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Boundary value problems for quasi-linear and higher-order elliptic operators and application to bifurcation and stabilization

THÈSE DE DOCTORAT

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Abstract/Résumé/Riassunto

Abstract

In this thesis, we are interested in the study of nonlinear eigenvalue problem and the controllability of partial differential equations in a smooth bounded domain with boundary.
The first part is devoted to the analysis of an eigenvalue problem for quasi-linear elliptic operators involving homogeneous Dirichlet boundary conditions. We investigate the asymptotic behaviour of the spectrum of the related problem by showing on the one hand the bifurcation results from trivial solutions using the Krasnoselski bifurcation theorem and bifurcation from infinity using the Leray-Schauder degree on the other hand. We also prove the existence of multiple critical points using variational methods and the Krasnoselski genus.
At last, we show a stabilization result for the damped plate equation with logarithmic decay of the associated energy. The proof of this result is achieved by means of a proper Carleman estimate for the fourth-order elliptic operators involving the so-called Lopatinskiĭ-Šapiro boundary conditions and a resolvent estimate for the generator of the damped plate semigroup associated with these boundary conditions.

Keywords: quasi-linear operators, bifurcation, bifurcation from infinity, multiple solutions, Carleman estimates; stabilization, Lopatinskiĭ-Šapiro, resolvent estimate.

Résumé

Dans cette thèse, on s'intéresse à l'étude des problèmes aux valeurs propres nonlinéaires et à la contrôlabilité des équations aux dérivées partielles dans un domaine borné, régulier avec bord.
La première partie est consacrée à l'analyse d'un problème aux valeurs propres pour des opérateurs elliptiques quasi-linéaires avec des conditions aux limites homogènes de Dirichlet. Nous étudions le comportement asymptotique du spectre du problème correspondant en montrant d'une part les résultats de bifurcation à partir de solutions triviales en utilisant le théorème de bifurcation de Krasnoselski et d'autre part la bifurcation à l'infini en utilisant le degré de Leray-Schauder. Nous prouvons également l'existence de points critiques multiples en utilisant des méthodes variationnelles et le genre de Krasnoselski.
Enfin, nous montrons un résultat de stabilisation pour l'équation des plaques amorties avec une décroissance logarithmique de l'énergie associée. La preuve de

ce résultat est réalisée au moyen d'une estimation de Carleman pour les opérateurs elliptiques d'ordre quatre avec les conditions au bord dites de Lopatinskiĭ-Šapiro et d'une estimation de la résolvante pour le générateur du semigroupe de la plaque amortie associé à ces conditions aux limites.

Mots clés: operateurs quasi-linéaires, bifurcation, bifurcation à l'infini, solutions multiples, stabilisation, inégalités de Carleman, inégalité de la reslovente, Lopatinskiĭ-Šapiro.

Riassunto

In questa tesi, siamo interessati allo studio di un problema non lineare agli autovalori e alla controllabilità delle equazioni differenziali alle derivate parziali in un dominio liscio e limitato. La prima parte è dedicata all'analisi di un problema agli autovalori per operatori ellittici quasi lineari che coinvolgono condizioni al contorno omogenee di Dirichlet. Indaghiamo il comportamento asintotico dello spettro associato al problema, mostrando da un lato risultati di biforcazione da soluzioni banali usando il teorema di biforcazione di Krasnoselski, e dall'atro la biforcazione da infinito usando il grado di Leray-Schauder. Proviamo anche l'esistenza di punti critici multipli usando metodi variazionali e il genere di Krasnoselski. Infine, mostriamo un risultato di stabilizzazione per l'equazione della piastra incostrata con decadimento logaritmico dell'energia associata. La dimostrazione di questo risultato è ottenuta per tramite di una stima di Carleman appropriata per operatori ellittici del quarto ordine che coinvolgono le cosiddette condizioni al contorno di Lopatinskiĭ-Šapiro e una stima del resolvente per il generatore del semigruppo della piastra incostrata associato a tali condizioni al contorno.

Parole chiave: operatori quasi lineari, biforcazione, biforcazione da infinito, soluzioni multipli, stabilizzazione, stima di Carleman, stima del resolvente, Lopatinskiĭ-Šapiro.

Contents

1. General introduction

This thesis is done in cotutelle and consists of two parts: Part A and Part B. The first part is related to the analysis of a nonlinear eigenvalue value problem for quasi-linear operators and its applications to bifurcation. Those results were obtained during my stay at the Università degli Studi di Milano in Italy. The second part is related to control theory, more especially the stabilization of the damped plate equation. The latter results were obtained during my stay at the Université Sorbonne Paris Nord in France.
The goal of Part A is to study the asymptotic behavior of the spectrum and the existence of multiple solutions of the following eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.0.1}$$

where $p \in (1, \infty)\backslash\{2\}$ is a real number.
The goal of Part B is to answer the following question: How fast does the energy of the following hyperbolic equation decrease?

$$\begin{cases} \partial_t^2 y + \Delta^2 y + \alpha(x)\partial_t y = 0 \quad (t, x) \in \mathbb{R}_+ \times \Omega, \\ B_1 y_{|\mathbb{R}_+ \times \partial\Omega} = B_2 y_{|\mathbb{R}_+ \times \partial\Omega} = 0, \\ y_{|t=0} = y^0, \ \partial_t y_{|t=0} = y^1, \end{cases}$$

where $\alpha \geq 0$ and where B_1 and B_2 denote two boundary differential operators.

Part A: Nonlinear eigenvalue problems

Nonlinear eigenvalue problems arise in many areas of computational science and engineering, including acoustics, control theory, fluid mechanics and structural engineering. We give the formulation of such a problems.

1.1 Description

Nonlinear eigenvalue problems are in general described by equation of the form

$$G(\lambda, u) = 0 \quad \lambda \in \mathbb{K},\ u \in X, \tag{1.1.1}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and X is a Banach space that can be the n-space $\mathbb{R}^n$ of $\mathbb{C}^n$. In equation (1.1.1), G is a continuous map of $\mathbb{K} \times X$ into X. It is assumed that $G(\lambda, 0) = 0$ for all λ, that is, $u = 0$ solves trivially equation (1.1.1) for all scalars λ. Then, one looks for those λ's, i.e, the eigenvalues of G, such that equation (1.1.1) has a solution $u \neq 0$ (an eigenvector or eigenfunction of G corresponding to λ).
A typical example of nonlinear eigenvalue problem is the so-called p-Laplace equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1.2}$$

where $p > 1$ and Ω is a bounded open set in $\mathbb{R}^n$. Let X be the Sobolev space $W_0^{1,p}(\Omega)$. Let $X' = W^{-1,p'}(\Omega)$ be the dual space of X. A weak solution of the p-Laplace equation (1.1.2) is a function $u \in X$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \lambda \int_\Omega |u|^{p-2}uv \, dx,$$

for all $v \in W_0^{1,p}(\Omega)$. The proof of the existence of countably many eigenvalues and eigenfunctions of equation (1.1.2) relies on the Lusternik-Schnirelmann theory of critical points for an even functional on a symmetric manifold. Presentations of this theory in both finite and infinite dimensional spaces can be found among others in [6, 55].

1.2 Bifurcation from the eigenvalues of the p-Laplacian

For instance to study the bifurcation phenomena associated with the p-Laplacian under Dirichlet boundary conditions, one considers the following problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + h(x,u,\lambda) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where $h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition in the two variables (x,t) and

$$h(x,t,\lambda) = o(|t|^{p-1})$$

near $t = 0$, uniformly a.e with respect to x and uniformly with respect to λ on bounded sets. We recall that $-\Delta_p u := -\text{div}(|\nabla u|^{p-2}\nabla u)$ for all $p > 1$. By a solution of (1.2.1) we understand a couple $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega)$ satisfying the following integral equality in the weak sense,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \lambda \int_\Omega |u|^{p-2}uv \, dx + \int_\Omega h(x,u,\lambda)v dx,$$

for all $v \in W_0^{1,p}(\Omega)$. We say that $(\lambda^*, 0)$ is a bifurcation point of (1.2.1) if in any neighborhood of $(\lambda^*, 0)$ in $\mathbb{R} \times W_0^{1,p}(\Omega)$ there is a nontrivial solution of (1.2.1). One speaks of bifurcation from trivial solution.

In [68] Proposition 2.1, it is shown with a compactness argument that a necessary condition for $(\lambda^*, 0)$ to be a bifurcation point of (1.2.1) is that λ^* be an eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\lambda_1(p)$ denote the first eigenvalue of (1.1.2). We note that $\lambda_1(p)$ can be characterized variationally as

$$\lambda_1(p) = \inf\left\{\int_\Omega |\nabla u|^p dx : \ u \in W_0^{1,p}(\Omega), \ \int_\Omega |u|^p dx = 1\right\}.$$

In [68] Theorem 1.1, it is shown that $(\lambda_1(p), 0)$ is a bifurcation point of (1.2.1). This result is well known in the case $p = 2$ (see, [78]). The key ingredient in the proof is the index formula which is proved via a suitable homotopic deformation from a general $p > 1$ to $p = 2$.

Introducing the following change of variable $w = u/\|u\|_{1,p}^2$ for $u \in W_0^{1,p}(\Omega)$ with

$u \neq 0$, into equation (1.2.1), this leads to the equation

$$\begin{cases} -\Delta_p w = \lambda |w|^{p-2} w + \|w\|_{1,p}^{2(p-1)} h(x, w, \lambda) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.2}$$

With this transformation, we have that the pair (λ^*, ∞) is a bifurcation point for the problem (1.2.1) if and only if the pair $(\lambda^*, 0)$ is a bifurcation point for the problem (1.2.2). One says that the point (λ^*, ∞) is a bifurcation point from infinity for problem (1.2.1).

We write equation (1.1.1) in the form

$$G(\lambda, u) = u - \lambda L u - K(\lambda, u), \tag{1.2.3}$$

where L is a compact linear operator and $K : \mathbb{R} \times X \to X$ is compact with $K = o(\|u\|_X)$ at $u = 0$ uniformly on bounded λ intervals. In this context, Krasnoselski [53] has shown that if μ is a real characteristic value of L of odd multiplicity, then $(\mu, 0)$ is a bifurcation point for G. With a suitable change of variable, one shows that if L is compact and linear, μ is a real characteristic value of L of odd multiplicity, $K = o(\|u\|_X)$ at $u = \infty$ uniformly on bounded λ intervals and is appropriately compact, then μ is a bifurcation point for $u = \lambda L u + K(\lambda, u)$. We will address a problem in the form (1.2.3), which corresponds to the first results concerning Part A given in Section 3.3.1.

1.3 Main results of Part A

In this section we give the main results concerning the bifurcation from the trivial solutions and from infinity that are fully presented in Section 3.7. In addition we state the result about the existence of multiple solutions, also presented in Section 3.8.

1.3.1 Bifurcation results

We set $S_\lambda(u) = u - \lambda(-\Delta_p - \Delta)u$, for $u \in L^2(\Omega) \subset W^{-1,p'}(\Omega)$ and $\lambda > 0$. We recall that λ_k^D stands for the $k-$th Dirichlet eigenvalue of the Laplacian.

Theorem 1.3.1. (*bifurcation from zero*) *Let $p > 2$. Then every eigenvalue λ_k^D with odd multiplicity is a bifurcation point in $\mathbb{R}_+^* \times W_0^{1,p}(\Omega)$ of $S_\lambda(u) = 0$, in the sense that in any neighborhood of $(\lambda_k^D, 0)$ in $\mathbb{R}_+^* \times W_0^{1,p}(\Omega)$ there exists a nontrivial solution of $S_\lambda(u) = 0$.*

Introducing a suitable change of variable in equation (1.0.1), leads us to set $\tilde{S}_\lambda(u) = u - \lambda(-\|u\|_{1,2}^{\gamma}\Delta_p - \Delta)u$ with $\gamma = 4 - p$ for $1 < p < 2$, $\lambda > 0$ and $u \in B_r(0) \subset L^2(\Omega) \subset W^{-1,2}(\Omega)$.

Theorem 1.3.2. (bifurcation from infinity) *The pair $(\lambda_1^D, 0)$ is a bifurcation point from infinity for the problem (1.0.1).*

With the change of variable, Theorem 1.3.2 is equivalent to the following theorem.

Theorem 1.3.3. *The pair $(\lambda_1^D, 0)$ is a bifurcation point in $\mathbb{R}^+ \times L^2(\Omega)$ of $\tilde{S}_\lambda(u) = 0$, for $1 < p < 2$.*

A more general result of Theorem 1.3.3 is the following.

Theorem 1.3.4. *The pair $(\lambda_k^D, 0)$ $(k > 1)$ is a bifurcation point of $\tilde{S}_\lambda(u) = 0$ for $1 < p < 2$ if λ_k^D is of odd multiplicity.*

1.3.2 Multiple solutions

We have obtained the following result.

Theorem 1.3.5. *Let $1 < p < 2$ or $2 < p < \infty$, and suppose that $\lambda \in (\lambda_k^D, \lambda_{k+1}^D)$ for any $k \geq 1$. Then the equation (1.0.1) has at least k pairs of nontrivial solutions.*

Part B: Control theory

Control theory is an important subject in science engineering. Control theory deals with the behaviour of dynamic systems and how to control such systems. More precisely it deals with a dynamical system on which one can act by using suitable controls.
We refer to the book of Jean-Michel Coron [26] for more details on the notions of control theory.

1.4 Some different notions of control

We consider a system of differential or partial differential equation of the form

$$\begin{cases} \partial_t u = \mathcal{K}(u, f) \\ u(0) = u_0, \end{cases} \tag{1.4.1}$$

where $\forall t \geq 0$, $u(t)$ is the sought solution belonging to a certain state-space H (a Banach or Hilbert space) $\mathcal{K}$ an operator that describes the system, and f a source term acting on the system. Natural questions are : knowing f and the initial data u_0, is it possible to recover the solution u? Is this continuous with respect to the data of the problem ? In control theory, problematics are different. Here, we consider the question of controllability: given the initial data u_0 as well as a couple of state-time target (u_T, T), can we find a control f belonging to some control space $\mathcal{X}$ such that the solution u to (1.4.1) satisfies $u(T) = u_T$? This means, controlling the solution so that it reaches a desired state at the desired time.
We recall some notions of controllability.

Definition 1.4.1. (exact controllability) Let $T > 0$. We say that the control system (1.4.1) is exactly controllable in time T, if for every $u_0 \in H$ and for any target state $u_T \in H$, there exists a control $f \in L^2((0,T); \mathcal{X})$ such that the solution u of the Cauchy problem (1.4.1) satisfies $u(T) = u_T$.

Definition 1.4.2. (approximate controllability) Let $T > 0$. We say that the control system (1.4.1) is approximately controllable in time T, if for every $u_0 \in H$ and for any target state $u_T \in H$, and for every $\varepsilon > 0$, there exists a control $f \in L^2((0,T); \mathcal{X})$ such that the solution u of the Cauchy problem (1.4.1) satisfies $\|u(T) - u_T\|_H < \varepsilon$.

Definition 1.4.3. (null controllability) Let $T > 0$. We say that the control system (1.4.1) is null controllable in time T if for every $u_0 \in H$, there exists a control $f \in L^2((0,T);\mathcal{X})$ such that the solution u of the Cauchy problem (1.4.1) associated to f satisfies $u(T) = 0$.

In the case of a system of linear ordinary differential equations in finite dimension of the form

$$\begin{cases} \partial_t u = Au + Bf \\ u(0) = u_0, \end{cases} \tag{1.4.2}$$

where A and B are matrices of respective sizes $n \times n$ and $n \times m$, $u(t) \in \mathbb{R}^n$, and $f(t) \in \mathbb{R}^m$, these three notions of controllability are equivalent. This is not the case in the context of partial differential equations. A necessary and sufficient exact controllability criterion for the finite-dimensional system (1.4.2) exists: the so-called Kalman criterion.

Theorem 1.4.4. (Kalman rank condition in finite dimension) *The time invariant linear control system (1.4.2) is controllable in time T if and only if*

$$\operatorname{rank}\big(B \mid AB \mid \ \dots \mid A^{n-1}B\big) = n.$$

In this theorem, we remark that the criterion does not depend on the chosen time $T > 0$, so we can deduce the following: if the system is controllable in some time $T > 0$, then it is controllable for any time $T' > 0$. However, this criterion does not specify how the control depends on the time. In the context of partial differential equations, some control systems are not exactly controllable in any time T. Some are exactly controllable if T is chosen sufficiently large. Then one speaks of minimal time of control. This is in articular the case of hyperbolic equations since the influence of the control is limited by the finite speed of propagation.
These notions of control have some weakness: they depend strongly on the initial data, and do not depend on the state of the system when $0 < t < T$. It is often important to consider problems with feedback, where the control acts on the system by responding continuously to the system (which does not undergo any interruption in time). It is particularly the case when studying *stablization* problems. Such systems can be written as

$$\begin{cases} \partial_t u = \mathcal{K}(u, f(u)) \\ u(0) = u_0. \end{cases} \tag{1.4.3}$$

Defining an energy depending on the solution u of (1.4.3), we are then interested in the decay properties of this energy.
When the control function f depends on the solution u and when the system becomes dissipative (for instance if absorbing boundary conditions or damped

terms are involved), the energy is a positive time-decreasing function. Therefore, we study the long time asymptotic behavior of the energy. In particular, the choice of various regularity for the initial data and/or geometrical hypotheses gives different estimates for the decay rate of the energy.

In general the proof of the controllability of a linear system relies on the proof of an observability criterium.

1.5 From control to observability

Here, we present how the null controllability can be reduced to obtaining a functional inequality, known as observability, by a duality argument. These inequalities play a central role in the study of control and stabilization of partial differential equations. Indeed, constructing a control is in general a difficult problem. On the other hand, there are many existing tools that can be used to show functional inequalities, such as energy inequalities, Carleman inequalities, Ingham inequalities, or Gårding's inequalities.

We place ourselves in the following case

$$\begin{cases} \partial_t u = Au + Bf, \\ u(0) = u_0 \in H, \end{cases} \tag{1.5.1}$$

where A is an operator acting on a Hilbert space H with domain $D(A) \subset H$. We suppose that A is the generator of a strongly continuous semigroup, denoted $\mathbb{S}(t)$. We assume that the control operator B acts on the control space $\mathcal{X}$ (also a Hilbert space) and in addition we assume that $B \in \mathcal{L}(\mathcal{X}, H)$ for simplicity. Note that this condition can be relaxed, then we must replace the continuity of the operator B by the so-called admissibility condition (see [95] for more details). This is particularly useful if one wishes to control a partial differential equation from the boundary of the domain. One acts on the system by means of the operator B, and in general, it restricts the possibilities of action. The definition of the null controllability of system (1.5.1) is given by Definition 1.4.3.
With the assumptions made on $\mathbb{S}(t)$ and B, we can write the solution of (1.5.1) with the Duhamel formula

$$u(t) = \mathbb{S}(t)u_0 + \int_0^t \mathbb{S}(t-s)Bf(s)ds. \tag{1.5.2}$$

Let $B^* \in \mathcal{L}(H, \mathcal{X})$ be the adjoint of B (the dual spaces H' and $\mathcal{X}'$ are identified to H and $\mathcal{X}$ respectively). We introduce here the following dual system on

$(0,T)$,

$$\begin{cases} \partial_t v = -A^* v, \quad \text{on} \quad (0,T) \\ v(T) = v_T \in H, \end{cases} \tag{1.5.3}$$

where, A^* is the adjoint operator of A in H. The generator of the adjoint semigroup $\mathbb{S}^*(t) = (e^{tA})^*$ is A^*, and the solution of (1.5.3) can be written $v(t) = e^{(T-t)A^*}v_T$, where $e^{(T-t)A^*} = (e^{(T-t)A})^*$. Note that (1.5.3) is homogeneous.
The question of the observability is the following : Is it possible, by observing only the quantity $B^*v(t)$, to know the energy of the system (1.5.3) at time $t = 0$, i.e, $\|v(0)\|_H^2$? The observability notion of system (1.5.3) is given by the following definition.

Definition 1.5.1. One says that the system (1.5.3) is observable at time $T > 0$ if there exists a constant $C_{Obs,T} > 0$ such that for every $v_T \in H$, the solution of (1.5.3) satisfies

$$\|\mathbb{S}^*(T)v_T\|_H \leq C_{Obs,T}\|B^*v(t)\|_{L^2((0,T);\mathcal{X})} = C_{Obs,T}\left(\int_0^T \|B^*v(t)\|_{\mathcal{X}}^2 dt\right)^{1/2}. \tag{1.5.4}$$

We note that $\mathbb{S}^*(t)v_T = v(0)$, where v is the solution of (1.5.3). The constant $C_{Obs,T}$ is called the constant of observability. This notion of observability has its own interest because it appears in many concrete situations when one would like to know the state of a system on which one can only make partial measurements. This is the case for instance in meteorology, in imaging or, more generally, in the field of inverse problems.
Another interest of the observability for system (1.5.3) lies in its link with the null controllability of the initial system (1.5.1). We then have the following result, proved by S. Dolecki and D. L. Russell [31], and J.-L. Lions [66].

Theorem 1.5.2. *The null controllability at time $T > 0$ of the system (1.5.3) is equivalent to the inequality (1.5.4) with a constant $C_{Obs,T} > 0$. Moreover if (1.5.4) holds with constant $C_{Obs,T}$, then one can find a control $f \in L^2((0,T),\mathcal{X})$ satisfying $\|f\|_{L^2((0,T),\mathcal{X})} \leq C_{Obs,T}\|u_0\|_H$.*

We also note that the exact controllability of system (1.5.1) is equivalent to the observability of system (1.5.3). More precisely, in this case the operator $\mathbb{S}^*(T)$ is replaced by the identity Id in the left part of inequality (1.5.4).

1.6 Example of stabilization: case of the wave equation

We give an example concerning the stabilization of the wave equation. Let Ω be a smooth open bounded subset of $\mathbb{R}^d$ and ω an open subset of Ω. We consider the following wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + \alpha(x)\partial_t u = 0 \quad \text{on } [0,\infty) \times \Omega \\ u_{|t=0} = u_0, \ \partial_t u_{|t=0} = u_1, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.6.1}$$

where $\alpha(x)$ is a nonnegative bounded function that satisfies $\alpha(x) \geq C > 0$, for $x \in \omega$. Multiplying (1.6.1) by $\partial_t u$ and integrating over Ω, we obtain

$$\int_\Omega \partial_t^2 u \ \partial_t u \ dx + \int_\Omega \nabla_x u \nabla_x \partial_t u \ dx + \int_\Omega \alpha(x) \partial_t u \ \partial_t u \ dx = 0,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\left[\|\partial_t u\|^2_{L^2(\Omega)} + \|\nabla_x u\|^2_{L^2(\Omega)}\right] = -\|\alpha(x)^{1/2}\partial_t u\|^2_{L^2(\Omega)}.$$

Introducing the H^1-energy

$$\mathcal{E}(u,t) := \frac{1}{2}\left(\|\partial_t u\|^2_{L^2(\Omega)} + \|\nabla_x u\|^2_{L^2(\Omega)}\right),$$

one finds

$$\frac{d}{dt}\mathcal{E}(u,t) = -\|\alpha(x)^{1/2}\partial_t u\|^2_{L^2(\Omega)} \leq 0.$$

One calls $\alpha(x)\partial_t u$ the *damping term*; it is the responsible for the decay of the energy. One refers to (1.6.1) as the damped wave equation. One proves that the energy decays to zero. A natural question is thus the study of the rate of convergence of this energy, i.e., to obtain an estimate of the type

$$\mathcal{E}(u,t) \leq h(t) G(u_0, u_1),$$

where h is a decreasing function that tends to zero at infinity and G a function. The weak stabilization consists in showing that for any (u_0, u_1) in a suitable space, $\lim_{t\to+\infty} \mathcal{E}(u,t) = 0$ and the strong stabilization consists to show, under suitable conditions the existence of $C > 0$ and $K > 0$ such that for any (u_0, u_1) in a suitable space, we have a uniform and exponential decay rate

$$\mathcal{E}(u,t) \leq K e^{-Ct}\mathcal{E}(u,0).$$

The exponential decay of the energy can be achieved if the so-called geometrical control condition (GCC) is fulfilled. This result was first proven in dimension one or on manifolds without boundary by [80, J. Rauch and M. Taylor]. The geometrical control condition expresses that all bicharacteristics (or rays of geometrical optics) reach the damping region in a finite time. The generalization of this exponential decay result to domains with boundaries in the case of homogeneous Dirichlet and Neumann conditions was proven [57,58, C. Bardos , G. Lebeau and J. Rauch]. Bicharacteristics are then replaced by so-called generalized bicharacteristics that obey the laws of reflection at a boundary. There are introduced by R. Melrose and J. Sjöstrand [70, 71] to describe the propagation of singularities. The proof in [57] is based on this description of propagation of singularities.
Note that exponential decay of the energy is equivalent to having an observability estimate for the wave equation (without damping), the observation being located in the region where the damping acts; we refer to the work of A. Haraux [38]. For the wave equation, stabilization (and equivalently observability) can also be expressed by means of the so-called Hautus test for the resolvent; we refer to the works of D. Russel and G. Weiss [84] and L. Miler [75]. Under weaker geometrical conditions one can obtain a polynomial decay rate of the damped wave equation. We shall be interested in such weaker decay rates here.

1.7 Carleman estimates

In 1939, T. Carleman introduced some energy estimates with exponential weights to prove a uniqueness result for some elliptic partial differential equations (PDE) with smooth coefficients in dimension two [21]. This type of estimate, now referred to as Carleman estimates, were generalized by L. Hörmander and others for a large class of differential operators in arbitrary dimensions (see, [42,43,98]). Carleman estimates are weighted a priori inequalities for the solutions of a partial partial differential equation of the form

$$\|e^{\tau\varphi}u\|_{L^2(\Omega)} \leq \|e^{\tau\varphi}Pu\|_{L^2(\Omega)}, \tag{1.7.1}$$

where P is a differential operator, u a function, φ a function called the weight function and $\tau > 0$ a large parameter.
The interest of such inequalities is the presence of the weight function φ which allows to "propagate" information from areas where φ is large to the whole domain, by means of the large parameter $\tau > 0$, known as large Carleman parameter. Additional terms on the left-hand side of the inequality can be obtained, including higher-order derivatives of the function u, depending of

course of the order of the operator P itself. For a second-order elliptic operator such as the Laplace operator one has

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2(\Omega)} + \tau^{1/2}\|e^{\tau\varphi}Du\|_{L^2(\Omega)} + \tau^{-1/2}\sum_{|\beta|=2}\|e^{\tau\varphi}D^\beta u\|_{L^2(\Omega)} \leq C\|e^{\tau\varphi}\Delta u\|_{L^2(\Omega)},$$

under the so-called sub-ellipticity condition; see Chapter 3 in [62]. Note that the power of the large parameter τ adds to $3/2$ with the order of the derivative in each term on the left-hand side. In fact, in the calculus used to derive such estimates one power of τ is equivalent to a derivative of order one. Thus with this $3/2$ compared with the order two of the operator one says that one looses a half-derivative in the estimate.

In more recent years, the field of applications of Carleman estimates has gone beyond the original domain; they are also used in the study of :

- Inverse problems, where Carleman estimates are used to obtain stability estimates for the unknown sought quantity (e.g. coefficient, source term) with respect to norms on measurements performed on the solution of the PDEs see e.g. [18, 45, 54, 96]; Carleman estimates are also fundamental in the construction of complex geometrical optic solutions that lead to the resolution of inverse problems such as the Calderón problem with partial data [85, 86].

- Control theory for PDEs; through unique continuation properties, Carleman estimates are used for the exact controllability of hyperbolic equations [57]. They also yield the null controllability of linear parabolic equations [60] and the null controllability of classes of semi-linear parabolic equations [9, 33, 36].

For a function supported near a point at the boundary, in normal geodesic coordinates where Ω is locally given by $\{x_d > 0\}$ (see Section 4.5.2 below) the estimate can take the form

$$\sum_{|\beta|\leq 2}\tau^{3/2-|\beta|}\|e^{\tau\varphi}D^\beta u\|_{L^2(\Omega)} + \sum_{|\beta|\leq 1}\tau^{3/2-|\beta|}|e^{\tau\varphi}D^\beta u_{|x_d=0^+}|_{L^2(\Omega)} \leq C\|e^{\tau\varphi}\Delta u\|_{L^2(\Omega)}.$$

This is the type of estimate we seek here for the operator $P_\sigma = \Delta^2 - \sigma^4$, with some uniformity with respect to σ. The main results concerning Part B are the following.

1.8 Main results of Part B

We state the main Carleman estimate for the operator P_σ in normal geodesic coordinates as presented in Section 4.5.2.

1.8.1 Carleman estimate

A point $x^0 \in \partial\Omega$ is considered and a weight function φ is assumed to be defined locally and such that

(1) $\partial_d \varphi \geq C > 0$ locally.

(2) $(-\Delta \pm \sigma^2, \varphi)$ satisfies the sub-ellipticity condition of Definition 4.9.1 locally. This is a necessary and sufficient condition for a Carleman estimate to hold for a second-order operator $-\Delta \pm \sigma^2$ to hold regardless of boundary conditions [62, Chapters 3 and 4].

(3) $(P_\sigma, B_1, B_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.7.1 at $\varrho' = (x^0, \xi', \tau, \sigma)$ for all $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ such that $\tau \geq \kappa_0 \sigma$, for some $\kappa_0 > 0$. This means that the Lopatinskiĭ-Šapiro condition holds after the conjugation of the operator P_σ and the boundary operators B_1 and B_2 by the weight function $\exp(\tau\varphi)$.

Theorem 1.8.1 (Carleman estimate)**.** *Let $\kappa_0' > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$. Let φ be such that the properties above hold locally. Then, there exists W^0 a neighborhood of x^0, $C > 0$, $\tau_0 > 0$ such that*

$$\tau^{-1/2} \|e^{\tau\varphi} u\|_{4,\tau} + |\operatorname{tr}(e^{\tau\varphi} u)|_{3,1/2,\tau} \leq C\big(\|e^{\tau\varphi} P_\sigma u\|_+ + \sum_{j=1}^{2} |e^{\tau\varphi} B_j u_{|x_d=0^+}|_{7/2-k_j,\tau}\big), \tag{1.8.1}$$

for $\tau \geq \tau_0$, $\kappa_0\sigma \leq \tau \leq \kappa_0'\sigma$, and $u \in \overline{\mathscr{C}}_c^\infty(W_+^0)$.

The volume norm is given by

$$\|e^{\tau\varphi} u\|_{4,\tau} = \sum_{|\beta| \leq 4} \tau^{4-|\beta|} \|e^{\tau\varphi} D^\beta u\|_{L^2(\Omega)}.$$

The trace norm is given by

$$|\operatorname{tr}(e^{\tau\varphi} u)|_{3,1/2,\tau} = \sum_{0 \leq n \leq 3} |\partial_\nu^n (e^{\tau\varphi} u)_{|x_d=0^+}|_{7/2-n,\tau},$$

where the norm $|.|_{7/2-n,\tau}$ is the L^2-norm in $\mathbb{R}^{d-1}$ after applying the Fourier multiplier $(\tau^2 + |\xi'|^2)^{7/4-n/2}$.

Observe that the Carleman estimate of Theorem 1.8.1 exhibits a loss of a half-derivative. A more precise statement is given in Theorem 4.10.3 in Section 4.10.2.

1.8.2 Stabilization result

Let $(\mathsf{P}_0, D(\mathsf{P}_0))$ be the unbounded operator on $L^2(\Omega)$ given by the domain

$$D(\mathsf{P}_0) = \big\{u \in H^4(\Omega);\ B_1 u_{|\partial\Omega} = B_2 u_{|\partial\Omega} = 0\big\} \tag{1.8.2}$$

given by $\mathsf{P}_0 u = \Delta^2 u$ for $u \in D(\mathsf{P}_0)$. As written above the two boundary differential operators are such that $(\mathsf{P}_0, D(\mathsf{P}_0))$ is self-adjoint and nonnegative.

Let $y(t)$ be a strong solution of the plate equation (4.5.1). A precise definition of strong solutions is given in Section 4.12.3. One has $y^0 \in D(\mathsf{P}_0)$ and $y^1 \in D(\mathsf{P}_0^{1/2})$. Its energy is defined as

$$\mathcal{E}(y)(t) = \frac{1}{2}\big(\|\partial_t y(t)\|^2_{L^2(\Omega)} + (\mathsf{P}_0 y(t), y(t))_{L^2(\Omega)}\big).$$

Theorem 1.8.2 (logarithmic stabilization for the damped plate equation)**.** *There exists $C > 0$ such that for any such strong solution to the damped plate equation (4.5.1) one has*

$$\mathcal{E}(y)(t) \leq \frac{C}{\big(\log(2+t)\big)^4}\big(\|\mathsf{P}_0 y^0 + \alpha y^1\|^2_{L^2(\Omega)} + \|\mathsf{P}_0^{1/2} y^1\|^2_{L^2(\Omega)}\big).$$

A more precise and more general statement is given in Theorem 4.13.3 in Section 4.13.2.

2. Some elements of functional analysis

2.1 General facts

We deal with functions u defined almost everywhere (a.e.) on an open bounded domain $\Omega \subset \mathbb{R}^d$. A point $x \in \Omega$ is written $x = (x_1, \cdots, x_d)$. The boundary of Ω will be denoted by $\partial\Omega$. We suppose that $\partial\Omega$ is sufficiently smooth, let say at least of class $\mathscr{C}^2$.

2.1.1 Functions space and definitions.

2.1.1.1 Functions spaces

Definition 2.1.1. A normed space X is a called a Banach space if it is complete, i.e., if every Cauchy sequence is convergent, that is,

$$\{u_n\}_{n\in\mathbb{N}} \text{ is a Cauchy in } X \Rightarrow \exists u \in X \text{ such that } u_n \to u \text{ as } n \to \infty.$$

For functions u (measurable) defined a.e. in Ω, the following spaces will be considered.

(i) The Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$, where

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \; ; \int_\Omega |u|^p \; dx < \infty \right\},$$

for $1 \leq p < \infty$. We denote by

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \; dx \right)^{\frac{1}{p}} \text{ the norm on } L^p(\Omega).$$

For $p = \infty$, by $L^\infty(\Omega)$ we denote the space of measurable functions u which are essentially bounded over Ω equipped with the norm

$$\|u\|_\infty = \inf\{M \in \mathbb{R}^+ \; : |u(x)| \leq M \; a.e. \text{ in } \Omega\}. \tag{2.1.1}$$

The norms that make $L^p(\Omega)$ Banach space are respectively $\|.\|_p$ and $\|.\|_\infty$. The space $L^p(\Omega)$ is reflexive for $1 < p < \infty$ with its dual denoted by $L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

(ii) By $\mathscr{C}_c^\infty(\Omega)$ we denote the set of all functions u defined and infinitely differentiable on Ω such that their support $\mathrm{supp}(u)$ is compact and satisfies

$\text{supp}(u) \subset \Omega$. Recall that $\text{supp}(u)$ is the closure of the set $\{x \in \Omega; u(x) \neq 0\}$.

(iii) For α a multi-index, $|\alpha| \geq 1$, the function v_α is called a weak (or distributional) derivative of u (of order α) if the identity

$$\int_\Omega v_\alpha(x)\varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \varphi(x)\, dx$$

holds true for every $\varphi \in \mathscr{C}_c^\infty(\Omega)$. Then v_α is denoted by $D^\alpha u$.

(iv) Sobolev spaces. For $k \in \mathbb{N}$ and $1 \leq p < \infty$ we denote by $W^{k,p}(\Omega)$ the set of all functions $u \in L^p(\Omega)$ for which the weak derivatives $D^\alpha u$ with $|\alpha| \leq k$ exists in the weak sense and belongs to $L^p(\Omega)$ as well.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space (uniformly convex and hence reflexive if $1 < p < \infty$) if equipped with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}. \tag{2.1.2}$$

Further, the space $W_0^{k,p}(\Omega)$ is defined as the closure of $\mathscr{C}_c^\infty(\Omega)$ with respect to the norm $\|.\|_{k,p}$. For Ω bounded the expression

$$\|u\|_{k,p} = \left(\sum_{|\alpha| = k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}$$

is a norm on $W_0^{k,p}(\Omega)$ equivalent to the one defines in (2.1.2). This is based on the Poincaré inequality, see below Proposition 2.1.4.

2.1.1.2 Some definitions

Definition 2.1.2. Let X be a Banach space. A sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ converges weakly to $u \in X$, in which case we write $u_n \rightharpoonup u$ in X if $f(u_n) \to f(u)$ for all $f \in X'$ (the dual space of X). In addition if $u_n \rightharpoonup u$ in X, then u_n is bounded in X and

$$\|u\|_X \leq \liminf_{n\to\infty} \|u_n\|_X.$$

Definition 2.1.3. One says that the sequence u_n converges strongly to u in X, in which case we note $u_n \to u$ in X, if $u_n, u \in X$ and if

$$\lim_{n\to\infty} \|u_n - u\|_X = 0.$$

Definition 2.1.4. Let x be an element of the open subset $U \subset X$. The mapping $F : U \to Y$ is Fréchet-differentiable at $x \in U$ if there exists a linear operator $A \in \mathscr{L}_c(X, Y)$, such that

$$F(x+h) - F(x) - Ah = o(\|h\|) \text{ with } F'(x) = A.$$

We use the notation $r(h) = o(\|h\|_X)$ for the mapping $r : X \to Y$ if and only if

$$\lim_{h \to 0} \frac{\|r(h)\|_Y}{\|h\|_X} = 0,$$

which means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|h\|_X < \delta$ then $\|r(h)\|_Y < \varepsilon \|h\|_X$.

Definition 2.1.5. Let X, Y be a Banach spaces and $U \subset X$ a sub-open set of X. Let $F : U \to Y$ be a mapping and $x \in X$. We say that F is Gâteaux-differentiable at x if there exists $A \in \mathscr{L}_c(X, Y)$, such that

$$\lim_{t \to 0} \frac{F(x+th) - F(x)}{t} = Ah, \text{ for all } h \in U.$$

The mapping A is uniquely determined. It is call Gâteaux-derivative of F at x and is denoted by $F'_G(x)$. If $Y = \mathbb{R}$, F is said to be a functional.

Definition 2.1.6. Let X be a Banach space, $U \subseteq X$ an open set and assume that $I : U \to \mathbb{R}$ is differentiable. A critical point of I is a point $u \in U$ such that $I'(u) = 0$. As $I'(u)$ is an element of the dual space X', this means of course $\langle I'(u), v \rangle = 0$ for all $v \in X$.

2.1.1.3 well-known results

Theorem 2.1.7. (Gauss-Green)
Let $U \subset \mathbb{R}^d$ be a given open set and Ω any smooth subregion within U. Then we have

$$\int_\Omega div(F)\ dx = \int_{\partial\Omega} F \cdot \nu\ dS,$$

where F denote the flux density ($F \in C^1(\overline{U}, \mathbb{R}^d)$) and ν the unit outer normal field.

Corollary 2.1.8. (Integration by parts) *Let Ω be a regular open set of class C^1. Let u and v be two $C^1(\overline{\Omega})$ functions with bounded support in the closed set $\overline{\Omega}$. Then they satisfy the integration by parts formula*

$$\int_\Omega u(x) \frac{\partial v}{\partial x_i}(x)\ dx = -\int_\Omega v(x) \frac{\partial u}{\partial x_i}(x)\ dx + \int_{\partial\Omega} u(x) v(x) \nu_i(x)\ dS. \qquad (2.1.3)$$

Proposition 2.1.9. (Hölder inequality) *Let Ω be an open subset of $\mathbb{R}^d$ and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in L^p(\Omega), g \in L^q(\Omega)$, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Proposition 2.1.10. (Poincaré inequality) *Let Ω be a bounded domain. There exists a positive constant C_p such that, for every $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_p \leq C_p \|\nabla u\|_p. \tag{2.1.4}$$

Theorem 2.1.11. (Rellich-Kondrachov embedding) *Suppose that $\Omega \subset \mathbb{R}^d$ is bounded and of class C^1. Then we have following compact injections.*

i. $W^{1,p}(\Omega) \subset\subset L^q(\Omega), \ \forall \ q \in [1, p^\star)$, *where* $\frac{1}{p^\star} = \frac{1}{p} - \frac{1}{d}$, *if* $p < d$.

ii. $W^{1,p}(\Omega) \subset\subset L^q(\Omega), \ \forall \ q \in [p, \infty)$, *if* $p = d$.

iii. $W^{1,p}(\Omega) \subset\subset C(\overline{\Omega})$ *if* $p > d$.

We have the following theorem.

Theorem 2.1.12. Suppose that $f : U \to \mathbb{R}$ has a continuous Gâteaux-derivative on U. Then f is Fréchet-differentiable and $f \in C^1(U, \mathbb{R})$.

Lemma 2.1.13. (Fatou's lemma)
Let $\{f_n\}$ be a sequence of functions in $L^1(\Omega)$ that satisfy

i. for all n, $f_n \geq 0$ a.e

ii. $\sup_n \displaystyle\int_\Omega f_n < \infty$.

For almost all $x \in \Omega$ we set $f(x) = \liminf_{n\to\infty} f_n(x) < \infty$. Then $f \in L^1(\Omega)$ and

$$\int_\Omega f \ dx \leq \liminf_{n\to\infty} \int_\Omega f_n \ dx.$$

2.1.2 Linear operators in Banach spaces

Here, X and Y will denote Banach spaces with their norms denoted by $\|\cdot\|_X$, $\|\cdot\|_Y$ or simply $\|\cdot\|$ if no confusion is possible.

An operator A from X to Y is a linear map on its domain. One denotes by $D(A)$ the domain of A. An operator from X to Y is thus characterized by its domain and how it acts on this domain. Operators defined this way are usually referred to as unbounded operators. One writes $(A, D(A))$ to denote the operator along with its domain. The set of linear operators from X to Y is denoted by $\mathscr{L}(X,Y)$. The set of linear and continuous operator from X to Y will be denoted by $\mathscr{L}_c(X,Y)$

If $D(A)$ is dense in X the operator A is said to be densely defined. If $D(A) = X$ one says that the operator A is on X to Y.

The range of the operator A is denoted by $\mathrm{Ran}(A)$, that is

$$\mathrm{Ran}(A) = \{Ax : \ x \in D(A)\} \subset Y$$

and its kernel, $\mathrm{Ker}(A)$, is the set of all $x \in D(A)$ such that $Ax = 0$.

The graph of A, $\mathscr{G}(A)$ is given by

$$\mathscr{G}(A) = \{(x, Ax) : \ x \in D(A)\} \subset X \times Y.$$

We naturally endow $X \times Y$ with the norm $\|(x,y)\|_{X\times Y} = \|x\|_X^2 + \|y\|_Y^2$ which makes $X \times Y$ a Banach space. One says that A is a closed operator if its graph $\mathscr{G}(A)$ is a closed subset of $X \times Y$ for this norm. The so-called graph norm on $D(A)$ is given by

$$\|x\|_{D(A)}^2 = \|(x, Ax)\|_{X\times Y}^2 = \|x\|_X^2 + \|Ax\|_Y^2.$$

The operator A is closed if and only if the space $D(A)$ is complete for the graph norm $\|\cdot\|_{D(A)}$.

If a linear operator A from X to Y is injective, one can define the operator A^{-1} from Y to X such that

$$D(A^{-1}) = \mathrm{Ran}(A), \quad \mathrm{Ran}(A^{-1}) = D(A), \quad A^{-1}A = \mathrm{Id}_{D(A)}, \quad AA^{-1} = \mathrm{Id}_{\mathrm{Ran}(A)}.$$

One says that A is invertible and A^{-1} is called the inverse operator.

Definition 2.1.14. An linear operator A from X to Y is said to be continuous if it is continuous at every $x \in D(A)$ or equivalently if it is continuous at $x = 0$. This is equivalent to having $\mathsf{K} > 0$ such that $\|Ax\|_Y \leq \mathsf{K}\|x\|_X$ for all $x \in D(A)$. One says that A is a bounded operator.

The positive number

$$\mathsf{K} = \sup_{\substack{x\in D(A)\\ x\neq 0}} \frac{\|Ax\|_Y}{\|x\|_X}$$

is called the bound of A, and denoted by $\|A\|_{\mathscr{L}(X,Y)}$ or simply $\|A\|$. Note that linear operator from X to Y that fails to be continuous are such that

$$\sup_{\substack{x\in D(A)\\ x\neq 0}} \frac{\|Ax\|_Y}{\|x\|_X} = +\infty.$$

This justifies the name unbounded for general linear operators from X to Y .

Theorem 2.1.15. (closed-graph theorem) *Let A be such that $D(A)$ is a closed linear subspace in X. Then, A is bounded if and only if A is a closed operator.*

For a proof we refer to [49].

2.1.3 Spectrum of linear operator in a Banach space

We consider here a linear operator from X to itself. One says that $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(A)$ of an linear operator A from X to X if the operator $\lambda \mathrm{Id} - A$ is injective and the inverse operator $(\lambda \mathrm{Id} - A)^{-1}$ has a dense domain $D((\lambda \mathrm{Id} - A)^{-1}) = \mathrm{Ran}(\lambda \mathrm{Id} - A)$ in X and is bounded. If $\lambda \in \rho(A)$ then we set $R_\lambda(A) = (\lambda \mathrm{Id} - A)^{-1}$. The spectrum is then simply the complement set of $\rho(A)$ in $\mathbb{C}$. We denote by $\mathrm{sp}(A)$ the spectrum of A.
The spectrum of a linear operator is othen separated in three disjoints sets

(a) The point spectrum that gathers all $\lambda \in \mathbb{C}$ such that the operator $\lambda \mathrm{Id} - A$ is not injective. Such a complex number λ is called an eigenvalue of A and the (finite or infinite) dimension of the kernel $\ker(\lambda \mathrm{Id} - A)$ is the the geometric multiplicity associated with this eigenvalue. An element of $\ker(\lambda \mathrm{Id} - A)$ is called an eigenvector or an eigenfunction in the case the Banach X is a function space.

(b) The continuous spectrum that gathers all $\lambda \in \mathbb{C}$ such that the operator $\lambda \mathrm{Id} - A$ is injective, has a dense domain but its inverse $R_\lambda(A)$ is not bounded.

(c) The residual spectrum that gathers all $\lambda \in \mathbb{C}$ such that the operator $\lambda \mathrm{Id} - A$ is injective but does not have a dense image.

2.1.4 Monotone mappings

Here X' denote the dual space of X.

Definition 2.1.16. Let X be a real Banach space. A mapping $B : X \to X'$ is called hemi-continuous at x_0 in X , if for all y in X, $\forall s_n \downarrow 0$ with $x_0 + s_n y \in X$, imply that $B(x_0 + s_n y) \rightharpoonup Bx_0$. It is called demi-continuous at $x_0 \in X$, if for all $x_n \in E$, $x_n \to x_0$ in X implies that $Bx_n \rightharpoonup Bx_0$, where $\rightharpoonup$ is the weak convergence.

We observe that “continuous” $\Rightarrow$ “demi-continuous” $\Rightarrow$ “hemi-continuous” .

Definition 2.1.17. Let X be a real Banach space. A mapping $B : X \to X'$ is called monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \text{for all} \quad x, y \in X.$$

An important property for monotone mappings reads as follows.

Lemma 2.1.18. *Let E be a convex subset of a real Banach space X. If $B : E \subset X \to X'$ is hemi-continuous and monontone, then for any sequence $x_n \in E$ with $x_n \rightharpoonup x \in E$ and $\overline{\lim}\langle Bx_n, x_n - x\rangle \leq 0$, we have*

$$\underline{\lim}\langle Bx_n, x_n - y\rangle \geq \langle Bx, x - y\rangle \quad \forall y \in E.$$

Proof. Since A is monotone, we have $\langle Bx_n - Bx, x_n - x\rangle \geq 0$ and this implies,

$$0 = \underline{\lim}\langle Bx, x_n - x\rangle \leq \underline{\lim}\langle Bx_n, x_n - x\rangle \leq \overline{\lim}\langle Bx_n, x_n - x\rangle \leq 0 \text{ (by assumption)}.$$

We then obtain that

$$\lim\langle Bx_n, x_n - x\rangle = 0. \tag{2.1.5}$$

Again by monotonicity and (2.1.5), for all $z \in E$ we have

$$\underline{\lim}\langle Bx_n, x - z\rangle = \underline{\lim}\langle Bx_n, x_n - z\rangle \geq \underline{\lim}\langle Bz, x_n - z\rangle = \langle Bz, x - z\rangle. \tag{2.1.6}$$

For all $z \in E$ and $s_k \in (0, 1)$ we set $z = z_k := (1 - s_k)x + s_k y$ and substituting this in (2.1.6), it follows that

$$\underline{\lim}\langle Bx_n, x - y\rangle \geq \langle Bz_k, x - y\rangle. \tag{2.1.7}$$

Now thanks to the hemi-continuity, the right hand side of (2.1.7) converges to $\langle Bx, x - y\rangle$ as $s_k \to 0$. Combining (2.1.5)-(2.1.7) together with the last fact, it follows that

$$\underline{\lim}\langle Bx_n, x_n - y\rangle \geq \langle Bx, x - y\rangle.$$

$\square$

The following notion on pseudo-monotonicity is abstracted from the combination of the monotonicity and the hemi-continuity.

Definition 2.1.19. Let X be a reflexive Banach space and let $E \subset X$ be a nonempty closed convex subset. An operator $B : E \to X'$ is called pseudo-monotone, if

(a) For all finite-dimensional linear subspace $\mathsf{L} \subset X$, $B|_{\mathsf{L}\cap E} : \mathsf{L} \cap E \to X'$ is demi-continuous.

(b) For all sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x \in E$, the condition $\overline{\lim}\langle Bx_n, x_n - x\rangle \leq 0$ implies $\underline{\lim}\langle Bx_n, x_n - y\rangle \geq \langle Bx, x - y\rangle, \quad \forall y \in E$.

One sees that a hemi-continuous monotone operator is pseudo-monotone. Moreover, a completely continuous mapping $B : X \to X'$(that is, for any $x_n \rightharpoonup x$ in X, we have $Bx_n \to Bx$ in X') is pseudo-monotone.

Theorem 2.1.20. (F. Browder) *Suppose that X is a reflexive Banach space, and that $B : X \to X'$ is pseudo-monotone and coercive, i.e., $\lim\limits_{\|x\|\to+\infty} \frac{\langle Bx,x\rangle}{\|x\|} = +\infty$. Then A is surjective.*

Proof. We need to show that $\forall z \in X'$, $\exists x_0 \in X$ such that $Bx_0 = z$. We define $T : x \mapsto Bx - z$. Then, T is hemi-continous and monotone, and so T is pseudo-monotone. In addition T satisfies

$$\langle Tx, x\rangle > 0 \;\text{ as } \|x\| > R_0,$$

for some $R_0 > 0$, provided by the coerciveness of B. We apply the Hartman-Stampacchia theorem [55, Theorem 2.5.7] to conclude the existence of $x_0 \in X$ satisfying $\langle Tx_0, x_0 - y\rangle = \langle Bx_0 - z, x_0 - y\rangle \leq 0$ for all $y \in X$. Since y is arbitrary in the linear space X, it follows that $Bx_0 = z$. □

Corollary 2.1.21. *Suppose that H is a real Hilbert space, and that B is a continuous strongly operator, i.e., $\exists C > 0$ such that*

$$(Bx - By, x - y) \geq C\|x - y\|^2 \quad \forall x, y \in H. \tag{2.1.8}$$

Then B is a homeomorphism.

Proof. We clearly have that B is pseudo-monotone and coercive. As the consequence of Theorem 2.1.20, B is surjective. The injectivity of B as well as the continuity of B^{-1} follows from the inequality (2.1.8). □

2.1.5 Fredholm operators

We shall denote by $\mathscr{B}(X, Y)$ the set of bounded operators A on X to Y, that is, such that $D(A) = X$. Once we speak of a bounded operator $A : X \to Y$ without any mention of its domain, this means that $D(A) = X$, that is, A is on X to Y. Let A be a linear closed operator from X to Y. The nullity of A, denoted $\mathrm{nul}(A)$, is defined as the dimension of $\ker(A)$ and the deficiency of A, denoted $\mathrm{def}(A)$, is defined as the dimension of $Y/\overline{\mathrm{Ran}(A)}$. Both $\mathrm{nul}(A)$ and $\mathrm{def}(A)$ take value in $\mathbb{N} \cup \{\infty\}$.

Definition 2.1.22. A linear operator A from X to Y is said to be Fredholm if

i. it is closed,

ii. Ran(A) is closed,

iii. both $\mathrm{nul}(A)$ and $\mathrm{def}(A)$ are finite.

One then sets the index of A as $\mathrm{ind}(A) = \mathrm{nul}(A) - \mathrm{def}(A)$.

2.1.6 Characterization of bounded Fredholm operators

We denote by $Fr\mathscr{B}(X,Y)$ the space of Fredholm operators that are bounded on X into Y. The following result states that those operators are the operators in $\mathscr{B}(X,Y)$ that have an inverse up to remainder operators that are compact.

Theorem 2.1.23. *Let $A \in \mathscr{B}(X,Y)$. It is Fredholm if and only if there exists $S \in \mathscr{B}(X,Y)$ such that*

$$SA = \mathrm{Id}_X + K^p, \quad AS = \mathrm{Id}_Y + K^q, \tag{2.1.9}$$

where $K^p \in \mathscr{B}(X,X)$ and $K^q \in \mathscr{B}(Y,Y)$ are compact operators. In particular, S is Fredholm and $\mathrm{ind}(A) = -\mathrm{ind}(S)$.

For a proof we refer to [62, Theorem 11.7].

2.2 Semigroup theory

Semigroup theory is at the centre of the understanding of many evolution equations that can be put in the following form

$$\frac{d}{dt}x(t) + Ax(t) = f(t), \quad t > 0, \quad x(0) = u_0, \tag{2.2.1}$$

with $x(t)$ and x_0 in a proper function space, usually a Banach space, denoted by X, if not a Hilbert, with A an unbounded operator on X, with dense domain, and f a function of the time variable t taking values in $[0,\infty)$. We will only review the case of a homogeneous equation, that is $f \equiv 0$. For general references on semigroups we refer to [28, 40, 76].

2.2.1 Strongly continuous semigroups

Consider the following homogeneous equation associated with the evolution equation problem (2.2.1),

$$\frac{d}{dt}x(t) + Ax(t) = 0, \quad t > 0, \quad x(0) = u_0. \tag{2.2.2}$$

Under proper assumption on A we can write the solution in the form $x(t) = \mathbb{S}(t)x_0$, where $\mathbb{S}(t) : X \to X$ is a bounded operator. Since some sort of differentiation with respect to time is expected in (2.2.2), a minimal assumption is then that

$$\mathbb{S}(0)x = x \quad \text{and} \quad t \mapsto \mathbb{S}(t)x \text{ be continuous for all } \ x \in X. \tag{2.2.3}$$

With $t \mapsto x(t)$ solution to (2.2.2), if the evolution problem is *well posed*, we expect from uniqueness that solving the following problem, for some $t_0 \geq 0$,

$$\frac{d}{dt}y(t) + Ay(t) = 0, \quad t > 0, \quad y(t_0) = x(t_0), \tag{2.2.4}$$

yield a solution that satisfies $y(t) = x(t)$ for $t \geq t_0$. In particular, this implies the following property:

$$\mathbb{S}(t + t') = \mathbb{S}(t) \circ \mathbb{S}(t'), \quad \text{for } t, t' \in [0, +\infty). \tag{2.2.5}$$

Properties (2.2.3) and (2.2.5) are precisely the starting point of semigroup theory in Banach spaces.

2.2.2 Definition and basic properties

Let X be a Banach space.

Definition 2.2.1. A family $\mathbb{S}(t)$ of bounded operators on X, with $t \in [0, +\infty)$ is called a semigroup if:

$$\mathbb{S}(0) = \mathrm{Id}_X \quad \text{and} \quad \mathbb{S}(t + t') = \mathbb{S}(t) \circ \mathbb{S}(t) \text{ for } t, t' \in [0, +\infty). \tag{2.2.6}$$

The semigroup is called strongly continuous if, moreover, for all $x \in X$ we have $\lim\limits_{t \to 0^+} \mathbb{S}(t)x = x$. One says that $\mathbb{S}(t)$ is a C_0-semigroup for short.

Theorem 2.2.2. *Let $\mathbb{S}(t)$ be a C_0-semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}, \qquad \textit{for } \ 0 \leq t < \infty. \tag{2.2.7}$$

Proof. We first show that $\|\mathbb{S}(t)\|_{\mathscr{L}(X)}$ is bounded. More precisely, we show that there is an $\mu > 0$ such that

$$\sup_{t \in [0, \mu]} \|\mathbb{S}(t)\|_{\mathscr{L}(X)} < \infty.$$

Suppose that $\|\mathbb{S}(t)\|_{\mathscr{L}(X)}$ is not bounded, that is, $\sup\limits_{t\in[0,\mu]} \|\mathbb{S}(t)\|_{\mathscr{L}(X)} = +\infty$, for all $\mu = \frac{1}{k}$ with $k \in \mathbb{N}$. Therefore for all $k \in \mathbb{N}$, there exists $t_k \in [0, \frac{1}{k}]$ such that $\sup\limits_{k\geq 1} \|\mathbb{S}(t_k)\|_{\mathscr{L}(X)} = +\infty$. From the Uniform boundedness theorem $\exists x \in X :$ $\sup\limits_{k\geq 1} \|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)} = +\infty$, i.e, $\|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)}$ is unbounded. On the other hand, for all $x \in X$, $\mathbb{R}^+ \ni t \mapsto \mathbb{S}(t)x \in X$ is continuous at 0; that is,

$$\forall \varepsilon > 0,\ \exists N > 0 :\ |t| < N \Rightarrow \|\mathbb{S}(t)u - u\|_{\mathscr{L}(X)} < \varepsilon.$$

In particular, letting $\varepsilon = 1$, it follows that $\|\mathbb{S}(t)x - x\|_{\mathscr{L}(X)} < 1$ and hence we have the estimates

$$\|\mathbb{S}(t)x\|_{\mathscr{L}(X)} - \|x\|_{\mathscr{L}(X)} \leq \left| \|\mathbb{S}(t)x\|_{\mathscr{L}(X)} - \|x\|_{\mathscr{L}(X)} \right| \leq \|\mathbb{S}(t)x - x\|_{\mathscr{L}(X)} < 1.$$

This implies that $\|\mathbb{S}(t)x\|_{\mathscr{L}(X)} \leq 1 + \|x\|_{\mathscr{L}(X)}$. But we have $0 \leq t_k \leq \frac{1}{k}$ and then $t_k \to 0$ as $k \to \infty$, and taking $N = \varepsilon$,

$$\exists k_0 \in \mathbb{N} :\ |t_k| < N;\ \forall k \geq k_0 \Rightarrow \|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)} \leq 1 + \|x\|_{\mathscr{L}(X)}. \tag{2.2.8}$$

Therefore $\sup\limits_{k\geq k_0} \|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)} \leq 1 + \|x\|_{\mathscr{L}(X)}$. For $k \in \{1, \ldots, k_0 - 1\}$ set $K = \max \|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)}$ since we have a finite number of $\mathbb{S}(t_k)x$. Then, for $k \in \{1, \ldots, k_0 - 1\}$ we have

$$\sup \|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)} \leq K. \tag{2.2.9}$$

Combining (2.2.8) and (2.2.9) we get $\sup\limits_{\geq 1} \|\mathbb{S}(t_k)x\|_{\mathscr{L}(X)} \leq 1 + \|x\|_{\mathscr{L}(X)} + K < \infty$ and this a contradiction. Hence there is an $\mu > 0$ such that

$$\sup_{t\in[0,\mu]} \|\mathbb{S}(t)\|_{\mathscr{L}(X)} < \infty.$$

Let $M = \sup\limits_{t\in[0,\mu]} \|\mathbb{S}(t)\|_{\mathscr{L}(X)}$, we have $M \geq 1$ since $\|\mathbb{S}(0)\|_{\mathscr{L}(X)} = 1$.
Set $\omega = \frac{\log M}{\mu} \geq 0$. Given $t \geq 0$ with $t > \delta$, we have $t = k\mu + \delta$, where $0 \leq \delta < \mu$ and therefore by the semigroup property

$$\|\mathbb{S}(t)\|_{\mathscr{L}(X)} = \|\mathbb{S}(\mu)^k \mathbb{S}(\delta)\|_{\mathscr{L}(X)} \leq \|\mathbb{S}(\mu)^k\|_{\mathscr{L}(X)} \|\mathbb{S}(\delta)\|_{\mathscr{L}(X)} \leq M M^k = M M^{\frac{t-\delta}{\mu}}.$$

This implies that

$$\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq M M^{\frac{t}{\mu}} = M e^{\omega\mu \times \frac{t}{\mu}} = M e^{\omega t}.$$

$\square$

Theorem 2.2.3. *If $(\mathbb{S}(t))_{t\geq 0}$ is a C_0-semigroup then for all $u \in X$, $(t, u) \mapsto \mathbb{S}(t)u$ is continous from $[0, \infty) \times X$ into X.*

A C_0-semigroup is said to be bounded if there exists $M \geq 1$ such that $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq M$, for $t \geq 0$. In the case $M = 1$, one says that the the C_0-semigroup is of contraction.
We define the unbounded linear operator A from X to X, with domain

$$D(A) = \{x \in X;\ \lim_{t\to 0^+} t^{-1}(x - \mathbb{S}(t)x) \text{ exists}\}, \tag{2.2.10}$$

and given by

$$Ax = \lim_{t\to 0^+} t^{-1}(x - \mathbb{S}(t)x), \quad x \in D(A). \tag{2.2.11}$$

The domain $D(A)$ is equiped with the graph norm

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X.$$

Since A is closed one finds that $(D(A), \|\cdot\|_{D(A)})$ is complete. This operator $(A, D(A))$ is called the generator of the C_0-semigroup .

We have the following theorem whose proof can be found in [76].

Theorem 2.2.4. *Let $\mathbb{S}(t)$ be a C_0-semigroup and A its generator. Then*

a) For $x \in X$,

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathbb{S}(s)x\ ds = \mathbb{S}(t)x. \tag{2.2.12}$$

b) For $x \in X$, $\int_0^t \mathbb{S}(t)x\ ds \in D(A)$ and

$$A\left(\int_0^t \mathbb{S}(t)x\ ds\right) = \mathbb{S}(t)x - x. \tag{2.2.13}$$

c) For $x \in D(A)$, $\mathbb{S}(t)x \in D(A)$ and

$$\frac{d}{dt}\mathbb{S}(t)x = A\mathbb{S}(t)x = \mathbb{S}(t)Ax. \tag{2.2.14}$$

d) For $x \in D(A)$,

$$\mathbb{S}(t)x - \mathbb{S}(s) = \int_s^t \mathbb{S}(\theta)Ax\ d\theta = \int_s^t A\mathbb{S}(\theta)x\ d\theta. \tag{2.2.15}$$

Corollary 2.2.5. *If A is a generator of a C_0-semigroup $(\mathbb{S}(t))_{t\geq 0}$ then the domain $D(A)$ of A, is dense in X and A is closed operator.*

For a proof we refer to [76].

Observe that if $\mathbb{S}(t)$ is a C_0-semigroup and $z \in \mathbb{C}$ then $e^{zt}\mathbb{S}(t)$ statisfies (2.2.6). We have the following proposition.

Proposition 2.2.6. *Let $\mathbb{S}(t)$ be a C_0-semigroup and $z \in \mathbb{C}$. Then $e^{zt}\mathbb{S}(t)$ is also a C_0-semigroup and its generator is $A - z\mathrm{Id}_X$.*

Note that because of the uniqueness of the generator of a C_0-semigroup [76, Theorem 1.2.6], conversely, if A generates a C_0-semigroup, then $A - z\mathrm{Id}_X$ is the generator of a C_0-semigroup, namely $e^{zt}\mathbb{S}(t)$.

The Hille-Yosida theorem

The natural question is to wonder if an unbounded operator on X is the generator of C_0-semigroup. The Hille-Yosida theorem is central in the semigroup theory, providing a clear answer to this question. We refer to [76, Theorem 1.3.1] for a proof.

Theorem 2.2.7. *Let $(A, D(A))$ be a linear unbounded operator on a Banach space X. It generates a C_0-semigroup of contraction if and only if*

(a) A is closed and $D(A)$ is dense in X.

(b) The resolvent set $\rho(A)$ of A contains $(-\infty, 0)$ and we have the following estimate

$$\|R_\lambda(A)\|_{\mathscr{L}(X)} \leq 1/|\lambda|, \quad \lambda < 0, \quad R_\lambda(A) = (\lambda\mathrm{Id}_X - A)^{-1}.$$

This result is limited to contraction C_0-semigroups. The following corollary provides a charcaterization of all generators of C_0-semigroups, we refer to [76, Theorem 1.5.3] for a proof.

Corollary 2.2.8. *Let $(A, D(A))$ be a linear unbounded operator on a Banach space X. It generates a C_0-semigroup $\mathbb{S}(t)$ such that $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq Me^{\omega t}$, for some $M \geq 1$ and $\omega \in \mathbb{R}$, if and only if*

i. A is closed and $D(A)$ is dense in X.

ii. The resolvent set $\rho(A)$ of A contains $(-\infty, -\omega)$ and we have the following estimate

$$\|R_\lambda(A)^n\|_{\mathscr{L}(X)} \leq 1/|\omega + \lambda|^n, \quad \lambda < -\omega, \quad n \in \mathbb{N}^*, \quad R_\lambda(A) = (\lambda\mathrm{Id}_X - A)^{-1}.$$

The Hille-Yosida theorem has the following simple consequence.

Corollary 2.2.9. *Let $(A, D(A))$ be the generator of a bounded C_0-semigroup $\mathbb{S}(t)$, that is, $\|\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq M$, for $t \geq 0$, for some $M > 0$. Then, its spectrum satisfies $sp(A) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.*

Proof. Let $c \in \mathbb{R}$, the C_0-semigroup $e^{ict}\mathbb{S}(t)$ is generated by $A - ic\mathrm{Id}_X$. Since $e^{ict}\mathbb{S}(t)$ satisfies $\|e^{ict}\mathbb{S}(t)\|_{\mathscr{L}(X)} \leq M$, for $t \geq 0$, the conclusion follows from Corollary 2.2.8 in the case $\omega = 0$. □

The Lumer-Phillips theorem

The Lumer-Phillips theorem provides another characterization of generators of contraction semigroups.

Let X' be the dual space of X equipped with the strong topology. For $x \in X$ we set

$$\mathsf{F}(x) = \{\varphi \in X' : \ \varphi(x) = \langle \varphi, x\rangle_{X',X} = \|\varphi\|_{X'}^2 = \|x\|_X^2\},$$

which is not empty by the Hahn-Banach theorem.

Definition 2.2.10. A linear unbounded operator $(A, D(A))$ on X is said to be monotone (or accretive) if for all $x \in D(A)$, $x \neq 0$, there exists $\varphi \in \mathsf{F}(x)$ such that $\operatorname{Re}\langle \varphi, Ax\rangle_{X',X} \geq 0$.

Definition 2.2.11. A linear unbounded operator $(A, D(A))$ on X is said to be maximal monotone if it is monotone and if moreover there exists $\lambda_0 > 0$ such that the range of $\lambda_0\mathrm{Id}_X + A$, $\operatorname{Ran}(\lambda_0\mathrm{Id}_X + A) = X$.

The Lumer-Phillips theorem reads as follows.

Theorem 2.2.12. *Let $(A, D(A))$ be a linear unbounded operator. It generates a C_0-semigroup of contraction if and only if*

1) A has a dense domain.

2) A is maximal monotone

A proof based on the Hille-Yosida theorem directly follows from the two lemmata below.

Remark 2.2.13. Observe that there is no need to assume that the operator A is closed in the converse part of the Lumer-Phillips theorem as in the Hille-Yosida theorem. In fact, as proven below, a maximal monotone operator is closed. In the case of a reflexive Banach space, the dense domain assumption may be dropped in the converse part of the Lumer-Phillips theorem: a maximal monotone operator has a dense domain, see [76, Theorem 1.4.6] and also [15, Proposition 7.1] for Hilbert space case.

This lemma gives a characterization of monotone operators. We refer to [62, Lemma 12.13] for a proof.

Lemma 2.2.14. *An unbounded operator* $(A, D(A))$ *on* X *is monotone if and only if*

$$\|(\lambda \mathrm{Id}_X + A)x\|_X \geq \lambda \|x\|_X, \quad x \in D(A) \quad \textit{and } \lambda > 0. \tag{2.2.16}$$

The value of $\lambda_0 > 0$ in Definition 2.2.11 is not of great significance. In fact, we have the following result.

Lemma 2.2.15. *Let* A *be a maximal monotone operator on* X*. Then,* A *is closed and for all* $\lambda > 0$ *the operator* $\lambda \mathrm{Id}_X + A$ *is bijective from* $D(A)$ *onto* X*. Moreover, if* $\lambda > 0$*, its inverse* $(\lambda \mathrm{Id}_X + A)^{-1}$ *is a bounded operator and we have the following estimation* $\|(\lambda \mathrm{Id}_X + A)^{-1}\|_{\mathscr{L}(X)} \leq 1/\lambda$.

Proof. Let $\lambda > 0$, for $x \in X$, we have $(\lambda \mathrm{Id} + A)x = 0$ and $\|(\lambda \mathrm{Id}_X + A)x\|_X = 0$ implies that $\lambda\|x\|_X = 0$ thanks to Lemma 2.2.14. It follows that $x = 0$ and $\lambda \mathrm{Id}_X + A$ is injective. As A is maximal monotone, there exists $\lambda_0 > 0$ such that $\mathrm{Ran}(\lambda_0 \mathrm{Id}_X + A) = X$, so $\lambda_0 \mathrm{Id}_X + A$ is surjective. Its inverse $(\lambda_0 \mathrm{Id}_X + A)^{-1}$ is thus well defined on X. By Lemma 2.2.14, we have $\|(\lambda_0 \mathrm{Id}_X + A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda_0}$. By the closed-graph theorem (see, Theorem 2.1.15), the graph of $(\lambda_0 \mathrm{Id}_X + A)^{-1}$ is closed in $X \times X$ and so is the graph of A.
We now prove that if $\lambda \mathrm{Id}_X + A$ is surjective then so is $\lambda' \mathrm{Id}_X + A$ for any λ' such that $\lambda > \lambda'/2 > 0$. By induction, starting with $\lambda = \lambda_0$ we then reach to the conclusion that $\lambda \mathrm{Id}_X + A$ is surjective for any $\lambda > 0$ and then the boundedness of its inverse follows from Lemma 2.2.14.
Let $\lambda, \lambda' > 0$ be such that $\lambda \mathrm{Id}_X + A$ is surjective and $2\lambda > \lambda' > 0$. Let $y \in X$. We want to find $x \in X$ such that $(\lambda' \mathrm{Id}_X + A)x = y$. This reads as $(\lambda \mathrm{Id}_X + A)x = y + (\lambda - \lambda')x$ and thus we have $x = (\lambda \mathrm{Id}_X + A)^{-1}(y + (\lambda - \lambda')x)$, meaning that we seek fixed point for the bounded map $M : x \mapsto (\lambda \mathrm{Id}_X + A)^{-1}(y + (\lambda - \lambda')x)$. By the computation above, we have $\|(\lambda \mathrm{Id}_X + A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{1}{\lambda}$, we then find $\|M(x) - M(x')\|_X \leq |1 - \frac{\lambda'}{\lambda}|\|x - x'\|_X$. Since $0 < |1 - \frac{\lambda'}{\lambda}| < 1$, the Banach contraction fixed point theorem applies. □

3. Part A: Nonlinear eigenvalue problems

3.1 Calculus of variations

The calculus of variations uses variations, which are small changes in functions and functionals, to find extrema(maxima, minima or other critical values) of functionals: mappings from a set of functions to the real numbers. Functionals are often expressed as definite integrals involving functions and their derivatives.

3.1.1 Functionals and critical points

Let X be a Banach space. A functional on X is a continuous real valued map $I : X \to \mathbb{R}$. In general, one could consider functionals defined on open subsets of X. But we will deal with functionals defined on all of X. In the applications, critical points turn out to be weak solutions of differential equations.

Definition 3.1.1. We say that $u \in X$ is a local minimum, respectively maximum of the functional $I \in C(E, \mathbb{R})$ if there exists a neighbourhood $\mathscr{V}$ of u such that

$$I(u) \leq I(v), \quad \text{respectively} \quad I(u) \geq I(v), \quad \forall\, v \in \mathscr{V} \backslash \{u\}. \tag{3.1.1}$$

If the inequalities in (3.1.1) are strict we say that u is a strict local minimum respectively maximum. If (3.1.1) holds for every $u \in X$, not only on $\mathscr{V} \backslash \{u\}$, u is a global minimum respectively maximum.

Next, we state some results dealing with the existence of maxima or minima. We restrict ourself to classical result dealing with functionals which are coercive and wealky lower-semi-continuous (shortly : w.l.s.c.).

Let us recall that $I \in C(X, \mathbb{R})$ is coercive if

$$\lim_{\|u\|_X \to +\infty} I(u) = +\infty.$$

The functional I is w.l.s.c. if for every sequence $u_n \in X$ such that $u_n \rightharpoonup u$ one has that

$$I(u) \leq \liminf_{n \to +\infty} I(u_n).$$

Lemma 3.1.2. Let X be a reflexive Banach space and let $I : X \to \mathbb{R}$ be coercive and w.l.s.c. Then I is bounded from below on X, namely there exists $a \in \mathbb{R}$ such that $I(u) \geq a$ for all $u \in X$.

Proof. Suppose by contradiction that there exists $u_n \in X$ such that $I(u_n) \to -\infty$. Since I is coercive there exists $M > 0$ such that $\|u\|_X \leq M$. Hence $(u_n)_n$ is bounded in X. But since X is a reflexive Banach space $(u_n)_n$ has a subsequence u_n (without relabelling) which converges weakly to some $u \in X$. Since I is w.l.s.c. we infer that $0 < I(u) \leq \liminf_{n\to+\infty} I(u_n) = -\infty$, contradiction, proving the lemma. □

Remark 3.1.3. The same arguments show that a w.l.s.c. functional is bounded from below on any ball $B_\rho = \{u \in X \ : \|u\|_X \leq \rho\}$.

Theorem 3.1.4. Let X be a reflexive Banach space and let $I : X \to \mathbb{R}$ be coercive and w.l.s.c. Then I has a global minimum, namely there exists $u \in X$ such that $I(u) = \min\{I(v) \ : \ v \in X\}$. If I is differentiable at u, then $I'(u) = 0$.

Proof. From Lemma 3.1.2, it follows that $m = \inf\{I(u) \ : u \in X\}$ is finite. Let u_n be a minimizing sequence, namely such that $I(u_n) \to m$. Again, the coercivity of I implies that $\|u_n\|_X \leq M'$, and $u_n \rightharpoonup u$ for some $u \in X$. Since I is w.l.s.c. it follows that $I(u) \leq \liminf_{n\to+\infty} I(u_n) = m$. Thus I achieves its infimum at $u : I(u) = m$. □

3.1.2 Bifurcation: definition and necessary conditions

Let X, Y be Banach spaces. We deal with an equation like

$$S(\lambda, u) = 0 \tag{3.1.2}$$

where $S : \mathbb{R} \times X \to Y$ is such that

$$S(\lambda, 0) = 0 \qquad \forall\, \lambda \in \mathbb{R}.$$

The solution $u = 0$ will be called trivial solution of (3.1.2). The set

$$\Sigma = \{(\lambda, u) \in \mathbb{R} \times X : \ u \neq 0, \ S(\lambda, u) = 0\}$$

will be called the set of nontrivial solutions of (3.1.2). The following phenomenon has been observed: a branch of solutions $u(\lambda)$ depending on λ, either disappeared or split into several branches, as λ approaches some critical values. This kind of phenomenon is called bifurcation. Many problems arising in applications can be modelled in this way.

For example, the following algebraic equation

$$u^3 - \lambda u = 0, \quad \lambda \in \mathbb{R}$$

has a solution $u = 0$ for all $\lambda \in \mathbb{R}$. For $\lambda \leq 0$, this is the unique solution, but for $\lambda > 0$ we have two more branches of solutions $u = \pm\sqrt{\lambda}$.
Bifurcation phenomena occur frequently in nature. Early in 1744, Euler observed the bending of a rod pressed along the direction of its axis. Let θ be the angle between the real axis and the tangent of the central line of the rod, and let λ be the pressure. The length of the rod is normalized to be π. We obtain the following differential equation with the two free end point conditions

$$\begin{cases} \ddot{\theta} + \lambda \sin\theta = 0 \\ \dot{\theta}(0) = \dot{\theta}(\pi) = 0. \end{cases} \tag{3.1.3}$$

Obviously, $\theta = 0$ is always a solution of the ordinary differential equation (3.1.3). Actually the solution is unique, if λ is not large. As λ increasingly passes through a certain value λ_0, it is shown by experiment that there exists a bending solution $\theta \neq 0$.
The same phenomenon occurs in the bending of plates, shells etc. Also, bifurcation occurs in the study of thermodynamics, rotation of fluids, solitary waves, superconductivity and lasers, etc.
Mathematically, we describe the bifurcation by the following:

Definition 3.1.5. (bifurcation from trivial solution)
A bifurcation point for (3.1.2) is a number $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, 0)$ belongs to the closure of Σ. In other words, λ^* is a bifurcation point if there exist sequences $\lambda_n \in \mathbb{R}$, $u_n \in X\backslash\{0\}$ such that

(i) $S(\lambda_n, u_n) = 0$,

(ii) $(\lambda_n, u_n) \to (\lambda^*, 0)$.

The main purpose of the theory of bifurcation is to establish conditions for finding bifurcation points and in general, to study the structure of Σ. If $S \in C^1(\mathbb{R} \times X, Y)$ a necessary condition for λ^* to be a bifurcation point can be immediately deduced from the implicit function theorem.

Definition 3.1.6. (bifurcation from infinity)
We say that λ^*_∞ is a bifurcation point from infinity for (3.1.2) if there exist $\lambda_n \to \lambda^*_\infty$ and $u_n \in X$, such that $\|u_n\|_X \to \infty$ and $(\lambda_n, u_n) \in \Sigma$.

We state a remarkable bifurcation result due to M. A. Krasnoselski [53].

Theorem 3.1.7. (Krasnoselski bifurcation theorem)
Let X be a Banach space and let $T \in C^1(X, X)$ be a compact operator such that $T(0) = 0$ and $T'(0) = 0$. Moreover, let $A \in \mathscr{L}(X)$ also be compact. Then every characteristic value λ^* of A with odd (algebraic) multiplicity is a bifurcation point for $u = \lambda Au + T(u)$.

For the proof of Theorem 3.1.7, we refer to [6].

3.1.3 Palais-Smale condition

Definition 3.1.8. Let X be a Banach space and $F : X \to \mathbb{R}$ be a differentiable functional. A sequence $\{u_n\}_n \subseteq X$ such that

$\{F(u_n)\}_n$ is bounded (in $\mathbb{R}$) and

$F'(u_n) \to 0$ (in X') as $n \to \infty$,

is called a Palais-Smale sequence for F.

Remark 3.1.9. In a Hilbert space E we can identify the differential with the gradient through the scalar product. Therefore, the second property of a Palais-Smale sequence reads $\nabla F(u_n) \to 0$ in E.

We recall that the convergence takes place in the strong topology of E.

Definition 3.1.10. Let X be a Banach space and let $F : X \to \mathbb{R}$ be a differentiable functional. We say that F satisfies the Palais-Smale condition (shortly: F satisfies (PS)) if every Palais-Smale sequence for F has a converging subsequence (in X).

The following lemma shows that the search for critical points can be split into two independent parts : the existence of Palais-Smale sequences, which will follow from topological reasons, and the convergence of these sequences, which is a compactness problem.

Lemma 3.1.11. *Let X be a Banach space and let $F : X \to \mathbb{R}$ be C^1 functional. If there exists a Palais-Smale sequence for F and F satisfies (PS), then F has a critical point.*

In the following section we discuss the notion of degree.

3.2 Topological degree

The reader who meets the notion of topological degree (shortly, degree) for the first time, could maybe start by asking the following question : *what is the topological degree ?* As a rough answer, the degree is a tool, precisely a number which gives information about the solution of particular equations. The degree was introduced by L. Brouwer in finite dimensional spaces and extended by J. Leray and J. Schauder to infinite dimensional spaces. The Leray–Schauder degree is an important topological tool in the study of nonlinear partial differential equations while the Brouwer degree is a powerful tool in algebraic topology. The nontriviality of the degree ensures the existence of a fixed point of the compact mapping in the domain. It enjoys the properties of homotopy invariance and additivity, which make the topological tool more convenient in application, and provides more information on fixed points. There is a very broad literature dealing with degree, among those we cite the following books [5, 6, 16, 29, 55].

3.2.1 Brouwer degree and its properties

Let us assume that

(1) Ω is an open bounded set in $\mathbb{R}^d$, with boundary $\partial\Omega$,

(2) h is a continuous function map from $\overline{\Omega}$ to $\mathbb{R}^d$, the components of h will be denoted by h_i,

(3) p is a point in $\mathbb{R}^d$ such that $p \notin h(\partial\Omega)$.

To each triple (h, Ω, p) satisfying (1)-(3), one can associate an integer $\deg(h, \Omega, p)$, called the degree of h (with respect to Ω and p), with the following properties.

(**P1**) Normalization: if $\mathrm{Id}_{\mathbb{R}^d}$ denotes the identity map in $\mathbb{R}^d$, then

$$\deg(\mathrm{Id}_{\mathbb{R}^d}, \Omega, p) = \begin{cases} 1 & \text{if } p \in \Omega, \\ \\ 0 & \text{if } p \notin \Omega. \end{cases}$$

(**P2**) Solution property: if $\deg(h, \Omega, p) \neq 0$ then there exists $y \in \Omega$ such that $h(y) = p$.

(**P3**) $\deg(h, \Omega, p) = \deg(h - p, \Omega, 0)$.

(**P4**) Decomposition: if $\Omega_1 \cap \Omega_2 = \emptyset$ then

$$\deg(h, \Omega_1 \cup \Omega_2, p) = \deg(h, \Omega_1, p) + \deg(h, \Omega_2, p).$$

An outline of the procedure usually followed to define the degree, omitting the consistency of the definition and the verification of (P1)-(P4) is given by the following.

Consider a C^1 map h and a regular value p. Let us recall that p is said to be regular value for h, if the Jacobian $J_h(x) \neq 0$ for every $x \in h^{-1}(p)$. The Jacobian is the determinant of the matrix $h'(x)$ with entries

$$e_{ij} = \frac{\partial h_i}{\partial x_j}.$$

If p is a regular value then the set $h^{-1}(p)$ is finite and one can define the degree by setting

$$\deg(h,\Omega,p)=\sum_{x\in h^{-1}(p)}\operatorname{sign}[J_h(x)], \tag{3.2.1}$$

where, for $a\in\mathbb{R}\backslash\{0\}$, we set

$$\operatorname{sign}[\mathrm{a}]=\begin{cases}+1 & \text{if } a>0,\\ -1 & \text{if } a<0.\end{cases}$$

We see that the degree defined in (3.2.1) verifies the properties of (P1)-(P4) of the degree defined above.
An important property of the degree defined above is the invariance by homotopy. An homotopy is a map $H=H(\lambda,x)$ such that $H\in C([0,1]\times\overline{\Omega},\mathbb{R}^d)$. An homotopy is admissible (with respect to Ω an p), if $H(\lambda,x)\neq p$ for all $(\lambda,x)\in[0,1]\times\partial\Omega$.

(**P5**) Homotopy invariance : if H is an admissible homotopy, then $\deg(H(\lambda,\cdot),\Omega,p)$ is constant with respect to $\lambda\in[0,1]$. In particular, if $f(x)=H(0,x)$ and $g(x)=H(1,x)$ then $\deg(f,\Omega,p)=\deg(g,\Omega,p)$.

As an immediate consequence of the homotopy invariance, we can deduce the following

Theorem 3.2.1. (Dependence on the boundary values)
Let $f,g\in C(\Omega,\mathbb{R}^d)$ be such that $f(x)=g(x)$ for all $x\in\partial\Omega$ and let $p\in f(\partial\Omega)=g(\partial\Omega)$. Then $\deg(f,\Omega,p)=\deg(g,\Omega,p)$.

Proof. Consider the homotopy defined by

$$H(\lambda,x)=\lambda g(x)+(1-\lambda)f(x).$$

One has $g(x)=f(x)$ for all $x\in\partial\Omega$ and thus $H(\lambda,x)=g(x)\neq p$. Hence H is admissible and the homotopy invariance yields :

$$\deg(f,\Omega,p)=\deg(H(0,\cdot),\Omega,p)=\deg(H(1,\cdot),\Omega,p)=\deg(g,\Omega,p).$$

□

We list below some further properties of the degree.

(P6) Continuity : if $h_k\to h$ uniformly in $\overline{\Omega}$, then $\deg(h_k,\Omega,p)\to\deg(h,\Omega,p)$. Moreover, $\deg(h,\Omega,p)$ is continuous with respect to p.

(P7) Excision property: let $\Omega_0\subset\Omega$ be an open set such that $f(x)\neq p$, for all $x\in\Omega\backslash\Omega_0$. Then $\deg(f,\Omega,p)=\deg(f,\Omega_0,p)$.

3.2.2 The Leray-Schauder degree

The Leray-Schauder degree is defined for mappings of the form $\mathrm{Id} - \mathsf{C}$, where C is a compact mapping from the closure of an open bounded subset of a Banach space X.

Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity in a Banach space X. If $\Omega \subset X$ is an open bounded set, $h : \overline{\Omega} \to X$ is compact, and $p \notin (\mathrm{Id} - h)(\partial\Omega)$, the Leray-Schauder degree $\deg_{LS}(\mathrm{Id} - h, \Omega, p)$ of $\mathrm{Id} - h$ in Ω over p is constructed from the Brouwer degree by approximating the compact map h over $\overline{\Omega}$ by mappings h_ε with range in a finite-dimensional subspace X_ε (containing p) of X, and showing that the Brouwer degrees $\deg_B((\mathrm{Id} - h_\varepsilon)|X_\varepsilon, \Omega \cap X_\varepsilon, p)$ stabilize for sufficiently small positive ε to a common value defining $\deg_{LS}(\mathrm{Id} - h, \Omega, p)$. This topological degree "algebraically counts" the number of fixed point of $h(\cdot) - p$ in Ω, and for h of class C^1, and $\mathrm{Id} - h'(a)$ invertible for each fixed point $h(\cdot) - p$ in Ω, Leray and Schauder show that

$$\deg_{LS}(\mathrm{Id} - h, \Omega, p) = \sum_{a \in (\mathrm{Id} - h)^{-1}(p)} (-1)^{\sigma_j(a)},$$

where $\sigma_j(a)$ is the sum of the algebraic multiplicities of the eigenvalues $h'(a)$.

The Leray-Schauder degree conserves the basic properties of Brouwer degree.

Theorem 3.2.2. The Leray-Schauder degree has the following properties.

(a) (Additivity) If $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are open and disjoint, and if $p \notin (\mathrm{Id} - h)(\partial\Omega_1) \cup (\mathrm{Id} - h)(\partial\Omega_2)$, then

$$\deg_{LS}(\mathrm{Id} - h, \Omega, p) = \deg_{LS}(\mathrm{Id} - h, \Omega_1, p) + \deg_{LS}(\mathrm{Id} - h, \Omega_2, p).$$

(b) (Existence) If $\deg_{LS}(\mathrm{Id} - h, \Omega, p) \neq 0$, then $p \in (\mathrm{Id} - h)(\Omega)$.

(c) (Homotopy invariance) Let $\Omega \subset X$ be a bounded open set, and let $F : \mathbb{R} \times \overline{\Omega} \to X$ be compact. If $x - F(\lambda, x) \neq p$ for all $(\lambda, x) \in \mathbb{R} \times \partial\Omega$, then $\deg_{LS}(\mathrm{Id} - F(\lambda, \cdot), \Omega, p)$ is independent of λ.

3.3 The Krasnoselski genus

The genus was introduced by M. A. Krasnoselski [53]. Let E be a infinite dimensional Hilbert space. We say that a subset $O \subset E$ is symmetric if it is symmetric with respect to the origin of E, namely

$$u \in O \Rightarrow -u \in O.$$

Let Γ be the class of all the symmetric subsets $A \subseteq E\backslash\{0\}$ which are closed in $E\backslash\{0\}$.

Definition 3.3.1. Let $A \in \Gamma$. The genus of A is defined as the least integer number $n \in \mathbb{N}$ such that there exists $\psi : A \to \mathbb{R}^n$ continuous, odd and such that

$$\psi(x) \neq 0 \quad \text{for all } x \in A.$$

The genus of A is usually denoted by $\gamma(A)$. If such a number does not exists, we set $\gamma(A) = \infty$ and, if $A = \emptyset$, we conventionally set $\gamma(A) = 0$.

Remark 3.3.2. An equivalent way to define the genus $\gamma(A)$ is to take the minimal integer d such that there exists an odd map $\psi \in C(A, \mathbb{R}^d\backslash\{0\})$. Actually, such a ψ can be extended to a map $\hat{\psi} \in C(E, \mathbb{R}^d)$. If ψ^* is the odd part of $\hat{\psi}$, namely

$$\psi^*(u) = \frac{1}{2}\left(\hat{\psi}(u) - \hat{\psi}(-u)\right),$$

ψ^* verifies the properties required in the above definition.

Remark 3.3.3. The definition of the genus does not change if we require ψ to be function with values in the sphere $\mathbb{S}^{n-1}$ instead of $\mathbb{R}^n\backslash\{0\}$ since we can compose with the projection

$$\text{proj}(x) := \frac{x}{|x|}.$$

Lemma 3.3.4. Let $E = L^2(\mathbb{R}^d)$ and let $A = \mathbb{S}_E$ be the unit sphere in $L^2(\mathbb{R}^d)$. Then $\gamma(A) = +\infty$.

Proof. Let $n \in \mathbb{N}$ be any positive integer, and let $\psi : A \to \mathbb{R}^n$ be continuous and odd map. The infinite dimensional sphere contains the k-sphere $\mathbb{S}^k \subset \mathbb{R}^{k+1}$; thus by Borsuk-Ulam theorem it follows that, for $k > n$

$$0 \in \psi(\mathbb{S}^k) \Rightarrow 0 \in \psi(A).$$

Since A contains every finite dimensional sphere, for every $n \in \mathbb{N}$ we can take $k = n + 1$ and obtain that 0 is in the image. This proves that the genus is $+\infty$. $\square$

Remark 3.3.5. In a similar way, one shows that $\gamma(\partial\Omega) = n$, where $\Omega \subset \mathbb{R}^n$ is an open bounded symmetric subset such that $0 \in \Omega$. In particular,

$$\gamma(\mathbb{S}^{n-1}) = n.$$

The following proposition gives some properties of the genus.

Proposition 3.3.6. Let A and B be elements of the class Γ.

(i) The set A is empty if and only if the genus $\gamma(A) = 0$.

(ii) If $\psi : A \to B$ is a continuous odd map, then $\gamma(A) \leq \gamma(B)$. In particular

$$A \subseteq B \Rightarrow \gamma(A) \leq \gamma(B).$$

(iii) The genus is subadditive, namely, $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

(iv) If A is compact then $\gamma(A) < +\infty$ and there exists a symmetric neighbourhood U_A of A such that $\gamma(\overline{U}_A) = \gamma(A)$.

3.3.1 Existence of multiple critical points of functionals

The genus can be used to prove existence results of critical points of functional provided that the functional is even and $M \in \Gamma$, where M is a the constraint set. Here, we consider the functional $J \in C^1(E, \mathbb{R})$. For any positive integer, we define

$$\Gamma_m = \{A \subset M : \; A \in \Gamma, A \text{ is compact and } \gamma(A) \geq m\}$$

and

$$\sigma_m = \inf_{A \in \Gamma_m} \sup_{u \in A} J(u).$$

We explicitly remark that $\sigma_m < +\infty$ and $\sigma_m \leq \sigma_{m+1}$. Moreover, if J is bounded from below on M, then $\sigma_1 > -\infty$ and hence any σ_m is finite. If we deal with problems without constraints, namely if we are looking for stationary points of $J \in C^1(E, \mathbb{R})$ on E, we understand $\Gamma = \{A \in E \backslash \{0\} : \; A \text{ is symmetric}\}$, that

$$\Gamma_m = \{A \subset \Gamma, \; A \text{ is compact and } \gamma(A) \geq m\},$$

and that σ_m is defined as above.
We state the following general result which holds both in the case of critical points of J constrained on M and in the case without constraints. We refer to [6, Proposition 10.8] for a proof.

Proposition 3.3.7. *Each finite σ_m is a critical level for $J \in C^1(E, \mathbb{R})$ (or a critical level for J on M) provided $(PS)_{\sigma_m}$ holds. Moreover, if $\tilde{\sigma} = \sigma_m = \sigma_{m+1} = \cdots = \sigma_{m+n} \in \mathbb{R}$ for some integer $n \geq 1$, then $\gamma(Z_\sigma) \geq n + 1$, where Z_σ denotes the set of critical points of J at the critical level $\tilde{\sigma}$.*

The following theorem, known as Clark's theorem asserts the existence of a sequence of negative critical values tending to 0 for even coercive functionals.

Theorem 3.3.8. (Clark, [24])
Let X be a Banach space and $G \in C^1(X, \mathbb{R})$ satisfying the Palais-Smale condition with $G(0) = 0$. Let $\Gamma_k = \{ \ A \in \Sigma \ : \ \gamma(A) \geq k \ \}$ with $\Sigma = \{ \ A \subset X \ ; \ A = -A$ and A closed $\}$. If $c_k = \inf\limits_{A \in \Gamma_k} \sup\limits_{u \in A} G(u) \in (-\infty, 0)$, then c_k is a critical value.

Nonlinear eigenvalue problems for quasi-linear operators and applications to bifurcation

In this part we present the results obtained in [97].

3.4 Setting of the problem

We consider $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ an open bounded domain with smooth boundary $\partial\Omega$. A classical result in the theory of eigenvalue problems guarantees that the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4.1}$$

possesses a nondecreasing sequence of eigenvalues and a sequence of corresponding eigenfunctions which define a Hilbert basis in $L^2(\Omega)$ [see, [39]]. Moreover, it is known that the first eigenvalue of problem (3.4.1) is characterized in the variational point of view by,

$$\lambda_1^D := \inf_{u \in W_0^{1,2}(\Omega) \setminus \{0\}} \left\{ \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} \right\}.$$

Suppose that $p > 1$ is a given real number and consider the nonlinear eigenvalue problem with Neumann boundary condition

$$\begin{cases} -\Delta_p u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4.2}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ stands for the p-Laplace operator and $\lambda \in \mathbb{R}$. This problem was considered in [74], and using a direct method in calculus of variations (if $p > 2$) or a mountain-pass argument (if $p \in (\frac{2N}{N+2}, 2)$) it was shown that the set of eigenvalues of problem (3.4.2) is exactly the interval $[0, \infty)$. Indeed, it is sufficient to find one positive eigenvalue, say $-\Delta_p u = \lambda u$. Then a continuous family of eigenvalues can be found by the reparametrization $u = \alpha v$, satisfying $-\Delta_p v = \mu(\alpha) v$, with $\mu(\alpha) = \frac{\lambda}{\alpha^{p-2}}$.
We consider the so-called $(p, 2)$-Laplace operator [see, [37]] with Dirichlet boundary conditions. More precisely, we analyze the following nonlinear eigenvalue problem,

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4.3}$$

where $p \in (1, \infty)\backslash\{2\}$ is a real number. We recall that if $1 < p < q$, then $L^q(\Omega) \subset L^p(\Omega)$ and as a consequence, one has $W_0^{1,q}(\Omega) \subset W_0^{1,p}(\Omega)$. We will say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (3.4.3) if there exists $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ (if $p > 2$), $u \in W_0^{1,2}(\Omega)\backslash\{0\}$ (if $1 < p < 2$) such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \; dx + \int_\Omega \nabla u \cdot \nabla v \; dx = \lambda \int_\Omega u \; v \; dx, \tag{3.4.4}$$

for all $v \in W_0^{1,p}(\Omega)$ (if $p > 2$), $v \in W_0^{1,2}(\Omega)$ (if $1 < p < 2$). In this case, such a pair (u, λ) is called an eigenpair, and $\lambda \in \mathbb{R}$ is called an eigenvalue and $u \in W_0^{1,p}(\Omega)\backslash\{0\}$ is an eigenfunction associated to λ. We say that λ is a "first eigenvalue", if the corresponding eigenfunction u is positive or negative.
The operator $-\Delta_p - \Delta$ appears in quantum field theory [see, [35]], where it arises in the mathematical description of propagation phenomena of solitary waves. We recall that a solitary wave is a wave which propagates without any temporal evolution in shape.
The operator $-\Delta_p - \Delta$ is a special case of the so called (p, q)-Laplace operator given by $-\Delta_p - \Delta_q$ which has been widely studied; for some results related to our studies, see e.g., [13, 14, 23, 69, 89] .
We investigate the nonlinear eigenvalue problem (3.4.3) when $p > 2$, and $1 < p < 2$ respectively. In particular, we show in section 3.5 that the set of the first eigenvalues is given by the interval (λ_1^D, ∞), where λ_1^D is the first Dirichlet eigenvalue of the Laplacian. We show that the first eigenvalue of (3.4.3) can be obtained variationally, using a Nehari set for $1 < p < 2$, and a minimization for $p > 2$. Also in the same section, we recall some results of [74], [72] and [73]. In section 3.6, we prove that the eigenfunctions associated to λ belong to $L^\infty(\Omega)$, the first eigenvalue λ_1^D of problem (3.4.3) is simple and the corresponding eigenfunctions are positive or negative. In addition, in section 3.6.3 we show a homeomorphism property related to $-\Delta_p - \Delta$.
In section 3.7, we prove that λ_1^D is a bifurcation point for a branch of first eigenvalues from zero if $p > 2$, and λ_1^D is a bifurcation point from infinity if $p < 2$. Also the higher Dirichlet eigenvalues λ_k^D are bifurcation points (from 0 if $p > 2$, respectively from infinity if $1 < p < 2$), if the multiplicity of λ_k^D is odd. Finally in section 3.8, we prove by variational methods that if $\lambda \in (\lambda_k^D, \lambda_{k+1}^D)$ then there exist at least k nonlinear eigenvalues using Krasnoselski's genus.

3.5 The spectrum of the nonlinear problem

We start with the discussion of the properties of the spectrum of the nonlinear eigenvalues problem (3.4.3).

Remark 3.5.1. Any $\lambda \leq 0$ is not an eigenvalue of problem (3.4.3).

Indeed, suppose by contradiction that $\lambda = 0$ is an eigenvalue of equation (3.4.3), then relation (3.4.4) with $v = u_0$ gives

$$\int_\Omega |\nabla u_0|^p \, dx + \int_\Omega |\nabla u_0|^2 \, dx = 0.$$

Consequently $|\nabla u_0| = 0$, therefore u_0 is constant on Ω and $u_0 = 0$ on Ω. And this contradicts the fact that u_0 is a nontrivial eigenfunction. Hence $\lambda = 0$ is not an eigenvalue of problem (3.4.3).
Now it remains to show that any $\lambda < 0$ is not an eigenvalue of (3.4.3). Suppose by contradiction that $\lambda < 0$ is an eigenvalue of (3.4.3), with $u_\lambda \in W_0^{1,p}(\Omega) \backslash \{0\}$ the corresponding eigenfunction. The relation (3.4.4) with $v = u_\lambda$ implies

$$0 \leq \int_\Omega |\nabla u_\lambda|^p \, dx + \int_\Omega |\nabla u_\lambda|^2 \, dx = \lambda \int_\Omega u_\lambda^2 \, dx < 0.$$

Which yields a contradiction and thus $\lambda < 0$ cannot be an eigenvalue of problem (3.4.3).

Lemma 3.5.2. Any $\lambda \in (0, \lambda_1^D]$ is not an eigenvalue of (3.4.3).

For the proof see also [74].

Proof. Let $\lambda \in (0, \lambda_1^D)$, i.e., $\lambda_1^D > \lambda$. Let's assume by contradiction that there exists a $\lambda \in (0, \lambda_1^D)$ which is an eigenvalue of (3.4.3) with $u_\lambda \in W_0^{1,2}(\Omega) \backslash \{0\}$ the corresponding eigenfunction. Letting $v = u_\lambda$ in relation (3.4.4), we have on the one hand,

$$\int_\Omega |\nabla u_\lambda|^p \, dx + \int_\Omega |\nabla u_\lambda|^2 \, dx = \lambda \int_\Omega u_\lambda^2 \, dx$$

and on the other hand,

$$\lambda_1^D \int_\Omega u_\lambda^2 dx \leq \int_\Omega |\nabla u_\lambda|^2 \, dx. \tag{3.5.1}$$

By subtracting both side of (3.5.1) by $\lambda \int_\Omega u_\lambda^2 \, dx$, we obtain

$$(\lambda_1^D - \lambda) \int_\Omega u_\lambda^2 \, dx \leq \int_\Omega |\nabla u_\lambda|^2 \, dx - \lambda \int_\Omega u_\lambda^2 \, dx,$$

$$(\lambda_1^D - \lambda)\int_\Omega u_\lambda^2\, dx \le \int_\Omega |\nabla u_\lambda|^2\, dx - \lambda\int_\Omega u_\lambda^2\, dx + \int_\Omega |\nabla u_\lambda|^p\, dx = 0.$$

Therefore $(\lambda_1^D - \lambda)\int_\Omega u_\lambda^2\, dx \le 0$, which is a contradiction. Hence, we conclude that $\lambda \in (0, \lambda_1^D)$ is not an eigenvalue of problem (3.4.3). In order to complete the proof of the Lemma 3.5.2 we shall show that $\lambda = \lambda_1^D$ is not an eigenvalue of (3.4.3).

By contradiction we assume that $\lambda = \lambda_1^D$ is an eigenvalue of (3.4.3). So there exists $u_{\lambda_1^D} \in W_0^{1,2}(\Omega)\backslash\{0\}$ such that relation (3.4.4) holds true. Letting $v = u_{\lambda_1^D}$ in relation (3.4.4), it follows that

$$\int_\Omega |\nabla u_{\lambda_1^D}|^p\, dx + \int_\Omega |\nabla u_{\lambda_1^D}|^2\, dx = \lambda_1^D \int_\Omega u_{\lambda_1^D}^2\, dx.$$

But $\lambda_1^D \int_\Omega u_{\lambda_1^D}^2\, dx \le \int_\Omega |\nabla u_{\lambda_1^D}|^2\, dx$, therefore

$$\int_\Omega |\nabla u_{\lambda_1^D}|^p\, dx + \int_\Omega |\nabla u_{\lambda_1^D}|^2\, dx \le \int_\Omega |\nabla u_{\lambda_1^D}|^2\, dx \Rightarrow \int_\Omega |\nabla u_{\lambda_1^D}|^p\, dx \le 0.$$

Using relation (2.1.4), we have $u_{\lambda_1^D} = 0$, which is a contradiction since $u_{\lambda_1^D} \in W_0^{1,2}(\Omega)\backslash\{0\}$. So $\lambda = \lambda_1^D$ is not an eigenvalue of (3.4.3). $\square$

Theorem 3.5.3. Assume $p \in (1,2)$. Then the set of first eigenvalues of problem (3.4.3) is given by

$$(\lambda_1^D, \infty), \quad \text{where } \lambda_1^D \text{ denotes the first eigenvalue of } -\Delta \text{ on } \Omega.$$

Proof. Let $\lambda \in (\lambda_1^D, \infty)$, and define the energy functional

$$J_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R} \;\text{ by }\; J_\lambda(u) = \int_\Omega |\nabla u|^2\, dx + \frac{2}{p}\int_\Omega |\nabla u|^p\, dx - \lambda\int_\Omega u^2\, dx.$$

One shows that $J_\lambda \in C^1(W_0^{1,2}(\Omega), \mathbb{R})$ (see, [37]) with its derivatives given by

$$\langle J_\lambda'(u), v\rangle = 2\int_\Omega \nabla u\cdot\nabla v\, dx + 2\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\, dx - 2\lambda\int_\Omega u\, v\, dx \;,\; \forall\, v \in W_0^{1,2}(\Omega).$$

Thus we note that λ is an eigenvalue of problem (3.4.3) if and only if J_λ possesses a nontrivial critical point. Considering $J_\lambda(\rho e_1)$, where e_1 is the L^2-normalized first eigenfunction of the Laplacian, we see that

$$J_\lambda(\rho e_1) \le \lambda_1^D \rho^2 + C\rho^p - \lambda\rho^2 \to -\infty, \quad \text{as } \rho \to +\infty.$$

Hence, we cannot establish the coercivity of J_λ on $W_0^{1,2}(\Omega)$ for $p \in (1,2)$, and consequently we cannot use a direct method in calculus of variations in order to determine a critical point of J_λ. To overcome this difficulty, the idea will be to analyze the functional J_λ on the so called Nehari manifold defined by

$$\mathcal{N}_\lambda := \left\{ u \in W_0^{1,2}(\Omega)\backslash\{0\} \ : \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla u|^p \, dx = \lambda \int_\Omega u^2 \, dx \right\}.$$

Note that all non-trivial solutions of (3.4.3) lie on $\mathcal{N}_\lambda$. On $\mathcal{N}_\lambda$ the functional J_λ takes the form

$$\begin{aligned} J_\lambda(u) &= \int_\Omega |\nabla u|^2 \, dx + \frac{2}{p} \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega u^2 \, dx \\ &= (\frac{2}{p} - 1) \int_\Omega |\nabla u|^p \, dx > 0. \end{aligned}$$

We have seen in Lemma 3.5.2, that any $\lambda \in (0, \lambda_1^D]$ is not an eigenvalue of problem (3.4.3); see also [74]. It remains to prove the :

Claim : Every $\lambda \in (\lambda_1^D, \infty)$ is a first eigenvalue of problem (3.4.3). Indeed, we will split the proof of the claim into four steps.

Step 1. Here we will show that $\mathcal{N}_\lambda \neq \emptyset$ and every minimizing sequence for J_λ on $\mathcal{N}_\lambda$ is bounded in $W_0^{1,2}(\Omega)$. Since $\lambda > \lambda_1^D$ there exists $v_\lambda \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla v_\lambda|^2 \, dx < \lambda \int_\Omega v_\lambda^2 \, dx.$$

Then there exists $t > 0$ such that $tv_\lambda \in \mathcal{N}_\lambda$. In fact

$$\int_\Omega |\nabla (tv_\lambda)|^2 \, dx + \int_\Omega |\nabla (tv_\lambda)|^p \, dx = \lambda \int_\Omega (tv_\lambda)^2 \, dx \Rightarrow$$

$$t^2 \int_\Omega |\nabla v_\lambda|^2 \, dx + t^p \int_\Omega |\nabla v_\lambda|^p \, dx = t^2 \lambda \int_\Omega v_\lambda^2 \, dx \Rightarrow$$

$$t = \left(\frac{\int_\Omega |\nabla v_\lambda|^p \, dx}{\lambda \int_\Omega v_\lambda^2 \, dx - \int_\Omega |\nabla v_\lambda|^2 \, dx} \right)^{\frac{1}{2-p}} > 0.$$

With such t we have $tv_\lambda \in \mathcal{N}_\lambda$ and $\mathcal{N}_\lambda \neq \emptyset$.

Note that for $u \in B_r(v_\lambda)$, $r > 0$ small, the inequality $\lambda \int_\Omega |u|^2 dx > \int_\Omega |\nabla u|^2 dx$ remains valid, and then $t(u)u \in \mathcal{N}_\lambda$ for $u \in B_r(v_\lambda)$. Since $t(u) \in C^1$ we conclude that $\mathcal{N}_\lambda$ is a C^1-manifold.
Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence of $J_\lambda|_{\mathcal{N}_\lambda}$, i.e. $J_\lambda(u_k) \to m =$

$\inf_{w\in\mathcal{N}_\lambda} J_\lambda(w)$. Then

$$\lambda\int_\Omega u_k^2\,dx - \int_\Omega |\nabla u_k|^2\,dx = \int_\Omega |\nabla u_k|^p\,dx \to \left(\frac{2}{p}-1\right)^{-1} m \text{ as } k\to\infty. \tag{3.5.2}$$

Assume by contradiction that $\{u_k\}$ is not bounded in $W_0^{1,2}(\Omega)$, i.e. $\int_\Omega |\nabla u_k|^2\,dx \to \infty$ as $k\to\infty$. It follows that $\int_\Omega u_k^2\,dx \to \infty$ as $k\to\infty$, thanks to relation (3.5.2). We set $v_k = \frac{u_k}{\|u_k\|_2}$. Since $\int_\Omega |\nabla u_k|^2\,dx < \lambda\int_\Omega u_k^2\,dx$, we deduce that $\int_\Omega |\nabla v_k|^2\,dx < \lambda$, for each k and $\|v_k\|_{1,2} < \sqrt{\lambda}$. Hence $\{v_k\}\subset W_0^{1,2}(\Omega)$ is bounded in $W_0^{1,2}(\Omega)$. Therefore there exists $v_0\in W_0^{1,2}(\Omega)$ such that $v_k \rightharpoonup v_0$ in $W_0^{1,2}(\Omega)\subset W_0^{1,p}(\Omega)$ and $v_k\to v_0$ in $L^2(\Omega)$. Dividing relation (3.5.2) by $\|u_k\|_2^p$, we get

$$\int_\Omega |\nabla v_k|^p\,dx = \frac{\lambda\int_\Omega u_k^2\,dx - \int_\Omega |\nabla u_k|^2\,dx}{\|u_k\|_2^p} \to 0 \text{ as k} \to\infty,$$

since $\lambda\int_\Omega u_k^2\,dx - \int_\Omega |\nabla u_k|^2\,dx \to \left(\frac{2}{p}-1\right)^{-1} m < \infty$ and $\|u_k\|_2^p\to\infty$ as $k\to\infty$. On the other hand, since $v_k\rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$, we have $\int_\Omega |\nabla v_0|^p\,dx \le \liminf_{k\to\infty}\int_\Omega |\nabla v_k|^p\,dx = 0$ and consequently $v_0=0$. It follows that $v_k\to 0$ in $L^2(\Omega)$, which is a contradiction since $\|v_k\|_2=1$. Hence, $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$.

Step 2. $m = \inf_{w\in\mathcal{N}_\lambda} J_\lambda(w) > 0$. Indeed, assume by contradiction that $m=0$. Then, for $\{u_k\}$ as in step 1, we have

$$0<\lambda\int_\Omega u_k^2\,dx - \int_\Omega |\nabla u_k|^2\,dx = \int_\Omega |\nabla u_k|^p\,dx \to 0, \text{as } k\to\infty. \tag{3.5.3}$$

By Step 1, we deduce that $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. Therefore there exists $u_0\in W_0^{1,2}(\Omega)$ such that $u_k\rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $u_k\to u_0$ in $L^2(\Omega)$.

Thus $\int_\Omega |\nabla u_0|^p\,dx \le \liminf_{k\to\infty}\int_\Omega |\nabla u_k|^p\,dx = 0$. And consequently $u_0=0$, $u_k\rightharpoonup 0$ in $W_0^{1,2}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $u_k\to 0$ in $L^2(\Omega)$. Writing again

$v_k = \frac{u_k}{\|u_k\|_2}$ we have

$$0 < \frac{\lambda \int_\Omega u_k^2 \, dx - \int_\Omega |\nabla u_k|^2 \, dx}{\|u_k\|_2^2} = \|u_k\|_2^{p-2} \int_\Omega |\nabla v_k|^p \, dx,$$

therefore

$$\begin{aligned} \int_\Omega |\nabla v_k|^p \, dx &= \|u_k\|_2^{2-p} \left(\frac{\lambda \|u_k\|_2^2}{\|u_k\|_2^2} - \frac{\int_\Omega |\nabla u_k|^2 \, dx}{\|u_k\|_2^2} \right) \\ &= \|u_k\|_2^{2-p} \left(\lambda - \int_\Omega |\nabla v_k|^2 \, dx \right) \to 0 \text{ as } k \to \infty, \end{aligned}$$

since $\|u_k\|_2 \to 0$ and $p \in (1,2)$, and $\{v_k\}$ is bounded in $W_0^{1,2}(\Omega)$. Next since $v_k \rightharpoonup v_0$ in $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$, we deduce that $\int_\Omega |\nabla v_0|^p \, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla v_k|^p \, dx = 0$ and we have $v_0 = 0$. And it follows that $v_k \to 0$ in $L^2(\Omega)$ which is a contradiction since $\|v_k\|_2 = 1$ for each k. Hence, m is positive.

Step 3. There exists $u_0 \in \mathcal{N}_\lambda$ such that $J_\lambda(u_0) = m$.

Let $\{u_k\} \subset \mathcal{N}_\lambda$ be a minimizing sequence, i.e., $J_\lambda(u_k) \to m$ as $k \to \infty$. Thanks to Step 1, we have that $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. It follows that there exists $u_0 \in W_0^{1,2}(\Omega)$ such that $u_k \rightharpoonup u_0$ in $W_0^{1,2}(\Omega)$ and $W_0^{1,p}(\Omega)$ and strongly in $L^2(\Omega)$. The results in the two steps above guarantee that $J_\lambda(u_0) \leq \liminf_{k\to\infty} J_\lambda(u_k) = m$. Since for each k we have $u_k \in \mathcal{N}_\lambda$, we have

$$\int_\Omega |\nabla u_k|^2 \, dx + \int_\Omega |\nabla u_k|^p \, dx = \lambda \int_\Omega u_k^2 \, dx \quad \text{for all } k. \tag{3.5.4}$$

Assuming $u_0 \equiv 0$ on Ω implies that $\int_\Omega u_k^2 \, dx \to 0$ as $k \to \infty$, and by relation (3.5.4) we obtain that $\int_\Omega |\nabla u_k|^2 \, dx \to 0$ as $k \to \infty$. Combining this with the fact that u_k converges weakly to 0 in $W_0^{1,2}(\Omega)$, we deduce that u_k converges strongly to 0 in $W_0^{1,2}(\Omega)$ and consequently in $W_0^{1,p}(\Omega)$. Hence we infer that

$$\lambda \int_\Omega u_k^2 \, dx - \int_\Omega |\nabla u_k|^2 \, dx = \int_\Omega |\nabla u_k|^p \, dx \to 0, \text{as } k \to \infty.$$

Next, using similar argument as the one used in the proof of Step 2, we will reach to a contradiction, which shows that $u_0 \not\equiv 0$. Letting $k \to \infty$ in relation (3.5.4), we deduce that

$$\int_\Omega |\nabla u_0|^2 \, dx + \int_\Omega |\nabla u_0|^p \, dx \leq \lambda \int_\Omega u_0^2 \, dx.$$

If there is equality in the above relation then $u_0 \in \mathcal{N}_\lambda$ and $m \leq J_\lambda(u_0)$. Assume by contradiction that

$$\int_\Omega |\nabla u_0|^2 \, dx + \int_\Omega |\nabla u_0|^p \, dx < \lambda \int_\Omega u_0^2 \, dx. \tag{3.5.5}$$

Let $t > 0$ be such that $tu_0 \in \mathcal{N}_\lambda$, i.e.,

$$t = \left(\frac{\lambda \displaystyle\int_\Omega u_0^2 \, dx - \int_\Omega |\nabla u_0|^2 \, dx}{\displaystyle\int_\Omega |\nabla u_0|^p \, dx} \right)^{\frac{1}{p-2}}.$$

We note that $t \in (0,1)$ since $1 < t^{p-2}$ (thanks to (3.5.5)). Finally, since $tu_0 \in \mathcal{N}_\lambda$ with $t \in (0,1)$ we have

$$\begin{aligned} 0 < m \leq J_\lambda(tu_0) &= \left(\frac{2}{p} - 1\right) \int_\Omega |\nabla(tu_0)|^p \, dx = t^p \left(\frac{2}{p} - 1\right) \int_\Omega |\nabla u_0|^p \, dx \\ &= t^p J_\lambda(u_0) \\ &\leq t^p \liminf_{k\to\infty} J_\lambda(u_k) = t^p m < m \text{ for } t \in (0,1), \end{aligned}$$

and this is a contradiction which assures that relation (3.5.5) cannot hold and consequently we have $u_0 \in \mathcal{N}_\lambda$. Hence $m \leq J_\lambda(u_0)$ and $m = J_\lambda(u_0)$.

Step 4. We conclude the proof of the claim. Let $u \in \mathcal{N}_\lambda$ be such that $J_\lambda(u) = m$ (thanks to Step 3). Since $u \in \mathcal{N}_\lambda$, we have

$$\int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla u|^p \, dx = \lambda \int_\Omega u^2 \, dx,$$

and

$$\int_\Omega |\nabla u|^2 \, dx < \lambda \int_\Omega u^2 \, dx.$$

Let $v \in \partial B_1(0) \subset W_0^{1,2}(\Omega)$ and $\varepsilon > 0$ be very small such that $u + \delta v \not\equiv 0$ in Ω for all $\delta \in (-\varepsilon, \varepsilon)$ and

$$\int_\Omega |\nabla(u + \delta v)|^2 \, dx < \lambda \int_\Omega (u + \delta v)^2 \, dx;$$

this is equivalent to

$$\lambda \int_\Omega u^2\,dx - \int_\Omega |\nabla u|^2\,dx > \delta\left(2\int_\Omega \nabla u\cdot\nabla v\,dx - 2\lambda\int_\Omega uv\,dx\right) + \delta^2\left(\int_\Omega |\nabla v|^2\,dx - \lambda\int_\Omega v^2\,dx\right),$$

which holds true for δ small enough since the left hand side is positive while the function

$$h(\delta) := |\delta|\left|2\int_\Omega \nabla u\cdot\nabla v\,dx - 2\lambda\int_\Omega uv\,dx\right| + \delta^2\left|\int_\Omega |\nabla v|^2\,dx - \lambda\int_\Omega v^2\,dx\right|$$

dominates the term from the right hand side and $h(\delta)$ is a continuous function (polynomial in δ) which vanishes in $\delta = 0$. For each $\delta \in (-\varepsilon, \varepsilon)$, let $t(\delta) > 0$ be given by

$$t(\delta) = \left(\frac{\lambda\displaystyle\int_\Omega (u+\delta v)^2\,dx - \int_\Omega |\nabla(u+\delta v)|^2\,dx}{\displaystyle\int_\Omega |\nabla(u+\delta v)|^p\,dx}\right)^{\frac{1}{p-2}},$$

so that $t(\delta)\cdot(u+\delta v) \in \mathcal{N}_\lambda$. We have that $t(\delta)$ is of class $C^1(-\varepsilon,\varepsilon)$ since $t(\delta)$ is the composition of some functions of class C^1. On the other hand, since $u \in \mathcal{N}_\lambda$ we have $t(0) = 1$.

Define $\iota : (-\varepsilon,\varepsilon) \to \mathbb{R}$ by $\iota(\delta) = J_\lambda(t(\delta)(u+\delta v))$ which is of class $C^1(-\varepsilon,\varepsilon)$ and has a minimum at $\delta = 0$. We have

$$\iota'(\delta) = [t'(\delta)(u+\delta v) + vt(\delta)]J'_\lambda(t(\delta)(u+\delta v)) \Rightarrow$$

$$0 = \iota'(0) = J'_\lambda(t(0)(u))[t'(0)u + vt(0)] = \langle J'_\lambda(u), v\rangle$$

since $t(0) = 1$ and $t'(0) = 0$.
This shows that every $\lambda \in (\lambda_1^D, \infty)$ is an eigenvalue of problem (3.4.3).

□

In the next theorem we consider the case $p > 2$. For similar results for the Neumann case, [see, [72]].

Theorem 3.5.4. For $p > 2$, the set of first eigenvalues of problem (3.4.3) is given by (λ_1^D, ∞).

The proof of Theorem 3.5.4 will follow as a direct consequence of the lemmas proved below:

Lemma 3.5.5. Let

$$\lambda_1(p) := \inf_{u \in W_0^{1,p} \setminus \{0\}} \left\{ \frac{\frac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \, dx + \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx}{\frac{1}{2}\displaystyle\int_\Omega u^2 \, dx} \right\}.$$

Then $\lambda_1(p) = \lambda_1^D$, for all $p > 2$.

Proof. We clearly have $\lambda_1(p) \geq \lambda_1^D$ since a positive term is added. On the other hand, consider $u_n = \frac{1}{n} e_1$ (where e_1 is the first eigenfunction of $-\Delta$), we get

$$\lambda_1(p) \leq \frac{\frac{1}{2n^2}\int_\Omega |\nabla e_1|^2 dx + \frac{1}{pn^p}\int_\Omega |\nabla e_1|^p dx}{\frac{1}{2n^2}\int_\Omega |e_1|^2 dx} \to \lambda_1^D \quad \text{as } n \to \infty.$$

$\square$

Lemma 3.5.6. For each $\lambda > 0$, we have

$$\lim_{\|u\|_{1,p} \to \infty} \left(\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{2}\int_\Omega u^2 \, dx \right) = \infty.$$

Proof. Clearly

$$\frac{1}{p}\int_\Omega |\nabla u|^p \, dx + \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx \quad \geq \quad \frac{1}{p}\int_\Omega |\nabla u|^p \, dx.$$

On the one hand, using Poincaré's inequality with $p = 2$,
we have $\displaystyle\int_\Omega u^2 \, dx \leq C_2(\Omega)\int_\Omega |\nabla u|^2 \, dx, \forall u \in W_0^{1,p}(\Omega) \subset W_0^{1,2}(\Omega)$ and then applying the Hölder inequality to the right hand side term of the previous estimate, we obtain

$$\int_\Omega |\nabla u|^2 \, dx \leq |\Omega|^{\frac{p-2}{p}} \|u\|_{1,p}^2,$$

so $\displaystyle\int_\Omega u^2 \, dx \leq D\|u\|_{1,p}^2$, where $D = C_2(\Omega)|\Omega|^{\frac{p-2}{p}}$. Therefore for $\lambda > 0$,

$$\frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{2}\int_\Omega u^2 \, dx \geq C\|u\|_{1,p}^p - \frac{\lambda}{2} D\|u\|_{1,p}^2, \quad (3.5.6)$$

and the the right-hand side of (3.5.6) tends to ∞, as $\|u\|_{1,p} \to \infty$, since $p > 2$. $\square$

Lemma 3.5.7. Every $\lambda \in (\lambda_1^D, \infty)$ is a first eigenvalue of problem (3.4.3).

Proof. For each $\lambda > \lambda_1^D$ define $F_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{2}\int_\Omega u^2\, dx \quad, \forall u \in W_0^{1,p}(\Omega).$$

Standard arguments shows that $F_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ [see, [37]] with its derivative given by

$$\langle F_\lambda'(u), \varphi\rangle = \int_\Omega (|\nabla u|^{p-2} + 1)\nabla u \cdot \nabla \varphi\, dx - \lambda \int_\Omega u\varphi\, dx,$$

for all $u, \varphi \in W_0^{1,p}(\Omega)$. Estimate (3.5.6) shows that F_λ is coercive in $W_0^{1,p}(\Omega)$. On the other hand, F_λ is also weakly lower semi-continuous on $W_0^{1,p}(\Omega)$ since F_λ is a continuous convex functional, (see [8, Proposition 1.5.10 and Theorem 1.5.3]) . Then we can apply a calculus of variations result, in order to obtain the existence of a global minimum point of F_λ, denoted by θ_λ, i.e., $F_\lambda(\theta_\lambda) = \min\limits_{W_0^{1,p}(\Omega)} F_\lambda$. Note that for any $\lambda > \lambda_1^D$ there exists $u_\lambda \in W_0^{1,p}(\Omega)$ such that $F_\lambda(u_\lambda) < 0$. Indeed, taking $u_\lambda = re_1$, we have

$$F_\lambda(re_1) = \frac{r^2}{2}(\lambda_1^D - \lambda) + \frac{r^p}{p}\int_\Omega |\nabla e_1|^p\, dx < 0 \quad \text{for} \quad r > 0 \quad \text{small.}$$

But then $F_\lambda(\theta_\lambda) \le F_\lambda(u_\lambda) < 0$, which means that $\theta_\lambda \in W_0^{1,p}(\Omega)\backslash\{0\}$. On the other hand, we have $\langle F_\lambda'(\theta_\lambda), \varphi\rangle = 0, \forall \varphi \in W_0^{1,p}(\Omega)$ (θ_λ is a critical point of F_λ) with $\theta_\lambda \in W_0^{1,p}(\Omega)\backslash\{0\} \subset W_0^{1,2}(\Omega)\backslash\{0\}$. Consequently each $\lambda > \lambda_1^D$ is an eigenvalue of problem (3.4.3). □

Proposition 3.5.8. *The first eigenfunctions u_1^λ associated to $\lambda \in (\lambda_1^D, \infty)$ are positive or negative in Ω.*

Proof. Let $u_1^\lambda \in W_0^{1,p}(\Omega)\backslash\{0\}$ be an eigenfunction associated to $\lambda \in (\lambda_1^D, \infty)$, then
$\int_\Omega |\nabla u_1^\lambda|^p\, dx + \int_\Omega |\nabla u_1^\lambda|^2\, dx = \lambda \int_\Omega |u_1^\lambda|^2\, dx$, which means u_1^λ achieves the infimum in the definition of λ_1^D. On the other hand we have $\|\nabla |u_1^\lambda|\|_{1,p} = \|\nabla u_1^\lambda\|_{1,p}$ and $\|\nabla |u_1^\lambda|\|_{1,2} = \|\nabla u_1^\lambda\|_{1,2}$, since $|\nabla |u_1^\lambda|| = |\nabla u_1^\lambda|$ almost everywhere. It follows that $|u_1^\lambda|$ achieves also the infimum in the definition of λ_1^D, and therefore by the Harnack inequality [see, [32]], we have $|u_1^\lambda(x)| > 0\ \forall x \in \Omega$ and consequently u_1^λ is either positive or negative in Ω. □

A similar result of Theorem 3.6.1 was proved in [56] in the case of the p-Laplacian.

3.6 Properties of eigenfunctions and the operator $-\Delta_p - \Delta$

3.6.1 Boundedness of the eigenfunctions

We shall prove boundedness of eigenfunctions and use this fact to obtain $C^{1,\alpha}$ smoothness of all eigenfunctions of the quasi-linear problem (3.4.3). The latter result is due to [56, Theorem 4.4], which originates from [12] and [92].

Theorem 3.6.1. Let $(u, \lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R}_+^\star$ be an eigensolution of the weak formulation (3.4.4). Then $u \in L^\infty(\Omega)$.

Proof. By Morrey's embedding theorem it suffices to consider the case $p \leq N$. Let us assume first that $u > 0$. For $M \geq 0$ define $w_M(x) = \min\{u(x), M\}$. Letting

$$g(x) = \begin{cases} x & \text{if } x \leq M \\ M & \text{if } x > M \end{cases} \tag{3.6.1}$$

we have $g \in C(\mathbb{R})$ piecewise smooth function with $g(0) = 0$. Since $u \in W_0^{1,p}(\Omega)$ and $g' \in L^\infty(\Omega)$, then $g \circ u \in W_0^{1,p}(\Omega)$ and $w_M \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see, Theorem B.3 in [56]). For $k > 0$, define $\varphi = w_M^{kp+1}$ then $\nabla\varphi = (kp+1)\nabla w_M w_M^{kp}$ and $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.
Using φ as a test function in (3.4.4), one obtains

$$(kp+1)\left[\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_M w_M^{kp}\, dx + \int_\Omega \nabla u \cdot \nabla w_M w_M^{kp}\, dx\right] = \lambda \int_\Omega u\, w_M^{kp+1}\, dx.$$

On the other hand using the fact that $w_M^{kp+1} \leq u^{kp+1}$, it follows that

$$(kp+1)\left[\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w_M w_M^{kp}\, dx + \int_\Omega \nabla u \cdot \nabla w_M w_M^{kp}\, dx\right] \leq \lambda \int_\Omega |u|^{(k+1)p}\, dx.$$

We have $\nabla(w_M^{k+1}) = (k+1)\nabla w_M w_M^k \Rightarrow |\nabla w_M^{k+1}|^p = (k+1)^p w_M^{kp}|\nabla w_M|^p$. Since the integrals on the left are zero on $\{x : u(x) > M\}$ we can take $u = w_M$ in the previous inequality, and it follows that

$$(kp+1)\left[\int_\Omega |\nabla w_M|^p w_M^{kp}\, dx + \int_\Omega |\nabla w_M|^2 w_M^{kp}\, dx\right] \leq \lambda \int_\Omega |u|^{(k+1)p}\, dx.$$

Replacing $|\nabla w_M|^p w_M^{kp}$ by $\frac{1}{(k+1)^p}|\nabla w_M^{k+1}|^p$, we have

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla w_M^{k+1}|^p\, dx + (kp+1)\int_\Omega |\nabla w_M|^2 w_M^{kp}\, dx \leq \lambda \int_\Omega |u|^{(k+1)p}\, dx,$$

which implies that

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla w_M^{k+1}|^p\,dx \le \lambda \int_\Omega |u|^{(k+1)p}\,dx$$

and then

$$\int_\Omega |\nabla w_M^{k+1}|^p\,dx \le \left(\lambda\frac{(k+1)^p}{kp+1}\right)\int_\Omega |u|^{(k+1)p}\,dx. \tag{3.6.2}$$

By Sobolev's embedding theorem, there is a constant $c_1 > 0$ such that

$$\|w_M^{k+1}\|_{p^\star} \le c_1 \|w_M^{k+1}\|_{1,p}. \tag{3.6.3}$$

where $p^\star$ is the Sobolev critical exponent. Consequently, we have

$$\|w_M\|_{(k+1)p^\star} \le \|w_M^{k+1}\|_{p^\star}^{\frac{1}{k+1}}, \tag{3.6.4}$$

and therefore

$$\|w_M\|_{(k+1)p^\star} \le \left(c_1\|w_M^{k+1}\|_{1,p}\right)^{\frac{1}{k+1}} = c_1^{\frac{1}{k+1}}\|w_M^{k+1}\|_{1,p}^{\frac{1}{k+1}}. \tag{3.6.5}$$

But by (3.6.2),

$$\|w_M^{k+1}\|_{1,p} \le \left(\lambda\frac{(k+1)^p}{kp+1}\right)^{\frac{1}{p}}\|u\|_{(k+1)p}^{k+1} \tag{3.6.6}$$

and we note that we can find a constant $c_2 > 0$ such that
$\left(\lambda\frac{(k+1)^p}{kp+1}\right)^{\frac{1}{p\sqrt{k+1}}} \le c_2$, independently of k and consequently

$$\|w_M\|_{(k+1)p^\star} \le c_1^{\frac{1}{k+1}} c_2^{\frac{1}{\sqrt{k+1}}}\|u\|_{(k+1)p}. \tag{3.6.7}$$

Letting $M \to \infty$, Fatou's lemma implies

$$\|u\|_{(k+1)p^\star} \le c_1^{\frac{1}{k+1}} c_2^{\frac{1}{\sqrt{k+1}}}\|u\|_{(k+1)p}. \tag{3.6.8}$$

Choosing k_1, such that $(k_1+1)p = p^\star$, then $\|u\|_{(k_1+1)p^\star} \le c_1^{\frac{1}{k_1+1}} c_2^{\frac{1}{\sqrt{k_1+1}}}\|u\|_{p^\star}$. Next we choose k_2 such that $(k_2+1)p = (k_1+1)p^\star$, then taking $k_2 = k$ in inequality (3.6.8), it follows that

$$\|u\|_{(k_2+1)p^\star} \le c_1^{\frac{1}{k_2+1}} c_2^{\frac{1}{\sqrt{k_2+1}}}\|u\|_{(k_1+1)p^\star}. \tag{3.6.9}$$

By induction we obtain

$$\|u\|_{(k_n+1)p^\star} \le c_1^{\frac{1}{k_n+1}} c_2^{\frac{1}{\sqrt{k_n+1}}}\|u\|_{(k_{n-1}+1)p^\star}, \tag{3.6.10}$$

where the sequence $\{k_n\}$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^\star, k_0 = 0$. One gets $k_n + 1 = (\frac{p^\star}{p})^n$. As $\frac{p}{p^\star} < 1$, there is $C > 0$ (which depends on c_1 and c_2) such that for any $n = 1, 2, \dots$

$$\|u\|_{r_n} \le C\|u\|_{p^\star} \tag{3.6.11}$$

with $r_n = (k_n+1)p^\star \to \infty$ as $n \to \infty$. We note that (3.6.11) follows by iterating the previous inequality (3.6.10). We will indirectly show that $u \in L^\infty(\Omega)$. Suppose $u \notin L^\infty(\Omega)$, then there exists $\varepsilon > 0$ and a set A of positive measure in Ω such that $|u(x)| > C\|u\|_{p^\star} + \varepsilon = K$, for all $x \in A$. We then have,

$$\lim_{n\to\infty} \inf \|u\|_{r_n} \ge \lim_{n\to\infty} \inf \left(\int_A K^{r_n}\right)^{1/r_n} = \lim_{n\to\infty} \inf K|A|^{1/r_n} = K > C\|u\|_{p^\star}, \tag{3.6.12}$$

which contradicts (3.6.11). If u changes sign, we consider $u = u^+ - u^-$ where

$$u^+ = \max\{u, 0\} \quad \text{and} \quad u^- = \max\{-u, 0\}. \tag{3.6.13}$$

We have $u^+, u^- \in W_0^{1,p}(\Omega)$. For each $M > 0$ define $w_M = \min\{u^+(x), M\}$ and take again $\varphi = w_M^{kp+1}$ as a test function in (3.4.4). Proceeding the same way as above we conclude that $u^+ \in L^\infty(\Omega)$. Similarly we have $u^- \in L^\infty(\Omega)$. Therefore $u = u^+ - u^-$ is in $L^\infty(\Omega)$. □

3.6.2 Simplicity of the eigenvalues

We prove an auxiliary result which will imply uniqueness of the first eigenfunction. Let

$$\begin{aligned} I(u,v) &= \langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\rangle + \langle -\Delta u, \frac{u^2 - v^2}{u}\rangle \\ &+ \langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}}\rangle + \langle -\Delta v, \frac{v^2 - u^2}{v}\rangle, \end{aligned}$$

for all $(u, v) \in D_I$, where

$$D_I = \{(u_1, u_2) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \ : \ u_i > 0 \text{ in } \Omega \text{ and } u_i \in L^\infty(\Omega) \text{ for } i = 1, 2\} \text{ if } p > 2,$$

and

$$D_I = \{(u_1, u_2) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \ : \ u_i > 0 \text{ in } \Omega \text{ and } u_i \in L^\infty(\Omega) \text{ for } i = 1, 2\} \text{ if } 1 < p < 2.$$

Proposition 3.6.2. For all $(u, v) \in D_I$, we have $I(u, v) \ge 0$. Furthermore $I(u, v) = 0$ if and only if there exists $\alpha \in \mathbb{R}_+^\star$ such that $u = \alpha v$.

Proof. We first show that $I(u,v) \geq 0$. We recall that (if $2 < p < \infty$)

$$\langle -\Delta_p u, w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \; dx \quad \text{for all } w \in W_0^{1,p}(\Omega)$$

$$\langle -\Delta u, w\rangle = \int_\Omega \nabla u \cdot \nabla w \; dx \quad \text{for all } w \in W_0^{1,p}(\Omega).$$

and (if $1 < p < 2$)

$$\langle -\Delta_p u, w\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \; dx \quad \text{for all } w \in W_0^{1,2}(\Omega)$$

$$\langle -\Delta u, w\rangle = \int_\Omega \nabla u \cdot \nabla w \; dx \quad \text{for all } w \in W_0^{1,2}(\Omega).$$

Let us consider $\beta = \frac{u^p - v^p}{u^{p-1}}$, $\eta = \frac{v^p-u^p}{v^{p-1}}$, $\xi = \frac{u^2-v^2}{u}$ and $\zeta = \frac{v^2-u^2}{v}$ as test functions in (3.4.4) for any $p > 1$. Straightforward computations give,

$$\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \left\{1 + (p-1)\left(\frac{v}{u}\right)^p\right\}\nabla u - p\left(\frac{v}{u}\right)^{p-1}\nabla v$$

$$\nabla\left(\frac{v^p - u^p}{v^{p-1}}\right) = \left\{1 + (p-1)\left(\frac{u}{v}\right)^p\right\}\nabla v - p\left(\frac{u}{v}\right)^{p-1}\nabla u$$

$$\nabla\left(\frac{u^2 - v^2}{u}\right) = \left\{1 + \left(\frac{v}{u}\right)^2\right\}\nabla u - 2\left(\frac{v}{u}\right)\nabla v$$

$$\nabla\left(\frac{v^2 - u^2}{v}\right) = \left\{1 + \left(\frac{u}{v}\right)^2\right\}\nabla v - 2\left(\frac{u}{v}\right)\nabla u.$$

Therefore

$$\begin{aligned}
\langle -\Delta_p u, \frac{u^p - v^p}{u^{p-1}}\rangle &= \int_\Omega \left\{-p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-2}\nabla u \cdot \nabla v + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right)|\nabla u|^p\right\} dx \\
&= \int_\Omega \left\{p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-2}\left(|\nabla u||\nabla v| - \nabla u \cdot \nabla v\right) + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right)|\nabla u|^p\right\} dx \\
&- \int_\Omega p\left(\frac{v}{u}\right)^{p-1}|\nabla u|^{p-1}|\nabla v| \; dx
\end{aligned}$$

and

$$\langle -\Delta u, \frac{u^2 - v^2}{u}\rangle = \int_\Omega \left\{2\left(\frac{v}{u}\right)\left(|\nabla u||\nabla v| - \nabla u \cdot \nabla v\right) + \left(1 + \left(\frac{v}{u}\right)^2\right)|\nabla u|^2 - 2\left(\frac{v}{u}\right)|\nabla u||\nabla v|\right\} dx.$$

By symmetry we have

$$\begin{aligned}
\langle -\Delta_p v, \frac{v^p - u^p}{v^{p-1}} \rangle &= \int_\Omega \left\{ -p \left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2} \nabla v \cdot \nabla u + \left(1 + (p-1)\left(\frac{u}{v}\right)^p\right) |\nabla v|^p \right\} \, dx \\
&= \int_\Omega \left\{ p \left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2} \left(|\nabla v||\nabla u| - \nabla v \cdot \nabla u \right) + \left(1 + (p-1)\left(\frac{u}{v}\right)^p\right) |\nabla v|^p \right\} \, dx \\
&- \int_\Omega p \left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-1} |\nabla u| \, dx
\end{aligned}$$

and

$$\langle -\Delta v, \frac{v^2 - u^2}{v} \rangle = \int_\Omega \left\{ 2 \left(\frac{u}{v}\right) \left(|\nabla v||\nabla u| - \nabla v \cdot \nabla u \right) + \left(1 + \left(\frac{u}{v}\right)^2\right) |\nabla v|^2 - 2 \left(\frac{u}{v}\right) |\nabla v||\nabla u| \right\} \, dx.$$

Thus

$$\begin{aligned}
I(u,v) &= \int_\Omega \left\{ p \left(\frac{v}{u}\right)^{p-1} |\nabla u|^{p-2} \left(|\nabla u||\nabla v| - \nabla u \cdot \nabla v \right) + \left(1 + (p-1)\left(\frac{v}{u}\right)^p\right) |\nabla u|^p \right\} \, dx \\
&- p \left(\frac{v}{u}\right)^{p-1} |\nabla u|^{p-1} |\nabla v| \, dx \\
&+ \int_\Omega \left\{ p \left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2} \left(|\nabla v||\nabla u| - \nabla v \cdot \nabla u \right) + \left(1 + (p-1)\left(\frac{u}{v}\right)^p\right) |\nabla v|^p \right\} \, dx \\
&- p \left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-1} |\nabla u| \, dx \\
&+ \int_\Omega \left\{ 2 \left(\frac{v}{u}\right) \left(|\nabla u||\nabla v| - \nabla u \cdot \nabla v \right) + \left(1 + \left(\frac{v}{u}\right)^2\right) |\nabla u|^2 - 2 \left(\frac{v}{u}\right) |\nabla u||\nabla v| \right\} \, dx \\
&+ \int_\Omega \left\{ 2 \left(\frac{u}{v}\right) \left(|\nabla v||\nabla u| - \nabla v \cdot \nabla u \right) + \left(1 + \left(\frac{u}{v}\right)^2\right) |\nabla v|^2 - 2 \left(\frac{u}{v}\right) |\nabla v||\nabla u| \right\} \, dx.
\end{aligned}$$

So

$$I(u,v) = \int_\Omega F(\frac{v}{u}, \nabla v, \nabla u) \, dx + \int_\Omega G(\frac{v}{u}, |\nabla v|, |\nabla u|) \, dx,$$

where

$$\begin{aligned}
F(t,S,R) &= p\left\{ t^{p-1} |R|^{p-2} \left(|R||S| - R \cdot S \right) + t^{1-p} |S|^{p-2} \left(|R||S| - R \cdot S \right) \right\} \\
&+ 2\left\{ t \left(|R||S| - R \cdot S \right) \right\} + 2\left\{ t^{-1} \left(|R||S| - R \cdot S \right) \right\}
\end{aligned}$$

and

$$\begin{aligned}
G(t,s,r) &= \left(1 + (p-1)t^p\right) r^p + \left(1 + (p-1)t^{-p}\right) s^p + (1+t^2) r^2 \\
&+ (1 + t^{-2}) s^2 - p t^{p-1} r^{p-1} s - p t^{1-p} s^{p-1} r - 2trs - 2t^{-1} rs,
\end{aligned}$$

for all $t = \frac{v}{u} > 0, R = \nabla u, S = \nabla v \in \mathbb{R}^N$ and $r = |\nabla u|, s = |\nabla v| \in \mathbb{R}^+$. We clearly have that F is non-negative. Now let us show that G is non-negative.

Indeed, we observe that

$$G(t,s,0) = \big(1+(p-1)t^{-p}\big)\, s^p + (1+t^{-2})s^2 \geq 0$$

and $G(t,s,0)=0 \Rightarrow s=0$. If $r \neq 0$, by setting $z = \frac{s}{tr}$ we obtain

$$\begin{aligned} G(t,s,r) &= t^p r^p(z^p - pz + (p-1)) + r^p((p-1)z^p - pz^{p-1} + 1) \\ &+ t^2r^2(z^2-2z+1) + r^2(z^2-2z+1), \end{aligned}$$

and G can be written as

$$G(t,s,r) = r^p(t^p f(z) + g(z)) + r^2(t^2 h(z) + k(z)),$$

with $f(z) = z^p - pz + (p-1)$, $g(z) = (p-1)z^p - pz^{p-1} + 1$, $h(z) = k(z) = z^2 - 2z + 1$ $\forall p > 1$. We can see that f, g, h and k are non-negative. Hence G is non-negative and thus $I(u,v) \geq 0$ for all $(u,v) \in D_I$. In addition since f, g, h and k vanish if and only if $z = 1$, then $G(t,s,r) = 0$ if and only if $s = tr$. Consequently, if $I(u,v) = 0$ then we have

$$\nabla u \cdot \nabla v = |\nabla u||\nabla v| \quad \text{and} \quad u|\nabla v| = v|\nabla u|$$

almost everywhere in Ω. This is equivalent to $(u\nabla v - v\nabla u)^2 = 0$, which implies that $u = \alpha v$ with $\alpha \in \mathbb{R}_+^\star$. □

Theorem 3.6.3. *The first eigenvalues λ of equation (3.4.3) are simple, i.e. if u and v are two positive first eigenfunctions associated to λ, then $u = v$.*

Proof. By proposition 3.6.2, we have $u = \alpha v$. Inserting this into the equation (3.4.3) implies that $\alpha = 1$. □

3.6.3 Invertibility of the operator $-\Delta_p - \Delta$

To simplify some notations, here we set $X = W_0^{1,p}(\Omega)$ and its dual $X^\star = W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.
For the proof of the following lemma, we refer to [65].

Lemma 3.6.4. *Let $p > 2$. Then there exist two positive constants c_1, c_2 such that, for all $x_1, x_2 \in \mathbb{R}^n$, we have :*

(i) $(x_2 - x_1) \cdot (|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1) \geq c_1|x_2 - x_1|^p$

(ii) $||x_2|^{p-2}x_2 - |x_1|^{p-2}x_1| \leq c_2(|x_2| + |x_1|)^{p-2}|x_2 - x_1|$

Proposition 3.6.5. *For $p > 2$, the operator $-\Delta_p - \Delta$ is a global homeomorphism.*

The proof is based on the previous Lemma 3.6.4.

Proof. Define the nonlinear operator $A : X \to X^\star$ by
$\langle Au, v\rangle = \displaystyle\int_\Omega \nabla u \cdot \nabla v \; dx + \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \; dx$ for all $u, v \in X$.
To show that $-\Delta_p - \Delta$ is a homeomorphism, it is enough to show that A is a continuous strongly monotone operator, [see Theorem 2.1.21].

For $p > 2$, for all $u, v \in X$, by (i), we get

$$\begin{aligned}
\langle Au - Av, u - v\rangle &= \int_\Omega |\nabla(u-v)|^2 dx + \int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla(u-v) \; dx \\
&\geq \int_\Omega |\nabla(u-v)|^2 dx + c_1 \int_\Omega |\nabla(u-v)|^p dx \\
&\geq c_1 \|u - v\|^p_{1,p}
\end{aligned}$$

Thus A is a strongly monotone operator.
We claim that A is a continuous operator from X to $X^\star$. Indeed, assume that $u_n \to u$ in X. We have to show that $\|Au_n - Au\|_{X^\star} \to 0$ as $n \to \infty$. Indeed, using (ii) and Hölder's inequality and the Sobolev embedding theorem, one has

$$\begin{aligned}
|\langle Au_n - Au, w\rangle| &\leq \int_\Omega \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right| |\nabla w| \; dx + \int_\Omega |\nabla(u_n - u)||\nabla w| \; dx \\
&\leq c_2 \int_\Omega (|\nabla u_n| + |\nabla u|)^{p-2} |\nabla(u_n - u)||\nabla w| \; dx + \int_\Omega |\nabla(u_n - u)||\nabla w| \; dx \\
&\leq c_2 \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^p \; dx\right)^{p-2/p} \left(\int_\Omega |\nabla(u_n - u)|^p dx\right)^{1/p} \left(\int_\Omega |\nabla w|^p dx\right)^{1/p} \\
&+ c_3 \|u_n - u\|_{1,2} \|w\|_{1,2} \\
&\leq c_4 (\|u_n\|_{1,p} + \|u\|_{1,p})^{p-2} \|u_n - u\|_{1,p} \|w\|_{1,p} + c_5 \|u_n - u\|_{1,p} \|w\|_{1,p}.
\end{aligned}$$

Thus $\|Au_n - Au\|_{X^\star} \to 0$, as $n \to +\infty$, and hence A is a homeomorphism. $\square$

3.7 Bifurcation of eigenvalues

In the next subsection we show that for equation (3.4.3) there is a branch of first eigenvalues bifurcating from $(\lambda_1^D, 0) \in \mathbb{R}^+ \times W_0^{1,p}(\Omega)$.

3.7.1 Bifurcation from zero : the case $p > 2$

By proposition 3.6.5, equation (3.4.3) is equivalent to

$$u = \lambda(-\Delta_p - \Delta)^{-1} u \quad \text{for} \quad u \in W^{-1,p'}(\Omega). \tag{3.7.1}$$

We set
$$S_\lambda(u) = u - \lambda(-\Delta_p - \Delta)^{-1}u, \tag{3.7.2}$$
$u \in L^2(\Omega) \subset W^{-1,p'}(\Omega)$ and $\lambda > 0$. By $\Sigma = \{(\lambda, u) \in \mathbb{R}^+ \times W_0^{1,p}(\Omega)/\ u \neq 0\ , S_\lambda(u) = 0\}$, we denote the set of nontrivial solutions of (4.1). A bifurcation point for (4.1) is a number $\lambda^\star \in \mathbb{R}^+$ such that $(\lambda^\star, 0)$ belongs to the closure of Σ. This is equivalent to say that, in any neighbourhood of $(\lambda^\star, 0)$ in $\mathbb{R}^+ \times W_0^{1,p}(\Omega)$, there exists a nontrivial solution of $S_\lambda(u) = 0$.
Our goal is to apply the Krasnoselski bifurcation Theorem 3.1.7. We state our bifurcation result.

Theorem 3.7.1. Let $p > 2$. Then every eigenvalue λ_k^D with odd multiplicity is a bifurcation point in $\mathbb{R}^+ \times W_0^{1,p}(\Omega)$ of $S_\lambda(u) = 0$, in the sense that in any neighbourhood of $(\lambda_k^D, 0)$ in $\mathbb{R}^+ \times W_0^{1,p}(\Omega)$ there exists a nontrivial solution of $S_\lambda(u) = 0$.

Proof. We write the equation $S_\lambda(u) = 0$ as
$$u = \lambda A u + T_\lambda(u),$$
where $Au = (-\Delta)^{-1}u$ and $T_\lambda(u) = [(-\Delta_p - \Delta)^{-1} - (-\Delta)^{-1}](\lambda u)$, where we consider
$$(-\Delta_p - \Delta)^{-1} : L^2(\Omega) \subset W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$$
and $(-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega) \subset\subset L^2(\Omega)$.

For $p > 2$, the mapping
$$(-\Delta_p - \Delta)^{-1} - (-\Delta)^{-1} : L^2(\Omega) \subset W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega) \subset\subset L^2(\Omega)$$
is compact thanks to Rellich-Kondrachov theorem. We clearly have $A \in \mathcal{L}(L^2(\Omega))$ and $T_\lambda(0) = 0$. Now we have to show that

(1) $T_\lambda \in C^1$.

(2) $T'_\lambda(0) = 0$.

In order to show (1) and (2), it suffices to show that

(a) $-\Delta_p - \Delta : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuously differentiable in a neighborhood $u \in W_0^{1,p}(\Omega)$.

(b) $(-\Delta_p - \Delta)^{-1}$ is a continuous inverse operator.

According to Proposition 3.6.5, $-\Delta_p - \Delta$ is a homeomorphism, hence $(-\Delta_p - \Delta)^{-1}$ is continuous and this shows (b). We also recall that in section 3.6.2, we have shown that λ_1^D is simple.

Let us show (a). We claim that $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is Gâteaux differentiable. Indeed, for $\varphi \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}
\langle -\Delta_p(u+\delta v), \varphi\rangle - \langle -\Delta_p u, \varphi\rangle &= \langle |\nabla(u+\delta v)|^{p-2}\nabla(u+\delta v), \nabla\varphi\rangle - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&= \langle \left(|\nabla(u+\delta v)|^2\right)^{\frac{p-2}{2}} \nabla(u+\delta v), \nabla\varphi\rangle - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&= \langle \left(|\nabla u|^2 + 2\delta\langle\nabla u, \nabla v\rangle + \delta^2|\nabla v|^2\right)^{\frac{p-2}{2}} \nabla(u+\delta v), \nabla\varphi\rangle \\
&- \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&= \langle [|\nabla u|^{p-2} + (p-2)|\nabla u|^{2(\frac{p-2}{2}-1)}\delta\langle\nabla u, \nabla v\rangle \\
&+ O(\delta^2)]\nabla(u+\delta v), \nabla\varphi\rangle - \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&= \langle [|\nabla u|^{p-2} + (p-2)|\nabla u|^{p-4}\delta\langle\nabla u, \nabla v\rangle + O(\delta^2)]\nabla(u+\delta v), \nabla\varphi\rangle \\
&- \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \\
&= (p-2)\delta|\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle + \delta\langle |\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle + O(\delta^2) \\
&= \delta[(p-2)|\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle + \langle |\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle + O(\delta)].
\end{aligned}$$

Define

$$\langle B(u)v, \varphi\rangle = (p-2)|\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle + \langle |\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle$$

and let $(u_n)_{n\geq 0} \subset W_0^{1,p}(\Omega)$. Assume that $u_n \to u$, as $n \to \infty$ in $W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}
\langle B(u_n)v - B(u)v, \varphi\rangle &= (p-2)\left[|\nabla u_n|^{p-4}\langle\nabla u_n, \nabla v\rangle\langle\nabla u_n, \nabla\varphi\rangle - |\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle\right] \\
&+ \langle |\nabla u_n|^{p-2}\nabla v, \nabla\varphi\rangle - \langle |\nabla u|^{p-2}\nabla v, \nabla\varphi\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle B(u_n)v - B(u)v, \varphi\rangle| \leq (p-2) &\left||\nabla u_n|^{p-4}\langle\nabla u_n, \nabla v\rangle\langle\nabla u_n, \nabla\varphi\rangle - |\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle\right| \\
+ &\left||\nabla u_n|^{p-2} - |\nabla u|^{p-2}\right| |\langle\nabla v, \nabla\varphi\rangle|.
\end{aligned}$$

By assumption, we can assume that, up to subsequences,

$(*)$ $\nabla u_n \to \nabla u$ in $(L^p(\Omega))^N$ as $n \to \infty$ and

$(**)$ $\nabla u_n(x) \to \nabla u(x)$ almost everywhere as $n \to \infty$.

Then $|\nabla u_n|^{p-4}\langle\nabla u_n, \nabla v\rangle\langle\nabla u_n, \nabla\varphi\rangle \to |\nabla u|^{p-4}\langle\nabla u, \nabla v\rangle\langle\nabla u, \nabla\varphi\rangle$ as $n \to \infty$ and consequently $\langle B(u_n)v, \varphi\rangle \to \langle B(u)v, \varphi\rangle$as $n \to \infty$. Thus, we find that

$-\Delta_p - \Delta \in C^1$ and thanks to the Inverse function theorem $(-\Delta_p - \Delta)^{-1}$ is differentiable in a neighborhood of $u \in W_0^{1,p}(\Omega)$. Therefore according to the Krasnoselski bifurcation Theorem, we obtain that λ_k^D is a bifurcation point at zero. □

3.7.2 Bifurcation from infinity : the case $1 < p < 2$

We recall the nonlinear eigenvalue problem we are studying,

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7.3}$$

Under a solution of (3.7.3) (for $1 < p < 2$), we understand a pair $(\lambda, u) \in \mathbb{R}_\star^+ \times W_0^{1,2}(\Omega)$ satisfying the integral equality,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx + \int_\Omega \nabla u \cdot \nabla\varphi \, dx = \lambda \int_\Omega u\varphi \, dx \;\; \text{for every } \varphi \in W_0^{1,2}(\Omega). \tag{3.7.4}$$

We now state the main theorem concerning the bifurcation from infinity.

Theorem 3.7.2. The pair (λ_1^D, ∞) is a bifurcation point from infinity for the problem (3.7.3).

For $u \in W_0^{1,2}(\Omega),\; u \neq 0$, we set $v = u/\|u\|_{1,2}^{2-\frac{1}{2}p}$. We have $\|v\|_{1,2} = \frac{1}{\|u\|_{1,2}^{1-\frac{1}{2}p}}$ and

$$|\nabla v|^{p-2}\nabla v = \frac{1}{\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-1)}}|\nabla u|^{p-2}\nabla u.$$

Introducing this change of variable in (3.7.4), we find that,

$$\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-2)} \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi \, dx + \int_\Omega \nabla v \cdot \nabla\varphi \, dx = \lambda \int_\Omega v\varphi \, dx \;\; \text{for every } \varphi \in W_0^{1,2}(\Omega). \tag{3.7.5}$$

But, on the other hand, we have

$$\|v\|_{1,2}^{p-4} = \frac{1}{\|u\|_{1,2}^{(1-\frac{1}{2}p)(p-4)}} = \frac{1}{\|u\|_{1,2}^{(2-\frac{1}{2}p)(p-2)}}.$$

Consequently it follows that equation (3.7.5) is equivalent to

$$\|v\|_{1,2}^{4-p} \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi \, dx + \int_\Omega \nabla v \cdot \nabla\varphi \, dx = \lambda \int_\Omega v\varphi \, dx \;\; \text{for every } \varphi \in W_0^{1,2}(\Omega). \tag{3.7.6}$$

This leads to the following nonlinear eigenvalue problem (for $1 < p < 2$)

$$\begin{cases} -\|v\|_{1,2}^{4-p}\Delta_p v - \Delta v = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7.7}$$

The proof of Theorem 3.7.2 follows immediately from the following remark, and the proof that $(\lambda_1^D, 0)$ is a bifurcation of (3.7.7).

Remark 3.7.3. With this transformation, we have that the pair (λ_1^D, ∞) is a bifurcation point for the problem (3.7.3) if and only if the pair $(\lambda_1^D, 0)$ is a bifurcation point for the problem (3.7.7).

Let us consider a small ball $B_r(0) := \{\ w \ \in W_0^{1,2}(\Omega)/ \quad \|w\|_{1,2} < r\ \}$, and consider the operator

$$T := -\|\cdot\|_{1,2}^{4-p}\Delta_p - \Delta : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega).$$

Proposition 3.7.4. Let $1 < p < 2$. There exists $r > 0$ such that the mapping $T : B_r(0) \subset W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is invertible, with a continuous inverse.

Proof. In order to prove that the operator T is invertible with a continuous inverse, we again rely on Theorem 2.1.21. We show that there exists $\delta > 0$ such that

$$\langle T(u) - T(v), u - v\rangle \geq \delta\|u - v\|_{1,2}^2, \quad \text{for} \quad u, v \in B_r(0) \subset W_0^{1,2}(\Omega)$$

with $r > 0$ sufficiently small.
Indeed, using that $-\Delta_p$ is strongly monotone on $W_0^{1,p}(\Omega)$ on the one hand and the Hölder inequality on the other hand, we have

$$\begin{aligned}
\langle T(u) - T(v), u - v\rangle &= \|\nabla u - \nabla v\|_2^2 + \left(\|u\|_{1,2}^{4-p}(-\Delta_p u) - \|v\|_{1,2}^{4-p}(-\Delta_p v), u - v\right) \\
&= \|u - v\|_{1,2}^2 + \|u\|_{1,2}^{4-p}\left((-\Delta_p u) - (-\Delta_p v), u - v\right) \\
&+ \left(\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right)(-\Delta_p v, u - v) \\
&\geq \|u - v\|_{1,2}^2 - \left|\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right| \|\nabla v\|_p^{p-1}\|\nabla(u - v)\|_p \\
&\geq \|u - v\|_{1,2}^2 - \left|\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right| C\|v\|_{1,2}^{p-1}\|u - v\|_{1,2}
\end{aligned} \tag{3.7.8}$$

Now, we obtain by the Mean Value Theorem that there exists $\theta \in [0,1]$ such that

$$\begin{aligned}
\left|\|u\|_{1,2}^{4-p} - \|v\|_{1,2}^{4-p}\right| &= \left|\frac{d}{dt}\left(\|u+t(v-u)\|_{1,2}^2\right)^{2-\frac{1}{2}p}|_{t=\theta}(v-u)\right| \\
&= \left|(2-\frac{1}{2}p)\left(\|u+\theta(v-u)\|_{1,2}^2\right)^{1-\frac{1}{2}p} 2\left(u+\theta(v-u), v-u\right)_{1,2}\right| \\
&\leq (4-p)\|u+\theta(v-u)\|_{1,2}^{2-p}\|u+\theta(v-u)\|_{1,2}\|u-v\|_{1,2} \\
&= (4-p)\|u+\theta(v-u)\|_{1,2}^{3-p}\|u-v\|_{1,2} \\
&\leq (4-p)\left((1-\theta)\|u\|_{1,2}+\theta\|v\|_{1,2}\right)^{3-p}\|u-v\|_{1,2} \\
&\leq (4-p)r^{3-p}\|u-v\|_{1,2}.
\end{aligned}$$

Hence, continuing with the estimate of equation (3.7.8), we get

$$\langle T(u)-T(v), u-v\rangle \geq \|u-v\|_{1,2}^2(1-(4-p)r^{3-p}Cr^{p-1}) = \|u-v\|_{1,2}^2(1-C'r^2),$$

and thus the claim, for $r>0$ small enough.
Hence, the operator T is strongly monotone on $B_r(0)$ and it is continuous, and hence the claim follows. □

Clearly the mappings

$$T_\tau = -\Delta - \tau\|\cdot\|_{1,2}^\gamma\Delta_p : B_r(0) \subset W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega), \quad 0 \leq \tau \leq 1$$

are also local homeomorphisms for $1<p<2$ with $\gamma = 4-p > 0$. Consider now the homotopy maps

$$H(\tau, y) := (-\tau\|\cdot\|_{1,2}^\gamma\Delta_p - \Delta)^{-1}(y), \quad y \in T_\tau(B_r(0)) \subset W^{-1,2}(\Omega).$$

Then we can find a $\rho > 0$ such that the ball

$$B_\rho(0) \subset \bigcap_{0\leq\tau\leq1} T_\tau(B_r(0))$$

and

$$H(\tau,\cdot) : B_\rho(0)\cap L^2(\Omega) \mapsto W_0^{1,2}(\Omega) \subset\subset L^2(\Omega)$$

are compact mappings. Set now

$$\tilde{S}_\lambda(u) = u - \lambda(-\|u\|_{1,2}^\gamma\Delta_p - \Delta)^{-1}u.$$

Notice that $\tilde{S}_\lambda$ is a compact perturbation of the identity in $L^2(\Omega)$. We have $0 \notin H([0,1]\times\partial B_r(0))$. So it makes sense to consider the Leray-Schauder topological degree of $H(\tau,\cdot)$ on $B_r(0)$. And by the property of the invariance by homotopy, one has

$$\deg(H(0,\cdot), B_r(0), 0) = \deg(H(1,\cdot), B_r(0), 0). \tag{3.7.9}$$

Theorem 3.7.5. The pair $(\lambda_1^D, 0)$ is a bifurcation point in $\mathbb{R}^+ \times L^2(\Omega)$ of $\tilde{S}_\lambda(u) = 0$, for $1 < p < 2$.

Proof. Suppose by contradiction that $(\lambda_1^D, 0)$ is not a bifurcation for $\tilde{S}_\lambda$. Then, there exist $\delta_0 > 0$ such that for all $r \in (0, \delta_0)$ and $\varepsilon \in (0, \delta_0)$,

$$\tilde{S}_\lambda(u) \neq 0 \quad \forall\ |\lambda_1^D - \lambda| \leq \varepsilon,\ \forall\ u \in L^2(\Omega),\ \ \|u\|_2 = r. \tag{3.7.10}$$

Taking into account that (3.7.10) holds, it follows that it make sense to consider the Leray-Schauder topological degree $\deg(\tilde{S}_\lambda, B_r(0), 0)$ of $\tilde{S}_\lambda$ on $B_r(0)$.

We observe that

$$\big(I - (\lambda_1^D - \varepsilon) H(\tau, \cdot)\big)\,|_{\partial B_r(0)} \neq 0 \ \text{ for } \ \tau \in [0,1]. \tag{3.7.11}$$

Proving (3.7.11) garantee the well posedness of $\deg(I - (\lambda_1^D \pm \varepsilon) H(\tau,\cdot), B_r(0), 0)$ for any $\tau \in [0,1]$. Indeed, by contradiction suppose that there exists $v \in \partial B_r(0) \subset L^2(\Omega)$ such that
$v - (\lambda_1^D - \varepsilon) H(\tau, v) = 0$, for some $\tau \in [0,1]$.
One concludes that then $v \in W_0^{1,2}(\Omega)$, and then that

$$-\Delta v - \tau \|v\|_{1,2}^\gamma \Delta_p v = (\lambda_1^D - \varepsilon) v.$$

However, we get the contradiction,

$$(\lambda_1^D - \varepsilon)\|v\|_2^2 = \|\nabla v\|_2^2 + \tau \|v\|_{1,2}^\gamma \|\nabla v\|_p^p \geq \|\nabla v\|_2^2 \geq \lambda_1^D \|v\|_2^2.$$

By the contradiction assumption, we have

$$\deg(I - (\lambda_1^D + \varepsilon) H(1,\cdot), B_r(0), 0) = \deg(I - (\lambda_1^D - \varepsilon) H(1,\cdot), B_r(0), 0). \tag{3.7.12}$$

By homotopy using (3.7.9), we have

$$\begin{aligned} \deg(I - (\lambda_1^D - \varepsilon) H(1,\cdot), B_r(0), 0) &= \deg(I - (\lambda_1^D - \varepsilon) H(0,\cdot), B_r(0), 0) \\ &= \deg(I - (\lambda_1^D - \varepsilon)(-\Delta)^{-1}, B_r(0), 0) = 1 \end{aligned} \tag{3.7.13}$$

Now, using (3.7.13) and (3.7.12), we find that

$$\deg(I - (\lambda_1^D + \varepsilon) H(1,\cdot), B_r(0), 0) = \deg(I - (\lambda_1^D - \varepsilon) H(0,\cdot), B_r(0), 0) = 1 \tag{3.7.14}$$

Furthermore, since λ_1^D is a simple eigenvalue of $-\Delta$, it is well-known [see [6]] that

$$\deg(I-(\lambda_1^D+\varepsilon)(-\Delta)^{-1},B_r(0),0)=\deg(I-(\lambda_1^D+\varepsilon)H(0,\cdot),B_r(0),0)=-1 \tag{3.7.15}$$

In order to get contradiction (to relation (3.7.14)), it is enough to show that,

$$\deg(I-(\lambda_1^D+\varepsilon)H(1,\cdot),B_r(0),0)=\deg(I-(\lambda_1^D+\varepsilon)H(0,\cdot),B_r(0),0), \tag{3.7.16}$$

$r>0$ sufficiently small. We have to show that

$$\big(I-(\lambda_1^D+\varepsilon)H(\tau,\cdot)\big)\,|_{\partial B_r(0)}\neq 0 \quad \text{for} \quad \tau\in[0,1].$$

Suppose by contradiction that there is $r_n\to 0$, $\tau_n\in[0,1]$ and $u_n\in\partial B_{r_n}(0)$ such that

$$u_n-(\lambda_1^D+\varepsilon)H(\tau_n,u_n)=0$$

or equivalently

$$-\tau_n\|u_n\|_{1,2}^{\gamma}\Delta_p u_n-\Delta u_n=(\lambda_1^D+\varepsilon)u_n. \tag{3.7.17}$$

Dividing the equation (3.7.17) by $\|u_n\|_{1,2}$, we obtain

$$-\tau_n\|u_n\|_{1,2}^{\gamma+p-1}\Delta_p\left(\frac{u_n}{\|u_n\|_{1,2}}\right)-\Delta\left(\frac{u_n}{\|u_n\|_{1,2}}\right)=(\lambda_1^D+\varepsilon)\frac{u_n}{\|u_n\|_{1,2}},$$

and by setting $v_n=\frac{u_n}{\|u_n\|_{1,2}}$, it follows that

$$-\tau_n\|u_n\|_{1,2}^{\gamma+p-1}\Delta_p v_n-\Delta v_n=(\lambda_1^D+\varepsilon)v_n. \tag{3.7.18}$$

But since $\|v_n\|_{1,2}=1$, we have $v_n\rightharpoonup v$ in $W_0^{1,2}(\Omega)$ and $v_n\to v$ in $L^2(\Omega)$. Furthermore, the first term in the left hand side of equation (3.7.18) tends to zero in $W^{-1,p'}(\Omega)$ as $r_n\to 0$ and hence in $W^{-1,2}(\Omega)$. Equation (3.7.17) then implies that $v_n\to v$ strongly in $W_0^{1,2}(\Omega)$ since $-\Delta: W_0^{1,2}(\Omega)\to W^{-1,2}(\Omega)$ is a homeomorphism and thus v with $\|v\|_{1,2}=1$ solves $-\Delta v=(\lambda_1^D+\varepsilon)v$, which is impossible because $\lambda_1^D+\varepsilon$ is not the first eigenvalue of $-\Delta$ on $W_0^{1,2}(\Omega)$ for $\varepsilon>0$.

Therefore, by homotopy it follows that

$$\deg(I-(\lambda_1^D+\varepsilon)H(1,\cdot),B_r(0),0)=\deg(I-(\lambda_1^D+\varepsilon)H(0,\cdot),B_r(0),0).$$

Now, thanks to (3.7.15), we find that

$$\deg(I-(\lambda_1^D+\varepsilon)H(1,\cdot),B_r(0),0)=-1,$$

which contradicts equation (3.7.14). $\square$

Theorem 3.7.6. The pair $(\lambda_k^D, 0)$ $(k > 1)$ is a bifurcation point of $\tilde{S}_\lambda(u) = 0$, for $1 < p < 2$ if λ_k^D is of odd multiplicity.

Proof. Suppose by contradiction that $(\lambda_k^D, 0)$ is not a bifurcation for $\tilde{S}_\lambda$. Then, there exist $\delta_0 > 0$ such that for all $r \in (0, \delta_0)$ and $\varepsilon \in (0, \delta_0)$,

$$\tilde{S}_\lambda(u) \neq 0 \quad \forall\ |\lambda_k^D - \lambda| \leq \varepsilon,\ \forall\ u \in L^2(\Omega),\ \|u\|_2 = r. \tag{3.7.19}$$

Taking into account that (3.7.19) holds, it follows that it make sense to consider the Leray-Schauder topological degree $\deg(\tilde{S}_\lambda, B_r(0), 0)$ of $\tilde{S}_\lambda$ on $B_r(0)$.
We show that

$$\big(I - (\lambda_k^D - \varepsilon)H(\tau, \cdot)\big)\,|_{\partial B_r(0)} \neq 0 \quad \text{for} \quad \tau \in [0, 1]. \tag{3.7.20}$$

Proving (3.7.20) garantee the well posedness of $\deg(I - (\lambda_k^D \pm \varepsilon)H(\tau, \cdot), B_r(0), 0)$ for any $\tau \in [0, 1]$. Indeed, consider the projections P^- and P^+ onto the spaces $\text{span}\{e_1, \dots, e_{k-1}\}$ and $\text{span}\{e_k, e_{k+1}, \dots\}$, respectively, where $e_1 \dots, e_k, e_{k+1}, \dots$ denote the eigenfunctions associated to the Dirichlet problem (3.4.1).
Suppose by contradiction that relation (3.7.20) does not hold. Then there exists $v \in \partial B_r(0) \subset L^2(\Omega)$ such that $v - (\lambda_k^D - \varepsilon)H(\tau, v) = 0$, for some $\tau \in [0, 1]$. This is equivalent of having

$$-\Delta v - (\lambda_k^D - \varepsilon)v = \tau\|v\|_{1,2}^\gamma \Delta_p v. \tag{3.7.21}$$

Replacing v by $P^+v + P^-v$, and multiplying equation (3.7.21) by $P^+v - P^-v$ in the both sides, we obtain

$$\langle[-\Delta - (\lambda_k^D - \varepsilon)](P^+v + P^-v), P^+v - P^-v\rangle = \tau\|P^+v + P^-v\|_{1,2}^\gamma\langle\Delta_p[P^+v + P^-v], P^+v - P^-v\rangle$$

$$\Updownarrow$$

$$\begin{aligned} -\left[\|\nabla P^-v\|_2^2 - (\lambda_k^D - \varepsilon)\|P^-v\|_2^2\right] + \|\nabla P^+v\|_2^2 - (\lambda_k^D - \varepsilon)\|P^+v\|_2^2 \;&=\; \tau\|P^+v + P^-v\|_{1,2}^\gamma \\ &\times\; \langle\Delta_p[P^+v + P^-v], P^+v - P^-v\rangle. \end{aligned}$$

But

$$\langle\Delta_p[P^+v + P^-v], P^+v - P^-v\rangle = -\int_\Omega |\nabla(P^+v + P^-v)|^{p-2}\nabla(P^+v + P^-v)\cdot\nabla(P^+v - P^-v)\,dx,$$

and using the Hölder inequality, the embedding $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$ and the fact that P^+v and P^-v don't vanish simultaneously, there is some positive constant $C' > 0$ such that

$\|P^+v - P^-v\|_{1,2} \leq C'(\|P^+v\|_{1,2}^2 + \|P^-v\|_{1,2}^2) = C'\|P^+v - P^-v\|_{1,2}^2$, since $(P^+v, P^-v)_{1,2} = 0$, we have

$$\begin{aligned}
\left|\langle \Delta_p[P^+v + P^-v], P^+v - P^-v\rangle\right| &\leq \|P^+v + P^-v\|_{1,p}^{p-1}\|P^+v - P^-v\|_{1,p} \\
&\leq C'\|P^+v + P^-v\|_{1,2}^{p-1}\|P^+v - P^-v\|_{1,2}^2 \\
&\leq C'\|P^+v + P^-v\|_{1,2}^{p+1}, \quad \text{since} \quad \|P^+v - P^-v\|_{1,2}^2 = \|P^+v + P^-v\|_{1,2}^2.
\end{aligned}$$

On the other hand, thanks to the Poincaré inequality as well as the variational characterization of eigenvalues we find

$$-\left[\|\nabla P^-v\|_2^2 - (\lambda_k^D - \varepsilon)\|P^-v\|_2^2\right] \geq 0$$

and

$$\|\nabla P^+v\|_2^2 - (\lambda_k^D - \varepsilon)\|P^+v\|_2^2 \geq 0,$$

we can bound from below these two inequalities together by $\|\nabla P^+v\|_2^2 + \|\nabla P^-v\|_2^2$. Finally, we have

$$\|v\|_{1,2}^2 = \|\nabla P^+v\|_2^2 + \|\nabla P^-v\|_2^2 \leq \tau C'\|P^+v + P^-v\|_{1,2}^{\gamma+p+1}, \quad \text{with} \quad \gamma = 4 - p,$$

$$\Updownarrow$$

$$\|v\|_{1,2}^2 \leq C''\|v\|_{1,2}^{\gamma+p+1} \Leftrightarrow 1 \leq C''r^3 \to 0,$$

for r taken small enough. This shows that (3.7.20) holds.
By the contradiction assumption, we have

$$\deg(I - (\lambda_k^D + \varepsilon)H(1,\cdot), B_r(0), 0) = \deg(I - (\lambda_k^D - \varepsilon)H(1,\cdot), B_r(0), 0). \quad (3.7.22)$$

By homotopy using (3.7.20), we have

$$\begin{aligned}
\deg(I - (\lambda_k^D - \varepsilon)H(1,\cdot), B_r(0), 0) &= \deg(I - (\lambda_k^D - \varepsilon)H(0,\cdot), B_r(0), 0) \quad (3.7.23) \\
&= \deg(I - (\lambda_k^D - \varepsilon)(-\Delta)^{-1}, B_r(0), 0) = (-1)^{\beta},
\end{aligned}$$

where β is the sum of algebraic multiplicities of the eigenvalues $\lambda_k^D - \varepsilon < \lambda$. Similarly, if β' denotes the sum of the algebraic multiplicities of the characteristic values of $(-\Delta)^{-1}$ such that $\lambda > \lambda_k^D + \varepsilon$, then

$$\deg(I - (\lambda_k^D + \varepsilon)H(1,\cdot), B_r(0), 0) = (-1)^{\beta'} \quad (3.7.24)$$

But since $[\lambda_k^D-\varepsilon,\lambda_k^D+\varepsilon]$ contains only the eigenvalue λ_k^D, it follows that $\beta'=\beta+\alpha$, where α denotes the algebraic multiplicity of λ_k^D. Consequently, we have

$$\begin{aligned}\deg(I-(\lambda_k^D+\varepsilon)H(1,\cdot),B_r(0),0) &= (-1)^{\beta+\alpha}\\ &= (-1)^{\alpha}\deg(I-(\lambda_k^D+\varepsilon)H(1,\cdot),B_r(0),0)\\ &= -\deg(I-(\lambda_k^D+\varepsilon)H(1,\cdot),B_r(0),0),\end{aligned}$$

since λ_k^D is with odd multiplicity. This contradicts (3.7.22). $\square$

3.8 Multiple solutions

In this section we prove multiciplity results by distinguishing again the two cases $1<p<2$ and $p>2$. Let Γ be the class of all the symmetric subsets $A\subseteq X\backslash\{0\}$ which are closed in $X\backslash\{0\}$.

Theorem 3.8.1. *Let $1<p<2$ or $2<p<\infty$, and suppose that $\lambda\in(\lambda_k^D,\lambda_{k+1}^D)$ for any $k\in\mathbb{N}^*$. Then equation (3.4.3) has at least k pairs of nontrivial solutions.*

Proof. Case 1: $1<p<2$.
In this case we will avail of [6, Proposition 10.8]. We consider the energy functional $I_\lambda: W_0^{1,2}(\Omega)\backslash\{0\}\to\mathbb{R}$ associated to the problem (3.4.3) defined by

$$I_\lambda(u)=\frac{2}{p}\int_\Omega|\nabla u|^p\,dx+\int_\Omega|\nabla u|^2\,dx-\lambda\int_\Omega u^2\,dx.$$

The functional I_λ is not bounded from below on $W_0^{1,2}(\Omega)$, so we consider again the natural constraint set, the Nehari manifold on which we minimize the functional I_λ. The Nehari manifold is given by

$$\mathcal{N}_\lambda:=\{u\in W_0^{1,2}(\Omega)\backslash\{0\}:\ \langle I'_\lambda(u),u\rangle=0\}.$$

On $\mathcal{N}_\lambda$, we have $I_\lambda(u)=(\frac{2}{p}-1)\displaystyle\int_\Omega|\nabla u|^p\,dx>0$. We clearly have that, I_λ is even and bounded from below on $\mathcal{N}_\lambda$.
Now, let us show that every (PS) sequence for I_λ has a converging subsequence on $\mathcal{N}_\lambda$. Let $(u_n)_n$ be a (PS) sequence, i.e, $|I_\lambda(u_n)|\le C$, for all n, for some $C>0$ and $I'_\lambda(u_n)\to 0$ in $W^{-1,2}(\Omega)$ as $n\to+\infty$.
We first show that the sequence $(u_n)_n$ is bounded on $\mathcal{N}_\lambda$. Suppose by contradiction that this is not true, so $\displaystyle\int_\Omega|\nabla u_n|^2\,dx\to+\infty$ as $n\to+\infty$. Since $I_\lambda(u_n)=(\frac{2}{p}-1)\displaystyle\int_\Omega|\nabla u_n|^p\,dx$ we have $\displaystyle\int_\Omega|\nabla u_n|^p\,dx\le c$. On $\mathcal{N}_\lambda$, we have

$$0<\int_\Omega|\nabla u_n|^p\,dx=\lambda\int_\Omega u_n^2\,dx-\int_\Omega|\nabla u_n|^2\,dx,\tag{3.8.1}$$

and hence $\displaystyle\int_\Omega u_n^2\,dx \to +\infty$. Let $v_n = \frac{u_n}{\|u_n\|_2}$ then $\displaystyle\int_\Omega |\nabla v_n|^2\,dx \leq \lambda$ and hence v_n is bounded in $W_0^{1,2}(\Omega)$. Therefore there exists $v_0 \in W_0^{1,2}(\Omega)$ such that $v_n \rightharpoonup v_0$ in $W_0^{1,2}(\Omega)$ and $v_n \to v_0$ in $L^2(\Omega)$. Dividing (3.8.1) by $\|u_n\|_2^p$, we have

$$\frac{\lambda \displaystyle\int_\Omega u_n^2\,dx - \int_\Omega |\nabla u_n|^2\,dx}{\|u_n\|_2^p} = \int_\Omega |\nabla v_n|^p\,dx \to 0,$$

since $\lambda \displaystyle\int_\Omega u_n^2\,dx - \int_\Omega |\nabla u_n|^2\,dx = (\frac{2}{p} - 1)^{-1} I_\lambda(u_n)$, $|I_\lambda(u_n)| \leq C$ and $\|u_n\|_2^p \to +\infty$. Now, since $v_n \rightharpoonup v_0$ in $W_0^{1,2}(\Omega) \subset W_0^{1,p}(\Omega)$, we infer that

$$\int_\Omega |\nabla v_0|^p\,dx \leq \liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^p\,dx = 0,$$

and consequently $v_0 = 0$. So $v_n \to 0$ in $L^2(\Omega)$ and this is a contradiction since $\|v_n\|_2 = 1$. So $(u_n)_n$ is bounded on $\mathcal{N}_\lambda$.
Next, we show that u_n converges strongly to u in $W_0^{1,2}(\Omega)$.
To do this, we will use the following vectors inequality for $1 < p < 2$

$$(|x_2|^{p-2}x_2 - |x_1|^{p-2}x_1)\cdot(x_2 - x_1) \geq C'(|x_2| + |x_1|)^{p-2}|x_2 - x_1|^2,$$

for all $x_1, x_2 \in \mathbb{R}^N$ and for some $C' > 0$, [see [65]].
We have $\displaystyle\int_\Omega u_n^2\,dx \to \int_\Omega u^2\,dx$ and since $I'_\lambda(u_n) \to 0$ in $W^{-1,2}(\Omega)$, $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$, we also have $I'_\lambda(u_n)(u_n - u) \to 0$ and $I'_\lambda(u)(u_n - u) \to 0$ as $n \to +\infty$.
On the other hand, one has

$$\begin{aligned}
\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle &= 2\left[\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n - u)\,dx\right] \\
&+ 2\int_\Omega |\nabla(u_n - u)|^2\,dx - 2\lambda\int_\Omega |u_n - u|^2\,dx \\
&\geq C'\int_\Omega (|\nabla u_n| + |\nabla u|)^{p-2}\,|\nabla(u_n - u)|^2\,dx \\
&+ 2\int_\Omega |\nabla(u_n - u)|^2\,dx - 2\lambda\int_\Omega |u_n - u|^2\,dx \\
&\geq 2\int_\Omega |\nabla(u_n - u)|^2\,dx - 2\lambda\int_\Omega |u_n - u|^2\,dx \\
&\geq \|u_n - u\|_{1,2}^2 - \lambda\int_\Omega |u_n - u|^2\,dx.
\end{aligned}$$

Therefore $\|u_n - u\|_{1,2} \to 0$ as $n \to +\infty$ and u_n converges strongly to u in $W_0^{1,2}(\Omega)$.

Let $\Sigma' = \{A \subset \mathcal{N}_\lambda : \ A \text{ closed and } -A = A\}$ and $\Gamma_j = \{A \in \Sigma' : \ \gamma(A) \geq j\}$, where $\gamma(A)$ denotes the Krasnoselski's genus. We show that $\Gamma_j \neq \emptyset$.
Set $E_j = \operatorname{span}\{e_i, \ \ i = 1, \dots, j\}$, where e_i are the eigenfunctions associated to the problem (3.4.1). Let $\lambda \in (\lambda_j^D, \lambda_{j+1}^D)$, and consider $v \in S_j := \{v \in E_j : \ \int_\Omega |v|^2 \, dx = 1\}$. Then set

$$\rho(v) = \left[\frac{\int_\Omega |\nabla v|^p \, dx}{\lambda \int_\Omega v^2 \, dx - \int_\Omega |\nabla v|^2 \, dx}\right]^{\frac{1}{2-p}}.$$

Then $\lambda \int_\Omega v^2 \, dx - \int_\Omega |\nabla v|^2 \, dx \geq \lambda \int_\Omega v^2 \, dx - \sum_{i=1}^{j} \int_\Omega \lambda_i |e_i|^2 \, dx \geq (\lambda - \lambda_j) \int_\Omega |v|^2 \, dx > 0$. Hence, $\rho(v)v \in \mathcal{N}_\lambda$, and then $\rho(S_j) \in \Sigma'$, and $\gamma(\rho(S_j)) = \gamma(S_j) = j$ for $1 \leq j \leq k$, for any $k \in \mathbb{N}^*$.
It is then standard (see [6, Proposition 10.8]) to conclude that

$$\sigma_{\lambda,j} = \inf_{\gamma(A) \geq j} \sup_{u \in A} I_\lambda(u), \quad 1 \leq j \leq k, \quad \text{for any} \quad k \in \mathbb{N}^*$$

yields k pairs of nontrivial critical points for I_λ, which gives rise to k nontrivial solutions of problem (3.4.3).

Case 2: $p > 2$.
In this case, we will rely on Theorem 3.3.8.

Let us consider the C^1 energy functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$I_\lambda(u) = \frac{2}{p} \int_\Omega |\nabla u|^p \, dx + \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega |u|^2 \, dx.$$

We want to show that

$$-\infty < \sigma_j = \inf_{\{A \in \Sigma', \gamma(A) \geq j\}} \sup_{u \in A} I_\lambda(u) \tag{3.8.2}$$

is a critical point for I_λ, where $\Sigma' = \{A \subseteq S_j\}$, where $S_j = \{v \in E_j \ : \ \int_\Omega |v|^2 \, dx = 1\}$.
We clearly have that $I_\lambda(u)$ is an even functional for all $u \in W_0^{1,p}(\Omega)$, and also $I_\lambda(u)$ is bounded from below on $W_0^{1,p}(\Omega)$ since $I_\lambda(u) \geq C\|u\|_{1,p}^p - C'\|u\|_{1,p}^2$.

We show that $I_\lambda(u)$ satisfies the (PS) condition. Let $\{u_n\}$ be a Palais-Smale sequence, i.e., $|I_\lambda(u_n)| \leq M$ for all n, $M > 0$ and $I'_\lambda(u_n) \to 0$ in $W^{-1,p'}(\Omega)$ as $n \to \infty$. We first show that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. We have

$$M \ \geq \ |C\|u_n\|_{1,p}^p - C'\|u_n\|_{1,p}^2| \geq \left(C\|u_n\|_{1,p}^{p-2} - C'\right)\|u_n\|_{1,p}^2,$$

and so $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, $u \in W_0^{1,p}(\Omega)$ exists such that, up to subsequences that we will denote by $(u_n)_n$ we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$.
We will use the following inequality for $v_1, v_2 \in \mathbb{R}^N$: there exists $R > 0$ such that

$$|v_1 - v_2|^p \le R\left(|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2\right)(v_1 - v_2),$$

for $p > 2$ [see [65]]. Then we obtain

$$\begin{aligned}
\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle &= 2\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot\nabla(u_n - u)\, dx + 2\int_\Omega |\nabla u_n - \nabla u|^2\, dx \\
&\quad - 2\lambda\int_\Omega |u_n - u|^2\, dx \\
&\ge \frac{2}{R}\int_\Omega |\nabla u_n - \nabla u|^p\, dx + 2\int_\Omega |\nabla u_n - \nabla u|^2\, dx - 2\lambda\int_\Omega |u_n - u|^2\, dx \\
&\ge \frac{2}{R}\|u_n - u\|_{1,p}^p - 2\lambda\int_\Omega |u_n - u|^2\, dx.
\end{aligned}$$

Therefore $\|u_n - u\|_{1,p} \to 0$ as $n \to +\infty$, and so u_n converges to u in $W_0^{1,p}(\Omega)$.

Next, we show that there exists sets A_j of genus $j = 1, \ldots, k$ such that $\sup\limits_{u\in A_j} I_\lambda(u) <$ 0.

Consider $E_j = \operatorname{span}\{e_i,\ i = 1, \ldots, j\}$ and $S_j = \{v \in E_j :\ \int_\Omega |v|^2\, dx = 1\}$. For any $s \in (0, 1)$, we define the set $A_j(s) := s(S_j \cap E_j)$ and so $\gamma(A_j(s)) = j$ for $j = 1, \ldots, k$. We have, for any $s \in (0, 1)$

$$\begin{aligned}
\sup_{u\in A_j} I_\lambda(u) &= \sup_{v\in S_j\cap E_j} I_\lambda(sv) \\
&\le \sup_{v\in S_j\cap E_j}\left\{\frac{s^p}{p}\int_\Omega |\nabla v|^p dx + \frac{s^2}{2}\int_\Omega |\nabla v|^2 dx - \frac{\lambda s^2}{2}\int_\Omega |v|^2 dx\right\} \\
&\le \sup_{v\in S_j\cap E_j}\left\{\frac{s^p}{p}\int_\Omega |\nabla v|^p dx + \frac{s^2}{2}(\lambda_j - \lambda)\right\} < 0
\end{aligned}$$

for $s > 0$ sufficiently small, since $\displaystyle\int_\Omega |\nabla v|^p\, dx \le c_j$, where c_j denotes some positive constant.
Finally, we conclude that $\sigma_{\lambda,j}$ $(j = 1, \ldots, k)$ are critical values thanks to Clark's Theorem. □

4. Part B: Control theory

For more details about what we recall in this chapter we refer to [62].

Pseudo-differential operators, oscillatory integrals, parametrices

4.1 Pseudo-differential operators with large parameter

We shall use the notations $a \lesssim b$ for $a \leq Cb$ and $a \gtrsim b$ for $a \geq Cb$, with a constant $C > 0$ that may change from one line to another. We also write $a \asymp b$ to denote $a \lesssim b \lesssim a$. For functions norms we also use the notation $\|\cdot\|$ for functions defined in the interior of the domain and $|\cdot|$ for functions defined on the boundaries.

To motivate the form of the pseudo-differential operators, which we will present below, we first formulate differential operators by the means of the Fourier transformation. Suppose that $q(x, \xi, \tau)$ is a polynomial in (ξ, τ) of order less than or equal to m, with $x, \xi \in \mathbb{R}^d$, and $\tau \geq 1$. We write it in the form

$$q(x, \xi, \tau) = \sum_{|\alpha|+k \leq m} a_\alpha(x) \xi^\alpha \tau^k,$$

and we set $q(x, D, \tau)u = \sum_{|\alpha|+k \leq m} a_\alpha(x) \tau^k D^\alpha u$. For u in the Schawrtz space $\mathscr{S}(\mathbb{R}^d)$, we denote by $\hat{u}$ the Fourier transform, that is

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x) dx.$$

Observing that

$$D^\alpha u = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \xi^\alpha \hat{u}(\xi)\, d\xi \ \text{ for } u \in \mathscr{S}(\mathbb{R}^d), \quad \text{we write}$$

$$q(x, D, \tau)u = \frac{1}{(2\pi)^d} \sum_{|\alpha|+k \leq m} \tau^k \int_{\mathbb{R}^d} e^{ix\cdot\xi} a_\alpha(x) \xi^\alpha \hat{u}(\xi) d\xi, \quad \text{that is}$$

$$q(x, D, \tau)u = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} q(x, \xi, \tau) \hat{u}(\xi) d\xi.$$

We also note that

$$|q(x,\xi,\tau)| \lesssim (\tau+|\xi|)^m \tag{4.1.1}$$

and for all $\alpha,\beta\in\mathbb{N}^d$,

$$|\partial_x^\alpha\partial_\xi^\beta q(x,\xi,\tau)| \leq C_{\alpha\beta}(\tau+|\xi|)^{m-|\beta|}, \tag{4.1.2}$$

for $|\beta|\leq m$ and $\partial_x^\alpha\partial_\xi^\beta q(x,\xi,\tau)=0$ for $|\beta|>m$. We wish to generalize such differential operators $Q(x,D,\tau)$ that involves a large parameter such as τ, to the case of more general functions $q(x,\xi,\tau)$.

Semi-classical calculus acting on $\mathbb{R}^d$

Here, we recall some notions on semi-classical pseudo-differential operators with large parameter $\tau\geq 1$. We denote by S^m_τ the space of smooth functions $a(x,\xi,\tau)$ defined on $\mathbb{R}^d\times\mathbb{R}^d$, with $\tau\geq 1$ as a large parameter, that satisfies the following : for all multi-indices $\alpha,\beta\in\mathbb{N}^d$ and $m\in\mathbb{R}$, there exists $C_{\alpha,\beta}>0$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi,\tau)|\leq C_{\alpha,\beta}\lambda_\tau^{m-|\beta|}, \quad \text{where} \quad \lambda_\tau^2=\tau^2+|\xi|^2,$$

for all $(x,\xi,\tau)\in\mathbb{R}^d\times\mathbb{R}^d\times[1,\infty)$. For $a\in S^m_\tau$, the define pseudo-differential operator of order m, denoted by $A=\mathrm{Op}(a)$ is

$$a(x,D,\tau)u(x)=Au(x):=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}a(x,\xi,\tau)\hat{u}(\xi)d\xi \tag{4.1.3}$$

for $u\in\overline{\mathscr{S}}(\mathbb{R}^d_+)$.

We say that a is the symbol of A. We denote Ψ^m_τ the set of pseudo-differential operators of order m. We shall denote by $\mathscr{D}^m_\tau$ the space of semi-classical differential operators, i.e, the case when the symbol $a(x,\xi,\tau)$ is a polynomial function of order m in (ξ,τ).

We have the following properties for symbols.

Proposition 4.1.1. *i. If $a\in S^m_\tau$, then $\partial_x^\alpha\partial_\xi^\beta a\in S^{m-|\beta|}_\tau$.*

ii. If $a\in S^m_\tau$, and $b\in S^{m'}_\tau$ then $ab\in S^{m+m'}_\tau$.

iii. If $m\leq m'$, then $S^m_\tau\subset S^{m'}_\tau$.

Proof. Let $\alpha,\beta,\alpha',\beta'\in\mathbb{N}^d$.

i. We have

$$\begin{aligned}\left|\partial_x^{\alpha'}\partial_\xi^{\beta'}\left(\partial_x^{\alpha}\partial_\xi^{\beta}a(x,\xi,\tau)\right)\right| &= \left|\partial_x^{\alpha'}\partial_\xi^{\beta'}\partial_x^{\alpha}\partial_\xi^{\beta}a(x,\xi,\tau)\right| \\ &= \left|\partial_x^{\alpha'+\alpha}\partial_\xi^{\beta'+\beta}a(x,\xi,\tau)\right| \leq C_{\alpha'',\beta''}\lambda_\tau^{m-(|\beta|+|\beta'|)} \\ &\leq C_{\alpha'+\alpha,\beta'+\beta}\lambda_\tau^{(m-|\beta|)-|\beta'|}.\end{aligned}$$

Hence $\partial_x^{\alpha}\partial_\xi^{\beta}a \in S^{m-|\beta|}$.

ii. For all multi-indices δ, γ there exists $C_{\delta,\gamma}, C'_{\delta,\gamma} > 0$ such that

$$\left|\partial_x^{\delta}\partial_\xi^{\gamma}a(x,\xi,\tau)\right| \leq C_{\delta,\gamma}\lambda_\tau^{m-|\gamma|},$$

$$\left|\partial_x^{\delta}\partial_\xi^{\gamma}b(x,\xi,\tau)\right| \leq C'_{\delta,\gamma}\lambda_\tau^{m'-|\gamma|}.$$

In addition, using Leibniz formula, we have

$$\begin{aligned}\left|\partial_x^{\alpha}\partial_\xi^{\beta}\left(a(x,\xi,\tau)b(x,\xi,\tau)\right)\right| &\leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\left|\partial_x^{\alpha}\left(\partial_\xi^{\beta-\gamma}a(x,\xi,\tau)\partial_\xi^{\gamma}b(x,\xi,\tau)\right)\right| \\ &\leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}\sum_{\delta\leq\alpha}\binom{\alpha}{\delta}\left|\partial_x^{\alpha-\delta}\partial_\xi^{\beta-\gamma}a(x,\xi,\tau)\right|\left|\partial_x^{\delta}\partial_\xi^{\gamma}b(x,\xi,\tau)\right| \\ &\leq \sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C_{\alpha-\delta,\beta-\gamma}\lambda_\tau^{m-|\beta-\gamma|}C'_{\delta,\gamma}\lambda_\tau^{m'-|\gamma|} \\ &\leq \sum_{\gamma\leq\beta}\sum_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C'_{\delta,\gamma}C_{\alpha-\delta,\beta-\gamma}\lambda_\tau^{m+m'-|\beta|}, \quad \text{since} \quad |\beta| \leq |\beta-\gamma|+|\gamma|, \\ &\leq C\lambda_\tau^{m+m'-|\beta|},\end{aligned}$$

with $C = \sum\limits_{\gamma\leq\beta}\sum\limits_{\delta\leq\alpha}\binom{\beta}{\gamma}\binom{\alpha}{\delta}C'_{\delta,\gamma}C_{\alpha-\delta,\beta-\gamma}$. Thus $ab \in S_\tau^{m+m'}$.

Let $m \leq m'$. We have $\lambda_\tau^m \leq \lambda_\tau^{m'}$ and $\lambda_\tau^{m-|\beta|} \leq \lambda_\tau^{m'-|\beta|}$. So, if $a \in S_\tau^m$, it follows that for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha,\beta} > 0$ such that $\left|\partial_x^{\alpha}\partial_\xi^{\beta}a(x,\xi,\tau)\right| \leq C_{\alpha,\beta}\lambda^{m-|\beta|} \leq C_{\alpha,\beta}\lambda_\tau^{m'-|\beta|}$. Thus $a \in S_\tau^m$. □

Remark 4.1.2. Formally we can define

$$Au(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}a(x,\xi,\tau)u(y,x_d)dy\, d\xi.$$

Such a double integral may not have a meaning in the classical sense, e.g. Lebesgue integration. Yet it has a very precise definition and meaning in the sense of so-called oscillatory integrals.

Remark 4.1.3. The pseudo-differential operators defined above apply to functions defined in the whole $\mathbb{R}^d$, through the use of the Fourier transformation.

Below, we shall introduce tangential pseudo-differential operators that can act on functions defined on a half-space.

Proposition 4.1.4. *Let $a \in S^m_\tau$. We have* $\mathrm{Op}(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ *continuously.*

Proof. For all $1 \le \ell \le d$, we have

$$D_{x_\ell}(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}(\underbrace{\xi_\ell a(x,\xi,\tau) + D_{x_\ell}a(x,\xi,\tau)}_{\in S^{m+1}_\tau})\hat{u}(\xi)d\xi,$$

and by induction we find that for α multi-index,

$$D^\alpha_x(\mathrm{Op}(a)u)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\underbrace{b(x,\xi,\tau)}_{\in S^{m+|\alpha|}_\tau}\hat{u}(\xi)\ d\xi.$$

We also have

$$\begin{aligned} x_\ell(\mathrm{Op}(a)u)(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\left(D_{\xi_\ell}e^{ix\cdot\xi}\right)a(x,\xi,\tau)\hat{u}(\xi)\ d\xi \\ &= -(2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\left(D_{\xi_\ell}a(x,\xi,\tau)\hat{u}(\xi) + a(x,\xi,\tau)D_{\xi_\ell}\hat{u}(\xi)\right)\ d\xi, \end{aligned}$$

and by induction,

$$x^\alpha(\mathrm{Op}(a)u)(x) = (-1)^{|\alpha|}(2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\underbrace{D^\beta_{\xi_\ell}a(x,\xi,\tau)}_{\in S^{m-|\beta|}_\tau}D^{\alpha-\beta}_{\xi_\ell}\hat{u}(\xi)\ d\xi.$$

Combining the two formulae, for all multi-indices α et β, we obtain the existence of semi-norms $p_j \in S^m(\mathbb{R}^d\times\mathbb{R}^d)$ and q_j, q'_j in $\mathscr{S}(\mathbb{R}^d)$ such that

$$|x^\alpha D^\alpha_x(\mathrm{Op}(a)u)(x)| \le C_{\alpha,\beta}(\tau)\sum_j p_j(a)q_j(\hat{u}) \le C_{\alpha,\beta}(\tau)\sum_j p_j(a)q'_j(\hat{u}),$$

by Proposition 4.1.1 and this shows the continuity result. □

The action of pseudo-differential operators can extended to temperate distributions.

4.2 Oscillatory integrals

Oscillatory integral are useful for the definition of the Schwartz kernel of pseudo-differential operators (and many other operators) and also for the understanding of the pseudo-differential calculus. For a review of Schwartz kernel we refer to [62]. We shall give a precise meaning to integrals of the form

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(x-y)\cdot\xi}a(x,\xi,\tau)u(y)\ dyd\xi,\quad u\in\mathscr{S}(\mathbb{R}^d),\quad x\in\mathbb{R}^d,$$

for $a\in S_\tau^{+\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ and view the Schwartz kernel of $\mathrm{Op}(a)$,

$$\mathcal{K}_\tau(x,y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}a(x,\xi,\tau)u(y)\ d\xi,$$

as a distribution. Actually these two integrals are perfectly well defined if

$$|a(x,\xi,\tau)|\lesssim\langle\xi\rangle^m,$$

with $m<-d$ where $\langle\cdot\rangle=(1+|\cdot|^2)^{1/2}$. This holds in particular if $a\in S_\tau^m(\mathbb{R}^d\times\mathbb{R}^d)$ and $m<-d$. Yet, for $m\geq -d$, the meaning of the two integrals may not be clear according to the classical integration theories. This type of integral is called *oscillatory* because of the phase term $e^{i(x-y)\cdot\xi}$.
We shall in fact introduce more general phase functions, to be denoted by Φ here, and give a precise meaning to the following type of integral

$$\int_{\mathbb{R}^d} e^{i\Phi(x,\xi)}a(x,\xi,\tau)d\xi,$$

in the sense of distributions for $a\in S_\tau^m(\mathbb{R}^p\times\mathbb{R}^d)$ possibly with $p\neq d$.
For a proof of the following theorem we refer to [3].

Theorem 4.2.1. *Let $p,d\in\mathbb{N}$ and let $\Phi:\mathbb{R}^p\times\mathbb{R}^d\to\mathbb{C}$ be $\mathscr{C}^\infty$ and such that*

(1) $\mathrm{Im}\,\Phi\geq 0$,

(2) *Φ is homogeneous of degree 1 in ξ, for $|\xi|\geq 1$,*

(3) *for all α,β, there exists $C_{\alpha,\beta}>0$ such that*

$$|\xi|^{|\beta|}|\partial_x^\alpha\partial_\xi^\beta\Phi(x,\xi)|\leq C_{\alpha,\beta}|\xi|,\quad x\in\mathbb{R}^p,\ \xi\in\mathbb{R}^d.$$

(4) *there exists $C>0$ such that*

$$|d_x\Phi|^2+|\xi|^2|d_\xi\Phi|^2\geq C|\xi|^2,\quad x\in\mathbb{R}^p,\ \xi\in\mathbb{R}^d.$$

Then, the functional

$$I_\Phi(a,u,\tau)=\int_{\mathbb{R}^p\times\mathbb{R}^d} e^{i\Phi(x,\xi)}a(x,\xi,\tau)u(x)d\xi dx,$$

that is well defined for $u \in \mathscr{S}(\mathbb{R}^p)$ *and* $a \in S_\tau^{-d-\varepsilon}(\mathbb{R}^p \times \mathbb{R}^d)$, $\varepsilon > 0$, *can be extended in a unique manner by continuity to all* $a \in S_\tau^m(\mathbb{R}^p \times \mathbb{R}^d)$, *for all* $m \in \mathbb{R}$. *Moreover as a distribution in* $\mathscr{S}'(\mathbb{R}^p)$, *the map* $u \mapsto I_\Phi(a, u, \tau)$, *is of order less or equal* k *for all* $k > m + d$.

Note that

$$\begin{aligned} \Phi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d &\to \mathbb{R} \\ (x, y, \xi) &\mapsto (x - y) \cdot \xi \end{aligned}$$

satisfies the assumption made on the phase function in Theorem 4.2.1. Thus for $a \in S_\tau^m(\mathbb{R}^d \times \mathbb{R}^d)$ the map

$$w \mapsto (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x, \xi, \tau) w(x, y) \ d\xi dy dx$$

is a distribution in $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d)$ in the x, y variables of order less or equal k for all $k > m + d$. This allows one to write the Schwartz kernel of the operator $\mathrm{Op}(a)$ as

$$\mathcal{K}_\tau(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x, \xi, \tau) u(y) \ d\xi,$$

and with the kernel theorem (see [62, Theorem 8.45]) we have for $u, v \in \mathscr{S}(\mathbb{R}^d)$,

$$\begin{aligned} \langle \mathrm{Op}(a)u(x), v(y) \rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} &= \langle \mathcal{K}_\tau, u \otimes v \rangle_{\mathscr{S}'(\mathbb{R}^{2d}), \mathscr{S}(\mathbb{R}^{2d})} \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x, \xi, \tau) \ v(x) u(y) \ d\xi dx dy. \end{aligned}$$

We recall that the kernel theorem states that $\mathrm{Op}(a)u(x) = \langle \mathcal{K}_\tau(x, \cdot), u(\cdot) \rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} \in \mathscr{S}'(\mathbb{R}^d)$ for $u \in \mathscr{S}(\mathbb{R}^d)$. Actually, in the present case we have $\mathrm{Op}(a)u(x) \in \mathscr{S}(\mathbb{R}^d)$ by Proposition 4.1.4. Observe that Theorem 4.2.1 gives a precise meaning to the formula

$$\begin{aligned} \mathrm{Op}(a)u(x) &= \langle \mathcal{K}_\tau(x, \cdot), u(\cdot) \rangle_{\mathscr{S}'(\mathbb{R}^d), \mathscr{S}(\mathbb{R}^d)} \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x, \xi, \tau) u(y) \ d\xi \ dy, \end{aligned}$$

for any given value x by considering the phase $\Phi_x(y, \xi) = (x - y) \cdot \xi$ with the variable x as a parameter.

Remark 4.2.2. Let $a \in S^m(\mathbb{R}^p \times \mathbb{R}^d)$ and $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ be such that $\chi(0) = 1$. By [62, Proposition 2.58] one has that $\chi(\varepsilon\lambda_\tau)$ converges to a in $S_\tau^{m'}(\mathbb{R}^p \times \mathbb{R}^d)$

for $m < m'$ and we conclude by Theorem 4.2.1 that we have

$$\begin{aligned} I_\Phi(a,u,\tau) &= \int_{\mathbb{R}^p\times\mathbb{R}^d} e^{i\Phi(x,\xi)} a(x,\xi,\tau)u(x)d\xi dx \\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^p\times\mathbb{R}^d} e^{i\Phi(x,\xi)}\chi(\varepsilon\lambda_\tau)a(x,\xi,\tau)u(x)d\xi dx \end{aligned}$$

for $u \in \mathscr{S}(\mathbb{R}^d)$, i.e,

$$\int_{\mathbb{R}^d} e^{i\Phi(x,\xi)}a(x,\xi,\tau)d\xi = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} e^{i\Phi(x,\xi)}\chi(\varepsilon\lambda_\tau)a(x,\xi,\tau)d\xi,$$

in the sense of distributions.

We say that the oscillatory integral is regularized in this limiting process.

Remark 4.2.3. Regularization allows one to generalize to oscillatory integrals the usual calculus rules for absolutely convergent integrals: integration by parts, homogeneous change of variables, the Fubini theorem, limits and differentiations under the sum sign.

Tangential semi-classical calculus

Here, we consider pseudo-differential operators that only act in the tangential direction x' with x_d as a parameter. We shall denote by $S^m_{\mathsf{T},\tau}$, the set of smooth functions $b(x,\xi',\tau)$ defined for $\tau \geq 1$ as a large parameter, satisfying the following: for all multi-indices $\alpha \in \mathbb{N}^d$, $\beta \in \mathbb{N}^{d-1}$ and $m \in \mathbb{R}$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_{\xi'}^\beta b(x,\xi',\tau)| \leq C_{\alpha,\beta}\lambda_{\mathsf{T},\tau}^{m-|\beta|}, \quad \text{where} \quad \lambda_{\mathsf{T},\tau}^2 = \tau^2 + |\xi'|^2, \tag{4.2.1}$$

for all $(x,\xi',\tau) \in \mathbb{R}^d \times \mathbb{R}^{d-1} \times [1,\infty)$. For $b \in S^m_{\mathsf{T},\tau}$, we define a tangential pseudo-differential operator $B = \mathrm{Op}_{\mathsf{T}}(b)$ of order m by

$$\begin{aligned} b(x,D',\tau)u(x) = Bu(x) \;&:=\; \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'} b(x,\xi',\tau)\hat{u}(\xi',x_d)d\xi' \qquad (4.2.2) \\ &=\; \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}} e^{i(x'-y')\cdot\xi'} b(x,\xi',\tau)u(y',x_d)dy'd\xi' \end{aligned}$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$. We define $\Psi^m_{\mathsf{T},\tau}$ as the set of tangential pseudo-differential operators of order m, and $\mathscr{D}^m_{\mathsf{T},\tau}$ the set of tangential differential operators of order m. We also set

$$\Lambda^m_{\mathsf{T},\tau} = \mathrm{Op}_{\mathsf{T}}(\lambda^m_{\mathsf{T},\tau}).$$

Additional classes of symbols.

Here, we shall often write $\varrho = (x, \xi, \tau)$, $\varrho' = (x, \xi', \tau)$ and $X = \mathbb{R}^d$ or $\overline{\mathbb{R}}^d_+$.

Definition 4.2.4. Let $a(\varrho) \in \mathscr{C}^\infty(X \times \mathbb{R}^d)$, with τ as a large parameter in $[1, +\infty)$, and $m \in \mathbb{N}$, $r \in \mathbb{R}$. We say that $a \in S^{m,r}_\tau(X \times \mathbb{R}^d)$ if

$$a(\varrho) = \sum_{j=0}^{m} a_j(\varrho')\xi_d^j, \quad a_j \in S^{m-j+r}_{\mathsf{T},\tau}(X \times \mathbb{R}^{d-1}),$$

for $x \in X$, $\xi \in \mathbb{R}^d$, $\tau \in [1, +\infty)$, and $\xi_d \in \mathbb{R}$.

We also simply write $a \in S^{m,r}_\tau$. If $\mathscr{U}$ is a conic open set of $X \times \mathbb{R}^d \times [1, +\infty)$ we say that $a \in S^{m,r}_\tau$ microlocally for $\varrho' \in \mathscr{U}$ if each a_j is in $S^{m-j+r}_{\mathsf{T},\tau}$ microlocally in $\mathscr{U}$, $j = 0, \dots, m$.
Note that we have

$$S^{m,r}_\tau \subset S^{m+m',r-m'}_\tau, \quad m, m' \in \mathbb{N}, \quad r \in \mathbb{R}.$$

The principal symbol of a denoted by $\sigma(a)$ is

$$\sigma(a)(\varrho) = \sum_{j=0}^{m} \sigma(a_j)(\varrho')\xi_d^j,$$

which is a representative of the class of a in $S^{m,r}_\tau / S^{m,r-1}_\tau$. Note that $S^{m,r}_\tau \not\subset S^{m+r}_\tau$. Indeed, consider $a(x, \xi, \tau) = \lambda_{\mathsf{T},\tau}\xi_d$ for $\lambda_{\mathsf{T},\tau} \geq 1$. We have $a \in S^{1,1}_\tau \subset S^{2,0}_\tau$ and yet $a \notin S^2_\tau$. In fact observe that differentiating with respect to ξ' yields

$$|\partial^\alpha_{\xi'} a(x, \xi, \tau)| \leq C_\alpha \lambda^{1-|\alpha|}_{\mathsf{T},\tau} |\xi_d|.$$

An estimate of the form of (4.2.1) is however not achieved for $|\alpha| \geq 2$.
We recall that $a \sim \sum_{j \in \mathbb{N}} a_{m-j} \in S^m_{\tau,ph}$ if $a_{m-j} \in S^{m-j}_\tau$ is homogeneous of degree $m - j$ with respect to (ξ, τ) for $j \in \mathbb{N}$. Additionally we give the definition of tangential polyhomogeneous symbols that are characterized by an asymptotic expansion where each term is positively homogeneous with respect to (ξ', τ).

Definition 4.2.5. We shall say that $a \in S^m_{\mathsf{T},\tau,ph}(X \times \mathbb{R}^{d-1})$ or simply $S^m_{\mathsf{T},\tau,ph}$ if there exists $a_j \in S^m_{\mathsf{T},\tau}$, homogeneous of degree $m - j$ in (ξ', τ) for $|(\xi', \tau)| \geq r_0$, with $r_0 \geq 0$, such that

$$a \sim \sum_{j \geq 0} a_j, \quad \text{in the sense that} \quad a - \sum_{j=0}^{N} a_j \in S^{m-N-1}_{\mathsf{T},\tau}.$$

A representative of the principal symbol is then given by the first term in the expansion. We denote it by $\sigma(a)$. We have

$$S^m_{\tau,ph} \subset S^m_\tau, \quad S^m_{\mathsf{T},\tau,ph} \subset S^m_{\mathsf{T},\tau}.$$

Thus, for $m \in \mathbb{N}$ and $r \in \mathbb{R}$, we say that $a(\varrho) \in S^{m,r}_{\tau,ph}(X \times \mathbb{R}^d)$ or simply $S^{m,r}_{\tau,ph}$, if

$$a(\varrho) = \sum_{j=0}^{m} a_j(\varrho')\xi_d^j, \quad \text{with } a_j \in S^{m-j+r}_{\tau,ph}.$$

A representative of the principal symbol is given by $\sum_{j=0}^{m} \sigma(a_j)(\varrho')\xi_d^j$ and is homogeneous of degree m in (ξ, τ).

We recall that the Poisson bracket of two smooth functions is given by

$$\{f, g\} = \sum_{j=1}^{d} \left(\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g\right).$$

The canonical inner product in $\mathbb{C}^m$ is denoted by $(z, z')_{\mathbb{C}^m} = \sum_{k=0}^{m-1} z_k \bar{z}'_k$, for $z = (z_0, \cdots, z_{m-1}) \in \mathbb{C}^m$, $z' = (z'_0, \cdots, z'_{m-1}) \in \mathbb{C}^m$. The associated norm will be denoted $|z|^2_{\mathbb{C}^m} = \sum_{k=0}^{m-1} |z_k|^2$.

Sobolev norms with parameter

We introduce the following norms, for $m \in \mathbb{N}$ and $m' \in \mathbb{R}$,

$$\|u\|_{m,m',\tau} \asymp \sum_{j=0}^{m} \|\Lambda^{m+m'-j}_{\mathsf{T},\tau} D^j_{x_d} u\|_+;$$

$$\|u\|_{m,\tau} = \|u\|_{m,0,\tau} \asymp \sum_{j=0}^{m} \|\Lambda^{m-j}_{\mathsf{T},\tau} D^j_{x_d} u\|_+,$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$, where $\|\cdot\|_+ = \|\cdot\|_{L^2(\mathbb{R}^d_+)}$. We also denote $(u, v)_+ = (u, v)_{L^2(\mathbb{R}^d_+)}$ and $(u_{|_{x_d=0^+}}, v_{|_{x_d=0^+}})_\partial = (u_{|_{x_d=0^+}}, v_{|_{x_d=0^+}})_{L^2(\mathbb{R}^{d-1})}$. We have

$$\|u\|_{m,\tau} \asymp \sum_{|\alpha| \le m} \tau^{m-|\alpha|} \|D^\alpha u\|_+,$$

and in the case $m' \in \mathbb{N}$ we have

$$\|u\|_{m,m',\tau} \asymp \sum_{\substack{\alpha_d \leq m \\ |\alpha| \leq m+m'}} \tau^{m+m'-|\alpha|} \|D^\alpha u\|_+,$$

with $\alpha = (\alpha', \alpha_d) \in \mathbb{N}^d$.

Remark 4.2.6. We have that for some $C > 0$,

$$\|u\|_{m,s,\tau} \leq C\tau^{-\ell} \|u\|_{m,s+\ell,\tau}, \quad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+),$$

for $m \in \mathbb{N},\ s \in \mathbb{R}$ and $\ell \geq 0$. This implies that $\|u\|_{m,s,\tau} \ll \|u\|_{m,s+\ell,\tau}$ for τ sufficiently large.

The following argument will be used on numerous occasions: for $m \in \mathbb{N}$, $m', \ell \in \mathbb{R}$, with $\ell \geq 0$,

$$\|u\|_{m,m',\tau} \ll \|u\|_{m,m'+\ell,\tau},$$

if τ is chosen sufficiently large.
For a sufficiently smooth function u defined in $\overline{\mathbb{R}}^d_+$ we set, for $m \in \mathbb{N}$,

$$\operatorname{tr}(u) = (u_{|x_d=0^+}, D_d u_{|x_d=0^+}, \ldots, D_d^m u_{|x_d=0^+})$$

on $\{x_d = 0\}$ and we define the following norm for $m \in \mathbb{N}$ and $m' \in \mathbb{R}$,

$$|\operatorname{tr}(u)|^2_{m,m',\tau} = \sum_{j=0}^m |\Lambda^{m+m'-j}_{\mathsf{T},\tau} D^j_{x_d} u|_{x_d=0^+}|^2_\partial, \quad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+),$$

where $|\cdot|_\partial = |\cdot|_{L^2(\mathbb{R}^{d-1})}$.

Proposition 4.2.7. (trace inequality)
Let $s > 0$. There exists $C > 0$ such that $|u_{|x_d=0^+}|_{s,\tau} \leq C\|u\|_{s+1/2,\tau}, \quad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$.

Corollary 4.2.8. (trace inequality)
Let $m \in \mathbb{N}$ and $s \in \mathbb{R}$. For some $C > 0$, we have

$$|\operatorname{tr}(u)|_{m,s,\tau} \leq C\|u\|_{m+1,s-1/2,\tau}, \quad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+).$$

From pseudo-differential symbol calculus we obtain the following inequalities.

Proposition 4.2.9. *If $a \in S^{m,r}_\tau$, with $m \in \mathbb{N}$ and $r \in \mathbb{R}$, then for $m' \in \mathbb{N}$ and $r \in \mathbb{R}$ there exists $C > 0$ such that*

$$\|\operatorname{Op}(a)u\|_{m',r',\tau} \leq C\|u\|_{m+m',r+r',\tau}, \quad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+).$$

A consequence of this result is the following corollary.

Corollary 4.2.10. *Let $m, m' \in \mathbb{N}$ and $r \in \mathbb{R}$. There exists $C > 0$ such that*

$$\|u\|_{m,r,\tau} \leq \|u\|_{m+m',r-m',\tau}, \quad u \in \overline{\mathscr{S}}(\mathbb{R}^d_+).$$

Proof. Set $u = \Lambda^{m'}_{\mathsf{T},\tau} v$ with $v = \Lambda^{-m'}_{\mathsf{T},\tau} u$. Since we have $\lambda^{m'}_{\mathsf{T},\tau} \in S^{m'}_{\mathsf{T},\tau} = S^{0,m'}_{\tau} \subset S^{m',0}_{\tau}$, it follows that

$$\|u\|_{m,r,\tau} = \|\Lambda^{m'}_{\mathsf{T},\tau} v\|_{m,r,\tau} \leq C\|v\|_{m+m',r,\tau} = C\|\Lambda^{-m'}_{\mathsf{T},\tau} u\|_{m+m',r,\tau} = C\|u\|_{m+m',r-m',\tau},$$

thanks to Proposition (4.2.9). ∎

4.2.1 Differential quadratic forms

Quadratic forms in a half space

Definition 4.2.11 (interior differential quadratic form)**.** Let $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$. We say that

$$Q(u) = \sum_{s=1}^{N} (A^s u, B^s u)_+, \qquad A^s = \mathrm{Op}(a^s),\ B^s = \mathrm{Op}(b^s), \tag{4.2.3}$$

is an interior differential quadratic form of type (m, r) with smooth coefficients, if for each $s = 1, \ldots N$, we have $a^s(\varrho) \in S^{m,r'}_{\tau}$ and $b^s(\varrho) \in S^{m,r''}_{\tau}$, with $r' + r'' = 2r$, $\varrho = (x, \xi, \tau)$.

The principal symbol of the quadratic form Q is defined as the class of

$$q(\varrho) = \sum_{s=1}^{N} a^s(\varrho)\overline{b^s}(\varrho) \tag{4.2.4}$$

in $S^{2m,2r}_{\tau} / S^{2m,2r-1}_{\tau}$.

A result we shall use is the following microlocal Gårding inequality.

Proposition 4.2.12 (microlocal Gårding inequality)**.** *Let K be a compact set of $\overline{\mathbb{R}^d_+}$ and let $\mathscr{U}$ be a conic open set of $\overline{\mathbb{R}^d_+} \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ contained in $K \times \mathbb{R}^{d-1} \times \mathbb{R}_+$. Let also $\chi \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0, be such that $\mathrm{supp}(\chi) \subset \mathscr{U}$. Let Q be an interior differential quadratic form of type (m, r) with homogeneous principal symbol $q \in S^{2m,2r}_{\tau}$ satisfying, for some $C_0 > 0$ and $r_0 > 0$,*

$$\mathrm{Re}\, q(\varrho) \geq C_0 \lambda^{2m}_{\tau} \lambda^{2r}_{\mathsf{T},\tau}, \quad \text{for } \tau \geq r_0, \quad \varrho = (\varrho', \xi_d),\ \varrho' = (x, \xi', \tau) \in \mathscr{U},\ \xi_d \in \mathbb{R}.$$

For $0 < C_1 < C_0$ and $N \in \mathbb{N}$ there exist τ_*, $C > 0$, and $C_N > 0$ such that

$$\mathrm{Re}\, Q(\mathrm{Op}_{\mathsf{T}}(\chi)u) \geq C_1 \|\mathrm{Op}_{\mathsf{T}}(\chi)u\|^2_{m,r,\tau} - C|\,\mathrm{tr}(\mathrm{Op}_{\mathsf{T}}(\chi)u)|^2_{m-1,r+1/2,\tau} - C_N\|u\|^2_{m,-N,\tau},$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ *and* $\tau \geq \tau_*$.

We refer to [11, Proposition 3.5] and [63, Theorem 6.17] for a proof. A local version of the result is the following one that follows from Proposition 4.2.12.

Proposition 4.2.13 (Gårding inequality)**.** *Let* U_0 *be a bounded open subset of* $\overline{\mathbb{R}^d_+}$ *and let* Q *be an interior differential quadratic form of type* (m, r) *with homogeneous principal symbol* $q \in S^{2m,2r}_\tau$ *satisfying, for some* $C_0 > 0$ *and* $r_0 > 0$,

$$\operatorname{Re} q(\varrho) \geq C_0 \lambda_\tau^{2m} \lambda_{\mathsf{T},\tau}^{2r}, \quad \textit{for } \tau \geq r_0, \ \ \varrho = (\varrho', \xi_d), \ \varrho' = (x, \xi', \tau) \in U_0 \times \mathbb{R}^{d-1} \times \mathbb{R}^+, \ \xi_d \in \mathbb{R}.$$

For $0 < C_1 < C_0$ *there exist* τ_*, $C > 0$*such that*

$$\operatorname{Re} Q(u) \geq C_1 \|u\|^2_{m,r,\tau} - C |\operatorname{tr}(u)|^2_{m-1,r+1/2,\tau},$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ *and* $\tau \geq \tau_*$.

Boundary differential quadratic forms

Definition 4.2.14. Let $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$. We say that

$$Q(u) = \sum_{s=1}^{N} (A^s u, B^s u)_\partial, \qquad A^s = a^s(x, D, \tau), \ B^s = b^s(x, D, \tau), \tag{4.2.5}$$

is a boundary differential quadratic form of type $(m-1, r)$ with $\mathscr{C}^\infty$ coefficients, if for each $s = 1, \ldots N$, we have $a^s(\varrho) \in S^{m-1,r'}_\tau(\overline{\mathbb{R}^d_+} \times \mathbb{R}^d)$, $b^s(\varrho) \in S^{m-1,r''}_\tau(\overline{\mathbb{R}^d_+} \times \mathbb{R}^d)$ with $r' + r'' = 2r$, $\varrho = (\varrho', \xi_d)$ with $\varrho' = (x, \xi', \tau)$. The symbol of the boundary differential quadratic form Q is defined by

$$B(\varrho', \xi_d, \tilde{\xi}_d) = \sum_{s=1}^{N} a^s(\varrho', \xi_d) \overline{b^s}(\varrho', \tilde{\xi}_d).$$

For $\mathbf{z} = (z_0, \ldots, z_{m-1}) \in \mathbb{C}^m$ and $a(\varrho) \in S^{m-1,\tilde{r}}_\tau$, of the form $a(\varrho', \xi_d) = \sum_{j=0}^{m-1} a_j(\varrho') \xi_d^j$ with $a_j(\varrho') \in S^{m-1+\tilde{r}-j}_{\mathsf{T},\tau}$ we set

$$\Sigma_a(\varrho', \mathbf{z}) = \sum_{j=0}^{\ell-1} a_j(\varrho') z_j. \tag{4.2.6}$$

From the boundary differential quadratic form Q we introduce the following bilinear symbol $\Sigma_Q : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$

$$\Sigma_Q(\varrho', \mathbf{z}, \mathbf{z}') = \sum_{s=1}^{N} \Sigma_{a^s}(\varrho', \mathbf{z}) \overline{\Sigma_{b^s}}(\varrho', \overline{\mathbf{z}}'), \quad \mathbf{z}, \mathbf{z}' \in \mathbb{C}^m. \tag{4.2.7}$$

Definition 4.2.15. Let Q be a boundary differential quadratic form of type $(m-1, r)$ with homogeneous principal symbol and associated with the bilinear symbol $\Sigma_Q(\varrho', \mathbf{z}, \mathbf{z}')$. We say that Q is positive definite in $\mathscr{W}$ if there exist $C > 0$ and $R > 0$ such that

$$\operatorname{Re} \Sigma_Q(\varrho'', x_d = 0^+, \mathbf{z}, \mathbf{z}) \geq C \sum_{j=0}^{m-1} \lambda_{\mathsf{T},\tau}^{2(m-1-j+r)} |z_j|^2,$$

for $\varrho'' = (x', \xi', \tau) \in \mathscr{W}$, and $\mathbf{z} = (z_0, \ldots, z_{m-1}) \in \mathbb{C}^m$.

Proposition 4.2.16. *Let Q be a boundary differential quadratic form of type $(m-1, r)$, positive definite in $\mathscr{W}$, an open conic set in $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1} \times \mathbb{R}_+$, with bilinear symbol $\Sigma_Q(\varrho', \mathbf{z}, \mathbf{z}')$. Let $\chi \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0, with $\operatorname{supp}(\chi_{|x_d=0^+}) \subset \mathscr{W}$ and let $N \in \mathbb{N}$. Then there exist $\tau_* \geq 1$, $C > 0$, $C_N > 0$ such that*

$$\operatorname{Re} Q(\operatorname{Op}_{\mathsf{T}}(\chi)u) \geq C |\operatorname{tr}(\operatorname{Op}_{\mathsf{T}}(\chi)u)|^2_{m-1,r,\tau} - C_N |\operatorname{tr}(u)|^2_{m-1,r-N,\tau},$$

for $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geq \tau_*$.

Parametrices

Elliptic operators can be inverted up to some regularizing operator.

Proposition 4.2.17. *Let $m \in \mathbb{R}$ and let $p \in S^m_\tau$ be elliptic, i.e, for some $C > 0$ and $R > 0$,*

$$|p(x, \xi, \tau)| \geq C \lambda_\tau^m, \quad x \in \mathbb{R}^d, \ \xi \in \mathbb{R}^d, \ \tau \in [\tau_0, +\infty), \quad \lambda_\tau \geq R.$$

For any $N \in \mathbb{N}$ there exist $q_N \in S^{-m}_\tau$ and $r_N, r'_N \in S^{-N}_\tau$ such that

$$q_N \circ p = 1 + r_N, \qquad p \circ q_N = 1 + r'_N.$$

Moreover q_N is unique in $S^{-m}_\tau / S^{-m-N}_\tau$. There exist also $p \in S^{-m}_\tau$ and $r_\infty, r'_\infty \in S^{-\infty}_\tau$ such that

$$q \circ p = 1 + r_\infty, \qquad p \circ q = 1 + r'_\infty,$$

with q unique in $S^{-m}_\tau / S^{-\infty}_\tau$.

This proposition is a particular case of the microlocal version [62, Proposition 2.34] whose proof can be found in [62].

4.3 Standard pseudo-differential operators

We refer to as standard pseudo-differential operators, the operators that do not depend on a large parameter.

Definition 4.3.1. (standard symbols) Let $d \in \mathbb{N}$ and let $a(x,\xi) \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with $m \in \mathbb{R}$, be such that for all multi-indices α, β we have

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d, \tag{4.3.1}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$. We then write $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$.

We also define $S^{-\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \cap_{r\in\mathbb{R}} S^r(\mathbb{R}^d \times \mathbb{R}^d)$ and $S^{+\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \cup_{r\in\mathbb{R}} S^r(\mathbb{R}^d \times \mathbb{R}^d)$. We shall often simply write S^m (respectively $S^{-\infty}, S^{+\infty}$) when no confusion is possible. For $a \in S^m$ we call principal symbol, $\sigma(a)$ the equivalence class of a in S^m/S^{m-1}. With this symbol classes we can define standard pseudo-differential operators.

Definition 4.3.2. (standard pseudo-differentiel operators) If $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, we set

$$\begin{aligned} a(x,D)u(x) = \mathrm{Op}(a)u(x) &:= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi) d\xi \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi \end{aligned}$$

for $u \in \mathscr{S}(\mathbb{R}^d)$.

We denote by Ψ^m the set of these pseudo-differential operators.

Remark 4.3.3. Note that $a(x,\xi,1) \in S^m$ if $a(x,\xi,\tau) \in S^m_\tau$. So we can recover standard pseudo-differential operators by setting τ to be a fix value (e.g $\tau = 1$) in the case of the pseudo-differential operators with large parameter τ.

Definition 4.3.4. (Tangential symbols and operators) We say that $a(x,\xi') \in S^m_{\mathsf{T}}(\mathbb{R}^d \times \mathbb{R}^{d-1})$ if we have

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(x,\xi')| \leq C_{\alpha\beta} \langle \xi' \rangle^{m-|\beta|}, \quad x \in \mathbb{R}^d, \ \xi' \in \mathbb{R}^{d-1},$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$.

We denote by Ψ^m_{T} the set of associated operators, that is

$$\begin{aligned} a(x,D')u(x) = \mathrm{Op}_{\mathsf{T}}(a)u(x) &:= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}} e^{ix'\cdot\xi'} a(x,\xi')\hat{u}(\xi',x_d) d\xi' \\ &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}} e^{i(x'-y')\cdot\xi'} a(x,\xi') u(y',x_d) dy' d\xi'. \end{aligned}$$

4.4 Fredholm properties for fourth order elliptic operators

It is well-known that elliptic problems are well-posed only if the boundary conditions are chosen appropriately. By well-posedness one usually means that the solution exists and is unique in some space, and it depends continuously on data and parameters, or more generally that the associated operator is at least Fredholm. The property which the boundary conditions should satisfy to have a well-posed problem in some Sobolev spaces for a elliptic boundary value problem is called the Lopatinskiĭ-Šapiro condition.
In this section we will point out the relation between the operators which are Fredholm type and that satisfy the Lopatinskiĭ-Šapiro condition.

On a smooth compact Riemmannian manifold $(\mathcal{M}, g)$, with boundary, we consider $P = \Delta_g^2$, where Δ_g denotes the Laplace-Beltrami operator. We denote by $p(x,\omega)$ its principal symbol for $(x,\omega) \in T^*\mathcal{M}$. One defines the following polynomial in z,

$$\tilde{p}(x, \omega', z) = p(x, \omega' - zn_x),$$

for $x \in \partial\mathcal{M}$, $\omega' \in T_x^*\partial\mathcal{M}$, $z \in \mathbb{R}$ and where n_x denotes the outward pointing conormal vector at x, unitary in the sense of the metric g. We denote by $\rho_j(x,\omega')$, $1 \le j \le 4$ the complex roots of $\tilde{p}$. One sets

$$\tilde{p}^+(x,\omega',z) = \prod_{\operatorname{Im}\rho_j(x,\omega')\ge 0} (z - \rho_j(x,\omega')).$$

Given boundary operators B_1, B_2 in a neighborhood of $\partial\mathcal{M}$, with principal symbols $b_j(x,\omega)$, $j = 1,2$ one also sets $\tilde{b}_j(x,\omega',z) = b_j(x,\omega' - zn_x)$. We note that the boundary operators B_1 and B_2 are of order k_1 and k_2 respectively.

Definition 4.4.1. (Lopatinskiĭ-Šapiro condition) Let $(x,\omega') \in T^*\partial\mathcal{M}$ with $\omega' \neq 0$. One says that the Lopatinskiĭ-Šapiro condition condition holds for (P, B_1, B_2) at (x,ω') if for any polynomial function $f(z)$ with complex coefficients, there exists $c_1, c_2 \in \mathbb{C}$ and a polynomial function $g(z)$ with complex coefficients such that, for all $z \in \mathbb{C}$,

$$f(z) = \sum_{1\le j\le 2} c_j \tilde{b}_j(x,\omega',z) + g(z)\tilde{p}^+(x,\omega',z).$$

We say that the Lopatinskiĭ-Šapiro condition holds for (P, B_1, B_2) at $x \in \partial\mathcal{M}$ if it holds at (x,ω') for all $\omega' \in T_x^*\partial\mathcal{M}$ with $\omega' \neq 0$.

Observe that the Lopatinskiĭ-Šapiro condition is written here without any use of local coordinates. It is then a geometrical condition.

The general boundary operators B_1 and B_2 are then given by

$$B_\ell(x,D) = \sum_{0\le j\le \min(3,k_\ell)} B_\ell^{k_\ell - j}(x,D')(i\partial_\nu)^j, \qquad \ell = 1,2,$$

with $B_\ell^{k_\ell-j}(x,D')$ differential operators acting in the tangential variables. We denote by $b_1(x,\omega)$ and $b_2(x,\omega)$ the principal symbols of B_1 and B_2 respectively. For $(x,\omega')\in T^*\partial\Omega$, we set

$$\tilde{b}_\ell(x,\omega',z) = \sum_{0\le j\le \min(3,k_\ell)} b_\ell^{k_\ell - j}(x,\omega')z^j, \qquad \ell = 1,2.$$

For $m\in\mathbb{N}$, we study the Fredholm property of the operator

$$\begin{aligned} L : H^{m+4}(\mathcal{M}) &\to H^m(\mathcal{M}) \oplus H^{(m+7/2)}(\partial\mathcal{M}) \\ u &\mapsto (Pu, B_1 u|_{\partial\mathcal{M}}, B_2 u|_{\partial\mathcal{M}}), \end{aligned} \tag{4.4.1}$$

where $H^{(m+7/2)}(\partial\mathcal{M}) = H^{m+7/2-k_1}(\partial\mathcal{M}) \oplus H^{m+7/2-k_2}(\partial\mathcal{M})$. We state the following useful theorem. To prove it, one can adapt the proof of Theorem 15.1 in [63].

Theorem 4.4.2. *The operator L is Fredholm if and only if (P,B_1,B_2) fulfills the Lopatinskiĭ-Šapiro condition on $\partial\mathcal{M}$.*

In order to show Theorem 4.4.2 one needs to establish the following result. We note that by Theorem 2.1.23 this implies that the Lopatinskiĭ-Šapiro condition is sufficient for the Fredholm property of L to hold.

Proposition 4.4.3. *Let $m\in\mathbb{N}$. Assume that (P,B_1,B_2) fulfills theLopatinskiĭ-Šapiro condition on $\partial\mathcal{M}$. There exists a bounded linear operator*

$$M : H^m(\mathcal{M}) \oplus H^{(m+7/2)}(\partial\mathcal{M}) \to H^{m+4}(\mathcal{M})$$

such that

$$ML = \mathrm{Id}_{H^{m+4}(\mathcal{M})} + K^s \quad \textit{and} \quad LM = \mathrm{Id}_{H^m(\mathcal{M})\oplus H^{(m+7/2)}(\partial\mathcal{M})} + K^r,$$

where both operators

$$\begin{aligned} &K^s : H^{m+4}(\mathcal{M}) \to H^{m+5}(\mathcal{M}) \\ &K^r : H^m(\mathcal{M}) \oplus H^{(m+7/2)}(\partial\mathcal{M}) \to H^{m+1}(\mathcal{M}) \oplus H^{(m+9/2)}(\partial\mathcal{M}) \end{aligned} \tag{4.4.2}$$

are bounded.

By the Rellich- Kondrachov theorem (see [62, Theorem 30.7]) K^s is compact from $H^{m+4}(\mathcal{M})$ into itself and K^r from $H^m(\mathcal{M}) \oplus H^{(m+7/2)}(\partial\mathcal{M})$ into itself.

Proposition 4.4.3 is contained in Theorem 20.1.7 of [44] and its proof is performed based on the analysis of the fourth order operator in a half-space. For instance the case of the Laplacian we refer to Proposition 15.2 in [63]. Theorem 4.4.2 states in particular that L is not Fredholm if the Lopatinskiĭ-Šapiro condition does not hold.

Stabilization of the damped plate equation under general boundary conditions

Here, we present the results obtained in [83] concerning the Part B.

4.5 Setting of the problem and some notations

Let Ω be a bounded connected open subset in $\mathbb{R}^d$, or a smooth bounded connected d-dimensional manifold, with smooth boundary $\partial\Omega$, where we consider a damped plate equation

$$\begin{cases} \partial_t^2 y + \Delta^2 y + \alpha(x)\partial_t y = 0 & (t,x) \in \mathbb{R}_+ \times \Omega, \\ B_1 y_{|\mathbb{R}_+\times\partial\Omega} = B_2 y_{|\mathbb{R}_+\times\partial\Omega} = 0, \\ y_{|t=0} = y^0, \quad \partial_t y_{|t=0} = y^1, \end{cases} \tag{4.5.1}$$

where $\alpha \geq 0$ and where B_1 and B_2 denote two boundary differential operators. The damping property is provided by $+\alpha(x)\partial_t$ thus referred as the *damping term*. As introduced below Δ^2 is the bi-Laplace operator, that is, the square of the Laplace operator. Here, it is associated with a smooth metric g to be introduced below; it is thus rather the bi-Laplace-Beltrami operator. This equation appears in models for the description of mechanical vibrations of thin domains. The two boundary operators are of k_j, $j = 1, 2$ respectively, yet at most of order 3 in the direction normal to the boundary. They are chosen such that the two following properties are fulfilled:

(1) the Lopatinskiĭ-Šapiro boundary condition holds (this condition is fully described in what follows);

(2) along with the homogeneous boundary conditions given above the bi-Laplace operator is self-adjoint and nonnegative. This guarantees the preservation of the energy of the solution in the case of a damping free equation, that is, if $\alpha = 0$.

We are concerned with the decay of the energy of the solution in the case α is not identically zero. We shall prove that the damping term yields a stabilization property: the energy decays to zero as time $t \to \infty$ and we shall prove that the decay rate is at least logarithmic.

Among the existing results available in the literature for plate type equations, many of them concern the "hinged" boundary conditions, that is, $u_{|\partial\Omega} = 0$

and $\Delta u_{|\partial\Omega} = 0$. We first mention these result. An important result obtained in [46] on the controllability of the plate equation on a rectangle domain with an arbitrarily small control domain. The method relies on the generalization of Ingham type inequalities in [48]. An exponential stabilization result, in the same geometry, can be found in [88], using similar techniques. In [88] the localized damping term involves the time derivative $\partial_t y$ as in (4.5.1). Interior nonlinear feedbacks can be used for exponential stabilization [90]. There, feedbacks are localized in a neighborhood of part of the boundary that fulfills multiplier-type conditions. A general analysis of nonlinear damping that includes the plate equation is provided in [2] under multiplier-type conditions. For "hinged" boundary conditions also, with a boundary damping term, we cite [94] where, on a square domain, a necessary and sufficient condition is provided for exponential stabilization.

Note that under "hinged" boundary conditions the bi-Laplace operator is precisely the square of the Dirichlet-Laplace operator. This makes its mathematical analysis much easier, in particular where using spectral properties, and this explains why this type of boundary conditions appears very frequently in the mathematical literature.

A more challenging type of boundary condition is the so-called "clamped" boundary conditions, that is, $u_{|\partial\Omega} = 0$ and $\partial_\nu u_{|\partial\Omega} = 0$, for which few results are available. We cite [1], where a general analysis of nonlinearly damped systems that includes the plate equation under multiplier-type conditions is provided. In [77], the analysis of discretized general nonlinearly damped system is also carried out, with the plate equation as an application. In [91], a nonlinear damping involving the *p-Laplacian* is used also under multiplier-type conditions. In [30], an exponential decay is obtained in the case of "clamped" boundary conditions, yet with a damping term of the Kelvin-Voigt type, that is of the form $\partial_t \Delta y$, that acts over the whole domain. In the case of the "clamped" boundary conditions, the logarithmic-type stabilization result we obtain here was proven in [82]. The present result thus stands as a generalization of the stabilisation result of [82] if considering a whole class of boundary condition instead of specializing to a certain type. It contains in particular also the case of "hinged" boundary conditions.

4.5.1 Method

Following the works of [59, 60, 82] we obtain a logarithmic decay rate for the energy of the solution to (4.5.1) which is obtained by means of a resolvent estimate for the generator of the semigroup associated with the damped plate equation (4.5.1). This estimate follows from a Carleman inequality derived for

the operator $P_\sigma = \Delta^2 - \sigma^4$ where σ is a spectral parameter for the generator of the semigroup.
Our first goal is thus the derivation of the Carleman inequality for the operator P_σ near the boundary under the boundary conditions given by B_1 and B_2.

Then, from the Carleman estimate one deduces an observation inequality for the operator P_σ in the case of the prescribed boundary conditions. The resolvent estimate then follows from this observation inequality.

4.5.2 Geometrical setting

On Ω we consider a Riemannian metric $g_x = (g_{ij}(x))$, with associated cometric $(g^{ij}(x)) = (g_x)^{-1}$. It stands as a bilinear form that act on vector fields,

$$g_x(u_x, v_x) = g_{ij}(x) u_x^i v_x^j, \quad u_x = u_x^i \partial_{x_i}, \ v_x = v_x^i \partial_{x_i}.$$

For $x \in \partial\Omega$ we denote by ν_x the unit outward pointing *normal vector* at x, unitary in the sense of the metric g, that is

$$g_x(\nu_x, \nu_x) = 1 \ \text{ and } \ g_x(\nu_x, u_x) = 1 \quad \forall u_x \in T_x\partial\Omega.$$

We denote by ∂_ν the associated derivative at the boundary, that is, $\partial_\nu f(x) = \nu_x(f)$. We also denote by n_x the unit outward pointing *conormal vector* at x, that is, $n_x = \nu_x^\flat$, that is, $(n_x)_i = g_{ij}\nu_x^j$.

Near a boundary point we shall often use normal geodesic coordinates where Ω is locally given by $\{x_d > 0\}$ and the metric g takes the form

$$g = dx^d \otimes dx^d + \sum_{1 \le i,j \le d-1} g_{ij} dx^i \otimes dx^j.$$

Then, the vector field ν_x is locally given by $(0, \dots, 0, -1)$. The same for the one form n_x.

Normal geodesic coordinates allow us to locally formulate boundary problems in a half-space geometry. We write

$$\mathbb{R}^d_+ := \{x \in \mathbb{R}^d, \ x_d > 0\} \qquad \text{where } x = (x', x_d) \text{ with } x' \in \mathbb{R}^{d-1}, x_d \in \mathbb{R}.$$

We shall naturally denote its closure by $\overline{\mathbb{R}^d_+}$, that is, $\overline{\mathbb{R}^d_+} = \{x \in \mathbb{R}^d; x_d \ge 0\}$.

The Laplace-Beltrami operator is given by

$$(\Delta_g f)(x) = (\det g_x)^{-1/2} \sum_{1 \le i,j \le d} \partial_{x_i}\big((\det g_x)^{1/2} g^{ij}(x) \partial_{x_j} f\big)(x). \tag{4.5.2}$$

in local coordinates. Its principal part is given by $\sum_{1\leq i,j\leq d} g^{ij}(x)\partial_{x_i}\partial_{x_j}$ and its principal symbol by $\sum_{1\leq i,j\leq d} g^{ij}(x)\xi_i\xi_j$.

The bi-Laplace operator is $P = \Delta_g^2$. In what follows we shall write Δ, Δ^2 in place of Δ_g, Δ_g^2.

For an open set U of $\mathbb{R}^d$ we set $U_+ = U \cap \mathbb{R}^d_+$ and

$$\overline{\mathscr{C}}_c^\infty(U_+) = \{u = v_{|\mathbb{R}^d_+}; v \in \mathscr{C}_c^\infty(\mathbb{R}^d) \text{ and } \operatorname{supp}(v) \subset U\}. \tag{4.5.3}$$

We set $\overline{\mathscr{S}}(\mathbb{R}^d_+) = \{u_{|\mathbb{R}^d_+};\ u \in \mathscr{S}(\mathbb{R}^d)\}$ with $\mathscr{S}(\mathbb{R}^d)$ the usual Schwartz space in $\mathbb{R}^d$:

$$u \in \mathscr{S}(\mathbb{R}^d) \quad \Leftrightarrow \quad u \in \mathscr{C}^\infty(\mathbb{R}^d) \text{ and } \forall \alpha, \beta \in \mathbb{N}^d \sup_{x\in\mathbb{R}^d} |x^\alpha D_x^\beta u(x)| < \infty.$$

4.5.3 Observations concerning the need for boundary conditions

On $(0, +\infty)$ we consider the first-order differential operator $L = D_s - \lambda\rho$, with $\rho \in \mathbb{C}$ and $\lambda > 0$. The parameter λ is intended to become large.
Let $u \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$. One aims to achieve the following estimate

$$\|u\|_{L^2(\mathbb{R}_+)} \lesssim \|Lu\|_{L^2(\mathbb{R}_+)}. \tag{4.5.4}$$

First, we assume that $\operatorname{Im}\rho < 0$ and we compute

$$\begin{aligned} \operatorname{Re}(Lu, iu)_{L^2(\mathbb{R}_+)} &= \operatorname{Re}(D_s u, iu)_{L^2(\mathbb{R}_+)} - \lambda \operatorname{Im}\rho \|u\|^2_{L^2(\mathbb{R}_+)} \\ &= \operatorname{Re}(D_s u, iu)_{L^2(\mathbb{R}_+)} + \lambda |\operatorname{Im}\rho| \|u\|^2_{L^2(\mathbb{R}_+)} \\ &= \frac{1}{2}|u(0)|^2 + \lambda |\operatorname{Im}\rho| \|u\|^2_{L^2(\mathbb{R}_+)}, \end{aligned} \tag{4.5.5}$$

since

$$\begin{aligned} (D_s u, iu)_{L^2(\mathbb{R}_+)} = \int_{\mathbb{R}_+} D_s u\ \overline{(iu)}\ ds = -i \int_{\mathbb{R}_+} D_s u\ u\ ds &= -\int_{\mathbb{R}_+} \partial_s u\ u\ ds \\ &= -\frac{1}{2}\int_{\mathbb{R}_+} \partial_s |u|^2\ ds. \end{aligned}$$

Using the Cauchy-Schwartz inequality and the Young inequality we obtain for any $\varepsilon > 0$

$$|u(0)|^2 + \lambda\|u\|^2_{L^2(\mathbb{R}_+)} \lesssim \|Lu\|_{L^2(\mathbb{R}_+)}\|u\|_{L^2(\mathbb{R}_+)} \lesssim (\lambda\varepsilon)^{-1}\|Lu\|^2_{L^2(\mathbb{R}_+)} + (\lambda\varepsilon)\|u\|^2_{L^2(\mathbb{R}_+)}.$$

With $\varepsilon > 0$ chosen sufficiently small one concludes that

$$\lambda^{1/2}|u(0)| + \lambda\|u\|_{L^2(\mathbb{R}_+)} \lesssim \|Lu\|_{L^2(\mathbb{R}_+)}. \tag{4.5.6}$$

In this first case, we obtain (4.5.4) but, better yet, we estimate also the trace of u at $s = 0^+$.
Second, assume that $\operatorname{Im}\rho > 0$. Consider the cut-off $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ with $\chi(s) = 1$ for $|s| \le 1$ and $\chi(s) = 0$ for $s \ge 2$. We set $u(s) = \chi(s)e^{i\lambda\rho s}$ and we observe that $Lu(s) = -i\chi'(s)e^{i\lambda\rho s}$. On the one has

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}_+)} &= \int_0^2 \chi^2(s)e^{-2\lambda\operatorname{Im}\rho s}ds\\ &\ge \int_0^1 e^{-2\lambda\operatorname{Im}\rho s}ds = \frac{1}{2\lambda\operatorname{Im}\rho}\left(1 - e^{-2\lambda\operatorname{Im}\rho}\right).\end{aligned}$$

On the other hand, one has

$$\begin{aligned}\|Lu\|^2_{L^2(\mathbb{R}_+)} &= \int_0^2 (\chi'(s))^2 e^{-2\lambda\operatorname{Im}\rho s}ds\\ &\lesssim \int_1^2 e^{-2\lambda\operatorname{Im}\rho s}ds = \frac{1}{2\lambda\operatorname{Im}\rho}\left(e^{-2\lambda\operatorname{Im}\rho} - e^{-4\lambda\operatorname{Im}\rho}\right)\\ &= \frac{1}{2\lambda\operatorname{Im}\rho}\left(1 - e^{-2\lambda\operatorname{Im}\rho}\right)e^{-2\lambda\operatorname{Im}\rho}.\end{aligned}$$

This ruins any hope of having an estimate of the form (4.5.4). Yet, by computing $\operatorname{Re}(Lu, -iu)_{L^2(\mathbb{R}_+)}$ and arguing as above for (4.5.4)-(4.5.6), one obtains the following estimate

$$\lambda\|u\|_{L^2(\mathbb{R}_+)} \lesssim \|Lu\|_{L^2(\mathbb{R}_+)} + \lambda^{1/2}|u(0)|. \tag{4.5.7}$$

Consider now the Laplace-Beltrami operator $-\Delta_g$ in the normal geodesic coordinates, that is $-\Delta_g = D_d^2 + R(x, D_{x'})$, where $R(x, D_{x'})$ is a second-order differential operator. To simplify, we assume that $R(x, D_{x'})$ is a constant coefficient operator. Then, up to a Fourier transformation in the x' variables, one obtains for the principal part the operator

$$\hat{P} = D_d^2 + R(\xi') = D_d^2 + |\xi'|^2\tilde{R}(\xi'),$$

where $\tilde{R}(\xi') = R(\xi'/|\xi'|)$, which we write

$$\hat{P} = L^+L^- \quad L^\pm = D_d \pm |\xi'|\tilde{R}(\xi')^{1/2}.$$

With $\lambda = |\xi'| > 0$ and $x_d = s$, and the two cases considered above, one finds that for the factor L^- and estimate as in (4.5.6) can be obtained. For the operator

L^+, one can only obtain an estimation as in (4.5.7). Combined together one obtains an estimate either of the form

$$\lambda^2 \|u\|_{L^2(\mathbb{R}_+)} + \lambda^{1/2}|D_d u(0)| \lesssim \|\hat{P}u\|_{L^2(\mathbb{R}_+)} + \lambda^{3/2}|u(0)|,$$

or the form

$$\lambda^2 \|u\|_{L^2(\mathbb{R}_+)} + \lambda^{3/2}|u(0)| \lesssim \|\hat{P}u\|_{L^2(\mathbb{R}_+)} + \lambda^{1/2}|D_d u(0)|.$$

This simple example shows that if an elliptic operator can be written as a product of several factors of the form $D_d - \lambda\rho$ with $\operatorname{Im}\rho < 0$ yield an estimate without requiring any boundary term while factors of the form $D_d - \lambda\rho$ with $\operatorname{Im}\rho > 0$ require a boundary term. The number of factors of the second kind yield the number of required boundary conditions.
In that framework, if given some boundary operators, the Lopatinskiĭ-Šapiro condition states their compatibility with the different factors $D_d - \lambda\rho$ with $\operatorname{Im}\rho > 0$.

4.5.4 Symbols and operators with an additional large parameter

We shall often use operators with a symbol that depends on an additional large parameter σ, say $a(x, \xi, \tau, \sigma)$. They will satisfy estimate of the form

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, \tau, \sigma)| \leq C_{\alpha,\beta}(\tau^2 + |\xi|^2 + \sigma^2)^{(m-|\beta|)/2}.$$

We observe that if $\tau \gtrsim \sigma$ one has

$$\lambda_\tau^2 \leq \tau^2 + |\xi|^2 + \sigma^2 \lesssim \lambda_\tau^2.$$

Thus, as far as pseudo-differential calculus is concerned it is as if $a \in S_\tau^m$ and this property will be exploited in what follows.

Similarly if $a = a(x, \xi', \tau, \sigma)$ fulfills a tangential-type estimate of the form

$$|\partial_x^\alpha \partial_{\xi'}^\beta a(x, \xi', \tau, \sigma)| \leq C_{\alpha,\beta}(\tau^2 + |\xi'|^2 + \sigma^2)^{(m-|\beta|)/2},$$

if one has $\tau \gtrsim \sigma$ one will be able to apply techniques adapted to symbols in $S_{\mathsf{T},\tau}^m$ and associated operators, like for instance the results on differential quadratic forms listed in Section 4.2.1.

4.5.5 Outline

In Section 4.6, the Lopatinskiĭ-Šapiro boundary condition are properly defined for an elliptic operator, we give examples focusing on the Laplace and bi-Laplace operator and we give a formulation in local normal geodesic coordinated that we shall mostly use throughout the manuscript. For the bi-Laplace operator we provide a series of examples of boundary operators for which the Lopatinskiĭ-Šapiro boundary conditions holds and moreover the resulting operator is symmetric. We also show that the algebraic conditions that characterize the Lopatinskiĭ-Šapiro condition are robust under perturbation. This last aspect is key in the understanding of how the Lopatinskiĭ-Šapiro condition get preserved under conjugation and the introduction of a spectral parameter. This is done in Section 4.7, where an analysis of the configuration of the roots of the conjugated bi-Laplace operator is performed. In Section 4.7.5 the Lopatinskiĭ-Šapiro condition for the conjugated operator is exploited to obtain a symbol positivity for a quadratic form to prepare for the derivation of a Carleman estimate.

In Section 4.8 we derive a estimation of the boundary traces. This is precisely where the Lopatinskiĭ-Šapiro condition is used. The result is first obtained microlocally and we then apply a patching procedure.

To obtain the Carleman estimate for the bi-Laplace operator with spectral parameter $\Delta^2 - \sigma^4$ in Section 4.9 we first derive microlocal estimates for the operators $\Delta \pm \sigma^2$. Imposing σ to be non-zero, in the sense that $\sigma \gtrsim \tau$, the previous estimates exhibits losses in different microlocal regions. Thus concatenating the two estimates one derives an estimate for $\Delta^2 - \sigma^4$ where losses do not accumulates. A local Carleman estimate with only a loss of a half-derivative is obtained. This is done in Section 4.10. With the traces estimation obtained in Section 4.8 one obtains the local Carleman estimate of Theorem 1.8.1.

For the application to stabilization we have in mind, in Section 4.11 we use a global weight function and derive a global version of the Carleman estimate for $\Delta^2 - \sigma^4$ on the whole Ω. This leads to an observability inequality.

In Section 4.12 we recall aspects of strong and weak solutions to the damped plate equation, in particular through a semigroup formulation. With the observability inequality obtained in Section 4.11 we derive in Section 4.13 a resolvent estimate for the generator of the plate semigroup that in turn implies the stabilization result of Theorem 1.8.2.

4.6 Lopatinskiĭ-Šapiro boundary conditions for an elliptic operator

Let P be an elliptic differential operator of order $2k$ on Ω, $(k \geq 1)$, with principal symbol $p(x, \omega)$ for $(x, \omega) \in T^*\Omega$. One defines the following polynomial in z,

$$\tilde{p}(x, \omega', z) = p(x, \omega' - z n_x),$$

for $x \in \partial\Omega$, $\omega' \in T_x^*\partial\Omega$, $z \in \mathbb{R}$ and where n_x denotes the outward unit pointing conormal vector at x (see Section 4.5.2). Here x and ω' are considered to act as parameters. We denote by $\rho_j(x, \omega')$, $1 \leq j \leq 2k$ the complex roots of $\tilde{p}$. One sets

$$\tilde{p}^+(x, \omega', z) = \prod_{\operatorname{Im} \rho_j(x, \omega') \geq 0} (z - \rho_j(x, \omega')).$$

Given boundary operators $B_1, \cdots, B_k$ in a neighborhood of $\partial\Omega$, with principal symbols $b_j(x, \omega)$, $j = 1, \cdots, k$, one also sets $\tilde{b}_j(x, \omega', z) = b_j(x, \omega' - z n_x)$.

Definition 4.6.1 (Lopatinskiĭ-Šapiro boundary condition)**.** Let $(x, \omega') \in T^*\partial\Omega$ with $\omega' \neq 0$. One says that the Lopatinskiĭ-Šapiro condition holds for $(P, B_1, \cdots, B_k)$ at (x, ω') if for any polynomial function $f(z)$ with complex coefficients, there exists $c_1, \cdots, c_k \in \mathbb{C}$ and a polynomial function $g(z)$ with complex coefficients such that, for all $z \in \mathbb{C}$,

$$f(z) = \sum_{1 \leq j \leq k} c_j \tilde{b}_j(x, \omega', z) + g(z) \tilde{p}^+(x, \omega', z).$$

We say that the Lopatinskiĭ-Šapiro condition holds for $(P, B_1, \cdots, B_k)$ at $x \in \partial\Omega$ if it holds at (x, ω') for all $\omega' \in T_x^*\partial\Omega$ with $\omega' \neq 0$.

4.6.1 Some examples

For instance the Lopatinskiĭ-Šapiro condition holds in the following cases

- $P = -\Delta$ on Ω, with the Dirichlet boundary condition, $Bu_{|\partial\Omega} = u_{|\partial\Omega}$.
- $P = \Delta^2$ on Ω, along with the so-called clamped boundary conditions, i.e, $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_\nu u_{|\partial\Omega}$, where ν is the normal outward pointing unit vector to $\partial\Omega$; see Section 4.5.2.
- $P = \Delta^2$ on Ω, along with the so-called hinged boundary conditions, i.e, $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \Delta u_{|\partial\Omega}$.

4.6.2 Case of the bi-Laplace operator

With $P = \Delta^2$ on Ω, along with the general boundary operators B_1 and B_2 of orders k_1 and k_2 respectively, we give a matrix criterion of the Lopatinskiĭ-Šapiro condition. The general boundary operators B_1 and B_2 are then given by

$$B_\ell(x, D) = \sum_{0 \leq j \leq \min(3, k_\ell)} B_\ell^{k_\ell - j}(x, D')(i\partial_\nu)^j, \qquad \ell = 1, 2,$$

with $B_\ell^{k_\ell-j}(x, D')$ differential operators acting in the tangential variables. We denote by $b_1(x, \omega)$ and $b_2(x, \omega)$ the principal symbols of B_1 and B_2 respectively. For $(x, \omega') \in T^*\partial\Omega$, we set

$$\tilde{b}_\ell(x, \omega', z) = \sum_{0 \leq j \leq \min(3, k_\ell)} b_\ell^{k_\ell - j}(x, \omega') z^j, \qquad \ell = 1, 2.$$

We recall that the principal symbol of P is given by $p(x, \omega) = |\omega|_g^4$. One thus has

$$\tilde{p}(x, \omega', z) = p(x, \omega' - zn_x) = \left(z^2 + |\omega'|_g^2\right)^2.$$

Therefore $\tilde{p}(x, \omega', z) = (z - i|\omega'|_g)^2(z + i|\omega'|_g)^2$. According to the above definition we set $\tilde{p}^+(x, \omega', z) = (z - i|\omega'|_g)^2$. Thus, the Lopatinskiĭ-Šapiro condition holds at (x, ω') with $\omega' \neq 0$ if and only if for any function $f(z)$ the complex number $i|\omega'|_g$ is a root of the polynomial function $z \mapsto f(z) - c_1\tilde{b}_1(x, \omega', z) - c_2\tilde{b}_2(x, \omega', z)$ and its derivative for some $c_1, c_2 \in \mathbb{C}$. This leads to the following determinant condition.

Lemma 4.6.2. *Let $P = \Delta^2$ on Ω, B_1 and B_2 be two boundary operators. If $x \in \partial\Omega$, $\omega' \in T_x^*\partial\Omega$, with $\omega' \neq 0$, the Lopatinskiĭ-Šapiro condition holds at (x, ω') if and only if*

$$\det \begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z \tilde{b}_1 & \partial_z \tilde{b}_2 \end{pmatrix} (x, \omega', z = i|\omega'|_g) \neq 0. \tag{4.6.1}$$

Remark 4.6.3. With the determinant condition and homogeneity, we note that if the Lopatinskiĭ-Šapiro condition holds for (P, B_1, B_2) at (x, ω') it also holds in a conic neighborhood of (x, ω') by continuity. If it holds at $x \in \Omega$, it also holds in a neighborhood of x.

4.6.3 Formulation in normal geodesic coordinates

Near a boundary point $x \in \partial\Omega$, we shall use normal geodesic coordinates. These coordinates are recalled at the beginning of Section 4.5.2. Then the principal symbols of Δ and Δ^2 are given by $\xi_d^2 + r(x, \xi')$ and $(\xi_d^2 + r(x, \xi'))^2$ respectively, where $r(x,\xi')$ is the principal symbol of a tangential differential elliptic operator $R(x,D')$ of order 2, with

$$r(x,\xi') = \sum_{1\le i,j\le d-1} g^{ij}(x)\xi_i'\xi_j' \ \text{ and } \ r(x,\xi') \ge C|\xi'|^2.$$

Here g^{ij} is the inverse of the metric g_{ij}. Below, we shall often write $|\xi'|_x^2 = r(x,\xi')$ and we shall also write $|\xi|_x^2 = \xi_d^2 + r(x,\xi')$, for $\xi = (\xi',\xi_d)$.

If $b_1(x,\xi)$ and $b_2(x,\xi)$ are the principal symbols of the boundary operators B_1 and B_2 in the normal geodesic coordinates then the Lopatinskiĭ-Šapiro condition for (P,B_1,B_2) with $P=\Delta^2$ at (x,ξ') reads

$$\det\begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2\end{pmatrix}(x,\xi',\xi_d = i|\xi'|_x) \ne 0,$$

if $|\xi'|_x \ne 0$ according to Lemma 4.6.2. If the Lopatinskiĭ-Šapiro condition holds at some x^0, because of homogeneity, there exists $C_0>0$ such that

$$\left|\det\begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2\end{pmatrix}\right|(x^0,\xi',i|\xi'|_x) \ge C_0|\xi'|_x^{k_1+k_2-1}, \qquad \xi'\in\mathbb{R}^{d-1}. \tag{4.6.2}$$

4.6.4 Stability of the Lopatinskiĭ-Šapiro condition under perturbation

To prepare for the study of how the Lopatinskiĭ-Šapiro condition behaves under conjugation with Carleman exponential weight and the addition of a spectral parameter, we introduce some perturbations in the formulation of the Lopatinskiĭ-Šapiro condition for (P,B_1,B_2) as written in (4.6.2).

Lemma 4.6.4. *Let V^0 be a compact set of $\partial\Omega$ be such that the Lopatinskiĭ-Šapiro condition holds for (P,B_1,B_2) at every point x of V^0. There exist $C_1>0$ and $\varepsilon>0$ such that*

$$\left|\det\begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2\end{pmatrix}\right|(x,\xi'+\zeta',\xi_d = i|\xi'|_x+\delta) \ge C_1|\xi'|_x^{k_1+k_2-1}, \tag{4.6.3}$$

for $x\in V^0$, $\xi'\in\mathbb{R}^{d-1}$, $\zeta'\in\mathbb{C}^{d-1}$, and $\delta\in\mathbb{C}$, if $|\zeta'|+|\delta|\le\varepsilon|\xi'|_x$. Moreover one has

$$\left|\det\begin{pmatrix} b_1(x,\xi'+\zeta',\xi_d = i|\xi'|_x+\delta) & b_2(x,\xi'+\zeta',\xi_d = i|\xi'|_x+\delta) \\ b_1(x,\xi'+\zeta',\xi_d = i|\xi'|_x+\tilde\delta) & b_2(x,\xi'+\zeta',\xi_d = i|\xi'|_x+\tilde\delta)\end{pmatrix}\right| \ge C_1|\delta-\tilde\delta|\,|\xi'|_x^{k_1+k_2-1}, \tag{4.6.4}$$

for $x\in V^0$, $\xi'\in\mathbb{R}^{d-1}$, $\zeta'\in\mathbb{C}^{d-1}$, and $\delta,\tilde\delta\in\mathbb{C}$, if $|\zeta'|+|\delta|+|\tilde\delta|\le\varepsilon|\xi'|_x$.

Proof. From (4.6.2), since V^0 is compact having the Lopatinskiĭ-Šapiro condition holding at every point x of V^0 means there exists $C_0 > 0$ such that

$$\left|\det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix}\right| (x, \xi', i|\xi'|_x) \geq C_0 |\xi'|_x^{k_1+k_2-1}, \qquad x \in V^0, \xi' \in \mathbb{R}^{d-1}. \tag{4.6.5}$$

The first part is a consequence of the mean value theorem, homogeneity and (4.6.5) with say $C_1 = C_0/2$.

For the second part it is sufficient to assume that $\delta \neq \tilde{\delta}$ since the result is obvious otherwise. For $j = 1, 2$ one writes the Taylor formula

$$\begin{aligned} b_j(x, \xi' + \zeta', i|\xi'|_x + \tilde{\delta}) &= b_j(x, \xi' + \zeta', i|\xi'|_x + \delta) + (\tilde{\delta} - \delta)\partial_{\xi_d} b_j(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ &\quad + (\tilde{\delta} - \delta)^2 \int_0^1 (1-s)\partial^2_{\xi_d} b_j(x, \xi' + \zeta', i|\xi'|_x + \delta_s)\, ds, \end{aligned}$$

with $\delta_s = (1-s)\delta + s\tilde{\delta}$, yielding

$$\begin{aligned} &\frac{1}{\tilde{\delta} - \delta} \det \begin{pmatrix} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta) & b_2(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ b_1(x, \xi' + \zeta', i|\xi'|_x + \tilde{\delta}) & b_2(x, \xi' + \zeta', i|\xi'|_x + \tilde{\delta}) \end{pmatrix} \\ &\qquad = \det \begin{pmatrix} b_1 & b_2 \\ \partial_{\xi_d} b_1 & \partial_{\xi_d} b_2 \end{pmatrix} (x, \xi' + \zeta', i|\xi'|_x + \delta) \\ &\qquad\quad + (\tilde{\delta} - \delta) \int_0^1 (1-s) \det \begin{pmatrix} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta) & b_2(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ \partial^2_{\xi_d} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta_s) & \partial^2_{\xi_d} b_2(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \end{pmatrix} ds. \end{aligned}$$

With homogeneity, if $|\zeta'| + |\delta| + |\tilde{\delta}| \lesssim |\xi'|_x$ one finds

$$\left|\det \begin{pmatrix} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta) & b_2(x, \xi' + \zeta', i|\xi'|_x + \delta) \\ \partial^2_{\xi_d} b_1(x, \xi' + \zeta', i|\xi'|_x + \delta_s) & \partial^2_{\xi_d} b_2(x, \xi' + \zeta', i|\xi'|_x + \delta_s) \end{pmatrix}\right| \lesssim |\xi'|_x^{k_1+k_2-2},$$

Thus with $|\delta - \tilde{\delta}| \leq \varepsilon |\xi'|_x$, for $\varepsilon > 0$ chosen sufficiently small, using the first part of the lemma one obtains the second result. □

4.6.5 Examples of boundary operators yielding symmetry

We give some examples of pairs of boundary operators B_1, B_2 that fulfill (1) the Lopatinskiĭ-Šapiro condition and (2) yield symmetry for the bi-Lalace operator $P = \Delta^2$, that is,

$$(Pu, v)_{L^2(\Omega)} = (u, Pv)_{L^2(\Omega)}$$

for $u, v \in H^4(\Omega)$ such that $B_j u_{|\partial\Omega} = B_j v_{|\partial\Omega} = 0$, $j = 1, 2$.

We first recall that following Green formula

$$(\Delta u, v)_{L^2(\Omega)} = (u, \Delta v)_{L^2(\Omega)} + (\partial_n u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)}, \quad (4.6.6)$$

which applied twice gives $(Pu, v)_{L^2(\Omega)} = (u, Pv)_{L^2(\Omega)} + T(u, v)$ with

$$\begin{aligned} T(u, v) = {} & (\partial_n \Delta u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - (\Delta u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} \\ & + (\partial_n u_{|\partial\Omega}, \Delta v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, \partial_n \Delta v_{|\partial\Omega})_{L^2(\partial\Omega)}. \end{aligned} \quad (4.6.7)$$

Using normal geodesic coordinates in a neighborhood of the whole boundary $\partial\Omega$ allows one to write $\Delta = \partial_n^2 + \Delta'$ where Δ' is the resulting Laplace operator on the boundary, that is, associated with the trace of the metric on $\partial\Omega$. Since Δ' is selfadjoint on $\partial\Omega$ this allows one to write formula (4.6.7) in the alternative forms

$$\begin{aligned} T(u, v) = {} & (\partial_n^3 u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - ((\partial_n^2 + 2\Delta') u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} \\ & + (\partial_n u_{|\partial\Omega}, (\partial_n^2 + 2\Delta') v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, \partial_n^3 v_{|\partial\Omega})_{L^2(\partial\Omega)}, \end{aligned} \quad (4.6.8)$$

or

$$\begin{aligned} T(u, v) = {} & ((\partial_n^3 + 2\Delta'\partial_n) u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} - (\partial_n^2 u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} \\ & + (\partial_n u_{|\partial\Omega}, \partial_n^2 v_{|\partial\Omega})_{L^2(\partial\Omega)} - (u_{|\partial\Omega}, (\partial_n^3 + 2\Delta'\partial_n) v_{|\partial\Omega})_{L^2(\partial\Omega)}. \end{aligned} \quad (4.6.9)$$

We start our list of examples with the most basics ones.

Example 4.6.5 (Hinged boundary conditions)**.** This type of conditions refers to $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \Delta u_{|\partial\Omega}$. With (4.6.7) one finds $T(u, v) = 0$ in the case of homogeneous conditions, hence symmetry.

Note that the hinged boundary conditions are equivalent to having $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n^2 u_{|\partial\Omega}$. With the notation of Section 4.6 this gives $\tilde{b}_1(x, \omega', z) = 1$ and $\tilde{b}_2(x, \omega', z) = (-iz)^2 = -z^2$. It follows that

$$\det \begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z \tilde{b}_1 & \partial_z \tilde{b}_2 \end{pmatrix} (x, \omega', z = i|\omega'|_g) = \det \begin{pmatrix} 1 & |\omega'|_g^2 \\ 0 & -2i|\omega'|_g \end{pmatrix} = -2i|\omega'|_g \neq 0,$$

if $\omega' \neq 0$ and thus the Lopatinskiĭ-Šapiro condition holds by Lemma 4.6.2.

With the hinged boundary conditions observe that the bi-Laplace operator is precisely the square of the Dirichlet-Laplace operator. This makes its analysis quite simple and this explains why this type of boundary condition is often chosen in the mathematical literature. Observe that symmetry is then obvious without invoking the above formulae.

Example 4.6.6 (Clamped boundary conditions)**.** This type of conditions refers to $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n u_{|\partial\Omega}$. With (4.6.8) one finds $T(u,v) = 0$ in the case of homogeneous conditions, hence symmetry. With the notation of Section 4.6 this gives $\tilde{b}_1(x,\omega',z) = 1$ and $\tilde{b}_2(x,\omega',z) = -iz$. It follows that

$$\det\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z \tilde{b}_1 & \partial_z \tilde{b}_2 \end{pmatrix}(x,\omega',z=i|\omega'|_g) = \det\begin{pmatrix} 1 & |\omega'|_g \\ 0 & -i \end{pmatrix} = -i \neq 0.$$

Thus the Lopatinskiĭ-Šapiro condition holds by Lemma 4.6.2.

Note that with the clamped boundary conditions the bi-Laplace operator cannot be seen as the square of the Laplace operator with some well chosen boundary condition as opposed to the case of the hinged boundary conditions displayed above.

Examples 4.6.7 (More examples)**.**

i. Take $B_1 u_{|\partial\Omega} = \partial_n u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n \Delta u_{|\partial\Omega}$. With these boundary conditions the bi-Laplace operator is precisely the square of the Neumann-Laplace operator. The symmetry property is immediate and so is the Lopatinskiĭ-Šapiro condition.

ii. Take $B_1 u_{|\partial\Omega} = (\partial_n^2 + 2\Delta')u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = \partial_n^3 u_{|\partial\Omega}$. With (4.6.8) one finds $T(u,v) = 0$ in the case of homogeneous conditions, hence symmetry. We have $\tilde{b}_1(x,\omega',z) = -z^2 - 2|\omega'|_g^2$ and $\tilde{b}_2(x,\omega',z) = iz^3$ and

$$\det\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z \tilde{b}_1 & \partial_z \tilde{b}_2 \end{pmatrix}(x,\omega',z=i|\omega'|_g) = \det\begin{pmatrix} -|\omega'|_g^2 & |\omega'|_g^3 \\ -2i|\omega'|_g & -3i|\omega'|_g^2 \end{pmatrix} = 5i|\omega'|_g^4 \neq 0,$$

if $\omega' \neq 0$ and thus the Lopatinskiĭ-Šapiro condition holds by Lemma 4.6.2.

iii. Take $B_1 u_{|\partial\Omega} = \partial_n u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = (\partial_n^3 + A')u_{|\partial\Omega}$, with A' a symmetric differential operator of order less than or equal to three on $\partial\Omega$, with homogeneous principal symbol $a'(x,\omega')$ such that $a'(x,\omega') \neq 2|\omega'|_g^3$ for $\omega' \neq 0$, that is, $a'(x,\omega') \neq 2$ for $|\omega'|_g = 1$.

With (4.6.8) one finds

$$T(u,v) = (-A'u_{|\partial\Omega}, v_{|\partial\Omega})_{L^2(\partial\Omega)} + (u_{|\partial\Omega}, A'v_{|\partial\Omega})_{L^2(\partial\Omega)} = 0,$$

in the case of homogeneous conditions, hence symmetry for P.

We have $\tilde{b}_1(x,\omega',z) = -iz$ and $\tilde{b}_2(x,\omega',z) = iz^3 + a'(x,\omega')$ with a' the principal symbol of A'.

$$\det\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z\tilde{b}_1 & \partial_z\tilde{b}_2 \end{pmatrix}(x,\omega',z=i|\omega'|_g) = \det\begin{pmatrix} |\omega'|_g & |\omega'|_g^3 + a'(x,\omega') \\ -i & -3i|\omega'|_g^2 \end{pmatrix} = i\big(a'(x,\omega') - 2|\omega'|_g^3\big) \neq 0,$$

if $\omega' \neq 0$ since $a'(x,\omega') \neq 2|\omega'|_g^3$ by assumption implying that the Lopatinskiĭ-Šapiro condition holds by Lemma 4.6.2.

iv. Take $B_1 u_{|\partial\Omega} = u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = (\partial_n^2 + A'\partial_n)u_{|\partial\Omega}$ with A' a symmetric differential operator of order less than or equal to one on $\partial\Omega$, with homogeneous principal symbol $a'(x,\omega')$ such that $a'(x,\omega') \neq -2|\omega'|_g$ for $\omega' \neq 0$, that is, $a'(x,\omega') \neq -2$ for $|\omega'|_g = 1$. This is a refinement of the case of hinged boundary conditions given in Example 4.6.5 above.

With (4.6.8) one finds

$$T(u,v) = (A'\partial_n u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} + (\partial_n u_{|\partial\Omega}, -A'\partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} = 0,$$

in the case of homogeneous conditions, hence symmetry for P.

We have $\tilde{b}_1(x,\omega',z) = 1$ and $\tilde{b}_2(x,\omega',z) = -z^2 - iza'(x,\omega')$ with a' the principal symbol of A'.

$$\det\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z\tilde{b}_1 & \partial_z\tilde{b}_2 \end{pmatrix}(x,\omega',z=i|\omega'|_g) = \det\begin{pmatrix} 1 & |\omega'|_g^2 + |\omega'|_g a'(x,\omega') \\ 0 & -2i|\omega'|_g - ia'(x,\omega') \end{pmatrix} = -i\big(a'(x,\omega') + 2|\omega'|_g\big) \neq 0,$$

if $\omega' \neq 0$ since $a'(x,\omega') \neq -2|\omega'|_g$ by assumption implying that the Lopatinskiĭ-Šapiro condition holds by Lemma 4.6.2.

v. Take $B_1 u_{|\partial\Omega} = (\partial_n^2 + A'\partial_n)u_{|\partial\Omega}$ and $B_2 u_{|\partial\Omega} = (\partial_n^3 + 2\partial_n\Delta')u_{|\partial\Omega}$, with A' a symmetric differential operator of order less than or equal to one on $\partial\Omega$, with homogeneous principal symbol $a'(x,\omega')$ such that $2a'(x,\omega') \neq -3|\omega'|_g$ for $\omega' \neq 0$, that is, $a'(x,\omega') \neq -3/2$ for $|\omega'|_g = 1$. With (4.6.9) one finds

$$T(u,v) = (A'\partial_n u_{|\partial\Omega}, \partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} + (\partial_n u_{|\partial\Omega}, -A'\partial_n v_{|\partial\Omega})_{L^2(\partial\Omega)} = 0,$$

in the case of homogeneous conditions, hence symmetry for P.

We have $\tilde{b}_1(x,\omega',z) = -z^2 - iza'(x,\omega')$ and $\tilde{b}_2(x,\omega',z) = iz^3 + 2iz|\omega'|_g^2$ and

$$\det\begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 \\ \partial_z\tilde{b}_1 & \partial_z\tilde{b}_2 \end{pmatrix}(x,\omega',z=i|\omega'|_g) = \det\begin{pmatrix} |\omega'|_g^2 + |\omega'|_g a'(x,\omega') & -|\omega'|_g^3 \\ -2i|\omega'|_g - ia'(x,\omega') & -i|\omega'|_g^2 \end{pmatrix}$$
$$= -i|\omega'|_g^3\big(2a'(x,\omega') + 3|\omega'|_g\big) \neq 0,$$

if $\omega' \neq 0$ since $2a'(x,\omega') + 3|\omega'|_g \neq 0$ by assumption implying that the Lopatinskiĭ-Šapiro condition holds by Lemma 4.6.2.

4.7 Lopatinskiĭ-Šapiro condition for the conjugated bi-Laplacian with spectral parameter

Set $P_\sigma = \Delta^2 - \sigma^4$ with $\sigma \in [0,+\infty)$ and denote by $P_{\sigma,\varphi} = e^{\tau\varphi} P_\sigma e^{-\tau\varphi}$ the conjugate operator of P_σ with $\tau \geq 0$ a large parameter and $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d,\mathbb{R})$. We shall refer to φ as the weight function. The principal symbol of P_σ in normal geodesic coordinates is given by

$$p_\sigma(x,\xi) = (\xi_d^2 + r(x,\xi'))^2 - \sigma^4.$$

Observe that $e^{\tau\varphi} D_j e^{-\tau\varphi} = D_j + i\tau\partial_j\varphi \in \mathscr{D}^1_\tau$. So, after conjugation, the principal symbol becomes

$$\begin{aligned}
p_{\sigma,\varphi}(x,\xi,\tau) &= p_\sigma(x,\xi + i\tau d_x\varphi)\\
&= \big((\xi_d + i\tau\partial_d\varphi)^2 + r(x,\xi' + i\tau d_{x'}\varphi)\big)^2 - \sigma^4\\
&= \big((\xi_d + i\tau\partial_d\varphi)^2 + r(x,\xi' + i\tau d_{x'}\varphi) - \sigma^2\big)\,\big((\xi_d + i\tau\partial_d\varphi)^2 + r(x,\xi' + i\tau d_{x'}\varphi) + \sigma^2\big)
\end{aligned}$$

We write $p_{\sigma,\varphi}(x,\xi,\tau) = q^1_{\sigma,\varphi}(x,\xi,\tau) q^2_{\sigma,\varphi}(x,\xi,\tau)$ with

$$q^j_{\sigma,\varphi}(x,\xi,\tau) = (\xi_d + i\tau\partial_d\varphi)^2 + r(x,\xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2, \quad j = 1,2.$$

We consider two boundary operators B_1 and B_2 of order k_1 and k_2 with $b_j(x,\xi)$ for principal symbol, $j = 1,2$. The associated conjugated operators

$$B_{j,\varphi} = e^{\tau\varphi} B_j e^{-\tau\varphi},$$

have respective principal symbols

$$b_{j,\varphi}(x,\xi,\tau) = b_j(x,\xi + i\tau d\varphi), \quad j = 1,2.$$

We assume that the Lopatinskiĭ-Šapiro condition holds for (P_0, B_1, B_2) as in Definition 4.6.1 for any point $(x,\omega') \in T^*_x\partial\Omega$. We wish to know if the Lopatinskiĭ-Šapiro condition can hold for $(P_\sigma, B_1, B_2, \varphi)$, as given by the following definition (in local coordinates for simplicity).

Definition 4.7.1. Let $(x,\xi',\tau,\sigma) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0,+\infty) \times [0,+\infty)$ with $(\xi',\tau,\sigma) \neq 0$. One says that the Lopatinskiĭ-Šapiro condition holds for $(P_\sigma, B_1, B_2, \varphi)$ at (x,ξ',τ,σ) if for any polynomial function $f(\xi_d)$ with complex coefficients there exist $c_1, c_2 \in \mathbb{C}$ and a polynomial function $\ell(\xi_d)$ with complex coefficients such that, for all $\xi_d \in \mathbb{C}$

$$f(\xi_d) = c_1 b_{1,\varphi}(x, \xi', \xi_d, \tau) + c_2 b_{2,\varphi}(x, \xi', \xi_d, \tau) + \ell(\xi_d) p^+_{\sigma,\varphi}(x, \xi', \xi_d, \tau),$$

with

$$p^+_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = \prod_{\operatorname{Im} \rho_j(\xi',\tau,\sigma) \geq 0} (\xi_d - \rho_j(\xi', \tau, \sigma)),$$

where $\rho_j(x, \xi', \tau, \sigma)$, $j = 1, \cdots, 4$, denote the complex roots of $p_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ viewed as a polynomial in ξ_d.

In what follows, we shall assume that $\partial_d \varphi > 0$. Locally, one has $\partial_d \varphi \geq C_1 > 0$, for some $C_1 > 0$.

4.7.1 Discussion on the Lopatinskiĭ-Šapiro condition according to the position of the roots

With the assumption that $\partial_d \varphi > 0$, for any point (x, ξ', τ, σ) at most two roots lie in the upper complex closed half-plane (this is explained below). We then enumerate the following cases.

- Case 1 : No root lying in the upper complex closed half-plane, then $p^+_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = 1$ and the Lopatinskiĭ-Šapiro condition of Definition 4.7.1 holds trivially at (x, ξ', τ, σ).

- Case 2 : One root lying in the upper complex closed half-plane. Let us denote by ρ^+ that root, then $p^+_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = \xi_d - \rho^+$. With Definition 4.7.1, for any choice of f, the polynomial function $\xi_d \mapsto f(\xi_d) - c_1 b_{1,\varphi}(x, \xi', \xi_d, \tau) - c_2 b_{2,\varphi}(x, \xi', \xi_d, \tau)$ admits ρ^+ as a root for $c_1, c_2 \in \mathbb{C}$ well chosen. Hence, the Lopatinskiĭ-Šapiro condition holds at (x, ξ', τ, σ) if and only if

$$b_{1,\varphi}(x, \xi', \xi_d = \rho^+, \tau) \neq 0 \quad \text{or} \quad b_{2,\varphi}(x, \xi', \xi_d = \rho^+, \tau) \neq 0.$$

  Note that it then suffices to have

$$\det \begin{pmatrix} b_{1,\varphi} & b_{2,\varphi} \\ \partial_{\xi_d} b_{1,\varphi} & \partial_{\xi_d} b_{2,\varphi} \end{pmatrix} (x, \xi', \xi_d = \rho^+, \tau) \neq 0.$$

- Case 3 : Two different roots lying in the upper complex closed half-plane. Let denote by ρ_1^+ and ρ_2^+ these roots. One has $p^+_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = (\xi_d - \rho_1^+)\xi_d - \rho_2^+)$. The Lopatinskiĭ-Šapiro condition holds at (x, ξ', τ, σ) if and only if the complex numbers ρ_1^+ and ρ_2^+ are the roots of the polynomial function $\xi_d \mapsto f(\xi_d) - c_1 b_{1,\varphi}(x, \xi', \xi_d, \tau) - c_2 b_{2,\varphi}(x, \xi', \xi_d, \tau)$, for c_1, c_2 well chosen. This reads

$$\begin{cases} f(\rho_1^+) = c_1 b_{1,\varphi}(x, \xi', \xi_d = \rho_1^+, \tau) + c_2 b_{2,\varphi}(x, \xi', \xi_d = \rho_1^+, \tau), \\ f(\rho_2^+) = c_1 b_{1,\varphi}(x, \xi', \xi_d = \rho_2^+, \tau) + c_2 b_{2,\varphi}(x, \xi', \xi_d = \rho_2^+, \tau). \end{cases}$$

Being able to solve this system in c_1 and c_2 for any f is equivalent to having

$$\det \begin{pmatrix} b_{1,\varphi}(x, \xi', \xi_d = \rho_1^+, \tau) & b_{2,\varphi}(x, \xi', \xi_d = \rho_1^+, \tau) \\ b_{1,\varphi}(x, \xi', \xi_d = \rho_2^+, \tau) & b_{2,\varphi}(x, \xi', \xi_d = \rho_2^+, \tau) \end{pmatrix} \neq 0. \tag{4.7.1}$$

- Case 4 : A double root lying in the upper complex closed half-plane. Denote by ρ^+ this root; one has $p_{\sigma,\varphi}^+(x, \xi', \xi_d, \tau) = (\xi_d - \rho^+)^2$. The Lopatinskiĭ-Šapiro condition holds at (x, ξ', τ, σ) if and only if for any choice of f, the complex number ρ^+ is a double root of the polynomial function $\xi_d \mapsto f(\xi_d) - c_1 b_{1,\varphi}(x, \xi', \xi, \tau) - c_2 b_{2,\varphi}(x, \xi', \xi, \tau)$ for some $c_1, c_2 \in \mathbb{C}$. Thus having the Lopatinskiĭ-Šapiro condition is equivalent of having the following determinant condition,

$$\det \begin{pmatrix} b_{1,\varphi}(x, \xi', \xi_d = \rho^+, \tau) & b_{2,\varphi}(x, \xi', \xi_d = \rho^+, \tau) \\ \partial_{\xi_d} b_{1,\varphi}(x, \xi', \xi_d = \rho^+, \tau) & \partial_{\xi_d} b_{2,\varphi}(x, \xi', \xi_d = \rho^+, \tau) \end{pmatrix} \neq 0. \tag{4.7.2}$$

Observe that case 4 can only occur if $\sigma = 0$ (then one has $(\xi', \tau) \neq (0, 0)$). If $\sigma > 0$ then only cases 1, 2, and 3 are possible. This is precisely stated in Lemma 4.7.7. This will be an important point in what follows.

We now state the following important proposition.

Proposition 4.7.2. *Let $x^0 \in \partial\Omega$. Assume that the Lopatinskiĭ-Šapiro condition holds for (P_0, B_1, B_2) at x^0 and thus in a compact neighborhood V^0 of x^0 (by Remark 4.6.3). Assume also that $\partial_d \varphi \geq C_1 > 0$ in V^0. There exist $\mu_0 > 0$ and $\mu_1 > 0$ such that if $(x, \xi', \tau, \sigma) \in V^0 \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ with $(\xi', \tau, \sigma) \neq (0, 0, 0)$,*

$$|d_{x'} \varphi(x)| \leq \mu_0 \partial_d \varphi(x) \quad \textit{and} \quad \sigma \leq \mu_1 \tau \partial_d \varphi(x),$$

then the Lopatinskiĭ-Šapiro condition holds for $(P_\sigma, B_1, B_2, \varphi)$ at (x, ξ', τ, σ).

The proof of Proposition 4.7.2 is given below. We first need to analyze the configuration of the roots of the symbol $p_{\sigma,\varphi}$ starting with each factor $q_{\sigma,\varphi}^j$, $j = 1, 2$.

4.7.2 Root configuration for each factor

We consider either factors $\xi_d \mapsto q^j_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$. We recall that

$$q^j_{\sigma,\varphi}(x, \xi, \tau) = (\xi_d + i\tau\partial_d\varphi)^2 + r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2, \quad j = 1, 2.$$

First, we consider the case $r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \in \mathbb{R}^-$, that is, equal to $-\beta^2$ with $\beta \in \mathbb{R}$. Then, the roots of $\xi_d \mapsto q^j_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ are given by

$$-i\tau\partial_d\varphi + \beta \quad \text{and} \quad -i\tau\partial_d\varphi - \beta.$$

Both lie in the lower complex open half-plane.

Second, we consider the case $r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \in \mathbb{C} \setminus \mathbb{R}^-$. There exists a unique $\alpha_j \in \mathbb{C}$ such that $\operatorname{Re}\alpha_j > 0$ and

$$\begin{aligned} \alpha_j^2 &= r(x, \xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \\ &= r(x, \xi') - \tau^2 r(x, d_{x'}\varphi) + (-1)^j\sigma^2 + i2\tau\tilde{r}(x, \xi', d_{x'}\varphi)^2, \end{aligned} \tag{4.7.3}$$

where $\tilde{r}(x, ., .)$ denotes the symmetric bilinear form associated with the quadratic form $r(x, .)$. Then, the two roots of $\xi_d \mapsto q^j_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ are given by

$$\pi_{j,1} = -i\tau\partial_d\varphi - i\alpha_j \quad \text{and} \quad \pi_{j,2} = -i\tau\partial_d\varphi + i\alpha_j. \tag{4.7.4}$$

One has $\operatorname{Im}\pi_{j,1} < 0$ since $\partial_d\varphi \geq C_1 > 0$. With $\operatorname{Im}\pi_{j,2} = -\tau\partial_d\varphi + \operatorname{Re}\alpha_j$ one sees that the sign of $\operatorname{Im}\pi_{j,2}$ may change. The following lemma gives an algebraic characterization of the sign of $\operatorname{Im}\pi_{j,2}$.

Lemma 4.7.3. *Assume that $\partial_d\varphi > 0$. Having $\operatorname{Im}\pi_{j,2} < 0$ is equivalent to having*

$$(\partial_d\varphi)^2 r(x, \xi') + \tilde{r}(x, \xi', d_{x'}\varphi)^2 < \tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2.$$

Proof. From 4.7.4 one has $\operatorname{Im}\pi_{j,2} < 0$ if and only if $\operatorname{Re}\alpha_j < \tau\partial_d\varphi = |\tau\partial_d\varphi|$, that is, if and only if

$$4(\tau\partial_d\varphi)^2 \operatorname{Re}\alpha_j^2 - 4(\tau\partial_d\varphi)^4 + (\operatorname{Im}\alpha_j^2)^2 < 0,$$

by Lemma 4.7.4 below. With (4.7.3) this gives the result. □

Lemma 4.7.4. *Let $z \in \mathbb{C}$ such that $m = z^2$. For $x_0 \in \mathbb{R}$ such that $x_0 \neq 0$, we have*

$$|\operatorname{Re} z| \lesseqqgtr |x_0| \qquad \Longleftrightarrow \qquad 4x_0^2 \operatorname{Re} m - 4x_0^4 + (\operatorname{Im} m)^2 \lesseqqgtr 0.$$

Proof. Let $z = x + iy \in \mathbb{C}$. On the one hand we have $z^2 = x^2 - y^2 + 2ixy = m$ and $\operatorname{Re} m = x^2 - y^2$, $\operatorname{Im} m = 2xy$. On the other hand we have

$$\begin{aligned} 4x_0^2 \operatorname{Re} m - 4x_0^4 + (\operatorname{Im} m)^2 &= 4x_0^2(x^2 - y^2) - 4x_0^4 + 4x^2y^2 \\ &= 4(x_0^2 + y^2)(x^2 - x_0^2), \end{aligned}$$

thus with the same sign as $(x^2 - x_0^2)$. Since $|\operatorname{Re} z| \lesseqqgtr |x_0| \;\Leftrightarrow\; x^2 - x_0^2 \lesseqqgtr 0$ the conclusion follows. $\square$

With the following two lemmata we now connect the sign of $\operatorname{Im} \pi_{j,2}$ with the low frequency regime $|\xi'| \lesssim \tau$.

Lemma 4.7.5. *Assume there exists $K_0 > 0$ such that $|d_{x'}\varphi| \leq K_0|\partial_d\varphi|$. Then, there exists $C_{K_0} > 0$ such that $\operatorname{Im} \pi_{j,2} < 0$ if $C_{K_0}|\xi'| + \sigma \leq \tau\partial_d\varphi$, $j = 0, 1$.*

Proof. With Lemma 4.7.3 having $\operatorname{Im} \pi_{j,2} < 0$ reads

$$(\partial_d\varphi)^2 r(x, \xi') + \tilde{r}(x, \xi', d_{x'}\varphi)^2 < \tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2. \tag{4.7.5}$$

On the one hand, since $|d_{x'}\varphi| \leq K_0|\partial_d\varphi|$ one has

$$(\partial_d\varphi)^2 r(x, \xi') + \tilde{r}(x, \xi', d_{x'}\varphi)^2 \leq K(\partial_d\varphi)^2|\xi'|^2,$$

for some $K > 0$ that depends on K_0, using that $|\xi'|_x \eqsim |\xi'|$. On the other hand one has

$$\tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2 \geq \tau^2(\partial_d\varphi)^4 - \sigma^2(\partial_d\varphi)^2.$$

Thus (4.7.5) holds if one has

$$\tau^2(\partial_d\varphi)^4 - \sigma^2(\partial_d\varphi)^2 \geq K(\partial_d\varphi)^2|\xi'|^2,$$

that is, $\tau^2(\partial_d\varphi)^2 \geq K|\xi'|^2 + \sigma^2$. $\square$

Lemma 4.7.6. *Let W be a bounded open set of $\mathbb{R}^d$ and $x^0 \in W$. Assume that $\partial_d\varphi > 0$ in $\overline{W}$ and let $\kappa_0 > 0$. Then, there exists $C > 0$ such that*

$$|\xi'| \leq C\tau \quad \textit{if } \operatorname{Im} \pi_{j,2}(x, \xi', \tau, \sigma) < 0 \textit{ and } \kappa_0\sigma \leq \tau, \quad x \in W.$$

Proof. With Lemma 4.7.3 having $\operatorname{Im} \pi_{j,2} < 0$ reads

$$(\partial_d\varphi)^2 r(x, \xi') + \tilde{r}(x, \xi', d_{x'}\varphi)^2 < \tau^2(\partial_d\varphi)^2|d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2(\partial_d\varphi)^2.$$

In particular, this implies

$$r(x,\xi') < \tau^2 |d_x\varphi|_x^2 + (-1)^{j+1}\sigma^2 \leq (\sup_W |d_x\varphi|_x^2 + 1/\kappa_0^2)\tau^2.$$

The result follows since $|\xi'| \asymp r(x,\xi')$. □

As mentioned in Section 4.7.1, we have the following result.

Lemma 4.7.7. *Assume that $\sigma > 0$. Then, $\pi_{1,2} \neq \pi_{2,2}$. Moreover, the roots $\pi_{1,2}$ and $\pi_{2,2}$ cannot be both real.*

Proof. With the forms of the roots given in (4.7.4) if $\pi_{1,2} = \pi_{2,2}$ then $\alpha_1 = \alpha_2$, thus $\alpha_1^2 = \alpha_2^2$ implying $\sigma^2 = 0$.

Assume now that $\pi_{1,2} \in \mathbb{R}$ and $\pi_{2,2} \in \mathbb{R}$, that is, $\operatorname{Im}\pi_{1,2} = \operatorname{Im}\pi_{2,2} = 0$. This reads $\operatorname{Re}\alpha_j = \tau\partial_d\varphi$, giving $|\operatorname{Re}\alpha_j| = |\partial_d\varphi|$, for $j = 1$ and 2. With Lemma 4.7.4 one has

$$4(\tau\partial_d\varphi)^2 \operatorname{Re}\alpha_j^2 - 4(\tau\partial_d\varphi)^4 + (\operatorname{Im}\alpha_j^2)^2 = 0, \qquad j = 1,2.$$

Observing that $\operatorname{Im}\alpha_1^2 = \operatorname{Im}\alpha_2^2$ one thus obtains $\operatorname{Re}\alpha_1^2 = \operatorname{Re}\alpha_2^2$, and the conclusion follows as for the first part. □

4.7.3 Proof of Proposition 4.7.2

Here, according to the statement of Proposition 4.7.2 we consider

$$|d_{x'}\varphi| \leq \mu_0\partial_d\varphi \quad \text{and} \quad \sigma \leq \mu_1\tau\partial_d\varphi.$$

First, we choose $0 < \mu_0 \leq 1$ and $0 < \mu_1 \leq 1/2$. Below both may be chosen much smaller. According to Lemma 4.7.5, with $K_0 = 1$ therein, for some $C_2 = 2C_{K_0} > 0$ if one has $C_2|\xi'| \leq \tau\partial_d\varphi$ then all four roots of $\xi_d \mapsto p_{\sigma,\varphi}(x,\xi',\xi_d,\tau)$ lie in the lower complex open half-plane. If so, we face Case 1 as in the discussion of Section 4.7.1 and the Lopatinskiĭ-Šapiro condition holds. To carry on with the proof of Proposition 4.7.2 we now only have to consider having

$$\tau\partial_d\varphi \leq C_2|\xi'|. \tag{4.7.6}$$

Our proof of Proposition 4.7.2 relies on the following lemma.

Lemma 4.7.8. *There exists $C_3 > 0$ such that, for $j = 1$ or 2, for $0 < \mu_0 \leq 1$, $0 < \mu_1 \leq 1/2$, and for all $(x,\xi',\tau,\sigma) \in \overline{V^0} \times \mathbb{R}^{d-1} \times [0,+\infty) \times [0,+\infty)$, one has*

$$|d_{x'}\varphi| \leq \mu_0\partial_d\varphi, \;\; \sigma \leq \mu_1\tau\partial_d\varphi \;\; \textit{and} \;\; \operatorname{Im}\pi_{j,2} \geq 0 \implies \big|\alpha_j - |\xi'|_x\big| + \tau|d_{x'}\varphi| \leq |\xi'|_x C_3(\mu_0 + \mu_1^2).$$

Proof. With (4.7.6) one has

$$\tau|d_{x'}\varphi| \leq \mu_0\tau\partial_d\varphi \lesssim \mu_0|\xi'|_x. \tag{4.7.7}$$

With the first-order Taylor formula one has

$$\begin{aligned}\alpha_j^2 &= r(x,\xi'+i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \\ &= r(x,\xi') + \int_0^1 d_{\xi'}r(x,\xi'+i\tau s\, d_{x'}\varphi)(i\tau d_{x'}\varphi)ds + (-1)^j\sigma^2.\end{aligned}$$

With (4.7.7) and homogeneity one has

$$\left|d_{\xi'}r(x,\xi'+i\tau s\, d_{x'}\varphi)(i\tau d_{x'}\varphi)\right| \lesssim \mu_0|\xi'|_x^2.$$

One also has $\sigma \leq \mu_1\tau\partial_d\varphi \lesssim \mu_1|\xi'|_x$. Since $r(x,\xi') = |\xi'|_x^2$, this yields $\alpha_j^2 = |\xi'|_x^2\big(1+O(\mu_0+\mu_1^2)\big)$ and hence $\alpha_j = |\xi'|_x\big(1+O(\mu_0+\mu_1^2)\big)$. This and (4.7.7) gives the result. □

Before proceeding, we make the following computation. For $j = 1,2$ and $\ell = 1,2$ we write

$$\begin{aligned}b_{\ell,\varphi}(x,\xi',\xi_d=\pi_{j,2},\tau) &= b_\ell(x,\xi'+i\tau d_{x'}\varphi,\pi_{j,2}+i\tau\partial_d\varphi) = b_\ell(x,\xi'+i\tau d_{x'}\varphi,i\alpha_j) \\ &= b_\ell(x,\xi'+i\tau d_{x'}\varphi,i|\xi'|_x+i(\alpha_j-|\xi'|_x)).\end{aligned} \tag{4.7.8}$$

We use Lemma 4.6.4 and the value of $\varepsilon > 0$ given therein. We choose $0 < \mu_0 \leq 1$ and $0 < \mu_1 \leq 1/2$ such that

$$C_3(\mu_0+\mu_1^2) \leq \varepsilon, \tag{4.7.9}$$

with $C_3 > 0$ as given by Lemma 4.7.8.

We now consider the root configurations that remain to consider according to the discussion in Section 4.7.1.

Case 2.

In this case, one root of $p_{\sigma,\varphi}$ lies in the upper complex closed half-plane. We denote this root by ρ^+. According to the discussion in Section 4.7.1 it suffices to prove that

$$\det\begin{pmatrix} b_{1,\varphi} & b_{2,\varphi} \\ \partial_{\xi_d}b_{1,\varphi} & \partial_{\xi_d}b_{2,\varphi}\end{pmatrix}(x,\xi',\xi_d=\rho^+,\tau) \neq 0. \tag{4.7.10}$$

In fact, one has $\rho^+ = \pi_{j,2}$ with $j = 1$ or 2. We use the first part of Lemma 4.6.4 with $\zeta' = i\tau d_{x'}\varphi$ and $\delta = i(\alpha_j - |\xi|_x)$. With (4.7.8) and (4.7.9) with Lemma 4.7.8 and the first part of Lemma 4.6.4 one obtains (4.7.10).

Case 3.
In this case $\operatorname{Im}\pi_{1,2} > 0$ and $\operatorname{Im}\pi_{2,2} > 0$. According to the discussion in Section 4.7.1 it suffices to prove that

$$\det\begin{pmatrix} b_{1,\varphi}(x,\xi',\xi_d=\pi_{1,2},\tau) & b_{2,\varphi}(x,\xi',\xi_d=\pi_{1,2},\tau) \\ b_{1,\varphi}(x,\xi',\xi_d=\pi_{2,2},\tau) & b_{2,\varphi}(x,\xi',\xi_d=\pi_{2,2},\tau) \end{pmatrix} \neq 0. \tag{4.7.11}$$

We use the second part of Lemma 4.6.4 with $\zeta' = i\tau d_{x'}\varphi$, $\delta = i(\alpha_1 - |\xi|_x)$, and $\tilde{\delta} = i(\alpha_2 - |\xi|_x)$. With (4.7.8) and (4.7.9) with Lemma 4.7.8 and the second part of Lemma 4.6.4 one obtains (4.7.11).
Case 4.
In this case (that only occurs if $\sigma = 0$) the Lopatinskiĭ-Šapiro condition holds also if one has (4.7.10). The proof is thus the same as for Case 2. This concludes the proof of Proposition 4.7.2. ☐

4.7.4 Local stability of the algebraic conditions associated with the Lopatinskiĭ-Šapiro condition

In Section 4.6 we saw that the Lopatinskiĭ-Šapiro condition for (P_σ, B_1, B_2) in Definition 4.6.1 exhibits some stability property. This was used in the statement of Proposition 4.7.2 that states how the Lopatinskiĭ-Šapiro condition for (P_σ, B_1, B_2) can imply the Lopatinskiĭ-Šapiro condition of Definition 4.7.1 for $(P_\sigma, B_1, B_2, \varphi)$, that is, the version of this condition for the conjugated operators.

A natural question would then be: does the Lopatinskiĭ-Šapiro condition for the conjugated operators enjoy the same stability property? The answer is yes. Yet, this is not needed in what follows. In fact, below one exploits the algebraic conditions listed in Section 4.7.1 once the Lopatinskiĭ-Šapiro condition is know to hold at a point $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$ in tangential phase space. One thus rather needs to know that these algebraic conditions are stable. Here also the answer is positive and is the subject of the present section.

As in Definition 4.7.1 for $\varrho' = (x, \xi', \tau, \sigma)$ one denotes by $\rho_j(\varrho')$ the roots of $p_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ viewed as a polynomial in ξ_d.

Let $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$. One sets

$$J^+ = \{j \in \{1,2,3,4\};\ \operatorname{Im}\rho_j(\varrho^{0\prime}) \geq 0\}, \qquad J^- = \{j \in \{1,2,3,4\};\ \operatorname{Im}\rho_j(\varrho^{0\prime}) < 0\}$$

and, for $\varrho' = (x, \xi', \tau, \sigma)$,

$$\kappa^+_{\varrho^{0\prime}}(\varrho') = \prod_{j \in J^+} \big(\xi_d - \rho_j(\varrho')\big), \qquad \kappa^-_{\varrho^{0\prime}}(\varrho') = \prod_{j \in J^-} \big(\xi_d - \rho_j(\varrho')\big).$$

Naturally, one has $\kappa^+_{\varrho^{0\prime}}(\varrho^{0\prime},\xi_d) = p^+_{\sigma,\varphi}(x^0,\xi^{0\prime},\xi_d,\tau^0)$ and $\kappa^-_{\varrho^{0\prime}}(\varrho^{0\prime},\xi_d) = p^-_{\sigma,\varphi}(x^0,\xi^{0\prime},\xi_d,\tau^0)$. Moreover, there exists a conic neighborhood $\mathscr{U}_0$ of $\varrho^{0\prime}$ where both $\kappa^+_{\varrho^{0\prime}}(\varrho')$ and $\kappa^-_{\varrho^{0\prime}}(\varrho')$ are smooth with respect to ϱ'. One has

$$p_{\sigma,\varphi} = p^+_{\sigma,\varphi} p^-_{\sigma,\varphi} = \kappa^+_{\varrho^{0\prime}} \kappa^-_{\varrho^{0\prime}}.$$

Note however that for $\varrho' = (x,\xi',\tau,\sigma) \in \mathscr{U}_0$ it may very well happen that $p^+_{\sigma,\varphi}(x,\xi',\xi_d,\tau) \neq \kappa^+_{\varrho^{0\prime}}(\varrho',\xi_d)$ and $p^-_{\sigma,\varphi}(x,\xi',\xi_d,\tau) \neq \kappa^-_{\varrho^{0\prime}}(\varrho',\xi_d)$.

The following proposition can be found in [11, proposition 1.8].

Proposition 4.7.9. *Let the Lopatinskiĭ-Šapiro condition hold at $\varrho^{0\prime} = (x^0,\xi^{0\prime},\tau^0,\sigma^0) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0,+\infty) \times [0,+\infty)$ for $(P_\sigma, B_1, B_2, \varphi)$. Then,*

i. The polynomial $\xi_d \mapsto p^+_{\sigma,\varphi}(x^0,\xi^{0\prime},\xi_d,\tau^0)$ is of degree less than or equal to two.

ii. There exists a conic neighborhood $\mathscr{U}$ of $\varrho^{0\prime}$ such that $\{b^1_\varphi(\varrho',\xi_d), b^2_\varphi(\varrho',\xi_d)\}$ is complete modulo $\kappa^+_{\varrho^{0\prime}}(\varrho',\xi_d)$ at every point $\varrho' = (x,\xi',\tau,\sigma) \in \mathscr{U}$, namely for any polynomial function $f(\xi_d)$ with complex coefficients there exist $c_1, c_2 \in \mathbb{C}$ and a polynomial function $\ell(\xi_d)$ with complex coefficients such that, for all $\xi_d \in \mathbb{C}$

$$f(\xi_d) = c_1 b_{1,\varphi}(x,\xi',\xi_d,\tau) + c_2 b_{2,\varphi}(x,\xi',\xi_d,\tau) + \ell(\xi_d)\kappa^+_{\varrho^{0\prime}}(\varrho',\xi_d). \quad (4.7.12)$$

We emphasize again that the second property in Proposition 4.7.9 looks very much like the statement of Lopatinskiĭ-Šapiro condition for $(P_\sigma, B_1, B_2, \varphi)$ at ϱ' in Definition 4.7.1. Yet, it differs by having $p^+_{\sigma,\varphi}(x,\xi',\xi_d,\tau)$ that only depends on the root configuration at ϱ' replaced by $\kappa^+_{\varrho^{0\prime}}(\varrho',\xi_d)$ whose structure is based on the root configuration at $\varrho^{0\prime}$.

Let m^+ be the common degree of $p^+_{\sigma,\varphi}(\varrho^{0\prime},\xi_d)$ and $\kappa^+_{\varrho^{0\prime}}(\varrho',\xi_d)$ and m^- be the common degree of $p^-_{\sigma,\varphi}(\varrho^{0\prime},\xi_d)$ and $\kappa^-_{\varrho^{0\prime}}(\varrho',\xi_d)$ for $\varrho' \in \mathscr{U}$. One has $m^+ + m^- = 4$ and thus $m^- \geq 2$ for $\varrho' \in \mathscr{U}$ since $m^+ \leq 2$.

Invoking the Euclidean division of polynomials, one sees that it is sufficient to consider polynomials f of degree less than or equal to $m^+ - 1 \leq 1$ in (4.7.12). Since the degree of $b_{j,\varphi}(\varrho',\xi_d)$ can be as high as $3 > m^+ - 1$ it however makes sense to consider f of degree less than or equal to $m = 3$. Then, the second property in Proposition 4.7.9 is equivalent to having

$$\{b_{1,\varphi}(x,\xi',\xi_d,\tau), b_{2,\varphi}(x,\xi',\xi_d,\tau)\} \cup \bigcup_{0 \leq \ell \leq 3-m^+} \{\kappa^+_{\varrho^{0\prime}}(\varrho',\xi_d)\xi_d^\ell\}$$

be a complete in the set of polynomials of degree less than or equal to $m = 3$. Note that this family is made of $m' = 6 - m^+ = 2 + m^-$ polynomials.

We now express an inequality that follows from Proposition 4.7.9 that will be key in the boundary estimation given in Proposition 4.8.1 below.

4.7.5 Symbol positivity at the boundary

The symbols $b_{j,\varphi}$, $j = 1, 2$, are polynomial in ξ_d of degree $k_j \leq 3$ and we may thus write them in the form

$$b_{j,\varphi}(\varrho', \xi_d) = \sum_{\ell=0}^{k_j} b^{\ell}_{j,\varphi}(\varrho')\xi_d^{\ell},$$

with $b^{\ell}_{j,\varphi}$ homogeneous of degree $k_j - \ell$.

The polynomial $\xi_d \mapsto \kappa^{+}_{\varrho^{0\prime}}(\varrho', \xi_d)$ is of degree $m^+ \leq 2$ for $\varrho' \in \mathscr{U}$ with $\mathscr{U}$ given by Proposition 4.7.9. Similarly, we write

$$\kappa^{+}_{\varrho^{0\prime}}(\varrho', \xi_d) = \sum_{\ell=0}^{m^+} \kappa^{+,\ell}_{\varrho^{0\prime}}(\varrho')\xi_d^{\ell},$$

with $\kappa^{+,\ell}_{\varrho^{0\prime}}$ homogeneous of degree $m^+ - \ell$. We introduce

$$e_{j,\varrho^{0\prime}}(\varrho', \xi_d) = \begin{cases} b_{j,\varphi}(\varrho', \xi_d) & \text{if } j = 1, 2, \\ \kappa^{+}_{\varrho^{0\prime}}(\varrho', \xi_d)\xi_d^{j-3} & \text{if } j = 3, \ldots, m'. \end{cases}$$

As explained above, all these polynomials are of degree less than or equal to three. If we now write

$$e_{j,\varrho^{0\prime}}(\varrho', \xi_d) = \sum_{\ell=0}^{3} e^{\ell}_{j,\varrho^{0\prime}}(\varrho')\xi_d^{\ell},$$

for $j = 1, 2$ one has $e^{\ell}_{j,\varrho^{0\prime}}(\varrho') = b^{\ell}_{j,\varphi}(\varrho')$, with $\ell = 0, \ldots, k_j$ and $e^{\ell}_{j,\varrho^{0\prime}}(\varrho') = 0$ for $\ell > k_j$, and

$$\text{for } j = 3, \ldots, m', \quad e^{\ell}_{j,\varrho^{0\prime}}(\varrho') = \begin{cases} 0 & \text{if } \ell < j - 3, \\ \kappa^{+,\ell+3-j}_{\varrho^{0\prime}}(\varrho') & \text{if } \ell = j - 3, \ldots, m^+ + j - 3 \leq m^+ + m' - 3 = 3, \\ 0 & \text{if } \ell > m^+ + j - 3. \end{cases}$$

In particular $e^{\ell}_{j,\varrho^{0\prime}}(\varrho')$ is homogeneous of degree $m^+ + j - \ell - 3$. We thus have the following result.

Lemma 4.7.10. *Set the $m' \times (m + 1)$ matrix $M(\varrho') = (M_{j,\ell}(\varrho'))_{\substack{1 \leq j \leq m' \\ 0 \leq \ell \leq m}}$ with $M_{j,\ell}(\varrho') = e^{\ell}_{j,\varrho^{0\prime}}(\varrho')$. Then, the second point in Proposition 4.7.9 states that $M(\varrho')$ is of rank $m + 1 = 4$ for $\varrho' \in \mathscr{U}$.*

Recall that $m' = m^- + 2 \geq 4$.

We now set

$$\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}) = \sum_{\ell=0}^{3} e^{\ell}_{j,\varrho^{0\prime}}(\varrho') z_\ell = \sum_{\ell=0}^{3} M_{j,\ell}(\varrho') z_\ell, \qquad \mathbf{z} = (z_0, \dots, z_3). \tag{4.7.13}$$

in agreement with the notation introduced in (4.2.6) in Section 4.2.1. One has the following positivity result.

Lemma 4.7.11. *Let the Lopatinskiĭ-Šapiro condition hold at $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0) \in \partial\Omega \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ for $(P_\sigma, B_1, B_2, \varphi)$ and let $\mathscr{U}$ be as given by Proposition 4.7.9. Then, if $\varrho' \in \mathscr{U}$ there exists $C > 0$ such that*

$$\sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 \geq C|\mathbf{z}|^2_{\mathbb{C}^4}, \qquad \mathbf{z} = (z_0, \dots, z_3) \in \mathbb{C}^4.$$

Proof. In $\mathbb{C}^4$ define the bilinear form $\Sigma_{\mathscr{B}}(\mathbf{z}, \mathbf{z}') = \sum_{j=1}^{m'} \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}) \overline{\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}')}$. With (4.7.13) one has

$$\Sigma_{\mathscr{B}}(\mathbf{z}, \mathbf{z}') = \big(M(\varrho')\mathbf{z}, M(\varrho')\mathbf{z}'\big)_{\mathbb{C}^{m'}} = \big({}^t\overline{M(\varrho')} M(\varrho')\mathbf{z}, \mathbf{z}'\big)_{\mathbb{C}^4}.$$

As $\operatorname{rank} {}^t\overline{M(\varrho')} M(\varrho') = \operatorname{rank} M(\varrho') = 4$ by Lemma 4.7.10 one obtains the result. ☐

4.8 Estimate for the boundary norm under Lopatinskiĭ-Šapiro condition

Near $x^0 \in \partial\Omega$ we consider two boundary operators B_1 and B_2. As in Section 4.7 the associated conjugated operators are denoted by $B_{j,\varphi}$, $j = 1, 2$ with respective principal symbols $b_{j,\varphi}(x, \xi, \tau)$.

The main result of this section is the following proposition for the fourth-order conjugated operator $P_{\sigma,\varphi}$. It is key in the final result of the present article. It states that all traces are controlled by norms of $B_{1,\varphi} v_{|x_d=0^+}$ and $B_{2,\varphi} v_{|x_d=0^+}$ if the Lopatinskiĭ-Šapiro condition holds for (P, B_1, B_2, φ).

Proposition 4.8.1. *Let $\kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$. Assume that $(P_\sigma, B_1, P_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.7.1 at $\varrho' = (x^0, \xi', \tau, \sigma)$ for all $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ such that $\tau \geq \kappa_0 \sigma$.*

Then, there exist W^0 a neighborhood of x^0, $C > 0$, $\tau_0 > 0$ such that

$$|\operatorname{tr}(v)|_{3,1/2,\tau} \leq C\big(\|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|v\|_{4,-1,\tau}\big),$$

for $\sigma \geq 0$, $\tau \geq \max(\tau_0, \kappa_0\sigma)$ and $v \in \overline{\mathscr{C}}_c^\infty(W_+^0)$.

The notation of the function space $\overline{\mathscr{C}}_c^\infty(W_+^0)$ is introduced in (4.5.3).

For the proof of Proposition 4.8.1 we start with a microlocal version of the result.

4.8.1 A microlocal estimate

Proposition 4.8.2. *Let $\kappa_1 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ nonvanishing with $\tau^0 \geq \kappa_1\sigma^0$ and such that $(P_\sigma, B_1, P_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.7.1 at $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$.*

Then, there exists $\mathscr{U}$ a conic neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0\sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$|\operatorname{tr}(\operatorname{Op}_{\mathsf{T}}(\chi)v)|_{3,1/2,\tau} \leq C\Big(\sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau}\Big),$$

for $\sigma \geq 0$, $\tau \geq \max(\tau_0, \kappa_0\sigma)$ and $v \in \overline{\mathscr{C}}_c^\infty(W_+)$.

Proof. We choose a conic neighborhood $\mathscr{U}_0$ of $\varrho^{0\prime}$ according to Proposition 4.7.9 and such that $\mathscr{U}_0 \subset W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$. Assume moreover that $\tau \geq \kappa_0\sigma$ in $\mathscr{U}_0$.

In Section 4.7.5 we introduced the symbols $e_{j,\varrho^{0\prime}}(\varrho', \xi_d)$, $j = 1, \dots, m' = m^- + 2 = 6 - m^+$. Set $\mathbb{S}_{\overline{\mathscr{U}_0}} = \{\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}_0};\ |(\xi', \tau, \sigma)| = 1\}$.

Consequence of the Lopatinskiĭ-Šapiro condition holding at $\varrho^{0\prime}$ for all $\varrho' \in \mathbb{S}_{\overline{\mathscr{U}_0}}$, by Lemma 4.7.11 there exists $C > 0$ such that

$$\sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 \geq C|\mathbf{z}|^2_{\mathbb{C}^4}, \qquad \mathbf{z} = (z_0, \dots, z_3) \in \mathbb{C}^4.$$

Since $\mathbb{S}_{\overline{\mathscr{U}_0}}$ is compact (recall that W is bounded), there exists $C_0 > 0$ such that

$$\sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 \geq C_0|\mathbf{z}|^2_{\mathbb{C}^4}, \qquad \mathbf{z} = (z_0, \dots, z_3) \in \mathbb{C}^4,\ \varrho' \in \mathbb{S}_{\overline{\mathscr{U}_0}}.$$

Introducing the map $N_t\varrho' = (x, t\xi', t\tau, t\sigma)$, for $\varrho' = (x, \xi', \tau, \sigma)$ with $t = |(\xi', \tau, \sigma)|^{-1}$ one has

$$\sum_{j=1}^{m'} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(N_t\varrho', \mathbf{z})\big|^2 \geq C_0 |\mathbf{z}|^2_{\mathbb{C}^4}, \qquad \mathbf{z} = (z_0, \dots, z_3) \in \mathbb{C}^4, \ \varrho' \in \mathscr{U}_0, \tag{4.8.1}$$

since $N_t\varrho' \in \mathbb{S}_{\overline{\mathscr{U}_0}}$. Now, for $j = 1, 2$ one has

$$\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}) = \sum_{\ell=0}^{k_j} e^{\ell}_{j,\varrho^{0\prime}}(\varrho') z_\ell,$$

with $e^{\ell}_{j,\varrho^{0\prime}}(\varrho')$ homogeneous of degree $k_j - \ell$, and for $3 \leq j \leq m'$ one has

$$\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}) = \sum_{\ell=0}^{3} e^{\ell}_{j,\varrho^{0\prime}}(\varrho') z_\ell,$$

with $e^{\ell}_{j,\varrho^{0\prime}}(\varrho')$ homogeneous of degree $m^+ + j - \ell - 3$. We define $\mathbf{z}' \in \mathbb{C}^4$ by $z'_\ell = t^{\ell - 7/2} z_\ell$, $\ell = 0, \dots, 3$. One has

$$\Sigma_{e_{j,\varrho^{0\prime}}}(N_t\varrho', \mathbf{z}') = t^{k_j - 7/2} \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}), \qquad j = 1, 2,$$

and

$$\Sigma_{e_{j,\varrho^{0\prime}}}(N_t\varrho', \mathbf{z}') = t^{m^+ + j - 13/2} \Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z}), \qquad j = 3, \dots, m'.$$

Thus from (4.8.1) we deduce

$$\sum_{j=1}^{2} \lambda_{\mathsf{T},\tau}^{2(7/2 - k_j)} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 + \sum_{j=3}^{m'} \lambda_{\mathsf{T},\tau}^{2(13/2 - m^+ - j)} \big|\Sigma_{e_{j,\varrho^{0\prime}}}(\varrho', \mathbf{z})\big|^2 \geq C_0 \sum_{\ell=0}^{3} \lambda_{\mathsf{T},\tau}^{2(7/2 - \ell)} |z_\ell|^2, \tag{4.8.2}$$

for $\mathbf{z} = (z_0, \dots, z_3) \in \mathbb{C}^4$, and $\varrho' \in \mathscr{U}_0$, since $t \asymp \lambda_{\mathsf{T},\tau}^{-1}$ as $\tau \gtrsim \sigma$ in $\mathscr{U}_0$.

We now choose $\mathscr{U}$ a conic open neighborhood of $\varrho^{0\prime}$, such that $\overline{\mathscr{U}} \subset \mathscr{U}_0$. Let $\chi \in S^0_{\mathsf{T},\tau}$ be as in the statement and let $\tilde{\chi} \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0, with $\operatorname{supp}(\tilde{\chi}) \subset \mathscr{U}_0$ and $\tilde{\chi} \equiv 1$ in a neighborhood of $\overline{\mathscr{U}}$, and thus in a neighborhood of $\operatorname{supp}(\chi)$.

For $j = 3, \dots, m'$ one has $e_{j,\varrho^{0\prime}}(\varrho', \xi_d) = \kappa^+_{\varrho^{0\prime}}(\varrho', \xi_d) \xi_d^{j-3} \in S^{m^+ + j - 3, 0}_{\tau}$. Set $E_j = \operatorname{Op}(\tilde{\chi} e_{j,\varrho^{0\prime}})$. The introduction of $\tilde{\chi}$ is made such that $\tilde{\chi} e_{j,\varrho^{0\prime}}$ is defined on the whole tangential phase-space. Observe that

$$\mathscr{B}(w) = \sum_{j=1}^{2} |B_{j,\varphi} w_{|x_d=0^+}|^2_{7/2-k_j,\tau} + \sum_{j=3}^{m'} |E_j w_{|x_d=0^+}|^2_{13/2-m^+-j,\tau}$$
$$= \sum_{j=1}^{2} |\Lambda_{\mathsf{T},\tau}^{7/2-k_j} B_{j,\varphi} w_{|x_d=0^+}|^2_{\partial} + \sum_{j=3}^{m'} |\Lambda_{\mathsf{T},\tau}^{13/2-m^+-j} E_j w_{|x_d=0^+}|^2_{\partial}$$

is a boundary quadratic form of type $(3, 1/2)$ as in Definition 4.2.14. From Proposition 4.2.16 and (4.8.2) we have

$$|\operatorname{tr}(u)|^2_{3,1/2,\tau} \lesssim \sum_{j=1}^{2} |B_{j,\varphi} u_{|x_d=0^+}|^2_{7/2-k_j,\tau} + \sum_{j=3}^{m'} |E_j u_{|x_d=0^+}|^2_{13/2-m^+-j,\tau} + |\operatorname{tr}(v)|^2_{3,-N,\tau}. \tag{4.8.3}$$

for $u = \operatorname{Op}_{\mathsf{T}}(\chi) v$ and $\tau \geq \kappa_0 \sigma$ chosen sufficiently large.

In $\mathscr{U}_0$ one can write

$$p_{\sigma,\varphi} = p^+_{\sigma,\varphi} p^-_{\sigma,\varphi} = \kappa^+_{\varrho^{0\prime}} \kappa^-_{\varrho^{0\prime}},$$

with $\kappa^+_{\varrho^{0\prime}}$ of degree m^+ and $\kappa^-_{\varrho^{0\prime}}$ of degree m^-. In fact we set

$$\tilde{\kappa}^+_{\varrho^{0\prime}}(\varrho') = \prod_{j \in J^+} \big(\xi_d - \tilde{\chi}\rho_j(\varrho')\big), \qquad \tilde{\kappa}^-_{\varrho^{0\prime}}(\varrho') = \prod_{j \in J^-} \big(\xi_d - \tilde{\chi}\rho_j(\varrho')\big),$$

with the notation of Section 4.7.4, thus making the two symbols defined on the whole tangential phase-space. In $\mathscr{U}$, one has also

$$p_{\sigma,\varphi} = \tilde{\kappa}^+_{\varrho^{0\prime}} \tilde{\kappa}^-_{\varrho^{0\prime}}.$$

The factor $\tilde{\kappa}^-_{\varrho^{0\prime}}$ is associated with roots with negative imaginary part. With Lemma 6.1.1 given in Appendix 6.1 one has the following microlocal elliptic estimate

$$\|\operatorname{Op}_{\mathsf{T}}(\chi) w\|_{m^-,\tau} + |\operatorname{tr}(\operatorname{Op}_{\mathsf{T}}(\chi) w)|_{m^--1,1/2,\tau} \lesssim \|\operatorname{Op}_{\mathsf{T}}(\tilde{\kappa}^-_{\varrho^{0\prime}}) \operatorname{Op}_{\mathsf{T}}(\chi) w\|_+ + \|w\|_{m^-,-N,\tau},$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geq \kappa_0 \tau$ chosen sufficiently large. We apply this inequality to $w = \operatorname{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}}) v$. Since

$$\operatorname{Op}_{\mathsf{T}}(\tilde{\kappa}^-_{\varrho^{0\prime}}) \operatorname{Op}_{\mathsf{T}}(\chi) \operatorname{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}}) = \operatorname{Op}_{\mathsf{T}}(\chi) P_{\sigma,\varphi} \mod \Psi^{4,-1}_{\tau},$$

one obtains

$$|\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi)\mathrm{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}})v)|_{m^- -1,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau}.$$

With $[\mathrm{Op}_{\mathsf{T}}(\chi), \mathrm{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}})] \in \Psi_\tau^{m^+,-1}$ one then has

$$|\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}})u)|_{m^- -1,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau},$$

with $u = \mathrm{Op}_{\mathsf{T}}(\chi)v$ as above, using that $m^+ + m^- = 4$. Note that

$$\begin{aligned}
|\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}})u)|_{m^- -1,1/2,\tau} &\asymp \sum_{j=0}^{m^- -1} |D_d^j \mathrm{Op}_{\mathsf{T}}(\tilde{\kappa}^+_{\varrho^{0\prime}})u_{|x_d=0^+}|_{m^- -j-1/2,\tau} \\
&\gtrsim \sum_{j=3}^{m'} |E_j u_{|x_d=0^+}|_{5/2+m^- -j,\tau} - |\operatorname{tr}(v)_{|x_d=0^+}|_{3,-1/2,\tau},
\end{aligned}$$

using that $\xi_d^j \tilde{\kappa}^+_{\varrho^{0\prime}} = \tilde{\chi} e_{j+3,\varrho^{0\prime}}$ in a conic neighborhood of $\operatorname{supp}(\chi)$ and using that $m^- = m' - 2$. We thus obtain

$$\sum_{j=3}^{m'} |E_j u_{|x_d=0^+}|_{13/2-m^+ -j,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau},$$

since $13/2 - m^+ = 5/2 + m^-$. With (4.8.3) then one finds

$$|\operatorname{tr}(u)|_{3,1/2,\tau} \lesssim \sum_{j=1}^{2} |B_{j,\varphi} u_{|x_d=0^+}|^2_{7/2-k_j,\tau} + \|P_{\sigma,\varphi}v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau}.$$

In addition, observing that

$$|B_{j,\varphi} u_{|x_d=0^+}|_{7/2-k_j,\tau} \lesssim |B_{j,\varphi} v_{|x_d=0^+}|_{7/2-k_j,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau},$$

the result of Proposition 4.8.2 follows. $\square$

4.8.2 Proof of Proposition 4.8.1

As mentioned above the proof relies on a patching procedure of microlocal estimates given by Proposition 4.8.2.

Let $0 < \kappa_0' < \kappa_0$. We set

$$\Gamma^{d-1}_{+,\kappa_0} = \{(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty);\ \tau \geq \kappa_0 \sigma, \},$$

and

$$\mathbb{S}^{d-1}_{+,\kappa_0} = \{(\xi', \tau, \sigma) \in \Gamma^{d-1}_{+,\kappa_0};\ |(\xi', \tau, \sigma)| = 1\}.$$

Consider $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{S}^{d-1}_{+,\kappa_0}$. Since the Lopatinskiĭ-Šapiro condition holds at $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$, we can invoke Proposition 4.8.2:

(1) There exists a conic open neighborhood $\mathscr{U}_{\varrho^{0\prime}}$ of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0' \sigma$;

(2) For any $\chi_{\varrho^{0\prime}} \in S^0_{\mathsf{T},\tau}$ homogeneous of degree 0 supported in $\mathscr{U}_{\varrho^{0\prime}}$ the estimate of Proposition 4.8.2 applies to $\mathrm{Op}_{\mathsf{T}}(\chi_{\varrho^{0\prime}})v$ for $\tau \geq \max(\tau_{\varrho^{0\prime}}, \kappa_0\sigma)$.

Without any loss of generality we may choose $\mathscr{U}_{\varrho^{0\prime}}$ of the form $\mathscr{U}_{\varrho^{0\prime}} = \mathscr{O}_{\varrho^{0\prime}} \times \Gamma_{\varrho^{0\prime}}$, with $\mathscr{O}_{\varrho^{0\prime}} \subset W$ an open neighborhood of x^0 and $\Gamma_{\varrho^{0\prime}}$ a conic open neighborhood of $(\xi^{0\prime}, \tau^0, \sigma^0)$ in $\mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0'\sigma$.

Since $\{x^0\} \times \mathbb{S}^{d-1}_{+,\kappa_0}$ is compact we can extract a finite covering of it by open sets of the form of $\mathscr{U}_{\varrho^{0\prime}}$. We denote by $\tilde{\mathscr{U}}_i$, $i \in I$ with $|I| < \infty$, such a finite covering. This is also a finite covering of $\{x^0\} \times \Gamma^{d-1}_{+,\kappa_0}$.

Each $\tilde{\mathscr{U}}_i$ has the form $\tilde{\mathscr{U}}_i = \mathscr{O}_i \times \Gamma_i$, with $\mathscr{O}_i$ an open neighborhood of x^0 and Γ_i a conic open set in $\mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0'\sigma$.

We set $\mathscr{O} = \cap_{i \in I} \mathscr{O}_i$ and $\mathscr{U}_i = \mathscr{O} \times \Gamma_i$, $i \in I$.

Let W^0 be an open neighborhood of x^0 such that $W^0 \Subset \mathscr{O}$. The open sets $\mathscr{U}_i$ give also an open covering of $\overline{W^0} \times \mathbb{S}^{d-1}_{+,\kappa_0}$ and $\overline{W^0} \times \Gamma^{d-1}_{+,\kappa_0}$. With this second covering we associate a partition of unity χ_i, $i \in I$, of $\overline{W^0} \times \mathbb{S}^{d-1}_{+,\kappa_0}$, where each χ_i is chosen smooth and homogeneous of degree one for $|(\xi', \tau, \sigma)| \geq 1$, that is:

$$\sum_{i \in I} \chi_i(\varrho') = 1 \qquad \text{for } \varrho' = (x, \xi', \tau, \sigma) \text{ in a neighborhood of } \overline{W^0} \times \Gamma^{d-1}_{+,\kappa_0}, \text{ and } |(\xi', \tau, \sigma)| \geq 1.$$

Let $u \in \overline{\mathscr{C}}^\infty_c(W_0^+)$. Since each χ_i is in $S^0_{\mathsf{T},\tau}$ and supported in $\mathscr{U}_i$, Proposition 4.8.2 applies:

$$|\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi_i)v)|_{3,1/2,\tau} \leq C_i \Big(\sum_{j=1}^{2} |B_{j,\varphi} v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|P_{\sigma,\varphi} v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau} \Big), \tag{4.8.4}$$

for some $C_i > 0$, for $\sigma \geq 0$, $\tau \geq \max(\tau_i, \kappa_0\sigma)$ for some $\tau_i > 0$.

We set $\tilde{\chi} = 1 - \sum_{i \in I} \chi_i$. One has $\tilde{\chi} \in S^{-\infty}_{\mathsf{T},\tau}$ microlocally in a neighborhood of $\overline{W^0} \times \Gamma^{d-1}_{+,\kappa_0}$. Thus, considering the definition of $\Gamma^{d-1}_{+,\kappa_0}$, if one imposes $\tau \geq \kappa_0\sigma$, as we do, then $\tilde{\chi} \in S^{-\infty}_{\mathsf{T},\tau}$ locally in a neighborhood of $\overline{W^0}$.

For any $N \in \mathbb{N}$ using that $\operatorname{supp}(v) \subset W^0$ one has

$$
\begin{aligned}
|\operatorname{tr}(v)|_{3,1/2,\tau} &\leq \sum_{i \in I} |\operatorname{tr}(\operatorname{Op}_\mathsf{T}(\chi_i)v)|_{3,1/2,\tau} + |\operatorname{tr}(\operatorname{Op}_\mathsf{T}(\tilde{\chi})v)|_{3,1/2,\tau} \\
&\lesssim \sum_{i \in I} |\operatorname{tr}(\operatorname{Op}_\mathsf{T}(\chi_i)v)|_{3,1/2,\tau} + |\operatorname{tr}(v)|_{3,-N,\tau} \\
&\lesssim \sum_{i \in I} |\operatorname{tr}(\operatorname{Op}_\mathsf{T}(\chi_i)v)|_{3,1/2,\tau} + \|v\|_{4,-N,\tau}.
\end{aligned}
$$

Summing estimates (4.8.4) together for $i \in I$ we thus obtain

$$
|\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \sum_{j=1}^{2} |B_{j,\varphi} v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|P_{\sigma,\varphi} v\|_+ + \|v\|_{4,-1,\tau} + |\operatorname{tr}(v)|_{3,-1/2,\tau},
$$

for $\tau \geq \max(\max_i \tau_i, \kappa_0 \sigma)$. Therefore, by choosing $\tau \geq \kappa_0 \sigma$ sufficiently large one obtains the result of Proposition 4.8.1. $\square$

4.9 Microlocal estimate for each second-order factors composing $P_{\sigma,\varphi}$

We recall that $P_\sigma = \Delta^2 - \sigma^4 = (-\Delta - \sigma^2)(-\Delta + \sigma^2)$ with $\sigma \geq 0$. Set $Q_\sigma^j = -\Delta + (-1)^j \sigma^2$; then $P_\sigma = Q_\sigma^1 Q_\sigma^2$. We also set $Q = -\Delta$, that is, $Q = Q_0^1 = Q_0^2$.

The principal symbols of Q_σ^j and Q are given by

$$
q_\sigma^j(x,\xi) = \xi_d^2 + r(x,\xi') + (-1)^j \sigma^2 \quad \text{and} \quad q(x,\xi) = \xi_d^2 + r(x,\xi'), \tag{4.9.1}
$$

respectively. The conjugated operator $P_{\sigma,\varphi} = e^{\tau\varphi} P_\sigma e^{-\tau}$ reads

$$
P_{\sigma,\varphi} = Q_{\sigma,\varphi}^1 Q_{\sigma,\varphi}^2, \quad \text{with} \quad Q_{\sigma,\varphi}^j = e^{\tau\varphi} Q_\sigma^j e^{-\tau\varphi}.
$$

We set

$$
Q_s^j = \frac{Q_{\sigma,\varphi}^j + (Q_{\sigma,\varphi}^j)^*}{2} \quad \text{and} \quad Q_a = \frac{Q_{\sigma,\varphi}^j - (Q_{\sigma,\varphi}^j)^*}{2i},
$$

both formally selfadjoint and such that $Q_{\sigma,\varphi}^j = Q_s^j + iQ_a$. Note that Q_a is independent of σ. Their respective principal symbols are

$$
\begin{aligned}
&q_s^j(x,\xi,\tau,\sigma) = \xi_d^2 - (\tau \partial_d \varphi)^2 + r(x,\xi') + (-1)^j \sigma^2 - \tau^2 r(x, d_{x'}\varphi), \\
&q_a(x,\xi,\tau) = 2\tau \xi_d \partial_d \varphi + 2\tau \tilde{r}(x,\xi', d_{x'}\varphi).
\end{aligned}
$$

Note that Q_s^j and Q_a take the forms

$$Q_s^j = D_d^2 + T_s^j, \qquad Q_a = \tau(\partial_d\varphi D_d + D_d\partial_d\varphi) + T_a, \tag{4.9.2}$$

where $T_s^j, T_a \in \mathscr{D}^2_{\mathsf{T},\tau}$ are such that $(T_s^j)^* = T_s^j$ and $T_a^* = T_a$. Naturally, the principal symbol of $Q^j_{\sigma,\varphi}$ is

$$q^j_{\sigma,\varphi}(x,\xi,\tau) = q^j_s(x,\xi,\tau,\sigma) + iq_a(x,\xi,\tau).$$

The principal symbol of $Q^j_{\sigma,\varphi} = e^{\tau\varphi}Q^j_\sigma e^{-\tau\varphi}$ is

$$q^j_{\sigma,\varphi}(x,\xi,\tau,\sigma) = (\xi_d + i\tau\partial_d\varphi)^2 + r(x,\xi') + (-1)^j\sigma^2 - \tau^2 r(x,d_{x'}\varphi) + 2i\tau\tilde{r}(x,\xi',d_{x'}\varphi).$$

As in Section 4.7.2 we let $\alpha_j \in \mathbb{C}$ be such that

$$\begin{aligned}\alpha_j(x,\xi',\tau,\sigma)^2 &= r(x,\xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2\\ &= r(x,\xi') - \tau^2 r(x,d_{x'}\varphi) + 2i\tau\tilde{r}(x,\xi',d_{x'}\varphi) + (-1)^j\sigma^2,\end{aligned}$$

and $\operatorname{Re}\alpha_j \geq 0$. Note that uniqueness in the choice of α_j holds except if $r(x,\xi' + i\tau d_{x'}\varphi) + (-1)^j\sigma^2 \in \mathbb{R}^-$; this lack of uniqueness in that case is however not an issue in what follows. One has

$$\begin{aligned}q^j_{\sigma,\varphi}(x,\xi',\xi_d,\tau) &= (\xi_d + i\tau\partial_d\varphi)^2 + \alpha_j(x,\xi',\tau,\sigma)^2\\ &= \big(\xi_d + i\tau\partial_d\varphi + i\alpha_j(x,\xi',\tau,\sigma)\big)\big(\xi_d + i\tau\partial_d\varphi - i\alpha_j(x,\xi',\tau,\sigma)\big).\end{aligned}$$

We recall from (4.7.4) that we write $q^j_{\sigma,\varphi}(x,\xi',\xi_d,\tau) = (\xi_d - \pi_{j,1})(\xi_d - \pi_{j,2})$ with

$$\pi_{j,1} = -i\tau\partial_d\varphi - i\alpha_j(x,\xi',\tau,\sigma) \quad\text{and}\quad \pi_{j,2} = -i\tau\partial_d\varphi + i\alpha_j(x,\xi',\tau,\sigma).$$

The roots $\pi_{j,k}$, $k = 1,2$ are functions of x, ξ', τ and σ.

We denote by B a boundary operator of order k that takes the form

$$B(x,D) = B^k(x,D') + B^{k-1}(x,D')D_d,$$

with $B^k(x,D')$ and $B^{k-1}(x,D')$ tangential differential operators of order k and $k-1$ respectively. The boundary operator $B(x,D)$ has $b(x,\xi) = b^k(x,\xi') + b^{k-1}(x,\xi')\xi_d$ for principal symbol. The conjugate boundary operator $B_\varphi = e^{\tau\varphi}Be^{-\tau\varphi}$ is then given by

$$\begin{aligned}B_\varphi(x,D,\tau) &= B^k_\varphi(x,D',\tau) + B^{k-1}_\varphi(x,D',\tau)(D_d + i\tau\partial_d\varphi)\\ &= \hat{B}^k_\varphi(x,D',\tau) + B^{k-1}_\varphi(x,D',\tau)D_d,\end{aligned}$$

with $\hat{B}^k_\varphi(x, D', \tau) = B^k_\varphi(x, D', \tau) + i\tau B^{k-1}_\varphi(x, D', \tau)\partial_d\varphi$. The principal symbol of $B_\varphi(x, D, \tau)$ is

$$b_\varphi(x, \xi, \tau) = \hat{b}^k_\varphi(x, \xi', \tau) + b^{k-1}_\varphi(x, \xi', \tau)\xi_d,$$

where $b^{k-1}_\varphi(x, \xi', \tau)$ is homogeneous of degree $k-1$ in $\lambda_{\mathsf{T},\tau}$ and $\hat{b}^k_\varphi(x, \xi', \tau) = b^k_\varphi(x, \xi', \tau) + \tau b^{k-1}_\varphi(x, \xi', \tau)\partial_d\varphi$ is homogeneous of degree k in $\lambda_{\mathsf{T},\tau}$.

4.9.1 Sub-ellipticity

Set

$$q_s(x, \xi, \tau) = \xi_d^2 + r(x, \xi') - (\tau\partial_d\varphi)^2 - r(x, \tau d_{x'}\varphi) = |\xi|_x^2 - |\tau d\varphi|_x^2,$$

where $|\xi|_x^2 = \xi_d^2 + r(x, \xi')$. One has $q_s^j = q_s + (-1)^j\sigma^2$. Observe that $\{q_s^j, q_a\} = \{q_s, q_a\}$.

Definition 4.9.1 (Sub-ellipticity)**.** Let W be a bounded open subset of $\mathbb{R}^d$ and $\varphi \in \mathscr{C}^\infty(\overline{W})$ such that $|d_x\varphi| > 0$. Let $j = 1$ or 2. We say that the couple (Q^j_σ, φ) satisfies the sub-ellipticity condition in $\overline{W}$ if there exist $C > 0$ and $\tau_0 > 0$ such that for $\sigma > 0$

$$\forall\ (x, \xi) \in \overline{W} \times \mathbb{R}^d,\ \tau \geq \tau_0\sigma,\ q^j_{\sigma,\varphi}(x, \xi, \tau) = 0 \Rightarrow \{q_s^j, q_a\}(x, \xi, \tau) = \{q_s, q_a\}(x, \xi, \tau) \geq C > 0.$$

Remark 4.9.2. Note that with homogeneity the sub-ellipticity property also reads

$$\forall\ (x, \xi) \in \overline{W} \times \mathbb{R}^d,\ \tau \geq \tau_0\sigma,\ q^j_{\sigma,\varphi}(x, \xi, \tau) = 0 \Rightarrow \{q_s^j, q_a\}(x, \xi, \tau) \geq C\lambda_\tau^3.$$

Proposition 4.9.3. *Let W be a bounded open subset of $\mathbb{R}^d$ and $\psi \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $\psi \geq 0$ and $|d_x\psi| \geq C > 0$ on $\overline{W}$. Let $\tau_0 > 0$. Then, there exists $\gamma_0 \geq 1$ such that (Q^j_σ, φ) satisfies the sub-ellipticity condition on $\overline{W}$ for $\tau \geq \tau_0\sigma$ for $\varphi = e^{\gamma\psi}$, with $\gamma \geq \gamma_0$, for both $j = 1$ and 2.*

Proof. We note that $|d_x\varphi(x)| \neq 0$.

The proof is slightly different whether one considers the symbol $q^1_{\sigma,\varphi}$ or the symbol $q^2_{\sigma,\varphi}$.

Case 1: proof for $\boldsymbol{q^1_{\sigma,\varphi}}$. Assume that $q^1_{\sigma,\varphi} = 0$. Thus $|\xi|_x^2 - |\tau d\varphi|_x^2 - \sigma^2 = 0$ implying $|\xi| \sim \sigma + \gamma\tau\varphi$. On the one hand by Lemma 3.55 in [62], one has

$$\begin{aligned}\{q_s, q_a\}(x, \xi, \tau) = \tau(\gamma^2\varphi)(\gamma\varphi)^2 \left((H_q\psi(x, \beta))^2 + 4\tau^2 q(x, d\psi(x))^2\right) \\ + (\gamma\varphi)^3 \frac{1}{2i}\{\overline{q_\psi}, q_\psi\}(x, \beta, \tau),\end{aligned} \tag{4.9.3}$$

with $\beta = \xi/(\gamma\varphi)$, and where H_q denotes the Hamiltonian vector field associated with the symbol q as defined in (4.9.1). Here, q_ψ denotes the principal symbol of $e^{\tau\psi}Qe^{-\tau\psi}$, that is,

$$q_\psi(x,\xi,\tau) = q(x,\xi + i\tau d_x\psi(x)) = (\xi_d + i\tau\partial_d\psi(x))^2 + r(x,\xi' + i\tau d_{x'}\psi(x)).$$

On the other hand, one has $(H_q\psi(x,\beta))^2 + 4\tau^2 q(x,d\psi(x))^2 \gtrsim \tau^2$ and since $\frac{1}{2i}\{\overline{q_\psi},q_\psi\}(x,\beta,\tau)$ is homogeneous of degree 3 in (β,τ), we obtain

$$\{q_s,q_a\} \geq C\gamma(\gamma\tau\varphi)^3 - C'(\gamma\tau\varphi + |\beta|\gamma\varphi)^3 = C\gamma\tilde\tau^3 - C''(\tilde\tau + |\xi|)^3, \quad \text{with } \tilde\tau = \gamma\tau\varphi.$$

Yet one has $|\xi| \sim \sigma + \tilde\tau$ implying

$$\{q_s,q_a\} \geq C\gamma\tilde\tau^3 - C'''(\tilde\tau + |\xi|)^3 \geq C\gamma\tilde\tau^3 - C^{(4)}(\tilde\tau + \sigma)^3.$$

Since $\psi \geq 0$ and $\gamma \geq 1$ one has $\varphi \geq 1$ implying $\tau_0\sigma \leq \tau \lesssim \tilde\tau$ and thus

$$\{q_s,q_a\}(x,\xi,\tau) \geq \tilde\tau^3(C\gamma - C^{(5)}).$$

It follows that for γ chosen sufficiently large one finds $\{q_s,q_a\}(x,\xi,\tau) \geq C > 0$.

Case 2: proof for $q^2_{\sigma,\varphi}$. Assume that $q^2_{\sigma,\varphi} = 0$. Then $|\xi|_x^2 + \sigma^2 = |\tau d\varphi|_x^2$ implying $|\xi| + \sigma \sim \tau|d\varphi| \sim \tilde\tau$. The same computation as in Case 1 gives

$$\{q_s,q_a\}(x,\xi,\tau) \geq C\gamma\tilde\tau^3 - C'(\tilde\tau + |\xi|)^3$$

Here $|\xi| + \tilde\tau \lesssim \tilde\tau$ yielding

$$\{q_s,q_a\}(x,\xi,\tau) \geq (C\gamma - C'')\tilde\tau^3.$$

The remaining part of the proof is the same. □

Lemma 4.9.4. *Let $j = 1$ or 2. Let (Q^j_σ,φ) have the sub-ellipticity property of Definition 4.9.1 in $\overline{W}$. For $\mu > 0$ one sets $t(\varrho) = \mu((q^j_s)^2 + q^2_a)(\varrho) + \tau\{q^j_s,q_a\}(\varrho)$ with $\varrho = (x,\xi,\tau,\sigma) \in \overline{W} \times \mathbb{R}^d \times [0,\infty) \times [0,\infty)$. Let $\tau_0 > 0$. Then, for μ chosen sufficiently large and $\tau \geq \tau_0\sigma$ one has $t(\varrho) \geq C\lambda_\tau^4$ for some $C > 0$.*

The proof of Lemma 4.9.4 uses the following lemma.

Lemma 4.9.5. *Consider two continuous functions, f and g, defined in a compact set $\mathscr{L}$, and assume that $f \geq 0$ and moreover*

$$f(y) = 0 \Rightarrow g(y) > 0 \quad \textit{for all} \quad y \in \mathscr{L}.$$

Setting $h_\mu = \mu f + g$, we have $h_\mu \geq C > 0$ for $\mu > 0$ chosen sufficiently large.

Proof of Lemma 4.9.4. Consider the compact set

$$\mathscr{L} = \{(x, \xi, \tau, \sigma);\ x \in \overline{W},\ |\xi|^2 + \tau^2 + \sigma^2 = 1,\ \tau \geq \tau_0 \sigma\}.$$

Applying the result of Lemma 4.9.5 to $t(\varrho)$ on $\mathscr{L}$ with $f = (q_s^j)^2 + q_a^2$ and $g = \tau\{q_s^j, q_a\}$ we find for $t(\varrho) \geq C$ on $\mathscr{L}$ for some $C > 0$ for μ chosen sufficiently large. Since $t(\varrho)$ is homogeneous of degree 4 in the variables (ξ, τ, σ) it follows that $t(\varrho) \geq C(\sigma^2 + \tau^2 + |\xi|^2)^4 \gtrsim \lambda_\tau^4$. □

4.9.2 Lopatinskiĭ-Šapiro condition for the second-order factors

Above, in Section 4.7, the Lopatinskiĭ-Šapiro condition is addressed for the fourth-order operator $P_{\sigma,\varphi}$. Here, we consider the two second-order factors $Q^j_{\sigma,\varphi}$.

With the roots $\pi_{j,1}$ and $\pi_{j,2}$ defined in (4.7.4) one sets

$$q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = \prod_{\substack{\operatorname{Im} \pi_{j,k} \geq 0 \\ k=1,2}} \big(\xi_d - \pi_{j,k}(x, \xi', \tau, \sigma)\big).$$

Definition 4.9.6. Let $j = 1, 2$. Let $x \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$. Let $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ with $(\xi', \tau, \sigma) \neq 0$. One says that the Lopatinskiĭ-Šapiro condition holds for (Q^j_σ, B, φ) at $\varrho' = (x, \xi', \tau, \sigma)$ if for any polynomial function $f(\xi_d)$ with complex coefficients there exist $c \in \mathbb{C}$ and a polynomial function $\ell(\xi_d)$ with complex coefficients such that, for all $\xi_d \in \mathbb{C}$

$$f(\xi_d) = c b_\varphi(x, \xi', \xi_d, \tau) + \ell(\xi_d) q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau). \tag{4.9.4}$$

Remark 4.9.7. With the Euclidean division of polynomials, we see that it suffices to consider the polynomial function $f(\xi_d)$ to be of degree less than that of $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau)$ in (4.9.4). Thus, in any case, the degree of $f(\xi_d)$ can be chosen less than or equal to one.

Lemma 4.9.8. *Let $j = 1$ or 2. Let $x \in \partial\Omega$ and $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ with $(\xi', \tau, \sigma) \neq 0$. The Lopatinskiĭ-Šapiro condition holds for (Q^j_σ, B, φ) at (x, ξ', τ, σ) if and only if*

i. either $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = 1$;

ii. or $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = \xi_d - \pi$ and $b_\varphi(x, \xi', \pi, \tau) \neq 0$.

Proof. If $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = (\xi_d - \pi_{j,1})(\xi_d - \pi_{j,2})$, that is, both roots $\pi_{j,1}$ and $\pi_{j,2}$ are in the upper complex half-plane, then condition (4.9.4) cannot hold, since by Remark 4.9.7 it means that the vector space of polynomials of degree less than or equal to one would be generated by the single polynomial $b_\varphi(x, \xi', \xi_d, \tau)$. Suppose that $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = \xi_d - \pi$ that is one the root $\pi_{j,1}$ and $\pi_{j,2}$ has a nonnegative imaginary part and the other root has a negative imaginary part. Then, the Lopatinskiĭ-Šapiro condition holds at (x, ξ', σ, τ) if for any $f(\xi_d)$, the polynomial function $\xi_d \mapsto f(\xi_d) - cb_\varphi(x, \xi', \xi_d, \tau)$ admits π as a root for some $c \in \mathbb{C}$. A necessary and sufficient condition is then $b_\varphi(x, \xi', \xi_d = \pi, \tau) \neq 0$. Finally if $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = 1$, that is, both roots $\pi_{j,1}$ and $\pi_{j,2}$ lie in the lower complex half-plane, then condition (4.9.4) trivially holds. $\square$

4.9.3 Microlocal estimates for a second-order factor

Here, for $j = 1$ or 2, we establish estimates for the operator Q^j_σ in a microlocal neighborhood of point at the boundary where (Q^j_σ, B, φ) satisfies the Lopatinskiĭ-Šapiro condition (after conjugaison) of Definition 4.9.6.

The quality of the estimation depends on the position of the roots. We shall assume that $\partial_d\varphi > 0$. Thus, from the form of the roots $\pi_{j,1}$ and $\pi_{j,2}$ given in (4.7.4), the root $\pi_{j,1}$ always lies in the lower complex half-plane. The sign of $\operatorname{Im}\pi_{j,2}$ may however vary. Three cases can thus occur:

i. The root $\pi_{j,2}$ at the considered point lies in the upper complex half-plane.

ii. The root $\pi_{j,2}$ at the considered point is real.

iii. The root $\pi_{j,2}$ at the considered point lies in the lower complex half-plane.

Proposition 4.9.9. *Let $j = 1$ or 2 and $\kappa_1 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d\varphi \geq C > 0$ in W and such that (Q^j_σ, φ) satisfies the sub-ellipticity condition in $\overline{W}$. Let $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$ with $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ nonvanishing with $\tau^0 \geq \kappa_1\sigma^0$ and such that (Q^j_σ, B, φ) satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.9.6 at $\varrho^{0\prime}$.*

i. Assume that $\operatorname{Im}\pi_{j,2}(\varrho^{0\prime}) > 0$. Then, there exists $\mathscr{U}$ a conic neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0\sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$\|\operatorname{Op}_{\mathsf{T}}(\chi)v\|_{2,\tau} + |\operatorname{tr}(\operatorname{Op}_{\mathsf{T}}(\chi)v)|_{1,1/2,\tau} \leq C\big(\|Q^j_{\sigma,\varphi}v\|_+ + |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}\big), \tag{4.9.5}$$

for $\sigma \geq 0$, $\tau \geq \max(\tau_0, \kappa_0\sigma)$ and $v \in \overline{\mathscr{C}}^\infty_c(W_+)$.

ii. Assume that $\operatorname{Im}\pi_{j,2}(\varrho^{0\prime}) = 0$. Then, there exists $\mathscr{U}$ a conic neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0\sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, *homogeneous of degree* 0 *in* (ξ', τ, σ) *with* $\operatorname{supp}(\chi) \subset \mathscr{U}$, *there exist* $C > 0$ *and* $\tau_0 > 0$ *such that*

$$\tau^{-1/2} \|\mathrm{Op}_{\mathsf{T}}(\chi) v\|_{2,\tau} + |\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi) v)|_{1,1/2,\tau} \leq C\big(\|Q^j_{\sigma,\varphi} v\|_+ + |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}\big), \tag{4.9.6}$$

for $\sigma \geq 0$, $\tau \geq \max(\tau_0, \kappa_0 \sigma)$ *and* $v \in \overline{\mathscr{C}}^\infty_c(W_+)$.

iii. *Assume that* $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) < 0$. *Then, there exists* $\mathscr{U}$ *a conic neighborhood of* $\varrho^{0\prime}$ *in* $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ *where* $\tau \geq \kappa_0 \sigma$ *such that if* $\chi \in S^0_{\mathsf{T},\tau}$, *homogeneous of degree* 0 *in* (ξ', τ, σ) *with* $\operatorname{supp}(\chi) \subset \mathscr{U}$, *there exist* $C > 0$ *and* $\tau_0 > 0$ *such that*

$$\|\mathrm{Op}_{\mathsf{T}}(\chi) v\|_{2,\tau} + |\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi) v)|_{1,1/2,\tau} \leq C\big(\|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{2,-1,\tau}\big), \tag{4.9.7}$$

for $\sigma \geq 0$, $\tau \geq \max(\tau_0, \kappa_0 \sigma)$ *and* $v \in \overline{\mathscr{C}}^\infty_c(W_+)$.

The notation of the function space $\overline{\mathscr{C}}^\infty_c(W_+)$ is introduced in (4.5.3).

4.9.3.1 Case (i): one root lying in the upper complex half-plane.

One has $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) > 0$ and $\operatorname{Im} \pi_{j,1}(\varrho^{0\prime}) < 0$.

Since the Lopatinskiĭ-Šapiro condition holds for (Q^j_σ, B, φ) at $\varrho^{0\prime}$, by Lemma 4.9.8 one has

$$b_\varphi(x^0, \xi^{0\prime}, \xi_d = \pi_{j,2}(\varrho^{0\prime}), \tau^0) = b\big(x^0, \xi^{0\prime} + i\tau^0 d_{x'}\varphi(x^0), i\alpha_j(\varrho^{0\prime})\big) \neq 0.$$

As the roots $\pi_{j,1}$ and $\pi_{j,2}$ are locally smooth with respect to $\varrho' = (x, \xi', \tau, \sigma)$ and homogeneous of degree one in (ξ', τ, σ), there exists $\mathscr{U}$ a conic neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ and $C_1 > 0$, $C_2 > 0$ such that $\mathbb{S}_{\overline{\mathscr{U}}} = \{\varrho' \in \overline{\mathscr{U}};\, |\xi'|^2 + \tau^2 + \sigma^2 = 1\}$ is compact and

$$\tau \geq \kappa_0 \sigma, \quad \operatorname{Im} \pi_{j,2}(\varrho') \geq C_2 \lambda_{\mathsf{T},\tau}, \quad \text{and } \operatorname{Im} \pi_{j,1}(\varrho') \leq -C_1 \lambda_{\mathsf{T},\tau},$$

and

$$b_\varphi(x, \xi', \xi_d = \pi_{j,2}(\varrho'), \tau) \neq 0, \tag{4.9.8}$$

if $\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}$.

We let $\chi \in S^0_{\mathsf{T},\tau}$ and $\tilde{\chi} \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree zero in the variable (ξ', τ, σ) and be such that $\operatorname{supp}(\tilde{\chi}) \subset \mathscr{U}$ and $\tilde{\chi} \equiv 1$ on a neighborhood of $\operatorname{supp}(\chi)$. From the smoothness and the homogeneity of the roots, one has $\tilde{\chi}\pi_{j,k} \in S^1_{\mathsf{T},\tau}$, $k = 1, 2$. We set

$$L_2 = D_d - \mathrm{Op}_{\mathsf{T}}(\tilde{\chi}\pi_{j,2}) \quad \text{and } L_1 = D_d - \mathrm{Op}_{\mathsf{T}}(\tilde{\chi}\pi_{j,1}).$$

The proof of Estimate (4.9.6) is based on three lemmata that we now list. Their proofs are given at the end of this section.

The following lemma provides an estimate for L_2 and boundary traces.

Lemma 4.9.10. *There exist $C > 0$ and $\tau_0 > 0$ such that for any $N \in \mathbb{N}$, there exists $C_N > 0$ such that*

$$\big|\operatorname{tr}\big(\mathrm{Op}_{\mathsf{T}}(\chi)w\big)\big|_{1,1/2,\tau} \leq C\big(\big|B_\varphi \mathrm{Op}_{\mathsf{T}}(\chi)w_{|x_d=0^+}\big|_{3/2-k,\tau}+\big|L_2\mathrm{Op}_{\mathsf{T}}(\chi)w_{|x_d=0^+}\big|_{1/2,\tau}\big)+C_N|\operatorname{tr}(w)|_{1,-N,\tau},$$

for $\tau \geq \max(\tau_0, \kappa_0\sigma)$ and $w \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$.

The proof of Lemma 4.9.10 relies on the Lopatinskiĭ-Šapiro condition.

The following lemma gives an estimate for L_1.

Lemma 4.9.11. *Let $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, be such that $\operatorname{supp}(\chi) \subset \mathscr{U}$ and $s \in \mathbb{R}$. There exist $C > 0$, $\tau_0 > 0$ and $N \in \mathbb{N}$ such that*

$$\|\mathrm{Op}_{\mathsf{T}}(\chi)w\|_{1,s,\tau} + |\mathrm{Op}_{\mathsf{T}}(\chi)w_{|x_d=0^+}|_{s+1/2,\tau} \leq C\big(\|L_1\mathrm{Op}_{\mathsf{T}}(\chi)w\|_{0,s,\tau} + \|w\|_{0,-N,\tau}\big),$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geq \max(\tau_0, \kappa_0\sigma)$.

The proof of Lemma 4.9.11 is based on a multiplier method and relies on the fact that the root $\pi_{j,1}$ that appears in the principal symbol of L_1 lies in the lower complex half-plane.

The following lemma gives an estimate for L_2.

Lemma 4.9.12. *Let $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, be such that $\operatorname{supp}(\chi) \subset \mathscr{U}$ and $s \in \mathbb{R}$. There exist $C > 0$, $\tau_0 > 0$ and $N \in \mathbb{N}$ such that*

$$\|\mathrm{Op}_{\mathsf{T}}(\chi)w\|_{1,s,\tau} \leq C\big(\|L_2\mathrm{Op}_{\mathsf{T}}(\chi)w\|_{0,s,\tau} + |\mathrm{Op}_{\mathsf{T}}(\chi)w_{|x_d=0^+}|_{s+1/2,\tau} + \|w\|_{0,-N,\tau}\big),$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ and $\tau \geq \max(\tau_0, \kappa_0\sigma)$.

Note that this estimate is weaker than that of Lemma 4.9.11

Observing that

$$\begin{aligned} L_1\mathrm{Op}_{\mathsf{T}}(\chi)L_2 &= \mathrm{Op}_{\mathsf{T}}(\chi)L_1L_2 \mod \Psi^{1,0}_\tau \\ &= \mathrm{Op}_{\mathsf{T}}(\chi)Q^j_{\sigma,\varphi} \mod \Psi^{1,0}_\tau, \end{aligned}$$

and applying Lemma 4.9.11 to $w = L_2 v$ with $s = 0$, one obtains

$$\begin{aligned} \|\mathrm{Op}_{\mathsf{T}}(\chi)L_2v\|_{1,\tau} + |\mathrm{Op}_{\mathsf{T}}(\chi)L_2v_{|x_d=0^+}|_{1/2,\tau} &\lesssim \|L_1\mathrm{Op}_{\mathsf{T}}(\chi)L_2v\|_+ + \|v\|_{1,-N,\tau} \\ &\lesssim \|Q^j_{\sigma,\varphi}v\|_+ + \|v\|_{1,\tau}, \end{aligned}$$

for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large. We set $u = \operatorname{Op}_{\mathsf{T}}(\chi)v$, and using the trace inequality

$$|w_{|x_d=0^+}|_{s,\tau} \lesssim \|w\|_{s+1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d_+) \text{ and } s > 0,$$

we have

$$\begin{aligned}\|L_2 u\|_{1,\tau} + |L_2 u_{|x_d=0^+}|_{1/2,\tau} &\lesssim \|\operatorname{Op}_{\mathsf{T}}(\chi) L_2 v\|_{1,\tau} + |\operatorname{Op}_{\mathsf{T}}(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} + \|v\|_{1,\tau} + |v_{|x_d=0^+}|_{1/2,\tau} \\ &\lesssim \|\operatorname{Op}_{\mathsf{T}}(\chi) L_2 v\|_{1,\tau} + |\operatorname{Op}_{\mathsf{T}}(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} + \|v\|_{1,\tau}.\end{aligned}$$

Therefore, we obtain

$$\|L_2 u\|_{1,\tau} + |L_2 u_{|x_d=0^+}|_{1/2,\tau} \lesssim \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{1,\tau}.$$

With Lemma 4.9.10, one has the estimate

$$\big|\operatorname{tr}(u)\big|_{1,1/2,\tau} + \|L_2 u\|_{1,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{2,-1,\tau},$$

for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large using the following trace inequality

$$|\operatorname{tr}(w)|_{m,s,\tau} \lesssim \|w\|_{m+1,s-1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d_+) \text{ and } m \in \mathbb{N},\ s \in \mathbb{R}.$$

With Lemma 4.9.12 for $s = 1$ one obtains

$$\|u\|_{1,1,\tau} + \big|\operatorname{tr}(u)\big|_{1,1/2,\tau} + \|L_2 u\|_{1,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{2,-1,\tau},$$

for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large. Finally, we write

$$\|D_d u\|_{1,\tau} \leq \|L_2 u\|_{1,\tau} + \|\operatorname{Op}_{\mathsf{T}}(\tilde{\chi}\pi_{j,2})u\|_{1,\tau} \lesssim \|L_2 u\|_{1,\tau} + \|u\|_{1,1,\tau},$$

yielding

$$\|u\|_{2,\tau} + \big|\operatorname{tr}(u)\big|_{1,1/2,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{2,-1,\tau}.$$

As $u = \operatorname{Op}_{\mathsf{T}}(\chi)v$, with a commutator argument we obtain

$$\begin{aligned}|B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} &\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + |\operatorname{tr}(v)|_{1,-1/2,\tau} \\ &\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}.\end{aligned}$$

yielding (4.9.5) and thus concluding the proof of Proposition 4.9.9 in Case (i).
$\square$

We now provide the proofs the three key lemmata used above.

Proof of Lemma 4.9.10. Set

$$\mathcal{T}(w) = |B_\varphi w_{|x_d=0^+}|^2_{3/2-k,\tau} + |L_2 w_{|x_d=0^+}|^2_{1/2,\tau} = |\Lambda^{3/2-k}_{\mathsf{T},\tau} B_\varphi w_{|x_d=0^+}|^2_\partial + |\Lambda^{1/2}_{\mathsf{T},\tau} L_2 w_{|x_d=0^+}|^2_\partial.$$

This is a boundary differential quadratic form of type $(1, 1/2)$ in the sense of Definition 4.2.14. The associated bilinear symbol is given by

$$\begin{aligned}\Sigma_{\mathcal{T}}(\varrho', \mathbf{z}, \mathbf{z}') = {}& \lambda^{3-2k}_{\mathsf{T},\tau}\big(\hat{b}^k_\varphi(x,\xi',\tau) z_0 + b^{k-1}_\varphi(x,\xi',\tau) z_1\big)\big(\overline{\hat{b}^k_\varphi}(x,\xi',\tau)\bar{z}'_0 + \overline{b^{k-1}_\varphi}(x,\xi',\tau)\bar{z}'_1\big)\\ &+ \lambda_{\mathsf{T},\tau}\big(z_1 - \tilde{\chi}\pi_{j,2}(\varrho') z_0\big)\big(\bar{z}'_1 - \tilde{\chi}\overline{\pi_{j,2}(\varrho')}\bar{z}'_0\big),\end{aligned}$$

with $\mathbf{z} = (z_0, z_1) \in \mathbb{C}^2$ and $\mathbf{z}' = (z'_0, z'_1) \in \mathbb{C}^2$, yielding

$$\Sigma_{\mathcal{T}}(\varrho', \mathbf{z}, \mathbf{z}) = \lambda^{3-2k}_{\mathsf{T},\tau}\big|\hat{b}^k_\varphi(x,\xi',\tau) z_0 + b^{k-1}_\varphi(x,\xi',\tau) z_1\big|^2 + \lambda_{\mathsf{T},\tau}\big|z_1 - \tilde{\chi}\pi_{j,2}(\varrho') z_0\big|^2.$$

One has $\Sigma_{\mathcal{T}}(\varrho', \mathbf{z}, \mathbf{z}) \geq 0$. For $\mathbf{z} \neq (0,0)$ if $\Sigma_{\mathcal{T}}(\varrho', \mathbf{z}, \mathbf{z}) = 0$ then

$$\begin{cases} z_1 = \tilde{\chi}\pi_{j,2}(\varrho') z_0,\\ \hat{b}^k_\varphi(x,\xi',\tau) z_0 + b^{k-1}_\varphi(x,\xi',\tau) z_1 = 0,\end{cases}$$

implying that $z_0 \neq 0$ and

$$b_\varphi\big(x,\xi',\xi_d = \tilde{\chi}\pi_{j,2}(\varrho'),\tau\big) = \hat{b}^k_\varphi(x,\xi',\tau) + b^{k-1}_\varphi(x,\xi',\tau)\tilde{\chi}\pi_{j,2}(\varrho') = 0.$$

Let $\mathscr{U}_1 \subset \mathscr{U}$ be a conic open set such that $\operatorname{supp}(\chi) \subset \mathscr{U}_1$ and $\tilde{\chi} = 1$ in a conic neighborhood of $\overline{\mathscr{U}_1}$. Then, for $\varrho' \in \overline{\mathscr{U}_1}$ one has

$$b_\varphi\big(x,\xi',\xi_d = \tilde{\chi}\pi_{j,2}(\varrho'),\tau\big) = b_\varphi\big(x,\xi',\xi_d = \pi_{j,2}(\varrho'),\tau\big) \neq 0,$$

by (4.9.8). From the homogeneity of $b^{k-1}_\varphi(x,\xi',\tau)$ and $\hat{b}^k_\varphi(x,\xi',\tau)$ in ϱ', it follows that there exists some $C > 0$ such that

$$\Sigma_{\mathcal{T}}(\varrho', \mathbf{z}, \mathbf{z}) \geq C\big(\lambda^3_{\mathsf{T},\tau}|z_0|^2 + \lambda_{\mathsf{T},\tau}|z_1|^2\big),$$

if $\varrho' \in \mathscr{U}_1$. The result of Lemma 4.9.10 thus follows from Proposition 4.2.16, having in mind what is exposed in Section 4.5.4 since we have $\tau \geq \kappa_0\sigma$ here. $\square$

Proof of Lemma 4.9.11. We let $u = \operatorname{Op}_{\mathsf{T}}(\chi) w$. Performing an integration by parts, one has

$$\begin{aligned}2\operatorname{Re}\big(L_1 u, i\Lambda^{2s+1}_{\mathsf{T},\tau} u\big)_+ &= 2\operatorname{Re}\big(\big(D_d - \operatorname{Op}_{\mathsf{T}}(\tilde{\chi}\pi_{j,1})\big)u, i\Lambda^{2s+1}_{\mathsf{T},\tau} u\big)_+\\ &= \operatorname{Re}\big(i\big(\Lambda^{2s+1}_{\mathsf{T},\tau}\operatorname{Op}_{\mathsf{T}}(\tilde{\chi}\pi_{j,1}) - \operatorname{Op}_{\mathsf{T}}(\tilde{\chi}\pi_{j,1})^*\Lambda^1_{\mathsf{T},\tau}\big)u, u\big)_+\\ &\quad + \operatorname{Re}(\Lambda^{2s+1}_{\mathsf{T},\tau} u_{|x_d=0^+}, u_{|x_d=0^+})_\partial.\end{aligned}$$

Note that $\operatorname{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1} u_{|x_d=0^+}, u_{|x_d=0^+})_\partial = |u_{|x_d=0^+}|^2_{s+1/2,\tau}$.

Next, the operator $i\big(\Lambda_{\mathsf{T},\tau}^{2s+1}\operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,1}) - \operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,1})^*\Lambda_{\mathsf{T},\tau}^{2s+1}\big)$ has the following *real* principal symbol

$$\vartheta(\varrho') = -2\operatorname{Im}\pi_{j,1}(\varrho')\lambda_{\mathsf{T},\tau}^{2s+1}.$$

and since $\operatorname{Im}\pi_{j,1}(\varrho') \leq -C_1\lambda_{\mathsf{T},\tau} < 0$ in $\mathscr{U}$ one obtains $\vartheta(\varrho') \gtrsim \lambda_{\mathsf{T},\tau}^{2s+2}$ in $\mathscr{U}$. Since $\mathscr{U}$ is neigborhood of $\operatorname{supp}(\chi)$, the microlocal Gårding inequality of Theorem 2.49 in [62] (the proof adapts to the case with parameter σ as explained in Section 4.5.4 since $\sigma \lesssim \tau$) yields

$$2\operatorname{Re}\big(L_1 u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\big)_+ \geq |u_{|x_d=0^+}|^2_{s+1/2,\tau} + C\|\Lambda_{\mathsf{T},\tau}^{s+1}u\|_+^2 - C_N\|w\|^2_{0,-N,\tau},$$

for $\tau \geq \kappa_0\sigma$ chosen sufficiently large. With the Young inequality one obtains

$$|\big(L_1 u, i\Lambda_{\mathsf{T},\tau}^{2s+1}u\big)_+| \lesssim \frac{1}{\varepsilon}\|\Lambda_{\mathsf{T},\tau}^{s}L_1 u\|_+^2 + \varepsilon\|\Lambda_{\mathsf{T},\tau}^{s+1}u\|_+^2,$$

which yields for ε chosen sufficiently small,

$$|u_{|x_d=0^+}|_{s+1/2,\tau} + \|u\|_{0,s+1,\tau} \lesssim \|L_1 u\|_{0,s,\tau} + \|w\|_{0,-N,\tau}. \tag{4.9.9}$$

Finally, we write

$$\|D_d u\|_{0,s,\tau} \leq \|L_1 u\|_{0,s,\tau} + \|\operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,1})u\|_{0,s,\tau} \lesssim \|L_1 u\|_{0,s,\tau} + \|u\|_{0,s+1,\tau}. \tag{4.9.10}$$

Putting together (4.9.9) and (4.9.10), the result of Lemma 4.9.11 follows. □

Proof of Lemma 4.9.12. We let $u = \operatorname{Op}_\mathsf{T}(\chi)w$. Performing an integration by parts, one has

$$\begin{aligned} 2\operatorname{Re}\big(L_2 u, -i\Lambda_{\mathsf{T},\tau}^{2s+1}u\big)_+ &= 2\operatorname{Re}\big(\big(D_d - \operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,2})\big)u, -i\Lambda_{\mathsf{T},\tau}^{2s+1}u\big)_+ \\ &= \operatorname{Re}\big(i\big(\operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,2})^*\Lambda_{\mathsf{T},\tau}^{1} - \Lambda_{\mathsf{T},\tau}^{2s+1}\operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,2})\big)u, u\big)_+ \\ &\quad - \operatorname{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1}u_{|x_d=0^+}, u_{|x_d=0^+})_\partial. \end{aligned}$$

Note that $\operatorname{Re}(\Lambda_{\mathsf{T},\tau}^{2s+1} u_{|x_d=0^+}, u_{|x_d=0^+})_\partial = |u_{|x_d=0^+}|^2_{s+1/2,\tau}$.

Next, the operator $i\big(\operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,2})^*\Lambda_{\mathsf{T},\tau}^{2s+1} - \Lambda_{\mathsf{T},\tau}^{2s+1}\operatorname{Op}_\mathsf{T}(\tilde{\chi}\pi_{j,2})\big)$ has the following *real* principal symbol

$$\vartheta(\varrho') = 2\operatorname{Im}\pi_{j,2}(\varrho')\lambda_{\mathsf{T},\tau}^{2s+1}.$$

and since $\operatorname{Im} \pi_{j,2}(\varrho') \geq C_2 \lambda_{\mathsf{T},\tau} > 0$ in $\mathscr{U}$ one obtains $\vartheta(\varrho') \gtrsim \lambda_{\mathsf{T},\tau}^{2s+2}$ in $\mathscr{U}$. Since $\mathscr{U}$ is neighborhood of $\operatorname{supp}(\chi)$, the microlocal Gårding inequality of Theorem 2.49 in [62] (the proof adapts to the case with parameter σ as explained in Section 4.5.4 since $\sigma \lesssim \tau$) yields

$$2 \operatorname{Re} \left(L_2 u, i \Lambda_{\mathsf{T},\tau}^{2s+1} u \right)_+ \geq -|u_{|x_d=0^+}|^2_{s+1/2,\tau} + C \|\Lambda_{\mathsf{T},\tau}^{s+1} u\|_+^2 - C_N \|w\|^2_{0,-N,\tau},$$

for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large. The end of the proof is then similar to that of Lemma 4.9.11. $\square$

4.9.3.2 Case (ii): one real root.

One has $\operatorname{Im} \pi_{j,2}(\varrho^{0\prime}) = 0$ and $\operatorname{Im} \pi_{j,1}(\varrho^{0\prime}) < 0$.

Since the Lopatinskiĭ-Šapiro condition holds for (Q_σ^j, B, φ) at $\varrho^{0\prime}$, by Lemma 4.9.8 one has

$$b_\varphi(x^0, \xi^{0\prime}, \xi_d = \pi_{j,2}(\varrho^{0\prime}), \tau^0) = b\big(x^0, \xi^{0\prime} + i\tau^0 d_{x'} \varphi(x^0), i\alpha_j(\varrho^{0\prime})\big) \neq 0.$$

As the roots $\pi_{j,1}$ and $\pi_{j,2}$ are locally smooth with respect to $\varrho' = (x, \xi', \tau, \sigma)$ and homogeneous of degree one in (ξ', τ, σ), there exists $\mathscr{U}$ a conic neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ and $C_1 > 0$, $C_2 > 0$ such that $\mathbb{S}_{\overline{\mathscr{U}}} = \{\varrho' \in \overline{\mathscr{U}};\, |\xi'|^2 + \tau^2 + \sigma^2 = 1\}$ is compact and

$$\tau \geq \kappa_0 \sigma, \quad \pi_{j,1}(\varrho') \neq \pi_{j,2}(\varrho'), \quad \operatorname{Im} \pi_{j,2}(\varrho') \geq -C_2 \lambda_{\mathsf{T},\tau}, \quad \text{and } \operatorname{Im} \pi_{j,1}(\varrho') \leq -C_1 \lambda_{\mathsf{T},\tau},$$

and

$$b_\varphi(x, \xi', \xi_d = \pi_{j,2}(\varrho'), \tau) \neq 0, \tag{4.9.11}$$

if $\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}$.

We let $\chi \in S^0_{\mathsf{T},\tau}$ and $\tilde{\chi} \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree zero in the variable (ξ', τ, σ) and be such that $\operatorname{supp}(\tilde{\chi}) \subset \mathscr{U}$ and $\tilde{\chi} \equiv 1$ on $\operatorname{supp}(\chi)$. From the smoothness and the homogeneity of the roots, one has $\tilde{\chi} \pi_{j,k} \in S^1_{\mathsf{T},\tau}$, $k = 1, 2$. We set

$$L_2 = D_d - \operatorname{Op}_{\mathsf{T}}(\tilde{\chi} \pi_{j,2}) \quad \text{and } L_1 = D_d - \operatorname{Op}_{\mathsf{T}}(\tilde{\chi} \pi_{j,1}).$$

Lemma 4.9.10 and Lemma 4.9.11 also apply in Case (ii) and we shall use them. In addition to these two lemmata we shall need the following lemma.

Lemma 4.9.13. *There exist $C > 0$, $\tau_0 > 0$ such that*

$$\tau^{-1/2} \|w\|_{2,\tau} \leq C \big(\|Q^j_{\sigma,\varphi} w\|_+ + |\operatorname{tr}(w)|_{1,1/2,\tau} \big),$$

for $\tau \geq \max(\tau_0, \kappa_0\sigma)$ *and* $w \in \overline{\mathscr{C}}_c^\infty(W_+)$.

Proving Lemma 4.9.13 is fairly classical, based on writing $Q^j_{\sigma,\varphi} = Q^j_s + iQ_a$ and on an expansion of $\|Q^j_{\sigma,\varphi} w\|_+^2$ and some integration by parts. We provide the details in the proof below as the occurence of the parameter σ is not that classical. Lemma 4.9.13 expresses the loss of a half-derivative if one root, here $\pi_{j,2}$, is real.

Observing that

$$\begin{aligned} L_1 \mathrm{Op}_\mathsf{T}(\chi) L_2 &= \mathrm{Op}_\mathsf{T}(\chi) L_1 L_2 \mod \Psi^{1,0}_\tau \\ &= \mathrm{Op}_\mathsf{T}(\chi) Q^j_{\sigma,\varphi} \mod \Psi^{1,0}_\tau, \end{aligned}$$

and applying Lemma 4.9.11 to $w = L_2 v$, one obtains

$$\begin{aligned} |\mathrm{Op}_\mathsf{T}(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} &\lesssim \|L_1 \mathrm{Op}_\mathsf{T}(\chi) L_2 v\|_+ + \|v\|_{1,-N,\tau} \\ &\lesssim \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{1,\tau}, \end{aligned}$$

for $\tau \geq \kappa_0\sigma$ chosen sufficiently large. We set $u = \mathrm{Op}_\mathsf{T}(\chi)v$, and using the trace inequality

$$|w_{|x_d=0^+}|_{s,\tau} \lesssim \|w\|_{s+1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d_+) \text{ and } s > 0,$$

we have

$$\begin{aligned} |L_2 u_{|x_d=0^+}|_{1/2,\tau} &\lesssim |\mathrm{Op}_\mathsf{T}(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} + |v_{|x_d=0^+}|_{1/2,\tau} \\ &\lesssim |\mathrm{Op}_\mathsf{T}(\chi) L_2 v_{|x_d=0^+}|_{1/2,\tau} + \|v\|_{1,\tau}. \end{aligned}$$

Therefore, we obtain

$$|L_2 u_{|x_d=0^+}|_{1/2,\tau} \lesssim \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{1,\tau}.$$

On the one hand, together with Lemma 4.9.10, one has the estimate

$$\big|\operatorname{tr}(u)\big|_{1,1/2,\tau} \lesssim |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{2,-1,\tau}, \tag{4.9.12}$$

for $\tau \geq \kappa_0\sigma$ chosen sufficiently large using the following trace inequality

$$|\operatorname{tr}(w)|_{m,s,\tau} \lesssim \|w\|_{m+1,s-1/2,\tau}, \qquad w \in \overline{\mathscr{S}}(\mathbb{R}^d_+) \text{ and } m \in \mathbb{N},\ s \in \mathbb{R}.$$

On the other hand, with Lemma 4.9.13 one has

$$\tau^{-1/2}\|u\|_{2,\tau} \lesssim \|Q^j_{\sigma,\varphi} u\|_+ + |\operatorname{tr}(u)|_{1,1/2,\tau},$$

again for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large and since $[Q^j_{\sigma,\varphi}, \mathrm{Op}_\mathsf{T}(\chi)] \in \Psi^{1,0}_\tau$ one finds

$$\tau^{-1/2} \|u\|_{2,\tau} \lesssim \|Q^j_{\sigma,\varphi} v\|_+ + \|v\|_{1,\tau} + |\operatorname{tr}(u)|_{1,1/2,\tau}, \tag{4.9.13}$$

Now, with $\varepsilon > 0$ chosen sufficiently small if one computes (4.9.12) $+ \varepsilon \times$ (4.9.13) one obtains

$$\tau^{-1/2} \|u\|_{2,\tau} + |\operatorname{tr}(u)|_{1,1/2,\tau} \lesssim \|Q^j_{\sigma,\varphi} v\|_+ + |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}.$$

As $u = \mathrm{Op}_\mathsf{T}(\chi) v$, with a commutator argument we obtain

$$\begin{aligned} |B_\varphi u_{|x_d=0^+}|_{3/2-k,\tau} &\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + |\operatorname{tr}(v)|_{1,-1/2,\tau} \\ &\lesssim |B_\varphi v_{|x_d=0^+}|_{3/2-k,\tau} + \|v\|_{2,-1,\tau}. \end{aligned}$$

yielding (4.9.6) and thus concluding the proof of Proposition 4.9.9 in Case (ii). $\square$

We now provide a proof of Lemma 4.9.13.

Proof of Lemma 4.9.13. We recall that $Q^j_{\sigma,\varphi} = Q^j_s + iQ_a$, yielding

$$\|Q^j_{\sigma,\varphi} w\|_+^2 = \|Q^j_s w\|_+^2 + \|Q_a w\|_+^2 + 2\operatorname{Re}(Q^j_s w, iQ_a w)_+. \tag{4.9.14}$$

With the integration by parts formula $(f, D_d g)_+ = (D_d f, g)_+ - i(f_{|x_d=0^+}, g_{|x_d=0^+})_\partial$, and the forms of Q^j_s and Q_a given in (4.9.2) one has

$$(f, Q^j_s g)_+ = (Q^j_s f, g)_+ - i(f_{|x_d=0^+}, D_d g_{|x_d=0^+})_\partial - i(D_d f_{|x_d=0^+}, g_{|x_d=0^+})_\partial,$$

and

$$(f, Q_a g)_+ = (Q_a f, g)_+ - 2\tau i(\partial_d \varphi f_{|x_d=0^+}, g_{|x_d=0^+})_\partial,$$

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$$\begin{aligned} (Q_a w, Q^j_s w)_+ &= (Q^j_s Q_a w, w)_+ - i(Q_a w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial - i(D_d Q_a w_{|x_d=0^+}, w_{|x_d=0^+})_\partial \\ (Q^j_s w, Q_a w)_+ &= (Q_a Q^j_s w, w)_+ - 2i\tau(\partial_d \varphi Q^j_s w_{|x_d=0^+}, w_{|x_d=0^+})_\partial. \end{aligned}$$

This gives

$$2\operatorname{Re}(Q^j_s w, iQ_a w)_+ = i([Q^j_s, Q_a] w, w)_+ + \tau \mathcal{A}(w) \tag{4.9.15}$$

with

$$\mathcal{A}(w) = \tau^{-1}(Q_a w, D_d w)_\partial + \tau^{-1}\big((D_d Q_a - 2\tau\partial_d\varphi Q_s^j)w, w\big)_\partial. \tag{4.9.16}$$

We have the following lemma adapted from Lemma 3.25 in [62].

Lemma 4.9.14. *The operators $Q_a \in \tau\mathscr{D}^1$ and $D_d Q_a - 2\tau\partial_d\varphi Q_s^j \in \mathscr{D}_\tau^3$ can be cast in the following forms*

$$Q_a = 2\tau\partial_d\varphi D_d + 2\tilde{r}(x, D', \tau d_{x'}\varphi) \mod \tau\mathscr{D}^0,$$

and

$$\begin{aligned} D_d Q_a - 2\tau\partial_d\varphi Q_s^j = &-2\tau\partial_d\varphi\big(R(x,D') + (-1)^j\sigma^2 - (\tau\partial_d\varphi)^2 - r(x,\tau d_{x'}\varphi)\big) \\ &+ 2\tilde{r}(x, D', \tau d_{x'}\varphi)D_d \mod \tau\Psi_\tau^{1,0}. \end{aligned}$$

With this lemma we find

$$\begin{aligned} \mathcal{A}(w) = &\, 2(\partial_d\varphi D_d w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial + 2(\tilde{r}(x,D',d_{x'}\varphi)w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial \\ &+ 2\big(\tilde{r}(x,D',d_{x'}\varphi)D_d w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial \\ &- 2\big(\partial_d\varphi\big(R(x,D') + (-1)^j\sigma^2 - (\tau\partial_d\varphi)^2 - r(x,\tau d_{x'}\varphi)\big)w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial \\ &+ (\mathrm{Op}(c_0)w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial + \big((\mathrm{Op}(\tilde{c}_0)D_d + \mathrm{Op}(c_1))w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial, \end{aligned} \tag{4.9.17}$$

with $\mathrm{Op}(c_0), \mathrm{Op}(\tilde{c}_0) \in \mathscr{D}^0$ and $\mathrm{Op}(c_1) \in \mathscr{D}_{\mathrm{T},\tau}^1$. Observe that one has

$$|\mathcal{A}(w)| \lesssim |\operatorname{tr}(w)|_{1,0,\tau}^2. \tag{4.9.18}$$

From (4.9.14) and (4.9.15) one writes

$$\|Q_{\sigma,\varphi}^j w\|_+^2 + \tau|\operatorname{tr}(w)|_{1,0,\tau}^2 \gtrsim \|Q_s^j w\|_+^2 + \|Q_a w\|_+^2 + \mathrm{Re}(i[Q_s^j, Q_a]w, w)_+. \tag{4.9.19}$$

We now use the following lemma whose proof is given below.

Lemma 4.9.15. *There exists $C, C' > 0$, $\mu > 0$ and $\tau_0 > 0$ such that*

$$\mu\big(\|Q_s^j w\|_+^2 + \|Q_a w\|_+^2\big) + \tau\,\mathrm{Re}(i[Q_s^j, Q_a]w, w)_+ \geq C\|w\|_{2,\tau}^2 - C'|\operatorname{tr}(w)|_{1,1/2,\tau}^2$$

for $\tau \geq \max(\tau_0, \kappa_0\sigma)$ and $w \in \overline{\mathscr{C}}_c^\infty(W_+)$

Let $\mu > 0$ be as in Lemma 4.9.15 and let $\tau > 0$ be such that $\mu\tau^{-1} \leq 1$. From (4.9.19) one then writes

$$\|Q^j_{\sigma,\varphi} w\|^2_+ + \tau|\operatorname{tr}(w)|^2_{1,0,\tau} \gtrsim \tau^{-1}\Big(\mu\big(\|Q^j_s w\|^2_+ + \|Q_a w\|^2_+\big) + i\tau([Q^j_s, Q_a]w, w)_+\Big),$$

which with Lemma 4.9.15 yields the result of Lemma 4.9.13 using that $\tau|\operatorname{tr}(w)|_{1,0,\tau} \lesssim |\operatorname{tr}(w)|_{1,1/2,\tau}$. □

Proof of Lemma 4.9.15. One has $[Q^j_s, Q_a] \in \tau\mathscr{D}^2_\tau$. Writing

$$\tau \operatorname{Re}\big(i[Q^j_s, Q_a]w, w\big)_+ = \operatorname{Re}\big(i\tau^{-1}[Q^j_s, Q_a]w, \tau^2 w\big)_+,$$

it can be seen as a interior differential quadratic form of type $(2,0)$ as in Definition 4.2.11. Therefore

$$T(w) = \mu\big(\|Q^j_s w\|^2_+ + \|Q_a w\|^2_+\big) + \tau \operatorname{Re}\big(i[Q^j_s, Q_a]w, w\big)_+$$

is also an interior differential quadratic form of this type with principal symbol given by

$$t(\varrho) = \mu|q^j_{\sigma,\varphi}(\varrho)|^2 + \tau\{q^j_s, q_a\}(\varrho), \qquad \varrho = (x, \xi, \tau, \sigma).$$

Let $\tau_0 > 0$. By Lemma 4.9.4, the sub-ellipticity property of (Q^j_σ, φ) implies

$$t(\varrho) \gtrsim \lambda^4_\tau, \qquad \varrho \in \overline{W} \times \mathbb{R}^d \times [0, +\infty) \times [0, +\infty),\ \tau \geq \tau_0\sigma,$$

for $\mu > 0$ chosen sufficiently large. The Gårding inequality of proposition 4.2.13 yields

$$T(w) \geq C\|w\|^2_{2,\tau} - C'|\operatorname{tr}(w)|^2_{1,1/2,\tau},$$

for some $C, C' > 0$ and for $\tau \geq \kappa_0\sigma$ chosen sufficiently large). □

4.9.3.3 Case (iii): both roots lying in the lower complex half-plane.

The result in the present case is a simple consequence of the general result given in Lemma 6.1.1 whose proof can be found in [11]. In the second order case however, the proof does not require the same level of technicality.

One has $\operatorname{Im}\pi_{j,1}(\varrho^{0\prime}) < 0$ and $\operatorname{Im}\pi_{j,2}(\varrho^{0\prime}) < 0$. As the roots $\pi_{j,1}$ and $\pi_{j,2}$ depend continuously on the variable $\varrho' = (x, \xi', \tau, \sigma)$, there exists $\mathscr{U}$ a conic open neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ and $C_0 > 0$ such that

$$\tau \geq \kappa_0 \sigma, \quad \operatorname{Im} \pi_{j,1}(\varrho') \leq -C_0 \lambda_{\mathsf{T},\tau}, \quad \text{and } \operatorname{Im} \pi_{j,2}(\varrho') \leq -C_0 \lambda_{\mathsf{T},\tau},$$

if $\varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}$.

Let $\chi \in S^0_{\mathsf{T},\tau}$ be as in the statement of Proposition 4.9.9 and set $u = \operatorname{Op}_{\mathsf{T}}(\chi) v$.

We recall that $Q^j_{\sigma,\varphi} = Q^j_s + i Q_a$, yielding

$$\|Q^j_{\sigma,\varphi} u\|_+^2 = \|Q^j_s u\|_+^2 + \|Q_a u\|_+^2 + 2 \operatorname{Re}(Q^j_s u, i Q_a u)_+.$$

We set $L(u) = \|Q^j_s u\|_+^2 + \|Q_a u\|_+^2$. This is an interior differential quadratic form in the sense of Definition 4.2.11. Its principal symbol is given by

$$\ell(\varrho) = (q^j_s)(\varrho)^2 + q_a(\varrho)^2, \qquad \varrho = (x, \xi, \tau, \sigma).$$

For $\varepsilon \in (0, 1)$ we write

$$\|Q^j_{\sigma,\varphi} u\|_+^2 \geq \varepsilon L(u) + 2 \operatorname{Re}(Q^j_s u, i Q_a u)_+. \tag{4.9.20}$$

For concision we write $\varrho = (\varrho', \xi_d)$ with $\varrho' = (x, \xi', \tau, \sigma)$. The set

$$\mathscr{L} = \{\varrho = (\varrho', \xi_d);\, \varrho' \in \overline{\mathscr{U}},\ \xi_d \in \mathbb{R}, \text{ and } |\xi|^2 + \tau^2 + \sigma^2 = 1\}$$

is compact recalling that W is bounded. On $\mathscr{L}$ one has $|q^j_{\sigma,\varphi}(\varrho)| \geq C > 0$. By homogeneity one has

$$|q^j_{\sigma,\varphi}(\varrho)| \gtrsim \lambda_\tau^2, \qquad \varrho' \in \mathscr{U},\ \xi_d \in \mathbb{R}, \quad \text{if } \tau \geq \tau_0 \sigma, \tag{4.9.21}$$

for some $\tau_0 > 0$. Therefore

$$\ell(\varrho) \gtrsim \lambda_\tau^4, \qquad \varrho' \in \mathscr{U},\ \xi_d \in \mathbb{R}, \quad \text{if } \tau \geq \tau_0 \sigma. \tag{4.9.22}$$

By the Gårding inequality of Proposition 4.2.12 one obtains

$$\operatorname{Re} L(u) \geq C \|u\|_{2,\tau}^2 - C' |\operatorname{tr}(u)|_{1,1/2}^2 - C_N \|v\|_{2,-N,\tau}^2, \tag{4.9.23}$$

for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large.

From the proof of Lemma 4.9.13 one has

$$2 \operatorname{Re}(Q^j_s u, i Q_a u)_+ = i([Q^j_s, Q_a] u, u)_+ + \tau \mathcal{A}(u) \tag{4.9.24}$$

with the boundary quadratic form $\mathcal{A}$ given in (4.9.16)–(4.9.17).

On the one hand, one has $[Q_s^j, Q_a] \in \tau \mathscr{D}_\tau^2$ and therefore

$$|\operatorname{Re}([Q_s^j, Q_a]u, u)_+| \lesssim \tau \|u\|_{2,-1,\tau}^2 \lesssim \tau^{-1}\|u\|_{2,\tau}^2. \tag{4.9.25}$$

On the other hand, we have the following lemma that provides a microlocal positivity property for the boundary quadratic form $\mathcal{A}$. A proof is given below.

Lemma 4.9.16. *There exist C, C_N and $\tau_0 > 0$ such that*

$$\tau \operatorname{Re} \mathcal{A}(u) \geq C|\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C_N|\operatorname{tr}(v)|_{1,-N,\tau}^2, \quad \textit{for } u = \operatorname{Op}_\mathsf{T}(\chi)v,$$

for $\tau \geq \max(\tau_0, \kappa_0\sigma)$.

With (4.9.24)–(4.9.25), and Lemma 4.9.16 one obtains

$$\begin{aligned} 2\operatorname{Re}(Q_s^j u, iQ_a u)_+ &\geq C|\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C'\tau^{-1}\|u\|_{2,\tau}^2 - C_N|\operatorname{tr}(v)|_{1,-N,\tau}^2 \\ &\geq C|\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C'\tau^{-1}\|u\|_{2,\tau}^2 - C_N'\|v\|_{2,-N,\tau}^2, \end{aligned} \tag{4.9.26}$$

with a trace inequality, for $\tau \geq \kappa_0\sigma$ chosen sufficiently large.

With (4.9.20), (4.9.23), and (4.9.26) one obtains

$$\begin{aligned} \|Q_{\sigma,\varphi}^j u\|_+^2 \geq \varepsilon C\|u\|_{2,\tau}^2 &- C'\varepsilon|\operatorname{tr}(u)|_{1,1/2}^2 - C_N\varepsilon\|v\|_{2,-N,\tau}^2 \\ &+ C|\operatorname{tr}(u)|_{1,1/2,\tau}^2 - C'\tau^{-1}\|u\|_{2,\tau}^2 - C_N'\|v\|_{2,-N,\tau}^2. \end{aligned}$$

With ε chosen sufficiently small and $\tau \geq \kappa_0\sigma$ sufficiently large one obtains for any $N \in \mathbb{N}$

$$\|u\|_{2,\tau} + |\operatorname{tr}(u)|_{1,1/2,\tau} \lesssim \|Q_{\sigma,\varphi}^j u\|_+ + \|v\|_{2,-N,\tau}.$$

With a commutator argument, as $u = \operatorname{Op}_\mathsf{T}(\chi)v$ one finds $\|Q_{\sigma,\varphi}^j u\|_+ \lesssim \|Q_{\sigma,\varphi}^j v\|_+ + \|v\|_{2,-1,\tau}$, yielding estimate (4.9.7) and thus concluding the proof of Proposition 4.9.9 in Case (iii). $\square$

Proof of Lemma 4.9.16. With (4.9.17) one sees that it suffices to consider the following boundary quadratic form

$$\begin{aligned} \tilde{\mathcal{A}}(w) = \,& 2(\partial_d\varphi D_d w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial + 2(\tilde{r}(x, D', d_{x'}\varphi)w_{|x_d=0^+}, D_d w_{|x_d=0^+})_\partial \\ &+ 2\big(\tilde{r}(x, D', d_{x'}\varphi)D_d w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial \\ &- 2\big(\partial_d\varphi\big(R(x, D') + (-1)^j\sigma^2 - (\tau\partial_d\varphi)^2 - r(x, \tau d_{x'}\varphi)\big)w_{|x_d=0^+}, w_{|x_d=0^+}\big)_\partial, \end{aligned}$$

in place of $\mathcal{A}$. It is of type $(1,0)$ in the sense of Definition 4.2.14. Its principal symbol is given by $a_0(\varrho', \xi_d, \xi_d') = (1, \xi_d)A(\varrho')\,{}^t(1, \xi_d)$ with

$$A(\varrho') = \begin{pmatrix} -(\partial_d\varphi)\big(r(x,\xi') + (-1)^j\sigma^2 - (\tau\partial_d\varphi)^2 - r(x, \tau d_{x'}\varphi)\big)_{|x_d=0^+} & \tilde{r}(x, \xi', d_{x'}\varphi)_{|x_d=0^+} \\ \tilde{r}(x, \xi', d_{x'}\varphi)_{|x_d=0^+} & \partial_d\varphi_{|x_d=0^+} \end{pmatrix},$$

with $\varrho' = (x, \xi', \tau, \sigma)$. The associated bilinear symbol introduced in (4.2.7) is given by

$$\Sigma_{\mathcal{A}}(\varrho', \mathbf{z}, \mathbf{z}') = \mathbf{z} A(\varrho')\, {}^t\overline{\mathbf{z}}', \qquad \mathbf{z} = (z_0, z_1) \in \mathbb{C}^2, \ \mathbf{z}' = (z_0', z_1') \in \mathbb{C}^2.$$

One computes

$$\det A(\varrho') = -\Big((\partial_d \varphi)^2 \big(r(x, \xi') + (-1)^j \sigma^2 - (\tau \partial_d \varphi)^2 - r(x, \tau d_{x'} \varphi) \big) + \tilde{r}(x, \xi', d_{x'} \varphi)^2 \Big)_{|x_d = 0^+}.$$

With Lemma 4.7.3 one sees that $\operatorname{Im} \pi_{j,2} < 0$ is equivalent to having $\det A(\varrho') > 0$. We thus have

$$\det A(\varrho') \geq C > 0, \quad \text{for } \varrho' = (x, \xi', \tau, \sigma) \in \mathbb{S}_{\overline{\mathscr{U}}},$$

with $\mathbb{S}_{\overline{\mathscr{U}}} = \{\varrho' \in \overline{\mathscr{U}};\ \xi_d \in \mathbb{R},\ |\xi|^2 + \tau^2 + \sigma^2 = 1\}$ since $\mathbb{S}_{\overline{\mathscr{U}}}$ is compact. Since $\partial_d \varphi_{|x_d = 0^+} \geq C' > 0$ then one finds that

$$\operatorname{Re} \Sigma_{\mathcal{A}}(\varrho', \mathbf{z}, \mathbf{z}) \geq C(|z_0|^2 + |z_1|^2), \quad \varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}, \quad |(\xi', \tau, \sigma)| = 1.$$

By homogeneity one obtains

$$\operatorname{Re} \Sigma_{\mathcal{A}}(\varrho', \mathbf{z}, \mathbf{z}) \geq C(\lambda_{\mathsf{T},\tau}^2 |z_0|^2 + |z_1|^2), \quad \varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{U}}, \quad |(\xi', \tau, \sigma)| \geq 1.$$

With Proposition 4.2.16, having in mind what is exposed in Section 4.5.4 since we have $\tau \geq \kappa_0 \sigma$ here, one obtains

$$\operatorname{Re} \tilde{\mathcal{A}}(u) \geq C |\operatorname{tr}(u)|^2_{1,0,\tau} - C_N |\operatorname{tr}(v)|^2_{1,-N,\tau}, \quad \text{for } u = \operatorname{Op}_{\mathsf{T}}(\chi) v,$$

for $\tau \geq \kappa_0 \sigma$ chosen sufficiently large.

Here, we have $\operatorname{Im} \pi_{j,2} < 0$ and thus $|\xi'| \lesssim \tau$ by Lemma 4.7.6. Thus one has

$$\tau |\operatorname{tr}(u)|^2_{1,0,\tau} \gtrsim |\operatorname{tr}(u)|^2_{1,1/2,\tau} - |\operatorname{tr}(v)|^2_{1,-N,\tau},$$

by the microlocal Gårding inequality, for instance invoking Proposition 4.2.16 for a boundary quadratic form of type $(1, 1/2)$. This concludes the proof. $\square$

4.10 Local Carleman estimate for the fourth-order operator

4.10.1 A first estimate

Proposition 4.10.1. *Let $\kappa_0' > \kappa_1' > \kappa_1 > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d \varphi \geq C > 0$ in W and such that (Q_σ^j, φ) satisfies the sub-ellipticity condition in $\overline{W}$ for both $j = 1$ and 2. Let $\varrho^{0\prime} = (x^0, \xi^{0\prime}, \tau^0, \sigma^0)$ with $(\xi^{0\prime}, \tau^0, \sigma^0) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ nonvanishing with $\kappa_1 \sigma^0 \leq \tau^0 \leq \kappa_1' \sigma^0$.*

Then, there exists $\mathscr{U}$ a conic neighborhood of $\varrho^{0\prime}$ in $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\kappa_0 \sigma \leq \tau \leq \kappa_0' \sigma$ such that if $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$, there exist $C > 0$ and $\tau_0 > 0$ such that

$$\tau^{-1/2} \|\mathrm{Op}_{\mathsf{T}}(\chi) v\|_{4,\tau} \leq C\big(\|P_{\sigma,\varphi} v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau}\big), \qquad (4.10.1)$$

for $\tau \geq \tau_0$, $\kappa_0 \sigma \leq \tau \leq \kappa_0' \sigma$, and $v \in \overline{\mathscr{C}}_c^\infty(W_+)$.

An important aspect is that here we have $\sigma \gtrsim \tau$; this explains that only one root of $p_{\sigma,\varphi}$ can lie on the real axis and thus only one half derivative is lost in this estimate. The proof of Proposition 4.10.1 is based on the microlocal results of Proposition 4.9.9.

Proof. We shall concatenate the estimates of Proposition 4.9.9 for $Q^1_{\sigma,\varphi}$ and $Q^2_{\sigma,\varphi}$ with the boundary operator B simply given by the Dirichlet trace operator, $Bu_{|x_d=0^+} = u_{|x_d=0^+}$.

One has $b(x,\xi) = 1$ and $b_\varphi(x, \xi', \xi_d, \tau) = 1$. Since $\partial_d \varphi > 0$ then $\operatorname{Im} \pi_{j,1} < 0$. Thus, either $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = 1$ or $q^{j,+}_{\sigma,\varphi}(x, \xi', \xi_d, \tau) = \xi_d - \pi_{j,2}$. With Lemma 4.9.8 one sees that the Lopatinskiĭ-Šapiro holds for $(Q^1_{\sigma,\varphi}, B, \varphi)$ and $(Q^2_{\sigma,\varphi}, B, \varphi)$ at $\varrho^{0\prime}$.

Proposition 4.9.9 thus applies. Let $\mathscr{U}_j$ be the conic neighborhood of $\varrho^{0\prime}$ obtained invoking this proposition for $Q^j_{\sigma,\varphi}$, for $j = 1$ or 2. In $\mathscr{U}_j$ one has $\tau \geq \kappa_0 \sigma$. We set

$$\mathscr{U} = \mathscr{U}_1 \cap \mathscr{U}_2 \cap \{\tau \leq \kappa_0' \sigma\},$$

and we consider $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0 in (ξ', τ, σ) with $\operatorname{supp}(\chi) \subset \mathscr{U}$.

Since in $\mathscr{U}$ one has $\sigma > 0$ then $\pi_{1,2}$ and $\pi_{2,2}$ cannot be both real by Lemma 4.7.7. Proposition 4.9.9 thus implies that we necessarily have the following two estimates

$$\tau^{-\ell_1}\|\mathrm{Op}_{\mathsf{T}}(\chi)w\|_{2,\tau} \lesssim \|Q^1_{\sigma,\varphi}w\|_+ + |w_{|x_d=0^+}|_{3/2,\tau} + \|w\|_{2,-1,\tau}, \tag{4.10.2}$$

and

$$\tau^{-\ell_2}\|\mathrm{Op}_{\mathsf{T}}(\chi)w\|_{2,\tau} \lesssim \|Q^2_{\sigma,\varphi}w\|_+ + |w|_{3/2,\tau} + \|w\|_{2,-1,\tau}, \tag{4.10.3}$$

with either $(\ell_1,\ell_2) = (1/2,0)$ or $(\ell_1,\ell_2) = (0,1/2)$, for $w \in \overline{\mathscr{C}}^\infty_c(W_+)$ and $\tau \geq \kappa_0\sigma$ chosen sufficiently large.

Let us assume that $(\ell_1,\ell_2) = (1/2,0)$. The other case can be treated similarly. Writing $P_{\sigma,\varphi} = Q^2_{\sigma,\varphi}Q^1_{\sigma,\varphi}$, with (4.10.3) one has

$$\begin{aligned}\|\mathrm{Op}_{\mathsf{T}}(\chi)Q^1_{\sigma,\varphi}v\|_{2,\tau} &\lesssim \|P_{\sigma,\varphi}v\|_+ + |Q^1_{\sigma,\varphi}v_{|x_d=0^+}|_{3/2,\tau} + \|v\|_{4,-1,\tau}\\ &\lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{2,3/2,\tau} + \|v\|_{4,-1,\tau}.\end{aligned}$$

Since $[\mathrm{Op}_{\mathsf{T}}(\chi), Q^1_{\sigma,\varphi}] \in \Psi^{1,0}_\tau$ one finds

$$\|Q^1_{\sigma,\varphi}\mathrm{Op}_{\mathsf{T}}(\chi)v\|_{2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau}. \tag{4.10.4}$$

For $k = 0, 1$ or 2, one writes

$$\begin{aligned}\|Q^1_{\sigma,\varphi}\mathrm{Op}_{\mathsf{T}}(\chi)D_d^k\Lambda^{2-k}_{\mathsf{T},\tau}v\|_+ + |\operatorname{tr}(\mathrm{Op}_{\mathsf{T}}(\chi)D_d^k\Lambda^{2-k}_{\mathsf{T},\tau}v)|_{1,1/2,\tau} + \|\mathrm{Op}_{\mathsf{T}}(\chi)D_d^k\Lambda^{2-k}_{\mathsf{T},\tau}v\|_{2,-1,\tau}\\ \lesssim \|Q^1_{\sigma,\varphi}\mathrm{Op}_{\mathsf{T}}(\chi)v\|_{2,\tau} + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau},\end{aligned}$$

since $[Q^1_{\sigma,\varphi}\mathrm{Op}_{\mathsf{T}}(\chi), D_d^k\Lambda^{2-k}_{\mathsf{T},\tau}] \in \Psi^{4,-1}_\tau$.

Let $\tilde\chi \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree zero in the variable (ξ',τ,σ) and be such that $\operatorname{supp}(\tilde\chi) \subset \mathscr{U}$ and $\tilde\chi \equiv 1$ on a neighborhood of $\operatorname{supp}(\chi)$. With (4.10.2), from (4.10.4) one thus obtains

$$\tau^{-1/2}\|\mathrm{Op}_{\mathsf{T}}(\tilde\chi)\mathrm{Op}_{\mathsf{T}}(\chi)D_d^k\Lambda^{2-k}_{\mathsf{T},\tau}v\|_{2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau}.$$

Since $\mathrm{Op}_{\mathsf{T}}(\tilde\chi)\mathrm{Op}_{\mathsf{T}}(\chi)D_d^k\Lambda^{2-k}_{\mathsf{T},\tau} = \Lambda^{2-k}_{\mathsf{T},\tau}D_d^k\mathrm{Op}_{\mathsf{T}}(\chi) \mod \Psi^{2,-1}_\tau$ one deduces

$$\tau^{-1/2}\|D_d^k\mathrm{Op}_{\mathsf{T}}(\chi)v\|_{2,2-k,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau} + \|v\|_{4,-1,\tau}.$$

Using that $k = 0, 1$ or 2, the result follows. $\square$

Consequence of this microlocal result is the following local result by means of a patching procedure as for the proof of Proposition 4.8.1 in Section 4.8.2.

Proposition 4.10.2. *Let $\kappa_0' > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d \varphi \geq C > 0$ in W and such that (Q_σ^j, φ) satisfies the sub-ellipticity condition in $\overline{W}$ for both $j = 1$ and 2.*

Then, there exists W^0 a neighborhood of x^0, $C > 0$, $\tau_0 > 0$ such that

$$\tau^{-1/2} \|v\|_{4,\tau} \leq C\big(\|P_{\sigma,\varphi} v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau}\big), \tag{4.10.5}$$

for $\tau \geq \tau_0$, $\kappa_0 \sigma \leq \tau \leq \kappa_0' \sigma$, and $v \in \overline{\mathscr{C}}_c^\infty(W_+)$.

4.10.2 Final estimate

Combining the local results of Section 4.8 for the estimation of the boundary norm under the Lopatinskiĭ-Šapiro condition and the previous local result without any prescribed boundary condition we obtain the Carleman estimate of Theorem 1.8.1. For a precise statement we write the following theorem.

Theorem 4.10.3 (local Carleman estimate for P_σ)**.** *Let $\kappa_0' > \kappa_0 > 0$. Let $x^0 \in \partial\Omega$, with Ω locally given by $\{x_d > 0\}$ and let W be a bounded open neighborhood of x^0 in $\mathbb{R}^d$. Let φ be such that $\partial_d \varphi \geq C > 0$ in W and such that (Q_σ^j, φ) satisfies the sub-ellipticity condition in $\overline{W}$ for both $j = 1$ and 2.*

Assume that $(P_\sigma, B_1, B_2, \varphi)$ satisfies the Lopatinskiĭ-Šapiro condition of Definition 4.7.1 at $\varrho' = (x^0, \xi', \tau, \sigma)$ for all $(\xi', \tau, \sigma) \in \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ such that $\tau \geq \kappa_0 \sigma$.

Then, there exists W^0 a neighborhood of x^0, $C > 0$, $\tau_0 > 0$ such that

$$\tau^{-1/2} \|e^{\tau\varphi} u\|_{4,\tau} + |\operatorname{tr}(e^{\tau\varphi} u)|_{3,1/2,\tau} \leq C\Big(\|e^{\tau\varphi} P_\sigma u\|_+ + \sum_{j=1}^{2} |e^{\tau\varphi} B_j u_{|x_d=0^+}|_{7/2-k_j,\tau}\Big), \tag{4.10.6}$$

for $\tau \geq \tau_0$, $\kappa_0 \sigma \leq \tau \leq \kappa_0' \sigma$, and $u \in \overline{\mathscr{C}}_c^\infty(W_+^0)$.

The notation of the function space $\overline{\mathscr{C}}_c^\infty(W_+^0)$ is introduced in (4.5.3).

For the application of this theorem, one has to design a weight function that yields the two important properties: sub-ellipticity and the Lopatinskiĭ-Šapiro condition. Sub-ellipticity is obtained by means of Proposition 4.9.3; the Lopatinskiĭ-Šapiro condition by means of Proposition 4.7.2.

Proof of Theorem 4.10.3. Let $v \in \overline{\mathscr{C}}_c^\infty(W_+)$. The assumption of the theorem allows one to invoke both Propositions 4.8.1 and 4.10.2. With the first proposition one has

$$|\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|v\|_{4,-1,\tau}, \tag{4.10.7}$$

for $\sigma \geq 0$, $\tau \geq \max(\tau_1, \kappa_0\sigma)$ for some $\tau_1 > 0$. With the second proposition one has

$$\tau^{-1/2}\|v\|_{4,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + |\operatorname{tr}(v)|_{3,1/2,\tau}, \tag{4.10.8}$$

for $\tau \geq \tau_1'$ and $\kappa_0\sigma \leq \tau \leq \kappa_0'\sigma$ for some $\tau_1' > 0$.

Consider $\sigma > 0$ and $\tau \geq \max(\tau_1, \tau_1')$ such that $\kappa_0\sigma \leq \tau \leq \kappa_0'\sigma$. Combined together (4.10.7) and (4.10.8) yield

$$\tau^{-1/2}\|v\|_{4,\tau} + |\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau} + \|v\|_{4,-1,\tau}.$$

Since $\|v\|_{4,-1,\tau} \ll \tau^{-1/2}\|v\|_{4,\tau}$ for τ large one obtains

$$\tau^{-1/2}\|v\|_{4,\tau} + |\operatorname{tr}(v)|_{3,1/2,\tau} \lesssim \|P_{\sigma,\varphi}v\|_+ + \sum_{j=1}^{2} |B_{j,\varphi}v_{|x_d=0^+}|_{7/2-k_j,\tau}.$$

If we set $v = e^{\tau\varphi}u$ then the conclusion follows. $\square$

4.11 Global Carleman estimate and observability

Using the local Carleman estimate of Theorem 4.10.3 we prove a global version of this estimate. This allows us to obtain an observability inequality with observation in some open subset $\mathscr{O}$ of Ω. In turn in Section 4.13 we use this latter inequality to obtain a resolvent estimate for the plate semigroup generator that allows one to deduce a stabilization result for the damped plate equation.

4.11.1 A global Carleman estimate

Assume that the Lopatinskiĭ-Šapiro condition of Definition 4.6.1 holds for (P_0, B_1, B_2) on $\partial\Omega$.

Let $\mathscr{O}_0, \mathscr{O}_1, \mathscr{O}$ be open sets such that $\mathscr{O}_0 \Subset \mathscr{O}_1 \Subset \mathscr{O} \Subset \Omega$. With Proposition 3.31 and Remark 3.32 in [62] there exists $\psi \in \mathscr{C}^\infty(\overline{\Omega})$ such that

i. $\psi = 0$ and $\partial_\nu\psi < -C_0 < 0$ on $\partial\Omega$;

ii. $\psi > 0$ in Ω;

iii. $d\psi \neq 0$ in $\Omega \setminus \mathscr{O}_0$.

Then, by Proposition 4.9.3, for γ chosen sufficiently large, one finds that $\varphi = \exp(\gamma\psi)$ is such that a

i. $\varphi = 1$ and $\partial_\nu \varphi < -C_0 < 0$ on $\partial\Omega$;

ii. $\varphi > 1$ in Ω;

iii. (Q^j_σ, φ) satisfies the sub-ellipticity condition in $\Omega \setminus \mathscr{O}_0$, for $j = 1, 2$, for $\tau \geq \tau_0 \sigma$ for τ_0 chosen sufficiently large.

Then, with Proposition 4.7.2, for $\kappa_0 > 0$ chosen sufficiently large one finds that the Lopatinskiĭ-Šapiro condition holds for $(P_\sigma, B_1, B_2, \varphi)$ at any (x, ξ', τ, σ) for any $x \in \partial\Omega$, $\xi' \in T^*_x\partial\Omega \simeq \mathbb{R}^{d-1}$, $\tau > 0$, and $\sigma > 0$ such that $\tau \geq \kappa_0 \sigma$, for κ_0 chosen sufficiently large, using that $\partial\Omega$ is compact.

Thus for any $x \in \partial\Omega$ the local estimate of Theorem 4.10.3 applies. A similar result applies in the neighborhood of any point of $\Omega \setminus \mathscr{O}_0$.

With the weight function φ constructed above, following the patching procedure described in the proof of Theorem 3.34 in [62], one obtains the following global estimate

$$\tau^{-1/2} \| e^{\tau\varphi} u \|_{4,\tau} + | \operatorname{tr}(e^{\tau\varphi} u) |_{3,1/2,\tau} \lesssim \| e^{\tau\varphi} P_\sigma u \|_{L^2(\Omega)} + \sum_{j=1}^{2} | e^{\tau\varphi} B_j u_{|\partial\Omega} |_{7/2-k_j,\tau} + \tau^{-1/2} \| e^{\tau\varphi} \chi_0 u \|_{4,\tau}, \tag{4.11.1}$$

for $\tau \geq \tau_0$, $\kappa_0 \sigma \leq \tau \leq \kappa'_0 \sigma$, and $u \in \mathscr{C}^\infty(\overline{\Omega})$, and where $\chi_0 \in \mathscr{C}^\infty_c(\mathscr{O})$ such that $\chi_0 \equiv 1$ in a neighborhood of $\overline{\mathscr{O}_1}$. Here, $\|.\|_{s,\tau}$ and $|.|_{s,\tau}$, the Sobolev norms with the large parameter τ, are understood in Ω and $\partial\Omega$ respectively.

Remark 4.11.1. Observe that inequality (4.11.1) also holds for third-order perturbations of P_σ. Below, we shall use it for a second-order perturbation $P_\sigma - i\sigma^2\alpha = \Delta^2 - \sigma^4 - i\sigma^2\alpha$.

4.11.2 Observability inequality

By density one finds that inequality 4.11.1 holds for $u \in H^4(\Omega)$.

Let $C_0 > \sup_{\overline{\Omega}} \varphi - 1$. Since $1 \leq \varphi \leq \sup_{\overline{\Omega}} \varphi$ one obtains

$$\|u\|_{H^4(\Omega)} \lesssim e^{C_0 \tau} \big(\|P_\sigma u\|_{L^2(\Omega)} + \sum_{j=1}^{2} |B_j u_{|\partial\Omega}|_{H^{7/2-k_j}(\partial\Omega)} + \|u\|_{H^4(\mathscr{O}_1)} \big). \tag{4.11.2}$$

for $\tau \geq \tau_0$, $\kappa_0 \sigma \leq \tau \leq \kappa'_0 \sigma$.

With the ellipticity of P_0 one has

$$\|u\|_{H^4(\mathscr{O}_1)} \lesssim \|P_0 u\|_{L^2(\mathscr{O})} + \|u\|_{L^2(\mathscr{O})},$$

since $\mathscr{O}_1 \Subset \mathscr{O}$. This can be proven by the introduction of a parametrics for P_0. One thus obtain

$$\|u\|_{H^4(\mathscr{O}_1)} \lesssim \|P_\sigma u\|_{L^2(\Omega)} + (1+\sigma^4)\|u\|_{L^2(\mathscr{O})},$$

and thus with (4.11.2) one obtains the following observability result.

Theorem 4.11.2 (observability inequality)**.** *Let $P_\sigma = \Delta^2 - \sigma^4$ and let B_1 and B_2 be two boundary operators of order k_1 and k_2 as given in Section 4.6.2. Assume that the Lopatinskiĭ-Šapiro condition of Definition 4.6.1 holds. Let $\mathscr{O}$ be an open set of Ω. There exists $C > 0$ such that*

$$\|u\|_{H^4(\Omega)} \leq Ce^{C|\sigma|^{1/2}}\big(\|P_\sigma u\|_{L^2(\Omega)} + \sum_{j=1}^{2} |B_j u_{|\partial\Omega}|_{H^{7/2-k_j}(\partial\Omega)} + \|u\|_{L^2(\mathscr{O})}\big),$$

for $u \in H^4(\Omega)$.

Remark 4.11.3. With Remark 4.11.1 the result of Theorem 4.11.2 hold for $P_\sigma = \Delta^2 - \sigma^4$ replaced by $P_\sigma - i\sigma^2\alpha = \Delta^2 - \sigma^4 - i\sigma^2\alpha$.

4.12 Solutions to the damped plate equations

Here, we review some aspects of the solutions of the damped plate equation :

$$\begin{cases} \partial_t^2 y + Py + \alpha(x)\partial_t y = 0 & (t,x) \in \mathbb{R}_+ \times \Omega, \\ B_1 y_{|\mathbb{R}_+\times\partial\Omega} = B_2 u_{|\mathbb{R}_+\times\partial\Omega} = 0, & \\ y_{|t=0} = y^0, \quad \partial_t y_{|t=0} = y^1, & \end{cases} \tag{4.12.1}$$

where $P = \Delta^2$ and $\alpha \geq 0$, positive on some open subset of Ω. The boundary operators B_1 and B_2 of orders k_j, $j = 1, 2$, less than or equal to 3 in the normal direction are chosen so that

(i) the Lopatinskiĭ-Šapiro condition of Definition 4.6.1 is fulfilled for (P, B_1, B_2) on $\partial\Omega$;

(ii) the operator P is symmetric under homogeneous boundary conditions, that is,

$$(Pu, v)_{L^2(\Omega)} = (u, Pv)_{L^2(\Omega)}, \tag{4.12.2}$$

for $u, v \in H^4(\Omega)$ such that $B_j u_{|\partial\Omega} = B_j v_{|\partial\Omega} = 0$ on $\partial\Omega$, $j = 1, 2$. Examples of such conditions are given in Section 4.6.5.

With the assumed Lopatinskiĭ-Šapiro condition the operator

$$\begin{aligned} L : H^4(\Omega) &\to L^2(\Omega) \oplus H^{7/2-k_1}(\partial\Omega) \oplus H^{7/2-k_2}(\partial\Omega), \\ u &\mapsto (Pu, B_1 u_{|\partial\Omega}, B_2 u_{|\partial\Omega}), \end{aligned} \tag{4.12.3}$$

is Fredholm.

(iii) We shall further assume that the Fredholm index of the operator L is zero.

The previous symmetry property gives $(Pu, u)_{L^2(\Omega)} \in \mathbb{R}$. We further assume the following nonnegativity property:

(iv) For $u \in H^4(\Omega)$ such that $B_j u_{|\partial\Omega} = 0$ on $\partial\Omega$, $j = 1, 2$ one has

$$(Pu, u)_{L^2(\Omega)} \geq 0. \tag{4.12.4}$$

This last property is very natural to define a nonnegative energy for the plate equation given in (4.12.1).

We first review some properties of the unbounded operator associated with the bi-Laplace operator and the two homogeneous boundary conditions based on the assumptions made here. Second, the well-posedness of the plate equation is reviewed by means of the a semigroup formulation. This semigroup formalism is also central in the stabilization result in Sections 4.13.1–4.13.2.

4.12.1 The unbounded operator associated with the bi-Laplace operator

Associated with P and the boundary operators B_1 and B_2 is the operator $(\mathsf{P}_0, D(\mathsf{P}_0))$ on $L^2(\Omega)$, with domain

$$D(\mathsf{P}_0) = \big\{u \in L^2(\Omega);\ Pu \in L^2(\Omega),\ B_1 u_{|\partial\Omega} = B_2 u_{|\partial\Omega} = 0\big\},$$

and given by $\mathsf{P}_0 u = Pu \in L^2(\Omega)$ for $u \in D(\mathsf{P}_0)$. The definition of $D(\mathsf{P}_0)$ makes sense since having $Pu \in L^2(\Omega)$ for $u \in L^2(\Omega)$ implies that the traces $\partial_\nu^k u_{|\partial\Omega}$ are well defined for $k = 0, 1, 2, 3$.

Since the Lopatinskiĭ-Šapiro condition holds on $\partial\Omega$ one has $D(\mathsf{P}_0) \subset H^4(\Omega)$ (see for instance Theorem 20.1.7 in [44]) and thus one can also write $D(\mathsf{P}_0)$ as in (1.8.2). From the assumed nonnegativity in (4.12.4) above one finds that

$\mathsf{P}_0 + \mathrm{id}$ is injective. Since the operator

$$\begin{aligned} L' : H^4(\Omega) &\to L^2(\Omega) \oplus H^{7/2-k_1}(\partial\Omega) \oplus H^{7/2-k_2}(\partial\Omega) \\ u &\mapsto (Pu + u, B_1 u_{|\partial\Omega}, B_2 u_{|\partial\Omega}) \end{aligned}$$

is Fredholm and has the same zero index as L defined in (4.12.3), one finds that L' is surjective. Thus $\mathrm{Ran}(\mathsf{P}_0 + \mathrm{Id}) = L^2(\Omega)$. One thus concludes that P_0 is maximal monotone. From the assumed symmetry property (4.12.2) and one finds that P_0 is selfadjoint, using that a symmetric maximal monotone operator is selfadjoint (see for instance Proposition 7.6 in [15]).

The resolvent of $\mathsf{P}_0 + \mathrm{Id}$ being compact on $L^2(\Omega)$, P_0 has a sequence of eigenvalues with finite multiplicities. With the assumed nonnegativity (4.12.4) they take the form of a sequence

$$0 \leq \mu_0 \leq \mu_1 \leq \cdots \leq \mu_k \leq \cdots$$

that grows to $+\infty$. Associated with this sequence is $(\varphi_j)_{j\in\mathbb{N}}$ a Hilbert basis of $L^2(\Omega)$. Any $u \in L^2(\Omega)$ reads $u = \sum_{j\in\mathbb{N}} u_j \varphi_j$, with $u_j = (u, \varphi_j)_{L^2(\Omega)}$. We define the Sobolev-like scale

$$H^k_B(\Omega) = \{u \in L^2(\Omega); (\mu_j^{k/4} u_j)_j \in \ell^2(\mathbb{C})\} \quad \text{for } k \geq 0. \tag{4.12.5}$$

One has $D(\mathsf{P}_0) = H^4_B(\Omega)$ and $L^2(\Omega) = H^0_B(\Omega)$. Each $H^k_B(\Omega)$, $k \geq 0$, is equipped with the inner product and norm

$$(u, v)_{H^k_B(\Omega)} = \sum_{j\in\mathbb{N}} (1 + \mu_j)^{k/2} u_j \overline{v_j}. \qquad \|u\|^2_{H^k_B(\Omega)} = \sum_{j\in\mathbb{N}} (1 + \mu_j)^{k/2} |u_j|^2,$$

yielding a Hilbert space structure. The space $H^k_B(\Omega)$ is dense in $H^{k'}_B(\Omega)$ if $0 \leq k' \leq k$ and the injection is compact. Note that one uses $(1+\mu_j)^{k/2}$ in place of $\mu_j^{k/2}$ since $\ker(\mathsf{P}_0)$ may not be trivial. Note that if $k = 0$ one recovers the standard L^2-inner product and norm.

Using $L^2(\Omega)$ as a pivot space, for $k > 0$ we also define the space $H^{-k}_B(\Omega)$ as the dual space of $H^k_B(\Omega)$. One finds that any $u \in H^{-k}_B(\Omega)$ takes the form of the following limit of L^2-functions

$$u = \lim_{\ell\to\infty} \sum_{j=0}^{\ell} u_j \varphi_j,$$

for some $(u_j)_j \subset \mathbb{C}$ such that $\big((1+\mu_j)^{-k/4} u_j\big)_j \in \ell^2(\mathbb{C})$, with the limit occurring in $\big(H^k_B(\Omega)\big)'$ with the natural dual strong topology. Moreover, one has $u_j = \langle u, \overline{\varphi_j}\rangle_{H^{-k}_B, H^k_B}$. If $u = \sum_{j\in\mathbb{N}} u_j \varphi_j \in H^{-k}_B(\Omega)$ and $v = \sum_{j\in\mathbb{N}} v_j \varphi_j \in H^k_B(\Omega)$ one finds

$$\langle u, \overline{v}\rangle_{H_B^{-k}, H_B^k} = \sum_{j\in\mathbb{N}} u_j \overline{v_j}.$$

One can then extend (or restrict) the action of P_0 on any space $H_B^k(\Omega)$, $k \in \mathbb{R}$. One has $\mathsf{P}_0 : H_B^k(\Omega) \to H_B^{k-4}(\Omega)$ continuously with

$$\mathsf{P}_0 u = \sum_{j\in\mathbb{N}} \mu_j u_j \varphi_j, \quad \text{with convergence in } H_B^{k-4}(\Omega) \text{ for } u = \sum_{j\in\mathbb{N}} u_j \varphi_j \in H_B^k(\Omega). \tag{4.12.6}$$

In particular, for $u \in H_B^4(\Omega) = D(\mathsf{P}_0)$ and $v \in H_B^2(\Omega)$ one has

$$(\mathsf{P}_0 u, v)_{L^2(\Omega)} = \langle \mathsf{P}_0 u, \bar{v}\rangle_{H_B^{-2}, H_B^2} = \sum_{j\in\mathbb{N}} \mu_j u_j \overline{v_j} \tag{4.12.7}$$

and if $u, v \in H_B^2(\Omega)$ one has

$$(u, v)_{H_B^2(\Omega)} = (u, v)_{L^2(\Omega)} + \langle \mathsf{P}_0 u, \bar{v}\rangle_{H_B^{-2}, H_B^2} = \sum_{j\in\mathbb{N}} (1 + \mu_j) u_j \overline{v_j}. \tag{4.12.8}$$

Note that

$$\langle \mathsf{P}_0 u, \bar{v}\rangle_{H_B^{-2}, H_B^2} = (\mathsf{P}_0^{1/2} u, \mathsf{P}_0^{1/2} v)_{L^2(\Omega)}, \tag{4.12.9}$$

with the operator $\mathsf{P}_0^{1/2}$ easily defined by means of the Hilbert basis $(\varphi_j)_{j\in\mathbb{N}}$. In fact, $H_B^2(\Omega)$ is the domain of $\mathsf{P}_0^{1/2}$ viewed as un unbounded operator on $L^2(\Omega)$.

We make the following observations.

i. If $\ker(\mathsf{P}_0) = \{0\}$ then

$$(u, v) \mapsto \langle \mathsf{P}_0 u, \bar{u}\rangle_{H_B^{-2}, H_B^2},$$

is also an inner-product on $H_B^2(\Omega)$, that yields an equivalent norm.

ii. If 0 is an eigenvalue, that is, $\dim \ker(\mathsf{P}_0) = n \geq 1$ then $(\varphi_0, \dots, \varphi_{n-1})$ is a orthonormal basis of $\ker(\mathsf{P}_0)$ for the L^2-inner product. From a classical unique continuation property, since $\alpha(x) > 0$ for x in an open subset of Ω one sees that

$$(u, v) \mapsto (\alpha u, v)_{L^2(\Omega)} \tag{4.12.10}$$

is also an inner product on the finite dimensional space $\ker(\mathsf{P}_0) \subset L^2(\Omega)$. We introduce a second basis $(\varphi_0, \dots, \varphi_{n-1})$ of $\ker(\mathsf{P}_0)$ orthonormal with respect to this second inner product.

In what follows, we treat the more difficult case where $\dim \ker(\mathsf{P}_0) = n \geq 1$.

4.12.2 The plate semigroup generator

Set $\mathcal{H} = H^2_B(\Omega) \oplus L^2(\Omega)$ with natural inner product and norm

$$\big((u^0, u^1), (v^0, v^1)\big)_{\mathcal{H}} = (u^0, v^0)_{H^2_B(\Omega)} + (u^1, v^1)_{L^2(\Omega)}, \tag{4.12.11}$$

$$\big\|(u^0, u^1)\big\|^2_{\mathcal{H}} = \|u^0\|^2_{H^2_B(\Omega)} + \|u^1\|^2_{L^2(\Omega)}. \tag{4.12.12}$$

Define the unbounded operator

$$A = \begin{pmatrix} 0 & -1 \\ \mathsf{P}_0 & \alpha(x) \end{pmatrix}, \tag{4.12.13}$$

on $\mathcal{H}$ with domain given by $D(A) = D(\mathsf{P}_0) \oplus H^2_B(\Omega)$. This domain is dense in $\mathcal{H}$ and A is a closed operator. One has

$$\mathcal{N} = \ker(A) = \big\{{}^t(u^0, 0);\ u^0 \in \ker(\mathsf{P}_0)\big\}.$$

The important result of this section is the following proposition.

Proposition 4.12.1. *The operator $(A, D(A))$ generates a bounded semigroup $S(t) = e^{-tA}$ on $\mathcal{H}$.*

The understanding of this generator property relies on the introduction of a reduced function space associated with $\ker(\mathsf{P}_0)$, following for instance the analysis of [61]. It will be also important in the derivation of a precise resolvent estimate in Section 4.13.1. If $\ker(\mathsf{P}_0) = \{0\}$, that is, $\mu_0 > 0$, this procedure is not necessary. For $v \in \ker(\mathsf{P}_0)$, $v \neq 0$, we introduce the linear form

$$\begin{aligned} F_v &: \mathcal{H} \to \mathbb{C} \\ (u^0, u^1) &\mapsto (\alpha v, v)^{-1}_{L^2(\Omega)}\big((\alpha u^0, v)_{L^2(\Omega)} + (u^1, v)_{L^2(\Omega)}\big), \end{aligned} \tag{4.12.14}$$

We set

$$\dot{\mathcal{H}} = \bigcap_{\substack{v \in \ker(\mathsf{P}_0) \\ v \neq 0}} \ker(F_v) = \bigcap_{0 \leq j \leq n-1} \ker(F_{\varphi_j}), \tag{4.12.15}$$

with the basis $(\varphi_0, \dots, \varphi_{n-1})$ of $\ker(\mathsf{P}_0)$ introduced above. If $(v, 0) \in \ker(A)$, with $0 \neq v \in \ker(\mathsf{P}_0)$, note that $F_v(v, 0) = 1$. We set $\Theta_j = {}^t(\varphi_j, 0)$, $j =$

$0, \ldots, n-1$ and

$$\Pi_{\mathcal{N}} V = \sum_{j=0}^{n-1} F_{\varphi_j}(V)\Theta_j, \qquad \text{for } V \in \mathcal{H},$$

and $\Pi_{\dot{\mathcal{H}}} = \mathrm{id}_{\mathcal{H}} - \Pi_{\mathcal{N}}$. We obtain that $\Pi_{\mathcal{N}}$ and $\Pi_{\dot{\mathcal{H}}}$ are *continuous* projectors associated with the direct sum

$$\mathcal{H} = \dot{\mathcal{H}} \oplus \mathcal{N} \quad \text{and} \quad \dot{\mathcal{H}} = \ker(\Pi_{\mathcal{N}}). \tag{4.12.16}$$

Note that $\dot{\mathcal{H}}$ and $\mathcal{N}$ are *not orthogonal* in $\mathcal{H}$. Yet, it is important to note the following result.

Lemma 4.12.2. *We have* $\mathrm{Ran}(A) \subset \dot{\mathcal{H}}$.

Proof. Let $U = {}^t(u^0, u^1) = AV$ with $V = {}^t(v^0, v^1) \in D(A)$. One has $u^0 = -v^1 \in H^2_B(\Omega)$ and $u^1 = \mathsf{P}_0 v^0 + \alpha v_1 \in L^2(\Omega)$. If $0 \neq \varphi \in \ker(\mathsf{P}_0)$ one writes

$$\begin{aligned}(\alpha\varphi, \varphi)_{L^2(\Omega)} F_\varphi(U) &= (-\alpha v^1, \varphi)_{L^2(\Omega)} + (\mathsf{P}_0 v^0 + \alpha v^1, \varphi)_{L^2(\Omega)} \\ &= (\mathsf{P}_0 v^0, \varphi)_{L^2(\Omega)} = (v^0, \mathsf{P}_0 \varphi)_{L^2(\Omega)} = 0.\end{aligned}$$

using that $v^0, \varphi \in D(\mathsf{P}_0)$, that $(\mathsf{P}_0, D(\mathsf{P}_0))$ is selfadjoint, and that $\varphi \in \ker(\mathsf{P}_0)$. The conclusion follows from the definition of $\dot{\mathcal{H}}$ in (4.12.15). $\square$

The space $\dot{\mathcal{H}}$ inherits the natural inner product and norm of $\mathcal{H}$ given in (4.12.11). Yet one finds that the inner product

$$((u^0, u^1), (v^0, v^1))_{\dot{\mathcal{H}}} = \langle \mathsf{P}_0 u^0, \overline{v^0} \rangle_{H_B^{-2}, H_B^2} + (u^1, v^1)_{L^2(\Omega)}, \tag{4.12.17}$$

and associated norm

$$\|(u^0, u^1)\|^2_{\dot{\mathcal{H}}} = \langle \mathsf{P}_0 u^0, \overline{u^0} \rangle_{H_B^{-2}, H_B^2} + \|u^1\|^2_{L^2(\Omega)}, \tag{4.12.18}$$

yields an equivalent norm on $\dot{\mathcal{H}}$ by a Poincaré-like argument.

We introduce the unbounded operator $\dot{A}$ on $\dot{\mathcal{H}}$ given by the domain $D(\dot{A}) = D(A) \cap \dot{\mathcal{H}}$ and such that $\dot{A}V = AV$ for $V \in D(\dot{A})$. We then have $A = \dot{A} \circ \Pi_{\dot{\mathcal{H}}}$. Observe that $D(\dot{A}) = \Pi_{\dot{\mathcal{H}}}\big(D(A)\big)$ since $\mathcal{N} = \ker(A) \subset D(A)$. Thus, one has

$$D(A) = D(\dot{A}) \oplus \mathcal{N}. \tag{4.12.19}$$

As for the decomposition of $\mathcal{H}$ given in (4.12.16) note that $D(\dot{A})$ and $\mathcal{N}$ are not orthogonal.

Lemma 4.12.3. *Let $z \in \mathbb{C}$ be such that* $\operatorname{Re} z < 0$. *We have*

$$\|(zId_{\dot{\mathcal{H}}} - \dot{A})U\|_{\dot{\mathcal{H}}} \geq |\operatorname{Re} z|\, \|U\|_{\dot{\mathcal{H}}}, \quad U \in D(\dot{A}).$$

The proof of this lemma is quite classical. It is given in Appendix 6.2.

With the previous lemma, with the Hille-Yosida theorem one proves the following result.

Lemma 4.12.4. *The operator $(\dot{A}, D(\dot{A}))$ generates a semigroup of contraction $\dot{S}(t) = e^{-t\dot{A}}$ on $\dot{\mathcal{H}}$.*

If we set

$$S(t) = \dot{S}(t) \circ \Pi_{\dot{\mathcal{H}}} + \Pi_{\mathcal{N}}, \tag{4.12.20}$$

we find that $S(t)$ is a semigroup on $\mathcal{H}$ generated by $(A, D(A))$, thus proving Proposition 4.12.1. If $Y^0 \in D(A)$, the solution of the semigroup equation $\frac{d}{dt}Y(t) + AY(t) = 0$ reads

$$Y(t) = S(t)Y^0 = \dot{S}(t) \circ \Pi_{\dot{\mathcal{H}}} Y^0 + \Pi_{\mathcal{N}} Y^0. \tag{4.12.21}$$

We set $\dot{Y}(t) = \Pi_{\dot{\mathcal{H}}} Y(t) = \dot{S}(t) \circ \Pi_{\dot{\mathcal{H}}} Y^0$.

The adjoint of $\dot{A}$ has domain $D(\dot{A}^*) = D(A)$ and is given by

$$\dot{A}^* = \begin{pmatrix} 0 & 1 \\ -\mathsf{P}_0 & \alpha(x) \end{pmatrix}.$$

Similarly to Lemma 4.12.3 one has the following result with a similar proof.

Lemma 4.12.5. *Let $z \in \mathbb{C}$ be such that* $\operatorname{Re} z < 0$. *We have*

$$\|(zId_{\dot{\mathcal{H}}} - \dot{A}^*)U\|_{\dot{\mathcal{H}}} \geq |\operatorname{Re} z|\, \|U\|_{\dot{\mathcal{H}}}, \quad U \in D(\dot{A}^*) = D(\dot{A}).$$

4.12.3 Strong and weak solutions to the damped plate equation

For $y(t)$ a solution to the damped plate equation (4.12.1) one has $Y(t) = {}^t(y(t), \partial_t y(t))$ formally solution to $\frac{d}{dt}Y(t) + AY(t) = 0$ and conversely.

The semigroup $S(t)$ generated by A as given by Proposition 4.12.1 allows one to go beyond this formal observation and one obtains the following well-posedness result for strong solutions of the damped plate equation.

Proposition 4.12.6 (strong solutions of the damped plate equation)**.** *For* $(y^0, y^1) \in H_B^4(\Omega) \times H_B^2(\Omega)$ *there exists a unique*

$$y \in \mathscr{C}^0\big([0,+\infty); H_B^4(\Omega)\big) \cap \mathscr{C}^1\big([0,+\infty); H_B^2(\Omega)\big) \cap \mathscr{C}^2\big([0,+\infty); L^2(\Omega)\big)$$

such that

$$\partial_t^2 y + Py + \alpha \partial_t y = 0 \quad \text{in } L^\infty([0,+\infty); L^2(\Omega)), \qquad y_{|t=0} = y^0,\ \partial_t y_{|t=0} = y^1. \tag{4.12.22}$$

Moreover, there exists $C > 0$ *such that*

$$\|y(t)\|_{H_B^4(\Omega)} + \|\partial_t y(t)\|_{H_B^2(\Omega)} \leq C\big(\|y^0\|_{H_B^4(\Omega)} + \|y^1\|_{H_B^2(\Omega)}\big), \qquad t \geq 0. \tag{4.12.23}$$

With $Y(t)$ as above, for such a solution $y(t)$ one has

$$\frac{d}{dt} Y(t) + AY(t) = 0, \qquad Y(0) = Y^0 = {}^t(y^0, y^1),$$

that is,

$$Y(t) = S(t)Y^0 \in \mathscr{C}^0\big([0,+\infty); D(A)\big) \cap \mathscr{C}^1\big([0,+\infty); H_B^2(\Omega) \oplus L^2(\Omega)\big).$$

A weak solution to the damped plate equation is simply associated with an initial data $(y^0, y^1) \in H_B^2(\Omega) \times L^2(\Omega)$ and given by the first coordinate of $Y(t) = S(t)Y^0$. Then one has

$$Y(t) \in \mathscr{C}^0\big([0,+\infty); \mathcal{H}\big) \cap \mathscr{C}^1\big([0,+\infty); L^2(\Omega) \oplus H_B^{-2}(\Omega)\big).$$

or equivalently

$$y \in \mathscr{C}^0\big([0,+\infty); H_B^2(\Omega)\big) \cap \mathscr{C}^1\big([0,+\infty); L^2(\Omega)\big) \cap \mathscr{C}^2\big([0,+\infty); H_B^{-2}(\Omega)\big).$$

For a strong solution, the natural energy is given by

$$\mathcal{E}(y)(t) = \frac{1}{2}\big(\|\partial_t y(t)\|^2_{L^2(\Omega)} + (\mathsf{P}_0 y(t), y(t))_{L^2(\Omega)}\big). \tag{4.12.24}$$

Observe that if $y^0 \in \ker(\mathsf{P}_0)$ then $y(t) = y^0$ is solution to (4.12.1) with $y^1 = 0$. This is consistent with the form of the semigroup $S(t)$ given in (4.12.20). Such a solution is independent of the evolution variable t, and thus, despite damping, there is no decay. However, note that such a solution is 'invisible' for the energy defined in (4.12.24). In fact, for a strong solution to (4.12.1) as given by

Proposition 4.12.6 one has

$$\mathcal{E}(y)(t) = \frac{1}{2}\|\dot{Y}(t)\|_{\dot{\mathcal{H}}}^2, \tag{4.12.25}$$

with $\dot{Y}(t)$ as defined below (4.12.21) and $\|.\|_{\dot{\mathcal{H}}}$ defined in (4.12.18). For a strong solution, we write

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(y)(t) &= \operatorname{Re}(\partial_t y(t), \partial_t^2 y(t))_{L^2(\Omega)} + \frac{1}{2}\langle \mathsf{P}_0 \partial_t y(t), \overline{y(t)}\rangle_{H_B^{-2},H_B^2} + \frac{1}{2}(\mathsf{P}_0 y(t), \partial_t y(t))_{L^2(\Omega)} \\ &= \operatorname{Re}(\partial_t y(t), (\partial_t^2 + \mathsf{P}_0)y(t))_{L^2(\Omega)} = -\operatorname{Re}(\partial_t y(t), \alpha\partial_t y(t))_{L^2(\Omega)} \leq 0\end{aligned}$$

since $\alpha \geq 0$. Thus, the energy of a strong solution is nonincreasing. To understand the decay of the energy one has to focus on the properties of the semigroup $\dot{S}(t)$ and its generator $(\dot{A}, D(\dot{A}))$ on $\dot{\mathcal{H}}$. This is done in Section 4.13.1.

For a weak solution $y(t) \in \mathscr{C}^0\big([0,+\infty); H_B^2(\Omega)\big) \cap \mathscr{C}^1\big([0,+\infty); L^2(\Omega)\big)$ the energy is defined by

$$\mathcal{E}(y)(t) = \frac{1}{2}\big(\|\partial_t y(t)\|_{L^2(\Omega)}^2 + \langle \mathsf{P}_0 y(t), \overline{y(t)}\rangle_{H_B^{-2},H_B^2}\big)$$

that coincides with (4.12.24) for a strong solution. The stabilization result we are interested in only concerns strong solutions (see Section 4.13.2). Thus, we shall not mention weak solutions in what follows.

4.13 Resolvent estimates and applications to stabilization

Here we use the observability inequality of Theorem 4.11.2 to obtain a resolvent estimate for the plate semigroup generator that allows one to deduce a stabilization result for the damped plate equation. This a sequence of argument comes from the seminal works of Lebeau [59] and Lebeau-Robbiano [61].

4.13.1 Resolvent estimate

We prove a resolvent estimate for the unbounded operator $(\dot{A}, D(\dot{A}))$ that acts on $\dot{\mathcal{H}}$. First, we establish that $\{\operatorname{Re} z \leq 0\}$ lies in the resolvent set of $\dot{A}$.

Proposition 4.13.1. *The spectrum of* $(\dot{A}, D(\dot{A}))$ *is contained in* $\{z \in \mathbb{C}; \operatorname{Re}(z) > 0\}$.

The proof of this proposition is rather classical based on a unique continuation argument and a Fredholm index argument for a compact perturbation. It is given in Appendix 6.3.

Theorem 4.13.2. *Let $\mathscr{O}$ be an open subset of Ω such that $\alpha \geq \delta > 0$ on $\mathscr{O}$. Then, for $\sigma \in \mathbb{R}$ the unbounded operator $i\sigma Id - \dot{A}$ is invertible on $\dot{\mathcal{H}}$ and for there exist $C > 0$ such that*

$$\|(i\sigma Id - \dot{A})^{-1}\|_{\mathcal{L}(\dot{\mathcal{H}})} \leq Ce^{C|\sigma|^{1/2}}, \qquad \sigma \in \mathbb{R}. \tag{4.13.1}$$

Proof. By Proposition 4.13.1 $i\sigma\mathrm{Id} - \dot{A}$ is indeed invertible. Observe that it then suffices to prove the resolvent estimate (4.13.1) for $|\sigma| \geq \sigma_0$ for some $\sigma_0 > 0$.

Let $U = {}^t(u^0, u^1) \in D(\dot{A})$ and $F = {}^t(f^0, f^1) \in \dot{\mathcal{H}}$ be such that $(i\sigma\mathrm{Id} - \dot{A})U = F$. This reads

$$f^0 = i\sigma u^0 + u^1, \qquad f^1 = -\mathsf{P}_0 u^0 + (i\sigma - \alpha)u^1.$$

which gives

$$(\mathsf{P}_0 - \sigma^2 - i\sigma\alpha)u^0 = f$$

with $f = (i\sigma - \alpha)f^0 - f^1$. Computing the L^2-inner product with u^0 one finds

$$((\mathsf{P}_0 - \sigma^2)u^0, u^0)_{L^2(\Omega)} - i\sigma(\alpha u^0, u^0)_{L^2(\Omega)} = (f, u^0)_{L^2(\Omega)}.$$

As $\alpha \geq 0$, computing the imaginary part one obtains

$$\sigma\|\alpha^{1/2}u^0\|^2_{L^2(\Omega)} = -\operatorname{Im}(f, u^0)_{L^2(\Omega)}.$$

Since $\alpha \geq \delta > 0$ in $\mathscr{O}$ by assumption and since we consider $|\sigma| \geq \sigma_0$ one has

$$\delta\sigma^0\|u^0\|^2_{L^2(\mathscr{O})} \leq \|f\|_{L^2(\Omega)}\|u^0\|_{L^2(\Omega)}.$$

Applying Theorem 4.11.2 (with Remark 4.11.3) one has

$$\|u^0\|_{H^4(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\big(\|f\|_{L^2(\Omega)} + \|u^0\|_{L^2(\mathscr{O})}\big).$$

Thus, we obtain

$$\|u^0\|_{H^4(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\big(\|f\|_{L^2(\Omega)} + \|f\|^{1/2}_{L^2(\Omega)}\|u^0\|^{1/2}_{L^2(\Omega)}\big),$$

for $|\sigma| \geq \sigma_0$. With Young inequality we write, for $\varepsilon > 0$,

$$e^{C|\sigma|^{1/2}}\|f\|^{1/2}_{L^2(\Omega)}\|u^0\|^{1/2}_{L^2(\Omega)} \lesssim \varepsilon^{-1}e^{2C|\sigma|^{1/2}}\|f\|_{L^2(\Omega)} + \varepsilon\|u^0\|_{L^2(\Omega)}.$$

Thus, with ε chosen sufficiently small one obtains

$$\|u^0\|_{H^4(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\|f\|_{L^2(\Omega)}.$$

Since $u^1 = f^0 - i\sigma u^0$ and $f = (i\sigma - \alpha)f^0 - f^1$ we finally obtain that

$$\begin{aligned}\|u^0\|_{H^4(\Omega)} + \|u^1\|_{L^2(\Omega)} &\lesssim e^{C|\sigma|^{1/2}}\big(\|f^0\|_{L^2(\Omega)} + \|f^1\|_{L^2(\Omega)}\big)\\ &\lesssim e^{C|\sigma|^{1/2}}\|F\|_{\dot{\mathcal{H}}}.\end{aligned}$$

Since $u^0 \in H^4(\Omega)$ one has

$$\big|(\mathsf{P}_0 u^0, u^0)_{L^2(\Omega)}\big| \le \|u^0\|_{H^4(\Omega)}\|u^0\|_{L^2(\Omega)} \le \|u^0\|^2_{H^4(\Omega)}$$

and thus one finally obtains

$$\|U\|^2_{\dot{\mathcal{H}}} = (\mathsf{P}_0 u^0, u^0)_{L^2(\Omega)} + \|u^1\|_{L^2(\Omega)} \lesssim e^{C|\sigma|^{1/2}}\|F\|_{\dot{\mathcal{H}}},$$

which concludes the proof of the resolvent estimate (4.13.1). □

4.13.2 Stabilization result

As an application of the resolvent estimate of Theorem 4.13.2, we give a logarithmic stabilization result of the damped plate equation (4.5.1).

For the plate generator $(A, D(A))$ its iterated domains are inductively given by

$$D(A^{n+1}) = \{U \in D(A^n); AU \in D(A^n)\}.$$

With Proposition 4.12.6, for $Y^0 = {}^t(y^0, y^1) \in D(A^n)$ then the first component of $Y(t) = S(t)Y^0$ is precisely the solution to (4.12.1). One has $Y(t) = \dot{Y}(t) + \Pi_{\mathcal{N}} Y^0$ with $\dot{Y}(t) = \dot{S}(t)\Pi_{\dot{\mathcal{H}}} Y^0$ with the semigroup $\dot{S}(t)$ defined in Section 4.12.2. Moreover, by (4.12.25) the energy of $y(t)$ is given by the square of the $\dot{\mathcal{H}}$-norm of $\dot{Y}(t)$.

With the resolvent estimate of Theorem 4.13.2, with the result of Theorem 1.5 in [10] one obtains the following bound for the energy of $y(t)$:

$$\mathcal{E}(y)(t) = \|\dot{Y}(t)\|^2_{\dot{\mathcal{H}}} \le \frac{C}{\big(\log(2+t)\big)^{4n}}\|A^n Y^0\|_{\dot{\mathcal{H}}}. \tag{4.13.2}$$

We have thus obtain the following theorem.

Theorem 4.13.3 (logarithmic stabilisation for the damped plate equation)**.** *Assume that conditions (i) to (iv) of Section 4.12 hold. Let $n \in \mathbb{N}$, $n \ge 1$. Then, there exists $C > 0$ such that for any $Y^0 = {}^t(y^0, y^1) \in D(A^n)$ the associated solution $y(t)$ of the damped plate equation* (4.5.1) *has the logarithmic energy decay given by* (4.13.2).

Note that for $n = 1$ using the form of A and (4.12.9) one recovers the statement of Theorem 1.8.2 .

5. Perspectives

5.1 Fučik spectrum

As an extension of the results of Part A, we are interested of investigating the Fučik spectrum of the $(p, 2)$-Lapalcian. The Fučik spectrum was introduced by S. Fučik [22] and N. Dancer [27] in the 70's. For the Laplacian it is defined as the set $\Sigma \subset \mathbb{R} \times \mathbb{R}$ of the points (α, β) for which there exists a nontrivial solution of the problem

$$\begin{cases} -\Delta u = \alpha u^+ - \beta u^- \quad \text{in} \ \ \Omega \\ Bu = 0 \quad \text{on} \ \ \partial\Omega \end{cases} \tag{5.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$, Bu stands for the considered boundary conditions and $u^{\pm} = \max\{\pm u, 0\}$.
We aim to investigate the set of pairs $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} -\Delta_p u - \Delta u = \alpha u^+ - \beta u^- \quad \text{in} \ \ \Omega \\ u = 0 \quad \text{on} \ \ \partial\Omega \end{cases} \tag{5.1.2}$$

has nontrivial solution, with $p \in (1, \infty) \setminus \{2\}$. We clearly see that if $\alpha = \beta$ we recover the case of equation (3.4.3).
We expect solutions branches of (5.1.2) to bifurcate from the Fučik eigenvalues of (5.1.1).
Next, our goal is to study the Fučik spectrum of the following problem

$$\begin{cases} -\Delta_p u - \Delta u = \alpha m(x) u^+ - \beta n(x) u^- \quad \text{in} \ \ \Omega \\ u = 0 \quad \text{on} \ \ \partial\Omega \end{cases} \tag{5.1.3}$$

where m and n are positive bounded weights for $p \in (1, \infty) \setminus \{2\}$.

5.2 Controllability and stablization

On a bounded regular open set of $\mathbb{R}^d$ if given a positive fourth-order elliptic operator P (or on a Riemannian manifold $(\mathcal{M}, g)$ with $P = \Delta_g^2$), one can consider the following controlled parabolic equation

$$\partial_t u + Pu = 1_\omega v \ \text{ for } t \geq 0, \ B_1 u_{|[0,\infty)} = 0, \ B_2 u_{|[0,\infty)} = 0, \ u_{|t=0} = u_0 \in L^2(\Omega). \tag{5.2.1}$$

Here, ω is a nonempty open subset of ω, B_1 and B_2 are boundary operators chosen that satisfies the Lopatinskiĭ-Šapiro condition. The function v is the control and lies in $L^2((0,\infty)\times\Omega)$. It only acts on the solution u in ω. The question of null controllability for this controlled parabolic equation is the following:

For a given initial data $u^0 \in L^2(\Omega)$, for a given time $T > 0$, can one find $v \in L^2((0,T)\times\Omega)$ such $u(T) = 0$.

The answer to this question rely on the derivation of a spectral inequality. If the boundary operators B_1 and B_2 are well chosen, the bi-Laplace operator Δ_g^2 can be selfadjoint on $L^2(\Omega)$; see Section 4.12.1. Associated with the operator is then a Hilbert basis $(\varphi_j)_{j\in\mathbb{N}}$ of $L^2(\omega)$. In the case of "clamped" boundary condition the following spectral inequality was proven in [82].

Theorem 5.2.1. (Spectral inequality for the "clamped" bi-Laplace operator). *Let ω be an open subset of Ω. There exists $C > 0$ such that*

$$\|u\|_{L^2(\Omega)} \leq Ce^{C\mu^{1/4}}\|u\|_{L^2(\omega)}, \quad \mu > 0, \quad u \in \mathit{Span}\{\varphi_j;\ \mu_j \leq \mu\}.$$

The proof of this theorem is based on a Carleman inequality for the fourth-order elliptic operator $D_s^4 + \Delta_g^2$, that is, after the addition of a variable s. Extending this strategy to the type of boundary conditions treated in Part B was not successful so far because it is not guaranteed that having the Lopatinskiĭ-Šapiro condition for Δ_g^2, B_1, and B_2 implies that the Lopatinskiĭ-Šapiro condition holds for $D_s^4 + \Delta_g^2, B_1$, and B_2. Yet, the Lopatinskiĭ-Šapiro is at the heart of the proof of our Carleman estimate. Proving a spectral estimate as in the above statement for the general boundary conditions considered here is an open question.

Next, as a follow up of the result in Part B, we aim to address the polyharmonic case, that is $Q = P^k$, $k \in \mathbb{N}$ with k boundary operators $B_1, \dots, B_k$. This can lead to applications similar to the results of Theorem 1.8.2.

For the polyharmonic case, the Lopatinskiĭ-Šapiro condition can be formulated as follows.

Let $Q = (-\Delta_g)^k$ be an elliptic differential operator of order $2k$ on Ω, $(k \geq 1)$, with principal symbol $q(x,\omega)$ for $(x,\omega) \in T^*\mathcal{M}$. One defines the following polynomial in z,

$$\tilde{q}(x,\omega',z) = q(x,\omega' - zn_x),$$

for $x \in \partial\mathcal{M}$, $\omega' \in T_x^*\partial\mathcal{M}$, $z \in \mathbb{R}$ and n_x denotes the outward pointing conormal vector at x, unitary in the sense of the metric g. Here x and ω' act as parameters. We denote by $r_j(x,\omega')$, $1 \leq j \leq 2k$ the complex roots of $\tilde{q}$. One sets

$$\tilde{q}^+(x,\omega',z) = \prod_{\operatorname{Im} r_j(x,\omega')\geq 0} (z - r_j(x,\omega')).$$

Given boundary operators $B_1, \ldots, B_k$ in a neighborhood of $\partial\mathcal{M}$, with principal symbols $b_j(x,\omega)$, $j = 1, \ldots, k$, one also sets $\tilde{b}_j(x,\omega',z) = b_j(x,\omega' - zn_x)$. According to Definition 4.6.1, along with the general boundary operators $B_1, B_2 \ldots, B_k$ of orders $d_1, d_2, \ldots, d_\ell$ respectively for $\ell = 1, \ldots, k$, we give a matrix criterion of the Lopatinskiĭ-Šapiro condition. The general boundary operators $B_1, B_2 \ldots, B_k$ is then given by

$$B_\ell(x,D) = \sum_{0 \le j \le \min(2k-1, d_\ell)} (-i)^j B_\ell^{d_\ell - j}(x,D') D_d^j, \qquad \ell = 1 \ldots, k.$$

We denote by $b_1(x,\omega), \ldots, b_k(x,\omega)$ the principal symbols of $B_1, \ldots, B_k$ respectively. For $(x,\omega') \in T^*\partial\mathcal{M}$, we set

$$b_\ell(x,\omega',z) = \sum_{0 \le j \le \min(2k-1, d_\ell)} (-i)^j b_\ell^{d_\ell - j}(x,\omega') z_d^j, \qquad \ell = 1 \ldots, k.$$

We recall that the principal symbol of Q is given by $q(x,\omega) = |\omega|_g^{2k}$. We set

$$\tilde{q}(x,\omega',z) = q(x,\omega' - zn_x) = (|\omega' - zn_x|^2)^k = (z^2 + |\omega'|^2)^k,$$

where $(n_x,\omega')_g = 0$. Therefore $\tilde{q}(x,\omega',z) = (z - i|\omega'|)^k (z + |\omega'|)^k$ and we set $\tilde{q}^+(x,\omega',z) = (z + |\omega'|)^k$. Thus the Lopatinskiĭ-Šapiro condition holds at (x,ω') with $\omega' \neq 0$ if and only if for any polynomial function $f(z)$ the complex number $i|\omega'|_g$ is a root of the polynomial function $z \mapsto f(z) - c_1\tilde{b}_1(x,\omega',z) - \cdots - c_k\tilde{b}_k(x,\omega',z)$ and its derivative up to order $k-1$ for some $c_1, \ldots, c_k \in \mathbb{C}$. This leads to the following determinant condition

$$\det \begin{pmatrix} \tilde{b}_1 & \tilde{b}_2 & \ldots & \tilde{b}_k \\ \partial_z\tilde{b}_1 & \partial_z\tilde{b}_2 & \ldots & \partial_z\tilde{b}_k \\ \vdots & \vdots & \ldots & \vdots \\ \partial_z^{k-1}\tilde{b}_1 & \partial_z^{k-1}\tilde{b}_2 & \ldots & \partial_z^{k-1}\tilde{b}_k \end{pmatrix} (x,\omega', z = i|\omega'|_g) \neq 0. \tag{5.2.2}$$

6. Appendix

6.1 A perfect elliptic estimate

Here we consider $a(\varrho', \xi_d)$ polynomial in the ξ_d variable and such that its root have negative imaginary parts microlocally.

Lemma 6.1.1. *Let $\kappa_0 > 0$. Let $a(\varrho', \xi_d) \in S_\tau^{k,0}$, with $\varrho' = (x, \xi', \tau, \sigma)$ and with $k \geq 1$, that is, $a(\varrho', \xi_d) = \sum_{j=0}^{k} a_j(\varrho')\xi_d^{k-j}$, and where the coefficients a_j are homogeneous in (ξ', τ, σ). Moreover, assume that $a_0(\varrho') = 1$. Set $A = \mathrm{Op}(a)$.*

Let $\mathscr{U}$ be a conic open subset of $W \times \mathbb{R}^{d-1} \times [0, +\infty) \times [0, +\infty)$ where $\tau \geq \kappa_0\sigma$ and such that all the roots of $a(\varrho', \xi_d)$ have a negative imaginary part for $\varrho' \in \mathscr{U}$.

Let $\chi(\varrho') \in S_{\mathrm{T},\tau}^0$ be homogeneous of degree zero and such that $\mathrm{supp}(\chi) \subset \mathscr{U}$ and $N \in \mathbb{N}$. Then there exist $C > 0$, $C_N > 0$, and $\tau_0 > 0$ such that

$$\|\mathrm{Op}(\chi)v\|_{k,\tau} + |\operatorname{tr}(\mathrm{Op}(\chi)v)|_{k-1,1/2,\tau} \leq C\|A\mathrm{Op}(\chi)v\|_+ + \|v\|_{k,-N,\tau},$$

for $w \in \overline{\mathscr{S}}(\mathbb{R}_+^d)$ and $\tau \geq \max(\tau_0, \kappa_0\sigma)$.

Proof. Let $\mathscr{W}$ be a conic open set of $W \times \mathbb{R}^{d-1} \times [0,\infty) \times [0,\infty)$ such that $\mathscr{W} \subset \mathscr{U}$ and $\mathrm{supp}(\chi) \subset \mathscr{W}$. We write

$$a(\varrho', \xi_d) = p(\varrho', \xi_d) + iq(\varrho', \xi_d),$$

where p and q are both homogeneous with $p \in S_\tau^{k,0}$ and $q \in S_\tau^{k-1,0}$. We set $P = \mathrm{Op}(p)$ and $Q = \mathrm{Op}(q)$ and we introduce the following quadratic form of type $(k, 0)$

$$S(w) = \|Pw\|_+^2 + \|Qw\|_+^2$$

with principal symbol $s(\varrho', \xi_d) = |p(\varrho', \xi_d)|^2 + |q(\varrho', \xi_d)|^2 \in S_\tau^{2k,0}$.
The Hermite theorem (see, [11, Proposition 3.13]) implies that $a(\varrho', \xi_d)$ and $b(\varrho', \xi_d)$ have distinct real roots for all $\varrho' \in \mathscr{U}$. Hence, on the compact set

$$\mathscr{C} = \{\ \varrho = (x, \xi, \tau, \sigma);\ \varrho' = (x, \xi', \tau, \sigma) \in \overline{\mathscr{W}},\ \xi_d \in \mathbb{R},\ \ |\xi|^2 + \tau^2 + \sigma^2 = 1\ \},$$

we have $s \neq 0$ yielding by homogeneity that

$$s(\varrho', \xi_d) \geq C|(\xi, \tau, \sigma)|^{2k} \geq \tilde{C}|(\xi, \tau)|^{2k}, \text{ since } |(\xi, \tau, \sigma)|^2 = |\xi|^2 + \tau^2 + \sigma^2 \gtrsim |\xi|^2 + \tau^2$$

for $\tau \gtrsim \sigma$. Setting $\underline{w} = \mathrm{Op}(\chi)v$, the Gårding inequality ([11, Proposition 3.5]) gives, for any $N \in \mathbb{N}$,

$$s(\underline{w}) \gtrsim \|\underline{w}\|_{k,\tau} - \|\operatorname{tr}(\underline{w})\|_{k-1,1/2,\tau} - \|v\|_{k,-N,\tau}. \tag{6.1.1}$$

In addition, by the Generalized Green's formula (see, [11, Proposition 3.15]), we have

$$2\,\mathrm{Re}(P\underline{w}, iQ\underline{w})_+ = I_{p,q}(\underline{w}) + \mathscr{B}_{p,q}(\underline{w}) + R(v),$$

where $I_{p,q}$ is an interior quadratic form of type $(k,-1/2)$, $\mathscr{B}_{p,q}$ is the boundary quadratic form of type $(k-1,1/2)$ and $R(\underline{w})$ is the remainder term which is a quadratic form that satisfies

$$|R(\underline{w})| \lesssim \|\underline{w}\|_{k,-1,\tau}.$$

Therefore,

$$\begin{array}{rcl} |2\,\mathrm{Re}(P\underline{w}, iQ\underline{w})_+ - \mathscr{B}_{p,q}(\underline{w})| & = & |I_{p,q}(\underline{w}) + R(\underline{w})| \\ & \leq & |I_{p,q}(\underline{w})| + |R(\underline{w})| \\ & \lesssim & \|\underline{w}\|_{k,-1/2,\tau}, \end{array}$$

since $|I_{p,q}(\underline{w})| \leq C\|\underline{w}\|_{k,-1/2,\tau}$ for $w \in \overline{\mathscr{S}}(\mathbb{R}^d_+)$ by Lemma 3.3 in [11]. Then we deduce that

$$2\,\mathrm{Re}(P\underline{w}, iQ\underline{w})_+ \gtrsim \mathscr{B}_{p,q}(\underline{w}) - \|\underline{w}\|_{k,-1/2,\tau}.$$

Again by the Hermite theorem ([11, Proposition 3.13]), the bilinear form associated to $\mathscr{B}_{p,q}$ denoted $\Sigma_{\mathscr{B}_{p,q}}$ is positive. Then by homogeneity we find that

$$\Sigma_{\mathscr{B}_{a,b}}(\underline{\varrho}', z, z) \geq C\sum_{n=0}^{k-1} \lambda_{\mathsf{T},\tau,\sigma}^{2(k-1-n+1/2)}|z_n|^2,\ \underline{\varrho}' \in \overline{\mathscr{W}},\ \ z = (z_0, \cdots, z_{m-1}) \in \mathbb{C}^m,\ \lambda_{\mathsf{T},\tau,\sigma} = |(\xi', \tau, \sigma)|.$$

Then the Gårding inequality of Lemma 3.9 in [11] gives, for any $N \in \mathbb{N}$,

$$2\mathrm{Re}(P\underline{w}, iQ\underline{w})_+ \gtrsim |\operatorname{tr}(\underline{w})|_{k-1,1/2,\tau} - \|\underline{w}\|_{k,-1/2,\tau} - |\operatorname{tr}(v)|_{k-1,-N,\tau}. \qquad (6.1.2)$$

But on the other hand, we have

$$\|A\underline{w}\|_+^2 = \|P\underline{w} + iQ\underline{w}\|_+^2 = \|P\underline{w}\|_+^2 + \|Q\underline{w}\|_+^2 + 2\,\mathrm{Re}(P\underline{w}, iQ\underline{w})_+ = S(\underline{w}) + 2\,\mathrm{Re}(P\underline{w}, iQ\underline{w})_+.$$

So, by (6.1.1) and (6.1.2) we find,

$$\|A\underline{w}\|_+ \gtrsim \|\underline{w}\|_{k,\tau} - |\operatorname{tr}(\underline{w})|_{k-1,1/2,\tau} - \|\underline{w}\|_{k,-N,\tau} - |\operatorname{tr}(v)|_{k-1,-N,\tau}, \qquad (6.1.3)$$

for τ chosen sufficiently large.
However, observing that $S(\underline{w}) \geq 0$, we also find that

$$\|A\underline{w}\|_+ \gtrsim |\operatorname{tr}(\underline{w})|_{k-1,1/2,\tau} - \|\underline{w}\|_{k,-1/2,\tau} - |\operatorname{tr}(v)|_{k-1,-N,\tau}. \qquad (6.1.4)$$

By adding estimate (6.1.3) and (6.1.4) side by side, and taking τ to be large enough, we obtain

$$\|\underline{w}\|_{k,\tau} + |\operatorname{tr}(\underline{w})|_{k-1,1/2,\tau} \lesssim \|A\underline{w}\|_+ + \|v\|_{k,-N,\tau} + |\operatorname{tr}(v)|_{k-1,-N,\tau},$$

which ends the proof. $\square$

6.2 Basic resolvent estimation

Here we provide a proof of Lemma 4.12.3

Let $U = {}^t(u^0, u^1) \in D(\dot{A})$. With (4.12.17) We write

$$\begin{aligned}((z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}} &= \left(\begin{pmatrix} zu^0 + u^1 \\ zu^1 - \mathsf{P}_0 u^0 - \alpha u^1 \end{pmatrix}, \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}\right)_{\dot{\mathcal{H}}} \\ &= z\|U\|^2_{\dot{\mathcal{H}}} + \langle \mathsf{P}_0 u^1, \overline{u^0}\rangle_{H_B^{-2}, H_B^2} - (\mathsf{P}_0 u^0, u^1)_{L^2(\Omega)} - (\alpha u^1, u^1)_{L^2(\Omega)} \\ &= z\|U\|^2_{\dot{\mathcal{H}}} + 2i\operatorname{Im}(u^1, \mathsf{P}_0 u^0)_{L^2(\Omega)} - (\alpha u^1, u^1)_{L^2(\Omega)}.\end{aligned}$$

Computing the real part one obtains

$$-\operatorname{Re}((z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}} = -\operatorname{Re}(z)\|U\|^2_{\dot{\mathcal{H}}} + (\alpha u^1, u^1)_{L^2(\Omega)}. \tag{6.2.1}$$

As $\alpha \geq 0$ and $\operatorname{Re} z < 0$, this gives

$$|\operatorname{Re}((z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}}| \geq |\operatorname{Re}(z)|\, \|U\|^2_{\dot{\mathcal{H}}},$$

which yields the conclusion of Lemma 4.12.3. $\square$

6.3 Basic estimation for the resolvent set

Here we provide a proof of Proposition 4.13.1.

Let $z \in \mathbb{C}$. We consider the two cases.

Case 1: $\operatorname{Re} z < 0$. By Lemma 4.12.3 $z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}$ is injective. Moreover, as its adjoint $\overline{z}\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}^*$ is injective and satisfies $\|(\overline{z}\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}^*)U\|_{\dot{\mathcal{H}}} \gtrsim \|U\|_{\dot{\mathcal{H}}}$ for $U \in D(\dot{A})$ by Lemma 4.12.5 the map $z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}$ is surjective (see for instance [?, Theorem 2.20]). The estimation of Lemma 4.12.3 then gives the continuity of the operator $(z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})^{-1}$ on $\dot{\mathcal{H}}$.

Case 2: Re $z = 0$. We start by proving the injectivity of $z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}$. Let thus $U = {}^t(u^0, u^1) \in D(\dot{A})$ be such that $zU - \dot{A}U = 0$. This gives

$$zu^0 + u^1 = 0, \quad -\mathsf{P}_0 u^0 + (z - \alpha)u^1 = 0. \tag{6.3.1}$$

First, if $z = 0$ one has $u^1 = 0$, and then $\mathsf{P}_0 u^0 = 0$. Thus, $u^0 \in \ker(\mathsf{P}_0)$ given $U \in \mathcal{N} = \ker(A)$. From the definition of $\dot{H}$ this gives $U = 0$.

Second, if now $z \neq 0$, using (6.2.1) we obtain

$$0 = \mathrm{Re}((z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A})U, U)_{\dot{\mathcal{H}}} = -(\alpha u^1, u^1)_{L^2(\Omega)}.$$

As $\alpha \geq 0$, this implies that u^0 vanishes a.e on $\mathrm{supp}(\alpha)$. Observe that

$$\mathsf{P}_0 u^0 = zu^1 = -z^2 u^0.$$

The function u^0 is thus an eigenfunction for P_0 that vanishes on an open set. With the unique continuation property we obtain that u^0 vanishes in Ω and u^1 as well.

If we now prove that $z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}$ is surjective, the result then follows from the closed graph theorem as $\dot{A}$ is a closed operator. We write $z\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A} = T + \mathrm{Id}_{\dot{\mathcal{H}}}$ with $T = (z-1)\mathrm{Id}_{\dot{\mathcal{H}}} - \dot{A}$. By the first part of the proof, T is invertible with a bounded inverse. The operator T is unbounded on $\dot{\mathcal{H}}$. We denote by $\tilde{T}$ the restriction of T to $D(\dot{A})$ equipped with the graph-norm associated with $\dot{A}$. The operator $\tilde{T}$ is bounded. It is also invertible. It is thus a bounded Fredholm operator of index $\mathrm{ind}\,\tilde{T} = 0$. Similarly, we denote by ι the injection of $D(\dot{A})$ into $\dot{\mathcal{H}}$ and $\tilde{A}$ the restriction of $\dot{A}$ on $D(\dot{A})$ viewed as a bounded operator. We have $z\iota - \tilde{A} = \tilde{T} + \iota$. Since ι is a compact operator, we obtain that $z\iota - \tilde{A}$ is also a bounded Fredholm operator of index 0. Hence, $z\iota - \tilde{A}$ is surjective since $z\mathrm{Id}_{\mathcal{H}} - \dot{A}$ is injective as proven above. Consequently, $z\mathrm{Id}_{\mathcal{H}} - \dot{A}$ is surjective. This concludes the proof of Proposition 4.13.1. □

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