_{\substack{k'\neq k\\ 1\le k,k'\le m}} (\delta W_{ik'}^j \delta W_{ik'}^{j'} - \delta_{jj'}\frac{\Delta_n}{m})\delta W_{ik}^{j''}, \\ Z_{3,t}^{n,jj'j''} &= n\sum_{i=1}^{[nt]} \sum_{1\le k''<k'<k\le m} \delta W_{ik'}^{j'}(\delta W_{ik}^j \delta W_{ik''}^{j''} + \delta W_{ik''}^j \delta W_{ik}^{j''}). \end{aligned}$$

Lemma 4.5.2. *Let us consider the triangular arrays given by*

$$Z_{0,t}^n = \sum_{i=1}^{[nt]}\sum_{k=1}^{m} \delta W_{ik}\delta W_{ik}^\top, \quad \textit{with} \quad t\in[0,1].$$

$$\textit{As } n\to\infty, \textit{we have} \quad Z_0^n - Z_0 \xrightarrow{L^p} 0, \quad \textit{where} \quad Z_{0,t} = tI_q, \quad t\in[0,1].$$

Proof. For any fixed $t\in[0,1]$, we rewrite Z_0^n as follows

$$Z_{0,t}^n = \sum_{i=1}^{[nt]}\sum_{k=1}^{m} (\delta W_{ik}\delta W_{ik}^\top - \frac{\Delta_n}{m} I_q) + [nt]\Delta_n I_q.$$

As $\mathbb{E}(\sum_{k=1}^{m}(\delta W_{ik}\delta W_{ik}^\top - \frac{\Delta_n}{m}I_q)|\mathcal{F}_{\frac{i-1}{n}}) = 0$, then by the discrete BDG inequality and Jensen's inequality, there is a generic positive constant C such that

$$\begin{aligned} \mathbb{E}\left(\sup_{0\le t\le 1} |\sum_{i=1}^{[nt]}\sum_{k=1}^{m}(\delta W_{ik}\delta W_{ik}^\top - \frac{\Delta_n}{m}I_q)|^p\right) &= \mathbb{E}\left(\max_{0\le \ell\le n} |\sum_{i=1}^{\ell}\sum_{k=1}^{m}(\delta W_{ik}\delta W_{ik}^\top - \frac{\Delta_n}{m}I_q)|^p\right) \\ &\le C\mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=1}^{m} |\delta W_{ik}\delta W_{ik}^\top - \frac{\Delta_n}{m}I_q|^2\right)^{p/2} \end{aligned}$$

$$\leq C n^{p/2-1} \sum_{i=1}^{n} \sum_{k=1}^{m} \mathbb{E}|\delta W_{ik} \delta W_{ik}^{\top} - \frac{\Delta_n}{m} I_q|^p \leq C \Delta_n^{p/2}.$$

Then it follows that $\max_{0 \leq \ell \leq [nt]} \sum_{i=1}^{\ell} \sum_{k=1}^{m} (\delta W_{ik} \delta W_{ik}^{\top} - \frac{\Delta_n}{m} I_q) \xrightarrow{L^p} 0$. Thus, we get the convergence of Z^n using that $[nt]\Delta_n I_q \to t I_q$ as $n \to \infty$. □

Theorem 4.5.3. *Let us consider the scalar components of the triangular array triplet* (Z_1^n, Z_2^n, Z_3^n) *given by*

$\forall j, j', j'' \in \{1, \ldots, q\}$,

$$Z_{1,t}^{n,jj'j''} = \sum_{i=1}^{[nt]} \zeta_{i,1}^{n,jj'j''} \text{ where } \zeta_{i,1}^{n,jj'j''} = n \sum_{\substack{k \neq k' \\ 1 \leq k,k' \leq m}} \left(\delta W_{ik'}^{j} \delta W_{ik'}^{j'} - \delta_{jj'} \frac{\Delta_n}{m} \right) \delta W_{ik}^{j''},$$

$\forall j, j' \in \{1, \ldots, q\}$,

$$Z_{2,t}^{n,jj'} = \sum_{i=1}^{[nt]} \zeta_{i,2}^{n,jj'} \text{ where } \zeta_{i,2}^{n,jj'} = \sqrt{n} \sum_{1 \leq k < k' \leq m} \left(\delta W_{ik}^{j} \delta W_{ik'}^{j'} - \delta W_{ik'}^{j} \delta W_{ik}^{j'} \right),$$

$\forall j, j', j'' \in \{1, \ldots, q\}$,

$$Z_{3,t}^{n,jj'j''} = \sum_{i=1}^{[nt]} \zeta_{i,3}^{n,jj'j''} \text{ where } \zeta_{i,3}^{n,jj'j''} = n \sum_{1 \leq k'' < k' < k \leq m} \delta W_{ik'}^{j'} (\delta W_{ik}^{j} \delta W_{ik''}^{j''} + \delta W_{ik''}^{j} \delta W_{ik}^{j''}),$$

with $t \in [0,1]$ *and* $Z_{1,t}^n \in \mathbb{R}^{q^3}$, $Z_{2,t}^n \in \mathbb{R}^{q^2}$ *and* $Z_{3,t}^n \in \mathbb{R}^{q^3}$. *Then as* $n \to \infty$, *we have*

$$(W, Z_1^n, Z_2^n, Z_3^n) \overset{\text{stably}}{\Rightarrow} (W, Z_1, Z_2, Z_3),$$

where
$$Z_{1,t}^{jj'j''} = \begin{cases} \frac{\sqrt{m-1}}{m} B_{1,t}^{jj'j''} & , \ j > j' \\ \frac{\sqrt{2(m-1)}}{m} B_{1,t}^{jjj''} & , \ j = j' \\ \frac{\sqrt{m-1}}{m} B_{1,t}^{j'jj''} & , \ j < j' \end{cases}, \quad Z_{2,t}^{jj'} = \begin{cases} \sqrt{\frac{m-1}{m}} B_{2,t}^{jj'} & , \ j > j' \\ 0 & , \ j = j' \\ -\sqrt{\frac{m-1}{m}} B_{2,t}^{j'j} & , \ j < j' \end{cases}$$

$$\text{and} \quad Z_{3,t}^{jj'j''} = \begin{cases} \sqrt{\frac{(m-1)(m-2)}{3m^2}} B_{3,t}^{jj'j''} & , \ j > j'' \\ \sqrt{\frac{2(m-1)(m-2)}{3m^2}} B_{3,t}^{jj'j''} & , \ j = j'' \\ \sqrt{\frac{(m-1)(m-2)}{3m^2}} B_{3,t}^{j''j'j} & , \ j < j'' \end{cases}$$

with $(B_1^{jj'j''})_{\substack{1 \leq j,j',j'' \leq q \\ j \geq j'}}$ *and* $(B_3^{jj'j''})_{\substack{1 \leq j,j',j'' \leq q \\ j \geq j''}}$ *are two standard* $q^2(q+1)/2$*-dimensional Brownian motions and* $(B_2^{jj'})_{1 \leq j' < j \leq q}$ *is a standard* $q(q-1)/2$*-dimensional Brownian motion. Moreover, we have* B_1, B_2 *and* B_3 *are independent of* W *and also independent of each other. Furthermore, we have the* (UT) *of* Z_0^n, Z_1^n, Z_2^n *and* Z_3^n.

Remark 4.5.4. *It is worth noticing that when* $m = 2$ *the noise term* Z_3^n *vanishes at the limit.*

Proof of Theorem 4.5.3. We aim to use Theorem 3.2 of Jacod, 1997 (see in appendix Theorem 4.8.2) combined with some useful technical tools in the proof of Theorem 5.1 of Jacod and Protter, 1998. We split our proof into four main steps to check the four conditions of theorem 4.8.2.

Step 1 For all $j, j', j'' \in \{1, \ldots, q\}$, we have $\mathbb{E}(\zeta_{i,1}^{n,jj'j''}|\mathcal{F}_{\frac{i-1}{n}}) = \mathbb{E}(\zeta_{i,2}^{n,jj'}|\mathcal{F}_{\frac{i-1}{n}}) = \mathbb{E}(\zeta_{i,3}^{n,jj'j''}|\mathcal{F}_{\frac{i-1}{n}}) = 0$. Then the first condition (a) of theorem 4.8.2 is satisfied.

Step 2 For this step, we need to check the validity of condition (b) of theorem 4.8.2 for our three triangular arrays.

First triangular array. Using the symmetric structure of $\zeta_{i,1}^{n,jj'j''}$ it is sufficient to consider only the case $j \geq j'$. Now thanks to the independence between the increments, we have for all $i \in \{1, \ldots, n\}$,

$$\mathbb{E}((\zeta_{i,1}^{n,jj'j''})^2|\mathcal{F}_{\frac{i-1}{n}}) = \sum_{\substack{k \neq k' \\ 1 \leq k,k' \leq m}} n^2 \mathbb{E}\left(\left(\delta W_{ik'}^{j} \delta W_{ik'}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right)^2 (\delta W_{ik}^{j''})^2\right) +$$

$$\sum_{\substack{k_1 \neq k'_1, k_2 \neq k'_2 \\ 1 \leq k'_1, k'_2, k_1, k_2 \leq m \\ (k'_1, k_1) \neq (k'_2, k_2)}} n^2 \mathbb{E}\left(\left(\delta W_{ik'_1}^{j} \delta W_{i,k'_1}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \delta W_{ik_1}^{j''} \left(\delta W_{ik'_2}^{j} \delta W_{ik'_2}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \delta W_{ik_2}^{j''}\right).$$

Actually, the generic term of the above second sum is equal to zero. To check that we consider the three following subcases.

- If $k_1 \neq k_2$ and $k'_1 = k'_2$, then we can deduce that $k_1 \notin \{k'_1, k'_2, k_2\}$ and therefore the generic term is equal to

$$\mathbb{E}(\delta W_{ik_1}^{j''}) \mathbb{E}\left(\left(\delta W_{ik'_1}^{j} \delta W_{i,k'_1}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \left(\delta W_{ik'_2}^{j} \delta W_{ik'_2}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \delta W_{ik_2}^{j''}\right) = 0.$$

- If $k_1 = k_2$ and $k'_1 \neq k'_2$, then we can deduce that $k'_1 \notin \{k_1, k'_2, k_2\}$. Therefore, the generic term is equal to

$$\mathbb{E}\left(\delta W_{ik'_1}^{j} \delta W_{i,k'_1}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \mathbb{E}\left(\delta W_{ik_1}^{j''} \left(\delta W_{ik'_2}^{j} \delta W_{ik'_2}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \delta W_{ik_2}^{j''}\right) = 0.$$

- If $k_1 \neq k_2$ and $k'_1 \neq k'_2$, then we have two subsubcases:
  - If $k_1 = k'_2$, we have $k_1 \notin \{k'_1, k_2\}$. Then, the generic term is equal to

$$\mathbb{E}\left(\left(\delta W_{ik'_1}^{j} \delta W_{i,k'_1}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \delta W_{ik_2}^{j''}\right) \mathbb{E}\left(\delta W_{ik_1}^{j''} \left(\delta W_{ik_1}^{j} \delta W_{ik_1}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right)\right) = 0.$$

  - If $k_1 \neq k'_2$, we have $k_1 \notin \{k'_1, k'_2, k_2\}$. Then, the generic term is equal to

$$\mathbb{E}(\delta W_{ik_1}^{j''}) \mathbb{E}\left(\left(\delta W_{ik'_1}^{j} \delta W_{i,k'_1}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \left(\delta W_{ik'_2}^{j} \delta W_{ik'_2}^{j'} - \delta_{jj'}\frac{\Delta_n}{m}\right) \delta W_{ik_2}^{j''}\right) = 0.$$

It is worth noticing that the above arguments rely only on the independence between the increments without using the independence between the components of the Brownian vector. Concerning the generic term of the first sum, if $j = j'$ then it is equal to

$$\mathbb{E}\left((\delta W_{ik'}^{j})^2 - \frac{\Delta_n}{m}\right)^2 \mathbb{E}(\delta W_{ik}^{j''})^2 = \frac{2}{n^3 m^3}$$

and for $j > j'$ it is equal to $\mathbb{E}(\delta W^j_{ik'})^2\mathbb{E}(\delta W^{j'}_{ik'})^2\mathbb{E}(\delta W^{j''}_{ik})^2 = \frac{1}{n^3m^3}$. Thus, we get as $n \to \infty$

$$\sum_{i=1}^{[nt]} \mathbb{E}((\zeta^{n,jj'j''}_{i,1})^2|\mathcal{F}_{\frac{i-1}{n}}) \longrightarrow \begin{cases} \dfrac{m-1}{m^2}t, & j > j' \\ \dfrac{2(m-1)}{m^2}t & j = j'. \end{cases}$$

Now, it remains to check that for any $q \geq j \geq j' \geq 1$, $q \geq \bar{j} \geq \bar{j}' \geq 1$ and $j'', \bar{j}'' \in \{1,\dots,q\}$ s.t. $(j,j',j'') \neq (\bar{j},\bar{j}',\bar{j}'')$, we have $\sum_{i=1}^{[nt]} \mathbb{E}(\zeta^{n,jj'j''}_{i,1}\zeta^{n,\bar{j}\bar{j}'\bar{j}''}_{i,1}|\mathcal{F}_{\frac{i-1}{n}}) = 0$. To do so, we write

$$\begin{aligned}&\mathbb{E}(\zeta^{n,jj'j''}_{i,1}\zeta^{n,\bar{j}\bar{j}'\bar{j}''}_{i,1}|\mathcal{F}_{\frac{i-1}{n}}) \\ &= \sum_{\substack{1\leq k_1,k'_1,k_2,k'_2\leq m\\ k_1\neq k'_1, k_2\neq k'_2}} n^2\mathbb{E}\left(\left(\delta W^j_{ik'_1}\delta W^{j'}_{ik'_1} - \delta_{jj'}\frac{\Delta_n}{m}\right)\delta W^{j''}_{ik_1}\left(\delta W^{\bar{j}}_{ik'_2}\delta W^{\bar{j}'}_{ik'_2} - \delta_{\bar{j}\bar{j}'}\frac{\Delta_n}{m}\right)\delta W^{\bar{j}''}_{ik_2}\right).\end{aligned}$$

It is easy to check that the arguments given above to prove that this term vanishes remain valid for the particular case $(k_1,k'_1) \neq (k_2,k'_2)$ and this, as noticed above is independent of the choice of (j,j',j'') and $(\bar{j},\bar{j}',\bar{j}'')$. Thus, we only need to consider the case $(k_1,k'_1) = (k_2,k'_2)$. Therefore, by the independence between the increments we rewrite the generic term as follows

$$\mathbb{E}\left((\delta W^j_{ik'_1}\delta W^{j'}_{ik'_1} - \delta_{jj'}\frac{\Delta_n}{m})(\delta W^{\bar{j}}_{ik'_1}\delta W^{\bar{j}'}_{ik'_1} - \delta_{\bar{j}\bar{j}'}\frac{\Delta_n}{m})\right)\mathbb{E}\left(\delta W^{j''}_{ik_1}\delta W^{\bar{j}''}_{ik_1}\right). \tag{4.5.3}$$

Then, it is obvious that when $j'' \neq \bar{j}''$ this generic term vanishes. Now when $j'' = \bar{j}''$, thanks to its symmetric structure it is sufficient to consider only the case $j \neq \bar{j}$. For this we have three subcases.

- If $j > j'$ and $\bar{j} > \bar{j}'$ then $\delta_{jj'} = \delta_{\bar{j}\bar{j}'} = 0$ and we have two possibilities: either $j \neq \bar{j}'$ then the generic term rewrites $\mathbb{E}\left(\delta W^j_{ik'_1}\right)\mathbb{E}\left(\delta W^{j'}_{ik'_1}\delta W^{\bar{j}}_{ik'_1}\delta W^{\bar{j}'}_{ik'_1}\right) = 0$ or $j = \bar{j}'$ and then as $\bar{j} > \bar{j}' = j > j'$ the generic term rewrites $\mathbb{E}\left((\delta W^j_{ik'_1})^2\right)\mathbb{E}\left(\delta W^{j'}_{ik'_1}\delta W^{\bar{j}}_{ik'_1}\right) = 0$.
- If $j = j'$ and $\bar{j} > \bar{j}'$ or $j > j'$ and $\bar{j} = \bar{j}'$, by the symmetry we can consider only the first case and as $\bar{j} \notin \{j, j', \bar{j}'\}$ for which the generic term is equal to zero.
- If $j = j'$ and $\bar{j} = \bar{j}'$ then the generic term rewrites $\mathbb{E}(((\delta W^j_{ik'_1})^2 - \frac{\Delta_n}{m})((\delta W^{\bar{j}}_{ik'_1})^2 - \frac{\Delta_n}{m})) = 0$.

Second triangular array. Using the anti-symmetric structure of $\zeta^{n,jj'}_{i,2}$ it is also sufficient to consider only the case $j > j'$ as $\zeta^{n,jj}_{i,2} = 0$. Then, for $i \in \{1,\dots,n\}$, we have

$$\begin{aligned}\mathbb{E}((\zeta^{n,jj'}_{i,2})^2|\mathcal{F}_{\frac{i-1}{n}}) &= \sum_{1\leq k<k'\leq m} n\mathbb{E}\left(\left(\delta W^j_{ik}\delta W^{j'}_{ik'} - \delta W^j_{ik'}\delta W^{j'}_{ik}\right)^2\right) \\ &+ \sum_{\substack{1\leq k_1<k'_1\leq m\\ 1\leq k_2<k'_2\leq m\\ (k_1,k'_1)\neq(k_2,k'_2)}} n\mathbb{E}\left(\left(\delta W^j_{ik_1}\delta W^{j'}_{ik'_1} - \delta W^j_{ik'_1}\delta W^{j'}_{ik_1}\right)\left(\delta W^j_{ik_2}\delta W^{j'}_{ik'_2} - \delta W^j_{ik'_2}\delta W^{j'}_{ik_2}\right)\right).\end{aligned}$$

In the same way as the first triangular array, the generic term of the above second sum is also equal to zero. This follows easily by expanding this generic term and using the independence structure between the increments under conditions $(k_1, k'_1) \neq (k_2, k'_2)$, $k_1 < k'_1$ and $k_2 < k'_2$. Now, concerning the generic term of the first sum, as $j > j'$ it is easy to check that $\mathbb{E}\left(\left(\delta W^j_{ik}\delta W^{j'}_{ik'} - \delta W^j_{ik'}\delta W^{j'}_{ik}\right)^2\right) = \frac{2}{n^2m^2}$. Thus, as $n \to \infty$, $\sum_{i=1}^{[nt]} \mathbb{E}((\zeta^{n,jj'}_{i,2})^2|\mathcal{F}_{\frac{i-1}{n}}) \to \frac{m-1}{m}t$. Now, it remains to check that for any $q \geq j > j' \geq 1$, $q \geq \bar{j} > \bar{j}' \geq 1$ s.t. $(j,j') \neq (\bar{j},\bar{j}')$, $\sum_{i=1}^{[nt]} \mathbb{E}(\zeta^{n,jj'}_{i,2}\zeta^{n,\bar{j}\bar{j}'}_{i,2}|\mathcal{F}_{\frac{i-1}{n}}) = 0$. So, we have

$$\begin{aligned}
&\mathbb{E}(\zeta^{n,jj'}_{i,2}\zeta^{n,\bar{j}\bar{j}'}_{i,2}|\mathcal{F}_{\frac{i-1}{n}}) \\
&= \sum_{\substack{1\leq k_1<k'_1\leq m\\ 1\leq k_2<k'_2\leq m}} n\mathbb{E}((\delta W^j_{i,k_1}\delta W^{j'}_{i,k'_1} - \delta W^j_{i,k'_1}\delta W^{j'}_{i,k_1})(\delta W^{\bar{j}}_{ik_2}\delta W^{\bar{j}'}_{ik'_2} - \delta W^{\bar{j}}_{ik'_2}\delta W^{\bar{j}'}_{ik_2})).
\end{aligned}$$

When $(k_1, k'_1) \neq (k_2, k'_2)$ by similar arguments as for the first triangular array the generic term of the above sum is equal to zero thanks to the independence between the increments. When $(k_1, k'_1) = (k_2, k'_2)$ $(j, j') \neq (\bar{j}, \bar{j}')$, we need to treat two cases.

- If $j = \bar{j}$ and $j' \neq \bar{j}'$ as $j > j'$ we have $j' \notin \{j, \bar{j}, \bar{j}'\}$ and consequently the generic term is equal to zero. If $j \neq \bar{j}$ and $j' = \bar{j}'$ we use similar arguments to prove that the generic term is zero.
- If $j \neq \bar{j}$, $j' \neq \bar{j}'$ we have two possibilities: either $j \neq \bar{j}'$ then $j \notin \{j', \bar{j}, \bar{j}'\}$ or $\bar{j} > \bar{j}' = j > j'$ and in both cases it is obvious that the generic term also vanishes.

Third triangular array. Using the symmetric structure of $\zeta^{n,jj'j''}_{i,3}$ it is also sufficient to consider only the case $j \geq j''$. Then, for $i \in \{1, \ldots, n\}$, we have

$$\begin{aligned}
\mathbb{E}((\zeta^{n,jj'j''}_{i,3})^2|\mathcal{F}_{\frac{i-1}{n}}) &= \sum_{1\leq k''<k'<k\leq m} n^2\mathbb{E}\left(\delta W^{j'}_{ik'}(\delta W^j_{ik}\delta W^{j''}_{ik''} + \delta W^j_{ik''}\delta W^{j''}_{ik})\right)^2 \\
&+ \sum_{\substack{1\leq k''_1<k'_1<k_1\leq m\\ 1\leq k''_2<k'_2<k_2\leq m\\ (k_1,k'_1,k''_1)\neq(k_2,k'_2,k''_2)}} n^2\mathbb{E}\left(\delta W^{j'}_{ik'_1}(\delta W^j_{ik_1}\delta W^{j''}_{ik''_1} + \delta W^j_{ik''_1}\delta W^{j''}_{ik_1})\delta W^{j'}_{ik'_2}(\delta W^j_{ik_2}\delta W^{j''}_{ik''_2} + \delta W^j_{ik''_2}\delta W^{j''}_{ik_2})\right).
\end{aligned}$$

Similarly as for the first triangular array, we use the independence structure between the increments under conditions $(k_1, k'_1, k''_1) \neq (k_2, k'_2, k''_2)$, $1 \leq k''_1 < k'_1 < k_1 \leq m$ and $1 \leq k''_2 < k'_2 < k_2 \leq m$ to check that the generic term of the second sum is also equal to zero. Now, concerning the generic term of the first sum, if $j = j''$ then it is equal to

$$4\mathbb{E}(\delta W^{j'}_{ik'})^2\mathbb{E}(\delta W^j_{ik})^2\mathbb{E}(\delta W^j_{ik''})^2 = \frac{4}{n^3m^3}$$

and for $j > j'$ it is equal to

$$\mathbb{E}(\delta W^{j'}_{ik'})^2(\mathbb{E}(\delta W^j_{ik})^2\mathbb{E}(\delta W^{j''}_{ik''})^2 + \mathbb{E}(\delta W^j_{ik''})^2\mathbb{E}(\delta W^{j''}_{ik})^2) = \frac{2}{n^3m^3}.$$

Thus, we get as $n\to\infty$

$$\sum_{i=1}^{[nt]}\mathbb{E}((\zeta_{i,3}^{n,jj'j''})^2|\mathcal{F}_{\frac{i-1}{n}})\longrightarrow\begin{cases}\dfrac{(m-1)(m-2)}{3m^2}t & j>j''\\[2ex] \dfrac{2(m-1)(m-2)}{3m^2}t & j=j''.\end{cases}$$

Now, it remains to check that for any $q\geq j\geq j''\geq 1$, $q\geq\bar{j}\geq\bar{j}''\geq 1$ and $j',\bar{j}'\in\{1,\dots,q\}$ s.t. $(j,j',j'')\neq(\bar{j},\bar{j}',\bar{j}'')$, we have $\sum_{i=1}^{[nt]}\mathbb{E}(\zeta_{i,3}^{n,jj'j''}\zeta_{i,3}^{n,j\bar{j}'\bar{j}''}|\mathcal{F}_{\frac{i-1}{n}})=0$. To do so, we write

$$\begin{aligned}&\mathbb{E}(\zeta_{i,3}^{n,jj'j''}\zeta_{i,3}^{n,j\bar{j}'\bar{j}''}|\mathcal{F}_{\frac{i-1}{n}})\\&=n^2\sum_{\substack{1\leq k''_1<k'_1<k_1\leq m\\1\leq k''_2<k'_2<k_2\leq m}}\mathbb{E}\left(\delta W^{j'}_{ik'_1}(\delta W^{j}_{ik_1}\delta W^{j''}_{ik''_1}+\delta W^{j}_{ik''_1}\delta W^{j''}_{ik_1})\delta W^{j'}_{ik'_2}(\delta W^{j}_{ik_2}\delta W^{j''}_{ik''_2}+\delta W^{j}_{ik''_2}\delta W^{j''}_{ik_2})\right).\end{aligned}$$

When $(k''_1,k'_1,k_1)\neq(k''_2,k'_2,k_2)$, by similar arguments as for the first triangular array it is easy to check that the generic term of the above sum is equal to zero. When $(k''_1,k'_1,k_1)=(k''_2,k'_2,k_2)$, with the condition $(j,j',j'')\neq(\bar{j},\bar{j}',\bar{j}'')$, the generic term equals to

$$\mathbb{E}(\delta W^{j'}_{ik'_1}\delta W^{\bar{j}'}_{ik'_1})\mathbb{E}((\delta W^{j''}_{ik_1}\delta W^{j}_{ik''_1}+\delta W^{j}_{ik_1}\delta W^{j''}_{ik''_1})(\delta W^{\bar{j}''}_{i,k_1}\delta W^{\bar{j}}_{ik''_1}+\delta W^{\bar{j}}_{ik_1}\delta W^{\bar{j}''}_{ik''_1})).$$

By the same arguments used to treat (4.5.3) we easily deduce that the above generic term vanishes.

Covariance between of the different triangular arrays. For any j,j',j'' and $\bar{j},\bar{j}'$ in $\{1,\dots,q\}$ $j\geq j'$, $\bar{j}>\bar{j}'$, we have

$$\begin{aligned}&\mathbb{E}(\zeta_{i,1}^{n,jj'j''}\zeta_{i,2}^{n,\bar{j}\bar{j}'}|\mathcal{F}_{\frac{i-1}{n}})\\&=n\sqrt{n}\sum_{\substack{1\leq k_1,k'_1,k_2,k'_2\leq m\\k_1\neq k'_1\\k_2<k'_2}}\mathbb{E}((\delta W^{j}_{ik_1}\delta W^{j'}_{ik_1}-\delta_{jj'}\frac{\Delta_n}{m})\delta W^{j''}_{ik'_1}(\delta W^{\bar{j}}_{ik_2}\delta W^{\bar{j}'}_{ik'_2}-\delta W^{\bar{j}}_{ik'_2}\delta W^{\bar{j}'}_{ik_2})).\end{aligned}$$

For any j,j',j'' and $\bar{j},\bar{j}',\bar{j}''$ in $\{1,\dots,q\}$ $j\geq j'$, $\bar{j}\geq\bar{j}''$, we have

$$\begin{aligned}&\mathbb{E}(\zeta_{i,1}^{n,jj'j''}\zeta_{i,3}^{n,\bar{j}\bar{j}'\bar{j}''}|\mathcal{F}_{\frac{i-1}{n}})\\&=n^2\sum_{\substack{1\leq k_1,k'_1\leq m\\k_1\neq k'_1\\1\leq k''_2<k'_2<k_2\leq m}}\mathbb{E}((\delta W^{j}_{ik_1}\delta W^{j'}_{ik_1}-\delta_{jj'}\frac{\Delta_n}{m})\delta W^{j''}_{ik'_1}\delta W^{\bar{j}'}_{ik'_2}(\delta W^{\bar{j}}_{ik_2}\delta W^{\bar{j}''}_{ik''_2}+\delta W^{\bar{j}''}_{ik_2}\delta W^{\bar{j}}_{ik''_2})).\end{aligned}$$

For any j,j',j'' and $\bar{j},\bar{j}'\in\{1,\dots,q\}$ $j\geq j''$, $\bar{j}>\bar{j}'$, we have

$$\begin{aligned}&\mathbb{E}(\zeta_{i,3}^{n,jj'j''}\zeta_{i,2}^{n,\bar{j}\bar{j}'}|\mathcal{F}_{\frac{i-1}{n}})\\&=n\sqrt{n}\sum_{\substack{1\leq k''_1<k'_1<k_1\leq m\\1\leq k_2<k'_2\leq m}}\mathbb{E}(\delta W^{j'}_{ik'_1}(\delta W^{j}_{ik_1}\delta W^{j''}_{ik''_1}+\delta W^{j''}_{ik_1}\delta W^{j}_{ik''_1})(\delta W^{\bar{j}}_{ik_2}\delta W^{\bar{j}'}_{ik'_2}-\delta W^{\bar{j}'}_{ik'_2}\delta W^{\bar{j}'}_{ik_2})).\end{aligned}$$

When developing the above three generic terms, we notice that we always have a product of an odd number of increments of the Brownian motion. Then, combining this together with the independence structure between the increments, we easily get $\sum_{i=1}^{[nt]}\mathbb{E}(\zeta_{i,\alpha}^{n,jj'j''}\zeta_{i,\beta}^{n,\bar{j}\bar{j}'}|\mathcal{F}_{\frac{i-1}{n}})=0,$ for all $\alpha,\beta\in\{1,2,3\}$ with $\alpha\neq\beta$.

Step 3 Independence with respect to the original Brownian motion We check the condition (c) of theorem 4.8.2.

The first triangular array For any j,j',j'' and j_1 in $\{1,\dots,q\}$, $j\geq j'$, using the independence between the increments, we have

$$\begin{aligned}\mathbb{E}(\zeta_{i,1}^{n,jj'j''}\Delta W_i^{j_1}|\mathcal{F}_{\frac{i-1}{n}}) &= \sum_{\substack{1\leq k_1,k_2\leq m\\ k_1\neq k_2}} n\mathbb{E}((\delta W_{ik_1}^{j}\delta W_{ik_1}^{j'}-\delta_{jj'}\frac{\Delta_n}{m})\delta W_{ik_2}^{j''}(\delta W_{ik_1}^{j_1}+\delta W_{ik_2}^{j_1}))\\ &= \sum_{\substack{1\leq k_1,k_2\leq m\\ k_1\neq k_2}} n\mathbb{E}((\delta W_{ik_1}^{j}\delta W_{ik_1}^{j'}-\delta_{jj'}\frac{\Delta_n}{m})\delta W_{ik_1}^{j_1})\mathbb{E}(\delta W_{ik_2}^{j''})\\ &+\sum_{\substack{1\leq k_1,k_2\leq m\\ k_1\neq k_2}} n\mathbb{E}((\delta W_{ik_1}^{j}\delta W_{ik_1}^{j'}-\delta_{jj'}\frac{\Delta_n}{m}))\mathbb{E}(\delta W_{ik_2}^{j''}\delta W_{ik_2}^{j_1})=0.\end{aligned}$$

The second triangular array For any j,j' and j_1 in $\{1,\dots,q\}$, it is straight forward that

$$\mathbb{E}(\zeta_{i,2}^{n,jj'}\Delta W_i^{j_1}|\mathcal{F}_{\frac{i-1}{n}})=\sum_{\substack{1\leq k_1,k_2\leq m\\ k_1<k_2}}\sqrt{n}\mathbb{E}((\delta W_{ik_1}^{j}\delta W_{ik_2}^{j'}-\delta W_{ik_2}^{j}\delta W_{ik_1}^{j'})(\delta W_{ik_1}^{j_1}+\delta W_{ik_2}^{j_1}))=0$$

since when developping the generic term of the above sum we always have the expectation of a product of an odd number of the Brownian increments.

The third triangular array For any j,j',j'', $j\geq j''$ and j_1 in $\{1,\dots,q\}$, using the independence between the different increments we have

$$\begin{aligned}&\mathbb{E}(\zeta_{i,3}^{n,jj'j''}\Delta W_i^{j_1}|\mathcal{F}_{\frac{i-1}{n}})\\ &\quad=n\sum_{1\leq k''<k'<k\leq m}\mathbb{E}(\delta W_{ik'}^{j'}(\delta W_{ik}^{j}\delta W_{ik''}^{j''}+\delta W_{ik''}^{j}\delta W_{ik}^{j''})(\delta W_{ik'}^{j_1}+\delta W_{ik}^{j_1}+\delta W_{ik''}^{j_1}))=0.\end{aligned}$$

Step 4 (Lyapunov's condition) Now we check condition (d) of theorem 4.8.2.

First triangular array. For any $j,j',j''\in\{1,\dots,q\}$, $j\geq j'$, we prove that $\sum_{i=1}^{[nt]}\mathbb{E}(|\zeta_{i,1}^{n,jj'j''}|^4|\mathcal{F}_{\frac{i-1}{n}})$ tends to 0 when $n\to\infty$. In fact, using the convexity property of the function $x\mapsto x^4$ we note first that there is a constant $C_q>0$ depending only on q such that

$$\mathbb{E}(|\zeta_{i,1}^{n,jj'j''}|^4|\mathcal{F}_{\frac{i-1}{n}})\leq C_q\sum_{\substack{1\leq k_1,k_2\leq q\\ k_1\neq k_2}} n^4\mathbb{E}\left((\delta W_{ik_1}^{j}\delta W_{ik_1}^{j'}-\delta_{jj'}\Delta_n/m)^4(\delta W_{ik_2}^{j''})^4\right).$$

Then by the scaling property of the Brownian motion it is easy to check that there is a constant $C_m > 0$ depending only on m such that for all $j, j', j'' \in \{1, \dots, q\}$, $j \geq j'$ and $1 \leq k_1, k_2 \leq q$ with $k_1 \neq k_2$, we have $\mathbb{E}\left((\delta W^j_{ik_1}\delta W^{j'}_{ik_1} - \delta_{jj'}\Delta_n/m)^4(\delta W^{j''}_{ik_2})^4\right) \leq \frac{C_m}{n^6}$.

Second triangular array. Similarly, for any $j, j' \in \{1, \dots, q\}$, $j > j'$, there is a constant $C_q > 0$ depending only on q such that

$$\mathbb{E}(|\zeta^{n,jj'}_{i,2}|^4|\mathcal{F}_{\frac{i-1}{n}}) \leq C_q \sum_{\substack{1\leq k_1,k_2\leq q\\ k_1<k_2}} n^2\mathbb{E}\left((\delta W^j_{ik_1}\delta W^{j'}_{ik_2} - \delta W^{j'}_{ik_1}\delta W^j_{ik_2})^4\right)$$

and we deduce the result using the estimate $\mathbb{E}\left((\delta W^j_{ik_1}\delta W^{j'}_{ik_2} - \delta W^{j'}_{ik_1}\delta W^j_{ik_2})^4\right) \leq \frac{C_m}{n^4}$ where C_m is a positive constant depending only on m.

Third triangular array. In the same way we get that
$\mathbb{E}\left(\delta W^{j'}_{ik'}(\delta W^j_{ik}\delta W^{j''}_{ik''} + \delta W^j_{ik''}\delta W^{j''}_{ik})\right)^4 \leq \frac{C_m}{n^6}$ for $C_m > 0$. □

Now we are ready to prove the convergence in law of the couple of main terms $(\mathcal{M}^{n,1}, \mathcal{M}^{n,2})$ given by (4.5.1) and (4.5.2). The following proposition is the core of our main result Theorem 4.3.2.

Proposition 4.5.5. *As $n \to \infty$, we have*

$$(\sqrt{n}\mathcal{M}^{n,1}, n\mathcal{M}^{n,2}) \overset{\text{stably}}{\Rightarrow} (\mathcal{M}_1, \mathcal{M}_2), \tag{4.5.4}$$

where for $\ell \in \{1, \dots, d\}$, the ℓ^{th} components of $\mathcal{M}_1$ and $\mathcal{M}_2$ are given by

$$\mathcal{M}^\ell_{1,t} = -2\int_0^t h_{\ell\bullet\bullet}(X_s) \blacklozenge dZ_{2,s} \text{ and } \mathcal{M}^\ell_{2,t} = \sum_{r=1}^4 \bar{\Gamma}_{\ell,t}(r) + \tilde{N}_{\ell,t} + \tilde{M}^1_{\ell,t} - \frac{1}{2}\tilde{M}^3_{\ell,t}, \quad t \in [0,1],$$

with

$$\bar{\Gamma}_{\ell,t}(1) = \frac{m-1}{2m}\int_0^t \left(\nabla f_\ell^\top(X_s)f(X_s) + \frac{1}{2}\sum_{j,j'=1}^q g^\top_{\bullet j}(X_s)\nabla^2 f_\ell(X_s)g_{\bullet j'}(X_s)\right)ds$$

$$\bar{\Gamma}_{\ell,t}(2) = \frac{m-1}{2m}\sum_{j=1}^q\int_0^t \left(\nabla f_\ell^\top(X_s)g_{\bullet j}(X_s) + \nabla g^\top_{\ell j}(X_s)f(X_s) + \frac{1}{2}g^\top(X_s)\nabla^2 g_{\ell j}(X_s)g(X_s) \blacklozenge I_q\right)dW^j_s$$

$$\bar{\Gamma}_{\ell,t}(3) = \frac{1}{2}\sum_{j=1}^q\int_0^t \left[\nabla g^\top_{\ell j}(X_s)\mathbb{H}(X_s) + \frac{1}{2}g^\top(X_s)\nabla^2 g_{\ell j}(X_s)g(X_s) + \dot{h}^s_{\ell\bullet\bullet} \blacklozenge g_{\bullet j}(X_s)\right] \blacklozenge dZ^{\bullet\bullet j}_{1,s}$$

$$\bar{\Gamma}_{\ell,t}(4) = \frac{1}{2}\sum_{j,j'=1}^q\int_0^t \nabla g^\top_{\ell j}(X_s)[\dot{g}^s \blacklozenge g_{\bullet j''}(X_s)]dZ^{j\bullet j'}_{3,s} + \frac{1}{2}\sum_{j=1}^q\int_0^t [g^\top(X_s)\nabla^2 g_{\ell j}(X_s)g(X_s)] \blacklozenge dZ^{j\bullet\bullet}_{3,s}$$

$$\tilde{N}_{\ell,t} = \int_0^t \frac{1}{8}U_s^\top\nabla^2 f_\ell(X_s)U_s ds$$

$$\tilde{M}^1_{\ell,t} = \sum_{j=1}^q\int_0^t \frac{1}{8}U_s^\top\nabla^2 g_{\ell j}(X_s)U_s dW^j_s$$

$$\tilde{M}^3_{\ell,t} = \int_0^t (\dot{h}^s_{\ell\bullet\bullet} \blacklozenge U_s) \blacklozenge dZ_{2,s},$$

where $\dot{g}^s \in (\mathbb{R}^{d\times 1})^{d\times q}$ is a block matrix such that for $\ell \in \{1,\dots,d\}$, $j \in \{1,\dots,q\}$, the ℓj-th block is given by $(\dot{g}^s)_{\ell j} = \nabla g_{\ell j}(X_s)$, $s \in [0,t]$ and $\dot{h}^s_{\ell\bullet\bullet} \in (\mathbb{R}^{d\times 1})^{q\times q}$ is a random block matrix such that for j and $j' \in \{1,\dots,q\}$, the jj'-th block is given by $(\dot{h}^s_{\ell\bullet\bullet})_{jj'} = \nabla h_{\ell jj'}(X_s) \in \mathbb{R}^{d\times 1}$, $s \in [0,t]$. Here, Z_1, Z_2 and Z_3 are defined above in Theorem 4.5.3 and for any $j, j'' \in \{1,\dots,q\}$, for $r \in \{1,3\}$, we denote

$$Z^{\bullet\bullet j}_{r,s} = \begin{pmatrix} Z^{11j}_{r,s} & \dots & Z^{1qj}_{r,s} \\ \vdots & \ddots & \vdots \\ Z^{q1j}_{r,s} & \dots & Z^{qqj}_{r,s} \end{pmatrix} \quad \textit{and} \quad Z^{j\bullet j''}_{r,t} = (Z^{j1j''}_{r,t}, \dots, Z^{jqj''}_{r,t})^\top.$$

Proof. At first, let us denote $\rho^n = (W, Z^n_0, Z^n_1, Z^n_2, Z^n_3)$. From Lemma 4.5.2 and Theorem 4.5.3 combined with Lemma 4.8.4, we deduce that $\rho^n \overset{\text{stably}}{\Rightarrow} \rho$, as $n \to \infty$ with $\rho = (W, Z_0, Z_1, Z_2, Z_3)$. Besides, as the coefficients $\dot{f}^n_i$ and $\dot{g}^n_i$ are functions of vector points lying between $X^{nm}_{\frac{i-1}{n}}$ and $X^{nm,\sigma}_{\frac{i-1}{n}}$, the equation (4.4.37) can be rewritten into the following continuous form

$$\sqrt{n}U^n_t = \sum_{j=0}^{q} \int_0^{\eta_n(t)} \dot{F}^{n,j}_{\eta_n(s)} \blacklozenge \sqrt{n}U^n_{\eta_n(s)} dY^j_s - \int_0^{\eta_n(t)} (\mathbb{H}(X^{nm}_{\eta_n(s)}) + \mathbb{H}(X^{nm,\sigma}_{\eta_n(s)})) \blacklozenge dZ^n_{2,s} + \sqrt{n}\mathcal{R}^{n,1}_t,$$

where

$$\dot{F}^{n,j}_{\frac{i-1}{n}} = \begin{cases} \dot{f}^n_i, & j = 0 \\ (\dot{g}^n_i)_{\bullet j}, & j \in \{1,\dots,q\} \end{cases}, \quad \text{where } (\dot{g}^n_i)_{\bullet j} = ((\dot{g}^n_i)_{1j}, \dots, (\dot{g}^n_i)_{dj})^\top.$$

Here we used that $\int_{\frac{i-1}{n}}^{\frac{i}{n}} dZ^n_{2,s} = Z^n_{2,\frac{i}{n}} - Z^n_{2,\frac{i-1}{n}}$ and $Y_t = (t, W^1_t, \dots, W^q_t)^\top$. Thanks to lemmas 4.2.1 and 4.3.1, under assumption ($\mathbf{H}_{f,g}$) the process $(\mathbb{H}(X^{nm}) + \mathbb{H}(X^{nm,\sigma})) - (2\mathbb{H}(X)) \overset{L^p}{\to} 0$. Then, since ρ^n is (UT) (see Theorem 4.5.3) we deduce thanks to Theorem 4.8.5 that as $n \to \infty$

$$(\rho^n, \sqrt{n}\mathcal{M}^{n,1}) = (\rho^n, \int (\mathbb{H}(X^{nm}_{\eta_n(s)}) + \mathbb{H}(X^{nm,\sigma}_{\eta_n(s)})) \blacklozenge dZ^n_{2,s}) \overset{\text{stably}}{\Rightarrow} (\rho, \int 2\mathbb{H}(X_s) \blacklozenge dZ_{2,s}).$$

Moreover, under assumption ($\mathbf{H}_{f,g}$) by lemmas 4.2.1 and 4.3.1, it is straightforward that for any $j \in \{0,\dots,q\}$, $\int \dot{F}^{n,j}_{\eta_n(s)} \blacklozenge \mathbb{1}_d dY^j_s - \int \dot{F}^j_s \blacklozenge \mathbb{1}_d dY^j_s \overset{L^p}{\to} 0$, with $\mathbb{1}_d = (1,\dots,1)^\top$. Thus, by Lemma 4.8.4 we deduce that as $n \to \infty$

$$(\rho^n, \sqrt{n}\mathcal{M}^{n,1}, \int \dot{F}^{n,j}_{\eta_n(s)} \blacklozenge \mathbb{1}_d dY^j_s) \overset{\text{stably}}{\Rightarrow} (\rho, \int 2\mathbb{H}(X_s) \blacklozenge dZ_{2,s}, \int \dot{F}^j_s \blacklozenge \mathbb{1}_d dY^j_s),$$

with $\dot{F}^0_s = \nabla f(X_s)$ and for any $j \in \{1,\dots,q\}$, $\dot{F}^j_s = \nabla g_{\bullet j}(X_s)$. Therefore, by Lemma 4.4.11 and Theorem 4.8.6 we get that

$$(\rho^n, \sqrt{n}\mathcal{M}^{n,1}, \sqrt{n}U^n) \overset{\text{stably}}{\Rightarrow} (\rho, J, U), \quad \text{as } n \to \infty. \tag{4.5.5}$$

Now let us recall that (4.4.31), (4.4.32) and (4.4.34), can be rewritten into a continuous form

$$n\tilde{N}^{nm}_{\ell,t} = \int_0^{\eta_n(t)} \frac{1}{16} \sqrt{n}U^n_{\eta_n(t)}{}^\top \left(\nabla^2 f_\ell(\zeta^{n,1}_{\eta_n(t)}) + \nabla^2 f_\ell(\zeta^{n,2}_{\eta_n(t)})\right) \sqrt{n}U^n_{\eta_n(t)} ds,$$

$$n\tilde{M}^{nm,1}_{\ell,t} = \int_0^{\eta_n(t)} \frac{1}{16}\sum_{j'=1}^{q} \sqrt{n}U^n_{\eta_n(t)}{}^\top \left(\nabla^2 g_{\ell j'}(\zeta^{n,3}_{\eta_n(t)}) + \nabla^2 g_{\ell j'}(\zeta^{n,4}_{\eta_n(t)})\right) \sqrt{n}U^n_{\eta_n(t)} dW^{j'}_s,$$
$$n\tilde{M}^{nm,3}_{\ell,t} = \int_0^{\eta_n(t)} \left[\dot{h}^{n,n\eta_n(t)+1,2}_{\ell\bullet\bullet} \blacklozenge \sqrt{n}U^n_{\eta_n(t)}\right] \blacklozenge dZ^n_{2,s}.$$

Under the assumption ($\mathbf{H}_{f,g}$) and thanks to lemmas 4.2.1 and 4.3.1 combined with (4.5.5), we deduce by Theorem 4.8.5 that

$$(\rho^n, \sqrt{n}\mathcal{M}^{n,1}, n\tilde{N}^{nm}, n\tilde{M}^{nm,1}, n\tilde{M}^{nm,3}) \overset{\text{stably}}{\Rightarrow} (\rho, J, \tilde{N}, \tilde{M}^1, \tilde{M}^3) \quad \text{as } n \to \infty.$$

Similarly, by rewriting $\bar{\Gamma}^n(r)$, $r \in \{1,\ldots,4\}$ in continuous forms we deduce by Theorem 4.8.5

$$(\rho^n, \sqrt{n}\mathcal{M}^{n,1}, n\tilde{N}^{nm}, n\tilde{M}^{nm,1}, n\tilde{M}^{nm,3}, \bar{\Gamma}^n(1), \bar{\Gamma}^n(2), \bar{\Gamma}^n(3), \bar{\Gamma}^n(4))$$
$$\overset{\text{stably}}{\Rightarrow} (\rho, J, \tilde{N}, \tilde{M}^1, \tilde{M}^3, \bar{\Gamma}(1), \bar{\Gamma}(2), \bar{\Gamma}(3), \bar{\Gamma}(4)) \quad \text{as } n \to \infty,$$

where for $i = 1,\ldots,4$ and $1 \le \ell \le d$ the ℓ-th component of the process $\bar{\Gamma}(i)$ is given by the process $\bar{\Gamma}_\ell(i)$. This completes the proof. □

4.6 Appendix A: Proofs concerning analysis of U^n and V^n

Proof of Lemma 4.4.11. By (4.4.37), we recall that

$$\mathcal{R}^{n,1}_t = \sum_{i=1}^{[nt]} (\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n)$$
$$+ \sum_{i=1}^{[nt]} (M^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - M^{nm,\sigma}_{\frac{i-1}{n}}) + \sum_{i=1}^{[nt]} (N^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - N^{nm,\sigma}_{\frac{i-1}{n}}).$$

At first, it is obvious that $\mathbb{E}(\sqrt{n}\sum_{i=1}^{[nt]}(\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n) | \mathcal{F}_{\frac{i-1}{n}}) = 0$. Then, by the discrete BDG inequality combined with Lemma 4.3.1 and assumption ($\mathbf{H}_{f,g}$), there is a generic positive constant C such that

$$n^{p/2}\mathbb{E}\left(\sup_{0\le t\le 1} |\sum_{i=1}^{[nt]} (\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n)|^p\right)$$
$$\le Cn^{p/2}\mathbb{E}\left(\sum_{i=1}^{n} |\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})|^2 |\Delta W_i \Delta W_i^\top - I_q\Delta_n|^2\right)^{p/2}$$
$$\le Cn^{p-1}\sum_{i=1}^{n} \mathbb{E}|\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})|^p \mathbb{E}|\Delta W_i \Delta W_i^\top - I_q\Delta_n|^p \le C\Delta_n^{p/2}.$$

Then the process $\sqrt{n}\sum_{i=1}^{[n.]}(\mathbb{H}(X^{nm}_{\frac{i-1}{n}}) - \mathbb{H}(X^{nm,\sigma}_{\frac{i-1}{n}})) \blacklozenge (\Delta W_i \Delta W_i^\top - I_q\Delta_n) \overset{L^p}{\to} 0$. In the same way as above, we use the discrete BDG inequality and (4.4.11), there exists a positive constant C such that

$$\mathbb{E}(\sup_{0\le t\le 1} |\sum_{i=1}^{[nt]} M^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - M^{nm,\sigma}_{\frac{i-1}{n}}|^p) \le Cn^{p/2-1}\sum_{i=1}^{n} \mathbb{E}(|M^{nm,\mathrm{Id}}_{\frac{i-1}{n}} - M^{nm,\sigma}_{\frac{i-1}{n}}|^p) \le C\Delta_n^p.$$

Therefore, we obtain also the convergence of the process $\sqrt{n}\sum_{i=1}^{[n.]}(M^{nm,\mathrm{Id}}_{\frac{i-1}{n}}-M^{nm,\sigma}_{\frac{i-1}{n}}) \overset{L^p}{\to} 0$. Now, by (4.4.12), we have $\mathbb{E}(\sup_{0\le t\le 1}|\sum_{i=1}^{[nt]}(N^{nm,\mathrm{Id}}_{\frac{i-1}{n}}-N^{nm,\sigma}_{\frac{i-1}{n}})|^p)$ is bounded by

$$n^{p-1}\sum_{i=1}^{n}\mathbb{E}(|N^{nm,\mathrm{Id}}_{\frac{i-1}{n}}-N^{nm,\sigma}_{\frac{i-1}{n}}|^p)\le n^{p-1}\sum_{i=1}^{n}2K_p\Delta_n^{2p}=2K_p\Delta_n^p.$$

Thus, we get $\sqrt{n}\sum_{i=1}^{[n.]}(N^{nm,\mathrm{Id}}_{\frac{i-1}{n}}-N^{nm,\sigma}_{\frac{i-1}{n}}) \overset{L^p}{\to} 0$. Finally, we get the (UT) of Z_0^n, Z_1^n, Z_2^n and Z_3^n thanks to Lemma 4.8.1. □

Proof of Lemma 4.4.12. By (4.4.40), we recall that

$$\mathcal{R}^{n,2}_t=\frac{1}{2}\sum_{\tilde\sigma\in\{\mathrm{Id},\sigma\}}\sum_{i=1}^{[nt]}\left(\tilde R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(0)+\sum_{r=0}^{3}R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(r)\right)+\tilde M^{nm,2}_t$$
$$+\sum_{i=1}^{[nt]}(\mathbb{H}(\bar X^{nm,\sigma}_{\frac{i-1}{n}})-\mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n).$$

At first, thanks to Jensen's inequality and (4.4.15), there is a positive constant C such that

$$n^p\mathbb{E}(\sup_{0\le t\le 1}|\sum_{i=1}^{[nt]}\tilde R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(0)|^p)\le Cn^{2p-1}\sum_{i=1}^{n}\mathbb{E}|\tilde R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(0)|^p=o(1).$$

Then we get $n\sum_{i=1}^{[n.]}\tilde R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(0)\overset{L^p}{\to}0$ as $n\to\infty$. Now, we consider $n\sum_{i=1}^{[nt]}R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(r)$ for any $r\in\{0,\dots,3\}$. By the discrete BDG inequality, (4.4.14), (4.4.17), (4.4.21), (4.4.25) and Jensen's inequality, there is a generic constant $C>0$ such that for all $r\in\{0,\dots,3\}$ we have

$$n^p\mathbb{E}(\sup_{0\le t\le 1}|\sum_{i=1}^{[nt]}R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(r)|^p)\le Cn^p\mathbb{E}\Big(\sum_{i=1}^{n}|R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(r)|^2\Big)^{p/2}\le Cn^{3p/2-1}\sum_{i=1}^{n}\mathbb{E}|R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(r)|^p=o(1).$$

Then we get $n\sum_{i=1}^{[n.]}R^{nm,\tilde\sigma}_{\frac{i-1}{n}}(r)\overset{L^p}{\to}0$ as n$\to\infty$, for any $r\in\{0,\dots,3\}$. Next, we recall from (4.4.33) that for $\ell\in\{1,\dots,d\}$, the ℓ^{th} component of the generic term of the martingale triangular array $\tilde M^{nm,2}_t$ is given by

$$\tilde M^{nm,2}_{\ell,\frac{i-1}{n}}=\frac{1}{4}\left[\dot h^{n,i,1}_{\ell\bullet\bullet}\blacklozenge(X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}})\right]\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n).$$

Similarly, we use the discrete BDG and Jensen inequalities to get $\mathbb{E}(n^p\sup_{0\le t\le 1}|\sum_{i=1}^{[nt]}\tilde M^{nm,2}_{\frac{i-1}{n}}|^p)$ is bounded by $Cn^{3p/2-1}\sum_{i=1}^{n}\mathbb{E}|\tilde M^{nm,2}_{\frac{i-1}{n}}|^p$. Besides, according to (4.4.33), for j and $j'\in\{1,\dots,q\}$, the jj'-th block is given by $(\dot h^{n,i,1}_{\ell\bullet\bullet})_{jj'}=\nabla h_{\ell jj'}(\zeta^{n,5}_{\frac{i-1}{n}})-\nabla h_{\ell jj'}(\zeta^{n,6}_{\frac{i-1}{n}})\in\mathbb{R}^{d\times 1}$ where $\zeta^{n,5}_{\frac{i-1}{n}}\in(X^{nm}_{\frac{i-1}{n}},\bar X^{nm,\sigma}_{\frac{i-1}{n}})$ and $\zeta^{n,6}_{\frac{i-1}{n}}\in(X^{nm,\sigma}_{\frac{i-1}{n}},\bar X^{nm,\sigma}_{\frac{i-1}{n}})$. By using the independence between ΔW_i and $\mathcal{F}_{\frac{i-1}{n}}$, Cauchy-Schwarz inequality, Lemma 4.3.1, Corollary 4.4.9

and ($\mathbf{H}_{f,g}$), we have

$$\begin{aligned}
&\max_{1\leq i\leq n} \mathbb{E}|\tilde{M}^{nm,2}_{\frac{i-1}{n}}|^p \\
&\leq C\Big[\max_{1\leq i\leq n} \mathbb{E}|\dot{h}^{n,i,1}_{\ell\bullet\bullet}|^{2p} \max_{1\leq i\leq n} \mathbb{E}|X^{nm}_{\frac{i-1}{n}} - X^{nm,\sigma}_{\frac{i-1}{n}}|^{2p}\Big]^{1/2} \max_{1\leq i\leq n} \mathbb{E}|\Delta W_i \Delta W_i^\top - I_q\Delta_n|^p \\
&\leq C(\Delta_n^p\Delta_n^p)^{1/2}\Delta_n^p = C\Delta_n^{2p}.
\end{aligned}$$

Therefore, we get

$$\mathbb{E}\left(n^p \sup_{0\leq t\leq 1} |\sum_{i=1}^{[nt]} \tilde{M}^{nm,2}_{\frac{i-1}{n}}|^p\right) = O(\Delta_n^{p/2}).$$

Finally, similarly as above, since $\mathbb{E}((\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}})-\mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top-I_q\Delta_n)|\mathcal{F}_{\frac{i-1}{n}}) = 0$, by the discrete BDG and Jensen inequalities and as $\mathbb{E}|\Delta W_i\Delta W_i^\top - I_q\Delta_n|^p = O(\Delta_n^p)$, we get $n^p\mathbb{E}\big(\sup_{0\leq t\leq 1} |\sum_{i=1}^{[nt]}(\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}) - \mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top - I_q\Delta_n)|^p\big)$ is bounded up to a positive multiplicative constant by $n^{p/2-1}\sum_{i=1}^n \mathbb{E}|\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}) - \mathbb{H}(X^n_{\frac{i-1}{n}})|^p$. Next, thanks to Corollary 4.4.9 and assumption ($\mathbf{H}_{f,g}$), we deduce that this upper bound is $O(\Delta_n^{p/2})$. Then we get $n\sum_{i=1}^{[n.]}(\mathbb{H}(\bar{X}^{nm,\sigma}_{\frac{i-1}{n}}) - \mathbb{H}(X^n_{\frac{i-1}{n}}))\blacklozenge(\Delta W_i\Delta W_i^\top - I_q\Delta_n) \xrightarrow{L^p} 0$ as $n\to\infty$. □

4.7 Appendix B: Proof of essential lemmas

Proof of Lemma 4.4.1. By the tower property we have

$$\mathbb{E}(M^{nm,\tilde{\sigma},1}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}}) = \sum_{k=2}^m \nabla f_\ell(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top \sum_{k'=1}^{k-1}\mathbb{E}\Big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\mathbb{E}\Big(\delta W_{i\tilde{\sigma}(k')}\frac{\Delta_n}{m}|\mathcal{F}^{k'-1,\tilde{\sigma}}_{\frac{i-1}{n}}\Big)|\mathcal{F}_{\frac{i-1}{n}}\Big)$$

$$\begin{aligned}
\mathbb{E}(M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}}) = \sum_{k=2}^m\sum_{j=1}^q \mathbb{E}\Bigg(\Bigg[&\nabla g_{\ell j}^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\sum_{k'=1}^{k-1}\Big(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m}+ \\
&\Big[\dot{g}^n_{ik'}\blacklozenge(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big]\delta W_{i\tilde{\sigma}(k')} + \mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})\Big) \\
&+\frac{1}{2}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Bigg]\mathbb{E}(\delta W^j_{i\tilde{\sigma}(k)}|\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}})|\mathcal{F}_{\frac{i-1}{n}}\Bigg)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(M^{nm,\tilde{\sigma},3}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}}) = \sum_{k=2}^m\mathbb{E}\Bigg(&\Big[\dot{h}^{n,ik}_{\ell\bullet\bullet}\blacklozenge(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big] \\
&\blacklozenge\,\mathbb{E}\Big(\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}}\Big)|\mathcal{F}_{\frac{i-1}{n}}\Bigg).
\end{aligned}$$

Since $\delta W_{i,\tilde{\sigma}(k)}$ is independent of $\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}}$, $k\in\{1,\dots,m\}$, $\mathbb{E}(\delta W_{i,\tilde{\sigma}(k)}) = 0$ and $\mathbb{E}(\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}) = 0$, we get $\mathbb{E}(M^{nm,\sigma}_{\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}}) = 0$. Now, it remains to have upper bounds for $\mathbb{E}(|M^{nm,\sigma}_{\frac{i-1}{n}}|^p)$ and $\mathbb{E}(|N^{nm,\sigma}_{\frac{i-1}{n}}|^p)$. Thanks to our assumption ($\mathbf{H}_{f,g}$) it is easy to see the existence of $C>0$ s.t.

$$|M^{nm,\tilde{\sigma},1}_{\ell,\frac{i-1}{n}}| \leq C\sum_{k=2}^m\sum_{k'=1}^{k-1}\Big(1+\Big|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}\Big|\Big)\Big|\delta W_{i\tilde{\sigma}(k')}\frac{\Delta_n}{m}\Big|$$

$$|M^{nm,\tilde{\sigma},2}_{\ell,\frac{i-1}{n}}| \le C\sum_{k=2}^{m}\sum_{j=1}^{q}\left(\sum_{k'=1}^{k-1}\left[(1+|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|)(\frac{\Delta_n}{m}+|\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m}|)\right.\right.$$
$$\left.\left.+|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k')}|\right]+\frac{1}{2}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|^2\right)|\delta W^j_{i\tilde{\sigma}(k)}|$$
$$|M^{nm,\tilde{\sigma},3}_{\ell,\frac{i-1}{n}}| \le C\sum_{k=2}^{m}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)}-I_q\frac{\Delta_n}{m}|,$$

Here, the constant C is a generic positive constant whose values may vary from line to line. We obtain (4.4.11) using the independence between the above increments combined with Lemma 4.3.1 and the fact that $\mathbb{E}|\delta W_{i,\tilde{\sigma}(k)}|^p = O(\Delta_n^{p/2})$ and $\mathbb{E}|\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|^p = O(\Delta_n^p)$. Similar arguments give us inequality (4.4.12). □

Proof of Lemma 4.4.2. Thanks to equations (4.4.1) and (4.4.6) combined with (4.2.2), we deduce relation (4.4.13) with :

$$R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0) = \frac{\Delta_n}{m}\sum_{\substack{k,k'=1\\k'<k}}^{m}\nabla f_\ell^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m})$$
$$+\left[g^\top(X_{\frac{i-1}{n}})\nabla^2 f_\ell(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\right]\blacklozenge\sum_{\substack{k,k',k''=1\\k''<k'<k}}^{m}\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k'')}\frac{\Delta_n}{m}$$

and

$$\tilde{R}^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0) = \frac{\Delta_n}{m}\sum_{\substack{k,k'=1\\k'<k}}^{m}\nabla f_\ell^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\big(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-f(X_{\frac{i-1}{n}})\big)\frac{\Delta_n}{m}$$
$$+\frac{m-1}{2m}\big(\nabla f_\ell^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})-\nabla f_\ell^\top(X_{\frac{i-1}{n}})\big)f(X_{\frac{i-1}{n}})\Delta_n^2$$
$$+\frac{1}{2}\sum_{k=2}^{m}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top\big(\nabla^2 f_\ell(\xi^{1,n}_{ik})-\nabla^2 f_\ell(X_{\frac{i-1}{n}})\big)(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\frac{\Delta_n}{m}$$
$$+\frac{1}{2}\sum_{\substack{k,k'=1\\k'<k}}^{m}(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m}+\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m}))^\top$$
$$\times\nabla^2 f_\ell(X_{\frac{i-1}{n}})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\frac{\Delta_n}{m}$$
$$+\frac{1}{2}\sum_{\substack{k,k'=1\\k'<k}}^{m}\left(\big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-g(X_{\frac{i-1}{n}})\big)\delta W_{i\tilde{\sigma}(k')}\right)^\top\nabla^2 f_\ell(X_{\frac{i-1}{n}})(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\frac{\Delta_n}{m}$$
$$+\frac{1}{2}\sum_{k=2}^{m}\sum_{k'=1}^{k-1}\big(g(X_{\frac{i-1}{n}})\delta W_{i\tilde{\sigma}(k')}\big)^\top\nabla^2 f_\ell(X_{\frac{i-1}{n}})\left(\sum_{k'=1}^{k-1}f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m}+\sum_{k'=1}^{k-1}\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\right.$$
$$\left.\blacklozenge(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m})+\sum_{k'=1}^{k-1}\big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-g(X_{\frac{i-1}{n}})\big)\delta W_{i\tilde{\sigma}(k')}\right)\frac{\Delta_n}{m}.$$

By the tower property we have

$$
\begin{aligned}
&\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|\mathcal{F}_{\frac{i-1}{n}})\\
&= \mathbb{E}\Bigg(\frac{\Delta_n}{m}\sum_{\substack{k,k'=1\\k'<k}}^{m} \nabla f_\ell^\top(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge \mathbb{E}(\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|\mathcal{F}^{k'-1,\tilde{\sigma}}_{\frac{i-1}{n}})\\
&+ \Big[g(X_{\frac{i-1}{n}})^\top\nabla^2 f_\ell(X_{\frac{i-1}{n}})g(X_{\frac{i-1}{n}})\Big]\blacklozenge \sum_{\substack{k,k',k''=1\\k''<k'<k}}^{m} \mathbb{E}(\delta W_{i\tilde{\sigma}(k')}|\mathcal{F}^{k'-1,\tilde{\sigma}}_{\frac{i-1}{n}})\delta W^\top_{i\tilde{\sigma}(k'')}\frac{\Delta_n}{m}\Big|\mathcal{F}_{\frac{i-1}{n}}\Bigg).
\end{aligned}
$$

Since $\delta W_{i\tilde{\sigma}(k)}$ is independent of $\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}}$, $k\in\{1,\dots,m\}$, the fact that $\mathbb{E}(\delta W_{i,\tilde{\sigma}(k)})=0$ and $\mathbb{E}(\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m})=0$, we get $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|\mathcal{F}_{\frac{i-1}{n}})=0$. Thanks to our assumption ($\mathbf{H}_{f,g}$) it is easy to see the existence of $C>0$ s.t.

$$
\begin{aligned}
|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)| \le C\Delta_n &\sum_{\substack{k,k'=1\\k'<k}}^{m}(1+|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|)|\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})|\\
&+ C\Delta_n(1+|X_{\frac{i-1}{n}}|^2)\sum_{\substack{k,k',k''=1\\k''<k'<k}}^{m}|\delta W_{i\tilde{\sigma}(k')}||\delta W_{i\tilde{\sigma}(k'')}|,
\end{aligned}
$$

and

$$
\begin{aligned}
|\tilde{R}^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)| &\le C\Delta_n^2\sum_{\substack{k,k'=1\\k'<k}}^{m}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}| + C\Delta_n^2|X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}} - X_{\frac{i-1}{n}}|(1+|X_{\frac{i-1}{n}}|)\\
&+ C\Delta_n\sum_{k=2}^{m}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|^2|\nabla^2 f_\ell^\top(\xi^{1,n}_{ik}) - \nabla^2 f_\ell^\top(X_{\frac{i-1}{n}})|\\
&+ C\Delta_n\sum_{\substack{k,k'=1\\k'<k}}^{m}(1+|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|)(\frac{\Delta_n}{m} + |\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|)|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|\\
&+ C\Delta_n\sum_{\substack{k,k'=1\\k'<k}}^{m}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k')}||X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|\\
&+ C\Delta_n\sum_{k=2}^{m}\Bigg(\sum_{k'=1}^{k-1}(1+|X_{\frac{i-1}{n}}|)|\delta W_{i\tilde{\sigma}(k')}|\Bigg)\Bigg(\sum_{k'=1}^{k-1}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k')}|\\
&+ \sum_{k'=1}^{k-1}(1+|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|)(\frac{\Delta_n}{m} + |\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|)\Bigg).
\end{aligned}
$$

Here, the constant C is a generic positive constant whose values may vary from line to line. Next, using the independence between the increments, we get

$$
\mathbb{E}|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|^p \le C\Delta_n^p\sum_{\substack{k,k'=1\\k'<k}}^{m}(1+\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|^p)\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|^p
$$

$$+ C\Delta_n^p(1+\mathbb{E}|X_{\frac{i-1}{n}}|^{2p}) \sum_{\substack{k,k',k''=1 \\ k''<k'<k}}^{m} \mathbb{E}|\delta W_{i\tilde{\sigma}(k')}|^p \mathbb{E}|\delta W_{i\tilde{\sigma}(k'')}|^p,$$

and by Cauchy-Schwarz inequality combined with the independence between the increments, we also get

$$\mathbb{E}|\tilde{R}^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|^p \leq C\Delta_n^{2p} \sum_{\substack{k,k'=1 \\ k'<k}}^{m} \mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}|^p$$
$$+ C\Delta_n^{2p}\big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}} - X_{\frac{i-1}{n}}|^{2p}\big)^{1/2}(1+\mathbb{E}|X_{\frac{i-1}{n}}|^{2p})^{1/2}$$
$$+ C\Delta_n^p \sum_{k=2}^{m} \big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|^{4p}\big)^{1/2}\big(\mathbb{E}|\nabla^2 f_\ell^\top(\xi^{1,n}_{ik}) - \nabla^2 f_\ell^\top(X_{\frac{i-1}{n}})|^{2p}\big)^{1/2}$$
$$+ C\Delta_n^p \sum_{\substack{k,k'=1 \\ k'<k}}^{m} \left((1+\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|^{2p})(\Delta_n^{2p} + \mathbb{E}|\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|^{2p})\right)^{1/2}$$
$$\times \big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|^{2p}\big)^{1/2}$$
$$+ C\Delta_n^p \sum_{\substack{k,k'=1 \\ k'<k}}^{m} \big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}|^{2p}\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}|^{2p}\big)^{1/2}\big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|^{2p}\big)^{1/2}$$
$$+ C\Delta_n^p \sum_{k=2}^{m}\sum_{k'=1}^{k-1} \left[(1+\mathbb{E}|X_{\frac{i-1}{n}}|^{2p})\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}|^{2p}\right]^{1/2}\Bigg(\sum_{k'=1}^{k-1}(1+\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|^{2p})\Delta_n^{2p} +$$
$$\sum_{k'=1}^{k-1}(1+\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|^{2p})\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}\delta W^\top_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|^{2p}$$
$$+ \sum_{k'=1}^{k-1}\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}|^{2p}\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}|^{2p}\Bigg)^{1/2}.$$

Now, using Lemma 4.2.1 combined with Lemma 4.3.1 and the fact that $\mathbb{E}|\delta W_{i,\tilde{\sigma}(k)}|^p = O(\Delta_n^{p/2})$, $\mathbb{E}|\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|^p = O(\Delta_n^p)$, we obtain (4.4.14) and we have

$$\mathbb{E}|\tilde{R}^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(0)|^p = O(\Delta_n^{5p/2}) + O(\Delta_n^{2p})\sum_{k=2}^{m}\big(\mathbb{E}|\nabla^2 f_\ell^\top(\xi^{1,n}_{ik}) - \nabla^2 f_\ell^\top(X_{\frac{i-1}{n}})|^{2p}\big)^{1/2}.$$

We recall from relation (4.4.4) section 4 that $\xi^{1,n}_{ik} \in (X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}, X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}})$. Then, by using Lemma 4.2.1, Lemma 4.3.1 and the assumption $(\mathbf{H}_{f,g})$, we have $\mathbb{E}|\nabla^2 f_\ell^\top(\xi^{1,n}_{ik}) - \nabla^2 f_\ell^\top(X_{\frac{i-1}{n}})|^{2p} = O(\Delta_n^p)$, which yields us (4.4.15). □

Proof of Lemma 4.4.3. Thanks to equation (4.4.7), we deduce from relation (4.4.16) the exact form of $R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)$ that is given by

$$R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1) = \frac{\Delta_n}{m}\nabla f_\ell(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top \sum_{\substack{k,k'=1 \\ k'<k}}^{m}\big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) - g(X_{\frac{i-1}{n}})\big)\delta W_{i\tilde{\sigma}(k')}$$

$$+\frac{\Delta_n}{m}(\nabla f_\ell(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top - \nabla f_\ell(X_{\frac{i-1}{n}})^\top)g(X_{\frac{i-1}{n}})\sum_{\substack{k,k'=1\\k'<k}}^m \delta W_{i\tilde{\sigma}(k')}.$$

By the tower property we have

$$\begin{aligned}&\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|\mathcal{F}_{\frac{i-1}{n}})\\&=\mathbb{E}\Bigg(\frac{\Delta_n}{m}\nabla f_\ell(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top\sum_{\substack{k,k'=1\\k'<k}}^m\big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-g(X_{\frac{i-1}{n}})\big)\mathbb{E}(\delta W_{i\tilde{\sigma}(k')}|\mathcal{F}^{k'-1,\tilde{\sigma}}_{\frac{i-1}{n}})\\&+\frac{\Delta_n}{m}(\nabla f_\ell(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})^\top-\nabla f_\ell(X_{\frac{i-1}{n}})^\top)g(X_{\frac{i-1}{n}})\sum_{\substack{k,k'=1\\k'<k}}^m\mathbb{E}(\delta W_{i\tilde{\sigma}(k')}|\mathcal{F}^{k'-1,\tilde{\sigma}}_{\frac{i-1}{n}})|\mathcal{F}_{\frac{i-1}{n}}\Bigg).\end{aligned}$$

Since $\delta W_{i,\tilde{\sigma}(k)}$ is independent of $\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}}$, $k\in\{1,\dots,m\}$, $\mathbb{E}(\delta W_{i,\tilde{\sigma}(k)})=0$ and then $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|\mathcal{F}_{\frac{i-1}{n}})=0$. Now, thanks to our assumption ($\mathbf{H}_{f,g}$) it is easy to see the existence of $C>0$ s.t.

$$\begin{aligned}|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|\le C\Delta_n&\sum_{\substack{k,k'=1\\k'<k}}^m|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k')}|\\&+C\Delta_n|X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|(1+|X_{\frac{i-1}{n}}|)\sum_{\substack{k,k'=1\\k'<k}}^m|\delta W_{i\tilde{\sigma}(k')}|.\end{aligned}$$

Here, the constant C is a generic positive constant whose values may vary from line to line. Next, applying Cauchy-Schwarz inequality and using the independence between the increments, we get

$$\begin{aligned}\mathbb{E}|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|^p\le C\Delta_n^p&\sum_{\substack{k,k'=1\\k'<k}}^m\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^p\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}|^p\\&+C\Delta_n^p\big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\big(1+\big(\mathbb{E}|X_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\big)\sum_{\substack{k,k'=1\\k'<k}}^m\mathbb{E}|\delta W_{i\tilde{\sigma}(k')}|^p.\end{aligned}$$

By the fact that $\mathbb{E}|\delta W_{i,\tilde{\sigma}(k)}|^p=O(\Delta_n^{p/2})$ and $\mathbb{E}|\delta W_{i\tilde{\sigma}(k)}\delta W^\top_{i\tilde{\sigma}(k)}-I_q\frac{\Delta_n}{m}|^p=O(\Delta_n^p)$, we have

$$\begin{aligned}\mathbb{E}|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(1)|^p\le C\Delta_n^{3p/2}&\sum_{\substack{k,k'=1\\k'<k}}^m\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^p\\&+C\Delta_n^{3p/2}\big(\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\big(1+\big(\mathbb{E}|X_{\frac{i-1}{n}}|^{2p}\big).\end{aligned}$$

We obtain (4.4.17) using Lemma 4.2.1 and Lemma 4.3.1. □

Proof of Lemma 4.4.4. Thanks to equation (4.4.19) and (4.2.2), we deduce from relation (4.4.20) the exact form of $R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(2)$. We have

$$
\begin{aligned}
R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(2) &= \sum_{j=1}^{q}\Bigg[\nabla g_{\ell j}^{\top}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\sum_{\substack{k,k'=1\\k'<k}}^{m}\Bigg(\Big(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-f(X_{\frac{i-1}{n}})\Big)\frac{\Delta_n}{m}+\\
&\Big(\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-\mathbb{H}(X_{\frac{i-1}{n}})\Big)\blacklozenge\big(\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m}\big)\Bigg)\Bigg]\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\sum_{j=1}^{q}\Bigg[\Big(\nabla g_{\ell j}^{\top}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})-\nabla g_{\ell j}^{\top}(X_{\frac{i-1}{n}})\Big)\sum_{\substack{k,k'=1\\k'<k}}^{m}\Bigg(f(X_{\frac{i-1}{n}})\frac{\Delta_n}{m}+\\
&\mathbb{H}(X_{\frac{i-1}{n}})\blacklozenge\big(\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m}\big)\Bigg)\Bigg]\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\sum_{k=3}^{m}\sum_{j=1}^{q}\Big(\nabla g_{\ell j}^{\top}(X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})-\nabla g_{\ell j}^{\top}(X_{\frac{i-1}{n}})\Big)\sum_{k'=2}^{k-1}\Big[\dot{g}^{n}_{ik'}\blacklozenge\big(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}\big)\Big]\delta W_{i\tilde{\sigma}(k')}\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\sum_{k=3}^{m}\sum_{j=1}^{q}\nabla g_{\ell j}^{\top}(X_{\frac{i-1}{n}})\sum_{k'=2}^{k-1}\Big[\big(\dot{g}^{n}_{ik'}-\dot{g}^{n}_{i}\big)\blacklozenge\big(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}\big)\Big]\delta W_{i\tilde{\sigma}(k')}\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\sum_{k=3}^{m}\sum_{j=1}^{q}\nabla g_{\ell j}^{\top}(X_{\frac{i-1}{n}})\sum_{k'=2}^{k-1}\Bigg[\dot{g}^{n}_{i}\blacklozenge\Bigg(\sum_{k''=1}^{k'-1}f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k''-1}{nm}})\frac{\Delta_n}{m}\\
&+\sum_{k''=1}^{k'-1}\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k''-1}{nm}})\blacklozenge\big(\delta W_{i\tilde{\sigma}(k'')}\delta W^{\top}_{i\tilde{\sigma}(k'')}-I_q\frac{\Delta_n}{m}\big)\\
&+\sum_{k''=1}^{k'-1}\Big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k''-1}{nm}})-g(X_{\frac{i-1}{n}})\Big)\delta W_{i\tilde{\sigma}(k'')}\Bigg)\Bigg]\delta W_{i\tilde{\sigma}(k')}\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\sum_{k=2}^{m}\sum_{j=1}^{q}\frac{1}{2}\big(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}\big)^{\top}\big(\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})-\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})\big)\big(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}\big)\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\frac{1}{2}\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}\Big(f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m}+\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\blacklozenge\big(\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m}\big)\Big)^{\top}\\
&\times\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})\big(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}\big)\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\frac{1}{2}\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}\Bigg(\Big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-g(X_{\frac{i-1}{n}})\Big)\delta W_{i\tilde{\sigma}(k')}\Bigg)^{\top}\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})\big(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}\big)\delta W^{j}_{i\tilde{\sigma}(k)}\\
&+\sum_{j=1}^{q}\frac{1}{2}\sum_{k=2}^{m}\Big(\sum_{k'=1}^{k-1}g(X_{\frac{i-1}{n}})\delta W_{i\tilde{\sigma}(k')}\Big)^{\top}\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})\Bigg(\sum_{k'=1}^{k-1}f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m}+\\
&\sum_{k'=1}^{k-1}\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\blacklozenge\big(\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')}-I_q\frac{\Delta_n}{m}\big)\\
&+\sum_{k'=1}^{k-1}\Big(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})-g(X_{\frac{i-1}{n}})\Big)\delta W_{i\tilde{\sigma}(k')}\Bigg)\delta W^{j}_{i\tilde{\sigma}(k)}.
\end{aligned}
$$

By the tower property, the independence of $\delta W_{i,\tilde\sigma(k)}$ of $\mathcal{F}^{k-1,\tilde\sigma}_{\frac{i-1}{n}}$, $k\in\{1,\dots,m\}$, and $\mathbb{E}(\delta W_{i,\tilde\sigma(k)})=0$, we deduce $\mathbb{E}(R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(2)|\mathcal{F}_{\frac{i-1}{n}})=0$. Moreover, thanks to our assumption ($\mathbf{H}_{f,g}$) it is easy to see the existence of a generic positive constant C which does not depend on i s.t.

$$
\begin{aligned}
|R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(2)| \le\; & C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|\Big(\frac{\Delta_n}{m}+|\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|\Big)|\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{j=1}^{q}|X^{nm,\tilde\sigma}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|\sum_{\substack{k,k'=1\\k'<k}}^{m}(1+|X_{\frac{i-1}{n}}|)\Big(\frac{\Delta_n}{m}+|\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|\Big)|\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{k=3}^{m}\sum_{j=1}^{q}|X^{nm,\tilde\sigma}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|\sum_{k'=2}^{k-1}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}||\delta W_{i\tilde\sigma(k')}||\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{k=3}^{m}\sum_{j=1}^{q}\sum_{k'=2}^{k-1}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}||\dot g^{n}_{ik'}-\dot g^{n}_{i}||\delta W_{i\tilde\sigma(k')}||\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{k=3}^{m}\sum_{j=1}^{q}\sum_{k'=2}^{k-1}\Big[\sum_{k''=1}^{k'-1}(1+|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k''-1}{nm}}|)(\frac{\Delta_n}{m}+|\delta W_{i\tilde\sigma(k'')}\delta W^{\top}_{i\tilde\sigma(k'')}-I_q\frac{\Delta_n}{m}|)\\
&+\sum_{k''=1}^{k'-1}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k''-1}{nm}}-X_{\frac{i-1}{n}}||\delta W_{i\tilde\sigma(k'')}|\Big]|\delta W_{i\tilde\sigma(k')}||\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{k=2}^{m}\sum_{j=1}^{q}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^2|\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})-\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})||\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}(1+|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}|)(\frac{\Delta_n}{m}+|\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|)|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}||\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}||\delta W_{i\tilde\sigma(k')}||X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}||\delta W^{j}_{i\tilde\sigma(k)}|\\
&+C\sum_{j=1}^{q}\sum_{k=2}^{m}\Big(\sum_{k'=1}^{k-1}(1+|X_{\frac{i-1}{n}}|)|\delta W_{i\tilde\sigma(k')}|\Big)\Big(\sum_{k'=1}^{k-1}(1+|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}|)(\frac{\Delta_n}{m}+|\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|)\\
&+\sum_{k'=1}^{k-1}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}||\delta W_{i\tilde\sigma(k')}|\Big)|\delta W^{j}_{i\tilde\sigma(k)}|.
\end{aligned}
$$

Here, the values of the constant C may vary from line to line. Next, we apply Cauchy-Schwarz inequality and use the independence between the increments, we get

$$
\begin{aligned}
\mathbb{E}|R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(2)|^p \le\; & C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^p\Big(\frac{\Delta_n^p}{m^p}+\mathbb{E}|\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|^p\Big)\mathbb{E}|\delta W^{j}_{i\tilde\sigma(k)}|^p\\
&+C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}\big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|^{2p}(1+\mathbb{E}|X_{\frac{i-1}{n}}|^{2p})\big)^{1/2}\Big(\frac{\Delta_n^p}{m^p}+\mathbb{E}|\delta W_{i\tilde\sigma(k')}\delta W^{\top}_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|^p\Big)\mathbb{E}|\delta W^{j}_{i\tilde\sigma(k)}|^p
\end{aligned}
$$

$$+C\sum_{k=3}^{m}\sum_{j=1}^{q}\big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{i-1}{n}}-X_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\sum_{k'=2}^{k-1}\big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^p\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p$$

$$+C\sum_{k=3}^{m}\sum_{j=1}^{q}\sum_{k'=2}^{k-1}\Big[\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^{2p}\mathbb{E}|\dot g^n_{ik'}-\dot g^n_i|^{2p}\Big]^{1/2}\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^p\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p$$

$$+C\sum_{k=3}^{m}\sum_{j=1}^{q}\sum_{k'=2}^{k-1}\Big[\sum_{k''=1}^{k'-1}(1+\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k''-1}{nm}}|^p)(\frac{\Delta_n^p}{m^p}+\mathbb{E}|\delta W_{i\tilde\sigma(k'')}\delta W^\top_{i\tilde\sigma(k'')}-I_q\frac{\Delta_n}{m}|^p)$$

$$+\sum_{k''=1}^{k'-1}\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k''-1}{nm}}-X_{\frac{i-1}{n}}|^p\mathbb{E}|\delta W_{i\tilde\sigma(k'')}|^p\Big]\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^p\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p$$

$$+C\sum_{k=2}^{m}\sum_{j=1}^{q}\Big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^{4p}\mathbb{E}|\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})-\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})|^{2p}\Big)^{1/2}\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p$$

$$+C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}\Big((1+\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}|^{2p})(\frac{\Delta_n^{2p}}{m^{2p}}+\mathbb{E}|\delta W_{i\tilde\sigma(k')}\delta W^\top_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|^{2p})\Big)^{1/2}$$

$$\times\big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p$$

$$+C\sum_{j=1}^{q}\sum_{\substack{k,k'=1\\k'<k}}^{m}\Big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^{2p}\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^{2p}\Big)^{1/2}\big(\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}}-X^{nm,\tilde\sigma}_{\frac{i-1}{n}}|^{2p}\big)^{1/2}\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p$$

$$+C\sum_{j=1}^{q}\sum_{k=2}^{m}\Big(\sum_{k'=1}^{k-1}(1+\mathbb{E}|X_{\frac{i-1}{n}}|^{2p})\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^{2p}\Big)^{1/2}\Big(\sum_{k'=1}^{k-1}(1+\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}|^{2p})(\frac{\Delta_n^{2p}}{m^{2p}}$$

$$+\mathbb{E}|\delta W_{i\tilde\sigma(k')}\delta W^\top_{i\tilde\sigma(k')}-I_q\frac{\Delta_n}{m}|^{2p})+\sum_{k'=1}^{k-1}\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^{2p}\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^{2p}\Big)^{1/2}\mathbb{E}|\delta W^j_{i\tilde\sigma(k)}|^p.$$

By using Lemma 4.2.1 combined with Lemma 4.3.1 and the fact that $\mathbb{E}|\delta W_{i,\tilde\sigma(k)}|^p=O(\Delta_n^{p/2})$, $\mathbb{E}|\delta W_{i\tilde\sigma(k)}\delta W^\top_{i\tilde\sigma(k)}-I_q\frac{\Delta_n}{m}|^p=O(\Delta_n^p)$, we get

$$\mathbb{E}|R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(2)|^p=O(\Delta_n^{2p})+O(\Delta_n^{3p/2})\sum_{k=3}^{m}\sum_{k'=2}^{k-1}\Big[\mathbb{E}|\dot g^n_{ik'}-\dot g^n_i|^{2p}\Big]^{1/2}$$
$$+O(\Delta_n^{3p/2})\sum_{k=2}^{m}\sum_{j=1}^{q}\Big(\mathbb{E}|\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})-\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})|^{2p}\Big)^{1/2}.$$

Now, let us recall that from relation (4.4.19) we have $\dot g^n_{ik'}\in(\mathbb{R}^{d\times1})^{d\times q}$ and $\dot g^n_i\in(\mathbb{R}^{d\times1})^{d\times q}$, for $\ell\in\{1,\dots,d\}, j\in\{1,\dots,q\}$, $(\dot g^n_{ik'})_{\ell j}=\nabla g_{\ell j}(\xi'^{2,n}_{ik'})\in\mathbb{R}^{d\times1}$ and $(\dot g^n_i)_{\ell j}=\nabla g_{\ell j}(X_{\frac{i-1}{n}})$ where $\xi'^{2,n}_{ik'}\in(X^{nm,\tilde\sigma}_{\frac{i-1}{n}},X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}})$. We also recall that from (4.4.19) $\xi^{2,n}_{ik}\in(X^{nm,\tilde\sigma}_{\frac{i-1}{n}},X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})$. Then, by Lemma 4.2.1, Lemma 4.3.1 and assumption ($\mathbf{H}_{f,g}$), we get $\mathbb{E}|\nabla^2 g_{\ell j}(\xi^{2,n}_{ik})-\nabla^2 g_{\ell j}(X_{\frac{i-1}{n}})|^{2p}=\mathbb{E}|\dot g^n_{ik'}-\dot g^n_i|^{2p}=O(\Delta_n^p)$. Hence, we deduce (4.4.21). □

Proof of Lemma 4.4.7. Thanks to equation (4.4.23), we deduce from relation (4.4.24) the exact form of $R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)$. We have

$$R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3) = \sum_{k=2}^{m}\Big[(\dot{h}^{n,ik}_{\ell\bullet\bullet} - \dot{h}^{n,i}_{\ell\bullet\bullet}) \blacklozenge (X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big] \blacklozenge (\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m})$$
$$+ \sum_{k=2}^{m}\Big[\dot{h}^{n,i}_{\ell\bullet\bullet} \blacklozenge \Big(\sum_{k'=1}^{k-1} f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} + \sum_{k'=1}^{k-1}\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge (\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})$$
$$+ \sum_{k'=1}^{k-1}(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) - g(X_{\frac{i-1}{n}}))\delta W_{i\tilde{\sigma}(k')}\Big)\Big] \blacklozenge (\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}).$$

By the tower property we have

$$\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)|\mathcal{F}_{\frac{i-1}{n}})$$
$$= \mathbb{E}\Big(\sum_{k=2}^{m}\Big[(\dot{h}^{n,ik}_{\ell\bullet\bullet} - \dot{h}^{n,i}_{\ell\bullet\bullet}) \blacklozenge (X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}})\Big] \blacklozenge \mathbb{E}(\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}})$$
$$+ \sum_{k=2}^{m}\Big[\dot{h}^{n,i}_{\ell\bullet\bullet} \blacklozenge \Big(\sum_{k'=1}^{k-1} f(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}})\frac{\Delta_n}{m} + \sum_{k'=1}^{k-1}\mathbb{H}(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) \blacklozenge (\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m})$$
$$+ \sum_{k'=1}^{k-1}(g(X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}) - g(X_{\frac{i-1}{n}}))\delta W_{i\tilde{\sigma}(k')}\Big)\Big] \blacklozenge \mathbb{E}(\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}})|\mathcal{F}_{\frac{i-1}{n}}\Big).$$

Since $\delta W_{i,\tilde{\sigma}(k)}$ is independent of $\mathcal{F}^{k-1,\tilde{\sigma}}_{\frac{i-1}{n}}$, $k \in \{1,\dots,m\}$, $\mathbb{E}(\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}) = 0$, we get $\mathbb{E}(R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)|\mathcal{F}_{\frac{i-1}{n}}) = 0$. Now, thanks to our assumption ($\mathbf{H}_{f,g}$) it is easy to see the existence of constant $C > 0$ s.t.

$$|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)| \le C\sum_{k=2}^{m}|\dot{h}^{n,ik}_{\ell\bullet\bullet} - \dot{h}^{n,i}_{\ell\bullet\bullet}||X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|$$
$$+ C\sum_{k=2}^{m}\Big(\sum_{k'=1}^{k-1}(1 + |X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|)(\frac{\Delta_n}{m} + |\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|)$$
$$+ \sum_{k'=1}^{k-1}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}} - X_{\frac{i-1}{n}}||\delta W_{i\tilde{\sigma}(k')}|\Big)|\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|.$$

Here, the constant C is a generic positive constant whose values may vary from line to line. Next, we apply Cauchy-Schwarz inequality and use the independence between the increments, to get

$$\mathbb{E}|R^{nm,\tilde{\sigma}}_{\ell,\frac{i-1}{n}}(3)|^p$$
$$\le C\sum_{k=2}^{m}(\mathbb{E}|\dot{h}^{n,ik}_{\ell\bullet\bullet} - \dot{h}^{n,i}_{\ell\bullet\bullet}|^{2p}\mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k-1}{nm}} - X^{nm,\tilde{\sigma}}_{\frac{i-1}{n}}|^{2p})^{1/2}\mathbb{E}|\delta W_{i\tilde{\sigma}(k)}\delta W^{\top}_{i\tilde{\sigma}(k)} - I_q\frac{\Delta_n}{m}|^p$$
$$+ C\sum_{k=2}^{m}\Big(\sum_{k'=1}^{k-1}(1 + \mathbb{E}|X^{nm,\tilde{\sigma}}_{\frac{m(i-1)+k'-1}{nm}}|^p)(\frac{\Delta_n^p}{m^p} + \mathbb{E}|\delta W_{i\tilde{\sigma}(k')}\delta W^{\top}_{i\tilde{\sigma}(k')} - I_q\frac{\Delta_n}{m}|^p)$$

$$+\sum_{k'=1}^{k-1}\mathbb{E}|X^{nm,\tilde\sigma}_{\frac{m(i-1)+k'-1}{nm}}-X_{\frac{i-1}{n}}|^p\mathbb{E}|\delta W_{i\tilde\sigma(k')}|^p\Big)\mathbb{E}|\delta W_{i\tilde\sigma(k)}\delta W^{\top}_{i\tilde\sigma(k)}-I_q\frac{\Delta_n}{m}|^p.$$

By Lemma 4.2.1, Lemma 4.3.1 and the fact that $\mathbb{E}|\delta W_{i,\tilde\sigma(k)}|^p=O(\Delta_n^{p/2})$, $\mathbb{E}|\delta W_{i\tilde\sigma(k)}\delta W^{\top}_{i\tilde\sigma(k)}-I_q\frac{\Delta_n}{m}|^p=O(\Delta_n^p)$, we have

$$\mathbb{E}|R^{nm,\tilde\sigma}_{\ell,\frac{i-1}{n}}(3)|^p=O(\Delta_n^{2p})+O(\Delta_n^{3p/2})\sum_{k=2}^{m}\left(\mathbb{E}|\dot h^{n,ik}_{\ell\bullet\bullet}-\dot h^{n,i}_{\ell\bullet\bullet}|^{2p}\right)^{1/2}.$$

We recall from relation (4.4.23) in Section 4 that $\dot h^{n,ik}_{\ell\bullet\bullet}\in(\mathbb{R}^{d\times1})^{q\times q}$ and $\dot h^{n,i}_{\ell\bullet\bullet}\in(\mathbb{R}^{d\times1})^{q\times q}$, for j and $j'\in\{1,\dots,q\}$, $(\dot h^{n,ik}_{\ell\bullet\bullet})_{jj'}=\nabla h_{\ell jj'}(\xi^{3,n}_{ik})\in\mathbb{R}^{d\times1}$ and $(\dot h^{n,i}_{\ell\bullet\bullet})_{jj'}=\nabla h_{\ell jj'}(X_{\frac{i-1}{n}})$ where $\xi^{3,n}_{ik}\in(X^{nm,\tilde\sigma}_{\frac{i-1}{n}},X^{nm,\tilde\sigma}_{\frac{m(i-1)+k-1}{nm}})$. Then, by using Lemma 4.3.1 and the assumption $(\mathbf{H}_{f,g})$, we have $\mathbb{E}|\dot h^{n,ik}_{\ell\bullet\bullet}-\dot h^{n,i}_{\ell\bullet\bullet}|^{2p}=O(\Delta_n^p)$. Hence, we obtain (4.4.25). □

Proof of Lemma 4.4.8. From (4.4.32), (4.4.33) and (4.4.34), we use similar arguments as in the proof of Lemma 4.4.1 to get

$$\begin{aligned}\mathbb{E}(\tilde M^{nm,1}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=&\frac{1}{16}\sum_{j'=1}^{q}(X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}})^{\top}\left(\nabla^2 g_{\ell j'}(\zeta^{n,3}_i)+\nabla^2 g_{\ell j'}(\zeta^{n,4}_i)\right)(X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}})\\&\times\mathbb{E}(\Delta W^{j'}_i|\mathcal{F}_{\frac{i-1}{n}}),\\ \mathbb{E}(\tilde M^{nm,2}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=&\frac14\left[\dot h^{n,i,1}_{\ell\bullet\bullet}\blacklozenge(X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}})\right]\blacklozenge\mathbb{E}(\Delta W_i\Delta W_i-I_q\Delta_n|\mathcal{F}_{\frac{i-1}{n}}),\\ \mathbb{E}(\tilde M^{nm,3}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=&\sum_{\substack{k,k'=1\\k<k'}}^{m}\left[\dot h^{n,i,2}_{\ell\bullet\bullet}\blacklozenge(X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}})\right]\blacklozenge\mathbb{E}(\delta W_{ik}\delta W^{\top}_{ik'}-\delta W_{ik'}\delta W^{\top}_{ik}|\mathcal{F}_{\frac{i-1}{n}}).\end{aligned}$$

Now if we also consider (4.4.31), we get thanks to assumption $(\mathbf{H}_{f,g})$, the existence of a generic positive constant C s.t.

$$\begin{aligned}\mathbb{E}|\tilde N^{nm}_{\ell,\frac{i-1}{n}}|^p\le&C\mathbb{E}|X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}}|^{2p}\Delta_n,\\ \mathbb{E}|\tilde M^{nm,1}_{\ell,\frac{i-1}{n}}|^p\le&C\sum_{j'=1}^{q}\mathbb{E}|X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}}|^{2p}\mathbb{E}|\Delta W^{j'}_i|^p,\\ \mathbb{E}|\tilde M^{nm,2}_{\ell,\frac{i-1}{n}}|^p\le&C\mathbb{E}|X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}}|^{p}\mathbb{E}|\Delta W_i\Delta W_i-I_q\Delta_n|^p,\\ \mathbb{E}|\tilde M^{nm,3}_{\ell,\frac{i-1}{n}}|^p\le&C\sum_{\substack{k,k'=1\\k<k'}}^{m}\mathbb{E}|X^{nm}_{\frac{i-1}{n}}-X^{nm,\sigma}_{\frac{i-1}{n}}|^{p}\mathbb{E}|\delta W_{ik}\delta W^{\top}_{ik'}-\delta W_{ik'}\delta W^{\top}_{ik}|^p.\end{aligned}$$

Thus, we easily deduce that $\mathbb{E}(\tilde M^{nm,1}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=\mathbb{E}(\tilde M^{nm,2}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=\mathbb{E}(\tilde M^{nm,3}_{\ell,\frac{i-1}{n}}|\mathcal{F}_{\frac{i-1}{n}})=0$ and using Lemma 4.3.1 we get $\mathbb{E}|\tilde N^{nm}_{\ell,\frac{i-1}{n}}|^p=O(\Delta_n^{2p})$ and $\mathbb{E}|\tilde M^{nm,1}_{\ell,\frac{i-1}{n}}|^p=\mathbb{E}|\tilde M^{nm,2}_{\ell,\frac{i-1}{n}}|^p=\mathbb{E}|\tilde M^{nm,3}_{\ell,\frac{i-1}{n}}|^p=O(\Delta_n^{3p/2})$. Finally, combining the above estimates with the obtained bounds on $\mathbb{E}(|M^{nm,\tilde\sigma}_{\frac{i-1}{n}}|)$ and $\mathbb{E}(|N^{nm,\tilde\sigma}_{\frac{i-1}{n}}|)$ for $\tilde\sigma\in\{\mathrm{Id},\sigma\}$ (see (4.4.11) and (4.4.12)), we easily get the required bounds for the moments of $M_{\frac{i-1}{n}}$ and $N_{\frac{i-1}{n}}$. □

4.8 Appendix C: Theoretical tools

4.8.1 Uniform tightness

We first recall the uniform tightness property (UT) defined in Jakubowski, Mémin, and Pagès, 1989. Let $X^n = (X^{n,i})_{1\le i\le d}$ be a sequence of $\mathbb{R}^d$-valued continuous semimartingales with the decomposition

$$X_t^{n,i} = X_0^{n,i} + A_t^{n,i} + M_t^{n,i}, \qquad 0 \le t \le T,$$

where, for each $n \in \mathbb{N}$ and $1 \le i \le d$, $A^{n,i}$ is a predictable process with finite variation, null at 0 and $M^{n,i}$ is a martingale null at 0. We say that X^n has (UT) if for each i

$$\langle M^{n,i}\rangle_T + \int_0^T |dA_s^{n,i}| \text{ is tight.} \tag{UT}$$

4.8.2 Stable convergence

Let (X_n) be a sequence of random variables with values in a Polish space E defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be an extension of $(\Omega, \mathcal{F}, \mathbb{P})$, and let X be an E-valued random variable on the extension. We say that (X_n) converges in law to X stably and write $X_n \overset{\text{stably}}{\Rightarrow} X$, as $n \to \infty$ if

$$\mathbb{E}(Uh(X_n)) \to \tilde{\mathbb{E}}(Uh(X)), \quad \text{as } n \to \infty$$

for all $h : E \to R$ bounded continuous and all bounded random variable U on $(\Omega, \mathcal{F})$. This convergence is obviously stronger than convergence in law that we will denote here by "$\overset{\text{stably}}{\Rightarrow}$".

Now, we recall the Lemma 2.1 in Jacod, 2004 about the uniform tightness property. For this aim, we consider sums of triangular arrays of the form

$$\Gamma_t^n = \sum_{i=1}^{[nt]} \zeta_i^n,$$

where for each n we have $\mathbb{R}^d$-valued random variables $(\zeta_i^n)_{i\ge 1}$ such that each ζ_i^n is $\mathcal{F}_{i/n}$-measurable.

Lemma 4.8.1. *If ζ_i^n are i.i.d. random variables and Γ_1^n converges in law to a limit U, then there is a Lévy process Γ such that $\Gamma_1 = U$. This process Γ is unique in law and Γ^n converges in law to Γ (for the Skorokhod topology). Further, the sequence (Γ^n) has (UT).*

Next, we recall the convergence theorem 3.2. of Jacod, 1997 for an $\mathbb{R}^d$-semimartingale process without jumps of form

$$Z_t^n = \sum_{i=1}^{[nt]} \chi_i^n,$$

where χ_i^n is $\mathcal{F}_{\frac{i}{n}}$-measurable.

Theorem 4.8.2. *Assume that M is a square-integrable continuous martingale, and that each χ is square-integrable. Assume also that there are two continuous processes F and G and a continuous process b of bounded variation on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le 1}, \mathbb{P})$ such* *that*

$$\sup_t |\sum_{i=1}^{[nt]} \mathbb{E}(\chi_i^n | \mathcal{F}_{\frac{i-1}{n}}) - b_t| \xrightarrow{\mathbb{P}} 0, \tag{a}$$

$$\sum_{i=1}^{[nt]} \Big(\mathbb{E}(\chi_i^n \chi_i^{n\top} | \mathcal{F}_{\frac{i-1}{n}}) - \mathbb{E}(\chi_i^n | \mathcal{F}_{\frac{i-1}{n}}) \mathbb{E}(\chi_i^{n\top} | \mathcal{F}_{\frac{i-1}{n}}) \Big) \xrightarrow{\mathbb{P}} F_t, \qquad \forall t \in [0,1], \tag{b}$$

$$\sum_{i=1}^{[nt]} \mathbb{E}(\chi_i^n \Delta M_{\frac{i-1}{n}}^\top | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{\mathbb{P}} G_t \qquad \forall t \in [0,1], \tag{c}$$

$$\sum_{i=1}^{n} \mathbb{E}(|\chi_i^n|^2 1_{|\chi_i^n| > \epsilon} | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{\mathbb{P}} 0 \qquad \forall \epsilon > 0 \qquad \textit{(Lindeberg's condition)}. \tag{d}$$

Then assume further that $d\langle M^i, M^i \rangle_t \ll dt$ and $dF_t^{ii} \ll dt$, there are predictable processes u, v, w with values in $\mathbb{R}^{D\times D}$, $\mathbb{R}^{d\times D}$ and $\mathbb{R}^{d\times d}$ respectively, such that

$$\langle M, M^\top \rangle_t = \int_0^t u_s u_s^\top ds, \qquad G_t = \int_0^t v_s u_s u_s^\top ds,$$

$$F_t = \int_0^t (v_s u_s u_s^\top v_s^\top + w_s w_s^\top) ds,$$

we have

$$Z^n \overset{\text{stably}}{\Rightarrow} Z,$$

with the limit Z can be realized on the canonical d-dimensional Wiener extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le 1}, \mathbb{P})$, with the canonical Wiener process B as

$$Z_t = b_t + \int_0^t u_s dM_s + \int_0^t w_s dB_s.$$

Remark 4.8.3. *If in the theorem above, every χ_i^n, $i \in \{1, \ldots, n\}$ have moments of order $p > 2$, then the Lindeberg's condition can be obtained by the Lyapunov condition:*

$$\sum_{i=1}^{n} \mathbb{E}(|\chi_i^n|^p | \mathcal{F}_{\frac{i-1}{n}}) \xrightarrow{\mathbb{P}} 0.$$

Now, according to Section 2 of Jacod, 1997 and Lemma 2.1 of Jacod and Protter, 1998, we have the following result

Lemma 4.8.4. *Let V_n and V be defined on $(\Omega, \mathcal{F})$ with values in another metric space E. If $V_n \xrightarrow{\mathbb{P}} V$, $X_n \overset{\text{stably}}{\Rightarrow} X$ then $(V_n, X_n) \overset{\text{stably}}{\Rightarrow} (V, X)$.*
Conversely, if $(V, X_n) \Rightarrow (V, X)$ and V generates the σ-field $\mathcal{F}$, we can realize this limit as (V, X) with X defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ and $X_n \overset{\text{stably}}{\Rightarrow} X$.

Now, we recall a result on the convergence of stochastic integrals formulated from Theorem 2.3 in Jacod and Protter, 1998.

Theorem 4.8.5. *Assume that the sequence (X^n) has (UT). Let H^n and H be a sequence of adapted, right-continuous and left-hand side limited processes all defined on* *the same filtered probability space. If* $(H^n, X^n) \overset{\text{stably}}{\Rightarrow} (H, X)$ *then* X *is a semimartingale with respect to the filtration generated by the limit process* (H, X)*, and we have* $(H^n, X^n, \int H^n dX^n) \overset{\text{stably}}{\Rightarrow} (H, X, \int HdX)$.

Now, we recall the Theorem 2.5c in Jacod and Protter, 1998.

Theorem 4.8.6. *We consider a sequence of SDE's like*

$$X_t^n = J_t^n + \int_0^t X_{s-}^n H_s^n dY_s,$$

all defined on the same filtered probability space and with the same dimensions. Also let ρ^n *be an auxiliary sequence of random variables with values in some Polish space* E*, all defined on the same space again.*
Let $V_t^n = \int_0^t H_s^n dY_s$*. Suppose the sequence* $\sup_{t\le 1} \|H_t^n\|$ *is tight and the sequence* (J^n, V^n, ρ^n) *stably converges to the limit* (J, V, ρ) *defined on some extension of the space. Then* V *is a semimartingale on some extension and* (J^n, V^n, X^n, ρ^n) *stably converges to the limit* (J, V, X, ρ) *where* X *is a solution of*

$$X_t = J_t + \int_0^t X_{s-} H_s dY_s.$$

4.8.3 Lindeberg-Feller central limit theorem

We recall also the following central limit theorem for triangular array (see, e.g. Theorem 7.2 and 7.3 in Billingsley, 1968).

Theorem 4.8.7. *Let* $(k_n)_{n\in\mathbb{N}}$ *be a sequence such that* $k_n \to \infty$ *as* $n \to \infty$*. For each* n*, let* $X_{n,1}$*,...,*X_{n,k_n} *be* k_n *independent random variables with finite variance such that* $\mathbb{E}(X_{n,k}) = 0$ *for all* $k \in \{1, \dots, k_n\}$*. Suppose that the following conditions hold:*

(1) $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k}|^2 = \vartheta$, $\vartheta > 0$.

(2) *Lindeberg's condition: For all* $\epsilon > 0$, $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}(|X_{n,k}|^2 \mathbb{1}_{|X_{n,k}|>\epsilon}) = 0$*. Then*

$$\sum_{k=1}^{k_n} X_{n,k} \Rightarrow \mathcal{N}(0, \vartheta), \qquad \text{as } n \to \infty.$$

Moreover, if the $X_{n,k}$ *have moments of order* $p > 2$*, then the Lindeberg's condition can be obtained by the following one:*

(3) *Lyapunov's condition:* $\lim_{n\to\infty} \sum_{k=1}^{k_n} \mathbb{E}|X_{n,k}|^p = 0$.

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