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Suites spectrales rassemblées, Bocksteins, et applications à THH

Gathered spectral sequences, Bocksteins, and applications to THH

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Suites spectrales rassemblées, Bocksteins et applications à THH

Soit ku le spectre de la K-théorie complexe connective, localisée en un premier p, et soit ℓ la summand d'Adams connective. La suite spectrale de Bockstein associée à la multiplication par $v_1 \in \ell_*$, et qui calcule les groupes d'homologie de Hochschild topologique $\mathrm{THH}_*(\ell)$, est connue. Le but de cette thèse est, dans un premier temps, d'étendre ces résultats à la suite spectrale de Bockstein associée à la multiplication par $u \in ku_*$, qui calcule $\mathrm{THH}_*(ku)$; dans un second temps, d'étudier la composée

$$\Sigma^\infty_+ K(\mathbb{Z}, 3) \to \Sigma^\infty_+ BGL_1(ku) \to K(ku) \to \mathrm{THH}(ku)$$

capturant une partie des unités de la K-théorie algébrique de ku via la trace de Bökstedt dans THH.

Nous développons d'abord des outils généraux, qui relient une suite spectrale à ce que nous appellerons des suites spectrales *rassemblées* et *tronquées*. Nous étudions ensuite comment les extensions dans une suite spectrale de Bockstein sont parfois déterminées par la suite spectrale elle-même. Ce résultats généraux nous permettent de calculer la suite spectrale de Bockstein de $\mathrm{THH}_*(ku)$ à partir de celle de $\mathrm{THH}_*(\ell)$. Nous ferons ensuite un deuxième calcul de $\mathrm{THH}_*(ku)$ en utilisant THH logarithmique. Enfin nous donnons une présentation de $ku_*K(\mathbb{Z}, 3)$ et nous calculons la partie sans torsion de l'application $ku_*K(\mathbb{Z}, 3) \to \mathrm{THH}_*(ku)$.

Mots clefs : topologie algébrique, homotopie stable, K-théorie algébrique, suites spectrales, homologie de Hochschild topologique, K-théorie complexe

Gathered spectral sequences, Bocksteins and applications to THH

Let ku be the connective complex K-theory spectrum, localized at a prime p, and let ℓ be its connective Adams summand. The Bockstein spectral sequence, related to the multiplication by $v_1 \in \ell_*$, that compute the topological Hochschild homology groups $\mathrm{THH}_*(\ell)$, is known. The purpose of this thesis is, first, to extend these results to the Bockstein spectral sequence, related to the multiplication by $u \in ku_*$, that compute $\mathrm{THH}_*(ku)$; and second, to study the composition

$$\Sigma^\infty_+ K(\mathbb{Z}, 3) \to \Sigma^\infty_+ BGL_1(ku) \to K(ku) \to \mathrm{THH}(ku)$$

that captures part of the units in the algebraic K-theory of ku via the Bökstedt trace map into THH.

We first develop general tools, that relate a spectral sequence to what we call *gathered* and *truncated* spectral sequences. We then study how the extensions in a Bockstein spectral sequence can sometimes be recovered from the spectral sequence itself. We use these general results to compute the Bockstein spectral sequence for $\mathrm{THH}_*(ku)$ from the one for $\mathrm{THH}_*(\ell)$. We give a second computation of $\mathrm{THH}_*(ku)$ using logarithmic THH. We then give a presentation of $ku_*K(\mathbb{Z}, 3)$ and compute the non-torsion part of the map $ku_*K(\mathbb{Z}, 3) \to \mathrm{THH}_*(ku)$.

Keywords: algebraic topology, stable homotopy, algebraic K-theory, spectral sequences, topological Hochschild homology, complex K-theory

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Contents

Introduction

The algebraic K-theory groups of a ring are difficult to compute, and thus are often studied through so-called *trace maps*. The first of this kind, the Dennis trace map $K \to HH$, maps algebraic K-theory to Hochschild homology. It was conjectured by Goodwillie that the Dennis trace map factors through a version of Hochschild homology where the ground ring is not the integers $\mathbb{Z}$, but the sphere spectrum S; this hypothetical object was then named topological Hochschild homology. Such a construction was eventually carried out by Bökstedt in unpublished work [13], where the first definition of THH and of the trace map $K \to \mathrm{THH}$ appear.

However, Bökstedt lacked a sufficiently structured category of spectra to mimic the definition of Hochschild homology over the sphere spectrum; it was only after the description of the category of S-modules or the category of symmetric spectra that such a definition could be made. Topological Hochschild homology then offers a trace map from the K-theory not only of a ring, but of any $\mathbb{E}_1$ ring spectrum. Furthermore, THH can be seen to be equipped with an action of the circle S^1, and the study of the homotopy fixed points for this action led to the definition of topological cyclic homology and the cyclotomic trace $K \to \mathrm{TC}$ that factorizes the Bökstedt trace map.

Many methods used to compute topological Hochschilds homology are based on varying the coefficients; when A is commutative and B is a symmetric (A, A)-bimodule, a first important property to manipulate the coefficients is the equation

$$\mathrm{THH}(A; B) \cong B \wedge_A \mathrm{THH}(A) \tag{0.0.1}$$

This allows us to identify the E^1 terms in the Bockstein spectral sequence obtained by some multiplication by $q \in B$ map in $\mathrm{THH}(A; B)$ with the homotopy of the spectrum $\mathrm{THH}(A; B/q)$. This also identify the modulo p homotopy of $\mathrm{THH}(A; B)$ with $\mathrm{THH}(A; V(0) \wedge B)$ where $V(0)$ is the modulo p Moore spectrum; likewise, for the Smith-Toda complex $V(1)$, the $V(1)$ homology of $\mathrm{THH}(A; B)$ will be the homotopy of $\mathrm{THH}(A; V(1) \wedge B)$. This produces spectral sequences whose first page might be computable by virtue of having a hopefully simpler coefficients ring.

Another kind of manipulation on the coefficients will come from the equation

$$\begin{aligned} \mathrm{THH}(A; B) &\simeq B \wedge_{A^e} A \\ &\cong (B \wedge_A B) \wedge^L_{A^e} B \end{aligned} \tag{0.0.2}$$

which will produce a Brun spectral sequence computing the homotopy of $\mathrm{THH}(A; B)$ from that of $\mathrm{THH}(B; H\pi_*(B \wedge_A B))$, which once again is hopefully easier. This method was studied with the level of generality we will need by Höning in [23].

Examples of computations related to our present work are those of McClure and Staffeldt in [28]. For a prime $p \geq 3$, they computed $V(0)_* \mathrm{THH}(\ell)$, the modulo p homotopy of the topological Hochschild homology of the Adams summand ℓ of ku, the connective cover of topological complex K-theory. They also obtained the formula

$$\mathrm{THH}(L) \simeq L \vee (\Sigma L)_{\mathbb{Q}} \tag{0.0.3}$$

for the periodic Adams summand L.

The computation of $V(0)_* \operatorname{THH}(\ell)$ was extended to $p = 2$ by Angelveit and Rognes in [3]; a similar result for ku was computed by Ausoni in [5], and an analogous periodic formula

$$\operatorname{THH}(KU) \simeq KU \vee (\Sigma KU)_{\mathbb{Q}} \tag{0.0.4}$$

was given.

McClure and Staffeldt's work was aiming at computing $\operatorname{THH}_*(\ell)$ via an Adams spectral sequence; that computation never appeared. However, $\operatorname{THH}_*(\ell)$ was computed by Angelveit, Hill and Lawson in [2], using what they called *dueling Bockstein spectral sequences*: multiple spectral sequences having the same target must somehow agree, and the resulting constraints lead to the result. This idea and their results are the basis of much of the present work.

Another point of interest is the study of the algebraic K-theory spectrum $K(ku)$ through $\operatorname{THH}(ku)$ and trace method. It is conjectured (see [7]) that $K(ku)$ is an elliptic cohomology theory of chromatic filtration 2 – a theory with meaningful geometrical content that is suitable to study v_2-periodic phenomenons, as topological K-theory is suitable to study v_1-periodicity.

One way to constuct classes in K-theory is to use the so-called unit map $\Sigma^\infty_+ BGL_1(R) \to K(R)$. When R is a classical commutative ring, the unit map has a right inverse $K(R) \to \Sigma^\infty_+ BGL_1(R)$ called the determinant map. For ku, $GL_1(ku)$ is the product of infinite loop spaces $K(\mathbb{Z}, 2) \times \mathbb{Z}/2 \times BSU_\otimes$. Thus, there is a map

$$\Sigma^\infty_+ K(\mathbb{Z}, 3) \to K(ku) \tag{0.0.5}$$

that captures part of the units. The π_3 of this map is computed in [6], and a corollary of this computation is that there is no determinant map $K(ku) \to \Sigma^\infty_+ BGL_1(ku)$.

When computing topological Hochschild homology, it is possible and often necessary to combine multiple steps of computation to arrive to a result. We will compute the homotopy of $\operatorname{THH}(ku)$; equation (0.0.1) tautologically identify that with computing the ku homology in the category of ku-modules of $\operatorname{THH}(ku)$. This is still interesting; p-localized ku has coefficients ring $\mathbb{Z}_{(p)}[u]$, which give two possibilities for non-trivial Bockstein spectral sequences: multiplying by p or by u. Thus, to compute the ku-homology of a spectrum X (e.g. to compute $\operatorname{THH}(ku)$), we can take two different approaches that start with $\mathbb{F}_p$-homology, that we give in the following diagram of spectral sequences:

$$\begin{tikzcd}
(V(0) \wedge ku)_* X \otimes P(p) \arrow[r, Rightarrow] & ku_* X \\
H_*(X; \mathbb{F}_p) \otimes P(p) \otimes P(u) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & H_*(X; \mathbb{Z}_{(p)}) \otimes P(u) \arrow[u, Rightarrow]
\end{tikzcd} \tag{0.0.6}$$

We will show how that square of Bockstein spectral sequences contains information relative to the additive extension problems that might arise in computing $ku_* X$.

The existence of the Adams summand ℓ in relation to ku also offers another piece of the computation; ℓ has coefficients ring $\mathbb{Z}_{(p)}[v_1]$ where the map $\ell \to ku$ send v_1 to u^{p-1}. This equation makes the v_1-Bockstein spectral sequence

$$H_*(X; \mathbb{Z}_{(p)}) \otimes P(v_1) \Rightarrow \ell_* X \tag{0.0.7}$$

not map into the u-Bockstein spectral sequence computing ku_*X, but more nicely into another v_1-Bockstein spectral sequence

$$(ku/v_1)_*X \otimes P(v_1) \Rightarrow ku_*X \tag{0.0.8}$$

Moreover, the element $(ku/v_1)_*X$ can be computed from a truncated u-Bockstein spectral sequence:

$$H_*(X;\mathbb{Z}_{(p)}) \otimes P_{p-1}(u) \Rightarrow (ku/v_1)_*X \tag{0.0.9}$$

These three spectral sequences fit in a diagram

$$\begin{tikzcd} H_*(X;\mathbb{Z}_{(p)}) \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & H_*(X;\mathbb{Z}_{(p)}) \otimes P(u) \arrow[r, Rightarrow] & ku_*X \\ H_*(X;\mathbb{Z}_{(p)}) \otimes P_{p-1}(u) \otimes P(v_1) \arrow[rr, Rightarrow] & & (ku/v_1)_*X \otimes P(v_1) \arrow[u, Rightarrow] \end{tikzcd} \tag{0.0.10}$$

where any information on one of the path, top or bottom, can be translated into information on the other.

Both diagram (0.0.6) and (0.0.10) present situations that are not specific to ku. A similar square diagram can be written for ℓ or any integral Morava K-theory. We will also develop the theory translating between the two path in the triangular diagram for *any* spectral sequence coming from a tower of spectra, not just for the Bockstein spectral sequences obtained from an element in a ring and its powers. Thus, our results could be used to compute spectral sequences in any case where it would make sense to gather steps in the filtration (potentially in order to compare more easily with another spectral sequence, as in our computation). For example, the Lubin-Tate spectrum E_n share with some homotopy fixed point spectra the same relationship as ku_p has with ℓ. The Adams summand ℓ is obtained as the homotopy fixed point of a C_{p-1}-action on ku_p, and a similar result can be stated for the E_n at other chromatic levels, see the results of [19], recounted as theorem 5.4.4 of [32]. Computations with similar steps could then be carried out, starting with the Morava K-theory $K(n)$ (that can be seen to be computable for any n in some cases, e.g. [31]) and their connective covers $k(n)$.

This thesis is organized in two parts; the first one – chapters 1 to 3 – contains general results on spectral sequence; the second one – chapters 4 to 7 – deals with the topological Hochschild homology of ku and the trace map from $K(\mathbb{Z}, 3)$.

Chapter 1 results are on a generalized version of the situation of diagram (0.0.10). We provide a dictionnnary between the differentials of a spectral sequence coming from a tower of spectra, truncated versions of that spectral sequence and gathered versions. A tower of spectra is a functor from the poset of the integers to a category of spectra; when considered together with the cofibers of the maps constituting the tower, it provides an unrolled exact couple by taking homotopy, and thus a spectral sequence. For ku, the tower we will use is the Whitehead tower, that can be obtained by repeating the multiplication by u map

$$\cdots \to \Sigma^4 ku \wedge X \to \Sigma^2 ku \wedge X \to ku \wedge X. \tag{0.0.11}$$

The tower – and the exact couple and spectral sequence – can be truncated by setting all the morphisms outside some bounds to be the identity. In the diagram (0.0.10), this is how we get the spectral sequence

$$H_*(X;\mathbb{Z}_{(p)}) \otimes P_{p-1}(u) \Rightarrow (ku/v_1)_*X. \tag{0.0.12}$$

The tower can also be *gathered* along an increasing map $\mathbb{Z} \to \mathbb{Z}$ – in diagram (0.0.10), the map $x \mapsto |v_1|x$ gives the spectral sequence

$$(ku/v_1)_*X \otimes P(v_1) \Rightarrow ku_*X. \tag{0.0.13}$$

We will state general results, which for our example will specialize the following: differentials in the u-Bockstein spectral sequence

$$H_*(X;\mathbb{Z}_{(p)}) \otimes P(u) \Rightarrow ku_*X \tag{0.0.14}$$

that are smaller than $|v_1|$ result in differential in (0.0.12) (theorem 1.2.11); that the differentials longer than $|v_1|$, on the other hand, are related to differentials in (0.0.13) (theorem 1.2.21). We also provide results going the other way, from either the truncated or gathered spectral sequences to the base spectral sequence (theorem 1.2.11 and theorem 1.2.28), as well as results for null differentials that are sufficient to manage computations (theorem 1.2.35).

Chapter 2 explains how the additive extensions in a Bockstein spectral sequence can sometimes be recovered from the differentials in a generalized diagram (0.0.6) (theorem 2.4.1). We will not work with ku but with any homology theory whose coefficients are polynomial in two elements q and v, analogous to p and u. We will provide two sets of hypotheses on the four Bockstein spectral sequences of diagram (0.0.6) under which this is possible. The stronger set of hypotheses constrain $H_*(X;\mathbb{F}_p)$ to be of rank at most 1 in each degree, and constrain the length of the u-towers in the E^∞-page of the spectral sequence on the right side of diagram (0.0.6) so that the extensions that can occur are unique. The weaker set of hypotheses relax the rank 1 hypothesis to only the infinite cycle in the bottom and left side spectral sequences, and relax the previous unicity property; in order to still be able to recover the extensions from the differentials, we will have to remark that some divisibilities by p, and thus additive extensions, are visible through a pattern in the differentials. Then it will be necessary to constrain the length of the u-towers in the E^∞-page to ensure that all the possible divisibilities are visible through this pattern.

Chapter 3 provide a proof of folklore result – an isomorphism between the Atiyah-Hirzebruch spectral sequence obtained from a skeletal filtration and the spectral sequences obtained from the Whitehead tower or the Postnikov tower (theorem 3.2.5).

Chapter 4 introduce topological Hochschild homology and the results we will need for our following computation of $\mathrm{THH}_*(ku)$.

Chapter 5 compute $\mathrm{THH}_*(ku)$ as a ku-module from $\mathrm{THH}_*(\ell)$ and using the results from chapter 1 and 2 to compute the Bockstein spectral sequence

$$\mathrm{THH}_*(ku; H\mathbb{Z}_{(p)}) \otimes P(u) \Rightarrow \mathrm{THH}_*(ku). \tag{0.0.15}$$

We will first see how $\mathrm{THH}_*(ku; H\mathbb{Z})$ is generated as an abelian group by 1, the suspension σu of u, the generators μ_n of $\mathrm{THH}_*(\mathbb{Z})$ and their products (proposition 5.2.31). We then compute $\mathrm{THH}_*(ku)$ (theorem 5.7.14): the generators that lifts in $\mathrm{THH}_*(ku)$ are 1 and σu for the non-torsion part, σu becoming divisible by increasingly bigger integers as it is multiplied by u; the torsion must be analyzed one prime p at a time. It can be separated into increasingly large submodules T_n^k for any $n \geq 1$ and $1 \leq k \leq p-1$, generated by the classes

$$\sigma u \mu_{kp^n},\ \sigma u \mu_{kp^n+p},\ \sigma u \mu_{kp^n+2p},\ \dots \sigma u \mu_{kp^n-p} \tag{0.0.16}$$

such that for any k and k', T_n^k and $T_n^{k'}$ are isomorphic when forgetting the degree, and $T_n^\bullet$ contains $p-1$ copies of $T_{n-1}^\bullet$ as submodules, as well as one more copy as a quotient. This is of course very similar to the result on $\mathrm{THH}_*(\ell)$, but it must be noted that

$$\mathrm{THH}_*(ku) \neq \mathrm{THH}_*(\ell) \otimes_{P(v_1)} P(u). \tag{0.0.17}$$

Chapter 6 provide another computation of $\mathrm{THH}_*(ku)$ the using logarithmic topological Hochschild homology $\mathrm{THH}_*(ku, \langle u \rangle)$. That computation still requires the knowledge of $\mathrm{THH}_*(\ell)$ and of some fact on the suspension map $ku \to \mathrm{THH}(ku)$. Logarithmic topological Hochschild homology comes with a short exact sequence

$$0 \to \mathrm{THH}_*(\ell) \to \mathrm{THH}_*(\ell, \langle v_1 \rangle) \to \mathrm{THH}_{*-1}(H\mathbb{Z}_{(p)}) \to 0 \tag{0.0.18}$$

as well as one for ku, and a weak equivalence

$$\mathrm{THH}(ku, \langle u \rangle) \simeq ku \wedge_\ell \mathrm{THH}(\ell, \langle v_1 \rangle) \tag{0.0.19}$$

which state that ku is formally log-THH-étale. The sequence (0.0.18) allows us to compute $\mathrm{THH}_*(\ell, \langle v_1 \rangle)$ (theorem 6.2.4), and from equation (0.0.19), we can deduce $\mathrm{THH}_*(ku, \langle u \rangle)$ (theorem 6.2.6) as well as $\mathrm{THH}_*(ku)$.

Chapter 7 introduces the Bökstedt trace map into THH, and provides a computation of the non-torsion part of $ku_*K(\mathbb{Z}, 3)$ (proposition 7.2.51), a computation of the torsion part up to the additive extensions in the u-Bockstein spectral sequence (theorem 7.2.50), and a computation of the non-torsion part of the map $ku_*K(\mathbb{Z}, 3) \to \mathrm{THH}_*(ku)$ induced by the trace (theorem 7.3.34). The non-torsion part of $ku_*K(\mathbb{Z}, 3)$ is similar to that of $\mathrm{THH}_*(ku)$, with a class $\sigma\beta_{(0)}$ mapping to σu, but differs in that when mutiplied by u, $\sigma\beta_{(0)}$ become as divisible by integers as σu after one more mutiplication. Thus, the non-torsion part of $ku_*K(\mathbb{Z}, 3)$ injects into $\mathrm{THH}_*(ku)$.

Finally, appendix A contains a computer program that was used to generated pictures of some submodules of $\mathrm{THH}_*(ku)$.

Notations and conventions

We will use the following notations to describe various algebras:

- $P(x)$ is a polynomial algebra over a generator x
- $P_n(x)$ is a truncated polynomial algebra at height n, that is the quotient of $P(x)$ by the relation $x^n = 0$
- $\Gamma(x)$ is a divided power algebra, which is generated additively by the divided power of x, denoted $\gamma_i x$ for any $i \geq 0$, and with the multiplicative relations:
$$\gamma_i x \cdot \gamma_j x = \binom{i+j}{i} \gamma_{i+j} x.$$
- $E(x)$ is what we will call an exterior algebra, which will always mean $P_2(x)$; however, this is not what is usually called an *exterior algebra* when not in odd characteristic, since in that case the relation we have is $2x^2 = 0$.

The base ring for these algebras will be determined in most case by the context in which they appear. When computing homology with coefficient in $\mathbb{F}_p$ or modulo p homotopy, the base ring will be $\mathbb{F}_p$. When computing homology with coefficients in $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$ (the integers, the p-localized integers or the p-completed integers), it will be $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$. When computing THH, it will be the base ring for the coefficient spectrum. If we need to specify the base ring, we will note it in a subscript: $P_{\mathbb{Q}}(x)$, $E_{\mathbb{Q}}(x)$, etc.

When writing spectral sequences, we will use tensor products $\otimes$ of these algebras. One of these tensor product will be written $\bar{\otimes}$, it will separate the algebras generated by classes whose bidegree lies on the x-axis – on the left of $\bar{\otimes}$ – and those generated by classes whose bidegree lies on the y-axis – on the right of $\bar{\otimes}$.

Part I

Computational tools for diagrams of spectral sequences

Chapter 1

Spectral sequences from towers of spectra

Our vocabulary concerning spectral sequences will follow Boardman's in [12]. We will work in a stable homotopy category, that is to say the homotopy category of a category of spectra. The underlying category of spectra could be Boardman's spectra (see [1] or [36]), or S-module from [20]. What we really use is that we have a triangulated category, with a functor to the graded group that produces long exact sequences from the triangles, with some unicity on the maps between two triangles (arising from the unicity up to homotopy of the maps between cofiber sequences).

We study spectral sequences arising from a tower of spectra indexed by $\mathbb{Z}$:

$$... \longrightarrow Y_{n+1} \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow ... \tag{1.0.1}$$

Let Y_∞ be the limit of the tower and $Y_{-\infty}$ be the colimit. For any a and b integers or $\pm\infty$ with $a \leq b$, let Y_a^b be the cofiber of the map $Y_b \to Y_a$. For each $n \in \mathbb{Z}$, the cofiber sequence:

$$Y_{n+1} \longrightarrow Y_n \longrightarrow Y_n^{n+1} \tag{1.0.2}$$

gives a long exact sequence in homotopy. Pasting each of these sequences defines an unrolled exact couple, and a spectral sequence.

To ensure (weak) convergence of the spectral sequence, we quotient the tower of spectra by the limit. To this end, we need to discuss the maps between these cofibers.

1.1 The octahedral axiom and consequences

The octahedral axiom is assumed true in any triangulated category. Here we will use it in the homotopy category of spectra, which is triangulated by virtue of being the homotopy category of a stable model category.

Axiom 1.1.1 (Octahedral)**.** *Let $A \to B \to C$, $A \to D \to E$ and $B \to D \to F$*

be triangles such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "id"] & B \arrow[d] \\ A \arrow[r] & D \end{tikzcd} \tag{1.1.2}$$

commutes. Then there are six triangles and a commutative diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r] & F \arrow[r, "id"] & F \end{tikzcd} \tag{1.1.3}$$

where $*$ is the zero-object of the category.

Remark that in the specific case of the stable homotopy category, the maps $C \to E$ and $E \to F$ are unique, and thus are unique up to homotopy in the category of spectra.

Our first lemma is a reformulation of this axiom with our notations:

Lemma 1.1.4. *Let $a \leq b \leq c$ be integers or $\pm\infty$. There is a morphism of cofiber sequences, and commutative diagram:*

$$\begin{tikzcd} Y_c \arrow[r] \arrow[d, "id"] & Y_b \arrow[r] \arrow[d] & Y_b^c \arrow[d] \\ Y_c \arrow[r] & Y_a \arrow[r] & Y_a^c \end{tikzcd} \tag{1.1.5}$$

Then there is a cofiber sequence:

$$Y_b^c \longrightarrow Y_a^c \longrightarrow Y_a^b \tag{1.1.6}$$

and a weak equivalence $f : Y_a^b \to Y_a^b$ making the following diagram commute.

$$\begin{tikzcd} Y_b \arrow[r] \arrow[d] & Y_a \arrow[r] \arrow[d] & Y_a^b \arrow[d, "f"] \\ Y_b^c \arrow[r] & Y_a^c \arrow[r] & Y_a^b \end{tikzcd} \tag{1.1.7}$$

We can conclude the following, which ensure that our spectral sequences can converge to their colimit

Proposition 1.1.8. *For any $a \leq b$ integers, the cofiber of $Y_b^\infty \to Y_a^\infty$ is Y_a^b. Then the towers of spectra*

$$... \longrightarrow Y_{n+1}^\infty \longrightarrow Y_n^\infty \longrightarrow Y_{n-1}^\infty \longrightarrow ... \tag{1.1.9}$$

$$... \longrightarrow Y_{n+1} \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow ... \tag{1.1.10}$$

induce isomorphic spectral sequences, beginning from the E^1 pages.

Proof. This is lemma 1.1.4: we have a morphism of exact couple induced by the diagrams

$$\begin{tikzcd} Y_{n+1} \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d] & Y_n^{n+1} \arrow[d, "\simeq"] \\ Y_{n+1}^\infty \arrow[r] & Y_n^\infty \arrow[r] & Y_n^{n+1} \end{tikzcd} \tag{1.1.11}$$

that is an isomorphism on the E^1 pages. The induced morphisms on the derived exact couples are then automatically isomorphisms on the following pages, and thus we have two isomorphic spectral sequences. □

This corollary will be used with towers of spectra such that for some $m \in \mathbb{Z}$ and for all $k \geq m$, all the $Y_{k+1} \to Y_k$ are isomorphism – that is, Y_m is the limit of the tower; and thus ∞ will be replaced by m. In fact, we will mostly deal with towers quotiented by their limits, and we will need another version of the octahedral axiom.

In the following, whenever $i \leq j \leq k$ are integers or $\pm\infty$, the map $Y_j^k \to Y_i^k$ is the map coming from the morphism between the cofiber sequences $Y_k \to Y_j \to Y_j^k$ and $Y_k \to Y_i \to Y_i^k$, and the map $Y_i^k \to Y_i^j$ is from the cofiber sequence $Y_j^k \to Y_i^k \to Y_i^j$ of lemma 1.1.4. Both are unique up to homotopy.

Lemma 1.1.12. *Let $a \leq b \leq c \leq d$ be integers or $\pm\infty$. There are commutative diagrams, both of six cofiber sequences:*

$$\begin{tikzcd} Y_c^d \arrow[r] \arrow[d, "id"] & Y_b^d \arrow[r] \arrow[d] & Y_b^c \arrow[d] \\ Y_c^d \arrow[r] \arrow[d] & Y_a^d \arrow[r] \arrow[d] & Y_a^c \arrow[d] \\ * \arrow[r] & Y_a^b \arrow[r, "\simeq"] & Y_a^b \end{tikzcd} \qquad \begin{tikzcd} Y_c^d \arrow[r] \arrow[d] & Y_a^d \arrow[r] \arrow[d, "id"] & Y_a^c \arrow[d] \\ Y_b^d \arrow[r] \arrow[d] & Y_a^d \arrow[r] \arrow[d] & Y_a^b \arrow[d] \\ Y_b^c \arrow[r] & * \arrow[r] & \Sigma Y_b^c \end{tikzcd} \tag{1.1.13}$$

Proof. The left one is direct from the octahedral axiom. The right one must be shifted one time in the horizontal direction using Σ to have the same form as the octahedral axiom. The maps can be seen to be the canonical one since they are unique up to homotopy. □

We won't say anything on the convergence of such general spectral sequences, other than the quotient by the limit which is necessary for weak convergence. We will use the techniques we develop hereafter with spectral sequences that are otherwise know to converge, e.g. Bockstein spectral sequence or Atiyah-Hirzebruch spectral sequences.

1.2 Truncated and gathered spectral sequences

For any spectrum Γ, write $\Gamma_* = \pi_*(\Gamma)$ its homotopy groups. The tower

$$... \longrightarrow Y^\infty_{n+1} \longrightarrow Y^\infty_n \longrightarrow Y^\infty_{n-1} \longrightarrow ... \tag{1.2.1}$$

gives a spectral sequence of the form

$$(\mathcal{B}) : E^1 = \bigoplus_{n\in\mathbb{Z}} (Y^{n+1}_n)_* \Rightarrow (Y^\infty_{-\infty})_*. \tag{1.2.2}$$

For any integers $a \leq b$, we can truncate the tower at a and b, and thus the spectral sequence $(\mathcal{B})$. Let X be the tower such that:

$$X_n = \begin{cases} Y^\infty_b \text{ if } n \geq b \\ Y^\infty_a \text{ if } n \leq a \\ Y^\infty_n \text{ otherwise} \end{cases} \tag{1.2.3}$$

with identities when necessary and maps induced by the original tower. This defines a truncated spectral sequence:

$$(\mathcal{T}^b_a) : E^1 = \bigoplus_{a\leq n<b} (Y^{n+1}_n)_* \Rightarrow (Y^b_a)_*. \tag{1.2.4}$$

Remark that the tower quotiented by the limit has components:

$$X'_n = \begin{cases} Y^b_b \simeq * & \text{if } n \geq b \\ Y^b_a & \text{if } n \leq a \\ Y^b_n & \text{otherwise.} \end{cases} \tag{1.2.5}$$

For any strictly increasing map $\phi : \mathbb{Z} \to \mathbb{Z}$, consider the tower whose n-th level is $Y^\infty_{\phi(n)}$ and maps the composition of the maps in the original tower. This defines a gathered spectral sequence:

$$({}^\phi\mathcal{B}) : E^1 = \bigoplus_{n\in\mathbb{Z}} (Y^{\phi(n+1)}_{\phi(n)})_* \Rightarrow (Y^\infty_{-\infty})_*. \tag{1.2.6}$$

We have choosen the term *gathered* by analogy with bookbinding – we are, after all, talking about the *pages* of a spectral sequence. Our sequence $(\mathcal{B})$ is the book. If we let $\phi = id$, then $({}^\phi\mathcal{B})$ is a folio. If ϕ is the multiplication by 2, $({}^\phi\mathcal{B})$ is an uncut quarto: the pages are gathered together two-by-two; the first differential d^1 of $({}^\phi\mathcal{B})$ contains information about the d^2 and d^3 of $(\mathcal{B})$, the second about d^4 and d^5, etc. If ϕ is multiplication by 8, $({}^\phi\mathcal{B})$ is an uncut octavo; its d^1 contains information about d^4, d^5, d^6 and d^7 of $(\mathcal{B})$. We will provide the

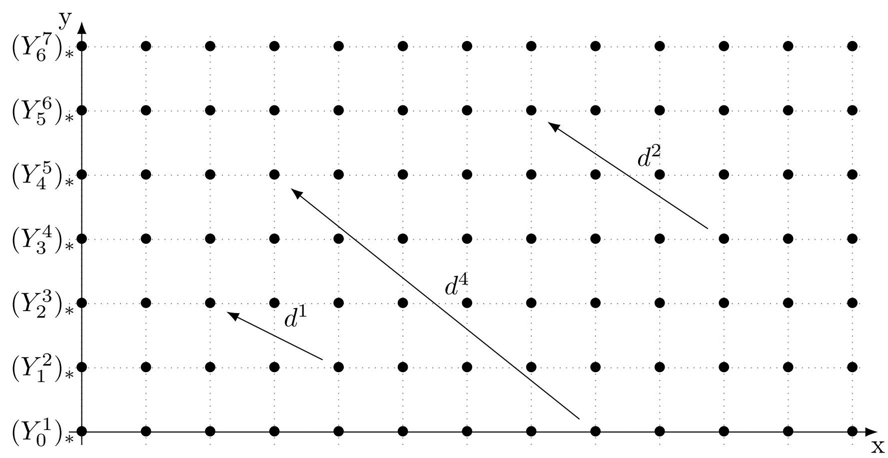


Figure 1.1: Example of the spectral sequence $(\mathcal{B})$.

necessary paper knife to recover $(\mathcal{B})$ from $({}^{\phi}\mathcal{B})$, but we will also say how to glue back the pages of $(\mathcal{B})$ into $({}^{\phi}\mathcal{B})$. It is left to the reader to choose a suitable name in latin when ϕ is a more complex function.

If one wants to compute $(Y_{-\infty}^{\infty})_*$, this gives two ways to do it: computing $(\mathcal{B})$, or computing each $(Y_{\phi(n)}^{\phi(n+1)})_*$ by means of $(\mathcal{T}_{\phi(n)}^{\phi(n+1)})$ and thereafter computing $({}^{\phi}\mathcal{B})$. These two computations are not independent. Let us represent our spectral sequences graphically with the following grading: the n in $(Y_n)_*$ (the filtration degree) is the y-coordinate, and the x-coordinate is such that $* = x + y$. With such bidegree, the differentials will have $|d^r| = (-r-1, r)$ when we let the exact couple given by the tower of spectra be the E^1 page. We will draw first quadrant spectral sequences, but our results apply to whole plane spectral sequences.

For each of figs. 1.1 to 1.5, a $\bullet$ represent a copy of a field $\mathbb{F}$ on the E^1-page, and the $\bullet^n$ in fig. 1.5 represent n copies of $\mathbb{F}$. On the fig. 1.1 we have figured 3 non-zero differentials of different size. We will choose our function $\phi : \mathbb{Z} \to \mathbb{Z}$ such that $\phi(0) = 0$, $\phi(1) = 3$ and $\phi(2) = 7$. Our first result is that the d^1 and d^2 figured will respectively be seen in $(\mathcal{T}_0^4)$ and $(\mathcal{T}_4^7)$, as seen in fig. 1.2 and fig. 1.3. Conversely, having such differentials in $(\mathcal{T}_0^4)$ or $(\mathcal{T}_4^7)$ will ensure a differential in $(\mathcal{B})$. This discussion is theorem 1.2.11.

However, the differentials d^4 is too long and is "jumping" from the area covered by $(\mathcal{T}_0^3)$ to that covered by $(\mathcal{T}_3^7)$, and thus is not visible in either of the truncated spectral sequences. When computing $(Y_0^3)_*$ with $(\mathcal{T}_0^3)$, in the end all the remaining classes are gathered on the $y = 0$ line (see fig. 1.4) to compute this line in the E^1-page of $({}^{\phi}\mathcal{B})$.

The d^4 differential will be visible in $({}^{\phi}\mathcal{B})$, as we will prove in theorem 1.2.21; in the fig. 1.5, we see that it gives a d^1 between the class in $(Y_0^3)_*$ represented by its source, and the class in $(Y_3^7)_*$ represented by its target. It is to be noted that differentials in $(\mathcal{B})$ between the zone covered by $(\mathcal{T}_0^3)$ and $(\mathcal{T}_3^7)$ all give d^1 in $({}^{\phi}\mathcal{B})$ regardless of their original length. Generally, differentials between the zone of $(\mathcal{T}_{\phi(n)}^{\phi(n+1)})$ and $(\mathcal{T}_{\phi(n+m)}^{\phi(n+m+1)})$ will be d^m in $({}^{\phi}\mathcal{B})$. Some regularity in the length of the differentials in $({}^{\phi}\mathcal{B})$ can be recovered when ϕ is linear; this is not the case in our example, but it will be later when comparing Bockstein spectral

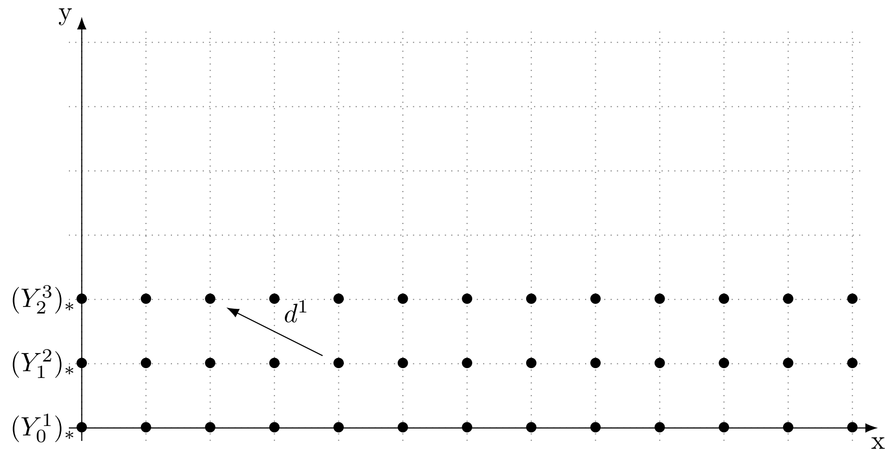


Figure 1.2: The spectral sequence $(\mathcal{T}_0^3)$ corresponding to the $(\mathcal{B})$ of fig. 1.1.

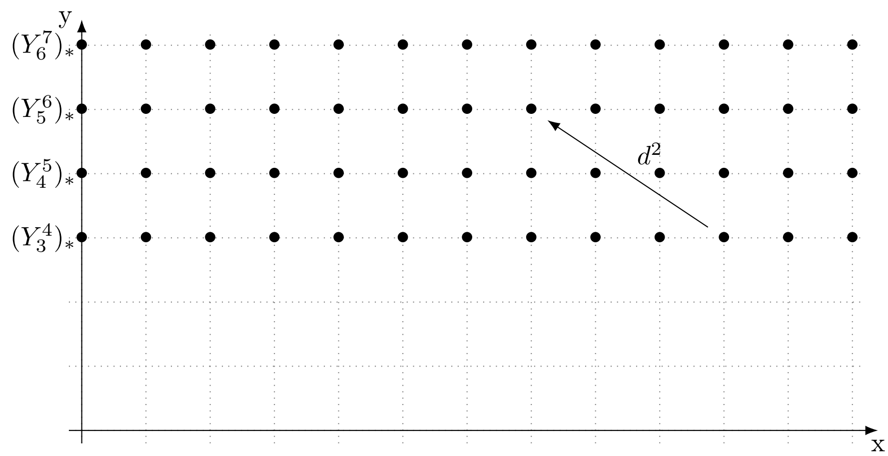


Figure 1.3: The spectral sequence $(\mathcal{T}_3^7)$ corresponding to the $(\mathcal{B})$ of fig. 1.1.

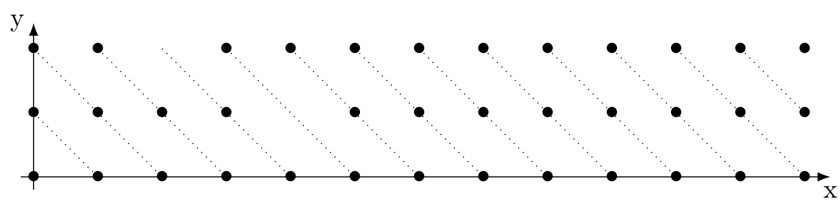


Figure 1.4: The E^∞ page of $(\mathcal{T}_0^3)$, isomorphic to $(Y_0^3)_*$. The lines fix the degree.

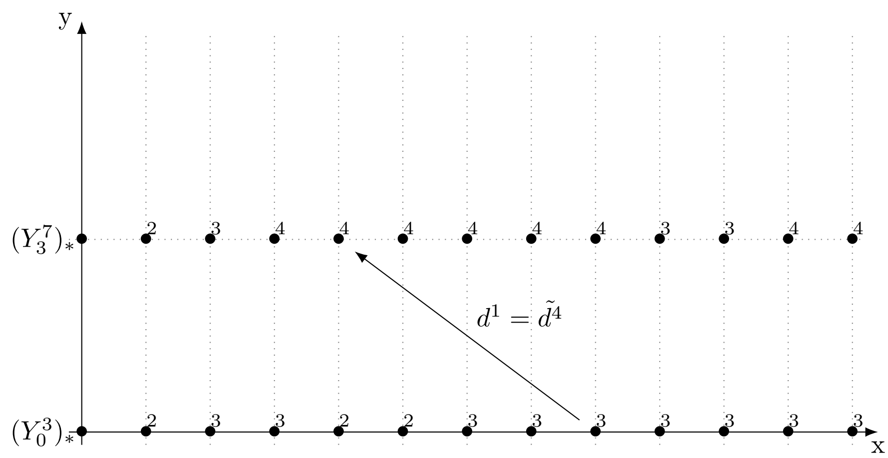


Figure 1.5: The spectral sequence $({}^{\phi}\mathcal{B})$ corresponding to the $(\mathcal{B})$ of fig. 1.1.

sequences obtained by filtering with multiplication by an element and by some power of the same element.

Finally, theorem 1.2.28 deals with the case of transferring a differential of $({}^{\phi}\mathcal{B})$ into $(\mathcal{B})$, and theorem 1.2.35 deals with the null differentials in $(\mathcal{B})$ and $({}^{\phi}\mathcal{B})$.

Consider an unrolled exact couple:

$$\begin{tikzcd} \ldots \arrow[r, "i"] & A_{n+1} \arrow[rr, "i"] & & A_n \arrow[r, "i"] \arrow[dl, "j"] & \ldots \\ & & E_n^1 \arrow[ul, "k"] & & \end{tikzcd} \tag{1.2.7}$$

For $r \geq 0$, let Z_n^r and B_n^r be the groups of r-cycles and of r-boundaries in E_n^1, that is:

$$\begin{aligned} Z_n^r &= k^{-1}(Im(i^{r-1} : A_{n+r} \to A_{n+1})) \\ B_n^r &= j(Ker(i^{r-1} : A_n \to A_{n-r+1})). \end{aligned} \tag{1.2.8}$$

We let E^r be the quotient Z^r/B^r for $r \geq 1$, and the differential d^r will be a map $E_n^r \to E_{n+r}^r$. We will write ${}^{\phi}Z_n^r$ and ${}^{\phi}B_n^r$ for the r-cycles and r-boundaries in the spectral sequence $({}^{\phi}\mathcal{B})$ to distinguish them from those in $(\mathcal{B})$.

Definition 1.2.9. *For $x \in E_n^r$ and $y \in E_{n+r}^r$, we write $d^r(x) = y$ when for some $\bar{x} \in Z_n^r$ representing x in the quotient and some $\bar{y} \in Z_{n+r}^r$ representing y, $k(\bar{x})$ can be lifted $r-1$ times through i, and the image of the $(r-1)$-th lift by j is $\bar{y}$.*

Let us also remark that stating $y \neq 0$ is stating that r is maximal for such lift of $k(\bar{x})$.

We can visualize this in the exact couple diagram:

$$\begin{tikzcd}
A_{n+r+1} \arrow[rr, "i"] & & A_{n+r} \arrow[dl, "j"] \arrow[r, "i"] & \ldots \arrow[r, "i"] & A_{n+1} \arrow[rr, "i"] & & A_n \arrow[dl, "j"] \\
& E^1_{n+r} \arrow[ul, "k"] & & & & E^1_n \arrow[ul, "k"] &
\end{tikzcd} \tag{1.2.10}$$

$$\begin{tikzcd}
& \alpha \arrow[r, mapsto] \arrow[dl, mapsto] & \ldots \arrow[r, mapsto] & i^{r-1}(\alpha) & \\
\bar{y} & & & & \bar{x} \arrow[ul, mapsto]
\end{tikzcd}$$

We now describe how the differential in the spectral sequences $(\mathcal{B})$, $({}^{\phi}\mathcal{B})$ and $(\mathcal{T}_{\phi(n)}^{\phi(n+1)})$ are interlinked.

First, we see how a differential in $(\mathcal{B})$ short enough to fit in $(\mathcal{T}_{\phi(n)}^{\phi(n+1)})$ will occur.

Theorem 1.2.11. *Let n, r and N be integers such that $\phi(N) \leq n \leq n+r < \phi(N+1)$, and let $x \in Z_n^r$ and $y \in Z_{n+r}^r$ in $(\mathcal{B})$.*

Then there is an equivalence between these propositions:

- $d^r(x) = y$ *in* $(\mathcal{B})$.
- $d^r(x) = y$ *in* $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})$.

where x and y stand for the quotients in the respective E^r-pages of the two spectral sequences.

Proof. This is seen directly in the differential diagram after definition 1.2.9. Remark that the cycles are not the same generally between $(\mathcal{B})$ and $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})$, but here we have $r < \phi(N+1) - \phi(N)$ so that the r-cycles are indeed the same. $\square$

We then need a technical lemma to describe the longer differentials.

Lemma 1.2.12. *For integers $a \leq b \leq c$, if the commutative diagram*

$$\begin{tikzcd}
(Y_b^{b+1})_* \arrow[d, "f"] & (Y_b^c)_* \arrow[l, "p"'] \arrow[r] \arrow[d, "e"] & (Y_a^c)_* \arrow[d] \\
(Y_{b+1}^{\infty})_{*-1} & (Y_c^{\infty})_{*-1} \arrow[l, "i^{c-b-1}"] \arrow[r, "id"] & (Y_c^{\infty})_{*-1}
\end{tikzcd} \tag{1.2.13}$$

can be populated with classes

$$\begin{tikzcd}
x \arrow[d, mapsto] & & \\
i^{c-b-1}(\beta) & \beta \arrow[l, mapsto] \arrow[r, mapsto] & \beta
\end{tikzcd} \tag{1.2.14}$$

then there exists lifts

$$\begin{tikzcd}
x & \tilde{x}-i(u) \arrow[l, mapsto] \arrow[r, mapsto] & \hat{x} \\
i^{c-b-1}(\beta) \arrow[from=u, mapsto] & \beta \arrow[l, mapsto] \arrow[r, mapsto] \arrow[from=u, mapsto] & \beta \arrow[from=u, mapsto]
\end{tikzcd} \tag{1.2.15}$$

Proof. The diagram of the statement is commutative because of lemma 1.1.12, which can also be used to check that the following diagram is commutative and has rows and column exact:

$$\begin{tikzcd}
 & (Y_b^\infty)_* \arrow[r, "id"] \arrow[d] & (Y_b^\infty)_* \arrow[d] & \\
(Y_{b+1}^c)_* \arrow[r, "i"] \arrow[d, "id"] & (Y_b^c)_* \arrow[r, "p"] \arrow[d, "e"] & (Y_b^{b+1})_* \arrow[r, "g"] \arrow[d, "f"] & (Y_{b+1}^c)_{*-1} \arrow[d, "id"] \\
(Y_{b+1}^c)_* \arrow[r, "\delta"] & (Y_c^\infty)_{*-1} \arrow[r, "i^{c-b-1}"] \arrow[d, "i^{c-b}"] & (Y_{b+1}^\infty)_{*-1} \arrow[r] \arrow[d, "i"] & (Y_{b+1}^c)_{*-1} \\
 & (Y_b^\infty)_{*-1} \arrow[r, "id"] & (Y_b^\infty)_{*-1} &
\end{tikzcd} \tag{1.2.16}$$

Here we can see that $x \in (Y_b^{b+1})_*$ can be lifted through p to $(Y_b^c)_*$: indeed, $f(x) = i^{c-b-1}(\beta)$ so $g(x) = 0$, and then there exists $\tilde{x} \in (Y_b^c)_*$ such that $p(\tilde{x}) = x$.

In the central square of the diagram, we have chosen two elements in $(Y_c^\infty)_{*-1}$, β and $e(\tilde{x})$, whose images by i^{c-b-1} are equal. By pushing $\beta - e(\tilde{x})$ in the bottom square, we can see that it is in the image of e, and thus so is β. Write $\tilde{x}'$ such that $e(\tilde{x}') = \beta$, and x' the image of $\tilde{x}'$ in $(Y_b^{b+1})_*$ by p.

Now in the central square, $i^{c-b-1}(e(\tilde{x} - \tilde{x}')) = 0$, so that there exists $u \in (Y_{b+1}^c)_*$ with $\delta(u) = e(\tilde{x} - \tilde{x}')$. But the map δ factors through $(Y_b^c)_*$ as $e \circ i$, and $i(u) \in (Y_b^c)_*$ has image 0 in $(Y_b^{b+1})_*$ by p since $u \in (Y_{b+1}^c)_*$.

Consider the element $\tilde{x} - i(u) \in (Y_b^c)_*$:

$$\begin{aligned}
e(\tilde{x} - i(u)) &= e(\tilde{x}) - \delta(u) \\
&= e(\tilde{x}) - e(\tilde{x} - \tilde{x}') \\
&= e(\tilde{x}') \\
&= \beta
\end{aligned} \tag{1.2.17}$$

$$\begin{aligned}
p(\tilde{x} - i(u)) &= p(\tilde{x}) \\
&= x.
\end{aligned} \tag{1.2.18}$$

It remains to push $\tilde{x} - i(u) \in (Y_b^c)_*$ into $(Y_a^c)_*$, and we have:

$$\begin{tikzcd}
x & \tilde{x}-i(u) \arrow[l, mapsto] \arrow[r, mapsto] & \hat{x} \\
i^{c-b-1}(\beta) \arrow[from=u, mapsto] & \beta \arrow[l, mapsto] \arrow[r, mapsto] \arrow[from=u, mapsto] & \beta \arrow[from=u, mapsto]
\end{tikzcd} \tag{1.2.19}$$

□

We now describe how a longer differential in $(\mathcal{B})$ occurs in the gathered spectral sequence $({}^{\phi}\mathcal{B})$. We need the following definition:

Definition 1.2.20. *An infinite cycle $x \in (Y_n^{n+1})_*$ in the spectral sequence $(\mathcal{B})$ is said to* represent *an element $\hat{x}$ of the target group $(Y_{-\infty}^{\infty})_*$ of $(\mathcal{B})$ when:*

- *x is not a boundary, i.e. is not the target of a differential.*
- *x lifts through the map $(Y_n^{\infty})_* \to (Y_n^{n+1})_*$ to an element $\tilde{x} \in (Y_n^{\infty})_*$ whose image in $(Y_{-\infty}^{\infty})_*$ is $\hat{x}$.*

Theorem 1.2.21. *Let n, m, N and M be integers such that*

$$\phi(N) \leq n < \phi(N+1) \leq \phi(M) \leq m < \phi(M+1) \tag{1.2.22}$$

and let $x \in Z_n^{m-n}$ and $y \in Z_m^{m-n}$ be classes in $(\mathcal{B})$ such that $d^{m-n}(x) = y \neq 0$.

Then:

- *x is an infinite cycle in $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})$, thus represent a class $\hat{x} \in (Y_{\phi(N)}^{\phi(N+1)})_*$.*
- *y is an infinite cycle in $(\mathcal{T}_{\phi(M)}^{\phi(M+1)})$, thus represent a class $\hat{y} \in (Y_{\phi(M)}^{\phi(M+1)})_{*-1}$.*
- *There is a differential $d^{M-N}(\hat{x}) = \hat{y}$ in $({}^{\phi}\mathcal{B})$.*

Proof. We see that x and y are infinite cycles in the truncated spectral sequences using definition 1.2.9.

The canonical maps assemble into a commutative diagram (it can be checked that each square is commutative using lemma 1.1.12):

$$\begin{tikzcd}
(Y_n^{n+1})_* \arrow[d, "f"] & (Y_n^{\phi(N+1)})_* \arrow[l, "p"'] \arrow[r] \arrow[d, "e"] & (Y_{\phi(N)}^{\phi(N+1)})_* \arrow[d] \\
(Y_{n+1}^{\infty})_{*-1} & (Y_{\phi(N+1)}^{\infty})_{*-1} \arrow[l] \arrow[r, "id"] & (Y_{\phi(N+1)}^{\infty})_{*-1} \\
(Y_m^{\infty})_{*-1} \arrow[u] \arrow[d] & (Y_m^{\infty})_{*-1} \arrow[l, "id"'] \arrow[r] \arrow[u] \arrow[d] & (Y_{\phi(M)}^{\infty})_{*-1} \arrow[u] \arrow[d] \\
(Y_m^{m+1})_{*-1} & (Y_m^{\phi(M+1)})_{*-1} \arrow[l] \arrow[r] & (Y_{\phi(M)}^{\phi(M+1)})_{*-1}
\end{tikzcd} \tag{1.2.23}$$

Remark that $x \in (Y_n^{n+1})_*$ and $y \in (Y_m^{m+1})_{*-1}$.

Having a differential $d^{m-n}(x) = y$ is having a class $\alpha \in (Y_m^{\infty})_{*-1}$ with

$$\begin{array}{ccccccc}
(Y_m^{m+1})_{*-1} & \longleftarrow & (Y_m^{\infty})_{*-1} & \longrightarrow & (Y_{n+1}^{\infty})_{*-1} & \longleftarrow & (Y_n^{n+1})_* \\
y & \longleftarrow\!\!\shortmid & \alpha & \longmapsto & i^{m-n-1}(\alpha) & \longleftarrow\!\!\shortmid & x.
\end{array} \tag{1.2.24}$$

This is the left column of our diagram.

Having y represent a class $\hat{y} \in (Y_{\phi(M)}^{\phi(M+1)})_*$ in $(\mathcal{T}_{\phi(M)}^{\phi(M+1)})$ is having an element $\tilde{y} \in (Y_m^{\phi(M+1)})_*$ such that

$$\begin{tikzcd}
(Y_m^{m+1})_{*-1} & (Y_m^{\phi(M+1)})_{*-1} \arrow[l] \arrow[r] & (Y_{\phi(M)}^{\phi(M+1)})_{*-1} \\
y & \tilde{y} \arrow[l, mapsto] \arrow[r, mapsto] & \hat{y}.
\end{tikzcd} \tag{1.2.25}$$

We choose $\hat{y}$ and $\tilde{y}$ by pushing α in the bottom right square.

We now have populated our commutative diagram with the elements

$$\begin{tikzcd}
x \arrow[d] & & \\
i^{m-n-1}(\alpha) & i^{m-\phi(N+1)}(\alpha) \arrow[l, mapsto] \arrow[r, mapsto] & i^{m-\phi(N+1)}(\alpha) \\
\alpha \arrow[u, mapsto] \arrow[d, mapsto] & \alpha \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & i^{m-\phi(M)}(\alpha) \arrow[u, mapsto] \arrow[d, mapsto] \\
y & \tilde{y} \arrow[l, mapsto] \arrow[r, mapsto] & \hat{y}
\end{tikzcd} \tag{1.2.26}$$

We use lemma 1.2.12 with $a = \phi(N)$, $b = n$ and $c = \phi(N+1)$, and with $\beta = i^{m-\phi(N+1)}(\alpha)$, that is on our first two rows. We thus get lifts:

$$\begin{tikzcd}
x \arrow[d] & \tilde{x} - i(u) \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d, mapsto] & \hat{x} \arrow[d, mapsto] \\
i^{m-n-1}(\alpha) & i^{m-\phi(N+1)}(\alpha) \arrow[l, mapsto] \arrow[r, mapsto] & i^{m-\phi(N+1)}(\alpha) \\
\alpha \arrow[u, mapsto] \arrow[d, mapsto] & \alpha \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & i^{m-\phi(M)}(\alpha) \arrow[u, mapsto] \arrow[d, mapsto] \\
y & \tilde{y} \arrow[l, mapsto] \arrow[r, mapsto] & \hat{y}
\end{tikzcd} \tag{1.2.27}$$

The right column states that $d^{M-N}(\hat{x}) = \hat{y}$ in $({}^{\phi}\mathcal{B})$. $\square$

The next result describes how differentials in $({}^{\phi}\mathcal{B})$ have counterparts in $(\mathcal{B})$.

Theorem 1.2.28. *Let $N < M$ be integers and let $x \in {}^{\phi}Z_N^{M-N}$ and $y \in {}^{\phi}Z_M^{M-N}$ be classes in $({}^{\phi}\mathcal{B})$ such that $d^{M-N}(x) = y \neq 0$. For some unique $\phi(N) \leq n < \phi(N+1)$ and $\phi(M) \leq m < \phi(M+1)$, x and y are represented by $\breve{x} \in (Y_n^{n+1})_*$ and $\breve{y} \in (Y_m^{m+1})_{*-1}$ in the spectral sequence $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})$ and $(\mathcal{T}_{\phi(M)}^{\phi(M+1)})$. Let $\breve{x}$ and $\breve{y}$ be fixed.*

Then there is a unique integer n' such that $\phi(N) \leq n \leq n' < \phi(N+1)$, and there is an element $x' \in (Y_{\phi(N)}^{\phi(N+1)})_*$ which is represented by $\breve{x}' \in (Y_{n'}^{n'+1})_*$ in the spectral sequence $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})$, that supports a differential $d^{M-N}(x') = y$ in

$({}^{\phi}\mathcal{B})$, *and such that there is a differential* $d^{m-n'}(\check{x}') = \check{y} \neq 0$ *in* $(\mathcal{B})$. *Moreover,* n' *does not depend on the choice of the representative* $\check{x}$ *and* $\check{y}$.

Proof. We work again in diagram (1.2.23). Remark that $x \in (Y_{\phi(N)}^{\phi(N+1)})_*$ and that $y \in (Y_{\phi(M)}^{\phi(M+1)})_{*-1}$.

First fix let's write $i^{m-\phi(N+1)}(\alpha)$ for the image of x in $(Y_{\phi(N+1)}^{\infty})_{*-1}$, with m maximal for such lift α in $(Y_m^{\infty})_{*-1}$. Necessarily, $\phi(M) \leq m < \phi(M+1)$. By definition, the image of $i^{m-\phi(M)}(\alpha)$ in $(Y_{\phi(M)}^{\phi(M+1)})_{*-1}$ is y up to a boundary of ${}^{\phi}B_M^{M-N}$; without loss of generality, we can suppose that it is y.

We can then push α to get $\tilde{y} \in (Y_m^{\phi(M+1)})_{*-1}$ and $\check{y} \in (Y_m^{m+1})_{*-1}$. By definition, n is such that x can be lifted to $(Y_n^{\phi(N+1)})_*$ but not to $(Y_{n+1}^{\phi(N+1)})_*$. Denote $\tilde{x}$ such a lift and $\check{x}$ its non-zero image in $(Y_n^{n+1})_*$.

Our diagram is populated as such:

$$\begin{tikzcd}
\check{x} \arrow[d, mapsto] & \tilde{x} \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
i^{m-n-1}(\alpha) & i^{m-\phi(N+1)}(\alpha) \arrow[l, mapsto] \arrow[r, mapsto] & i^{m-\phi(N+1)}(\alpha) \\
\alpha \arrow[u, mapsto] \arrow[d, mapsto] & \alpha \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & i^{m-\phi(M)}(\alpha) \arrow[u, mapsto] \arrow[d, mapsto] \\
\check{y} & \tilde{y} \arrow[l, mapsto] \arrow[r, mapsto] & y
\end{tikzcd} \tag{1.2.29}$$

It is however possible that $i^{m-n-1}(\alpha)$ is null.

Let n' be the biggest integer such that $i^{m-n'}(\alpha) = 0 \in (Y_n^{\infty})_{*-1}$. Since $i^{m-n-1}(\alpha) = f(\check{x})$, $i^{m-n}(\alpha) = 0$ so $n \leq n'$. We now work in diagram (1.2.23) with n replaced by n': $i^{m-n'-1}(\alpha)$ can be lifted to $(Y_{n'}^{n'+1})_*$ since $i^{m-n'}(\alpha) = 0$. Denote $\check{x}'$ such a lift. Again using lemma 1.2.12 on our first two rows we can construct classes $\tilde{x}' \in (Y_{n'}^{\phi(N+1)})_*$ and $x' \in (Y_{\phi(N)}^{\phi(N+1)})_*$ to complete the diagram and get the result. □

Remark that with this level of generality, the statement made cannot be ameliorated regarding the fact that we may have to change $\check{x}$ into $\check{x}'$ to get the differential in $(\mathcal{B})$. In fact, let us consider the tower of spectra such that:

$$Y_n = \begin{cases} * & \text{if } n \geq 3 \\ H\mathbb{Z} & \text{if } n = 2 \\ * & \text{if } n = 1 \\ \Sigma H\mathbb{Z} & \text{if } n \leq 0 \end{cases} \tag{1.2.30}$$

and the integer function ϕ such that:

$$\phi(n) = \begin{cases} n & \text{if } n \leq 0 \\ n+1 & \text{if } n \geq 1. \end{cases} \tag{1.2.31}$$

We will figure the interesting part the tower of spectra for each spectral sequence with the cofibers below. Remark that with $(\mathcal{T}_0^2)$ we quotient the tower by the limit which is Y_2, and that we put between braces the name of a generator for the homotopy.

$$(\mathcal{B}): \quad \begin{tikzcd} Y_3 \arrow[r] & Y_2 \arrow[r] \arrow[d] & Y_1 \arrow[r] \arrow[d] & Y_0 \arrow[d] \\ & Y_2^3 & Y_1^2 & Y_0^1 \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r] & H\mathbb{Z} \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma H\mathbb{Z} \arrow[d] \\ & H\mathbb{Z}\{\bar{y}\} & \Sigma H\mathbb{Z}\{\hat{x}'\} & \Sigma H\mathbb{Z}\{\hat{x}-\hat{x}'\} \end{tikzcd} \tag{1.2.32}$$

In $(\mathcal{B})$ there is a differential $d(\hat{x}') = \bar{y}$.

$$(\mathcal{T}_0^2): \quad \begin{tikzcd} Y_2^2 \arrow[r] & Y_1^2 \arrow[r] \arrow[d] & Y_0^2 \arrow[d] \\ & Y_1^2 & Y_0^1 \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r] & \Sigma H\mathbb{Z}\{\bar{x}'\} \arrow[r] \arrow[d] & \Sigma H\mathbb{Z}\{\bar{x}'\} \vee \Sigma H\mathbb{Z}\{\bar{x}-\bar{x}'\} \arrow[d] \\ & \Sigma H\mathbb{Z}\{\hat{x}'\} & \Sigma H\mathbb{Z}\{\hat{x}-\hat{x}'\} \end{tikzcd} \tag{1.2.33}$$

In $(\mathcal{T}_0^2)$ there is no non-zero differential.

$$({}^{\phi}\mathcal{B}): \quad \begin{tikzcd} Y_3 \arrow[r] & Y_2 \arrow[r] \arrow[d] & Y_0 \arrow[d] \\ & Y_2^3 & Y_0^2 \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r] & H\mathbb{Z} \arrow[r] \arrow[d] & \Sigma H\mathbb{Z} \arrow[d] \\ & H\mathbb{Z}\{\bar{y}\} & \Sigma H\mathbb{Z}\{\bar{x}'\} \vee \Sigma H\mathbb{Z}\{\bar{x}-\bar{x}'\} \end{tikzcd} \tag{1.2.34}$$

In $({}^{\phi}\mathcal{B})$ there are differentials $d(\bar{x}') = \bar{y}$, and $d(\bar{x}-\bar{x}') = 0$. But now, with slightly different notation from theorem 1.2.28, we have a class $\bar{x} = (\bar{x}-\bar{x}') + \bar{x}'$

such that $d(\bar{x}) = \bar{y}$ in $({}^{\phi}\mathcal{B})$, and that class is represented by $\hat{x} - \hat{x}'$ at the end of $(\mathcal{T}_0^2)$ since $\hat{x}'$ is of lower filtration. But in $(\mathcal{B})$, $d(\hat{x} - \hat{x}') = 0$, the differential is really supported by $\hat{x}'$. Thus, we cannot get a better result. However, this will not be an issue in the practical application following, since we will be able to prove a better result on the Bockstein spectral sequences we will compute.

Statements can also be made regarding null differentials.

Theorem 1.2.35. *(a) Let $x \in (Y_{\phi(N)}^{\phi(N+1)})_*$ be an $M - N$-cycle in $({}^{\phi}\mathcal{B})$, that is $d^i(x) = 0$ for $i \in \{1, \ldots, M - N\}$. Then any $\hat{x} \in (Y_n^{n+1})_*$ representing x in $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})_*$ is such that $d^{m-n}(\hat{x}) = 0$ in $(\mathcal{B})$ for any m such that $n < m \leq \phi(M + 1)$.*

(b) Let $\hat{x} \in (Y_n^{n+1})_*$ be an $m - n$-cycle in $(\mathcal{B})$. Then there exists a class $x \in (Y_{\phi(N)}^{\phi(N+1)})_*$ represented by $\hat{x}$ in $(\mathcal{T}_{\phi(N)}^{\phi(N+1)})_*$ such that x is an $M - N$-cycle in $({}^{\phi}\mathcal{B})$ for any M such that $\phi(N + 1) < \phi(M + 1) \leq m$.

Proof. First point is direct in diagram (1.2.23).

Second point is using lemma 1.2.12 to get a class represented by $\hat{x}$ whose image in $(Y_{\phi(N+1)}^{\infty})_{*-1}$ can be lifted as much as the image of $\hat{x}$ in $(Y_{n+1}^{\infty})_{*-1}$. □

Chapter 2

Recovering the extensions from the Bockstein spectral sequences

Let k be a ring spectrum such that $\mathrm{k}_* = \mathcal{Z}[v]$ is a polynomial ring on some generator v of positive even degree, and $\mathcal{Z}$ is a discrete valuation ring concentrated in degree zero, with a maximum ideal (q) and a quotient field denoted by $\mathbb{F}$. Let X be a bounded below spectrum. In this chapter, we will discuss the circumstances under which it is possible to compute the extensions, thus computing k_*X as a k_*-module, from the four Bockstein spectral sequences associated to q and v that we can construct for k.

Generally, a Bockstein spectral is determined from a ring spectrum k, an element v of k_* and any spectrum X. The usual point of view on the Bockstein is that of a *single column* spectral sequence, obtained by considering the exact sequence in homotopy associated to the cofiber sequence of the multiplication by v map:

$$\begin{tikzcd} \mathrm{k}_*X \arrow[rr, "v"] & & \mathrm{k}_*X \arrow[dl] \\ & \mathrm{k}/v_*X \arrow[ul] & \end{tikzcd} \tag{2.0.1}$$

That exact couple has only the homology degree, and thus cannot be unrolled – or is *already* unrolled.

We will use another point of view which will allow us to present our results more naturally. We need to consider only ring spectra k and $v \in \mathrm{k}_*$ such that k is bounded below and v is of non-negative degree. Hereafter, v can be substituted with q. We will consider the spectral sequence arising from the tower of spectra

$$\dots \xrightarrow{v} \Sigma^{2|v|}\mathrm{k} \wedge X \xrightarrow{v} \Sigma^{|v|}\mathrm{k} \wedge X \xrightarrow{v} \mathrm{k} \wedge X \xrightarrow{id} \mathrm{k} \wedge X \xrightarrow{id} \dots \tag{2.0.2}$$

The transition from the multiplication by v map and the identity will be on the piece of the tower indexed by 0, and the v map will decrease this index by 1. When taking homotopy groups, the colimit of the tower is k_*X and the limit is null since k is bounded below; in the case of multiplication by q, it is null since

$\mathcal{Z}$ is a discrete valuation ring. Thus, we get a spectral sequence of the type:

$$E^1_{s,t} = \mathrm{k}/v_s X \otimes P(v)_t \Rightarrow \mathrm{k}_{s+t} X \tag{2.0.3}$$

with differentials of degrees $|d^r| = (-r-1, r)$. We see this point of view as more convenient than the classical one, because the E^∞-page will have all the named classes needed to represent the target group, provided we can compute the extensions.

When $\mathrm{k}_* \cong P(v) \otimes A$ where A is a ring concentrated in degrees d such that $-|v| < d < |v|$, then the multiplication by v maps will weak equivalences between $\Sigma^{|v|}\mathrm{k}$ and $\mathrm{k}_{\geq |v|}$ and the tower will be a gathered (see chapter 1) Whitehead tower (see chapter 3). This is true for k_*, since $\mathrm{k}_* \cong P(v) \otimes \mathcal{Z}$ and $\mathcal{Z}$ is concentrated in degree 0.

$\mathcal{Z}$ is the homotopy ring of a ring spectrum k/v which is the cofiber of the multiplication by v map in k. Similarly, the cofiber k/q of the multiplication by q map has homotopy group the ring $\mathbb{F}[v]$. We can go further and quotient k/v and k/q by respectively q and v to get the Eilenberg-MacLane spectrum $H\mathbb{F}$, and we have four Bockstein spectral sequences of the form:

$$\begin{tikzcd} \mathrm{k}/q_* X \otimes P(q) \arrow[rr, Rightarrow, "(q.2)"] & & \mathrm{k}_* X \\ H_*(X;\mathbb{F}) \otimes P(q) \otimes P(v) \arrow[u, Rightarrow, "(v.1)"] \arrow[rr, Rightarrow, "(q.1)"] & & \mathrm{k}/v_* X \otimes P(v) \arrow[u, Rightarrow, "(v.2)"'] \end{tikzcd} \tag{2.0.4}$$

The names chosen reflect on which element the Bockstein spectral sequence is computed and its rank in the computation. As in (2.0.3), the elements will have bidegrees $(0,0)$ for q, $(0,|v|)$ for v and $(|x|,0)$ for any x in the specified homology group. Examples of such spectra are ku, ℓ and the others integral Morava K-theories with coefficients $\mathbb{Z}_p[v_n]$ for some $n \geq 2$. Our main example will be ku, the p-localized connective complex K-theory, with coefficients $ku_* \cong \mathbb{Z}_{(p)}[u]$ where u is in degree 2. The application we have in mind is the computation of $\mathrm{THH}_*(ku)$ done in chapter 5. The four Bockstein spectral sequences for ku will be:

$$\begin{tikzcd} (V(0) \wedge ku)_* X \otimes P(p) \arrow[rr, Rightarrow] & & ku_* X \\ H_*(X;\mathbb{F}_p) \otimes P(p) \otimes P(u) \arrow[u, Rightarrow] \arrow[rr, Rightarrow] & & H_*(X;\mathbb{Z}_{(p)}) \otimes P(u) \arrow[u, Rightarrow] \end{tikzcd} \tag{2.0.5}$$

We will give two sets of hypothesis under which the spectral sequences determine the k_*-module structure of the target group. The second set will be a simplified version of the first, less general but easier check. It is in fact this simplified hypothesis that we will use to compute $\mathrm{THH}_*(ku)$ as a ku_*-module. We will first provide an example of how the differentials can determine the extensions, and then an example where the spectral sequences will be shown not to determine the target module, which will provide some motivation for the hypothesis.

The following hypotheses and formulas are exact, but in applications, the differentials in the spectral sequences will often only be determined up to units; in what follows, it would mean that the $\pi(q_0^k a_i)$ might only be determined up to a unit, or equivalently, that the formulas obtained for the extensions might only be determined up a to a unit.

2.1 A note on extensions

We will begin by reviewing what are extension problems in spectral sequences. Strong convergence in the sense of [12] is relative to an abelian group G, called the target group, and a filtration $\cdots \subset F_{s+1} \subset F_s \subset \ldots$ of G, and implies that all the pieces E_s^∞ of the E^∞-page fit in short exact sequences

$$0 \to F_{s+1} \to F_s \to E_s^\infty \to 0. \tag{2.1.1}$$

For our Bockstein spectral sequences $(v.1)$ and $(v.2)$, G is respectively k/q_*X and k_*X and F_s for $s \geq 0$ is the image of the multiplication by v^s map in G. For $(q.1)$ and $(q.2)$, G is respectively k/v_*X and k_*X and F_s for $s \geq 0$ is the image of the multiplication by q^s map in G. If one of the F_r is known, and the E^∞-page of the spectral sequence is known, then the F_s for $s \leq r$ can be determined inductively from the short exact sequences by solving the *extension problems*, that is, knowing the two groups on both sides of the short exact sequence, what group can sit in the middle? Each F_s is said to be an extension of E_s^∞ by F_{s+1}. Since we are interested not only in the group structure but in an R-modules (with R being here k/v_*, k/q_* or $\mathbb{F}$), such extensions are classified by the extensions groups (see for example theorem 3.4.3 of [37])

$$\mathrm{Ext}^1_R(E_s^\infty; F_{s+1}). \tag{2.1.2}$$

In the case of $(v.1)$ or $(q.1)$, R is $\mathbb{F}$ and each E_s^∞ is a free $\mathbb{F}$-module, thus the $\mathrm{Ext}^1_{\mathbb{F}}$ are trivial and F_s is simply the sum $F_{s+1} \oplus E_s^\infty$ over $\mathbb{F}$. The structure of the exact couple defining our Bockstein spectral sequence ensure that the multiplication by respectively v or q on the E^∞-page is the same as in the target group, thus for $(v.1)$ and $(q.1)$ there is an isomorphism between the E^∞-page and the target group respectively as k/q_*-modules or k/v_*-modules.

There will, however, be extension problems in $(v.2)$ and $(q.2)$. We will make our statement about $(v.2)$. In that case, R is k/v_*, and each E_s^∞ is not free but a sum of some R and $R/(q^k)$ for various $k \geq 1$. Then the Ext^1_R will be a product of F_{s+1}/q^kF_{s+1} for each element $R/(q^k)$ of the initial sum. This can be computed using proposition 3.3.4 of [37] and using the straightforward resolution of $R/(q^k)$ with the map $q^k : R \to R$. Thus when an element $x \in E_s^\infty$ has $q^kx = 0$, it is possible that a lift $x' \in F_s$ of x is such that q^kx' is not zero, but is an element $y \in F_{s+1}$ determined up to q^kF_{s+1} by an element in the Ext^1_R group. Determining all these elements y is what we call solving the extension problems. When it is the case, we will say that q^kx *makes an extension* with y, or sometimes just that there is an extension between x and y. Moreover, as in the case of $(v.1)$, the multiplication by v is determined by the Bockstein spectral sequence, so solving the extension problems is the only things to do to determine the target group as a k_*-module from the E^∞-page. Finally, everything we just said about $(v.2)$ can also be stated about $(q.2)$ by replacing every v's with q's and vice versa.

2.2 Examples of computations

We will now present a basic example of extension and of the kind of reasoning we will use later to compute them. Assume given a spectrum X such that k/v_*X

$$\begin{tikzcd}[column sep=small, row sep=small]
b \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots \\
& a \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots
\end{tikzcd}$$

Figure 2.1: First example: the E^∞ page of (v.2).

$$\begin{tikzcd}[column sep=small, row sep=small]
b \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots \\
& a \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
b \arrow[r, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & \cdots \\
& a \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots
\end{tikzcd}$$

Figure 2.2: First example: two possibilities for k_*X.

is generated over k/v_* by two classes a and b such that $qa = 0$, $qb = 0$ and $|a| = |v| + |b|$. Assume also that the spectral sequence $(v.2)$ collapses at its first page, so that $E^\infty \cong E^1 \cong \mathbb{F}\{a, b\} \otimes P(v)$ as seen in fig. 2.1, where multiplication by v is horizontal and multiplication by q is vertical. From this description, we know that k_*X is generated by two classes a and b that are lifts of the classes of the same name in E^∞, and we know that $qb = 0$ since b is alone in its degree. However, we do not know what qa is: it is possible that $qa = 0$ as it is in E^∞, or that $qa = vb$ up to a unit, since vb is in higher filtration that a. These two possibilities, depicted in fig. 2.2, are not presenting isomorphic k_*-modules, and we cannot distinguish between them using $(v.2)$ alone; here we need to know $(v.1)$, whose E^∞-page is isomorphic to k/q_*X, and can be seen in fig. 2.3. If $qa = 0$, then v^kb is not divisible by q for any k, but if $qa = vb$, then v^kb is divisible by q for any $k \geq 1$. In the second case, $v^kb = 0$ in k/q_*X for $k \geq 1$, and the classes with these names in $(v.1)$ will be in the image of the differential; in the first case, they will not be in the image of the differential.

We will now work out a second example. Here, we began with a module that we know is determined from the spectral sequences. We have computed the spectral sequences from the module, but we will present the computation beginning with the spectral sequences and deriving the module, as if we were doing a real computation. We let X be a suitable space for what follows.

Let $H_*(X; \mathbb{F})$ be the free $\mathbb{F}$-module on sixteen generators

$$a,\ b,\ c,\ d,\ \alpha,\ \beta,\ \gamma,\ \delta,\ a',\ b',\ c',\ d',\ \alpha',\ \beta',\ \gamma',\ \delta' \tag{2.2.1}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
b \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots \\
& a \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
b & & & & \\
& a \arrow[r, dash] & \cdot \arrow[r, dash] & \cdot \arrow[r, dash] & \cdots
\end{tikzcd}$$

Figure 2.3: First example: two possibilities for k/q_*X.

$b \quad b' \quad c \quad c' \quad d \quad d' \quad a \quad a' \qquad\qquad \alpha \quad \alpha' \quad \delta \quad \delta' \quad \gamma \quad \gamma' \quad \beta \quad \beta'$

Figure 2.4: Second example: the $\mathbb{F}$-module $H_*(X;\mathbb{F})$.

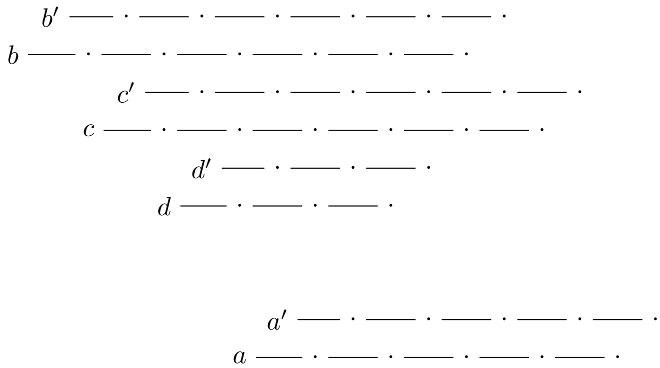


Figure 2.5: Second example: the k/q_*-module k/q_*X.

with b in lowest degree and

$$\begin{aligned} |a| &= |b| + 3|v| \\ |c| &= |b| + |v| \\ |d| &= |b| + 2|v| \\ |\alpha| &= |b| + 6|v| + 1 \\ |\beta| &= |b| + 9|v| + 1 \\ |\gamma| &= |b| + 8|v| + 1 \\ |\delta| &= |b| + 7|v| + 1 \\ |x'| &= |x| + 1 \text{ for any } x. \end{aligned} \tag{2.2.2}$$

We place them in fig. 2.4 by order of degrees, to scale with $|v| = 2$.

We now describe the spectral sequence (v.1) that compute k/q_*X:

$$\begin{aligned} d^4(\alpha) &= v^4 d \quad & d^4(\alpha') &= v^4 d' \\ d^6(\beta) &= v^6 a \quad & d^6(\beta') &= v^6 a' \\ d^7(\delta) &= v^7 b \quad & d^7(\delta') &= v^7 b' \\ d^8(\gamma) &= v^8 c \quad & d^8(\gamma') &= v^8 c' \end{aligned} \tag{2.2.3}$$

so that k/q_*X, which is isomorphic to the E^∞ page of (v.1), is given by fig. 2.5.

On the other side, we have the spectral sequence (q.1):

$$\begin{aligned} & d^1(x') = qx \text{ for } x \text{ equal to } b,\ c,\ d,\ \alpha,\ \delta \text{ and } \gamma. \\ & d^2(x') = q^2 x \text{ for } x \text{ equal to } a \text{ and } \beta. \end{aligned} \tag{2.2.4}$$

whose E^∞-page, isomorphic to k/v_*X, is given by fig. 2.6.

Figure 2.6: Second example: the k/v_*-module $\mathrm{k}/v_x X$.

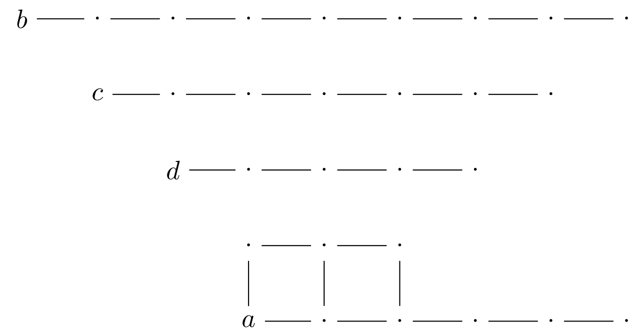


Figure 2.7: Second example: the E^∞-page of $(v.2)$.

Following that spectral sequence is $(v.2)$, whose target is k_*X. However, this time the E^∞-page need not be isomorphic to k_*X; there may be extensions. We begin with the differentials in $(v.2)$:

$$\begin{aligned} d^3(\alpha) &= v^3 qa \\ d^5(\delta) &= v^5 d \\ d^6(\beta) &= v^6 a \\ d^7(\gamma) &= v^7 c \\ d^9(q\beta) &= v^9 b. \end{aligned} \tag{2.2.5}$$

The E^∞-page is given in fig. 2.7. We now have to lift the remaining class in k_*X. We do so by using definition 1.2.9. The differential $d^5(\delta) = v^5 d$ implies that we can lift $d \in E^\infty$ into $d \in \mathrm{k}_*X$ such that $v^5 d = 0$. Similarly, we get a $c \in \mathrm{k}_*X$ such that $v^7 c = 0$. The differentials supported by β and $q\beta$ imply that we can lift a and b into $a\, b \in \mathrm{k}_*X$, with $v^6 a = 0$ and $v^9 b = 0$, but because of the linearity of the connecting map $\mathrm{k}/v_*X \to \mathrm{k}_{*-1}X$, we can also take our lifts such that $qv^5 a = v^8 b$. Later, we will write $\pi(a) = b$ when that kind of case occurs. This is the only relationship given by a multiplication by q that we now for sure on our lifts at the moment; it is possible, for example, that qd is not zero but is vc. The last differential allow us to lift $qa \in E^\infty$ into $q_0 a \in \mathrm{k}_*X$ such that $v^3 q_0 a = 0$. Here, we have chosen the notation $q_0 a$ to emphasize the fact that we do not know if, with the lifts chosen, qa is equal to $q_0 a$; q_0 is not an element in any ring, and $q_0 a$ is not a product but a name for a lift in k_*X of $qa \in \mathrm{k}/v_*X$. In fact, here, because of the relationship $qv^5 a = v^8 b$, q_0 cannot be equal to qa in k_*X. We write all the known properties of our lifts in fig. 2.8, where the dotted lines indicate that we do not know the corresponding multiplication by q.

We solve the extensions problems by going from the lowest degree generator to the highest. The class b must have $qb = 0$ since there is no element divisible

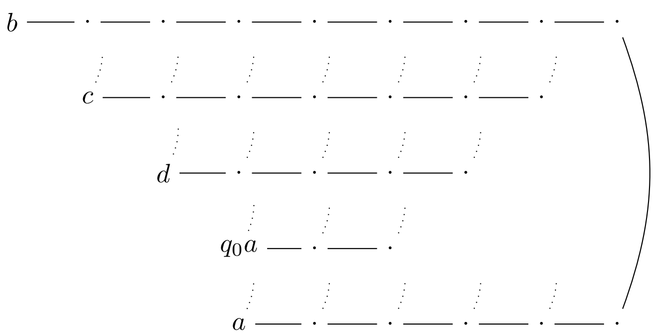


Figure 2.8: Second example: the known relationships between the lifts.

by v in its degree. Next, qc could be zero or vb up to a unit; but if it were vb, then vb would be divisible by q, and thus project to zero in k/q_*X; from our previous description of k/q_*X, this is not the case, so $qc = 0$. Similarly, if qd were not zero, one of the classes v^2b or vc would project to zero in k/q_*X, but this is impossible. So, $qd = 0$. Lastly, since neither vd, v^2c or v^3b project to zero modulo q, it must be that $qq_0a = 0$.

It remains to compute qa. Since q_0a is a lift of $qa \in E^\infty$, it must appear in the formula. From the already known relation $qv^5a = v^8b$, v^3b must also appear in the formula. Thus, we have:

$$qa = q_0a + v^3b + t \tag{2.2.6}$$

where t is such that $v^5t = 0$. Here, t can be any linear combination $\eta vd + \nu v^2c$. We use k/q_*X again: we know that for some y, $v^4d + v^5y$ is zero modulo q. Adding y is necessary because we only that $d \in \mathrm{k}_*X$ project to what we called $d \in \mathrm{k}/q_*X$ up to some element divisible by v. For this to be possible, it must be that η is not zero. This implies that $v^7b + \nu v^6c$ is divisible by q, but $v^6b + \nu v^5x$ is not. But no combination of v^6b and v^5c has this property in k/q_*X, excepted if $\nu = 0$. So we have determined that

$$qa = q_0a + v^3b + \eta vd. \tag{2.2.7}$$

The formula still has an undetermined unit η; later, by inspecting carefully the differentials, we will be able to determine η. Our result is presented in fig. 2.9

Finally, we review an example where the spectral sequences do not determine the module. After this following example, we will provide hypotheses under which it is always possible to recover the module: the hypothesis that is not verified in what follows is hypothesis 2.3.30 (Q_i) for $a_i = a$. We will present two non-isomorphic k_*-modules with the same four Bockstein spectral sequences. We can see that in both case, there is a relationship $qd \overset{d}{\simeq} v^2c$ since there is a differential $d^1(d') = qd$ in $(q.1)$ and a differential $d^2(d') = v^2c$ in $(v.1)$, but that

$$|d| < |v^{o_1(a)}q_0a|. \tag{2.2.8}$$

The module $\mathcal{M}_1$ of fig. 2.10 has a presentation with five generators a, q_0a, b,

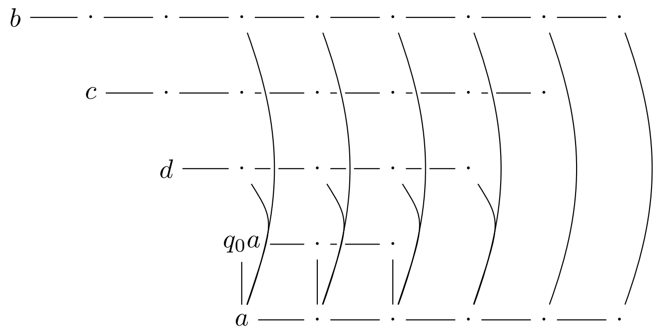


Figure 2.9: Second example: the k_*-module k_*X.

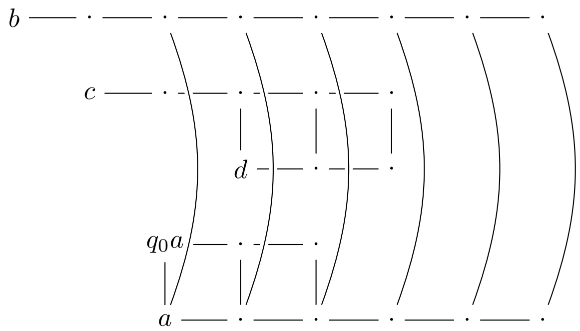


Figure 2.10: Third example: the k_*-module $\mathcal{M}_1 = k_*X_1$.

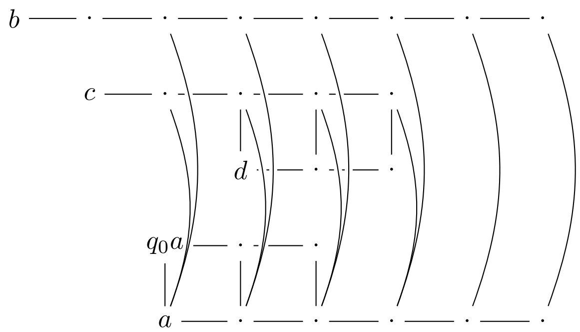


Figure 2.11: Third example: the k_*-module $\mathcal{M}_2 = k_*X_2$.

c and d over k_* and the relations:

$$\begin{aligned} qa &= q_0a + v^2b & v^6a &= 0 \\ qq_0a &= 0 & v^3q_0a &= 0 \\ qb &= 0 & v^8b &= 0 \\ qc &= 0 & v^5c &= 0 \\ qd &= v^2c & v^3d &= 0. \end{aligned} \tag{2.2.9}$$

The module $\mathcal{M}_2$ of fig. 2.11 has a presentation with the same five generators, but with relations:

$$\begin{aligned} qa &= q_0a + vc + v^2b & v^6a &= 0 \\ qq_0a &= 0 & v^3q_0a &= 0 \\ qb &= 0 & v^8b &= 0 \\ qc &= 0 & v^5c &= 0 \\ qd &= v^2c & v^3d &= 0. \end{aligned} \tag{2.2.10}$$

These two modules are not isomorphic, since $\mathcal{M}_1$ has an element (namely b) in degree $|b|$ and an element (namely a) in degree $|a|$ such that

$$v^3(qa - v^2b) = 0 \tag{2.2.11}$$

but no elements of $\mathcal{M}_2$ in these degrees verify this equation. However, if we realize $\mathcal{M}_1$ and $\mathcal{M}_2$ as some homologies k_*X_1 and k_*X_2, then the four Bockstein spectral sequences associated to these spectra are isomorphic. Indeed, for $X = X_1$ or $X = X_2$, the module k/v_*X will be the one of fig. 2.12, with generators b, a, c, δ, γ and α over k/v_* and relations

$$q^2a = 0, \quad qb = 0, \quad qc = 0, \quad q\delta = 0, \quad q^2\gamma = 0, \quad q\alpha = 0. \tag{2.2.12}$$

Figure 2.12: Third example: the k/v_*-module k/v_*X.

The non-zero differentials in the $(v.2)$ spectral sequence are given by:

$$\begin{aligned} d^3(\delta) &= v^3qa \\ d^3(\gamma) &= v^3d \\ d^5(q\gamma) &= v^5c \\ d^6(\alpha) &= v^6a \\ d^8(q\alpha) &= v^8b \end{aligned} \tag{2.2.13}$$

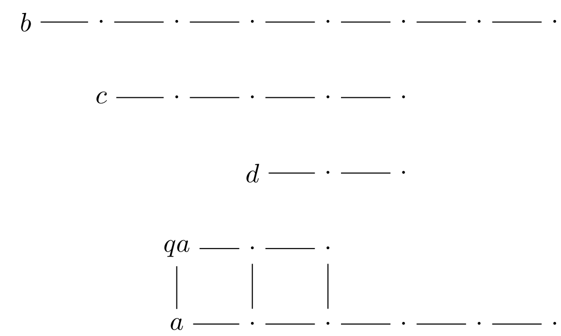


Figure 2.13: Third example: the E^∞-page of $(v.2)$.

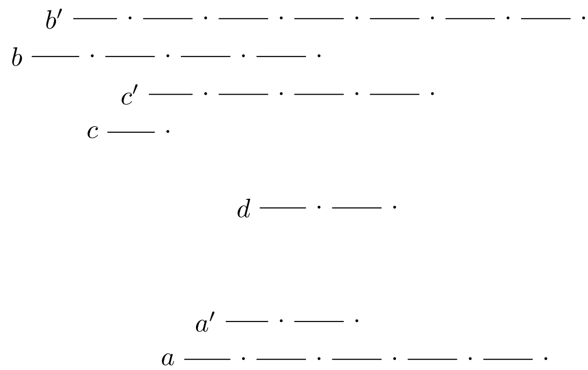


Figure 2.14: Third example: the k/q_*-module k/q_*X.

thus the E^∞-page of $(v.2)$ is given by fig. 2.13, and the difference between k_*X_1 and k_*X_2 is only produced by the extensions for the multiplication by q.

The module $H_*(X;\mathbb{F})$ has generators b, c, a, d, δ, γ, α and b', c' a', d', δ', γ', α' over $\mathbb{F}$, and the $(q.1)$ spectral sequence has differentials given by $d^1(x') = qx$ for any x excepted a, γ and α, which have $d^2(x') = q^2x$.

On the other side of the four spectral sequences, we have k/q_*X as seen on fig. 2.14 generated by b, b', c, c', a, a', and d over k/q_*, and with relations:

$$\begin{aligned} &v^5b = 0, \quad v^8b' = 0, \quad v^2c = 0, \quad v^5c' = 0 \\ &v^6a = 0, \quad v^3a' = 0, \quad v^4d = 0. \end{aligned} \tag{2.2.14}$$

The non-zero differentials in the spectral sequence $(q.2)$ are given by:

$$\begin{aligned} d^1(b') &= qb \\ d^1(c') &= qc \\ d^2(v^5b') &= q^2v^3a \\ d^2(v^2c') &= q^2d \\ d^3(a') &= q^3a \end{aligned} \tag{2.2.15}$$

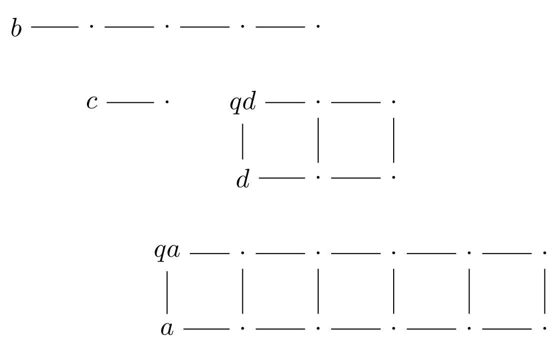


Figure 2.15: Third example: the E^∞-page of $(q.2)$.

thus the E^∞-page of $(q.2)$ is given by fig. 2.15 and the difference between k_*X_1 and k_*X_2 is only produced by the extensions for the multiplication by v.

With the same notation as before, the $(v.1)$ spectral has non-zero differentials given in both cases by:

$$\begin{aligned} d^2(d') &= v^2 c \\ d^3(\delta') &= v^3 a' \\ d^3(\gamma) &= v^3 d d^5(\delta) \quad = v^5 b \\ d^5(\gamma') &= v^5 c' \\ d^6(\alpha) &= v^6 a \\ d^8(\alpha') &= v^8 b'. \end{aligned} \tag{2.2.16}$$

2.3 Statements of the hypotheses and a lemma

We will recover k_*X when X is a bounded below spectrum from the computation of the spectral sequences $(q.1)$ followed by $(v.2)$. However the spectral sequence $(v.1)$ will also be used. Our first hypothesis will be used in lemma 2.3.18 and will provide some structure to the generators of k_*X that we will choose.

Recall that an infinite cycle in a spectral sequence is an element (in any page) whose differential in any subsequent page is zero. Thus, an infinite cycle can be projected in every page of the spectral sequence, is also an element of the E^∞-page and can be lifted in the target group, but might be zero if it is the target of a differential.

Hypothesis 2.3.1 (R1)**.** *In any degree $*$, the codimension over $\mathbb{F}$ of the subspace of the infinite cycles*

$$\bigcap_{r\geq 1} Z_0^r \subset E_0^1 = H_*(X;\mathbb{F}) \tag{2.3.2}$$

is at most 1 in both the spectral sequences $(v.1)$ and $(q.1)$.

Moreover, for any degree $*$, there exists $x \in \mathrm{k}/v_*X$ such that every non-zero differential going out of the bidegree $(*, 0)$ in $(v.2)$ is of the form

$$d^n(q^h x + y) = v^n a \tag{2.3.3}$$

for some n, h and y such that $d^n(y) = 0$.

We will use the first part of the hypothesis in lemma 2.3.18 the following way: there are two connecting homomorphism

$$\begin{tikzcd} H_*(X;\mathbb{F}) \arrow[r] \arrow[d] & \mathrm{k}/q_{*-|v|-1} \\ \mathrm{k}/v_{*-1} & \end{tikzcd} \tag{2.3.4}$$

and if we have two elements $\alpha \in \mathrm{k}/v_{*-1}$ and $\beta \in \mathrm{k}/q_{*-|v|-1}$ that we know can both be lifted to $H_*(X;\mathbb{F})$, then their lifts are not infinite cycles in one of the spectral sequences $(v.1)$ or $(q.1)$. Since from two hyperplanes, it is always possible to choose a third space that is in direct sum with both hyperplanes, we can choose a common lift for α and β up to a unit.

The name (R1) stands for rank 1, and is coming from the simplified version of this hypothesis:

Hypothesis 2.3.5 (sR1)**.** *$H_*(X;\mathbb{F})$ is of dimension at most 1 over $\mathbb{F}$ in any degree $*$.*

(sR1) is directly implying the first part of (R1), and the second part follow from the fact that under (sR1), k/v_*X is generated over $\mathcal{Z}$ by at most one element in each degree.

To complete the structure of the lift of the E^∞-page that we will use in k_*X, we add the following:

Hypothesis 2.3.6 (D)**.** *Assume (R1). We can choose a family $(a_i)_{i\in I}$ of non-divisible by q elements of k/v_*X, such that the $\mathrm{k}/v_* = \mathcal{Z}$-sub-modules of k/v_*X generated by the a_i decompose k/v_*X as a direct sum of summands of the type $\mathcal{Z}$ or $\mathcal{Z}/(q^m)$ for any m, and is such that the element x of hypothesis (R1) is in the family and carry differentials in $(v.2)$ of the form*

$$d^n(q^h x) = v^n q^k a_i \tag{2.3.7}$$

for any n and h such that $d^n(q^h x)$ is not zero.

Putting it differently, we have a family of elements of k/v_*X, that decompose it in a similar manner to the decomposition of finitely generated modules over an integral domain, such that any differential of $(v.2)$ is between two elements of the family. The integer n associated to k and a_i will be denoted by $o_k(a_i)$, it is the v-torsion order of $q^k a_i$ in the E^∞-page of $(v.2)$. When $q^k a_i$ doesn't receive any differential, we will put $o_k(a_i) = +\infty$ so that $1 \leq o_k(a_i) \leq +\infty$ for any k and i.

That hypothesis is *not* a consequence of (R1), as we will argue in section 2.5. However, (D) is a consequence of (sR1), since when there is a differential $d^n(q^h x) = v^n y$, then y has a well-defined degree.

We need to state a consequence of these hypotheses to state the rest of our hypothesis. We will make the following distinction:

Notation 2.3.8. *When we lift a class $q^k a \in \mathrm{k}/v_*X$ that survive to the E^∞-page of $(v.2)$, we will use the notation $q_0^k a$ for the class obtained in k_*X. The spectral sequence $(v.2)$ is a Bockstein spectral sequence associated to the multiplication by v, and thus, the classes lifted from the E^∞-page into k_*X will have the same properties for the multiplication by v in the E^∞-page and in k_*X, but might not have the same properties for the multiplication by q.*

The notation q will be used only for the multiplication by q in the spectral sequence or for the multiplication by q in k_*X, which is still unknown at this stage: if we have two lifts $q_0^k a$ and $q_0^{k+1}a$ of $q^k a$ and $q^{k+1}a$, it is possible that $q \cdot q_0^k a \neq q_0^{k+1} a$. The notation is analogous to our use of v_0 instead of p when computing over ℓ or ku.

Proposition 2.3.9. *Under (D), there are lifts $q_0^k a_i \in \mathrm{k}_*X$ of all the $q^k a_i \in \mathrm{k}/v_*X$ that are infinite cycles in (v.2) such that:*

(a) When $q^k a_i$ is of v-torsion in the E^∞-page of (v.2), then $v^{o_k(a_i)} q_0^k a_i = 0$, otherwise $q_0^k a_i$ is not of v-torsion either.

(b) when there are differentials

$$d(q^h x) = v^{o_k(a_i)} q^k a_i \quad \textit{and} \quad d(q^{h+1}x) = v^{o_\ell(a_j)} q^\ell a_j \tag{2.3.10}$$

in (v.2), then

$$q \cdot (v^{o_k(a_i)-1} q_0^k a_i) = v^{o_\ell(a_j)-1} q_0^\ell a_j \tag{2.3.11}$$

in k_*X. Note that it might be that $a_i = a_j$ and $\ell = k+1$.

(c) when

$$d(q^h x) = v^{o_k(a_i)} q^k a_i \quad \textit{and} \quad q^{h+1}x = 0 \tag{2.3.12}$$

in (v.2), then

$$q \cdot (v^{o_k(a_i)-1} q_0^k a_i) = 0 \tag{2.3.13}$$

in k_*X.

The $q_0^k a_i$ generate k_*X as a k_*-module.

Proof. We work in the exact couple diagram of definition 1.2.9 for our spectral sequence. If there is a non-zero differential $d(q^h x) = v^{o_k(a_i)} q^k a_i$, then:

$$\begin{tikzcd}
\mathrm{k}_*X \arrow[rr, "v"] & & \mathrm{k}_*X \arrow[r, "v"] \arrow[dl, "j"] & \dots \arrow[r, "v"] & \mathrm{k}_*X \arrow[rr, "v"] & & \mathrm{k}_*X \arrow[dl, "j"] \\
& \mathrm{k}/v_*X \arrow[ul, "\partial"] & & & & \mathrm{k}/v_*X \arrow[ul, "\partial"] &
\end{tikzcd}$$

$$\begin{tikzcd}
& q_0^k a_i \arrow[r, mapsto] \arrow[dl, mapsto] & \dots \arrow[r, mapsto] & v^{o_k(a_i)-1} q_0^k a_i \arrow[rr, mapsto] & & 0 \\
q^k a_i & & & & q^h x \arrow[ul, mapsto] &
\end{tikzcd} \tag{2.3.14}$$

Here $v^{o_k(a_i)-1} q_0^k a_i$ is obtained by setting $v^{o_k(a_i)-1} q_0^k a_i = \partial(q^h x)$; the differential ensure that it is divisible by $v^{o_k(a_i)-1}$, and that $j(q_0^k a_i) = q^k a_i$, so $q_0^k a_i$ is a lift of $q^k a_i$.

To get the second and third points of our claim, we use the fact that the connecting map ∂ is a map of k_*-module, and thus:

$$\partial(q^{h+1}x) = q \cdot \partial(q^h x) = q \cdot v^{o_k(a_i)-1} q_0^k a_i \tag{2.3.15}$$

which might be the lift of another element in case (b), or zero in case (c).

The case of the non-torsion classes remains. If $q^k a_i \in \mathrm{k}/v_* X$ is an infinite cycle, then $\partial(q^k a_i) = 0$ so that it lifts through j to $q_0^k a_i \in \mathrm{k}_* X$. Furthermore, if $q^k a_i$ is not of v-torsion in the E^∞-page, then $q_0^k a_i$ cannot be of v-torsion in $\mathrm{k}_* X$, otherwise $q^k a_i$ would be a boundary in some page of the spectral sequence. $\square$

We will write $\pi(q_0^k a_i) = q_0^\ell a_j$ when we are in the situation of (b), and $\pi(q_0^k a_i) = 0$ in the situation of (c), and note that when we will talk about degrees later we will use the convention that $|0| = -\infty$. Note that the $q_0^k a_i$ are not a minimal set of generators of $\mathrm{k}_* X$: if there is no extension to construct the target group from the E^∞-page, there should be a relation

$$q \cdot q_0^k a_i = q_0^{k+1} a_i. \tag{2.3.16}$$

This is possible only if $\pi(q_0^k a_i) = 0$ – if it is not the case, the relation of (b) is an obstruction. In general, there should be relations of the type

$$q \cdot q_0^k a_i = q_0^{k+1} a_i + v^\bullet \pi(q_0^k a_i) + ??? \tag{2.3.17}$$

where $\bullet$ is determined by homogeneity the unknown part of the formula will be determined by the rest of the hypotheses.

We now state the following result related to the divisibility by q in $\mathrm{k}_* X$; this result is central to the rest of the analysis.

Lemma 2.3.18 (divisibility by q)**.** *We assume (R1). Suppose given an element b of $\mathrm{k}_* X$ not divisible by v such that $v^m b$ is not divisible by q but $v^{m+1} b$ is, that there is an element a of $\mathrm{k}_* X$ not divisible by v such that $qv^n a = v^{m+1} b$. Then the modulo q reduction of $v^m b$ is non-zero, but that of $v^{m+1} b$ is zero, and there is a differential $d(x) = v^{m+1} b$ in $(v.1)$. Moreover,*

(a) if $n = 0$, then there is a differential $d(x) = q^{k+1} a'$ in $(q.1)$ for some $a' \in H_*(X; \mathbb{F})$ which is the reduction of $a' \in \mathrm{k}/v_* X$ such that $q^k a' = a$, and the relationship $qv^n a = v^{m+1} b$ in $\mathrm{k}_* X$ results from an extension in $(v.2)$.

(b) if $n > 0$, then $x \in H_*(X; \mathbb{F})$ lifts to $x' \in \mathrm{k}/v_* X$, which support a differential in $(v.2)$ that can be written either $d(-x') = v^n qa$ when $n < m+1$, $d(x') = v^{m+1} b$ when $m+1 < n$ or $d(x') = v^{m+1}(b - qa)$ when $n = m+1$.

Proof. We work in the following commutative diagram of cofiber sequences obtained from multiplication by v and q

$$\begin{tikzcd}
\Sigma^{-2} H\mathbb{F} \arrow[r] \arrow[d] & \Sigma^{|v|-1}\mathrm{k}/q \arrow[r] \arrow[d] & \Sigma^{-1}\mathrm{k}/q \arrow[r] \arrow[d] & \Sigma^{-1} H\mathbb{F} \arrow[d] \\
\Sigma^{-1}\mathrm{k}/v \arrow[r] \arrow[d] & \Sigma^{|v|}\mathrm{k} \arrow[r] \arrow[d] & \mathrm{k} \arrow[r] \arrow[d] & \mathrm{k}/v \arrow[d] \\
\Sigma^{-1}\mathrm{k}/v \arrow[r] \arrow[d] & \Sigma^{|v|}\mathrm{k} \arrow[r] \arrow[d] & \mathrm{k} \arrow[r] \arrow[d] & \mathrm{k}/v \arrow[d] \\
\Sigma^{-1} H\mathbb{F} \arrow[r] & \Sigma^{|v|}\mathrm{k}/q \arrow[r] & \mathrm{k}/q \arrow[r] & H\mathbb{F}
\end{tikzcd} \tag{2.3.19}$$

that we smash with X take homotopy groups to obtain long exact sequences.

On one hand, when $n = 0$, we write the relationship $qa = v^{m+1}b$ in k_*X in the central square. Using exactness, there exist x and y in $H_{*+1}(X;\mathbb{F})$ such that

$$\begin{tikzcd}
 & & & y \arrow[d, mapsto] \\
 & & a \arrow[r, mapsto] \arrow[d, mapsto] & q^k a' \neq 0 \arrow[d, mapsto] \\
 & v^m b \arrow[r, mapsto] \arrow[d, mapsto] & v^{m+1}b = qa \arrow[r, mapsto] \arrow[d, mapsto] & 0 \\
x \arrow[r, mapsto] & v^m b \arrow[r, mapsto] & 0 &
\end{tikzcd} \tag{2.3.20}$$

Since from two hyperplanes, we can always choose a third subspace of dimension 1 that is in direct sum with both hyperplanes, under the first part of (R1), we can choose x and y to be equal up to a unit, and its image through the vertical map yield the claimed differential in $(q.1)$.

On the other hand, when $n > 1$, we can start similarly from the central square with a supplemental lift

$$\begin{tikzcd}
 & v^{n-1}a \arrow[r, mapsto] & v^n a \arrow[d, mapsto] \\
 & v^m b \arrow[r, mapsto] \arrow[d, mapsto] & v^{m+1}b = qv^n a \arrow[d, mapsto] \\
x \arrow[r, mapsto] & v^m b \neq 0 \arrow[r, mapsto] & 0
\end{tikzcd} \tag{2.3.21}$$

and so starting to push from the top left corner we get

$$\begin{tikzcd}
v^{n-1}a \arrow[r, mapsto] \arrow[d, mapsto] & v^n a \arrow[d, mapsto] \\
qv^{n-1}a \arrow[r, mapsto] \arrow[d, mapsto] & v^{m+1}b = qv^n a \\
0 &
\end{tikzcd} \tag{2.3.22}$$

and subtracting the bottom part of the two diagrams

$$\begin{tikzcd}
x' \arrow[r, mapsto] \arrow[d, mapsto] & v^m b - qv^{n-1}a \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
x \arrow[r, mapsto] & v^m b \arrow[r, mapsto] & 0
\end{tikzcd} \tag{2.3.23}$$

which gives the claimed differentials depending on the highest filtration degree in $v^m b - qv^{n-1}a$. $\square$

The cases that will be particularly of interest to us will be (a) $(n = 0)$ and (b) with $n < m + 1$, since we are interested in the extensions, and in that case the class multiplied by q must be in lower filtration than the class receiving the multiplication. Thus, when a class becomes divisible by q because of an extension, it is always possible to determine from the spectral sequence the name of a class that will represent the quotient with this lemma, that is, to determines a from b.

To keep track of the possible extensions that would make a class a_i divisible by q, we introduce the following relation on the first page of the spectral sequence (v.2): when in the (a) case of lemma 2.3.18, precisely when there are differentials $d(x) = v^{m+1}a_i$ in (v.1) and $d(x) = q^{k+1}a_j$ in (q.1), we will write $q^{k+1}a_j \overset{d}{\simeq} v^{m+1}a_i$; when in the (b) case of lemma 2.3.18, that is there are differentials $d(x) = v^{m+1}a_i$ in (v.1) and $d(-x) = v^n q^k a_j$ in (v.2) with $n < m+1$, we will write $v^n q^k a_j \overset{d}{\simeq} v^{m+1}a_i$. That relation $\overset{d}{\simeq}$ lift as an equality in k_*X if we allow the classes on the left and right side to be replaced by classes represented by the same name in (v.2), that is we can add some $v^{\omega}c$ with $\omega > n$ on the left side and some $v^{\omega'}c'$ with $\omega' > m+1$ on the left side. We will be more interested in the converse: when the spectral sequences do not witness a relation $\overset{d}{\simeq}$, then no such equality can hold in k_*X. We also remark that under (R1) and (D), when $y \overset{d}{\simeq} v^{m+1}a_i$ then the name y is unique, since (D) results in unicity on the lift x' in the (b) case of lemma 2.3.18.

We will now state our last two hypotheses: (T_i) is related to the length of the tower for the multiplication by v for some a_i, and (Q_i) will prevent some divisibility by q to occur using $\overset{d}{\simeq}$ statements. These hypotheses depend on i and are not stated for all a_i. For the a_i such that (T_i) and (Q_i) hold, we will be able to compute the extensions. It might be the case that our target module can be split into $\mathcal{M}_1 \oplus \mathcal{M}_2$, where the hypothesis holds on $\mathcal{M}_1$ but not on $\mathcal{M}_2$; for $\mathrm{THH}_*(ku)$ the splitting is between the torsion and the non-torsion, and we will recover the extensions on the torsion using the techniques of this chapter, but the extensions on the non-torsion will be computed by other means.

Hypothesis 2.3.24 (T_i)**.** *For each k, when $\pi(q_0^k a_i) = q_0^h a_j$, with $i \neq j$ and a_j might be taken to be 0, the followings are true:*

(a) If ℓ is such that

$$\begin{gathered} |a_\ell| < |a_j| \\ |a_\ell| \equiv |a_i| \pmod{|v|} \\ |v^{o_0(a_\ell)}a_\ell| > |a_i| \end{gathered} \tag{2.3.25}$$

then

$$|v^{o_0(a_\ell)}a_\ell| \geq |v^{o_k(a_i)}q_0^k a_i|. \tag{2.3.26}$$

(b) If ℓ_1 and ℓ_2 are both such that (for $\ell = \ell_1$ or $\ell = \ell_2$)

$$\begin{gathered} |a_j| \leq |a_\ell| < |a_i| \\ |a_\ell| \equiv |a_i| \pmod{|v|} \\ |v^{o_{k+1}(a_i)}q_0^{k+1}a_i| < |v^{o_0(a_\ell)}a_\ell| < |v^{o_k(a_i)}q_0^k a_i| \end{gathered} \tag{2.3.27}$$

then

$$|a_{\ell_1}| < |a_{\ell_2}| \Rightarrow |v^{o_0(a_{\ell_2})}a_{\ell_2}| < |v^{o_0(a_{\ell_1})}a_{\ell_1}|. \tag{2.3.28}$$

When $a_j = 0$, we recall that we use the convention $|0| = -\infty$. If $q_0^k a_i$ is not of v-torsion, we will let $|v^{o_k(a_i)} q_0^k a_i|$ be $+\infty$ and for the purpose of (2.3.28), we will assume that the relation $+\infty < +\infty$ is false.

This hypothesis state nothing when $\pi(q_0^k a_i) = q_0^{k+1} a_i$.

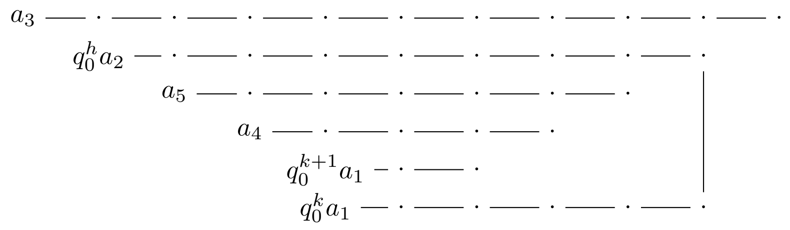


Figure 2.16: Structure of $\mathrm{k}_* X$ under (T_1).

We illustrate hypothesis (T_i) with fig. 2.16, where multiplication by q is denoted by going up and multiplication by v by going right (thus the horizontal axis denote the degree). In that example we have $\pi(q_0^k a_1) = q_0^h a_2$, so the only multiplication by q denoted is the one we already know about, that is the one at the end of the v-tower of $q_0^k a_1$ according to proposition 2.3.9. All the other formulas for multiplying by q are unknown, so we depict nothing. The part (a) of hypothesis (T_1) state that since a_3 is of degree lower than a_2, its v-tower must finish after those of $q_0^h a_2$ and $q_0^k a_1$. Remark that if $a_2 = 0$, then that first part is not constraining anything. The part (b) of the hypothesis state the v-tower of a_4 and a_5 must end in the reverse order compared to the degrees of a_4 and a_5.

Under (T_i), we can thus order the a_ℓ of part (b) of the hypothesis by the degree at which their v-tower ends. We will use the following notation:

Notation 2.3.29. *Let $q_0^k a_i$ bet such that $\pi(q_0^k a_i) = q_0^h a_j$ with $i \neq j$ (but a_j might be 0), and let $b_1^{k,i}, \dots b_n^{k,i}$ be the classes a_ℓ verifying* (2.3.27)*, ordered by increasing degree of $v^{o_0(a_\ell)} a_\ell$.*

It is possible, when $|b_\ell^{k,i}| = |b_{\ell+1}^{k,i}|$, that $|v^{o_0(b_k^{k,i})} b_\ell^{k,i}| = |v^{o_0(b_{\ell+1}^{k,i})} b_{\ell+1}^{k,i}|$, and in that case the ordering between the two classes can be chosen either ways. We are in the situation of fig. 2.17

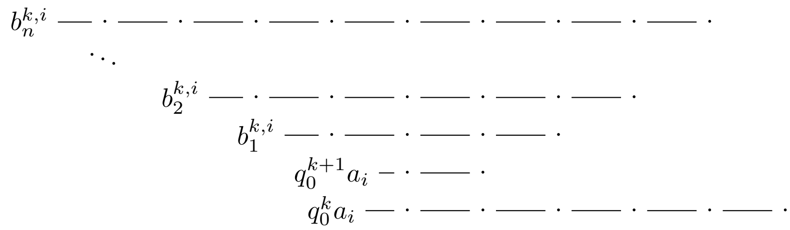


Figure 2.17: Ordering above $q_0^k a_i$ under (T_i).

Remark that (2.3.28) when $q_0^k a_i$ is not of v-torsion implies that there can be at most *one* other a_ℓ with $|a_\ell| \equiv |a_i| \pmod{|v|}$. This is a drastic condition on the periodic classes, but it is necessary for our purpose.

Hypothesis 2.3.30 (Q_i)**.** *Let $q_0^k a_i$ be such that $\pi(q_0^k a_i) = q_0^h a_j$ with $i \neq j$ (a_j might be 0).*

If it exists, let ℓ_1 be such that there is a relation $v^{o_{k+1}(a_i)} q^{k+1} a_i \overset{d}{\simeq} v^\bullet b_{\ell_1}^{k,i}$, and then inductively let ℓ_{j+1} be such that there is a $v^{o_0(b_{\ell_j}^{k,i})} b_{\ell_j}^{k,i} \overset{d}{\simeq} v^\bullet b_{\ell_{j+1}}^{k,i}$. We thus have an eventually empty subsequence $1 \leq \ell_1 < \cdots < \ell_j < \cdots \leq n$. The hypothesis state the following:

For all $1 \leq \ell \leq n$ there is no relation $y \overset{d}{\simeq} v^\bullet b_\ell^{k,i}$ with

$$|y| < |v^{o_{k+1}(a_i)} q_0^{k+1} a_i|. \tag{2.3.31}$$

For any j and ℓ such that $\ell_j < \ell \leq n$, there is no relation $y \overset{d}{\simeq} v^\bullet b_\ell^{k,i}$ with

$$|q_0^k a_i| < |y| < |v^{o_0(b_{\ell_j}^{k,i})} b_{\ell_j}^{k,i}|. \tag{2.3.32}$$

This will ensure that the $b_\ell^{k,i}$ that will appear in the formula for $q \cdot q_0^k a_i$ are correctly detected by the differentials, by not being divisible by q before they can be detected.

The simplified version of hypothesis (T_i) and (Q_i) will be that there are no classes $b_\ell^{k,i}$:

Hypothesis 2.3.33 (sT_i)**.** *For each k, when $\pi(q_0^k a_i) = q_0^h a_j$, with $i \neq j$ and a_j might be taken to be 0, and for each $\ell \neq j$ such that*

$$\begin{aligned} |a_\ell| &\equiv |a_i| \pmod{|v|} \\ |v^{o_0(a_\ell)} a_\ell| &> |a_i|. \end{aligned} \tag{2.3.34}$$

then

$$\begin{aligned} |a_\ell| &< |a_j| \\ |v^{o_0(a_\ell)} a_\ell| &\geq |v^{o_k(a_i)} q_0^k a_i|. \end{aligned} \tag{2.3.35}$$

That simplified hypothesis (sT_i) can then be seen to imply both (T_i) and (Q_i). Lastly, for J a subset of the indices of the family (a_i), we will write (T_J) for the hypothesis *(T_i) holds for all $i \in J$* and similarly (Q_J) and (sT_J).

2.4 Computing the module k_*X under the hypothesis

In this section, we will see how we can recover k_*X as a k_*-module from the spectral sequences when under the hypothesis (R1), (D), (T_I) and (Q_I), which are stated only using the spectral sequences. The same results can be obtained using the simplified hypothesis (sR1) and (sT_I), since they imply the previous one.

Theorem 2.4.1. *Let I be the set of indices appearing in the lifts of proposition 2.3.9.*

Under the hypothesis (R1), (D), (T_I) and (Q_I), we can choose the lifts $q_0^k a_i$ of proposition 2.3.9 such that k_*X is presented as a k_*-module by the $q_0^k a_i$ and the relations:

- $v^{o_k(a_i)} q_0^k a_i = 0$.
- *$q \cdot q_0^k a_i = q_0^{k+1} a_i$ whenever $\pi(q_0^k a_i) = q_0^{k+1} a_i$.*
- *otherwise*

$$q \cdot q_0^k a_i = v^\bullet \pi(q_0^k a_i) + \sum_{\ell=1}^{n} \beta_\ell^{k,i} v^\bullet b_\ell^{k,i} \tag{2.4.2}$$

where the $\bullet$ are simply determined to make the formula homogeneous in degree, the $b_\ell^{k,i}$ are those of 2.3.29 and the $\beta_\ell^{k,i}$ are in $\mathbb{F}$ with the ℓ_1, ℓ_2, ... such that $\beta_{\ell_j}^{k,i} \neq 0$ are determined by the existence of a relation

$$v^{o_{k+1}(a_i)} q^{k+1} a_i \overset{d}{\simeq} v^\bullet b_{\ell_1}^{k,i} \tag{2.4.3}$$

and then inductively by the existence of

$$v^{o_0(b_{\ell_j}^{k,i})} b_{\ell_j}^{k,i} \overset{d}{\simeq} v^\bullet b_{\ell_{j+1}}^{k,i}. \tag{2.4.4}$$

Proof. We will prove the formulas by considering the lifts $q_0^k a_i$ and ordering them by increasing $|a_i|$ and increasing k. Fix some $i \in I$. Let us denote by J the set of indices j such that $|a_j| < |a_i|$. Assume that our results is established for all $q_0^h a_j$ with $j \in J$ and any h, and for all $q_0^h a_i$ such that $h < k$. If there is some $i' \neq i$ such that $|a_{i'}| = |a_i|$, can consider them in either order.

First consider the case where $\pi(q_0^k a_i) = q_0^{k+1} a_i$. The convergence of the spectral sequence $(v.2)$ implies that

$$q \cdot q_0^k a_i = q_0^{k+1} a_i + \sum_{j \in J} \alpha_j v^\bullet q^{h_j} a_j \tag{2.4.5}$$

with the $\bullet$ determined only by homogeneity. Thus, by proposition 2.3.9,

$$v^{o_k(a_i)-1} \cdot \sum_{j \in J} \alpha_j v^\bullet q^{h_j} a_j = 0 \tag{2.4.6}$$

and we can simply change our lift $q_0^{k+1} a_i$ to be

$$q_0^{k+1} a_i' = q_0^{k+1} a_i + \sum_{j \in J} \alpha_j v^\bullet q^{h_j} a_j \tag{2.4.7}$$

to get the formula claimed. The lift $q_0^{k+1} a_i$ and $q_0^{k+1} a_i'$ of $q^{k+1} a_i$ have the exact same property with regard to proposition 2.3.9, so we can continue our induction.

The second case is where our hypothesis are really used. Assume now that $\pi(q_0^k a_i) = q_0^t a_\ell$ for some $\ell \neq i$ or that $\pi(q_0^k a_i) = 0$. We can write a similar formula:

$$q \cdot q_0^k a_i = q_0^{k+1} a_i + v^\bullet \pi(q_0^k a_i) + \sum_{j \in J \setminus \{\ell\}} \alpha_j v^\bullet q^{h_j} a_j \tag{2.4.8}$$

and the sum is still null before the end of the v-tower of $q_0^k a_i$

$$v^{o_k(a_i)-1} \cdot \sum_{j \in J \setminus \{\ell\}} \alpha_j v^\bullet q^{h_j} a_j = 0. \tag{2.4.9}$$

We can eliminate all the j with $h_j > 1$ of the formula by changing our lift $q_0^k a_i$ into

$$q_0^k a_i' = q_0^k a_i - \sum_{j \in J \setminus \{\ell\},\, h_j > 1} \alpha_j v^\bullet q^{h_j - 1} a_j \tag{2.4.10}$$

which once again have the same property from proposition 2.3.9. More generally we can eliminate the a_j that becomes divisible by q in a degree less than $|q_0^k a_i|$ by also subtracting them. We can eliminate the a_j whose v-tower ends before the degree $|v^{o_{k+1}(a_i)} q_0^{k+1} a_i|$ by adding them to $q_0^{k+1} a_i$, which also conserves all the relevant properties. Thus, from the hypothesis (T_i), we can assume without loss of generality that only the $b_\ell^{k,i}$ relative to $q_0^k a_i$ appear in the sum. We then have:

$$q \cdot q_0^k a_i = q_0^{k+1} a_i + v^\bullet \pi(q_0^k a_i) + \sum_{\ell=1}^{n} \beta_\ell^{k,i} v^\bullet b_\ell^{k,i} \tag{2.4.11}$$

and we only need to determine which $\beta_\ell^{k,i}$ are zero or a unit.

The formula implies that

$$q \cdot v^{o_{k+1}(a_i)} q_0^k a_i = v^\bullet \pi(q_0^k a_i) + \sum_{\ell=1}^{n} \beta_\ell^{k,i} v^\bullet b_\ell^{k,i}. \tag{2.4.12}$$

From our previous construction and hypothesis (Q_i), the sum on the right-hand side of the equation becomes divisible by q in the degree of the equation, and by lemma lemma 2.3.18 this will be visible in the differentials. However, we now need to prove some reciprocal to that lemma. In a first time, assume that $o_{k+1}(a_i) \geq 0$. We work again in the diagram (2.3.19) of the proof of lemma 2.3.18. Let the ℓ_j be defined as in the statement of our result. The element

$$v^{o_{k+1}(a_i)-1} q_0^{k+1} a_i = q \cdot v^{o_{k+1}(a_i)-1} q_0^k a_i - v^\bullet \pi(q_0^k a_i) - \sum_{\ell=1}^{n} \beta_\ell^{k,i} v^\bullet b_\ell^{k,i} \tag{2.4.13}$$

of k_*X has a null multiplication by v, and the relation $v^{o_{k+1}(a_i)} q^{k+1} a_i \overset{d}{\simeq} v^\bullet b_{\ell_1}^{k,i}$ implies that its image in k/q_*X is represented by $b_{\ell_1}^{k,i}$. Since by (Q_i), $b_\ell^{k,i}$ cannot be divisible by q up to that degree, ℓ_1 must be the first index of the sum with a non-zero $\beta_\ell^{k,i}$. We determine the rest of the formula with the same argument applied to the relations $v^{o_0(b_{\ell_j}^{k,i})} b_{\ell_j}^{k,i} \overset{d}{\simeq} v^\bullet b_{\ell_{j+1}}^{k,i}$.

In a second time, assume that $o_{k+1}(a_i) = 0$, that is to say $q_0^{k+1} a_i = 0$. The element

$$v^\bullet \pi(q_0^k a_i) + \sum_{\ell=1}^{n} \beta_\ell^{k,i} v^\bullet b_\ell^{k,i} \tag{2.4.14}$$

in degree $* = |q_0^k a_i| - |v|$ of k_*X cannot be divisible by q because of (Q_i), but its multiplication by v is divisible by q, the dividend being $q_0^k a_i$. Thus again the element of lowest filtration of the sum must appear in a $\overset{d}{\simeq}$ relation, and it must be $b_{\ell_1}^{k,i}$. □

Remark that if we otherwise know that k_*X is split as $\mathcal{M}_1 \oplus \mathcal{M}_2$ as a k_*-module, and that only $\mathcal{M}_1$ verify the hypothesis, the formulas given will be internal to $\mathcal{M}_1$, which is then entirely determined as a k_*-module.

2.5 The hypothesis (D) is not a consequence of (R1)

In this section we will see why, as stated earlier, the hypothesis (D) is not a consequence of (R1), and how this hypothesis can otherwise be reasoned about in terms of linear algebra on matrices. We begin by providing a k/v_*-module that verify (R1) but not (D). Our example has four generators over k/v_*, named a, b, x and y, and has relations:

$$\begin{aligned} q^2 a &= 0 \\ q^2 b &= 0 \\ qx &= 0 \\ qy &= 0. \end{aligned} \tag{2.5.1}$$

Assume that the $(v.2)$ spectral sequence from this module has non-zero differentials given by:

$$\begin{aligned} d^1(x) &= vqa \\ d^2(y) &= v^2(a - qb). \end{aligned} \tag{2.5.2}$$

This verifies (R1), but we cannot change the generator x and y other than up to a unit since they are alone in their respective degree, and we cannot change the generators a and b to make the formulas respect (D). We can take a supplementary subspace (generated by x and y) of the infinite cycles (generated by a and b) and write the differentials as

$$\begin{bmatrix} q & 1 \\ 0 & q \end{bmatrix} \tag{2.5.3}$$

in the basis (a, b) for the rows and (x, y) for the columns. This matrix cannot be diagonalized by using only operations on the rows (this represents changing the chosen generators a and b for the infinite cycles), and the operations on the columns of the form $C_j \leftarrow C_j + \alpha C_i$ for some $i < j$ (this represents the fact that after each differential we have quotiented the cycles by some boundaries). Thus, (D) cannot hold in that case.

Chapter 3

Isomorphisms between Whitehead, Postnikov and Atiyah-Hirzebruch spectral sequences

In this chapter, we give explicit isomorphisms between the spectral sequences coming from a Whitehead tower, a Postnikov tower and the Atiyah-Hirzebruch spectral sequence constructed from a skeletal filtration. These results are well known, and a proof for the cohomological case can be found in [25] and the appendix of [21]; however the author is not aware of them having a proof written down in the homological case.

We will work in the category of S-modules; all the proof in this chapter shall work in the homotopy category of any reasonable category of spectra. Let X and Y be spectra: Y will be our homology theory, and we want to compute Y_*X. First we will compare the spectral sequence coming from the Whitehead tower of Y to that coming from the Postnikov tower of Y. Then, X will need to have a CW structure (e.g. X is a CW-complex, a CW-R-module, ...); we will compare the Whitehead spectral sequences to the Atiyah-Hirzebruch spectral sequence defined by the CW structure on X.

3.1 Whitehead and Postnikov spectral sequences

A Whitehead tower for Y will be a tower of spectra $Y_{\geq n}$ for $n \in \mathbb{Z}$ with cofibers:

$$\begin{tikzcd} \dots \arrow[r] & Y_{\geq n+1} \arrow[r] \arrow[d] & Y_{\geq n} \arrow[r] \arrow[d] & Y_{\geq n-1} \arrow[r] \arrow[d] & \dots \\ & \Sigma^{n+1} HY_{n+1} & \Sigma^{n} HY_{n} & \Sigma^{n-1} HY_{n-1} & \end{tikzcd} \tag{3.1.1}$$

such that each $Y_{\geq n}$ is $(n-1)$-connected, Y is the colimit of the $Y_{\geq n}$, each map $Y_{\geq n} \to Y_{\geq n-1}$ is an isomorphism on homotopy group in degrees greater or equal to n, and HY_n is the Eilenberg-MacLane spectra associated to the group $\pi_n(Y)$.

A Postnikov tower for Y will be a tower of spectra $Y_{<n}$ for $n \in \mathbb{Z}$ with cofibers:

$$\begin{tikzcd} \dots \arrow[r] & Y_{<n+1} \arrow[r] \arrow[d] & Y_{<n} \arrow[r] \arrow[d] & Y_{<n-1} \arrow[r] \arrow[d] & \dots \\ & \Sigma^{n+2} HY_{n+1} & \Sigma^{n+1} HY_n & \Sigma^n HY_{n-1} & \end{tikzcd} \tag{3.1.2}$$

such that each $Y_{<n}$ is n-truncated, Y is the limit of the $Y_{<n}$, each map $Y_{<n+1} \to Y_{<n}$ is an isomorphism on homotopy group in degrees lesser than n, and HY_n is the Eilenberg-MacLane spectra associated to the group $\pi_n(Y)$.

By smashing with X for the Whitehead tower, and desuspending one time the Postnikov tower then smashing with X, we get two spectral sequences with

$$E^1_{p,q} = \pi_{p+q}(X \wedge \Sigma^q HY_q) = H_p(X; Y_q). \tag{3.1.3}$$

However, one would be computing its colimit and the other its limit. We already have the tool to compare them in the form of proposition 1.1.8. We just need to ensure that in our category of spectra, we can construct the Whitehead tower and the Postnikov tower of Y such that there are cofiber sequences:

$$Y \to Y_{<n} \to \Sigma Y_{\geq n} \tag{3.1.4}$$

for each n. The Whitehead tower is then the quotient of the Postnikov tower by its colimit.

Proposition 3.1.5. *With the hypothesis above, the first, non-derived exact couples defined by the Whitehead tower and the Postnikov tower are isomorphic.*

3.2 Whitehead and Atiyah-Hirzebruch spectral sequences

To define the Atiyah-Hirzebruch spectral sequence, we need X to have a CW structure, which for us will imply that is there is a tower with cofibers:

$$\begin{tikzcd} \dots \arrow[r] & X^{(p-1)} \arrow[r] \arrow[d] & X^{(p)} \arrow[r] \arrow[d] & X^{(p+1)} \arrow[r] \arrow[d] & \dots \\ & \bigvee S^{p-1} & \bigvee S^p & \bigvee S^{p+1} & \end{tikzcd} \tag{3.2.1}$$

such that $X^{(p)} \simeq *$ whenever $p < 0$, the cofibers are wedges of spheres of dimension p. Let (A^1, D^1) then be the exact couple obtained by smashing with Y and taking the homotopy:

$$\begin{aligned} A^1_{p,q} &= \pi_{p+q}(X^{(p)} \wedge Y) \\ D^1_{p,q} &= \pi_{p+q}(\bigvee S^p \wedge Y). \end{aligned} \tag{3.2.2}$$

We need also that in the derived exact couple (A^2, D^2) the wanted homology appears:

$$\begin{aligned} A^2_{p,q} &= Im(\pi_{p+q}(X^{(p-1)} \wedge Y) \to \pi_{p+q}(X^{(p)} \wedge Y)) \\ D^2_{p,q} &\cong H_p(X; Y_q). \end{aligned} \tag{3.2.3}$$

This will be the case for example for a CW-R-module or a CW-complex. This spectral sequence will converge to $\pi_*(X \wedge Y)$ under the right conditions – for X a CW-R-module, it might be that R is a connective cofibrant commutative S-algebra, and the spectral sequence will be strongly convergent. We suppose from this point that our construction gives a strongly convergent Atiyah-Hirzebruch spectral sequence.

Let us also explicit the exact couple (W^2, E^2) for the Whitehead spectral sequence. What was the first page in the previous section is now the second for convenience.

$$\begin{aligned} W^2_{p,q} &= \pi_{p+q}(X \wedge Y_{\geq q}) \\ E^2_{p,q} &= \pi_{p+q}(X \wedge \Sigma^q HY_q) = H_p(X; Y_q). \end{aligned} \tag{3.2.4}$$

Theorem 3.2.5. *The exact couples (A^2, D^2) and (W^2, E^2) are isomorphic, and thus define isomorphic spectral sequences.*

The rest of the chapter will be a proof of this theorem. We will introduce a third, intermediary exact couple (B^1, F^1), where

$$B^1_{p,q} = \pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \tag{3.2.6}$$

and its derived couple (B^2, F^2). We will see how (B^1, F^1) is isomorphic to (A^1, D^1), thus all their derived couples are also isomorphic, and then how (B^2, F^2) is isomorphic to (W^2, E^2), yielding our theorem by composition.

The bidegrees of the maps in the exact couples are as follows:

$$\begin{tikzcd} A^1 \arrow[rr, "(1,-1)"] & & A^1 \arrow[dl, "(0,0)"] \\ & D^1 \arrow[ul, "(-1,0)"] & \end{tikzcd} \qquad \begin{tikzcd} A^2 \arrow[rr, "(1,-1)"] & & A^2 \arrow[dl, "(-1,1)"] \\ & D^2 \arrow[ul, "(-1,0)"] & \end{tikzcd} \tag{3.2.7}$$

$$\begin{tikzcd} W^2 \arrow[rr, "(1,-1)"] & & W^2 \arrow[dl, "(0,0)"] \\ & E^2 \arrow[ul, "(-2,1)"] & \end{tikzcd} \tag{3.2.8}$$

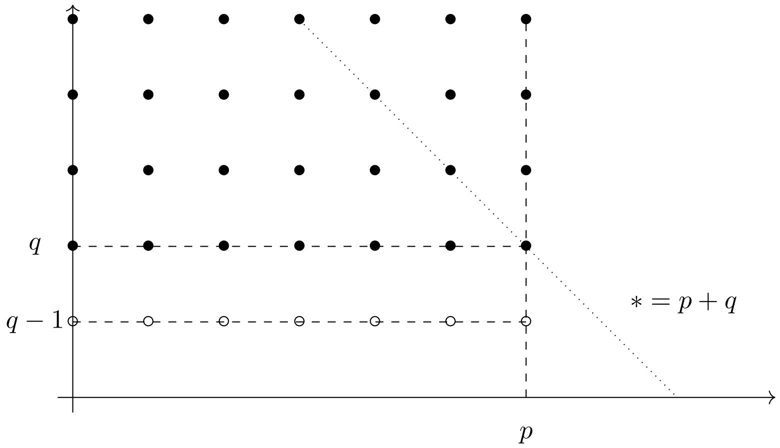
We are looking to construct a morphism of exact couples with $E^2 \to D^2$ of bidegree $(0,0)$, so it must be that the map $W^2 \to A^2$ have bidegree $(1,-1)$. Remark also that the differentials d^r in both sequences have bidegrees $(-r, r-1)$.

We will need the following lemma:

Lemma 3.2.9.
- *The map $\pi_*(X^{(p)} \wedge Y_{\geq q}) \to \pi_*(X^{(p)} \wedge Y_{\geq q-1})$ is an isomorphism when $* \geq p+q$ and is injective when $* = p+q-1$.*
- *The map $\pi_*(X^{(p)} \wedge Y_{\geq q}) \to \pi_*(X^{(p+1)} \wedge Y_{\geq q})$ is an isomorphism when $* \leq p+q-1$ and is surjective when $* = p+q$.*
- *The map $\pi_*(X^{(p)} \wedge Y_{\geq q}) \to \pi_*(X^{(p)} \wedge Y)$ is an isomorphism when $* \geq p+q$ and is injective when $* = p+q-1$.*
- *The map $\pi_*(X^{(p)} \wedge Y_{\geq q}) \to \pi_*(X \wedge Y_{\geq q})$ is an isomorphism when $* \leq p+q-1$ and is surjective when $* = p+q$.*

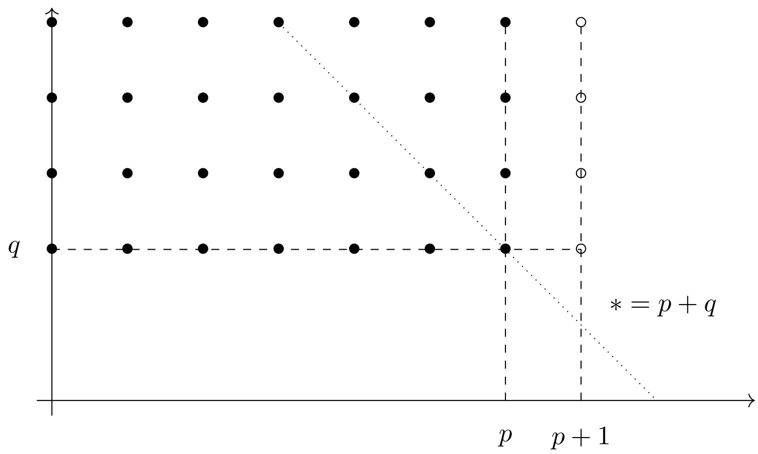
Proof. The third and fourth claims follow easily from respectively the first and second one.

To prove the first claim, let us consider the Atiyah-Hirzebruch spectral sequences computing $\pi_*(X^{(p)} \wedge Y_{\geq q})$ and $\pi_*(X^{(p)} \wedge Y_{\geq q-1})$ – the Whitehead-Postnikov spectral sequences would also work here. It follows from the hypothesis on X that the homology groups $H_*(X^{(p)}; Y_q)$ are concentrated in degrees between 0 and p. The E^2 pages are then as follows:



Each $\bullet$ is a group that is in both spectral sequences. Each $\circ$ is a group that is only in the spectral sequence computing $\pi_*(X^{(p)} \wedge Y_{\geq q-1})$. The differentials having source or target above or on the line $* = p+q$ can be seen to be the same in both sequences. Thus, the E^∞ pages are isomorphic in this zone, and the part about isomorphism in our claim follows. On the line below, $* = p+q-1$, the differentials are again the same, but there can be a non-zero group in bidegree $(p, q-1)$, and we can only get an injection. Remark that no result of this sort can be stated for all the lines $* < p+q-1$, since in that zone there might be differentials with source one the horizontal $q-1$ line.

To get the second claim, we proceed similarly with the spectral sequences computing $\pi_*(X^{(p)} \wedge Y_{\geq q})$ and $\pi_*(X^{(p+1)} \wedge Y_{\geq q})$. The E^2 pages are as follows:



We get isomorphisms for $* \leq p+q-1$, a surjection for $* = p+q$ since there can be differentials of source in bidegree $(p+1, q)$, and nothing can be said of $* > p+q$. $\square$

We will construct a third exact couple that we will use to compare the two already defined. For any integers p and q, let $\mathcal{F}_{p,q}$ be the fiber of the map $X^{(p)} \wedge Y_{\geq q} \to X^{(p+1)} \wedge Y_{\geq q-1}$. The groups for our exact couples will be:

$$\begin{aligned} B^1_{p,q} &= \pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \\ F^1_{p,q} &= \pi_{p+q}(\mathcal{F}_{p,q}). \end{aligned} \tag{3.2.10}$$

The maps will *not* be the one induced by the long exact sequence in homotopy of the fiber sequence. We will define them in the commutative diagram of fig. 3.1, with rows exact – the first two rows are to be ignored for the moment; the maps for the exact couple are going from the bottom left (sixth row) to the middle right (third row), and the bend arrow is simply the composition of the maps in the commutative square it crosses. The decorations of the arrows (isomorphisms and injections) are coming from lemma 3.2.9, except for the middle injection which come from half a five lemma.

The decorations are enough to check that we have indeed defined a long exact sequence, and thus an exact couple. The bidegrees in the exact couple and in the first derived exact couple are as follows:

$$\begin{tikzcd} B^1 \arrow[rr, "{(1,-1)}"] & & B^1 \arrow[dl, "{(-1,0)}"] \\ & F^1 \arrow[ul, "{(0,0)}"] & \end{tikzcd} \qquad \begin{tikzcd} B^2 \arrow[rr, "{(1,-1)}"] & & B^2 \arrow[dl, "{(-2,1)}"] \\ & F^2 \arrow[ul, "{(0,0)}"] & \end{tikzcd} \tag{3.2.11}$$

Proposition 3.2.12. *The exact couples* (A^1, D^1) *and* (B^1, F^1) *are isomorphic.*

Proof. This can be seen in fig. 3.1, this time paying attention to the whole diagram. Once again, the isomorphisms come from lemma 3.2.9, except for the middle one which is the five lemma. $\square$

The second part of our proof is to use the map $X^{(p)} \to X$ to compare the first derived exact couple (B^2, F^2) with the Whitehead exact couple (W^2, E^2).

Proposition 3.2.13. *For all p and q, there are isomorphisms* $B^2_{p+1,q-1} \cong W^2_{p,q}$ *that commutes with the exact couples maps:*

$$\begin{tikzcd} W^2_{p,q} \arrow[r] & W^2_{p+1,q-1} \\ B^2_{p+1,q-1} \arrow[r] \arrow[u, "\simeq"] \arrow[ur, phantom, "\circlearrowleft"] & B^2_{p+2,q-2} \arrow[u, "\simeq"'] \end{tikzcd} \tag{3.2.14}$$

Proof. There is a commutative diagram:

$$\begin{tikzcd} & \pi_{p+q}(X \wedge Y_{\geq q}) \\ \pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \arrow[r, two heads] \arrow[ur, two heads] \arrow[dr] & \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[u, "\simeq"] \arrow[d, hook] \\ & \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q-1}) \end{tikzcd} \tag{3.2.15}$$

$$\begin{tikzcd}
A^1_{p,q+1} \arrow[r] \arrow[d, equal] & A^1_{p+1,q} \arrow[r] \arrow[d, equal] & D^1_{p+1,q} \arrow[r] \arrow[d, equal] & A^1_{p,q} \arrow[r] \arrow[d, equal] & A^1_{p+1,q-1} \arrow[d, equal] \\
\pi_{p+q+1}(X^{(p)} \wedge Y) \arrow[r] & \pi_{p+q+1}(X^{(p+1)} \wedge Y) \arrow[r] & \pi_{p+q+1}(\bigvee S^{p+1} \wedge Y) \arrow[r] & \pi_{p+q}(X^{(p)} \wedge Y) \arrow[r] & \pi_{p+q}(X^{(p+1)} \wedge Y) \\
 & & F^1_{p,q} \arrow[r] \arrow[d, equal] & B^1_{p,q} \arrow[r] \arrow[d, equal] & B^1_{p+1,q-1} \arrow[d, equal] \\
\pi_{p+q+1}(X^{(p)} \wedge Y_{\geq q}) \arrow[r] \arrow[uu, "\cong"'] & \pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q-1}) \arrow[r] \arrow[uu, "\cong"'] & \pi_{p+q}(\mathcal{F}_{p,q}) \arrow[r] \arrow[uu, bend right, "\cong"'] & \pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \arrow[r] \arrow[uu, bend right, "\cong"'] & \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q-1}) \arrow[uu, bend right, "\cong"'] \\
\pi_{p+q+1}(X^{(p)} \wedge Y_{\geq q+1}) \arrow[r] \arrow[u, "\cong"'] \arrow[d, equal] & \pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[r] \arrow[u, "\cong"'] \arrow[d, equal] & \pi_{p+q}(\mathcal{F}_{p,q+1}) \arrow[r] \arrow[u, hook] & \pi_{p+q}(X^{(p)} \wedge Y_{\geq q+1}) \arrow[r] \arrow[u, hook] & \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[u, hook] \\
B^1_{p,q+1} \arrow[r] & B^1_{p+1,q} \arrow[uuur, bend right]
\end{tikzcd}$$

Figure 3.1: Definition of the third exact couple.

so that there are two isomorphisms:

$$\begin{tikzcd} W^2_{p,q} = \pi_{p+q}(X \wedge Y_{\geq q}) \\ Im(\pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \to \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q})) \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\ B^2_{p+1,q-1} = Im(\pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \to \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q-1})) \end{tikzcd} \tag{3.2.16}$$

which we compose to get the isomorphism claimed. The compatibility with the exact couples maps come from the commutative diagram:

$$\begin{tikzcd} \pi_{p+q}(X \wedge Y_{\geq q}) \arrow[r] & \pi_{p+q}(X \wedge Y_{\geq q-1}) \\ \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[r] \arrow[u] \arrow[d] & \pi_{p+q}(X^{(p+2)} \wedge Y_{\geq q-1}) \arrow[u] \arrow[d] \\ \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q-1}) \arrow[r] & \pi_{p+q}(X^{(p+2)} \wedge Y_{\geq q-2}) \end{tikzcd} \tag{3.2.17}$$

of which we considered restriction to images to get our isomorphisms. □

From the bidegree of our maps, we need to construct an isomorphism $F^2 \to E^2$ of bidegree $(1, 0)$ to complete our isomorphism between the exact couples.

Proposition 3.2.18. *For all p and q, there are isomorphisms $F^2_{p,q} \cong E^2_{p+1,q}$ that, together with the isomorphisms of the previous proposition, assemble into an isomorphism of exact couples between (B^2, F^2) and (W^2, E^2).*

Proof. Let us first remark that simply having maps $F^2_{p,q} \to E^2_{p+1,q}$, which together with the previous isomorphisms define a morphism of exact couple, would be sufficient for these maps to be isomorphism by the five lemma. The commutative diagram of fig. 3.2 allow us to construct such a map; it has rows exact, and the arrows are decorated according to lemma 3.2.9 and the five lemma.

We will define an application f from $Ker(d^1 : F^1_{p,q} \to F^1_{p-1,q})$ to $E^2_{p+1,q}$ first. Let $x \in Ker(F^1_{p,q} \to F^1_{p-1,q})$ (start chasing in $F^1_{p,q}$ near the middle of the diagram), and let its image in $\pi_{p+q}(X^{(p)} \wedge Y_{\geq q})$ be denoted by y. By pushing y into the bottom portion of the diagram, we see that y can be lifted to $\pi_{p+q}(X^{(p)} \wedge Y_{\geq q+1})$, which imply that x can be lifted to $x' \in \pi_{p+q}(\mathcal{F}_{p,q+1})$; such lift is unique, and its image $x'' \in E^2_{p+1q,q}$ allow us to define $f(x) = x''$.

Next we have to prove that $Im(d^1 : F^1_{p+1,q} \to F^1_{p,q}) \subset Ker\, f$ in order for our application f to be well-defined on the homology, that is on $F^2_{p,q}$. Let $a \in F^1_{p+1,q}$ – we begin our chase on the bottom left – and let $b \in \pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q})$ be its image; b can be pushed all the way up to $\pi_{p+q+1}(X \wedge Y_{\geq q})$, and following the other path using the big curved arrow, we see that its image must be 0 in $\pi_{p+q+1}(X \wedge Y_{\geq q})$. In order to do so, it is useful to remark that the two path

Figure 3.2: The map between the second pages.

$$
\begin{tikzcd}
 & W^2_{p+1,q} \arrow[d, equal] & E^2_{p+1,q} \arrow[d, equal] & W^2_{p-1,q+1} \arrow[d, equal] & \\
\pi_{p+q+1}(X \wedge Y_{\geq q+1}) \arrow[r] & \pi_{p+q+1}(X \wedge Y_{\geq q}) \arrow[r] & \pi_{p+q+1}(X \wedge \Sigma^q HY_q) \arrow[r] & \pi_{p+q}(X \wedge Y_{\geq q+1}) \arrow[r] & \pi_{p+q}(X \wedge Y_{\geq q}) \\
\pi_{p+q+1}(X^{(p)} \wedge Y_{\geq q+1}) \arrow[r] \arrow[u, two heads] \arrow[d, "\simeq"] & \pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[r] \arrow[u, two heads] \arrow[d, "\simeq"] & \pi_{p+q}(\mathcal{F}_{p,q+1}) \arrow[r] \arrow[u, two heads] \arrow[d, hook] & \pi_{p+q}(X^{(p)} \wedge Y_{\geq q+1}) \arrow[r] \arrow[u, "\simeq"'] \arrow[d, hook] & \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[u, "\simeq"'] \arrow[d, hook] \\
\pi_{p+q+1}(X^{(p)} \wedge Y_{\geq q}) \arrow[r] & \pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q-1}) \arrow[r] & \pi_{p+q}(\mathcal{F}_{p,q}) = F^1_{p,q} \arrow[r] & \pi_{p+q}(X^{(p)} \wedge Y_{\geq q}) \arrow[r] \arrow[ddd, "\simeq"] & \pi_{p+q}(X^{(p+1)} \wedge Y_{\geq q-1}) \\
\pi_{p+q+1}(\mathcal{F}_{p+1,q}) \arrow[r] \arrow[d, equal] & \pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[r] \arrow[u, "\simeq"] & \pi_{p+q+1}(X^{(p+2)} \wedge Y_{\geq q-1}) & & \\
F^1_{p+1,q} & & \pi_{p+q+1}(X^{(p+2)} \wedge Y_{\geq q}) \arrow[u, hook] \arrow[uuuul, bend left, "\simeq"] & & \\
 & & \pi_{p+q}(X^{(p-1)} \wedge Y_{\geq q}) \arrow[r] & \pi_{p+q}(X^{(p)} \wedge Y_{\geq q-1}) \arrow[r] & \pi_{p+q-1}(\mathcal{F}_{p-1,q}) \arrow[d, equal] \\
 & & \pi_{p+q}(X^{(p-1)} \wedge Y_{\geq q+1}) \arrow[u, "\simeq"] \arrow[uuuuur, two heads, bend right] & & F^1_{p-1,q}
\end{tikzcd}
$$

taken, when the isomorphism are inverted, lie in a commutative square:

$$\begin{tikzcd}
\pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[r] \arrow[d, "id"] & \pi_{p+q+1}(X^{(p+2)} \wedge Y_{\geq q}) \arrow[d] \\
\pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q}) \arrow[r] & \pi_{p+q+1}(X^{(p+2)} \wedge Y_{\geq q-1})
\end{tikzcd} \tag{3.2.19}$$

Then we see that $f(d^1(a)) = 0$, since to compute f we need the lift of $d^1(a)$ to $\pi_{p+q}(\mathcal{F}_{p,q+1})$, and b gives us such a lift whose image in $E^2_{p+1,q}$ must be zero. Thus, we have constructed an application as claimed.

To check commutativity, we work in the same diagram. We need to check that the following is commutative:

$$\begin{tikzcd}
B^2_{p+2,q-1} \arrow[r] \arrow[d, "\simeq"] & F^2_{p,q} \arrow[r] \arrow[d, "f"] & B^2_{p,q} \arrow[d, "\simeq"] \\
W^2_{p+1,q} \arrow[r] & E^2_{p+1,q} \arrow[r] & W^2_{p-1,q+1}
\end{tikzcd} \tag{3.2.20}$$

$F^2_{p,q}$ is a quotient of a subgroup of $F^1_{p,q}$; $B^2_{p+2,q-1} = Im(\pi_{p+q+1}(X^{(p+1)} \wedge Y_{\geq q}) \to \pi_{p+q+1}(X^{(p+2)} \wedge Y_{\geq q-1})$ (below $F^1_{p,q}$ in the diagram) and the isomorphism with $W^2_{p+1,q}$ is constructed using the injection and the big curved arrow; similarly, $B^2_{p,q} = Im(\pi_{p+q}(X^{(p-1)} \wedge Y_{\geq q+1}) \to \pi_{p+q}(X^{(p)} \wedge Y_{\geq q})$ and the isomorphism with $W^2_{p-1,q+1}$ is constructed using the injection and isomorphism in the column on the right of $F^1_{p,q}$. This part of the diagram is sufficient to check that we indeed have commutativity as claimed. □

Part II

Topological Hochschild Homology of ku and the Bökstedt trace map

Chapter 4

Topological Hochschild homology

In this chapter, we will define topological Hochschild homology and some of the tools, mostly spectral sequences, that we will later use in our computation.

The spectral sequence that appears with the first definition of Topological Hochschild homology by Bökstedt in [13] is of the following type:

$$HH_*(H_*(R;\mathbb{F}_p)) \Rightarrow H_*(\mathrm{THH}(R);\mathbb{F}_p) \tag{4.0.1}$$

where R is a ring spectrum and HH_* is the Hochschild homology. Both the source and the target of the spectral sequence can be seen to have the structure of a comodule over the dual Steenrod algebra and the structure of a commutative $H_*(R;\mathbb{F}_p)$-algebra, and the spectral sequence is compatible with these structures (see for example [3]), and this was used to compute $\mathrm{THH}_*(\mathbb{Z}_p)$ and $\mathrm{THH}_*(\mathbb{F}_p)$ in [13].

The existence of an algebra structure on $\mathrm{THH}(R)$ allows the construction of various Bockstein spectral sequences associated to the multiplication by some element of the algebra; as we studied in chapter 2, the exact couple of a Bockstein spectral sequence is obtained from the cofiber sequence of the multiplication by the chosen element. The Bökstedt spectral sequence can also be extended to compute other homology theories in situation where a Künneth formula holds; in [28], this is used to compute the first periodic Morava K-theory $K(1)_* \mathrm{THH}(\ell)$. That computation is extended to ku in [5] and this result is the basis to the computation of $V(0)_* \mathrm{THH}(ku)$ via the Bockstein spectral sequence for the multiplication by u.

Although multiple Bockstein spectral sequences can be constructed from an algebra, they must all compute the same thing. That fact yields a computation of $\mathrm{THH}_*(\ell)$ in [2] by making the Bockstein spectral sequences for multiplication by p and u compete. We will extend their result to $\mathrm{THH}_*(ku)$ and study some part of the computation with greater generality.

The spectral sequence of Brun compute THH of a ring A with coefficients in an A-algebra B from THH of B with coefficients in a generalized Tor group in the sense of [20]. In [14], that spectral sequence is introduced to compute $\mathrm{THH}_*(\mathbb{Z}/p^n)$. Modern categories of spectra allow us to express this spectral sequence as an Atiyah-Hirzebruch spectral sequence, as done in [23] to compute

$V(1)_*\,\mathrm{THH}(ku)$ and $V(0)_*\,\mathrm{THH}(K(\mathbb{F}_q);\mathbb{Z}_p)$. Switching the ring with the coefficients often yield a smaller object to compute; moreover, this can be repeated multiple times, and when the Tor groups is simple enough, the Brun spectral sequence will also be a Bockstein spectral sequence. Part of our computation of $\mathrm{THH}_*(ku)$ will use these facts.

Other techniques to compute THH include the use of results relating THH of a Thom spectrum $T(f)$ with a product of Thom spectra constructed from a map f classifying a spherical fibration. This can be seen in [9] or [10] to be able to compute $\mathrm{THH}(H\mathbb{Z})$, $\mathrm{THH}(H\mathbb{F}_p)$ or $\mathrm{THH}(MU)$ since these spectra can be described as Thom spectra.

We work in the category $\mathcal{M}_R$ of R-modules of [20], from which most of our definitions will come.

4.1 Simplicial spectra and their realization

Let Δ be the simplex category, whose object are the ordered sets of integers $[n] = \{0, \ldots, n\}$ and morphisms are the order preserving maps.

Definition 4.1.1. *A simplicial R-module is a functor $F : \Delta^{op} \to \mathcal{M}_R$.*

For such a functor, its geometric realization, denoted $|F|$, is the coend

$$\int^{\Delta} F \wedge (\Delta_\bullet)_+ \tag{4.1.2}$$

that is the coend of the functor $\Delta^{op} \times \Delta \to \mathcal{M}_R$ that sends (n, m) to $F(n) \wedge (\Delta_m)_+$, where $\Delta_\bullet$ is the topological simplex, viewed as a functor $\Delta \to Top$.

Similarly, a simplicial based space is a functor $F : \Delta^{op} \to Top_*$, and its geometric realization $|F|$ is the coend of the functor $F \wedge (\Delta_\bullet)_+$.

The geometric realization, as a coend, is in fact a coequalizer, and thus will commute with colimits. Other useful properties of the geometric realization are:

Proposition 4.1.3 (X.1.3 of [20]).

- *For a simplicial based space $X_\bullet$, there is a natural isomorphism*

$$\Sigma^\infty |X_\bullet| \cong |\Sigma^\infty X_\bullet|. \tag{4.1.4}$$

- *For a simplicial based space $X_\bullet$ and a simplicial spectrum $Y_\bullet$, a simplicial R-module $Y_\bullet \wedge X_\bullet$ can be obtained by composing the diagonal $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$ with the functor $\Delta^{op} \times \Delta^{op} \to \mathcal{M}_R$ sending $(n,\, m)$ to $Y_n \wedge X_m$, and there is a natural isomorphism*

$$|Y_\bullet \wedge X_\bullet| \cong |Y_\bullet| \wedge |X_\bullet|. \tag{4.1.5}$$

- *For two simplicial spectra $Y_\bullet$ and $Z_\bullet$, again using the diagonal structure, there is a natural isomorphism*

$$|Y_\bullet \wedge Z_\bullet| \cong |Y_\bullet| \wedge |Z_\bullet|. \tag{4.1.6}$$

A useful example of simplicial R-module is given by the bar construction:

Definition 4.1.7 (IV.7.2 of [20]). *For an S-algebra R, a right R-module M and a left R-module N, the bar construction of (M, R, N) is the simplicial S-module $B_\bullet(M, R, N)$ whose n-th simplicial level is*

$$B_n(M, R, N) = M \wedge R^{\wedge n} \wedge N \tag{4.1.8}$$

whose i-th face map is multiplication on the i-th $\wedge$, and whose i-th degeneracy map is given by adding an R between the i-th R and the $(i+1)$-th R via the unit $S \to R$.

Denote by $B(M, R, N)$ the realization $|B_\bullet(M, R, N)|$.

Proposition 4.1.9 (IV.7.5 of [20]). *For M a cell R-module and N any R-module, there is a natural weak equivalence*

$$B(M, R, N) \simeq M \wedge_R N. \tag{4.1.10}$$

If R is commutative, A is an R-algebra and M and N are right and left A-modules, one can also form the bar construction $B^R_\bullet(M, A, N)$ by replacing all the smash products by smash products over R. In that case:

Proposition 4.1.11 (X.1.2 and XII.1.2 of [20]). *There is a natural weak equivalence $B^R(A, A, N) \simeq N$.*

The next section will also define topological Hochschild homology as a simplicial spectrum.

4.2 Simplicial definition of THH and consequences

Let R be a cofibrant commutative S-algebra; A be a cofibrant R-algebra; M be an (A, A)-bimodule. Let

$$\phi : A \wedge_R A \to A \text{ and } \eta : R \to A \tag{4.2.1}$$

be the multiplication and unit of A. Let

$$\xi_\ell : A \wedge_R M \to M \text{ and } \xi_r : M \wedge_R A \to M \tag{4.2.2}$$

be the left and right action of A on M. Let

$$\tau : M \wedge_R A^{\wedge n} \wedge_R A \to A \wedge_R M \wedge_R A^{\wedge n} \tag{4.2.3}$$

be the map cyclically permuting the factors. Here and after all the smash products are over R.

Definition 4.2.4 (IX.2.1 of [20]). *The topological Hochschild homology of A with coefficients in M is the realization, denoted $\mathrm{THH}^R(A; M)$, of the simplicial R-module $\mathrm{THH}^R(A; M)_\bullet$ whose n-th simplicial level is given by*

$$\mathrm{THH}^R(A; M)_n = M \wedge_R A^{\wedge n} \tag{4.2.5}$$

with i-th face map given by $\xi_r \wedge id^{n-1}$ if $i = 0$, $id \wedge id^{i-1} \wedge \phi \wedge id^{n-i-1}$ if $0 < i < n$, $(\xi_\ell \wedge id^{n-1}) \circ \tau$ if $i = n$; and with i-th degeneracy map given by $id \wedge id^i \wedge \eta \wedge id^{n-1}$.

This construction is also called the cyclic bar construction; we will use it again in chapter 7 and write it as $B^{cy}(A;M)$.

When working over $R = S$, we will drop the ^{S} from the notation. When $M = A$, we will write $\mathrm{THH}^R(A) = \mathrm{THH}^R(A;A)$. When A is commutative, topological Hochschild homology has the following structure:

Proposition 4.2.6 (IX.2.2 of [20])**.** *Let A be a commutative R-algebra. Then $\mathrm{THH}^R(A)$ is naturally a commutative A-algebra with unit map the inclusion of the 0-th simplicial level $A \to \mathrm{THH}^R(A)$; $\mathrm{THH}^R(A;M)$ is a $\mathrm{THH}^R(A)$-module.*

From the cited properties of the geometric realization with respect to the smash product, and by seeing M as a constant simplicial spectrum, one can see that:

Proposition 4.2.7. *When A is commutative and M is a symmetric (A,A)-bimodule, there is a natural isomorphism of simplicial R-modules*

$$M \wedge_A \mathrm{THH}^R(A)_\bullet \cong \mathrm{THH}^R(A;M)_\bullet \tag{4.2.8}$$

and thus a natural isomorphism of R-modules

$$M \wedge_A \mathrm{THH}^R(A) \cong \mathrm{THH}^R(A;M). \tag{4.2.9}$$

We will use this mostly with the fact that for the Smith-Toda complex $V(0)$ (the modulo p sphere), we have $V(0) \wedge H\mathbb{Z} \cong V(0) \wedge H\mathbb{Z}_p \cong H\mathbb{F}_p$, so

$$V(0) \wedge \mathrm{THH}(A;H\mathbb{Z}) \cong V(0) \wedge \mathrm{THH}(A;H\mathbb{Z}_p) \cong \mathrm{THH}(A;H\mathbb{F}_p). \tag{4.2.10}$$

The simplicial construction of THH can also be linked with the bar construction. For an R-algebra A, let $A^e = A \wedge_R A^{op}$ be the enveloping algebra of A, where A^{op} is the R-algebra obtained by composing the multiplication $A \wedge_R A \to A$ of A with the map permuting the two factors $A \wedge A \to A \wedge A$.

Proposition 4.2.11 (IX.2.4 and IX.2.5 of [20])**.** *There is a natural isomorphism*

$$\mathrm{THH}^R(A;M) \cong M \wedge_{A^e} B^R(A,A,A) \tag{4.2.12}$$

that gives a natural weak equivalence

$$\mathrm{THH}^R(A;M) \simeq M \wedge_{A^e} A \tag{4.2.13}$$

when M is a cell A^e-module.

Proof. On the n-th simplicial level, by seeing M as a constant simplicial spectrum, here are natural isomorphism

$$M \wedge_R A^{\wedge n} \cong M \wedge_{A^e} (A^e \wedge_R A^{\wedge n}) \cong M \wedge_{A^e} (A \wedge_R A^{\wedge n} \wedge_R A) \tag{4.2.14}$$

and the simplicial maps can be seen to be that of $B^R_\bullet(A,A,A)$ on the right. The properties of the geometric realization yield the result.

The weak equivalence comes from proposition III.3.8 of [20] and the weak equivalence $B^R(A,A,A) \simeq A$. □

Thus, we could have defined $\mathrm{THH}^R(A;M)$ as the derived smash product $M \wedge^L_{A^e} A$, which is the second definition proposed in [20].

4.3 Spectral sequences computing THH

The original result of Brun was the following:

Theorem 4.3.1 (Brun). *When $R \to A$ is a ring homomorphism between (discrete) commutative rings, there is a multiplicative spectral sequence:*

$$E^2_{n,m} = \mathrm{THH}_n(HA; H\,\mathrm{Tor}^R_m(A, A)) \Rightarrow \mathrm{THH}(HR; HA). \tag{4.3.2}$$

That result was generalized by Höning in [23]

Theorem 4.3.3 (1.1 of [23]). *Let A be a cofibrant commutative S-algebra and B be a connective cofibrant commutative A-algebra. Let E be an S-ring spectrum. Then there is a multiplicative spectral sequence of the form*

$$E^2_{n,m} = \mathrm{THH}_n(B; HE^S_m(B \wedge_A B)) \Rightarrow E^S_{n+m}(\mathrm{THH}(A; B)). \tag{4.3.4}$$

The lemma below is in important step to the previous theorem.

Lemma 4.3.5 (4.8 of [23]). *Let $S \to A \to B$ be cofibration of commutative S-algebras. Then we have an isomorphism of A^e-ring spectra*

$$\mathrm{THH}(A; B) \cong (B \wedge_A B) \wedge^L_{A^e} B. \tag{4.3.6}$$

The theorem then comes from constructing a multiplicative Atiyah-Hirzebruch spectral sequence with the skeletal filtration on a CW model of B, and rewriting the E^2 page using the lemma again with the arising Eilenberg-MacLane spectra. The Atiyah-Hirzebruch spectral sequence is the following:

Theorem 4.3.7 (IV.3.7 of [20]). *Let R be a connective cofibrant commutative S-algebra. Let M be a connective R-module and let G be an arbitrary R-module. Then, we have a strongly convergent spectral sequence of the form*

$$(HG_*)^R_* M \Rightarrow G^R_* M. \tag{4.3.8}$$

These spectral sequences enjoys multiplicative property:

Lemma 4.3.9 (3.17 of [23]). *Let R be a connective cofibrant commutative S-algebra. Let M, N and L be connective R-modules and let G be a cell R-module. Let ${}_ME^*_{*,*}$, ${}_NE^*_{*,*}$ and ${}_LE^*_{*,*}$ be the Atiyah-Hirzebruch spectral sequences computing the G homology of M, N and L. Then maps $G \wedge^L_R G \to G$ and $M \wedge^L_R N \to L$ in the derived category of R-module induce a pairing of spectral sequences*

$$ {}_ME^*_{*,*} \otimes {}_NE^*_{*,*} \to {}_LE^*_{*,*} \tag{4.3.10}$$

that converge to the products

$$\pi_*(G \wedge^L_R M) \otimes \pi_*(G \wedge^L_R N) \to \pi_*(G \wedge^L_R L). \tag{4.3.11}$$

There is also another spectral sequence we will use to compute topological Hochschild homology:

Proposition 4.3.12 (Lemma 2.2 and corollary 2.3 of [2])**.** *Suppose $R \to Q$ is a map of S-algebras and M is a (Q, R)-bimodule, given an (R, R)-bimodule structure by pullback. Then there is a weak equivalence*

$$\mathrm{THH}(R; M) \simeq M \wedge^L_{Q \wedge R^{op}} Q \tag{4.3.13}$$

and thus a Künneth spectral sequence

$$\mathrm{Tor}^{Q_* R^{op}}_{*,*}(M_*, Q_*) \Rightarrow \mathrm{THH}_*(R; M). \tag{4.3.14}$$

The last spectral sequence we will use in our computation is the Bockstein spectral sequence, studied already in chapter 2. We will now specify our definition in the context if topological Hochschild homology.

Assume that A and M are R-algebra, that A is commutative, that M is a connective, symmetric (A, A)-bimodule and that there is a map of (A, A)-bimodule $m : \Sigma^n M \to M$ for some $n \geq 0$, which factorizes through a weak equivalence $\Sigma^n M \simeq M_{\geq n}$ with the nth Whitehead section. Let M/m be the cofiber

$$\Sigma^n M \xrightarrow{m} M \longrightarrow M/m. \tag{4.3.15}$$

We can define an exact couple from the tower of spectra with cofibers

$$\begin{tikzcd} \dots \arrow[r, "m"] & \Sigma^{2n} M \arrow[r, "m"] \arrow[d] & \Sigma^n M \arrow[r, "m"] \arrow[d] & M \arrow[d] \\ & \Sigma^{2n} M/m & \Sigma^n M/m & M/m \end{tikzcd} \tag{4.3.16}$$

after smashing it with $\wedge_A \mathrm{THH}(A)$. We prove in chapter 3 that these king of Whitehead spectral sequence is in fact an Atiyah-Hirzebruch spectral sequence, thus we can use theorem 4.3.7 and lemma 4.3.9 to get:

Proposition 4.3.17 (Bockstein spectral sequence)**.** *Under the hypotheses of the paragraph above, when $\mathrm{THH}(A; M/m)$ is connective, we have a strongly convergent spectral sequence*

$$\mathrm{THH}_*(A; M/m) \mathbin{\bar{\otimes}} P(m) \Rightarrow \mathrm{THH}_*(A; M). \tag{4.3.18}$$

Here, the map m need not be a multiplication; the $P(m)$ represent the different copies of $\mathrm{THH}_*(A; M/m)$ of the first page of the spectral sequence. Moreover, if M is and A-algebra, and M/m can also be realized as an A-algebra, then that spectral sequence is a spectral sequence of algebras.

4.4 Smashing localizations and THH

Let R be a cofibrant commutative S-algebra; A be a cofibrant R-algebra and M be an (A, A)-bimodule. Let E be a cell R-module. We will study the Bousfield localization at E, whose definition and useful properties can be found in chapter VIII of [20]. We suppose that the Bousfield localization at E of R-module is smashing, that is the localization of any R-module X, denoted X_E, can be realized as $R_E \wedge_R X$ where R_E is the Bousfield localization of R at E. Precisely, we can construct R_E to be an R-algebra and the localization map $\lambda : R \to R_E$ to be an algebra map. Then the localization map of A

$$\lambda : A \xrightarrow{\simeq} R \wedge_R A \xrightarrow{\lambda \wedge id} R_E \wedge_R A \tag{4.4.1}$$

can be seen to be an R-algebra map, where the multiplication on $R_E \wedge_R A$ is

$$R_E \wedge_R A \wedge_R R_E \wedge_R A \xrightarrow{id \wedge \tau \wedge id} R_E \wedge_R R_E \wedge_R A \wedge_R A \xrightarrow{\mu \wedge \mu} R_E \wedge_R A \tag{4.4.2}$$

where τ switch the two factors and μ are the multiplications. Similarly, B_E can be given both an (A, A)-bimodule such that λ is an (A, A)-bimodule map, and an (A_E, A_E)-bimodule structure.

Proposition 4.4.3. *If the condition above are meet, then there is an isomorphism*

$$\mathrm{THH}^R(A; B)_E \cong \mathrm{THH}^R(A; B_E) \tag{4.4.4}$$

and a weak equivalences

$$\mathrm{THH}^R(A; B_E) \simeq \mathrm{THH}^R(A_E; B_E). \tag{4.4.5}$$

Proof. $\mathrm{THH}^R(A; B)_E$ can be seen to be the realization of the simplicial object $R_E \wedge_R \mathrm{THH}^R(A; B)_\bullet$, which is also $\mathrm{THH}^R(A; B_E)_\bullet$. This yields the isomorphism.

The map $\lambda : R_E \to R_E \wedge_R R_E$ as defined above is an E-equivalence between E-local R-modules, and thus a weak equivalence. Define a simplicial map

$$\mathrm{THH}^R(A; B_E)_\bullet \to \mathrm{THH}^R(A_E; B_E)_\bullet \tag{4.4.6}$$

such that on the n-th simplicial level we have:

$$\begin{array}{c} B_E \wedge_R A^{\wedge n} = R_E \wedge_R B \wedge_R A^{\wedge n} \\ \Big\downarrow{\scriptstyle \simeq} \\ R_E \wedge_R R^{\wedge n} \wedge_R B \wedge_R A^{\wedge n} \\ \Big\downarrow{\scriptstyle id \wedge \lambda^n \wedge id} \\ R_E \wedge_R R_E^{\wedge n} \wedge_R B \wedge_R A^{\wedge n} \\ \Big\downarrow{\scriptstyle \tau} \\ R_E \wedge_R B \wedge_R (R_E \wedge_R A)^{\wedge n} = B_E \wedge_R A_E^{\wedge n}. \end{array} \tag{4.4.7}$$

Each of these maps is a weak equivalence, so by taking a suitable cellular replacement and by theorem X.1.2 of [20], we get a weak equivalence between the realizations. ∎

Chapter 5

Topological Hochschild homology of ku

In this chapter, we will compute $\mathrm{THH}_*(ku)$. The spectral sequences used in this computation are summed up in table 5.1 on the next page.

We first give a computation of $\mathrm{THH}_*(ku; H\mathbb{Z})$ using the Brun spectral sequence and some knowledge of the modulo 2 homotopy of that spectrum in section 5.2. Then the Bockstein spectral sequence (ℓ), computing $\mathrm{THH}_*(\ell)$, is known from [2]. We review this result in section 5.3. In order to lift this computation to the Bockstein spectral sequence (u), computing $\mathrm{THH}_*(ku)$, one must find another way to compare the sequences than the map induced by the inclusion $\ell \to ku$, since $\sigma v_1 \in \mathrm{THH}_{2p-1}(\ell; H\mathbb{Z}_p)$ should be compared to $u^{p-2}\sigma u$ which is not a class in $\mathrm{THH}_{2p-1}(ku; \mathbb{Z}_p)$. A solution is to consider the cofiber of the multiplication by v_1:

$$\Sigma^{2p-1} ku \xrightarrow{v_1} ku \longrightarrow ku/v_1. \tag{5.0.1}$$

We will have to work p-locally for an odd prime p, and we will see that $u^{p-2}\sigma u$ is indeed a class of $\mathrm{THH}_{2p-1}(ku; ku/v_1)$, that we compute in section 5.5 using a comparison between the Brun spectral sequences $(\ell_{\mathbb{Z}})$ and (u_{TB}) and the truncated Bockstein spectral sequence (u_T) – which has fewer classes and is easier to track.

The techniques we developed in chapter 1 can then be used to determine the u-Bockstein spectral sequence for ku, which is done in section 5.6. We can compare the v_1-Bockstein spectral sequences (ℓ) and (v_1), and the Bockstein spectral sequence (u) can be recovered from the truncated Bockstein spectral sequence (u_T) and the reindexed Bockstein spectral sequence (v_1).

Lastly, the extensions can be computed with the results of chapter 2 from the structure of the Bockstein spectral sequence (u), thus determining $\mathrm{THH}_*(ku)$ as ku_*-module.

Our q-cofibrant commutative S-algebra model for the connective complex K-theory spectrum ku will be the one of theorem VII.4.3 of [20]; notwithstanding, the E_∞ structure on ku can be seen to be unique (see [8]). In section 5.2 we will use this integral model for ku, but beginning with section 5.3 ku will denote p-localized connective complex K-theory and ℓ its Adams summand, unless otherwise stated. We obtain an S-algebra structure on the localization using

Table 5.1: Table of the spectral sequences used.

Name	Type	$E^1_{n,m}$	Target
$(\ell_{\mathbb{Z}})$	Brun	$\mathrm{THH}_n(H\mathbb{Z}_{(p)}; H(H\mathbb{Z}_{(p)} \wedge_\ell H\mathbb{Z}_{(p)})_m)$ $\cong \mathrm{THH}_n(H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, E(\sigma v_1)_m$	$\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})$
(ℓ)	Bockstein	$\mathrm{THH}_n(\ell; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P(v_1)_m$	$\mathrm{THH}_*(\ell)$
(u_V)	Bockstein	$V(0)_n\, \mathrm{THH}(ku; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P(u)_m$ $\cong (E(\sigma u, \lambda_1) \otimes P(\mu_1))_n \,\bar{\otimes}\, P(u)_m$	$V(0)_*\, \mathrm{THH}(ku)$
(u_{VT})	Truncated Bockstein	$V(0)_n\, \mathrm{THH}(ku; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P_{p-1}(u)_m$ $\cong (E(\sigma u, \lambda_1) \otimes P(\mu_1))_n \,\bar{\otimes}\, P_{p-1}(u)_m$	$V(0)_*\, \mathrm{THH}(ku; ku/v_1)$
(u_T)	Truncated Bockstein	$\mathrm{THH}_n(ku; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P_{p-1}(u)_m$ $\cong (\mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma u))_n \,\bar{\otimes}\, P_{p-1}(u)_m$	$\mathrm{THH}_*(ku; ku/v_1)$
(u_{TB})	Brun	$\mathrm{THH}_n(ku/v_1; H(ku/v_1 \wedge_{ku} ku/v_1)_m)$ $\cong (\mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma u) \otimes \Gamma(\varphi u))_n$ $\bar{\otimes}(E(\sigma v_1) \otimes P_{p-1}(u))_m$	$\mathrm{THH}_*(ku; ku/v_1)$
(L)	Bockstein	$\mathrm{THH}_n(\ell; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P(v_1, v_1^{-1})_m$	$\mathrm{THH}_*(\ell; L)$
(v_1)	Bockstein	$\mathrm{THH}_n(ku; ku/v_1) \,\bar{\otimes}\, P(v_1)_m$	$\mathrm{THH}_*(ku)$
(u)	Bockstein	$\mathrm{THH}_n(ku; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P(u)_m$	$\mathrm{THH}_*(ku)$

$|\mu_{kp}| = (2kp-1, 0)$, $k \geq 1$ the generators of $\mathrm{THH}_*(H\mathbb{Z}_{(p)})$

$|\sigma v_1| = (0, 2p-1)$

$|v_1| = (0, 2(p-1))$

$|\sigma u| = (3, 0)$

$|\lambda_1| = (2p-1, 0)$

$|\mu_1| = (2p, 0)$

$|u| = (0, 2)$

$|\varphi u| = (2p, 0)$

On the left side of the $\bar{\otimes}$, the generators have bidegrees lying on the horizontal axis; on the right, on the vertical axis.

the result on Bousfield localization stated in proposition VIII.1.8 of [20]. Our S-algebra model for the quotient of ku by v_1 will be

$$ku/v_1 = ku \wedge_\ell H\mathbb{Z}_{(p)} \tag{5.0.2}$$

which is also a q-cofibrant commutative S-algebra by remark VII.6.8 of [20].

5.1 The periodic case

The spectra ku and ℓ are the connective cover of the spectra KU and L, the (periodic) p-completed complex K-theory spectrum and its (periodic) Adams summand. Since we already defined the connective version, we will consider KU and L to be the spectra obtained by inverting the Bott element or v_1 and then p-completing. Inverting these elements is a smashing localization as stated in before theorem VIII.4.3 of [20]. This can also be seen to be the localization of ku and ℓ at the Johnson-Wilson spectrum $E(1)$. In either case, they have the structure of S-algebras. Moreover, what we proved earlier about smashing localization and THH applies.

The homotopy type of p-completed topological Hochschild homology of L was computed in [28] (theorem 8.1):

$$\mathrm{THH}(L)_p \simeq (L \vee \Sigma L_{\mathbb{Q}})_p \tag{5.1.1}$$

where the subscript p denotes p-completion and the subscript $\mathbb{Q}$ denotes rationalization. The argument was extended in [5] (proposition 7.13) to a compatible splitting with KU:

$$\mathrm{THH}(KU)_p \simeq (KU \vee \Sigma KU_{\mathbb{Q}})_p. \tag{5.1.2}$$

This periodic result allow us to prove the following important lemma on the structure of the connective case:

Lemma 5.1.3. *In* $\mathrm{THH}_*(ku)_{(p)}$ *and for any p prime, the p-torsion elements and the u-torsion elements are the same. Here, the subscript (p) denotes p-localization.*

Proof. We will work with the following commutative diagram where the maps are formally inverting the elements given:

$$\begin{array}{ccc}
\mathrm{THH}_*(ku)_{(p)} & \xrightarrow{\quad a \quad} & \mathrm{THH}_*(ku)_{(p)}[u^{-1}] \\
\big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle c} \\
\mathrm{THH}_*(ku)_{(p)}[p^{-1}] & \xrightarrow{\ d\ } & \mathrm{THH}_*(ku)_{(p)}[p^{-1}, u^{-1}]
\end{array} \tag{5.1.4}$$

The kernel of a is the u-torsion elements, the kernel of b is the p-torsion elements. To prove our claim, we only have to prove that c and d are monomorphisms.

In each degree, $\mathrm{THH}_*(ku)_{(p)}$ will be a p-local finitely generated abelian group; this can be seen from the E^1-page of the Bockstein spectral sequence (u). Thus, from the structure theorem of finitely generated abelian groups, we can see that to check if a map is a monomorphism, it is sufficient to check if the induced map on p-completion is a monomorphism.

$\mathrm{THH}_*(ku)_{(p)}[p^{-1}]$ is the rationalization $\mathrm{THH}_*(ku)_{\mathbb{Q}}$, which can be computed using the Künneth spectral sequence:

$$\mathrm{Tor}^{E_* A^e}(E_*A, E_*M) \Rightarrow E_* \mathrm{THH}^R(A; M). \tag{5.1.5}$$

Here, $E = H\mathbb{Q}$, $A = M = ku$ and R is the sphere spectrum, and we have:

$$\mathrm{Tor}^{ku_{\mathbb{Q}*} \otimes ku_{\mathbb{Q}*}}(ku_{\mathbb{Q}*}, ku_{\mathbb{Q}*}) \Rightarrow \mathrm{THH}_*(ku)_{\mathbb{Q}}. \tag{5.1.6}$$

$ku_{\mathbb{Q}*}$ has a resolution as a $ku_{\mathbb{Q}*} \otimes ku_{\mathbb{Q}*}$-module given by

$$0 \leftarrow ku_{\mathbb{Q}*} \leftarrow ku_{\mathbb{Q}*} \otimes ku_{\mathbb{Q}*}\{1\} \leftarrow ku_{\mathbb{Q}*} \otimes ku_{\mathbb{Q}*}\{\sigma u\} \leftarrow 0 \tag{5.1.7}$$

with $d(\sigma u) = 1 \otimes u - u \otimes 1$, thus the spectral sequence collapses at the E^2-page with

$$\mathrm{THH}_*(ku)_{\mathbb{Q}} \cong ku_{\mathbb{Q}*} \otimes E(\sigma u) \tag{5.1.8}$$

and $|\sigma u| = 3$. This is sufficient to see that the map d from the initial diagram is a monomorphism, and that

$$\mathrm{THH}_*(ku)_{(p)}[p^{-1}, u^{-1}] \cong KU_{\mathbb{Q}*} \otimes E(\sigma u). \tag{5.1.9}$$

On the other side, inverting u is a smashing localization (see lemma V.1.15 of [20]), so that our proposition 4.4.3 yields a weak equivalence

$$\mathrm{THH}_*(ku)_{(p)}[u^{-1}] \simeq \mathrm{THH}_*(KU)_{(p)}. \tag{5.1.10}$$

The previous result on p-completed $\mathrm{THH}(KU)$ and equation (5.1.9) allow us to conclude that c is also a monomorphism. □

5.2 Topological Hochschild homology of ku with coefficients in $H\mathbb{Z}$

In this section, we compute $\mathrm{THH}_*(ku; H\mathbb{Z})$ using the Brun spectral sequence:

$$E^2_{p,q} = \mathrm{THH}_p(H\mathbb{Z}; H\pi_q(H\mathbb{Z} \wedge_{ku} H\mathbb{Z})) \Rightarrow \mathrm{THH}_{p+q}(ku; H\mathbb{Z}) \tag{$u_{\mathbb{Z}}$}$$

whose differentials are of the form:

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}. \tag{5.2.1}$$

The Künneth spectral sequence can be used to compute the coefficients.

Proposition 5.2.2.

$$\pi_*(H\mathbb{Z} \wedge_{ku} H\mathbb{Z}) \cong E(\sigma u) \tag{5.2.3}$$

an exterior algebra over $\mathbb{Z}$ on the generator σu of degree 3.

Proof. $\mathbb{Z}$ has a resolution as a free ku_*-module given by $E(\sigma u)$, with σu of bidegree $(1, 2)$ and $d(\sigma u) = u$, so that $\mathrm{Tor}^{ku_*}_{*,*}(\mathbb{Z}, \mathbb{Z}) \cong E(\sigma u)$. Then the Künneth spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{ku_*}_{p,q}(\mathbb{Z}, \mathbb{Z}) \Rightarrow \pi_{p+q}(H\mathbb{Z} \wedge_{ku} H\mathbb{Z}) \tag{5.2.4}$$

collapses for bidegree reasons with no extensions possible. □

The E^2 page of our Brun spectral sequence will then be two copies of $\mathrm{THH}_*(H\mathbb{Z}; H\mathbb{Z}) = \mathrm{THH}_*(H\mathbb{Z})$, which was computed by Bökstedt in [13]:

$$\mathrm{THH}_k(H\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{if } k \geq 2 \text{ is even} \\ \mathbb{Z}/n & \text{if } k = 2n - 1 \geq 2. \end{cases} \tag{5.2.5}$$

Let μ_n be a generator of the $\mathbb{Z}/n$ in degree $2n - 1$.

The spectral sequence then begin with:

$$E^2_{p,q} = \mathrm{THH}_p(H\mathbb{Z}) \otimes E(\sigma u)_q. \tag{5.2.6}$$

For bidegree reason, the only possible non-zero differentials are the d^4 between μ_{n+2} and $\sigma u \mu_n$.

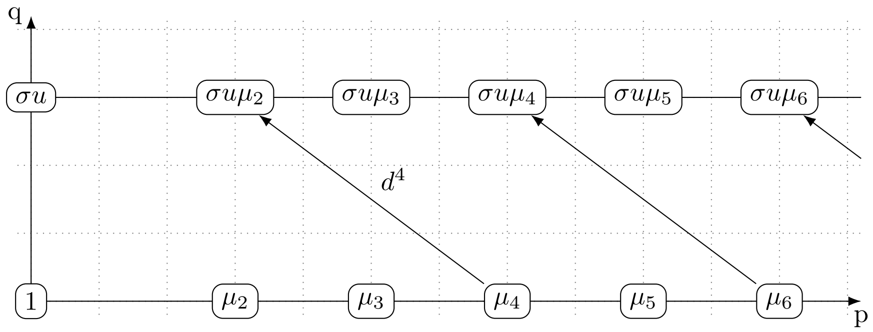


Figure 5.1: E^4 page of the Brun spectral sequence for $\mathrm{THH}_*(ku, H\mathbb{Z})$.

Proposition 5.2.7. *Let $n \geq 2$. When n is odd, $d^4(\mu_{n+2}) = 0$. When n is even, $d^4(\mu_{n+2})$ can only be 0 or $\frac{n}{2}\sigma u \mu_n$ up to a unit.*

Proof. μ_{n+2} must be sent to an element of order dividing $n + 2$ in the copy of $\mathbb{Z}/n$ generated by $\sigma u \mu_n$. But two consecutive odd integers are coprime, so that $d^4(\mu_{n+2}) = 0$ when n is odd. The greatest common divisor of two consecutive even integers is 2, so that $d^4(\mu_{n+2}) = 0$ or $d^4(\mu_{n+2}) = \frac{n}{2}\sigma u \mu_n$ when n is even. □

We will see that that even differentials are indeed all non-zero by computing the modulo 2 homotopy of $\mathrm{THH}_*(ku; H\mathbb{Z})$. Let $V(0)$ be the Moore spectrum for multiplication by 2. There is a Brun spectral sequence:

$$E^2_{p,q} = \mathrm{THH}_p(H\mathbb{Z}; H(V(0)_q(H\mathbb{Z} \wedge_{ku} H\mathbb{Z}))) \Rightarrow V(0)_{p+q}(\mathrm{THH}(ku; H\mathbb{Z})). \tag{5.2.8}$$

The Künneth spectral sequence, as in the integral case, can be used to compute:

$$V(0)_*(H\mathbb{Z} \wedge_{ku} H\mathbb{Z}) = E(\sigma u) \tag{5.2.9}$$

an exterior algebra over $\mathbb{F}_2$ on one generator σu of degree 3. We need to know $\mathrm{THH}_*(H\mathbb{Z}; \mathbb{F}_2) \cong V(0)_* \mathrm{THH}(H\mathbb{Z})$:

Proposition 5.2.10.

$$V(0)_* \mathrm{THH}(H\mathbb{Z}) \cong E(\lambda_1) \otimes P(\mu_1) \tag{5.2.11}$$

over $\mathbb{F}_2$, where $|\lambda_1| = 3$ and $|\mu_1| = 4$.

Proof. We can use the same method as theorem 5.7 of [5] for p=2. $V(0) \wedge \mathrm{THH}(H\mathbb{Z})$ is an $H\mathbb{F}_2$-module, thus the Hurewicz homomorphism

$$V(0)_* \mathrm{THH}(H\mathbb{Z}) \to H_*(V(0) \wedge \mathrm{THH}(H\mathbb{Z}); \mathbb{F}_2) \tag{5.2.12}$$

is an injection with image the A_*-comodule primitives, A_* being the dual Steenrod algebra. From theorem 5.12 of [3],

$$H_*(\mathrm{THH}(H\mathbb{Z}); \mathbb{F}_2) \cong H_*(H\mathbb{Z}; \mathbb{F}_2) \otimes E(\sigma\bar{\xi}_1^2) \otimes P(\sigma\bar{\xi}_2) \tag{5.2.13}$$

and then

$$H_*(V(0) \wedge \mathrm{THH}(H\mathbb{Z}); \mathbb{F}_2) \cong A_* \otimes E(\sigma\bar{\xi}_1^2) \otimes P(\sigma\bar{\xi}_2). \tag{5.2.14}$$

The primitives classes can be seen to be generated by $\sigma\bar{\xi}_1^2$ and $\sigma\bar{\xi}_2 - \xi_1\sigma\bar{\xi}_1^2$, and we denote their preimages respectively λ_1 and μ_1. □

We also evaluate $V(0)_* \mathrm{THH}(ku; H\mathbb{Z})$ non-multiplicatively using the spectral sequence from proposition 4.3.12:

$$\mathrm{Tor}_{*,*}^{H_*(ku;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \mathrm{THH}_*(ku; H\mathbb{F}_2). \tag{5.2.15}$$

We know $H_*(ku; \mathbb{F}_2)$, see for example proposition 5.3 of [3]:

$$H_*(ku; \mathbb{F}_2) \cong P(\bar{\xi}_1^2, \bar{\xi}_2^2, \bar{\xi}_k, k \geq 3) \tag{5.2.16}$$

thus

$$\mathrm{Tor}_{*,*}^{H_*(ku;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong E(\sigma\bar{\xi}_1^2, \sigma\bar{\xi}_2^2, \sigma\bar{\xi}_k, k \geq 3) \tag{5.2.17}$$

with bidegrees $|\sigma\bar{\xi}_1^2| = (1, 2)$, $|\sigma\bar{\xi}_2^2| = (1, 6)$ and $|\sigma\bar{\xi}_k| = (1, 2^k - 1)$. The spectral sequence collapses for bidegree reasons, and we now know the order of $V(0)_* \mathrm{THH}(ku; H\mathbb{Z})$ in each degree. In particular, $V(0)_3 \mathrm{THH}(ku; H\mathbb{Z}) \cong \mathbb{F}_2$.

Proposition 5.2.18. *In the Brun spectral sequence*

$$E(\sigma u, \lambda_1) \otimes P(\mu_1) \Rightarrow V(0)_* \mathrm{THH}(ku; H\mathbb{Z}) \tag{5.2.19}$$

the differentials are given by

$$\begin{aligned} &d^4(\mu_1^k) = \sigma u \mu_1^{k-1} \textit{ for } k \geq 1 \textit{ odd} \\ &d^4(\mu_1^k) = 0 \textit{ for } k \geq 0 \textit{ even} \\ &d^4(\lambda_1) = 0 \end{aligned} \tag{5.2.20}$$

up to multiplication by units.

Proof. λ_1 of bidegree $(3, 0)$ is an infinite cycle. It also cannot be in the image of a differential. The other generator σu of bidegree $(0, 3)$ must then vanish for $V(0)_3 \mathrm{THH}(ku; H\mathbb{Z})$ to be of dimension 1 over $\mathbb{F}_2$. Since it is also an infinite cycle, it must be the target of a differential, which can only be $d^4(\mu_1) = \sigma u$ up to a unit. The rest of the result is obtained multiplicatively. □

This is sufficient to compute the integral case.

Proposition 5.2.21. *In the Brun spectral sequence*

$$\mathrm{THH}_*(H\mathbb{Z}) \otimes E(\sigma u) \Rightarrow \mathrm{THH}_*(ku; H\mathbb{Z}) \tag{5.2.22}$$

the differentials are zero except for

$$d^4(\mu_{n+2}) = \frac{n}{2}\sigma u \mu_n \tag{5.2.23}$$

up to a unit when $n \geq 2$ is even. There is one non-trivial extension given by

$$2\mu_2 = \sigma u. \tag{5.2.24}$$

Proof. By inspecting the long exact sequence in homotopy given by the cofiber sequence

$$\mathrm{THH}(H\mathbb{Z}) \xrightarrow{\times 2} \mathrm{THH}(H\mathbb{Z}) \longrightarrow V(0) \wedge \mathrm{THH}(H\mathbb{Z}) \tag{5.2.25}$$

we can choose generator such that

$$\begin{aligned} f : \mathrm{THH}_*(H\mathbb{Z}) &\longrightarrow V(0)_* \mathrm{THH}(H\mathbb{Z}) \\ \mu_{2k} &\longmapsto \lambda_1 \mu_1^{k-1} \\ \sigma u &\longmapsto \sigma u \end{aligned} \tag{5.2.26}$$

and

$$\begin{aligned} g : V(0)_* \mathrm{THH}(H\mathbb{Z}) &\longrightarrow \mathrm{THH}_{*-1}(H\mathbb{Z}) \\ \mu_1^k &\longmapsto k\mu_{2k} \\ \sigma u \mu_1^k &\longmapsto k\sigma u \mu_{2k}. \end{aligned} \tag{5.2.27}$$

This also yield morphisms between the integral and mod 2 Brun spectral sequences. Depending on the parity of k, we use one of these maps to conclude. If k is even, then

$$f(d^4(\mu_{2k})) = d^4(\lambda_1 \mu_1^{k-1}) = \sigma u \lambda_1 \mu_1^{k-2} \tag{5.2.28}$$

so that

$$d^4(\mu_{2k}) = \sigma u \mu_{2k-2} \tag{5.2.29}$$

up to a unit, and $k-1$ is a unit. When k is odd,

$$d^4(\mu_{2k}) = g(d^4(\mu_1^k)) = g(\sigma u \mu_1^{k-1}) = (k-1)\sigma u \mu_{2k-2}. \tag{5.2.30}$$

These are the formulas we claimed.

Since σu is a boundary in the mod 2 spectral sequence, between the E^∞ pages we have $f(\sigma u) = 0$. Moreover, since σu is in the lowest filtration possible, this cannot be because of a shift of filtration, so that σu must be divisible by 2 in $\mathrm{THH}_3(ku; H\mathbb{Z})$. The only possibility is the extension claimed. There can be no other extensions for degree reasons. □

The result of this section can now be stated from the previous description of the spectral sequence.

Proposition 5.2.31.

$$\begin{aligned}\mathrm{THH}_*(ku; H\mathbb{Z}) \cong& \mathbb{Z}\{1,\, \mu_2\} \oplus \bigoplus_{n\geq 3\ odd} \mathbb{Z}/n\{\mu_n,\, \sigma u \mu_n\} \oplus \\ & \bigoplus_{n\geq 4\ even} \mathbb{Z}/(n/2)\{2\mu_n,\, \sigma u\mu_n\}\end{aligned} \tag{5.2.32}$$

where $|\mu_n| = 2n-1$ and the name of the generators have been chosen to reflect the multiplicative relations between them, with $\sigma u = 2\mu_2$.

5.3 Topological Hochschild homology of ℓ

In this section, we will review the results of [2] on $\mathrm{THH}_*(\ell)$. The results, relative to any prime p, will be stated about the following spectral sequences:

$$\begin{aligned}\mathrm{THH}_*(H\mathbb{Z}_{(p)}; H(H\mathbb{Z}_{(p)} \wedge_\ell H\mathbb{Z}_{(p)})_*) \cong \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \bar{\otimes} E(\sigma v_1) \\ \Rightarrow \mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})\end{aligned} \tag{$\ell_{\mathbb{Z}}$}$$

$$\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)}) \bar{\otimes} P(v_1) \Rightarrow \mathrm{THH}_*(\ell). \tag{ℓ}$$

The spectral sequence ($\ell_{\mathbb{Z}}$) is a Brun spectral sequence; (ℓ) is a Bockstein spectral sequence. We chose to name the written pages as the E^1 pages, so that the differentials have bidegrees $|d^r| = (-r-1, r)$. We have the bidegrees

$$\begin{aligned}&|\mu_{kp}| = (2kp-1, 0),\ k \geq 1 \text{ the generators of } \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \\ &|\sigma v_1| = (0, 2p-1) \\ &|v_1| = (0, 2(p-1)).\end{aligned} \tag{5.3.1}$$

When we will deem it necessary, for formulas in some discrete $\mathcal{R}$-algebra $\mathcal{A}$, we will use $x \cdot y$ for the $\mathcal{R}$-action of $x \in \mathcal{R}$ on $y \in \mathcal{A}$, and xy for the product of x, $y \in \mathcal{A}$. From [2], proposition 3.4, which compute $\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})$ we can deduce:

Proposition 5.3.2. *All the differentials in ($\ell_{\mathbb{Z}}$) are given by the formulas:*

$$d^{2p-1}(\mu_{(k+1)p}) = p^{\nu(k)} \cdot \sigma v_1 \mu_{kp} \tag{5.3.3}$$

up to a unit where $k \geq 1$ and ν is the p-adic valuation.

There is an extension given by $p\mu_p = \sigma v_1$.

(ℓ) is also computed in [2]. We will use the following notations:

$$\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}\{1,\, \mu_p\} \oplus \bigoplus_{k\geq 2} \mathbb{Z}/p^{\nu(k)}\{v_0\mu_{kp},\, \sigma v_1 \mu_{kp}\}. \tag{5.3.4}$$

Here from the Brun spectral sequence ($\ell_{\mathbb{Z}}$) we have $\sigma v_1 = p \cdot \mu_p$ and $v_0\mu_{kp}$ is a class represented by $p \cdot \mu_{kp}$. As in [2], we differentiate between the multiplication by p in the previous spectral sequence ($\ell_{\mathbb{Z}}$), denoted by v_0, and multiplication by p in the current spectral sequence (ℓ), denoted by p. This is the same distinction we made in chapter 2 between q and q_0: p denote the multiplication by $p \in \mathbb{Z}$ in a $\mathbb{Z}$-module, and v_0 will be used to name classes that are lifts of classes in the image of the multiplication by p.

Theorem 5.3.5 (Theorem 6.4 of [2]). *The differentials in* (ℓ) *are given by the formula:*

$$d^{p^{n+1}+\cdots+p}(p^n \cdot v_0\mu_{(k+1)p^{n+1}}) = kv_1^{p^{n+1}+\cdots+p}\sigma v_1\mu_{kp^{n+1}},\ k \geq 0,\ n \geq 0 \quad (5.3.6)$$

up to a unit and linearity with respect to multiplication by v_1.

There are extensions at the end of this spectral sequence. We now state the result with our notations:

Theorem 5.3.7 (different results in sections 6.2 and 6.3 of [2]). $\mathrm{THH}_*(\ell)$ *is a quotient of the* $\mathbb{Z}_{(p)}$*-module*

$$P(v_1) \otimes \Big(\mathbb{Z}_{(p)}\{1,\, \sigma v_1,\, v_0^n\mu_{p^{n+1}},\, n \geq 0\} \oplus P(v_0) \otimes \mathbb{Z}_{(p)}\{\sigma v_1\mu_{ap^n},\, n \geq 2,\, a \geq 1,\, a \text{ not divisible by } p\}\Big) \quad (5.3.8)$$

by the relations:

- $p \cdot \mu_p = \sigma v_1$.
- $p \cdot v_0^n\mu_{p^{n+1}} = v_1^{p^n} v_0^{n-1}\mu_{p^n}$ *for any* $n \geq 1$.
- $v_1^{p^{n-h-1}+p^{n-h-2}+\cdots+p} \cdot v_0^h\sigma v_1\mu_{ap^n} = 0$ *for any* h, a *and* n, a *not divisible by* p.
- $v_0^{n-1}\sigma v_1\mu_{ap^n} = 0$ *for any* a *and* n, a *not divisible by* p.
- $p \cdot \sigma v_1\mu_{(bp+p-1)p^n} = v_0\sigma v_1\mu_{(bp+p-1)p^n} + v_1^{p^n+p^{n-1}+\cdots+p}v_0^{\nu(b)}\sigma v_1\mu_{bp^{n+1}}$ *for any* $b \geq 1$ *not divisible by* p *and any* n.
- $p \cdot v_0^h\sigma v_1\mu_{ap^n} = v_0^{h+1}\sigma v_1\mu_{ap^n}$ *for any* a, n *and* $h \geq 1$ *or* $h = 0$ *not in the previous case.*

5.4 Review of the modulo p results

From here, and for the next sections of this chapter related to $\mathrm{THH}(ku)$, p is an odd prime. Remark that for $p = 2$, $ku = \ell$ so that the results of the previous section about $\mathrm{THH}(\ell)$ are results about $\mathrm{THH}(ku)$. The following Bockstein spectral sequence relative to the modulo p reduction of $\mathrm{THH}_*(ku)$ are known.

$$\begin{aligned} V(0)_*\,\mathrm{THH}(ku; H\mathbb{Z}_{(p)})\,\bar{\otimes}\,P(u) &\cong E(\sigma u, \lambda_1) \otimes P(\mu_1)\,\bar{\otimes}\,P(u) \\ &\Rightarrow V(0)_*\,\mathrm{THH}(ku) \qquad (u_V) \\ V(0)_*\,\mathrm{THH}(ku; H\mathbb{Z}_{(p)})\,\bar{\otimes}\,P_{p-1}(u) &\cong E(\sigma u, \lambda_1) \otimes P(\mu_1)\,\bar{\otimes}\,P_{p-1}(u) \\ &\Rightarrow V(0)_*\,\mathrm{THH}(ku; ku/v_1). \qquad (u_{VT}) \end{aligned}$$

Once again, these are the E^1 pages, with $|d^r| = (-r-1, r)$, and:

$$\begin{aligned} |\sigma u| &= (3, 0) \\ |\lambda_1| &= (2p-1, 0) \\ |\mu_1| &= (2p, 0) \\ |u| &= (0, 2). \end{aligned} \quad (5.4.1)$$

From theorem 1.2.11, we know that (u_V) determine (u_{VT}) entirely: the differentials in (u_{VT}) are those of (u_V) that are small enough to fit.

The spectral sequence (u_V) is known from [5]. We will now describe it.

For $n \in \mathbb{Z}$, let $a(n)$ and $b(n)$ be the integers

$$a(n) = \begin{cases} 0 & \text{if } n \le -1 \\ p^{n+1} - p^n + p^{n-1} - ... + p^2 - p & \text{if } n \ge 1 \text{ is odd} \\ p^{n+1} - p^n + p^{n-1} - ... + p^3 - p^2 + p - 2 & \text{if } n \ge 0 \text{ is even.} \end{cases} \tag{5.4.2}$$

$$b(n) = \begin{cases} 0 & \text{if } n \le 1 \\ p^{n-1} - p^{n-2} + ... + p^2 - p & \text{if } n \ge 3 \text{ is odd} \\ p^{n-1} - p^{n-2} + ... + p - 1 & \text{if } n \ge 2 \text{ is even.} \end{cases} \tag{5.4.3}$$

Proposition 5.4.4. *In (u_V), the differentials are determined by multiplicativity by the equations:*

$$d^{a(n)}(\mu_1^{p^n}) = \begin{cases} u^{a(n)} \sigma u \mu_1^{b(n)} & n \text{ even} \\ u^{a(n)} \lambda_1 \mu_1^{b(n)} & n \text{ odd.} \end{cases} \tag{5.4.5}$$

For $n \ge 1$, $a(n) \ge p - 1$ so that (u_{VT}) only see the first differentials with $n = 0$:

Proposition 5.4.6. *All the differentials in (u_{VT}) are given by the formulas:*

$$d^{p-2}(\lambda_1^\epsilon \mu_1^k) = k u^{p-2} \lambda_1^\epsilon \mu_1^{k-1} \tag{5.4.7}$$

where $\epsilon \in \{0, 1\}$ and $k \ge 1$.

5.5 Computation of $\mathrm{THH}_*(ku; ku/v_1)$

We will now compute $\mathrm{THH}_*(ku; ku/v_1)$ using both a Brun spectral sequence and a Bockstein spectral sequence. We first state results that allow us to compute the first page of the Brun spectral sequences.

Lemma 5.5.1. *(a) $(ku/v_1 \wedge_{ku} ku/v_1)_* \cong P_{p-1}(u) \otimes E(\sigma v_1)$ over $\mathbb{Z}_{(p)}$ with $|u| = 2$ and $|\sigma v_1| = 2p - 1$.*

(b) $V(0)_*(ku/v_1 \wedge_{ku} ku/v_1) \cong P_{p-1}(u) \otimes E(\sigma v_1)$ over $\mathbb{F}_p$ with $|u| = 2$ and $|\sigma v_1| = 2p - 1$.

(c) $\mathrm{THH}_*(ku/v_1; H\mathbb{Z}_{(p)}) \cong \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma u) \otimes \Gamma(\varphi u)$ over $\mathbb{Z}_{(p)}$ with $|\sigma u| = 3$ and $|\varphi u| = 2p$.

(d) $\mathrm{THH}_*(ku/v_1; H\mathbb{F}_p) \cong V(0)_* \mathrm{THH}(H\mathbb{Z}_{(p)}) \otimes E(\sigma u) \otimes \Gamma(\varphi u)$ over $\mathbb{F}_p$ with $|\sigma u| = 3$ and $|\varphi u| = 2p$.

Proof. The Künneth spectral sequence computing $(ku/v_1 \wedge_{ku} ku/v_1)_*$ has E^2 page $\mathrm{Tor}^{P(u)}_{*,*}(P_{p-1}(u), P_{p-1}(u)) = P_{p-1}(u) \otimes E(\sigma v_1)$ with $|u| = (0, 2)$ and $|\sigma v_1| = (1, 2p - 2)$. For degree reasons, the spectral sequence collapse with no possible extensions, yielding the result. The $V(0)$ result follows from the absence of p-torsion.

We use the Brun spectral sequence to compute $\mathrm{THH}_*(ku/v_1; H\mathbb{Z}_{(p)})$ and $\mathrm{THH}_*(ku/v_1; H\mathbb{F}_p)$:

$$\mathrm{THH}_*(H\mathbb{Z}_{(p)}; H(H\mathbb{Z}_{(p)} \wedge_{ku/v_1} H\mathbb{Z}_{(p)})_*) \Rightarrow \mathrm{THH}_*(ku/v_1; H\mathbb{Z}_{(p)}) \tag{5.5.2}$$

$$\mathrm{THH}_*(H\mathbb{F}_p; H(H\mathbb{F}_p \wedge_{ku/v_1} H\mathbb{F}_p)_*) \Rightarrow \mathrm{THH}_*(ku/v_1; H\mathbb{F}_p). \tag{5.5.3}$$

The Künneth spectral sequence computing $(H\mathbb{Z}_{(p)} \wedge_{ku/v_1} H\mathbb{Z}_{(p)})_*$ has E^2 page $Tor_{*,*}^{P_{p-1}(u)}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \cong E(\sigma u) \otimes \Gamma(\varphi u)$ with $|\sigma u| = (1, 2)$ and $|\varphi u| = (2, 2p-2)$. The indecomposables are σu and the divided power $\gamma_{p^i}\varphi u$. For degree reasons, they cannot support non-zero differentials, so the spectral sequence collapse with no possible extensions, and we have $(H\mathbb{Z}_{(p)} \wedge_{ku/v_1} H\mathbb{Z}_{(p)})_* \cong E(\sigma u) \otimes \Gamma(\varphi u)$. A similar argument yields $(H\mathbb{F}_p \wedge_{ku/v_1} H\mathbb{F}_p)_* \cong E(\sigma u) \otimes \Gamma(\varphi u)$, this time over $\mathbb{F}_p$.

Getting back to the Brun spectral sequences, when looking at the degrees modulo $2p$, we see that the indecomposables also cannot support non-zero differentials in both the integral and $V(0)$ case, so that the two spectral sequences collapse. The modulo p E^∞ page has exactly the right rank over $\mathbb{F}_p$ to fit into a long exact sequence of the multiplication by p for the integral E^∞ page. Having an extension in the integral spectral sequence would then mean that there is a non-zero differential in the modulo p one. We conclude that there is no extension in the integral spectral sequence, and there can also be none in the modulo p one. $\square$

Thus, we can write the following two spectral sequences computing THH of ku with coefficients in ku/v_1:

$$\begin{aligned} \mathrm{THH}_*(ku; H\mathbb{Z}_{(p)}) \bar{\otimes} P_{p-1}(u) &\cong \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma u) \bar{\otimes} P_{p-1}(u) \\ &\Rightarrow \mathrm{THH}_*(ku; ku/v_1) && (u_T) \\ \mathrm{THH}_*(ku/v_1; H(ku/v_1 \wedge_{ku} ku/v_1)_*) & \\ \cong \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma u) \otimes \Gamma(\varphi u) &\bar{\otimes} E(\sigma v_1) \otimes P_{p-1}(u) \\ &\Rightarrow \mathrm{THH}_*(ku; ku/v_1). && (u_{TB}) \end{aligned} \tag{5.5.4}$$

Here (u_T) is a truncated Bockstein spectral sequence, and (u_{TB}) is a Brun spectral sequence, and

$$\begin{aligned} &|\sigma u| = (3, 0) \\ &|\varphi u| = (2p, 0) \\ &|\mu_{kp}| = (2kp - 1, 0),\ k \geq 1 \text{ the generators of } \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \\ &|u| = (0, 2) \\ &|\sigma v_1| = (0, 2p - 1). \end{aligned} \tag{5.5.5}$$

For the following lemma, we will briefly use the non-truncated u-Bockstein spectral sequence (u) computing $\mathrm{THH}_*(ku)$ that we will study in the next section. It links the class σv_1 of $\mathrm{THH}_*(\ell)$ to a class of $\mathrm{THH}_*(ku)$. Another incomplete point of view on this result can be found in section 5.8.

Lemma 5.5.6. *The map $\mathrm{THH}_*(\ell) \to \mathrm{THH}_*(ku)$ sends σv_1 to a non-zero class represented up to a unit by $u^{p-2}\sigma u$ in the Bockstein spectral sequence computing $\mathrm{THH}_*(ku)$.*

Proof. Since L is the (smashing) localization of ℓ at the Johnson-Wilson spectrum $E(1)$, we can conclude from proposition 4.4.3 that there is a weak equivalence

$$\mathrm{THH}(L) \simeq \mathrm{THH}(\ell; L). \tag{5.5.7}$$

Similarly, there is a weak equivalence

$$\mathrm{THH}(KU) \simeq \mathrm{THH}(ku; KU). \tag{5.5.8}$$

$\mathrm{THH}(\ell; L)$ can be computed using a periodic Bockstein spectral sequence

$$\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)}) \bar{\otimes} P(v_1, v_1^{-1}) \Rightarrow \mathrm{THH}_*(\ell; L) \tag{L}$$

which is entirely determined by the map $(\ell) \to (L)$. In particular, we can see that σv_1 is a generator over $\mathbb{Q}$ and $P_{\mathbb{Q}}(v_1, v_1^{-1})$ of the summand $\Sigma L_{\mathbb{Q}}$ in the splitting

$$\mathrm{THH}(L)_p \simeq (L \vee \Sigma L_{\mathbb{Q}})_p. \tag{5.5.9}$$

Since the splitting on $\mathrm{THH}(L)$ and $\mathrm{THH}(KU)$ are compatible, it must be that $\sigma v_1 \in \mathrm{THH}_{2p-1}(\ell)$ is sent to a non-zero class in $\mathrm{THH}_{2p-1}(ku)$. There is also a relation $p\mu_p = \sigma v_1$, so that the only possibility is that the image of σv_1 in $\mathrm{THH}_{2p-1}(ku)$ is represented by $u^{p-2}\sigma$ to get both the extension with $p\mu_p$ and the splitting of $\mathrm{THH}(KU)$. □

Since ℓ/v_1 is just $H\mathbb{Z}_{(p)}$, we have a morphism between the Brun spectral sequences $(\ell_{\mathbb{Z}}) \to (u_{TB})$ induced by $i : \ell \to ku$. This allows us to prove:

Proposition 5.5.10. *In (u_{TB}), there are differentials*

$$d^{2p-4}(\gamma_k \varphi u) = u^{p-2}\sigma u \gamma_{k-1}\varphi u \tag{5.5.11}$$

up to a unit for all $k \geq 1$.

Proof. In the following commutative diagram:

$$\begin{tikzcd} \mathrm{THH}(\ell) \arrow[r, "i"] \arrow[d, "f"] & \mathrm{THH}(ku) \arrow[d, "f"] \\ \mathrm{THH}(\ell; H\mathbb{Z}_{(p)}) \arrow[r, "i"] & \mathrm{THH}(ku; ku/v_1) \end{tikzcd} \tag{5.5.12}$$

we have up to units, using lemma 5.5.6

$$f(i(\sigma v_1)) = f(u^{p-2}\sigma u) = u^{p-2} f(\sigma u) = i(f(\sigma v_1)) = i(\sigma v_1). \tag{5.5.13}$$

In order for this to be possible, there must be an extension $u \cdot u^{p-3}\sigma u = \sigma v_1$ in (u_{TB}), and it must be that $u^{p-2}\sigma u$ is either a boundary or not an infinite cycle. Since it is an infinite cycle for degree reasons, it is a boundary. The only class in degree $2p$ is φu, so up to a unit there is a differential $d^{2p-4}(\varphi u) = u^{p-2}\sigma u$ in (u_{TB}).

In the divided power algebra $\Gamma(\varphi u)$, $\varphi u\, \gamma_{k-1}\varphi u = k\, \gamma_k \varphi u$. We can then prove our formula by induction on k, using the facts that

$$k\, d(\gamma_k \varphi u) = d(\varphi u)\gamma_{k-1}\varphi u + \varphi u\, d(\gamma_{k-1}\varphi u) \tag{5.5.14}$$

and that $\mathbb{Z}_{(p)}$ is an integral domain. □

We can now get a description of all the differentials in the truncated Bockstein spectral sequence (u_T):

Proposition 5.5.15. *In the spectral sequence (u_T), the differentials are given by the formula:*

$$d^{2p-4}(\mu_{(k+1)p}) = p^{\nu(k)} u^{p-2} \sigma u \, \mu_{kp}, \quad k \geq 1 \tag{5.5.16}$$

up to a unit, where ν is the p-adic valuation.

Proof. The differentials given are the only possible in (u_T) for degree reasons; we only need to prove that they are indeed non-zero. We now know enough about (u_{TB}) to do so.

By looking at the degrees modulo $2p$, we can list the classes in E^1_{2kp-1} of (u_{TB}):

$$\mu_{kp}, \quad \gamma_{k-1}\varphi u \, \sigma v_1, \quad \gamma_{k-1}\varphi u \, u^{p-2}\sigma u, \quad \gamma_i \varphi u \, \mu_{(k-i)p}, \; 1 \leq i < k. \tag{5.5.17}$$

We know the following differentials in (u_{TB}):

$$d^{2p-4}(\gamma_i \varphi u) = u^{p-2} \sigma u \gamma_{i-1} \varphi u \tag{5.5.18}$$

for $i \geq 1$ from proposition 5.5.10;

$$d^{2p-1}(\mu_{(i+1)p}) = p^{\nu(i)} \sigma v_1 \mu_{ip} \tag{5.5.19}$$

from the map $(\ell_{\mathbb{Z}}) \to (u_{TB})$ and proposition 5.3.2;

To complete the multiplicative description, we also note that $d(\sigma v_1) = 0$ and that all the degreewise possible value for $d^{2p-4}(\mu_{(k-i)p})$ results in a non-zero $d^{2p-4}(\gamma_i \varphi u \, \mu_{(k-i)p})$.

From this description, after d^{2p-1} the only generator left in E^{2p}_{2kp-1} is $p\mu_{kp}$, so that $\mathrm{THH}_{2kp-1}(ku; ku/v_1)$ is isomorphic to $\mathbb{Z}/p^{\nu(k)}\mathbb{Z}$. This proves our claim about (u_T). □

We will now describe $\mathrm{THH}_*(ku; ku/v_1)$, and as in [2], we will use v_0 to denote multiplication by p in the spectral sequence (u_T), as opposed to p denoting the multiplication in the target group.

Proposition 5.5.20. $\mathrm{THH}_*(ku; ku/v_1)$ *is generated as a $\mathbb{Z}_{(p)}[u]/(u^{p-1})$-module by*

$$\begin{gathered} 1, \, \sigma u, \, \mu_p \\ v_0 \mu_{kp}, \, u\mu_{kp}, \, k \geq 2 \\ \sigma u \mu_{kp}, \, k \geq 1 \end{gathered} \tag{5.5.21}$$

with the relations:

$$\begin{aligned} u^{p-2} \cdot \sigma u &= p \cdot \mu_p \\ u \cdot v_0 \mu_{kp} &= p \cdot u\mu_{kp}, \, k \geq 2 \\ p^{\nu(k)+1} \cdot u\mu_{kp} &= 0, \, k \geq 2 \\ u^{p-3} \cdot u\mu_{kp} &= 0, \, k \geq 2 \\ p^{\nu(k)+1} \cdot \sigma u \mu_{kp} &= 0, \, k \geq 2 \\ p^{\nu(k)} \cdot v_0 \mu_{kp} &= 0, \, k \geq 2 \\ p^{\nu(k)} u^{p-2} \cdot \sigma u \mu_{kp} &= 0, \, k \geq 2. \end{aligned} \tag{5.5.22}$$

Proof. Except for the extension $p \cdot \mu_p = u^{p-2}\sigma u$ this is the E^∞ page of (u_T). This extension is present in $(\ell_{\mathbb{Z}})$, and since the map $i : \mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)}) \to \mathrm{THH}_*(ku; ku/v_1)$ is such that $i(\sigma v_1) = u^{p-2}\sigma u$, and on the E^∞ pages is such that $i(\mu_p) = \mu_p$, it must be that $u^{p-2}\sigma u$ is also divisible by p in $\mathrm{THH}_{2p-1}(ku; ku/v_1)$. The only possible extension is with μ_p, so we get our formula up to a unit. □

Without the module structure, writing all the classes, this is:

$$\begin{aligned}
&\mathbb{Z}_{(p)}\{1,\, u,\, \dots,\, u^{p-2},\, \sigma u,\, u\sigma u,\, \dots,\, u^{p-2}\sigma u,\, \mu_p\}\\
&\oplus \bigoplus_{k\geq 1} \mathbb{Z}/p^{\nu(k)+1}\{u\mu_{kp},\, u^2\mu_{kp},\, \dots,\, u^{p-2}\mu_{kp}\}\\
&\oplus \bigoplus_{k\geq 1} \mathbb{Z}/p^{\nu(k)+1}\{\sigma u\mu_{kp},\, u\mu_{kp},\, \dots,\, u^{p-3}\mu_{kp}\}\\
&\oplus \bigoplus_{k\geq 2} \mathbb{Z}/p^{\nu(k)}\{v_0\mu_{kp},\, u^{p-2}\sigma u\mu_{kp}\}
\end{aligned} \tag{5.5.23}$$

with relations $u^{p-2}\sigma u = p \cdot \mu_p$ and $u \cdot v_0\mu_{kp} = p \cdot u\mu_{kp}$.

5.6 Computation of the Bockstein spectral sequence for $\mathrm{THH}_*(ku)$

We know enough of these first three spectral sequences to compute the fourth:

$$\begin{aligned}
&\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P(v_1) && \Rightarrow \mathrm{THH}_*(\ell) && (\ell)\\
&\mathrm{THH}_*(ku; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P_{p-1}(u) && \Rightarrow \mathrm{THH}_*(ku; ku/v_1) && (u_T)\\
&\mathrm{THH}(ku; ku/v_1) \,\bar{\otimes}\, P(v_1) && \Rightarrow \mathrm{THH}_*(ku) && (v_1)\\
&\mathrm{THH}_*(ku; H\mathbb{Z}_{(p)}) \,\bar{\otimes}\, P(u) && \Rightarrow \mathrm{THH}_*(ku). && (u)
\end{aligned}$$

From the map $\mathrm{THH}(\ell; H\mathbb{Z}_{(p)}) \to \mathrm{THH}(k; ku/v_1)$ comes a morphism of spectral sequences $(\ell) \to (v_1)$, which determines some differentials in (v_1). These differentials, the one computed in the previous section in (u_T) and the lemmas relating a spectral sequence and its truncations yield a description of the differentials in (u).

Theorem 5.6.1. *The differentials in (u) are given by the formula:*

$$d^{p^{n+1}-2}(p^n\mu_{(k+1)p^{n+1}}) = ku^{p^{n+1}-2}\sigma u\mu_{kp^{n+1}},\ k \geq 0,\ n \geq 0 \tag{5.6.2}$$

up to a unit and linearity with respect to multiplication by u.

Proof. Here we make good use of our results on truncated spectral sequences.

First, the differentials in (u_T) from proposition 5.5.15 are lifted to (u) using theorem 1.2.11, that is in (u) there are differentials:

$$d^{2p-4}(\mu_{(k+1)p}) = p^{\nu(k)}u^{p-2}\sigma u\,\mu_{kp}, \quad k \geq 1 \tag{5.6.3}$$

repeated for each power of u. These are the only differentials d^r with $2 \leq r \leq 2p-4$ in (u) since these are the only differentials in (u_T), again using theorem 1.2.11.

We will now use theorem 1.2.21 and theorem 1.2.28. With regard to theorem 1.2.28, it is important to see that in our current computation, a statement stronger than the general case can be made. The general case would say that a differential $d(x) = y$ in (v_1) would result in the existence of an element x' such that $d(x') = y$ in (v_1), and such that this differential can be lifted to one in (u); but in (v_1), each generator is alone in its bidegree, so that necessarily $x = x'$. So each differential $d(x) = y$ in (v_1) can really be lifted to a differential $d(x) = y$ in (u).

Using theorem 1.2.21, the differentials of formula (5.6.3) results in (v_1) in

$$d^1(u^i\mu_{(k+1)p}) = p^{\nu(k)}v_1u^{i-1}\sigma u\,\mu_{kp}, \quad k \geq 1,\, 1 \leq i \leq p-2 \tag{5.6.4}$$

repeated for each power of v_1. These are the only differentials d^{2p-2} in (v_1) since having more differentials would result in more differentials d^r in (u) with $2 \leq r \leq 2p-4$. This gives the E^2 page of (v_1):

$$\begin{aligned}(v_1):\ E^2 \cong\, & \mathbb{Z}_{(p)}\{1,\, u,\, \ldots,\, u^{p-2},\, \sigma u,\, u\sigma u,\, \ldots,\, u^{p-3}\sigma u,\, \mu_p\} \otimes P(v_1)\\ & \oplus \bigoplus_{k\geq 2} \mathbb{Z}/p^{\nu(k)}\{v_0\mu_{kp}\} \otimes P_{p-1}(u) \otimes P(v_1)\\ & \oplus \bigoplus_{k\geq 1} \mathbb{Z}/p^{\nu(k)+1}\{\sigma u\mu_{kp},\, u\,\sigma u\mu_{kp},\, \ldots,\, u^{p-3}\sigma u\mu_{kp}\}\\ & \oplus \bigoplus_{k\geq 1} \mathbb{Z}/p^{\nu(k)}\{u^{p-2}\sigma u\mu_{kp},\, v_1\sigma u\mu_{kp},\, u\,v_1\sigma u\mu_{kp},\, \ldots\}.\end{aligned} \tag{5.6.5}$$

We have written all the generators $v_0\mu_{kp}$ with v_0 because we will now account for the differentials in (ℓ) of theorem 5.3.5:

$$d^{p^n+\cdots+p}(p^{n-1}\cdot v_0\mu_{kp^n}) = (k-1)v_1^{p^n+\cdots+p}\sigma v_1\mu_{(k-1)p^n},\ k \geq 1,\ n \geq 1. \tag{5.6.6}$$

That formula is also true in (v_1), and from theorem 1.2.28 we deduce the formula in (u) that was claimed (which also encompass the formula (5.6.3)).

It remains to prove that the classes $\sigma u\mu_{kp}$, $k \geq 1$ are infinite cycles in (u). The classes $u^{p-2}\sigma u\mu_{kp^2}$, $k \geq 1$ are in the image of $(\ell) \to (v_1)$ and so are infinite cycles in (v_1), thus also in (u) by theorem 1.2.35. Since in (u) the only u^{p-2}-torsion is in even degree, it must be that $\sigma u\mu_{kp^2}$ are infinite cycles in (u). The remaining classes to check are the $\sigma u\mu_{kp}$ with p not dividing k. Once again we now that these classes support no differentials of height up to u^{p-2}, and are of u^{p-2}-torsion after d^{p-2} by formula (5.6.3). If some $\sigma u\mu_{kp}$ supports a non-zero differential the target must be $p^{\nu(k-i)}u^{ip+1}\mu_{(k-i)p}$, $1 \leq i \leq k-1$ for degree reasons, and that target must be of u^{p-2}-torsion, that is to say some

$$\begin{gathered}p^{\nu(k-i)}u^{ip+2}\mu_{(k-i)p},\\ p^{\nu(k-i)}u^{ip+3}\mu_{(k-i)p},\\ \vdots\\ p^{\nu(k-i)}u^{ip+p-1}\mu_{(k-i)p}\end{gathered} \tag{5.6.7}$$

is already the target of a differential. But the only possible differentials still not accounted for are the one targeting $p^{\nu(k-i)}u^{ip+1}\mu_{(k-i)p}$, $1 \leq i \leq k-1$, and these are of height u^h with h reducing to 1 modulo p. □

We can change our generators so that the differentials are not given up to a unit but exactly.

Proposition 5.6.8. *We can change the generators μ_N and $\sigma u \mu_{N'}$ of $\mathrm{THH}_*(ku; H\mathbb{Z}_{(p)})$ with a multiplication by a unit so that the differentials in (u) are given by the formula:*

$$d^{p^{n+1}-2}(p^n \mu_{(k+1)p^{n+1}}) = p^{\nu(k)} u^{p^{n+1}-2} \sigma u \mu_{kp^{n+1}}, \ k \geq 0, \ n \geq 0 \tag{5.6.9}$$

Proof. Note that we have chosen $p^{\nu(k)}$ instead of k, but these are the same up to a unit. We could have written the same statement with k.

The differentials are making the μ_N and $\sigma u \mu_{N'}$ interact, and once we have chosen a specific unit for one of them, we have to use the same unit for all the $p^i \mu_N$ or $p^j \sigma u \mu_{N'}$. Consider the graph $\mathcal{G}$ whose vertices are the μ_N and $\sigma u \mu_{N'}$ and with an edge for each differential

$$d(p^i \mu_N) = p^j u^\bullet \sigma u \mu_{N'} \tag{5.6.10}$$

for any i and j, up to a unit, in the spectral sequence. The graph $\mathcal{G}$ is bipartite, since the differentials are always from a μ_N to a $\sigma u \mu_{N'}$. If we prove that $\mathcal{G}$ is a forest (as in a collection of trees), then we have proven our statement. Indeed, this is sufficient to choose a coherent set of μ_N and $\sigma u \mu_{N'}$, by choosing an arbitrary root for all the trees in the forest, and then changing each generation by a unit to verify the given formula.

We will reason on the p-adic valuation of N and N', denoted $\nu(N)$ and $\nu(N')$. There is an edge in $\mathcal{G}$ between μ_N and $\sigma u \mu_{N'}$ if and only if $N = (k+1)p^{n+1}$ and $N' = kp^{n+1}$ for some $k \geq 0$ and $n \geq 0$. In that case, $\nu(N) \geq \nu(N')$ if and only if $n+1 = \nu(N')$, so that for N' fixed, there is only one edge from $\sigma u \mu_{N'}$ to some μ_N that satisfy $\nu(N) \geq \nu(N')$. Moreover, $\nu(N') \geq \nu(N)$ if and only if $n+1 = \nu(N)$, so that for N fixed, there is only one edge from μ_N to some $\sigma u \mu_{N'}$ that satisfy $\nu(N') \geq \nu(N)$. Thus, if there is a cycle in $\mathcal{G}$, then it must be confined to vertices whose p-adic valuation are all equal. But any vertex can only have at most one edge going to another vertex of the same p-adic valuation, so that such cycles are impossible. $\square$

We can also state a result about the integral, non-local Bockstein spectral sequence

$$\mathrm{THH}_*(ku; H\mathbb{Z}) \Rightarrow \mathrm{THH}_*(ku). \tag{5.6.11}$$

We will use the following notations for the non-local classes:

$$\mathrm{THH}_*(ku; H\mathbb{Z}) \cong \mathbb{Z}\{1, \mu_2\} \oplus \bigoplus_{k \geq 3} \mathbb{Z}/f(k)\{2^{\epsilon(k)} \mu_k, \sigma u \mu_k\} \tag{5.6.12}$$

where

$$f(k) = \begin{cases} k & \text{when } k \text{ is odd} \\ k/2 & \text{when } k \text{ is even} \end{cases} \tag{5.6.13}$$

$$\epsilon(k) = \begin{cases} 0 & \text{when } k \text{ is odd} \\ 1 & \text{when } k \text{ is even} \end{cases} \tag{5.6.14}$$

and $2\mu_2 = \sigma u$.

In this section only – we later go back to p-local computation – let us write $\mu^{(p)}$ for the p-local generators. We can choose to lift them to the non-local case such that when k and p are coprime,

$$\mu^{(p)}_{kp^n} = k\mu_{kp^n}. \tag{5.6.15}$$

The p-local differentials can be rewritten as

$$d^{p^{n+1}-2}(p^n\mu^{(p)}_{kp^m}) = (kp^{m-n-1} - 1)u^{p^{n+1}-2}\sigma u\mu^{(p)}_{kp^m-p^{n+1}} \tag{5.6.16}$$

in the integral Bockstein spectral sequence, when k and p are coprime and $0 \leq n < m$.

Thus, we can conclude from all the maps $\mathrm{THH}_*(ku; H\mathbb{Z}) \to \mathrm{THH}_*(ku; H\mathbb{Z}_{(p)})$:

Proposition 5.6.17. *In the Bockstein spectral sequence computing not localized* $\mathrm{THH}_*(ku)$*, the differentials are given by:*

$$d^{p^{n+1}-2}(p^n k\mu_{kp^m}) = (kp^{m-n-1} - 1)u^{p^{n+1}-2}\sigma u\mu_{kp^m-p^{n+1}} \tag{5.6.18}$$

where p is any prime, $k \geq 1$, k and p coprime, $m \geq 1$, and $0 \leq n < m$. The formula is valid up to multiplication by a unit of $\mathbb{Z}/\frac{kp^m-p^{n+1}}{2}$.

5.7 Computing the extensions and a presentation of $\mathrm{THH}_*(ku)$

We first compute the extensions in the torsion-free part of the spectral sequence, from the knowledge that the p-torsion and the u-torsion must be the same in $\mathrm{THH}_*(ku)$.

Proposition 5.7.1. *The torsion-free part of* $\mathrm{THH}_*(ku)$ *is a quotient of*

$$P(u) \otimes \mathbb{Z}_{(p)}\{1, \sigma u, \mu_p, v_0\mu_{p^2}, v_0^2\mu_{p^3}, \dots\} \tag{5.7.2}$$

with relations

$$p \cdot \mu_p = u^{p-2}\sigma u \tag{5.7.3}$$

$$p \cdot v_0^n\mu_{p^{n+1}} = u^{p^{n+1}-p^n}v_0^{n-1}\mu_{p^n},\, n \geq 1. \tag{5.7.4}$$

Proof. From the differentials of theorem 5.6.1, the generators written are the only one not of u-torsion. We already know from lemma 5.1.3 that they must not be of p-torsion. We can give a second proof of this fact using the spectral sequences, by studying the connecting map of multiplication by p:

$$\delta : V(0)_*\,\mathrm{THH}(ku) \to \mathrm{THH}_{*-1}(ku) \tag{5.7.5}$$

which is a map of ku_*-modules.

Let $n \geq 0$. In order for $p \cdot v_0^n\mu_{p^{n+1}}$ to be zero, $v_0^n\mu_{p^{n+1}}$ needs to be in the image of δ, and since it is not divisible by u, it needs to be the image by δ of an element not divisible by u. Be there is no such element in $V(0)_{2p^{n+1}}\,\mathrm{THH}(ku)$: in the E^1 page of the Bockstein spectral sequence computing $V(0)_*\,\mathrm{THH}(ku)$, the only suitable element is $\mu_1^{p^n}$, but it is not an infinite cycle (see proposition 5.4.4). So $p \cdot v_0^n\mu_{p^{n+1}}$ is not zero.

We now have to prove that the extensions given are present. The u-tower over 1 is in even degree, so no extension are possible with the rest of the classes. We prove the rest of our formula by induction on n. Let us first observe that each extension must be with an element not already divisible by p; otherwise if $p \cdot v_0^n \mu_{p^{n+1}} = p \cdot u^k x$ for some $k \geq 1$ and x, then $v_0^n \mu_{p^{n+1}} - u^k x$ would be a non-divisible by u class in degree $2p^{n+1} - 1$ whose product with p is zero, which we already deemed impossible. Then $u^{p-2}\sigma u$ is the only choice (up to a unit) for $p \cdot \mu_p$. Let $n \geq 1$. If our formula holds up to rank $n-1$, then $p \cdot v_0^n \mu_{p^{n+1}}$ could degreewise be:

$$\begin{gathered} u^{p^{n+1}-2}\sigma u \\ u^{p^{n+1}-p}\mu_p \\ u^{p^{n+1}-p^2} v_0 \mu_{p^2} \\ \vdots \\ u^{p^{n+1}-p^n} v_0^{n-1} \mu_{p^n} \end{gathered} \tag{5.7.6}$$

but the only one not already divisible by p is $u^{p^{n+1}-p^n} v_0^{n-1} \mu_{p^n}$. □

Using the results of chapter 2, we can recover the torsion extensions.

Proposition 5.7.7. *The torsion ku_*-sub-module of* $\mathrm{THH}_*(ku)$ *is presented by the classes*

$$v_0^h \sigma u \mu_{ap^n} \tag{5.7.8}$$

in degree $2ap^n + 2$ where h, a and n are integers such that $h \geq 0$, $n \geq 1$, $a \geq 1$ and p does not divide a, together with the relations:

1) $u^{p^{n-h}-2} \cdot v_0^h \sigma u \mu_{ap^n} = 0$ *for any h, a and n.*
2) $v_0^n \sigma u \mu_{ap^n} = 0$ *for any a and n.*
3) $p \cdot \sigma u \mu_{(bp+p-1)p^n} = v_0 \sigma u \mu_{(bp+p-1)p^n} + u^{p^{n+1}-p^n} v_0^{\nu(b)} \sigma u \mu_{bp^{n+1}}$
   for any $b \geq 1$ not divisible by p and any n.
4) $p \cdot v_0^h \sigma u \mu_{ap^n} = v_0^{h+1} \sigma u \mu_{ap^n}$
   for any a, n and $h \geq 1$ or $h = 0$ not in the case of 3).

Proof. Here we will use the results of chapter 2 applied to the torsion elements of $\mathrm{THH}_*(ku)$. Hypothesis (sR1), which is in our case a statement about $V(0)_* \mathrm{THH}(ku; H\mathbb{Z}_{(p)})$ is easy to check from the modulo p results of section 5.4. Our lifts $q_0^k a_i$ of the E^∞-page will be the $v_0^{\nu(k)} \sigma u \mu_{kp^{n+1}}$ obtained from the differentials of proposition 5.6.8:

$$d^{p^{n+1}-2}(p^n \mu_{(k+1)p^{n+1}}) = p^{\nu(k)} k u^{p^{n+1}-2} \sigma u \mu_{kp^{n+1}}, \; k \geq 0, \; n \geq 0. \tag{5.7.9}$$

These differentials gives us the relations at the end of the u-towers

$$\begin{gathered} p \cdot u^{p^n-3} \sigma u \mu_{(bp+p-1)p^n} = u^{p^{n+1}-3} v_0^{\nu(b)} \sigma u \mu_{bp^{n+1}} \\ p \cdot u^{p^n-3} \sigma u \mu_{(bp+j)p^n} = 0 \text{ whenever } 0 < j < p-1 \\ p \cdot u^{p^{n-h}-3} v_0^h \sigma u \mu_{ap^n} = 0 \text{ whenever } h \geq 1. \end{gathered} \tag{5.7.10}$$

In the language of chapter 2, this is

$$\begin{aligned}\pi(\sigma u\mu_{(bp+p-1)p^n}) &= v_0^{\nu(b)}\sigma u\mu_{bp^{n+1}}\\ \pi(\sigma u\mu_{(bp+j)p^n}) &= 0 \text{ whenever } 0<j<p-1\\ \pi(v_0^h\sigma u\mu_{ap^n}) &= 0 \text{ whenever } h\geq 1.\end{aligned} \tag{5.7.11}$$

Let $n\geq 1$, $a\geq 1$ not divisible by p and $h\geq 0$. We will check (sT_i) for all classes $v_0^h\sigma u\mu_{ap^n}$, that is to say proving that the only other class that can appear in $p\cdot v_0^h\sigma u\mu_{ap^n}$ is $\pi(v_0^h\sigma u\mu_{ap^n})$, thus proving the formula. Hypothesis (sT_i) is proved by examining, for $m\geq 1$, $b\geq 1$ not divisible by p and $k\geq 0$, the classes $v_0^k\sigma u\mu_{bp^m}$ that can appear in the inequality

$$|v_0^k\sigma u\mu_{bp^m}|\leq|v_0^h\sigma u\mu_{ap^n}|<|u^{p^{m-k}-2}v_0^k\sigma u\mu_{bp^m}|\leq|u^{p^{n-h}-2}v_0^h\sigma u\mu_{ap^n}|. \tag{5.7.12}$$

This equation (5.7.12) is indeed equivalent to

$$\begin{aligned}&i)\ \sum_{i\geq 0}b_ip^{m+i}+1\leq\sum_{i\geq 0}a_ip^{n+i}+1\\ &ii)\ \sum_{i\geq 0}a_ip^{n+i}+1<\sum_{i\geq 0}b_ip^{m+i}+\sum_{i=0}^{m-k-1}(p-1)p^i\\ &iii)\ \sum_{i\geq 0}b_ip^{m+i}+\sum_{i=0}^{m-k-1}(p-1)p^i\leq\sum_{i\geq 0}a_ip^{n+i}+\sum_{i=0}^{n-h-1}(p-1)p^i.\end{aligned} \tag{5.7.13}$$

Here we have written $a=\sum_{i\geq 0}a_ip^i$ and $b=\sum_{i\geq 1}b_ip^i$ in base p.

If $m<n$, *i)* and *ii)* cannot hold together. If $m=n$, but then for *i)* and *ii)* to hold together, we must also have $a=b$. Then any k such that $h\leq k$ is suitable.

If $m>n$, then to have *i)* and *ii)*, bp^m must be a truncation of ap^n, and $m-k-1\geq n$. Then for *iii)* to hold, we must have $h=0$ and *iii)* is an equality, that is to say the firsts digits $a_0,\dots,a_\omega$ of a are $p-1$, and then we can have $n\leq m-k-1\leq n+\omega$, with $k=0$ except if $m-k-1=n+\omega$ and $a_{\omega+1}=0$. In that case, k must be the number of digits of a equal to zero after the position ω. For $n=m-1$, we get the class $v_0^k\sigma u\mu_{bp^m}=\pi(v_0^h\sigma u\mu_{ap^n})$, and for $m-1>n$ we get classes such that $|v_0^k\sigma u\mu_{bp^m}|<|\pi(v_0^h\sigma u\mu_{ap^n})|$ (remark that all these classes will be connected by a tower of extensions, one for every $p-1$ digits at the end of a). Thus, (sT_i) holds for all torsion classes. □

From the two previous results, we can give a presentation of $\mathrm{THH}_*(ku)$ as a ku_*-module.

Theorem 5.7.14. $\mathrm{THH}_*(ku)$ *is a quotient of the* $\mathbb{Z}_{(p)}[u]$*-module*

$$\begin{aligned}P(u)\otimes\Big(&\mathbb{Z}_{(p)}\{1,\,\sigma u,\,v_0^n\mu_{p^{n+1}},\,n\geq 0\}\\ &\oplus P(v_0)\otimes\mathbb{Z}_{(p)}\{\sigma u\mu_{ap^n},\,n\geq 1,\,a\geq 1,\,a\text{ not divisible by }p\}\Big)\end{aligned} \tag{5.7.15}$$

by the relations:

- $p\cdot\mu_p=u^{p-2}\sigma u.$

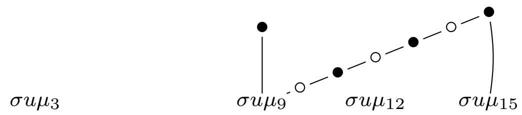


Figure 5.2: T_1 and T_2 for $p = 3$.

- $p \cdot v_0^n \mu_{p^{n+1}} = u^{p^{n+1}-p^n} v_0^{n-1} \mu_{p^n}$ *for any* $n \geq 1$.
- $u^{p^{n-h}-2} \cdot v_0^h \sigma u\mu_{ap^n} = 0$ *for any* h, a *and* n, a *not divisible by* p.
- $v_0^n \sigma u\mu_{ap^n} = 0$ *for any* a *and* n, a *not divisible by* p.
- $p \cdot \sigma u\mu_{(bp+p-1)p^n} = v_0 \sigma u\mu_{(bp+p-1)p^n} + u^{p^{n+1}-p^n} v_0^{\nu(b)} \sigma u\mu_{bp^{n+1}}$ *for any* $b \geq 1$ *not divisible by* p *and any* n.
- $p \cdot v_0^h \sigma u\mu_{ap^n} = v_0^{h+1} \sigma u\mu_{ap^n}$ *for any* a, n *and* $h \geq 1$ *or* $h = 0$ *not in the previous case.*

For the non-torsion part, we can state an integral, non-local version of this result; at each power u^{p^n-2} for $n \geq 1$, σu becomes divisible by p one more time. In what follows, ku is not localized at a prime.

Proposition 5.7.16. *The non-torsion part* $\mathrm{THH}_*(ku)$ *includes a tower* $\mathbb{Z}[u]$ *generated by* σu *where for each* $n \geq 1$, $u^{n-2}\sigma u$ *is divisible by the least common multiple of the integers* 1, 2, …, n. *That is, the non-torsion part is*

$$\mathbb{Z}[u]\{1\} \oplus \mathcal{Q} \tag{5.7.17}$$

where $\mathcal{Q}$ *is the sub-*$\mathbb{Z}$*-module of* $\mathbb{Q}[u]\{\sigma u\}$ *generated by the*

$$\frac{u^{n-1}\sigma\beta_{(0)}}{\mathrm{lcm}(1,\, 2,\, \ldots,\, n)} \tag{5.7.18}$$

for $n \geq 1$.

However, we are not able to provide such an integral description for the torsion part.

As studied in [2] for $\mathrm{THH}_*(\ell)$, the torsion modules of $\mathrm{THH}_*(ku)$ are divided into periodic submodules T_n for $n \geq 1$. Each T_n correspond to the submodules of the torsion elements of degrees between $|\sigma u\mu_{p^n}| = 2p^n + 2$ and $|\sigma u\mu_{2p^n}| - 1 = 2(2p^n) + 1$. Each of these appears $p - 1$ times, by replacing the leftmost class with $\sigma u\mu_{kp^n}$ for $1 \leq k \leq p - 1$, and p copies (as submodules or quotients) of T_n are present in T_{n+1}, so T_n appears an infinite numbers of times. In the following figures, the generators are named and placed on the bottom horizontal line; the rest of the non-zero class are indicated by a $\circ$ when they come from $\mathrm{THH}_*(\ell)$, a $\bullet$ otherwise; going straight up indicate a multiplication by p, and going upward and right is a multiplication by u; when two lines go up from a single class, it means the multiplication by p is the sum of the two elements reached. None of the named classes come from $\mathrm{THH}_*(\ell)$.

The code used to generate these pictures can be found in appendix A. We can see that $\mathrm{THH}_*(ku)$ is not $\mathrm{THH}_*(\ell)$ *étale*, by which we mean that

$$\mathrm{THH}_*(ku) \neq ku_* \otimes_{\ell_*} \mathrm{THH}_*(\ell). \tag{5.7.19}$$

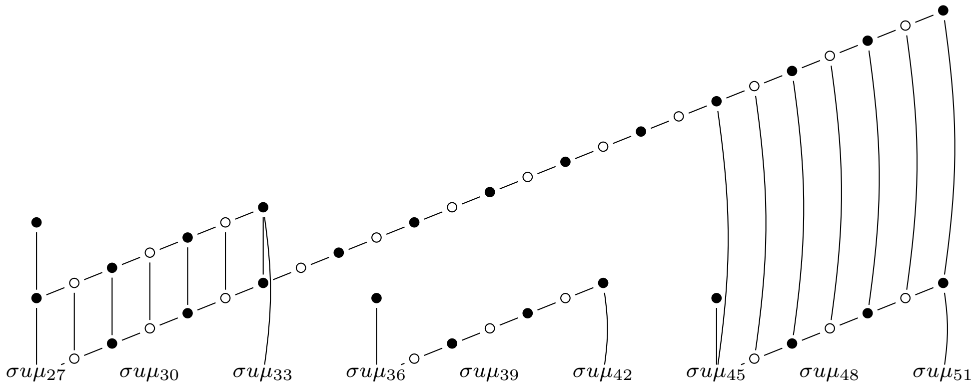


Figure 5.3: T_3 for $p = 3$.

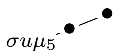


Figure 5.4: T_1 for $p = 5$.

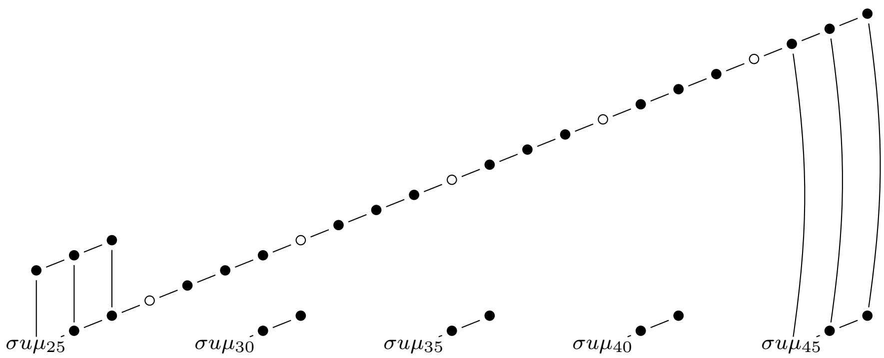


Figure 5.5: T_2 for $p = 5$.

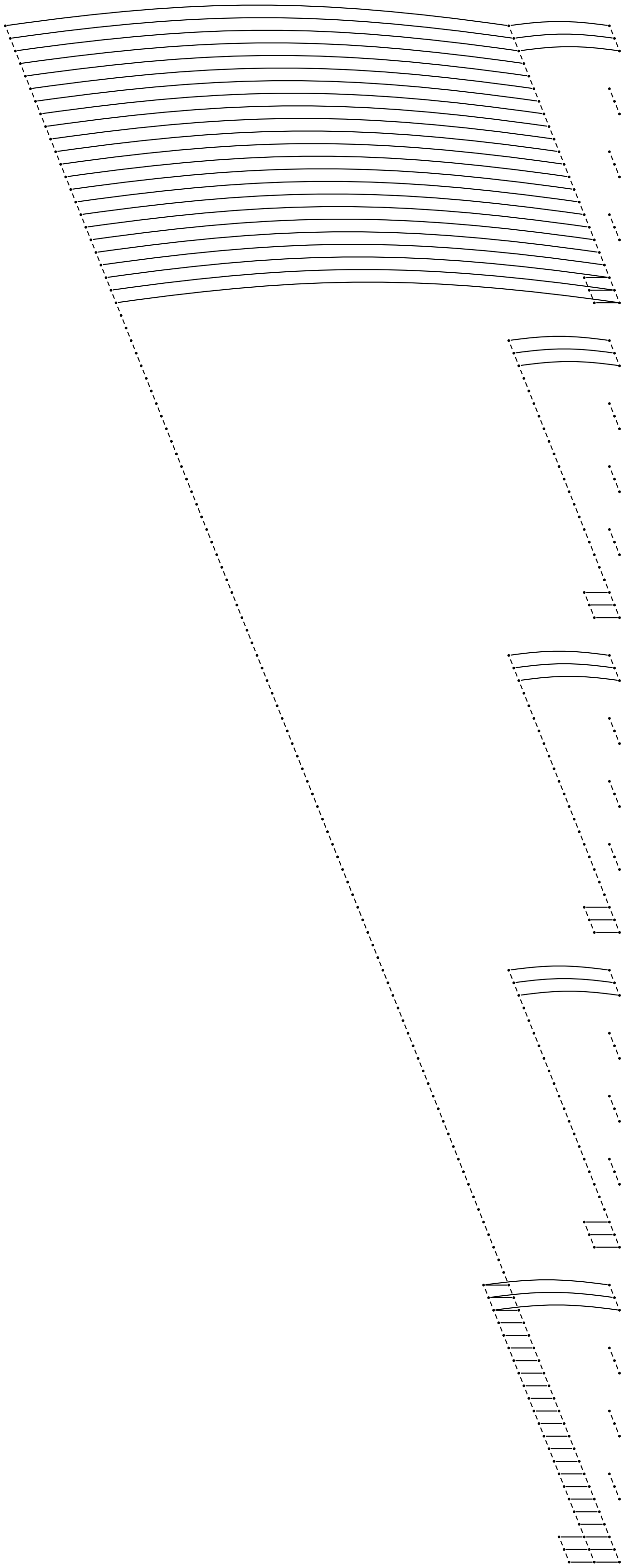


Figure 5.6: T_3 for $p = 5$.

Further discussions of these kinds of properties can be found in chapter 6. The extensions of scalars, however, does yield an injection, and in fact a short exact sequence

$$0 \longrightarrow ku_* \otimes_{\ell_*} \mathrm{THH}_*(\ell) \longrightarrow \mathrm{THH}_*(ku) \longrightarrow \mathcal{C} \longrightarrow 0 \qquad (5.7.20)$$

where the cokernel $\mathcal{C}$ can be presented as the quotient of the $\mathbb{Z}_{(p)}[u]$-module

$$P_{p-2}(u) \otimes \mathbb{Z}_{(p)}\{1,\, \sigma u,\, \sigma u \mu_{ap^n},\, n \geq 1,\, a \geq 1,\, a \text{ not divisible by } p\} \qquad (5.7.21)$$

by the relation $p^n \sigma u \mu_{ap^n} = 0$ for any a and n, a not divisible by p.

5.8 A remark on bisimplicial spectra and the suspension map

This section will offer a more general but incomplete point of view on lemma 5.5.6.

Topological Hochschild homology can be seen to have other structures in addition to the algebra structure. By viewing the simplicial construction of THH as a tensor product $S^1 \otimes A$ with a simplicial model of S^1, THH can be equipped with a Hopf algebra structure and an S^1 action; an account of such results can be found in [3]. Here we are interested in the suspension map $\sigma : \Sigma A \to \mathrm{THH}(A)$, which is constructed after remark 3.11 of [3] by splitting S^1_+ as $S^1 \vee S^0$, and composing $S^1 \wedge A \to S^1_+ \wedge A \to S^1 \otimes A$.

We will study the map σ simplicially, in order to show that it enjoys some compatibility with the Brun spectral sequence, but we will not get a result good enough to used as our lemma 5.5.6. To do so, we need to introduce bisimplicial S-modules.

Definition 5.8.1. *A bisimplicial S-module is a functor*

$$F : \Delta^{op} \times \Delta^{op} \to \mathcal{M}_S \qquad (5.8.2)$$

or, equivalently, a simplicial object in the category of simplicial S-modules. Its geometric realization $|F|$ is the coend of the functor $F \wedge (\Delta_\bullet)_+ \wedge (\Delta_\bullet)_+ : \Delta^{op} \times \Delta^{op} \times \Delta \times \Delta \to \mathcal{M}_S$ where $\Delta_\bullet$ is the topological simplex functor.

Since we have two simplicial directions, we can also realize F into a simplicial S-module in two different ways, that we will denote $|F|_1$ and $|F|_2$ for respectively realize following the first and second variable. These simplicial S-modules can then be realized a second time. As a consequence of the Fubini theorem for coend (see for example [24]), the following result holds:

Proposition 5.8.3. *There are natural isomorphism of S-modules*

$$|F| \cong ||F|_1| \cong ||F|_2|. \qquad (5.8.4)$$

Let $S^1_\bullet$ be the simplicial set that is the quotient of the 1-simplex $\Delta^1 = Hom(-, [1])$ by its boundary $\partial\Delta^1$, i.e. the coequalizer of

$$* \rightrightarrows \Delta^1 \qquad (5.8.5)$$

where the maps are the inclusion of the two 0-cells of Δ^1. One can think of that coequalizer as having n-th simplicial level:

$$\{f \in Hom([n],[1]) \text{ s.t. } 0 \text{ is in the image of } f\}. \tag{5.8.6}$$

Be wary though that this notation is not to be used with the simplicial maps, since the property "0 being in the image of f" is not stable by precomposing with a face map. It is only useful to have a representing set of our quotient and to count the number of cells. Here, our $S^1_\bullet$ has only two non-degenerate cells, one in dimension 0 and one in dimension 1. Hereafter, we consider that model to be a discrete based simplicial space, the base point being the zero map.

Let A and B be commutative S-algebras, with an algebra map $\eta : A \to B$. By considering A to be a constant simplicial S-module, we have a simplicial model for the suspension:

$$\Sigma A_\bullet = S^1_\bullet \wedge A. \tag{5.8.7}$$

Since $S^1_\bullet$ is discrete, each simplicial level is a wedge of copies of A, one for each cell in $S^1_\bullet$ that is not the base point. Thus:

$$\begin{aligned} \Sigma A_0 &\cong * \\ \Sigma A_1 &\cong A \\ \Sigma A_2 &\cong A \vee A. \end{aligned} \tag{5.8.8}$$

We can also write explicitly the simplicial model for $S^1_+ \wedge A$ coming from the same simplicial S^1, which have

$$\begin{aligned} (S^1_+ \wedge A)_0 &\cong A \\ (S^1_+ \wedge A)_1 &\cong A \vee A \\ (S^1_+ \wedge A)_2 &\cong A \vee A \vee A. \end{aligned} \tag{5.8.9}$$

However, we cannot control the map $\Sigma A \to S^1_+ \wedge A$ simplicially. This is the first issue to get a usable result mimicking lemma 5.5.6.

We will need a simplicial version of lemma 4.3.5, that can be proved similarly to proposition 4.2.11.

Lemma 5.8.10. *Let A and B be commutative S-algebras with a map of algebra $A \to B$. There is a natural isomorphism of S-modules*

$$\mathrm{THH}(A; B) \cong B \wedge_{B^e} B(B, A, B). \tag{5.8.11}$$

Let us define the three following bisimplicial spectra:

$$\begin{aligned} SA_{p,q} &= (S^1_+ \wedge A)_p \\ P_{p,q} &= B \wedge A^{\wedge p} \wedge B \\ T_{p,q} &= B \wedge B^{\wedge q} \wedge B \wedge_{B^e} B \wedge A^{\wedge p} \wedge B. \end{aligned} \tag{5.8.12}$$

SA is constantly $(S^1_+ \wedge A)_\bullet$ in the direction q. P is constantly $B_\bullet(B, A, B)$ in the direction q. T has a $B_\bullet(B, B, B)$ on the left of $\wedge_{B^e}$ and a $B_\bullet(B, A, B)$ on the right. Thus, $|SA|_1$ is the constant simplicial spectrum $S^1_+ \wedge A$, $|P|_1$ is the constant simplicial spectrum $B(B, A, B) \simeq B \wedge_A B$, and

$$\begin{aligned}
|T|_1 &\simeq B \wedge B^{\wedge q} \wedge B \wedge_{B^e} B \wedge_A B \\
&\simeq B^{\wedge q} \wedge B \wedge_A B \\
|T|_2 &\simeq B \wedge_{B^e} B \wedge A^{\wedge p} \wedge B \\
&\simeq B \wedge A^{\wedge p} \\
|T| &\simeq \mathrm{THH}(A; B) \\
&\simeq \mathrm{THH}(B; B \wedge_A B).
\end{aligned} \tag{5.8.13}$$

We have a map $\eta : P \to T$ which is the inclusion using the units of the component on the right of the $\wedge_{B^e}$. After realizing with $| - |_1$, this can be seen to be the inclusion

$$B \wedge_A B \to B \wedge B^{\wedge q} \wedge B \wedge_{B^e} B \wedge_A B \tag{5.8.14}$$

of the 0-cells into the simplicial construction of $\mathrm{THH}(B; B \wedge_A B)$. This is also the edge homomorphism of the Brun spectral sequence computing $\mathrm{THH}(A; B)$, since the Brun spectral sequence is constructed from a CW-structure on B, and in the construction in the proof of theorem III.2.10 (approximation by cell modules) of [20] can begin with a map that represents our 0-cells correctly.

The map $\sigma : \Sigma A \to \mathrm{THH}(A; B)$ is constructed from the map $\omega : S^1_+ \wedge A \to \mathrm{THH}(A; B)$, that can be seen simplicially to be defined as

$$SA_{0,q} = A \to B \wedge B^{\wedge q} \wedge B \wedge_{B^e} B \wedge B = T_{0,q} \tag{5.8.15}$$

from the unit $A \to B$ into the penultimate B on the right, and on the second non-degenerate cell – the A on the right of $A \vee A$ – to be

$$SA_{1,q} = A \vee A \to B \wedge B^{\wedge q} \wedge B \wedge_{B^e} B \wedge A \wedge B = T_{1,q} \tag{5.8.16}$$

the identity id_A into the only A on the right. The rest of the cells in S^1_+ are degenerate, so we have defined a bisimplicial map. That map can be seen to factorize through P, so that we have a commutative diagram of bisimplicial spectra maps

$$\begin{tikzcd} SA \arrow[rr] \arrow[dr] & & T \\ & P \arrow[ur] & \end{tikzcd} \tag{5.8.17}$$

that after realization gives

$$\begin{tikzcd} S^1_+ \wedge A \arrow[rr, "\omega"] \arrow[dr] & & \mathrm{THH}(A; B) \\ & B \wedge_A B \arrow[ur, "\eta"] & \end{tikzcd} \tag{5.8.18}$$

that gives after precomposing with the map $\Sigma A \to S^1_+ \wedge A$

$$\begin{tikzcd} \Sigma A \arrow[rr, "\sigma"] \arrow[dr] & & \mathrm{THH}(A; B) \\ & B \wedge_A B \arrow[ur, "\eta"] & \end{tikzcd} \tag{5.8.19}$$

If we were working with spaces, we could give an explicit point-set model of the realization of the bar construction (see e.g. (7.7) of [38]), and we could identify the image of the map $\Sigma A \to B(B, A, B)$ with the suspension in the bar resolution, so that we could also name that map σ and the classes we named σa in the Tor computing $\pi_*(B \wedge_A B)$ are indeed $\sigma(a)$. Thus, given that the map σ into THH is a derivation, we would automatically have the equation $\sigma v_1 = (p-1)u^{p-2}\sigma u$ in $\mathrm{THH}_*(ku)$ and $THH_*(ku; ku/v_1)$. But we don't have such a fine control on the realization of simplicial spectra.

Chapter 6

Logarithmic topological Hochschild homology

It is known – e.g. from $V(1)$ computations in [5] – that there is no weak equivalence between $\mathrm{THH}(ku)$ and $ku \wedge_\ell \mathrm{THH}(\ell)$. However, such weak equivalence is true for the so called logarithmic topological Hochschild homology. This, along with knowledge of $\mathrm{THH}_*(\ell)$ and a little input on its image in $\mathrm{THH}_*(ku)$, is sufficient to compute the latter.

There exist various constructions of logarithmic topological Hochschild homology. In [22], Hesselholt and Madsen construct it for discrete valuation rings. Here we will deal with the construction of Rognes, Sagave and Schlichtkrull in [33] and [34]. Another account of the logarithmic sequence for ku by Blumberg and Mandell can be found in [11].

6.1 Logarithmic topological Hochschild homology of ku

Logarithmic THH is constructed in [34] for symmetric spectra enticed with a “log ring structure”. We will only state what we need about this. Let E be a non-zero, d-periodic, positive fibrant commutative symmetric ring spectrum. Here d-periodic means that d is the smallest positive integer such that $\pi_*(E)$ has a unit x of degree d. Let e be the connective cover of E, also assumed to be positive fibrant. Then there is a log ring structure denoted by $(e, \langle x \rangle)$ on e. The following is theorem 4.4 of [34]:

Theorem 6.1.1. *There is a natural homotopy cofiber sequence*

$$\mathrm{THH}(e) \xrightarrow{\rho} \mathrm{THH}(e, \langle x \rangle) \xrightarrow{\partial} \Sigma\, \mathrm{THH}(e_{<d}) \tag{6.1.2}$$

of $\mathrm{THH}(e)$*-module with circle action, where ρ is a map of commutative symmetric ring spectra and $e_{<d}$ is the $(d-1)$-st Postnikov section of e.*

Here $\mathrm{THH}(e, \langle x \rangle)$ is logarithmic THH for the log ring structure coming from the unit x.

From that we can write diagram (8.1) of [34]:

Corollary 6.1.3. *There is a map of homotopy cofiber sequences:*

$$\begin{array}{ccccc} \mathrm{THH}(\ell) & \xrightarrow{\rho} & \mathrm{THH}(\ell,\langle v_1\rangle) & \xrightarrow{\partial} & \Sigma\,\mathrm{THH}(H\mathbb{Z}_{(p)}) \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle id} \\ \mathrm{THH}(ku) & \xrightarrow{\rho'} & \mathrm{THH}(ku,\langle u\rangle) & \xrightarrow{\partial'} & \Sigma\,\mathrm{THH}(H\mathbb{Z}_{(p)}) \end{array} \tag{6.1.4}$$

The last thing we need from [34] is theorem 6.3:

Theorem 6.1.5. *The map of pre-log ring spectra $(\ell,\langle v_1\rangle)\to(ku,\langle u\rangle)$ is formally log-THH-étale, i.e. there is a weak equivalence of ku-modules:*

$$\mathrm{THH}(ku,\langle u\rangle)\simeq ku\wedge_\ell \mathrm{THH}(\ell,\langle v_1\rangle). \tag{6.1.6}$$

The Künneth spectral sequence can then be used to conclude that additively,

$$\mathrm{THH}_*(ku,\langle u\rangle)\cong P_{p-1}(u)\otimes \mathrm{THH}_*(\ell,\langle v_1\rangle) \tag{6.1.7}$$

with the relation $u^{p-1}=v_1$. Remark that the inclusion of $\mathrm{THH}_*(\ell,\langle v_1\rangle)$ in $\mathrm{THH}_*(ku,\langle u\rangle)$ is the map induced by f'.

6.2 Deriving THH(ku) from THH(ℓ) and logarithmic THH

We need to know that in the image of the map $\mathrm{THH}_*(\ell)\to\mathrm{THH}_*(ku)$, the classes σv_1 and $\sigma v_1\mu_{kp^2}$, $k\geq 1$ are divisible by u; this is true from our previous computation of $\mathrm{THH}_*(ku)$, but it can be established only from the fact that $\sigma v_1=u^{p-2}\sigma u$ in $\mathrm{THH}_*(ku)$ (lemma 5.5.6), and the multiplicative properties of the map $\mathrm{THH}_*(\ell)\to\mathrm{THH}_*(ku)$. The previous results and the description of $\mathrm{THH}_*(\ell)$ of [2] are enough to compute $\mathrm{THH}_*(\ell,\langle v_1\rangle)$, $\mathrm{THH}_*(ku,\langle u\rangle)$ and $\mathrm{THH}_*(ku)$.

We work in the homotopy long exact sequences resulting from corollary 6.1.3. If we denote $g:\mathrm{THH}_*(H\mathbb{Z}_{(p)})\to\mathrm{THH}_*(\ell)$ the map induced in the long exact sequence, then $g(1)=0$ since $\mathrm{Im}\,\rho=\mathrm{Ker}\,g$ and ρ is a ring morphism, thus injective on $\mathrm{THH}_0(\ell)$. Moreover, $g(\mu_{kp})=0$ since $|\mu_{kp}|=2kp-1$ and there is no p-torsion in odd degree in $\mathrm{THH}_*(\ell)$. Thus $g=0$, and $f\circ g:\mathrm{THH}_*(H\mathbb{Z}_{(p)})\to\mathrm{THH}_*(ku)$ must also be 0. The long exact sequence splits into short exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{THH}_*(\ell) & \xrightarrow{\rho} & \mathrm{THH}_*(\ell,\langle v_1\rangle) & \xrightarrow{\partial} & \mathrm{THH}_{*-1}(H\mathbb{Z}_{(p)}) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathrm{THH}_*(ku) & \xrightarrow{\rho'} & \mathrm{THH}_*(ku,\langle u\rangle) & \xrightarrow{\partial'} & \mathrm{THH}_{*-1}(H\mathbb{Z}_{(p)}) & \longrightarrow & 0 \end{array} \tag{6.2.1}$$

Equation (6.1.7) ensures that f' is an injection, and then f must also be an injection. The maps are of THH(ℓ) or THH(ku)-modules, so of ℓ or ku-modules depending on the line. Let $x\in\mathrm{THH}_*(\ell)$ be any of the classes $\sigma v_1\alpha$, with $\alpha\in\{1,\,\mu_{kp^2},\,k\geq 1\}$. We know that $f(x)$ is divisible by u, and thus $\rho'(f(x))=f'(\rho(x))$ is divisible by u. But then from the eq. (6.1.7), $\rho(x)$ must be divisible by v_1. Let $y \in \mathrm{THH}_{*-2(p-1)}(\ell, \langle v_1 \rangle)$ be such that $v_1 y = \rho(x)$: y cannot be in the image of ρ since x is not divisible by v_1 in $\mathrm{THH}_*(\ell)$. Then $\partial(y) \neq 0$ and must be a multiple of the class named α in $\mathrm{THH}_{*-2p+1}(H\mathbb{Z}_{(p)})$. The coefficient must be a unit, since otherwise y would be divisible by p, and thus $\rho(x)$ too, but then this would mean that $\partial(p^{-1}\rho(x)) \neq 0$ which is impossible for degree reasons.

The remaining elements in $\mathrm{THH}_*(H\mathbb{Z}_{(p)})$ that we need to account for are the μ_{kp} with k not divisible by p. These can be lifted to classes we name $d\mu_{kp} \in \mathrm{THH}_{2kp}(\ell, \langle v_1 \rangle)$. If $v_1 d\mu_{kp} \neq 0$ then it must be equal for degree reason to an element divisible by v_1 coming from $\mathrm{THH}_{2kp+2(p-1)}(\ell)$; if $v_1 d\mu_{kp} = v_1\beta$ we can rename $d\mu_{kp}$ to be $d\mu_{kp} - \beta$. We still have $\partial(d\mu_{kp}) = \mu_{kp}$, but now $v_1 d\mu_{kp} = 0$.

Thus, we have proved:

Proposition 6.2.2. $\mathrm{THH}_*(\ell, \langle v_1 \rangle)$ *is equal to*

$$\mathrm{THH}_*(\ell) \oplus \mathbb{Z}_{(p)}\{d,\, d\mu_{kp},\, k \geq 1\} \tag{6.2.3}$$

quotiented by the relations $v_1 d = \sigma v_1$, $v_1 d\mu_{kp} = \sigma v_1 \mu_{kp}$ with the convention $\sigma v_1 \mu_{kp} = 0 \in \mathrm{THH}_*(\ell)$ when k is not divisible by p, and $p^{\nu(k)+1} d\mu_{kp} = 0$.

We now describe explicitly the structure of $\mathrm{THH}_*(\ell; \langle v_1 \rangle)$ and $\mathrm{THH}_*(ku, \langle u \rangle)$.

Theorem 6.2.4. $\mathrm{THH}_*(\ell, \langle v_1 \rangle)$ *is a quotient of the $\mathbb{Z}_{(p)}[v_1]$-module*

$$P(v_1) \otimes \Big(\mathbb{Z}_{(p)}\{1,\, d,\, v_0^n \mu_{p^{n+1}},\, n \geq 0\} \oplus P(v_0) \otimes \mathbb{Z}_{(p)}\{d\mu_{ap^n},\, n \geq 2,\, a \geq 1,\, a \text{ not divisible by } p\}\Big) \tag{6.2.5}$$

by the relations:

- $p \cdot \mu_p = v_1 d$.
- $p \cdot v_0^n \mu_{p^{n+1}} = v_1^{p^n} v_0^{n-1} \mu_{p^n}$ *for any* $n \geq 1$.
- $v_1^{p^{n-h-1}+p^{n-h-2}+\cdots+p+1} \cdot v_0^h d\mu_{ap^n} = 0$ *for any h, a and n, a not divisible by p.*
- $v_0^n d\mu_{ap^n} = 0$ *for any a and n, a not divisible by p.*
- $p \cdot d\mu_{(bp+p-1)p^n} = v_0 d\mu_{(bp+p-1)p^n} + v_1^{p^n+p^{n-1}+\cdots+p} v_0^{\nu(b)} d\mu_{bp^{n+1}}$ *for any $b \geq 1$ not divisible by p and any n.*
- $p \cdot v_0^h d\mu_{ap^n} = v_0^{h+1} d\mu_{ap^n}$ *for any a, n and $h \geq 1$ or $h = 0$ not in the previous case.*

Theorem 6.2.6. $\mathrm{THH}_*(ku, \langle u \rangle)$ *is a quotient of the $\mathbb{Z}_{(p)}[u]$-module*

$$P(u) \otimes \Big(\mathbb{Z}_{(p)}\{1,\, d,\, v_0^n \mu_{p^{n+1}},\, n \geq 0\} \oplus P(v_0) \otimes \mathbb{Z}_{(p)}\{d\mu_{ap^n},\, n \geq 1,\, a \geq 1,\, a \text{ not divisible by } p\}\Big) \tag{6.2.7}$$

by the relations:

- $p \cdot \mu_p = u^{p-1}d$.
- $p \cdot v_0^n \mu_{p^{n+1}} = u^{p^{n+1}-p^n} v_0^{n-1} \mu_{p^n}$ *for any* $n \geq 1$.
- $u^{p^{n-h}-1} \cdot v_0^h d\mu_{ap^n} = 0$ *for any* h, a *and* n, a *not divisible by* p.
- $v_0^n d\mu_{ap^n} = 0$ *for any* a *and* n, a *not divisible by* p.
- $p \cdot d\mu_{(bp+p-1)p^n} = v_0 d\mu_{(bp+p-1)p^n} + u^{p^{n+1}-p^n} v_0^{\nu(b)} d\mu_{bp^{n+1}}$ *for any* $b \geq 1$ *not divisible by* p *and any* n.
- $p \cdot v_0^h d\mu_{ap^n} = v_0^{h+1} d\mu_{ap^n}$ *for any* a, n *and* $h \geq 1$ *or* $h = 0$ *not in the previous case.*

From these descriptions of $\mathrm{THH}_*(\ell, \langle v_1 \rangle)$, $\mathrm{THH}_*(ku, \langle u \rangle)$, and of the maps f', ∂ and ∂', it is easy to derive a description of $\mathrm{THH}_*(ku)$ similar to that of theorem 5.7.14 with ρ' sending σu to ud.

We are also able to provide a logarithmic counterpart of the integral result of proposition 5.7.16. In what follows, ku is not localized at a prime. Once again, we cannot state anything of the sort for the torsion.

Proposition 6.2.8. *The non-torsion part* $\mathrm{THH}_*(ku, \langle u \rangle)$ *includes a tower* $\mathbb{Z}[u]$ *generated by* d *where for each* $n \geq 1$, $u^{n-1}d$ *is divisible by least common multiple of the integers* $1, 2, \ldots, n$. *All the elements can be written as such a quotient.*

We can revisit the figures of chapter 5 for logarithmic topological Hochschild homology. $\mathrm{THH}_*(\ell, \langle v_1 \rangle)$ and $\mathrm{THH}_*(ku, \langle u \rangle)$ are divided into submodules T'_n for $n \geq 1$ that are obtained from the T_n of $\mathrm{THH}_*(\ell)$ and $\mathrm{THH}_*(ku)$ by adding the extension $u \cdot d = \sigma u$.

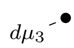


Figure 6.1: T'_1 for $p = 3$.

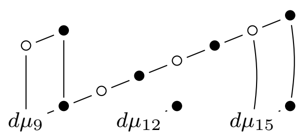


Figure 6.2: T'_2 for $p = 3$.

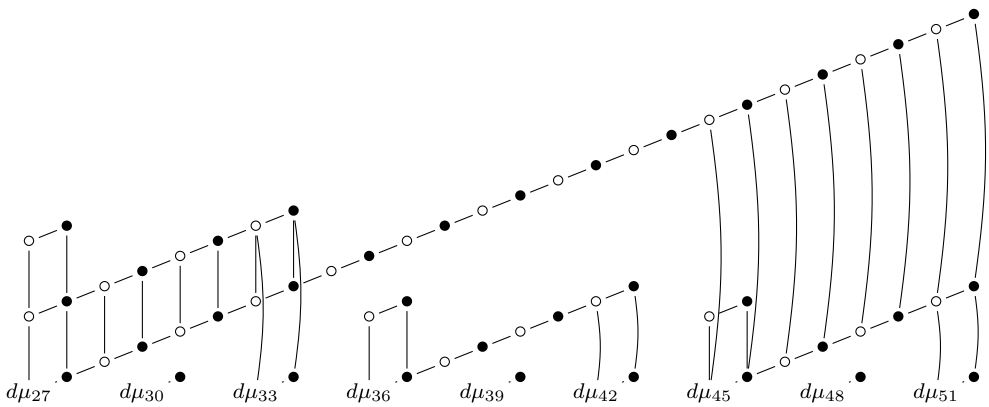


Figure 6.3: T_3' for $p = 3$.

Chapter 7

The $K(\mathbb{Z},3)$ units and the trace map

In this chapter, we study the trace in topological Hochschild homology of the map $\Sigma^\infty_+ K(\mathbb{Z},3) \to K(ku)$ induced by the map $\Sigma^\infty_+ K(\mathbb{Z},2) = \Sigma^\infty_+ \mathbb{CP}^\infty \to ku$.

This map capture part of the *units* of the ring spectrum ku. The units $GL_1(R)$ of a ring spectrum R can be defined as the homotopy pullback of the square

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R) \end{array} \tag{7.0.1}$$

The units of ku can thus be seen to be

$$GL_1(ku) \cong \mathbb{Z}/2 \times BU_\otimes. \tag{7.0.2}$$

The natural inclusion $BU(1) \to BU_\otimes$ and the fact that $U(1) = K(\mathbb{Z},1)$ define our map $K(\mathbb{Z},2) \to GL_1(ku)$. Furthermore, as seen in [26] (lemma V.3.1), the inclusion induces a splitting

$$BU_\otimes \cong BU(1) \times BSU_\otimes \tag{7.0.3}$$

The other ingredients in our discussion are the natural map from the ring units into algebraic K-theory and the Bökstedt trace map from algebraic K-theory to topological Hochschild homology. Both these maps are studied in e.g. [35] or [4].

Following [4] proposition 2.5, the composition of the unit map and the trace map

$$\Sigma^\infty_+ BGL_1(R) \to K(R) \to \mathrm{THH}(R) \tag{7.0.4}$$

is equivalent to the composition

$$\Sigma^\infty_+ BGL_1(R) \to \Sigma^\infty_+ B^{cy} GL_1(R) \simeq B^{cy} \Sigma^\infty_+ GL_1(R) \to B^{cy} R. \tag{7.0.5}$$

The first part is the map coming from the bar construction

$$\Sigma^\infty_+ BGL_1(R) \cong \Sigma^\infty_+ B^{cy}(GL_1(R); *) \to \Sigma^\infty_+ B^{cy} GL_1(R). \tag{7.0.6}$$

The second part comes from the counit of the adjunction

$$\Sigma^\infty_+ \dashv GL_1 \tag{7.0.7}$$

which gives a map

$$\Sigma^\infty_+ B^{cy} GL_1(R) \simeq B^{cy}\Sigma^\infty_+ GL_1(R) \to B^{cy}R = THH(R). \tag{7.0.8}$$

Thus, we have a composition

$$\Sigma^\infty_+ K(\mathbb{Z},3) \to \Sigma^\infty_+ BGL_1(ku) \simeq B^{cy}\Sigma^\infty_+ GL_1(ku) \to THH(ku) \tag{7.0.9}$$

that will factor through $ku \wedge K(\mathbb{Z},3)_+$ by extension of the scalars. The map

$$f : ku \wedge K(\mathbb{Z},3)_+ \to THH(ku) \tag{7.0.10}$$

is what we will partially compute. Our model from that map will be induced by the cyclic bar construction and the identification

$$\begin{aligned} ku \wedge K(\mathbb{Z},3)_+ &\simeq ku \wedge_{\Sigma^\infty_+ K(\mathbb{Z},2)^e} \Sigma^\infty_+ K(\mathbb{Z},2)^e \wedge B(S, \Sigma^\infty_+ K(\mathbb{Z},2), S) \\ &\simeq ku \wedge_{\Sigma^\infty_+ K(\mathbb{Z},2)^e} B(\Sigma^\infty_+ K(\mathbb{Z},2), \Sigma^\infty_+ K(\mathbb{Z},2), \Sigma^\infty_+ K(\mathbb{Z},2)) \\ &\simeq B^{cy}(\Sigma^\infty_+ K(\mathbb{Z},2); ku) \to B^{cy} ku \end{aligned} \tag{7.0.11}$$

obtained by proposition 4.2.11, the commutativity of $\Sigma^\infty_+ K(\mathbb{Z},2)$ and proposition II.1.2 of [20] applied to the simplicial constructions.

We can sumarize all the maps we have mentioned in the diagram of fig. 7.1 where ϕ_1 is the extension of scalars, f is the map we will compute, ϕ_2 is the inclusion of the units in K-theory and ϕ_3 is the Bökstedt trace map. Triangle (1) commute by functoriality, square (2) by naturality and square (3) by functoriality in the composition

$$S \to \Sigma^\infty_+ K(\mathbb{Z},2) \to \Sigma^\infty_+ GL_1(ku) \to ku. \tag{7.0.12}$$

The area (4) is the equivalence of proposition 2.5 of [4].

We will first compute the Bockstein spectral sequences

$$H_*(K(\mathbb{Z},3);\mathbb{F}_p) \otimes P(p) \Rightarrow H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)}) \tag{7.0.13}$$

and

$$H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)}) \otimes P(u) \Rightarrow ku_* K(\mathbb{Z},3). \tag{7.0.14}$$

Then, the identification $ku \wedge K(\mathbb{Z},3)_+ \simeq THH(\Sigma^\infty_+ K(\mathbb{Z},2); ku)$ will allow us to partially compute the map f.

7.1 Hopf rings and the external Bockstein of $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$

The groups $H_*(K(\mathbb{Z},3);\mathbb{Z})$, along with the homology of Eilenberg-MacLane spaces in general, were computed in the Cartan's seminar [16] (see also [29] for an overview). Here we will compute $H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})$ using Hopf rings techniques. What we will use about Hopf ring comes from part 2, section 7 of [38], along with a computation of $H_*(K(\mathbb{Z}/p,n);\mathbb{F}_p)$ for all $n \geq 0$ in part 2, section 8 that will be the basis of our own computation. Hopf rings techniques are also studied and used in [31].

$$
\begin{tikzcd}
& \Sigma^\infty_+ BK(\mathbb{Z},2) \arrow[r, phantom, "="] \arrow[dl, phantom, "\simeq" sloped] & \Sigma^\infty_+ K(\mathbb{Z},3) \arrow[dddll, bend right=60, "\phi_1"'] & & \\
\Sigma^\infty_+ B^{cy}(K(\mathbb{Z},2);*) \arrow[rr] \arrow[d, "\simeq"] \arrow[drr] & & \Sigma^\infty_+ B^{cy}(GL_1(ku);*) \arrow[rr, phantom, "\simeq"] \arrow[d] \arrow[dl, phantom, "(1)"] \arrow[drr, phantom, "(4)"] & & \Sigma^\infty_+ BGL_1(ku) \arrow[d, "\phi_2"] \\
B^{cy}(\Sigma^\infty_+ K(\mathbb{Z},2);S) \arrow[d] \arrow[drr] \arrow[rr, phantom, "(2)"] & & \Sigma^\infty_+ B^{cy} GL_1(ku) \arrow[d, "\simeq"] & & K(ku) \arrow[ddlll, bend left, "\phi_3"] \\
B^{cy}(\Sigma^\infty_+ K(\mathbb{Z},2);ku) \arrow[dr, "f"] \arrow[d, phantom, "\wr"] \arrow[rr, phantom, "(3)"] & & B^{cy}\Sigma^\infty_+ GL_1(ku) \arrow[dl] & & \\
ku \wedge K(\mathbb{Z},3)_+ & B^{cy} ku & & &
\end{tikzcd}
$$

Figure 7.1: The map f, the units and the trace.

7.1.1 Hopf rings

A Hopf ring is a graded ring object in a category of coalgebras. All our examples will be from the following constructions: let E_* be a homology satisfying a Künneth isomorphism, and let G be a group. The homology of the Eilenberg-MacLane spaces $E_*K(G,*)$ has a Hopf ring structure. Here, there are two graduations. The homology graduation is internal – we forget about it to get a coalgebra from our graded coalgebra – and the graduation corresponding to the level of the Eilenberg-MacLane spaces is the one we mentioned previously in "graded ring object".

Precisely, for each $n \geq 0$, the diagonal map

$$\Delta : K(G,n) \to K(G,n) \times K(G,n) \tag{7.1.1}$$

gives the homology $E_*K(G,n)$ the structure of a (graded) cocommutative coalgebra. There is also a first monoid map (the sum in our graded ring)

$$* : K(G,n) \times K(G,n) \to K(G,n) \tag{7.1.2}$$

induced by the group law on G. The product comes from the cup product in the cohomology theory represented by the $K(G,*)$, which gives a map

$$\circ : K(G,m) \times K(G,n) \to K(G,m+n) \tag{7.1.3}$$

for each m, $n \geq 0$, so that this is indeed a graded product in $E_*K(G,*)$. That map can be constructed to be compatible in some sense with the bar spectral sequence computing $K(G,n+1)$ as the classifying space $BK(G,n)$. This is the basis of the computation in [31] and [38], and the explicit construction can be found in section 1 of [31]. We will also simply write $*$, $\circ$ and Δ for the products and coproduct on $E_*K(G,*)$.

Both product $*$ and $\circ$ are maps of coalgebra, with the cocommutative coalgebra structure on $E_*K(G,*) \otimes E_*K(G,*)$ being given by $\Delta \otimes \Delta$, and the product being term by term. The distributivity of $\circ$ over $*$ correspond to a homotopy commutative diagram of spaces

$$\begin{tikzcd}
K(G,m) \times K(G,n) \times K(G,n) \arrow[r, "id\times *"] \arrow[d, "\Delta\times id\times id"] & K(G,m) \times K(G,n) \arrow[ddd, "\circ"] \\
K(G,m) \times K(G,m) \times K(G,n) \times K(G,n) \arrow[d, "id\times\tau\times id"] & \\
K(G,m) \times K(G,n) \times K(G,m) \times K(G,n) \arrow[d, "\circ\times\circ"] & \\
K(G,m+n) \times K(G,m+n) \arrow[r, "*"] & K(G,m+n)
\end{tikzcd} \tag{7.1.4}$$

which, applying E_*, results in the following distributivity formula:

$$x \circ (y * z) = \sum (-1)^{|x''|\,|y|} (x' \circ y) * (x'' \circ z) \tag{7.1.5}$$

where $\Delta(x) = \sum x' \otimes x''$ and the degrees are the graded ring one, i.e. n and m in the previous diagram.

The Bockstein spectral sequences we will study come from the cofiber sequence

$$H\mathbb{Z}_{(p)} \xrightarrow{\times p} H\mathbb{Z}_{(p)} \longrightarrow H\mathbb{F}_p \tag{7.1.6}$$

that induces a homology long exact sequence for our spaces. The homology Bockstein spectral sequence of an H-space is a spectral sequence of Hopf algebras, see for example chapter 10 of [27]. In our case, it means that the differentials β are derivations with respect to both products – $*$ gives an H-space structure to each $K(G,*)$, and $\circ$ gives an H-space structure to their union – and coderivation with respect to the coproduct.

7.1.2 Homologies of $K(\mathbb{Z},3)$

Our goal is this section is to compute $H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})$ from the Hopf ring $H_*(K(\mathbb{Z}/p,*);\mathbb{F}_p)$ and from $H_*(K(\mathbb{Z},2);\mathbb{Z}_{(p)})$. We first review the results of [38] regarding $H_*(K(\mathbb{Z}/p,*);\mathbb{F}_p)$.

In the bar spectral computing $H_*(K(\mathbb{Z}/p,n+1);\mathbb{F}_p)$ as $H_*(BK(\mathbb{Z}/p,n);\mathbb{F}_p)$, the E^2 page is

$$\mathrm{Tor}_{*,*}^{H_*(K(\mathbb{Z}/p,n);\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p). \tag{7.1.7}$$

That Tor yields a divided power algebra $\Gamma(\sigma x)$ over the suspension of x from an exterior algebra $E(x)$; an exterior algebra $E(\sigma x)$ from a polynomial algebra $P(x)$; an exterior algebra $E(\sigma x)$ and a divided power algebra $\Gamma(\phi x)$ over the transpotence of x from a truncated polynomial algebra $P_p(x)$. Remark that a divided power algebra $\Gamma(x)$ in characteristic p decomposes as the product $\bigotimes_{i\geq 0} P_p(\gamma_{p^i}x)$. Moreover, in that case, for bidegree reasons, all the spectral sequences collapse, so that we can write:

$$H_*(K(\mathbb{Z}/p,0);\mathbb{F}_p) \cong P_p([1]-[0]) \tag{7.1.8}$$

$$H_*(K(\mathbb{Z}/p,1);\mathbb{F}_p) \cong E(e_1)\otimes\Gamma(\alpha_1). \tag{7.1.9}$$

Here, e_1 is the suspension of $[1]-[0]$, in degree 1, and α_1 its transpotence, in degree 2. We will write $\alpha_{(i)}$ for the divided power $\gamma_{p^i}\alpha_1$, so that

$$\Gamma(\alpha_1) \cong \bigotimes_{i\geq 0} P_p(\alpha_{(i)}). \tag{7.1.10}$$

Thus, next we have:

$$H_*(K(\mathbb{Z}/p,2);\mathbb{F}_p) \cong \Gamma(\sigma e_1)\otimes \bigotimes_{i\geq 0} E(\sigma\alpha_{(i)})\otimes\Gamma(\phi\alpha_{(i)}). \tag{7.1.11}$$

This is when the Hopf ring structure becomes handy. It can be seen geometrically that the suspension of a class is in fact its $\circ$ product with e_1. Furthermore, the classes $\gamma_{p^j}\alpha_{(i)}$ can be rewritten modulo decomposables for the $*$ product as $\alpha_{(j)}\circ\alpha_{(i+j+1)}$. We denote $e_1\circ e_1$ by $\beta_1=\beta_{(0)}$, in degree 2, and let $\beta_{(i)}=\gamma_{p^i}\beta_1$, so that we can write:

$$H_*(K(\mathbb{Z}/p,2);\mathbb{F}_p) \cong \bigotimes_{i\geq 0} E(e_1\circ\alpha_{(i)})\otimes P_p(\beta_{(i)})\otimes \bigotimes_{i,j\geq 0} P_p(\alpha_{(j)}\circ\alpha_{(i+j+1)}). \tag{7.1.12}$$

That process continues to compute the Hopf ring $H_*(K(\mathbb{Z}/p,*);\mathbb{F}_p)$ entirely.

Theorem 7.1.13 (8.5 of [38])**.** *$H_*(K(\mathbb{Z}/p,*);\mathbb{F}_p)$ is the free Hopf ring on $H_*(K(\mathbb{Z}/p,0);\mathbb{F}_p)$ and the generator e_1, $\alpha_{(i)}$ and $\beta_{(i)}$ for $i \geq 0$ with the relation $e_1 \circ e_1 = \beta_{(0)}$.*

In fact, e_1 can be seen to be the image of the fundamental class of $H_*(K(\mathbb{Z},1);\mathbb{Z})$ and $\beta_{(0)}$ is the generator of the divided power algebra $H_*(K(\mathbb{Z},2);\mathbb{Z})$. We use the same notation for all these classes and their images into homologies with various coefficients, and with respect to the maps from $K(\mathbb{Z},*)$ into $K(\mathbb{Z}_{(p)},*)$ or $K(\mathbb{F}_p,*)$.

The specialized result for $K(\mathbb{Z}/p,3)$ can be seen in the local version of the previous theorem, which is 8.11 of [38]:

$$\begin{aligned} H_*(K(\mathbb{Z}/p,3);\mathbb{F}_p) \cong & \bigotimes_{i\geq 0} E(e_1 \circ \beta_{(i)}) \otimes \bigotimes_{i,j\geq 0} E(e_1 \circ \alpha_{(i)} \circ \alpha_{(i+j+1)}) \otimes P_p(\alpha_{(i)} \circ \beta_{(j)}) \\ & \otimes \bigotimes_{i,j,k\geq 0} P_p(\alpha_{(i)} \circ \alpha_{(i+j+1)} \circ \alpha_{(i+j+k+2)}). \end{aligned} \tag{7.1.14}$$

We now turn to the integral Eilenberg-MacLane spaces. With the previous notations, we have:

$$H_*(K(\mathbb{Z},1);\mathbb{F}_p) \cong E(e_1) \tag{7.1.15}$$

$$H_*(K(\mathbb{Z},2);\mathbb{F}_p) \cong \Gamma(\beta_{(0)}) \cong \bigotimes_{i\geq 0} P_p(\beta_{(i)}) \tag{7.1.16}$$

as sub-Hopf algebras of $H_*(K(\mathbb{Z}/p,1);\mathbb{F}_p)$ and $H_*(K(\mathbb{Z}/p,2);\mathbb{F}_p)$ with the reduction modulo p maps. This gives next:

$$H_*(K(\mathbb{Z},3);\mathbb{F}_p) \cong \bigotimes_{i\geq 0} E(\sigma\beta_{(i)}) \otimes \Gamma(\phi\beta_{(i)}). \tag{7.1.17}$$

That can be rewritten as a sub-Hopf algebra of $H_*(K(\mathbb{Z}/p,3);\mathbb{F}_p)$ using claim 8.16 of [38]:

$$H_*(K(\mathbb{Z},3);\mathbb{F}_p) \cong \bigotimes_{i\geq 0} E(e_1 \circ \beta_{(i)}) \otimes \bigotimes_{i,j\geq 0} P_p(\alpha_{(i)} \circ \beta_{(i+j+1)}) \tag{7.1.18}$$

where $\gamma_{p^i}\phi\beta_{(j)} = \alpha_{(i)} \circ \beta_{(i+j+1)}$ modulo $*$-decomposables. From the proof of claim 8.16 we see that this decomposable is zero when $i = 0$, so that there is an equality $\phi\beta_{(j)} = \alpha_{(0)} \circ \beta_{(j+1)}$.

To compute the Bockstein on $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, we need some input on the Bockstein on $K(\mathbb{Z},1)$ and $K(\mathbb{Z},2)$.

Proposition 7.1.19. *In $H_*(K(\mathbb{Z}/p,1);\mathbb{F}_p)$ the Bockstein are given by the formula:*

$$\beta^1(\gamma_k\alpha_{(0)}) = e_1 * \gamma_{k-1}\alpha_{(0)} \tag{7.1.20}$$

for all $k \geq 1$.

In $H_*(K(\mathbb{Z},2);\mathbb{F}_p)$ the classes $\beta_{(i)}$, $i \geq 0$ are all infinite cycles in the Bockstein spectral sequence.

Proof. We have:

$$H_*(K(\mathbb{Z}/p,1);\mathbb{Z}_{(p)}) \cong E(e_1) \tag{7.1.21}$$
$$H_*(K(\mathbb{Z},2);\mathbb{Z}_{(p)}) \cong \Gamma(\beta_{(0)}) \tag{7.1.22}$$

so that the $\gamma_k\alpha_{(0)}$ are not in the image of the integral homology and must support a Bockstein; the formula given is the only possible for degree reasons. We get the results up to a unit, and we can rename our classes when necessary to enforce the equality strictly.

Conversely, the $\beta_{(i)}$ are all in the image of the integral homology so that they must be infinite cycles. □

We will also need the coalgebra structure on $H_*(K(\mathbb{Z}/p,*);\mathbb{F}_p)$.

Proposition 7.1.23. *In $H_*(K(\mathbb{Z}/p,*);\mathbb{F}_p)$, the coproduct is generated with the Hopf ring properties and the formulas:*

$$\Delta(e_1) = e_1 \otimes 1 + 1 \otimes e_1 \tag{7.1.24}$$

$$\Delta(\gamma_n\alpha_{(0)}) = \sum_{k=0}^{n} \gamma_k\alpha_{(0)} \otimes \gamma_{n-k}\alpha_{(0)} \tag{7.1.25}$$

$$\Delta(\gamma_n\beta_{(0)}) = \sum_{k=0}^{n} \gamma_k\beta_{(0)} \otimes \gamma_{n-k}\beta_{(0)}. \tag{7.1.26}$$

The formulas are mentioned in the proof of 8.11 of [38]. This type of formula for the two transpotences $\alpha_{(0)}$ and $\beta_{(0)}$ are also valid for all the next transpotences. The only source for the formula for the coproduct of a transpotence seems to be Cartan's seminar sections [17], [18], and [15].

Proposition 7.1.27. *For any $x \in H_*(K(\mathbb{Z}/p,*-1);\mathbb{F}_p)$, the divided powers of its transpotence ϕx have coproducts*

$$\Delta(\gamma_n\phi x) = \sum_{k=0}^{n} \gamma_k\phi x \otimes \gamma_{n-k}\phi x. \tag{7.1.28}$$

Proof. We work from two claims established by Cartan: the transpotences are primitive classes (see [29] page 201) and the diagonal induces a morphism of algebras with divided powers (see [29] page 193). We then use the Leibniz formula for the divided power of a sum ((3) in [18]) and the formula for the divided power of a product ((4) in [18]).

$$\begin{aligned}
\Delta(\gamma_n\phi x) &= \gamma_n(\Delta\phi x)\\
&= \gamma_n(\phi x \otimes 1 + 1 \otimes \phi x)\\
&= \sum_{k=0}^{n} \gamma_k(\phi x \otimes 1)\cdot\gamma_{n-k}(1\otimes\phi x)\\
&= \sum_{k=0}^{n} \gamma_k\phi x \otimes 1 \cdot 1 \otimes \gamma_{n-k}\phi x\\
&= \sum_{k=0}^{n} \gamma_k\phi x \otimes \gamma_{n-k}\phi x
\end{aligned} \tag{7.1.29}$$

□

We need to establish the following claim, whose proof will also give an example on how to use the Hopf rings properties to compute the coproducts:

Proposition 7.1.30. *In $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, the coalgebra primitives are generated additively by the classes $\sigma\beta_{(i)}$ and $\phi\beta_{(i)}$ for $i \geq 0$.*

Proof. We use the following properties of Hopf rings: $1 \circ a = 0$ when a is not a unit, so that:

$$\Delta(e_1 \circ \beta_{(i)}) = (e_1 \otimes 1 + 1 \otimes e_1) \circ (\sum_{k=0}^{p^i} \gamma_k \beta_{(0)} \otimes \gamma_{p^i-k}\beta_{(0)}) \tag{7.1.31}$$

$$= (e_1 \circ \beta_{(i)}) \otimes 1 + 1 \otimes (e_1 \circ \beta_{(i)}). \tag{7.1.32}$$

Moreover,

$$\Delta(\gamma_n \phi\beta_{(i)}) = \sum_{k=0}^{n} \gamma_k \phi\beta_{(i)} \otimes \gamma_{n-k}\phi\beta_{(i)} \tag{7.1.33}$$

and other products can be seen not to be primitive. □

Proposition 7.1.34. *The differentials in the Bockstein spectral sequence on $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$ are given by*

$$\beta^1(\gamma_k\phi\beta_{(i)}) = \sigma\beta_{(i+1)}\gamma_{k-1}\phi\beta_{(i)} \tag{7.1.35}$$

and thus, all the p-torsion in $H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})$ is annihilated by p.

Proof. Let i be a non-negative integer. We will work inductively on the divided power using the coproduct. Proposition 7.1.19 implies that

$$\beta^1(\alpha_{(0)} \circ \beta_{(i+1)}) = e_1 \circ \beta_{(i+1)} = \sigma\beta_{(i+1)}. \tag{7.1.36}$$

Moreover, $\phi\beta_{(i)} = \alpha_{(0)} \circ \beta_{(i+1)}$, so our claim is established for $\phi\beta_{(i)}$. If it is established for

$$\phi\beta_{(i)},\ \gamma_p\phi\beta_{(i)},\ \ldots,\ \gamma_{p^{n-1}}\phi\beta_{(i)} \tag{7.1.37}$$

then it is established for all $\gamma_k\phi\beta_{(i)}$ with $0 \leq k \leq p^n - 1$ multiplicatively. Assume this is the case, and let us prove our claim for $\gamma_{p^n}\phi\beta_{(i)}$.

On one hand:

$$\begin{aligned}\Delta(\beta^1(\gamma_{p^n}\phi\beta_{(i)})) &= (1 \otimes \beta^1 + \beta^1 \otimes 1)(\Delta(\gamma_{p^n}\phi\beta_{(i)})) \\ &= (1 \otimes \beta^1 + \beta^1 \otimes 1)(\sum_{k=0}^{p^n} \gamma_k\phi\beta_{(i)} \otimes \gamma_{p^n-k}\phi\beta_{(i)}) \\ &= 1 \otimes \beta^1(\gamma_{p^n}\phi\beta_{(i)}) + \beta^1(\gamma_{p^n}\phi\beta_{(i)}) \otimes 1 \qquad (7.1.38) \\ &\quad + \sum_{k=1}^{p^n-1} (\gamma_k\phi\beta_{(i)} \otimes \sigma\beta_{(i+1)}\gamma_{p^n-k-1}\phi\beta_{(i)} \\ &\qquad + \sigma\beta_{(i+1)}\gamma_{k-1}\phi\beta_{(i)} \otimes \gamma_{p^n-k}\phi\beta_{(i)}).\end{aligned}$$

On the other hand:

$$\Delta(\sigma\beta_{(i+1)}\gamma_{p^n-1}\phi\beta_{(i)}) = (1\otimes\sigma\beta_{(i+1)} + \sigma\beta_{(i+1)}\otimes 1)\sum_{k=0}^{p^n-1}\gamma_k\phi\beta_{(i)}\otimes\gamma_{p^n-1-k}\phi\beta_{(i)} \quad (7.1.39)$$

so that $\beta^1(\gamma_{p^n}\phi\beta_{(i)}) - \sigma\beta_{(i+1)}\gamma_{p^n-1}\phi\beta_{(i)}$ is primitive, and is in degree $2p^{n+i+1} + 2p^n - 1$. But the non-zero primitives classes of odd degree are in degree $2p^k+1$ for some $k\geq 0$. This concludes our induction step.

Furthermore, $\beta^r(\sigma\beta_{(0)}) = 0$ for degree reasons. There can be no other differentials since now everything is either in the image of the Bockstein or supporting a Bockstein. □

This formula for the Bockstein also appears in Cartan's seminar (see again [29] page 201). This is sufficient to compute the homology of $K(\mathbb{Z},3)$ with $\mathbb{Z}_{(p)}$ coefficients.

Proposition 7.1.40.

$$H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}\{1,\,\sigma\beta_{(0)}\}\oplus T \quad (7.1.41)$$

where the torsion submodule T is isomorphic to $\mathrm{Im}(\beta^1)\subset H_*(K(\mathbb{Z},3);\mathbb{F}_p)$.

We won't try to give another description of the torsion part, but we will use the following notation: let $\delta : H\mathbb{F}_p \to \Sigma H\mathbb{Z}_{(p)}$ be the connecting map for the multiplication by p cofiber sequence; we will write $\delta(x)$ for the torsion element of $H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})$ corresponding to $\beta^1(x)$ in the previous isomorphism, when $x\in H_*(K(\mathbb{Z},3);\mathbb{F}_p)$. This way, we know that $\delta(x)=\delta(y)$ if and only if $\beta^1(x-y)=0$.

7.2 The u-Bockstein spectral sequence computing $ku_*K(\mathbb{Z},3)$

In the section, we compute $ku_*K(\mathbb{Z},3)$ using the Bockstein spectral sequence of the multiplication by u:

$$H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})\,\bar{\otimes}\,P(u) \Rightarrow ku_*K(\mathbb{Z},3). \quad (7.2.1)$$

To do so, we will use the map $\delta : H\mathbb{F}_p\to\Sigma H\mathbb{Z}_{(p)}$ and the Bockstein spectral sequence reduced modulo p:

$$H_*(K(\mathbb{Z},3);\mathbb{F}_p)\,\bar{\otimes}\,P(u) \Rightarrow (V(0)\wedge ku)_*K(\mathbb{Z},3). \quad (7.2.2)$$

Since all the torsion in the image of δ and the non torsion part is easy to study, the mod p sequence will determine the integral one.

To compute the mod p sequence, we will first compute the Bockstein spectral sequence associated to the multiplication by v_1 in the mod p Adams summand $V(0)\wedge\ell$, which is in fact the connective Morava K-theory $k(1)$ as remarked in [39] after theorem 1.3. The periodic Morava K-theory $K(1)_*K(\mathbb{Z},3)$ is computed in [31], along with all the $K(n)_*K(\mathbb{Z},m)$, which using the coalgebra structure allows us to recover the connective case.

7.2.1 First connective Morava K-theory of $K(\mathbb{Z},3)$

We begin by citing the periodic result, which is computed in [31] using the bar spectral sequence:

Theorem 7.2.3 (12.1 of [31])**.** *The first periodic Morava K-theory of $K(\mathbb{Z},3)$ is trivial.*

$$K(1)_*K(\mathbb{Z},3) \cong K(1)_*. \tag{7.2.4}$$

We are, however, interested in the v_1-Bockstein spectral sequence

$$H_*(K(\mathbb{Z},3);\mathbb{F}_p)\,\bar{\otimes}\,P(v_1,v_1^{-1}) \Rightarrow K(1)_*K(\mathbb{Z},3) \tag{7.2.5}$$

since it determines the connective sequence

$$H_*(K(\mathbb{Z},3);\mathbb{F}_p)\,\bar{\otimes}\,P(v_1) \Rightarrow k(1)_*K(\mathbb{Z},3). \tag{7.2.6}$$

We will compute the connective v_1-Bockstein sequence with some input from the connective bar spectral sequence.

Lemma 7.2.7. *There is a bar spectral sequence*

$$E^2 \cong \mathrm{Tor}^{k(1)_*K(\mathbb{Z},2)}(k(1)_*,k(1)_*) \Rightarrow k(1)_*K(\mathbb{Z},3) \tag{7.2.8}$$

and the Tor*-groups include classes $\sigma\beta_{(i)}$ for any $i \geq 0$, such that $v_1^{p^i}\sigma\beta_{(i)} = 0$ and that map to the classes with the same name in* $\mathrm{Tor}^{H_*(K(\mathbb{Z},2);\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)$.

Proof. We begin by stating theorem 5.6 from [31]: $K(1)_*K(\mathbb{Z},2)$ is generated as an algebra over $K(1)_*$ by the elements $\beta_{(i)}$ with $i \geq 0$ and the relations

$$\beta_{(i)}^p = v_1^{p^i}\beta_{(i)},\ i \geq 0. \tag{7.2.9}$$

Here, the name of the classes are chosen so that they reduce to the class of the same name in $H_*(K(\mathbb{Z},2);\mathbb{F}_p)$. We see that the periodic v_1-Bockstein spectral sequence computing $K(1)_*K(\mathbb{Z},2)$ has no non-zero differentials, but has multiplicative extensions giving the relations. Thus, the connective v_1-Bockstein spectral sequence computing $k(1)_*K(\mathbb{Z},2)$ also collapses and has the same extensions, and $k(1)_*K(\mathbb{Z},2)$ is generated as an algebra over $k(1)_*$ by the same classes with the same relations.

The bar spectral sequence exists for any homology theory, since it is constructed from the bar filtration of the space $K(\mathbb{Z},3)$. However, the identification of the second page with the Tor-groups is only possible when there is a Künneth isomorphism. Here, $k(1)_*$ does not generally have a Künneth isomorphism, but since $k(1)_*K(\mathbb{Z},2)$ is torsion free over $k(1)_*$, it has one for products of $K(\mathbb{Z},2)$, thus for the bar construction of $K(\mathbb{Z},3)$, and we have identified the E^2-page.

The $k(1)_*$-algebra $k(1)_*K(\mathbb{Z},2)$ splits as

$$k(1)_*K(\mathbb{Z},2) = \bigotimes_{i\geq 0} k(1)_*[\beta_{(i)}]\big/(\beta_{(i)}^p - v_1^{p^i}\beta_{(i)}) \tag{7.2.10}$$

where all the tensor products are over $k(1)_*$. Thus, if we prove that for each $i \geq 0$, the augmented complex

$$\begin{aligned}
0 \leftarrow k(1)_* \leftarrow & \ k(1)_*[\beta_{(i)}]\big/(\beta_{(i)}^p - v_1^{p^i}\beta_{(i)}) \otimes E(\sigma\beta_{(i)}) \otimes \Gamma(\phi\beta_{(i)}) \\
& d(\sigma\beta_{(i)}) = \beta_{(i)} \\
& d(\gamma_{k+1}\phi\beta_{(i)}) = (\beta_{(i)}^{p-1} - v_1^{p^i})\sigma\beta_{(i)}\gamma_k\phi\beta_{(i)},\ k \geq 0
\end{aligned} \tag{7.2.11}$$

is exact, then we have constructed a resolution of $k(1)_*$ as a free $k(1)_*K(\mathbb{Z},2)$-module.

In that complex, in odd degrees, the differentials are given by

$$d(\eta\sigma\beta_{(i)}\gamma_k\phi\beta_{(i)}) = \eta\beta_{(i)}\gamma_k\phi\beta_{(i)} \tag{7.2.12}$$

for any $\eta \in k(1)_*K(\mathbb{Z},2)$. This is zero if and only if $\eta\beta_{(i)} = 0$, that is

$$\eta = (\beta_{(i)}^{p-1} - v_1^{p^i})\eta' \tag{7.2.13}$$

for some $\eta' \in k(1)_*K(\mathbb{Z},2)$. In that case,

$$d(\eta'\gamma_{k+1}\phi\beta_{(i)}) = \eta\sigma\beta_{(i)}\gamma_k\phi\beta_{(i)}. \tag{7.2.14}$$

In even degrees, the differentials are given by

$$d(\eta\gamma_{k+1}\phi\beta_{(i)}) = \eta(\beta_{(i)}^{p-1} - v_1^{p^i})\sigma\beta_{(i)}\gamma_k\phi\beta_{(i)} \tag{7.2.15}$$

which is zero if and only if $\eta = \beta_{(i)}\eta'$ and in that case,

$$d(\eta'\sigma\beta_{(i)}\gamma_{k+1}\phi\beta_{(i)}) = \eta\gamma_{k+1}\phi\beta_{(i)} \tag{7.2.16}$$

Then, we have a resolution as claimed. After tensoring with $k(1)_*$ over $k(1)_*K(\mathbb{Z},2)$, the non-zero differentials are the

$$d(\gamma_{k+1}\phi\beta_{(i)}) = -v_1^{p^i}\sigma\beta_{(i)}\gamma_k\phi\beta_{(i)} \tag{7.2.17}$$

which yields the results, since our resolution of $k(1)_*$ is compatible with the resolution $\mathbb{F}_p$ and the map $k(1)_* \to \mathbb{F}_p$. □

Proposition 7.2.18. *In the v_1-Bockstein spectral sequence computing $K(1)_*K(\mathbb{Z},3)$, the differentials are given by the formula*

$$d^{p^i}(\gamma_k\phi\beta_{(i)}) = v_1^{p^i}\sigma\beta_{(i)}\gamma_{k-1}\phi\beta_{(i)} \tag{7.2.19}$$

up to a unit, with $i, k \geq 0$.

The differentials are given by the same formula in the v_1-Bockstein spectral sequence computing $k(1)_*K(\mathbb{Z},3)$.

Proof. Our periodic v_1-Bockstein spectral sequence is an Atiyah-Hirzebruch spectral sequence for a space (see chapter 3), and thus a spectral sequence of algebras from lemma 4.3.9. But, from the naturality of the diagonal map, it is also a spectral sequence of coalgebras, and thus a spectral sequence of Hopf algebras.

The connective v_1-Bockstein spectral sequence has by definition the same differentials. However, we can use lemma 7.2.7 to constraint its target: we know that for any $i \geq 0$, if there exists an antecedent of $\sigma\beta_{(i)} \in H_*(K(\mathbb{Z},3);\mathbb{F}_p)$ through the map $k(1)_*K(\mathbb{Z},3) \to H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, then it is of $v_1^{p^i}$ torsion, so that if $\sigma\beta_{(i)}$ survives to the connective v_1-Bockstein spectral sequence, then it must be of a torsion smaller than $v_1^{p^i}$ in its E^∞-page.

We will work by induction on i. Because of theorem 7.2.3, all the classes except the units in the periodic sequence must disappear somehow, and the differentials claimed are sufficient to do that; if we prove they are present, there can be no further differentials.

First, $\sigma\beta_{(0)}$ cannot be the source of a differential because of its degree, and thus must receive a differential

$$d^{2p-2}(x) = v_1\sigma\beta_{(0)}. \tag{7.2.20}$$

Since $\sigma\beta_{(0)}$ is in the lowest degree that can receive a differential, x must be indecomposable. The only possibility is that x is $\phi\beta_{(0)}$ up to a unit, so that

$$d^{2p-2}(\phi\beta_{(0)}) = v_1\sigma\beta_{(0)} \tag{7.2.21}$$

To get the result for all the divided power $\gamma_k\phi\beta_{(0)}$, we work with the coproduct in the same fashion as in the proof of proposition 7.1.34 for the Bockstein. For each $n \geq 1$, $d^1(\gamma_{p^n}\phi\beta_{(0)}) - v_1\gamma_{p^n-1}\phi\beta_{(0)}$ is primitive and in degree $2p^{n+1}+2p^n-1$, thus zero, and the result follows.

Now assume that up to a unit

$$d^{p^j}(\gamma_k\phi\beta_{(j)}) = v_1^{p^j}\sigma\beta_{(j)}\gamma_{k-1}\phi\beta_{(j)} \tag{7.2.22}$$

is true for all $k \geq 0$ and j such that $0 \leq j < i$. Consider the element $\sigma\beta_{(i)}$ in degree $2p^i+1$. The differentials are already determined for all element in degrees between 1 and $2p^{i+1}$. If $\sigma\beta_{(i)}$ is the source of a differential, its target must be a coalgebra primitive in even degree, that is one of the $\phi\beta_{(j)}$ with $j \leq i-2$. But these classes are already determined to hold differentials. Thus, $\sigma\beta_{(i)}$ survive to the $E^{2(p-1)p^i-1}$-page, and is the non-unit class of lowest degree in that page, so that if it is the target of a differential, it must be from an indecomposable and before the $E^{2(p-1)p^i+1}$-page. The only possibility is that up to a unit,

$$d^{2(p-1)p^i}(\phi\beta_{(i)}) = v_1^{p^i}\sigma\beta_{(i)}. \tag{7.2.23}$$

The rest of the divided power of $\phi\beta_{(i)}$ follow as in the $i=0$ case. □

Since $k(1) = V(0)\wedge \ell$, the map $\ell \to ku$ sending v_1 to u^{p-1} allow us to conclude the following:

Corollary 7.2.24. *In the u-Bockstein spectral sequence computing $(V(0)\wedge ku)_*K(\mathbb{Z},3)$, the differentials are given by the formula*

$$d^{(p-1)p^i}(\gamma_k\phi\beta_{(i)}) = u^{(p-1)p^i}\sigma\beta_{(i)}\gamma_{k-1}\phi\beta_{(i)} \tag{7.2.25}$$

up to a unit, with $i, k \geq 0$.

7.2.2 The connective complex K-theory of $K(\mathbb{Z},3)$

To deduce the integral result from the modulo p one, we use the connecting map $\delta : V(0)\wedge ku \to \Sigma ku$ of the cofiber sequence of the multiplication by p. That map reduces modulo u to the already similarly denoted $\delta : H\mathbb{F}_p \to \Sigma H\mathbb{Z}_{(p)}$, so that we have a morphism of spectral sequences:

$$\begin{array}{ccc} H_*(K(\mathbb{Z},3);\mathbb{F}_p)\,\bar{\otimes}\,P(u) & \Rightarrow & (V(0)\wedge ku)_*K(\mathbb{Z},3) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})\,\bar{\otimes}\,P(u) & \Rightarrow & ku_*K(\mathbb{Z},3) \end{array} \tag{7.2.26}$$

In the integral spectral sequence, the non-torsion generators 1 and $\sigma\beta_{(0)}$ cannot support differentials for degree reasons, and cannot be the target of a differential coming from the torsion. Thus, it remains only to compute the differentials internal to the torsion, which is entirely in the image of δ. The difficulty will then be to keep track of the different names given to a single element by writing it as an image $\delta(x)$; that is, to keep track of $\mathrm{Ker}(\delta) = \mathrm{Ker}(\beta^1)$.

We will use the following notation to denote all the products of the algebra generators of $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, i.e. all the additive generators. For I a finite subset of the non-negative integers $\mathbb{N}$, denote by σ_I the product taken in ascending order:

$$\prod_{i\in I}\sigma\beta_{(i)} = \sigma_I. \tag{7.2.27}$$

Let J be a finite multiset included in $\mathbb{N}$, that is to say an application

$$m_J : \mathbb{N} \to \mathbb{N} \tag{7.2.28}$$

whose support is finite, i.e. $m_J(n) \neq 0$ only for a finite number of n. We call $m_J(j)$ the multiplicity of j in J, and we write $j \in J$ when $m_J(j) > 0$. Generally, when taking a set operation on a multiset J, we mean taking the operation on the underlying set $\bar{J} = \{j \in J\}$, so that $\min J$ is the smallest integer n such that $m_J(n) > 0$. Denote by ϕ_J the product (taken in any order since the classes are of even degrees):

$$\prod_{j\in J}\gamma_{m_J(j)}\phi\beta_{(j)} = \phi_J. \tag{7.2.29}$$

When $j \in \mathbb{N}$, denote by $J[j^+]$ the multiset whose multiplicity function is given by

$$m_{J[j^+]} : n \mapsto \begin{cases} m_J(n) & \text{if } n \neq j \\ m_J(n)+1 & \text{if } n = j. \end{cases} \tag{7.2.30}$$

Similarly, when $j \in J$, denote by $J[j^-]$ the multiset whose multiplicity function is given by

$$m_{J[j^-]} : n \mapsto \begin{cases} m_J(n) & \text{if } n \neq j \\ m_J(n)-1 & \text{if } n = j. \end{cases} \tag{7.2.31}$$

We will also write $J[j^{++}]$ and $J[j^{--}]$ to increment or decrement by 2, and $J[j_1^+, j_2^+, \ldots]$ instead of $J[j_1^+][j_2^+]\ldots$.

The torsion elements in $H_*(K(\mathbb{Z},3,);\mathbb{Z}_{(p)})$ are then generated additively by the $\delta(\sigma_I\phi_J)$. These elements do not form a free family over $\mathbb{F}_p$, and some of them are even null:

Proposition 7.2.32. *In $H_*(K(\mathbb{Z},3);\mathbb{Z}_{(p)})$,*

$$\delta(\sigma_I\phi_J) = 0 \Leftrightarrow \forall j \in J,\, j+1 \in I. \tag{7.2.33}$$

Proof. Recall that

$$\delta(\sigma_I\phi_J) = 0 \Leftrightarrow \beta^1(\sigma_I\phi_J) = 0 \tag{7.2.34}$$

and from proposition 7.1.34 that

$$\beta^1(\sigma_I\phi_J) = \sum_{j\in J} \pm\sigma_I\sigma\beta_{(j+1)}\phi_{J[j^-]}. \tag{7.2.35}$$

The $\sigma_I\phi_J$ do form a basis of $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, and each term of the sum will vanish if and only if one the exterior classes appears two times in the product, that is if $j+1 \in I$. $\square$

We will now describe the differentials from the point of view of the targets.

Proposition 7.2.36. *In the u-Bockstein spectral sequence computing $ku_*K(\mathbb{Z},3)$, if $\delta(\sigma_I\phi_J) \neq 0$ is a $(p-1)p^{n-1}$-cycle and receive a differentials $d^{(p-1)p^n}(\delta(a)) = u^{(p-1)p^n}\delta(\sigma_I\phi_J) \neq 0$ for some $a \in H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, then we are in one of the following two cases:*

- *$n \in I$.*
- *$I \neq \emptyset$, $\min I > n$, $\min J = n-1$, $m_J(n-1) = 1$, $\forall j \in J$, $j \neq n-1 \Rightarrow j+1 \in I$ and the differentials can be realized by*

$$\begin{aligned} &d^{(p-1)p^n}\big(\delta(\sigma_{I\setminus\{i\}}\phi_{J[(i-1)^+,(n-1)^-,n^+)]}\big) \\ &\quad = u^{(p-1)p^n}\delta(\sigma_I\phi_J) \end{aligned} \tag{7.2.37}$$

  for any $i \in I$.

Proof. Assume that $n \notin I$, we need to prove that it implies that we are in the case of (7.2.37). Since δ is a morphism of spectral sequences,

$$\begin{aligned} d^{(p-1)p^n}(\delta(a)) &= u^{(p-1)p^n}\delta(\sigma_I\phi_J) \\ &\Leftrightarrow \delta(d^{(p-1)p^n}(a)) - u^{(p-1)p^n}\delta(\sigma_I\phi_J) = 0 \\ &\Leftrightarrow \beta^1(d^{(p-1)p^n}(a) - u^{(p-1)p^n}\sigma_I\phi_J) = 0 \end{aligned} \tag{7.2.38}$$

$d^{p^n}(a)$ is a sum in which $\sigma\beta_{(n)}$ can be factored, so that it can also be factored in $\beta^1(d^{p^n}(a))$. Thus all the terms of

$$\beta^1(\sigma_I\phi_J) = \sum_{\substack{j\in J \\ j+1\notin I}} \sigma_I\sigma\beta_{(j+1)}\phi_{J[j^-]} \tag{7.2.39}$$

not having $\sigma\beta_{(n)}$ as a factor must be zero, that is:

$$\forall j \in J,\ j \neq n-1 \Rightarrow j+1 \in I. \tag{7.2.40}$$

Moreover, since $\delta(\sigma_I\phi_J) \neq 0$ we know that $\beta^1(\sigma_I\phi_J) \neq 0$, and then $n-1 \in J$, otherwise all the terms of the sum are zero.

We now consider the different cases in comparing $\min I$, $\min J$ and n.

- Assume that $\min I < n$ and $\min I \leq \min J$. Then:

$$\begin{aligned} &\forall k < \min I,\ d^{(p-1)p^k}(\delta(\sigma_{I\setminus\{\min I\}}\phi_{J[\min I^+]})) = 0 \\ &d^{(p-1)p^{\min I}}(\delta(\sigma_{I\setminus\{\min I\}}\phi_{J[\min I^+]})) \\ &\quad = \pm u^{(p-1)p^{\min I}}\delta(\sigma_I\phi_J) \end{aligned} \tag{7.2.41}$$

  so that the differential $d^{(p-1)p^n}(\delta(a))$ we were considering must be zero.

- Assume that $\min J < \min I < n$. Then $\min I = \min J + 1$ since $n-1 \in J$, and
$$\begin{aligned} &\forall k < \min J,\ d^{(p-1)p^k}(\delta(\sigma_I\phi_J)) = 0 \\ &d^{(p-1)p^{\min J}}(\delta(\sigma_I\phi_J)) \\ &\qquad = \pm u^{(p-1)p^{\min J}}\delta(\sigma_I\sigma\beta_{(\min J)}\phi_{J[\min J^-]}) \\ &\qquad \neq 0 \end{aligned} \tag{7.2.42}$$
since $n-1 \in J[\min J^-]$ and $n \notin I$. Thus $\delta(\sigma_I\phi_J)$ is not a p^{n-1}-cycle.

- Assume that $\min I > n$. Then because of (7.2.40), $\min(J \setminus \{n-1\}) \geq n$ and $\min J = n-1$. We now refine this disjunction.

- Assume that $\min I > n$ and $m_J(n-1) \geq 2$. Then:
$$\begin{aligned} &\forall k < n-1,\ d^{(p-1)p^k}(\delta(\sigma_I\phi_J)) = 0 \\ &d^{(p-1)p^{n-1}}(\delta(\sigma_I\phi_J)) \\ &\qquad = \pm u^{(p-1)p^{n-1}}\delta(\sigma_I\sigma\beta_{(n-1)}\phi_{J[(n-1)^-]}) \\ &\qquad \neq 0 \end{aligned} \tag{7.2.43}$$
since $n-1 \in J[(n-1)^-]$ and $n \notin I \cup \{n-1\}$. Thus $\delta(\sigma_I\phi_J)$ is not a p^{n-1}-cycle.

- Finally, assume that $\min I > n$ and $m_J(n-1) = 1$. The previous formula is also valid, but this time $d^{(p-1)p^{n-1}}(\delta(\sigma_I\phi_J)) = 0$.

  If $I = \emptyset$, then (7.2.40) implies that $\sigma_I\phi_J = \phi\beta_{(n-1)}$, but then $\beta^1(\phi\beta_{(n-1)}) = \sigma\beta_{(n)}$ which is not a term in $\beta^1(d^{(p-1)p^n}(a))$, since those have all at least two different σ as a factor. Thus, the $\delta(\phi\beta_{(n-1)})$ are infinite cycles whose u-towers cannot be the target of a differential, and hereafter $I \neq \emptyset$.

  Now for any $i \in I$, $i \neq 0$ since $\min I > n \geq 0$, and we have:
$$\begin{aligned} &\beta^1(\sigma_{I\setminus\{i\}}\phi_{J[(i-1)^+]}) \\ &\qquad = \pm\sigma_{I\setminus\{i\}\cup\{n\}}\phi_{J[(i-1)^+,\,(n-1)^-]} \pm \sigma_I\phi_J \end{aligned} \tag{7.2.44}$$
  and there is indeed a differential
$$\begin{aligned} &d^{(p-1)p^n}(\delta(\sigma_{I\setminus\{i\}}\phi_{J[(i-1)^+,\,(n-1)^-,\,n^+]})) \\ &\quad = \pm u^{(p-1)p^n}\delta(\sigma_{I\setminus\{i\}\cup\{n\}}\phi_{J[(i-1)^+,\,(n-1)^-]}) \\ &\quad = \pm u^{(p-1)p^n}\delta(\sigma_I\phi_J) \end{aligned} \tag{7.2.45}$$
  as we claimed.

□

We can also partially describe the differentials from the point of view of the sources. When $\min J < \min I$, the first potentially non-zero differential supported by $\delta(\sigma_I\phi_J)$ is $d^{(p-1)p^{\min J}}$, and we have the following:

Proposition 7.2.46. *Let I and J be such that $\delta(\sigma_I\phi_J) \neq 0$. If $\min J < \min I$ and $d^{(p-1)p^{\min J}}(\delta(\sigma_I\phi_J)) = 0$, then we are either in the case of (7.2.37) and $\delta(\sigma_I\phi_J)$ is the target of a $d^{(p-1)p^{\min J+1}}$, or we have $\sigma_I\phi_J = \phi\beta_{(\min J)}$, an infinite cycle whose u-tower is not the target of any differential.*

Proof. We have

$$\begin{aligned} &d^{(p-1)p^{\min J}}(\delta(\sigma_I\phi_J)) \\ &\quad = u^{(p-1)p^{\min J}}\delta(\sigma_I\sigma\beta_{(\min J)}\phi_{J[\min J^-]}) \end{aligned} \tag{7.2.47}$$

and $\beta^1(\sigma_I\sigma\beta_{(\min J)}\phi_{J[\min J^-]})$ can be seen to be zero only in the cases we claimed when $\beta^1(\sigma_I\phi_J) \neq 0$.

The claim about $\sigma\beta_{(\min J)}$ was already established in the proof of proposition 7.2.36. □

We are now able to describe the E^∞ page of our spectral sequence. When $\min J < \min I$ or $I = \emptyset$, $\delta(\sigma_I\phi_J) \neq 0$ support a non-zero differential $d^{(p-1)p^{\min J}}$, or we are in the case of proposition 7.2.46, and then we have either an infinite cycle $\phi\beta_{(\min J)}$ which is not of u-torsion, or we have the differential of (7.2.47). But we saw in (7.2.45) in the proof of proposition 7.2.36 that the target of this differential can be rewritten as

$$\delta(\sigma_{I\setminus\{i\}\cup\{\min J+1\}}\phi_{J[(i-1)^+,\,(\min J)^-]}) \tag{7.2.48}$$

for any $i \in I$, which is a class with $\min(I \setminus \{i\} \cup \{\min J + 1\}) \leq \min(J[(i-1)^+, (\min J)^-])$ so that we won't need $\delta(\sigma_I\phi_J)$ in our following description of E^∞. Conversely, when $I \neq \emptyset$ and $\min I \leq \min J$, $\delta(\sigma_I\phi_J)$ is the target of the differential:

$$\begin{aligned} &d^{(p-1)p^{\min I}}(\delta(\sigma_{I\setminus\{\min I\}}\phi_{J[\min I^+]})) \\ &\quad = u^{(p-1)p^{\min I}}\delta(\sigma_I\phi_J) \end{aligned} \tag{7.2.49}$$

the u-tower of $\delta(\sigma_I\phi_J)$ is of infinite cycles, and we get an u-torsion class in the E^∞ page. Thus, we have proved:

Theorem 7.2.50. *In the E^∞ page of the u-Bockstein spectral sequence computing $ku_*K(\mathbb{Z},3)$, the non u-torsion part is generated as a $P(u)$-module by 1 and $\sigma\beta_{(0)}$, which give two copies of $\mathbb{Z}_{(p)}$, and by the $\delta(\phi\beta_{(n)})$ for all $n \geq 0$, which give copies of $\mathbb{F}_p$. The u-torsion part is generated as a $P(u)$-module by the $\delta(\sigma_I\phi_J)$ with $I \neq \emptyset$ and $\min I \leq \min J$, which all give copies of $\mathbb{F}_p$.*

There are some relations between the generators given for the torsion: the torsion submodule is the free $\mathbb{F}_p$-module over the $\delta(\sigma_I\phi_J)$ with $I \neq \emptyset$ and $\min I \leq \min J$, quotiented by $\mathrm{Ker}(\beta^1)$.

Degreewise, there can be extension both in the torsion and non-torsion part. We will now prove that, as for $\mathrm{THH}_*(ku)$, the p-torsion and the u-torsion are the same because of some of the extensions.

Proposition 7.2.51. *In $ku_*K(\mathbb{Z},3)$, there are relations*

$$\begin{aligned} p \cdot \delta(\phi\beta_{(0)}) &= u^{p-1}\sigma\beta_{(0)} \\ p \cdot \delta(\phi\beta_{(i)}) &= u^{(p-1)p^i}\delta(\phi\beta_{(i-1)}) \end{aligned} \tag{7.2.52}$$

for any $i \geq 1$.

Proof. Our computation of the connective u-Bockstein spectral sequence also determines the periodic u-Bockstein spectral sequence computing $KU_*K(\mathbb{Z},3)$, whose E^∞ page is generated as a $P(u,u^{-1})$-module by 1, $\sigma\beta_{(0)}$ and the $\delta(\phi\beta_{(i)})$. All our differentials comes from the v_1-Bockstein spectral sequence computing $L_*K(\mathbb{Z},3)$, so that its E^∞ page is generated as a $P(v_1,v_1^{-1})$-module by the same classes. However, reducing modulo p, we have:

$$(L\wedge V(0))_*K(\mathbb{Z},3)\cong K(1)_*K(\mathbb{Z},3)\cong K(1)_* \tag{7.2.53}$$

and the only possibility to get that from the E^∞ page is having the extensions

$$\begin{aligned} p\delta(\phi\beta_{(0)}) &= v_1\sigma\beta_{(0)} \\ p\delta(\phi\beta_{(i)}) &= v_1^{p^i}\delta(\phi\beta_{(i-1)}) \end{aligned} \tag{7.2.54}$$

so that

$$L_*K(\mathbb{Z},3)\simeq L\vee\Sigma^3L_{\mathbb{Q}}. \tag{7.2.55}$$

These extensions induces the one we claimed over ku. □

Here, we see a phenomenon – p-torsion and the u-torsion coincide – that occur similarly in $\mathrm{THH}_*(ku)$. From the E^∞ of the u-Bockstein spectral sequence, we now in both case that the u-torsion is included into the p-torsion. The formula for the periodic case then implies the converse, because inverting u get rid of the p-torsion too.

It is once again possible to describe the non-torsion part for integral ku as in proposition 5.7.16.

Proposition 7.2.56. *The non-torsion part $ku_*K(\mathbb{Z},3)$ includes a tower $\mathbb{Z}[u]$ generated by $\sigma\beta_{(0)}$ where for each $n\geq 1$, $u^{n-1}\sigma\beta_{(0)}$ is divisible by least common multiple of the integers 1, 2, …, n. That is, the non-torsion part is*

$$\mathbb{Z}[u]\{1\}\oplus\mathcal{Q} \tag{7.2.57}$$

where $\mathcal{Q}$ is the sub-$\mathbb{Z}$-module of $\mathbb{Q}[u]\{\sigma\beta_{(0)}\}$ generated by the

$$\frac{u^{n-1}\sigma\beta_{(0)}}{\mathrm{lcm}(1,\,2,\,\dots,\,n)} \tag{7.2.58}$$

for $n\geq 1$.

7.3 The trace of $K(\mathbb{Z},3)$ in $\mathrm{THH}(ku)$

We will compute partially the algebra map

$$f: ku\wedge K(\mathbb{Z},3)_+\to\mathrm{THH}(ku) \tag{7.3.1}$$

that factorizes the composition of the inclusion of part of the units in algebraic K-theory with the Bökstedt trace map

$$\begin{tikzcd} \Sigma^\infty_+K(\mathbb{Z},3) \arrow[r] \arrow[dr] & K(ku) \arrow[r] & \mathrm{THH}(ku) \\ & ku\wedge K(\mathbb{Z},3)_+ \arrow[ur, "f"] & \end{tikzcd} \tag{7.3.2}$$

as we have seen at the beginning of the chapter.

Our map reduces to maps we will also call f

$$\begin{aligned} H\mathbb{Z}_{(p)} \wedge K(\mathbb{Z},3)_+ &\to \mathrm{THH}(ku; H\mathbb{Z}_{(p)}) \\ H\mathbb{F}_p \wedge K(\mathbb{Z},3)_+ &\to \mathrm{THH}(ku; H\mathbb{F}_p) \end{aligned} \tag{7.3.3}$$

and we will study it from theses and the p and u Bockstein spectral sequences. The reduced maps are also algebra maps.

Our model for the modulo p map will be induced by

$$\Sigma^\infty_+ K(\mathbb{Z},2) \to ku \tag{7.3.4}$$

adjunct to the map

$$K(\mathbb{Z},2) \to GL_1(ku). \tag{7.3.5}$$

The spectral sequence from proposition 4.3.12 is in this case

$$\mathrm{Tor}^{H_*(K(\mathbb{Z},2);\mathbb{F}_p)}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \mathrm{THH}_*(\Sigma^\infty_+ K(\mathbb{Z},2); H\mathbb{F}_p). \tag{7.3.6}$$

That spectral sequence collapses for degree reasons, and since they have the same first page, it is formally the same as the bar spectral sequence computing $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$; we have an isomorphism

$$THH_*(\Sigma^\infty_+ K(\mathbb{Z},2); \mathbb{F}_p) \cong H_*(K(\mathbb{Z},3);\mathbb{F}_p). \tag{7.3.7}$$

Let us compare it with the same spectral sequence computing ku:

$$\mathrm{Tor}^{H_*(ku;\mathbb{F}_p)}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \mathrm{THH}_*(ku; H\mathbb{F}_p). \tag{7.3.8}$$

It is known (see for example [5]) that

$$H_*(ku;\mathbb{F}_p) \cong P_{p-1}(x) \otimes H_*(\ell;\mathbb{F}_p) \tag{7.3.9}$$

and the homology of ℓ can be written as a sub-Hopf algebra of the dual Steenrod algebra

$$H_*(\ell;\mathbb{F}_p) \cong E(\tau_i,\, i \geq 2) \otimes P(\xi_i,\, i \geq 1) \tag{7.3.10}$$

so that our spectral sequence begin with

$$\mathrm{Tor}^{H_*(ku;\mathbb{F}_p)}_{*,*}(\mathbb{F}_p, \mathbb{F}_p) \cong E(\sigma x) \otimes E(\sigma\xi_i,\, i \geq 1) \otimes \Gamma(\phi x) \otimes \Gamma(\sigma\tau_i,\, i \geq 2) \tag{7.3.11}$$

with bidegrees

$$\begin{aligned} |\sigma x| &= (1,2) \\ |\sigma\xi_i| &= (1, 2p^i - 2) \\ |\phi x| &= (2, 2p-2) \\ |\sigma\tau_i| &= (2, 2p^i - 1). \end{aligned} \tag{7.3.12}$$

Proposition 7.3.13.

$$\mathrm{THH}_*(ku; H\mathbb{F}_p) \cong E(\sigma x, \lambda_1) \otimes P(\mu_1) \tag{7.3.14}$$

with degrees

$$\begin{aligned} |\sigma x| &= 3 \\ |\lambda_1| &= 2p-1 \\ |\mu_1| &= 2p \end{aligned} \tag{7.3.15}$$

and in the previous spectral sequence, σx is represented by σx, λ_1 by $\sigma\xi_1$, μ_1 by ϕx and $\mu_1^{p^i}$ by $\sigma\tau_{i+1}$ when $i \geq 1$.

Proof. The claim about $\mathrm{THH}_*(ku; H\mathbb{F}_p) \cong V(0)_* \mathrm{THH}(ku; H\mathbb{Z}_{(p)})$ is theorem 6.8 of [5]

For degree reasons, σx is indeed σx, λ_1 must be $\sigma\xi_1$ and μ_1 must be ϕx. Then there must be a multiplicative extension since $\phi x^p = 0$ and $\mu_1^p \neq 0$. The only suitable class in lower filtration is $\sigma\tau_2$. Then for the same reasons, for each $i \geq 1$, $\mu_1^{p^i}$ is represented by $\sigma\tau_{i+1}$. ☐

This allows us to describe f on the multiplicative generators of $H_*(K(\mathbb{Z},3); \mathbb{F}_p)$.

Proposition 7.3.16. *In* $\mathrm{THH}_*(ku; H\mathbb{F}_p)$ *we have*

$$\begin{aligned} &f(\sigma\beta_{(0)}) = \sigma x \\ &f(\sigma\beta_{(i)}) = 0 \textit{ for any } i \geq 1 \\ &f(\phi\beta_{(i)}) = 0 \textit{ for any } i \geq 0 \\ &f(\gamma_{p^k}\phi\beta_{(i)}) = 0 \textit{ or } \sigma x \lambda_1 \mu_1^{p^{i+k}+p^{k-1}-1} \textit{ for any } i \geq 0 \textit{ and } k \geq 1. \end{aligned} \tag{7.3.17}$$

Proof. We denoted earlier

$$\mathrm{Tor}_{*,*}^{H_*(ku;\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \cong \bigotimes_{i \geq 0} E(\sigma\beta_{(i)}) \otimes \Gamma(\phi\beta_{(i)}) \tag{7.3.18}$$

with bidegrees

$$\begin{aligned} |\sigma\beta_{(i)}| &= (1, 2p^i) \\ |\phi\beta_{(i)}| &= (2, 2p^{i+1}). \end{aligned} \tag{7.3.19}$$

The elements $\beta_{(0)} \in H_2(K(\mathbb{Z},2); \mathbb{F}_p)$ is sent to $x \in H_2(ku; \mathbb{F}_p)$, thus $f(\sigma\beta_{(0)}) = \sigma x$. For bidegree reasons, whenever $j \geq 1$ and $i \geq 0$ the images by f of $\sigma\beta_{(j)}$ and $\phi\beta_{(i)}$ must be zero. Whenever $i \geq 0$ and $k \geq 1$ the $\gamma_{p^k}\phi\beta_{(i)}$ are generating truncated polynomial algebras, thus their images can only be zero or a square-zero class. The only possible square-zero class is

$$f : \gamma_{p^k}\phi\beta_{(i)} \stackrel{?}{\longmapsto} \sigma x \lambda_1 \mu_1^{p^{i+k}+p^{k-1}-1} \tag{7.3.20}$$

because of the degree. ☐

Now we are able to describe f on part of the torsion of $H_*(K(\mathbb{Z},3); \mathbb{Z}_{(p)})$ and on the class in degree 3.

Proposition 7.3.21. *In* $\mathrm{THH}_*(ku; H\mathbb{Z}_{(p)})$, *we have in the torsion*

$$f(\delta(\sigma\phi_{(i)})) = 0 \tag{7.3.22}$$

for any $i \geq 0$ *and*

$$f(\delta(\sigma_I \phi_J)) = 0 \tag{7.3.23}$$

for any I *and* J *such that* I *is neither empty nor* $\{0\}$.

The non torsion element $\sigma\beta_{(0)}$ *in degree 3 has image* σu *by* f.

Proof. We need to describe the exact couple

$$\begin{tikzcd} \mathrm{THH}_*(ku;H\mathbb{Z}_{(p)}) \arrow[rr, "m"] & & \mathrm{THH}_*(ku;H\mathbb{Z}_{(p)}) \arrow[dl, "\pi"] \\ & \mathrm{THH}_*(ku;H\mathbb{F}_p) \arrow[ul, "\delta"] & \end{tikzcd} \tag{7.3.24}$$

Here the map δ is reducing the degree by 1. Recall that

$$\mathrm{THH}_*(ku;H\mathbb{Z}_{(p)}) \cong E(\sigma u) \otimes \mathrm{THH}_*(H\mathbb{Z}_{(p)}) \tag{7.3.25}$$

since our prime p is odd. The map m is multiplication by p and we can choose our generator such that:

$$\begin{aligned} \pi(\sigma u) &= \sigma x \\ \pi(\mu_{kp}) &= \lambda_1 \mu_1^{k-1} \\ \delta(\mu_1^k) &= \nu(k)\mu_{kp} \end{aligned} \tag{7.3.26}$$

where ν is the p-adic valuation.

Since there is a commutative diagram

$$\begin{tikzcd} H_*(K(\mathbb{Z},3);\mathbb{F}_p) \arrow[r, "f"] \arrow[d, "\delta"] & \mathrm{THH}_*(ku;H\mathbb{F}_p) \arrow[d, "\delta"] \\ H_{*-1}(K(\mathbb{Z},3);\mathbb{Z}_{(p)}) \arrow[r, "f"] & \mathrm{THH}_{*-1}(ku;H\mathbb{Z}_{(p)}) \end{tikzcd} \tag{7.3.27}$$

we can see that whether equation (7.3.20) is true or not, for all integers i and k,

$$f(\delta(\gamma_{p^k}\phi\beta_{(i)})) = 0. \tag{7.3.28}$$

We also have

$$f(\delta(\sigma\beta_{(i)})) = 0 \tag{7.3.29}$$

since $\delta(\sigma\beta_{(i)}) = 0$, so that we now the map $f\delta$ on every multiplicative generators, and δ have a sufficient multiplicative property to determine f on part the torsion of $H_*(K(\mathbb{Z},3);H\mathbb{Z}_{(p)})$. This time, we work in the exact couple

$$\begin{tikzcd} H_*(K(\mathbb{Z},3);H\mathbb{Z}_{(p)}) \arrow[rr, "m"] & & H_*(K(\mathbb{Z},3);H\mathbb{Z}_{(p)}) \arrow[dl, "\pi"] \\ & H_*(K(\mathbb{Z},3);H\mathbb{F}_p) \arrow[ul, "\delta"] & \end{tikzcd} \tag{7.3.30}$$

in which we have chosen to represent the torsion by the image of δ. But in our case, since there is no higher p-Bockstein, the torsion is also isomorphic via π to the image of β^1, and π is an algebra map. Thus, for a and b in $H_*(K(\mathbb{Z},3);\mathbb{F}_p)$, we have:

$$\begin{aligned} \pi(\delta(a\beta^1(b))) &= \beta^1(a\beta^1(b)) \\ &= \beta^1(a)\beta^1(b) \\ &= \pi(\delta(a))\pi(\delta(b)) \\ &= \pi(\delta(a)\delta(b)) \end{aligned} \tag{7.3.31}$$

so that when composing with π^{-1},

$$\delta(a\beta^1(b)) = \delta(a)\delta(b). \tag{7.3.32}$$

When I is neither empty nor $\{0\}$, we can choose some element $i \geq 1$ in I, and then

$$\begin{aligned}\delta(\sigma_I\phi_J) &= \pm\delta(\sigma_{I\setminus\{i\}}\phi_J\beta^1(\phi\beta_{(i-1)})) \\ &= \pm\delta(\sigma_{I\setminus\{i\}}\phi_J)\delta(\phi\beta_{(i-1)})\end{aligned} \tag{7.3.33}$$

and thus $f(\delta(\sigma_I\phi_J)) = 0$.

Finally, the claim about the class of degree 3 is true modulo p, so it must be true up to the p-torsion, which is null in degree 3. □

Because of the extensions in $ku_*K(\mathbb{Z},3)$, this is enough to determines f on the non-torsion part. For the torsion, the way it is generated additively reduces the number of classes whose image by f we cannot say anything about, because the torsion is generated by the $\delta(\sigma_I\phi_J)$ such that I is not empty and $\min I \leq \min J$. The following comes from comparing the u-Bockstein spectral sequences computing $ku_*K(\mathbb{Z},3)$ and $\mathrm{THH}_*(ku)$.

Theorem 7.3.34. *In* $\mathrm{THH}_*(ku)$*, in the non-torsion part,*

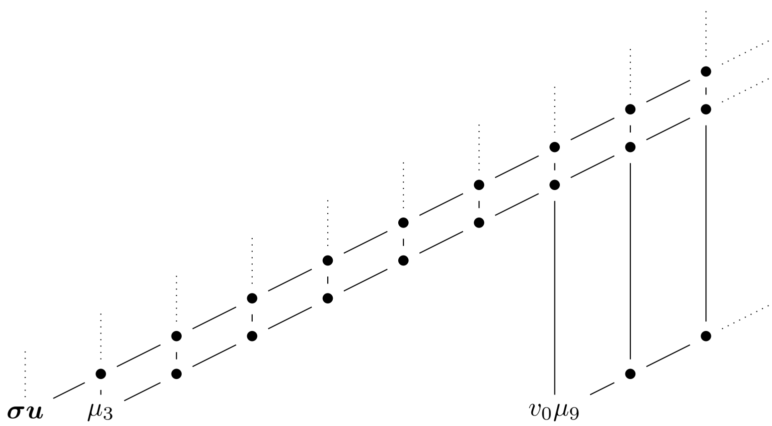
$$\begin{aligned} f(\sigma\beta_{(0)}) &= \sigma u \\ f(\delta(\phi\beta_{(i)})) &= uv_0^{p^i}\mu_{p^{i+1}} \end{aligned} \tag{7.3.35}$$

for any $i \geq 0$.

In the torsion, for any I *and* J *such that* I *is not neither empty nor* $\{0\}$*, and* $\min I \leq \min J$*,* $f(\delta(\sigma_I\phi_J))$ *is divisible by* u.

We still cannot say anything about $f(\delta(\sigma\beta_{(0)}\phi_J))$.

We can draw the tower above σu in $\mathrm{THH}_*(ku)$ for $p = 3$, with multiplication by p going up and multiplication by u going up and right.



Here the class σu in bold and the classes represented by • are in the image of f, but the rest of the named classes are not.

7.4 A remark on ku_* as a $ku_*K(\mathbb{Z},2)$-module

In this section we present some difficulties we had at computing a resolution of ku_* as a $ku_*K(\mathbb{Z},2)$-module. The goal was to compute $\mathrm{Tor}_{*,*}^{ku_*K(\mathbb{Z},2)}(ku_*, ku_*)$, as a first step for the bar spectral sequence computing $ku_*K(\mathbb{Z},3)$. We will first review some facts about the algebra $ku_*K(\mathbb{Z},2)$, and then fail at providing a proper resolution.

7.4.1 The algebra structure on $ku_*K(\mathbb{Z},2)$

As a complex-oriented homology theory, the connective complex K-theory ku of $\mathbb{CP}^\infty \cong K(\mathbb{Z},2)$ has an algebra structure determined by its formal group law. We recall the following facts, which can be found in [30] or [1]

Lemma 7.4.1 (3.3 of [30])**.** *When E is a complex-oriented homology theory, with complex orientation given by $x^E \in E^2\mathbb{CP}^\infty$, then in cohomology:*

- *$E^*\mathbb{CP}^\infty \cong E^*[[x^E]]$ the power series on x^E over E^*.*
- *$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*\mathbb{CP}^\infty \otimes_{E^*} E^*\mathbb{CP}^\infty$.*

In homology:

- *$E_*\mathbb{CP}^\infty$ is E_* free on $\beta_i \in E_{2i}\mathbb{CP}^\infty$ for $i \geq 0$, dual to x^i.*
- *$E_*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E_*\mathbb{CP}^\infty \otimes_{E_*} E_*\mathbb{CP}^\infty$.*

Moreover:

- *The diagonal $\mathbb{CP}^\infty \to \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ induces a coproduct ψ on $E_*\mathbb{CP}^\infty$ with $\psi(\beta_n) = \sum_{i=0}^n \beta_i \otimes \beta_{n-i}$.*
- *The H-space product $m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ induces a coproduct m^* on $E^*\mathbb{CP}^\infty$ with $m^*(x^E) = \sum_{i,j\geq 0} a_{ij} x^i \otimes x^j$ and $a_{ij} \in E^{-2(i+j+1)} = E_{2(i+j+1)}$.*
- *$F(y,z) = y +_F z = \sum_{i,j\geq 0} a_{ij} y^i z^j$ is a commutative associative formal group law over E^*, i.e.*

$$F(y,z) = F(z,y) \quad F(y,0) = y \quad F(y,F(z,w)) = F(F(y,z),w). \tag{7.4.2}$$

- *In the power series ring $E_*\mathbb{CP}^\infty[[s,t]]$,*

$$\beta(s)\beta(t) = \beta(s +_F t) \tag{7.4.3}$$

  where $\beta(r) = \sum_{i\geq 0} \beta_i r^i$ and the product is the H-space product.

- *Define $[1]_F(s) = s$ and inductively $[n]_F(s) = [n-1]_F(s) +_F s$, then*

$$\beta(s)^n = \beta([n]_F(s)). \tag{7.4.4}$$

The formal group law on KU is known

Proposition 7.4.5 (example II.2.9 of [1]). *The formal group law on KU is the multiplicative one*

$$F(y,z) = y + z + yz. \tag{7.4.6}$$

Since the map $ku^*\mathbb{CP}^\infty \to KU^*\mathbb{CP}^\infty$ is a map of Hopf algebra, the formal group law is the same on ku. Remark that it is convenient not to write the Bott element in the formal group law, which should really be written as

$$F(y,z) = uy + uz + yz \tag{7.4.7}$$

In what follows, we will not write the Bott element either; every computation we will do will be homogeneous, so that the suitable power of u should be inserted when needed.

We know that $ku_*K(\mathbb{Z},2)$ is a free ku_*-module over the $\beta_i \in ku_{2i}K(\mathbb{Z},2)$. We want to determine the algebra structure on $ku_*K(\mathbb{Z},2)$, that is the multiplicative relations between the β_i. This is entirely given by the formal group law. We know that the following is true in that algebra:

$$\begin{aligned}\beta(s)^2 &= \beta([2]_F(s)) \\ &= \beta(2s+s^2) \\ &= \sum_{j\geq 0}\beta_j(2s+s^2)^j \\ &= \sum_{j\geq 0}\beta_j\sum_{k=0}^{j}\binom{j}{k}2^k s^{k+2(j-k)}\end{aligned} \tag{7.4.8}$$

so when computing the coefficient of s^n of each term, we have the equality

$$\sum_{i=0}^{n}\beta_i\beta_{n-i} = \sum_{\frac{n}{2}\leq j\leq n}\binom{j}{2j-n}2^{2j-n}\beta_j \tag{7.4.9}$$

for any $n \geq 0$. The element β_0 is what we usually call 1, so now we see that these equations determine the β_n inductively from β_1, since on the left-hand side the coefficient of β_n is 2, and on the right-hand side it is 2^n.

Proposition 7.4.10. *In $ku_*K(\mathbb{Z},2)$, the following equivalent formulas are true:*

$$\begin{aligned}\beta_n &= \frac{1}{n!}\prod_{i=0}^{n-1}(\beta_1 - i) \\ \beta_1\beta_n &= n\beta_n + (n+1)\beta_{n+1} \\ \beta_{n+1} &= \frac{1}{n+1}(\beta_1 - n)\beta_n.\end{aligned} \tag{7.4.11}$$

Proof. Since the equations (7.4.9) determine the β_n inductively, we only need to prove that the formulas we propose make them hold for each $n \geq 0$. Let P_n be the polynomial in β_1 we get when inserting our first formula in the left-hand side of (7.4.9):

$$P_n(\beta_1) = \sum_{i=0}^{n}\frac{1}{i!(n-i)!}\prod_{k=0}^{i-1}(\beta_1 - k)\prod_{k=0}^{n-i-1}(\beta_1 - k). \tag{7.4.12}$$

Let Q_n be the one we get from the right-hand side:

$$Q_n(\beta_1) = \sum_{\frac{n}{2}\le j\le n} \binom{j}{2j-n} 2^{2j-n} \frac{1}{j!} \prod_{k=0}^{j-1} (\beta_1 - k). \tag{7.4.13}$$

For any positive integer k, we have

$$\begin{aligned} P_n(k) &= \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \frac{k!}{(k-i)!} \frac{k!}{(k-n+1)!} \\ &= \sum_{i=0}^{n} \binom{k}{i}\binom{k}{n-i} \end{aligned} \tag{7.4.14}$$

and

$$\begin{aligned} Q_n(k) &= \sum_{\frac{n}{2}\le j\le n} \binom{j}{2j-n} 2^{2j-n} \frac{1}{j!} \frac{k!}{(k-j)!} \\ &= \sum_{\frac{n}{2}\le j\le n} 2^{2j-n} \binom{j}{n-j}\binom{k}{j}. \end{aligned} \tag{7.4.15}$$

Combinatorially, $P_n(k)$ is the number of ways, in a set E with k elements, to choose an integers i, and to choose a subset A of E with i elements and a subset B of E with $n-i$ elements. On the other hand, $Q_n(k)$ is the number of ways to choose an integer j, to choose a subset U of E with j elements, to choose a subset V of U with $n-j$ elements, and a subset W of $U \setminus V$ (which has $j-(n-j) = 2j-n$ elements) of any cardinality. These two processes are equivalent, and we can go from one to the other by setting

$$\begin{aligned} U &= A \cup B \\ V &= A \cap B \\ W \cup V &= A \\ U \setminus W &= B \end{aligned} \tag{7.4.16}$$

so that for any integers n and k, $P_n(k) = Q_n(k)$; then $P_n(\beta_1) = Q_n(\beta_1)$, and we have proved our claim. □

From this formula we can deduce one for all the products between the β_n's. In what follows, we use the usual convention that $\binom{p}{k} = 0$ when $k < 0$ or $k > p$.

Proposition 7.4.17. *In $ku_*K(\mathbb{Z},2)$*

$$\beta_n\beta_m = \sum_{i=0}^{n} \binom{m+i}{n}\binom{n}{i}\beta_{m+i}. \tag{7.4.18}$$

The convention above makes that formula symmetrical, and the first term of the sum is really $\beta_{\max(n,m)}$.

Proof. Let m be a fixed integer. We prove the claim by induction on n. For $n = 1$, this is proposition 7.4.10. Assume that the formula for $\beta_n\beta_m$ holds. Then:

$$\begin{aligned}
&\beta_{n+1}\beta_m \\
&= \frac{1}{n+1}(\beta_1 - n)\beta_n\beta_m \\
&= \frac{1}{n+1}\sum_{i=0}^{n}\binom{m+i}{n}\binom{n}{i}((m+i)\beta_{m+i} + (m+i+1)\beta_{m+i+1} - n\beta_{m+i}) \\
&= \frac{1}{n+1}\sum_{i=0}^{n}\binom{m+i}{n}\binom{n}{i}(m+i-n)\beta_{m+i} \\
&\quad + \frac{1}{n+1}\sum_{i=1}^{n+1}\binom{m+i-1}{n}\binom{n}{i-1}(m+i)\beta_{m+i} \\
&= \frac{1}{n+1}\sum_{i=0}^{n+1}\left(\binom{m+i}{n}\binom{n}{i}(m+i-n)\right. \\
&\qquad\qquad \left. + \binom{m+i-1}{n}\binom{n}{i-1}(m+i)\right)\beta_{m+i} \\
&= \frac{1}{n+1}\sum_{i=0}^{n+1}\frac{(m+i)!}{(m+i-n-1)!}\left(\frac{n-i+1}{i!(n+1-i)!} + \frac{i}{i!(n+1-i)!}\right)\beta_{m+i} \\
&= \sum_{i=0}^{n+1}\frac{(m+i)!}{(m+i-n-1)!i!(n+1-i)!}\beta_{m+i} \\
&= \sum_{i=0}^{n+1}\binom{m+i}{n+1}\binom{n+1}{i}\beta_{m+i}
\end{aligned} \tag{7.4.19}$$

so that our formula holds for $\beta_{n+1}\beta_m$. This complete our induction. □

7.4.2 A non-resolution of ku_*

Let $\mathcal{C}$ be the augmented algebra complex presented as:

$$0 \leftarrow ku_* \leftarrow ku_*K(\mathbb{Z},2) \otimes E(\sigma\beta_n,\, n \geq 1) \otimes \Gamma(\phi_n,\, n \geq 2) \tag{7.4.20}$$

with bidegrees $|\sigma\beta_n| = (1, 2n)$ and $|\phi_n| = (2, 2n)$ and with differentials

$$\begin{aligned}
d(\sigma\beta_n) &= \beta_n \\
d(\phi_n) &= (\beta_1 - (n-1)u)\sigma\beta_{n-1} - n\sigma\beta_n.
\end{aligned} \tag{7.4.21}$$

Note that we now write the Bott element. Since in the previous section we saw that the relations

$$\beta_1\beta_n = nu\beta_n + (n+1)\beta_{n+1} \tag{7.4.22}$$

were enough to determine $ku_*K(\mathbb{Z},2)$ as an algebra, $\mathcal{C}$ is a candidate for a resolution of ku_*. This is not the case.

Proposition 7.4.23. *In $\mathcal{C}$,*

$$d(\beta_2\sigma\beta_2 - u^2\sigma\beta_2 - 6u\sigma\beta_3 - 6\sigma\beta_4) = 0 \tag{7.4.24}$$

but that element is not a boundary.

Proof. Proposition 7.4.17 implies that

$$\beta_2^2 = u^2\beta_2 + 6u\beta_3 + 6\beta_4 \tag{7.4.25}$$

in $ku_*K(\mathbb{Z}, 2)$, so that the differential of the claim is indeed zero. The element supporting that differentials is in bidegree $(1, 8)$. For it to be a boundary, we need an element in bidegree $(2, 8)$, that is a linear combination with integral coefficients of the elements:

$$\begin{gathered} u\sigma\beta_1\sigma\beta_2 \\ \beta_1\sigma\beta_1\sigma\beta_2 \\ \sigma\beta_1\sigma\beta_3 \\ u^2\phi_2 \\ u\beta_1\phi_2 \\ \beta_2\phi_2 \\ u\phi_3 \\ \beta_1\phi_3 \\ \phi_4. \end{gathered} \tag{7.4.26}$$

In order for $\sigma\beta_4$ to appear in the differential, the coefficient of ϕ_4 must not be zero. But

$$d(\phi_4) = (\beta_1 - 3u)\sigma\beta_3 - 4\sigma\beta_4 \tag{7.4.27}$$

so that the coefficient of $\sigma\beta_4$ is a multiple of 4, and cannot be 6. □

This is not the only difficulty, since we can make a similar argument for the product $\beta_n\beta_m$, where $\binom{m+n}{n}\beta_{n+m}$ appears, and $d(\phi_{m+n})$, whenever $m + n$ does not divide $\binom{m+n}{n}$. It seems to be the case a lot when n and $m + n$ are not coprime. Moreover, we can compute:

$$\begin{aligned} &d((\beta_1 + u)\phi_3 + 3\phi_4) \\ &\quad = (\beta_1 + u)(\beta_1 - 2u)\sigma\beta_2 - 3(\beta_1 + u)\sigma\beta_3 + 3(\beta_1 - 3u)\sigma\beta_3 - 12\sigma\beta_4 \\ &\quad = (u\beta_1 + 2\beta_2 - 2u\beta_1 + u\beta_1 - 2u^2)\sigma\beta_2 - 12u\sigma\beta_3 - 12\sigma\beta_4 \\ &\quad = 2\beta_2\sigma\beta_2 - 2u^2\sigma\beta_2 - 12u\sigma\beta_3 - 12\sigma\beta_4 \end{aligned} \tag{7.4.28}$$

so that if we add an element ψ to our complex such that

$$d(\psi) = \beta_2\sigma\beta_2 - u^2\sigma\beta_2 - 6u\sigma\beta_3 - 6\sigma\beta_4 \tag{7.4.29}$$

we also add the relation

$$d((\beta_1 + u)\phi_3 + 3\phi_4 - 2\psi) = 0. \tag{7.4.30}$$

Since this does not solve our problem, and that the divisibility of $\binom{m+n}{n}$ by $m+n$ is not an easy problem, this cannot be a reasonable way to obtain a resolution.

Remark that everything we said about $ku_*K(\mathbb{Z}, 2)$ is also an obstruction when substituting ku with $H\mathbb{Z}$, that is when putting $u = 0$ in all the equations.

Part III

Appendix

Appendix A

Some code to generate pictures of the torsion module of $\mathrm{THH}_*(ku)$

The following Haskell code compiles to a program that output to its standard output a minimal LaTeX document containing a tikz picture of the torsion part of $\mathrm{THH}_*(ku)$ of degree d such that $|\sigma u \mu_{p^n}| \leq d < |\sigma u \mu_{2p^n}|$. It needs to be called from the command line with exactly 5 argument:

```
$ ./prog p n u1 u2 p2
```

where p is an odd prime, $n \geq 1$ is an integer, $(u1, u2)$ is a couple of (decimal) numbers which determines the offset of multiplying by u in the plane, and p2 is a decimal number which determines the vertical offset of multiplying by p. Suggested parameters can be:

```
$ ./prog 3 3 0.5 0.5 1
```

Be wary though that tikz is a bad backend to draw big pictures, and that any attempt to up p or n too much will result in an uncompilable LaTeX document – e.g., $p = 5$ and $n = 3$ might fail depending on the sizes chosen.

```
import System.Environment
import Data.Maybe

-- return the p-adic valuation of x
valuation :: Int -> Int -> Int
valuation p x = val x 0
  where
    val y v
      | y `mod` p == 0 = val (y `div` p) (v+1)
      | otherwise      = v


-- return the smaller (right most) non zero digit of x in base p
premierChiffre :: Int -> Int -> Int
```

```
premierChiffre p x
  | x `mod` p == 0 = premierChiffre p (x `div` p)
  | otherwise      = x `mod` p


-- list of the element u^k v_0^q \sigma u \mu_{p^n+r} in the module
-- corresponding to \sigma u \mu_{p^n}
listeDesNœuds ::Int -> Int -> [(Int, (Int, Int))]
listeDesNœuds p n =
  [ (r, (q, k))
  | r <- [0, p .. p^n - p]
  , q <- [0 .. (valuation p (p^n+r)) - 1]
  , k <- [0 .. p^(valuation p (p^n+r) - q) - 3]]


-- tell if an element u^k v_0^q \sigma u \mu_{p^n+r} is a valid
-- element of the module corresponding to \sigma u \mu_{p^n}
nœudExiste :: Int -> Int -> Int -> Int -> Int -> Bool
nœudExiste p n r q k =
   r `mod` p == 0 && 0 <= r && r <= p^n - p &&
   0 <= q && q <= valuation p (p^n + r) - 1 &&
   0 <= k && k <= p^(valuation p (p^n + r) - q) - 3


-- return a list of the line needed in the drawing from the
-- node r q k, in a Maybe
-- a line is (Bool, (Node1, Node2)) going from Node1 to Node2
-- with Bool being true if the line need to be bended
traitsDuNœud :: Int -> Int -> (Int, (Int, Int))
             -> [Maybe (Bool, ((Int, (Int, Int)), (Int, (Int, Int))))]
traitsDuNœud p n (r, (q, k)) =
  [traitPossible1, traitPossible2, traitPossible3]
  where
    maybeFromBool t a = if t then Just a else Nothing
    traitPossible1 =
      maybeFromBool (nœudExiste p n r q (k + 1))
                    (False, ((r, (q, k)), (r, (q, k + 1))))
    traitPossible2 =
      maybeFromBool (nœudExiste p n r (q + 1) k)
                    (False, ((r, (q, k)), (r, (q + 1, k))))
    traitPossible3 =
      maybeFromBool (q == 0 && b == p - 1)
                    (True, ((r, (q, k)), (c - p^n,
                    (valuation p c - m - 1, k + p^(m + 1) - p^m))))
    b = premierChiffre p (p^n + r)
    c = p^n + r - (p - 1) * p^m
    m = valuation p (p^n + r)
```

```
-- return the list of all the edges to be drawn in the
-- module corresponding to \sigma u \mu_{p^n}
listeDesTraits :: Int -> Int
               -> [(Bool, ((Int, (Int, Int)), (Int, (Int, Int))))]
listeDesTraits p n = concat $ map (catMaybes . traitsDuNœud p n)
                     $ listeDesNœuds p n


-- write a string to be used as a label in the tikz drawing
écrireÉtiquette :: Int -> Int -> (Int, (Int, Int)) -> [Char]
écrireÉtiquette p n (r, (q, k)) =
  "u" ++ show k ++ "p" ++ show q ++ "m" ++ show (p^n + r)


-- write a string of latex to be shown at the node
écrireNomLatex :: Int -> Int -> (Int, (Int, Int)) -> [Char]
écrireNomLatex p n (r, (q, k))
  | k == 0 && q == 0 = "\\sigma u \\mu_{" ++ show (p^n + r) ++ "}"
  | otherwise        = "\\bullet"

-- write the line of tikz for a node
écrireNœudTikz :: Int -> Int -> (Float, Float) -> (Float, Float)
               -> (Int, (Int, Int)) -> [Char]
écrireNœudTikz p n vecteuru vecteurp no =
  "\\node[inner sep=1pt] (" ++ (écrireÉtiquette p n no) ++
  ") at " ++ (show  $ coordonnéesNœud vecteuru vecteurp no) ++
  " {$" ++ (écrireNomLatex p n no) ++ "$};"


-- coordinate of the node, p1 should really be 0
coordonnéesNœud :: (Float, Float) -> (Float, Float)
                -> (Int, (Int, Int)) -> (Float, Float)
coordonnéesNœud (u1, u2) (p1, p2) (r, (q, k)) =
  (p1 * fromIntegral q + u1 * fromIntegral k + u1 * fromIntegral r
  , p2 * fromIntegral q + u2 * fromIntegral k)


-- write the line of tikz for an edge
écrireTraitTikz :: Int -> Int
                -> (Bool, ((Int, (Int, Int)), (Int, (Int, Int))))
                -> [Char]
écrireTraitTikz p n (b, (no1, no2))
   | b =
       "\\draw (" ++ (écrireÉtiquette p n no1) ++
       ") to[bend right=8] (" ++ (écrireÉtiquette p n no2) ++ ");"
   | otherwise =
       "\\draw (" ++ (écrireÉtiquette p n no1) ++
       ") to (" ++ (écrireÉtiquette p n no2) ++ ");"
```

```
main :: IO ()
main = do
  a:b:c:d:e:[] <- getArgs
  let p = read a :: Int
  let n = read b :: Int
  let u1 = read c :: Float
  let u2 = read d :: Float
  let p2 = read e :: Float
  putStrLn "\\documentclass[tikz]{standalone}"
  putStrLn "\\usepackage[utf8]{inputenc}"
  putStrLn "\\usepackage[T1]{fontenc}"
  putStrLn "\\begin{document}"
  putStrLn "\\begin{tikzpicture}"
  mapM_ (putStrLn . écrireNœudTikz p n (u1,u2) (0,p2))
        (listeDesNœuds p n)
  mapM_ (putStrLn . écrireTraitTikz p n) (listeDesTraits p n)
  putStrLn "\\end{tikzpicture}"
  putStrLn "\\end{document}"
```

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