UNIVERSITÉ PARIS XIII - SORBONNE PARIS NORD
École Doctorale Sciences, Technologies, Santé Galilée

Compatibilité locale-globale du programme de Langlands modulo p pour certaines variétés de Shimura

THÈSE DE DOCTORAT

présentée par

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pour l'obtention du grade de

DOCTEUR EN <Mathématiques>

soutenue le 11, février 2022 devant le jury d'examen composé de :

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Résumé. Nous généralisons le résultat de compatibilité local-global dans [28] aux cas de dimension supérieure, en examinant la relation entre le foncteur de Scholze et la cohomologie des variétés de Shimura de type Kottwitz-Harris-Taylor. En chemin, nous prouvons un critère de cuspidalité de la théorie des types. Nous traitons également de la compatibilité des classes de torsion dans le cas des représentations semi-simples mod p Galois sans multiplicité, sous certaines hypothèses de platitude. Enfin, nous enlevons la condition sur semisimplicité et la remplaçons par la condition beaucoup plus faible d'être sans multiplicité. Ce dernier résultat est obtenu en collaboration avec Z. Qian.

Mots clés: *Compatibilité local-global, programme de Langlands, variétés de Shimura, représentations galoisiennes*

Abstract. We generalize the local-global compatibility result in [28] to higher dimensional cases, by examining the relation between Scholze's functor and cohomology of Kottwitz-Harris-Taylor type Shimura varieties. Along the way we prove a cuspidality criterion from type theory. We also deal with compatibility for torsion classes in the case of semisimple mod p Galois representations which are multiplicity free, under certain flatness hypotheses. Finally, we remove the semisimple condition and replace it by the much weaker condition of being multiplicity free. This last result is obtained in joint work with Z. Qian.

Key words: *Local-global compatibility, Langlands program, Shimura varieties, Galois representations*

Acknowledgement

First and foremost, I would like to thank sincerely my supervisors, Pascal Boyer and Stefano Morra. They bring me into this interesting subject and are always glad to share their insights on it. I thank them for their constant support and help, for their guidance and kind suggestions during the PhD period, as well as for the opportunities that they created which helped to broaden my horizons. Without them, nothing of these would be possible.

Next I am grateful to Christian Johansson and Benoît Stroh for accepting to be the referees of my thesis and for their careful reading of the manuscript together with their constructive remarks. I thank warmly Daniel Le and Ariane Mézard for kindly agreeing to be in the jury of my thesis defense.

During my PhD study, I have had many opportunities to attend to various kinds of seminars and Groupes de Travail where I could not only learn about new mathematics, but also see how the mathematical tradition and culture in France continue and develop; this used to be an interesting point to me since I got to know the history of mathematics of Europe. I would like to thank Vincent Sécherre, Laurent Fargues, Jacques Tilouine, Daniel Le for answering my questions and for their suggestions. I especially thank Zicheng Qian for the collaboration on the final section of this thesis, where he shared his opinions on the subject and gave me many helpful suggestions on that section as well as on other parts of the thesis.

I would like to thank the professors from whose lessons I increased my knowledge: Jean-Benoît Bost, Etienne Fouvry, François Charles, Benjamin Schraen, Bertrand Rémy, Sug Woo Shin, Eugen Hellmann, Bao Viet Le Hung and so on. Their own ways of lecturing mathematics and telling mathematical stories have benefited me in many aspects. On the other hand, my study in France would not be possible without the support of several professors at Shandong University; among them I thank in particular Shige Peng, Hualin Huang and Penghui Wang.

The mathematics department of Université Sornonne Paris Nord has always been a quiet place, and it provides a nice atmosphere for the doctorants to work. I thank my colleagues at LAGA, as well as the staff of the laboratory and Ecole Doctoral Galilée for their help in administrative procedures, and thank in particular, Francis Nier and Farid Mokrane for their suggestions and help on several academic procedures. In February 2019, I spent a few days at Università di Padova and it is during that period that some important progress in my project was achieved; I thank the Mathematics Department and the staff there for their hospitality.

It is the accompany of my friends that makes me not feel alone since my arrival in France. I cannot forget the joyful moments when I stay with them and discuss math or other stuffs, do cooking, go hiking... These helped me to discover more aspects of life in Paris, and also taught me to notice beauty in daily life. It is thus my pleasure to thank Sebastian Bartling, Angelot Behajaina, Yichang Cai, Nguyen Kieu Hieu, Yu Min, Zicheng Qian, Zhenlin Ran, Arnaud Vanhaecke, Xiaozong Wang, Zhixiang Wu, Xiaoyu Zhang, Yanbin Zhang and Jiandi Zou for helpful discussions in mathematics and for other activities together. It is also my pleasure to thank Nicolas Camps, Zhangchi Chen, Chen Gong, Ning Guo, Yi Pan, Changzhen Sun, Yisheng Tian, Xinyue Tu, Robin Witthaus, Shiyao Zhang, Yugang Zhang and Yuzhe Zhu for the good memories together. Special thanks go to Nicholas Camps for taking me to the beach with his family where we spent a great time when I went to Montpellier in 2018.

Finally, I wish to express my gratitude to my parents for their constant and unconditional support throughout the years.

Contents

1. Introduction

The existence of a p-adic local Langlands correspondence, as was first envisioned by Breuil (cf. [4]), is still widely open beyond the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. The work of Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin [6] provides a construction which attaches p-adic $\mathrm{GL}_n(F)$-representations to Galois representations for a p-adic field F. Their construction is global in nature. Later in [28], Scholze takes the other direction and constructs Galois representations from mod p (and p-adic) representations of $\mathrm{GL}_n(F)$, in a purely local way. Then Scholze proves the compatibility between his construction and the patching construction of Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin, as well as a local-global compatibility result, both for GL_2.

The purpose of the thesis is to generalize the local-global compatibility result of Scholze to GL_n for $n > 2$. For this we follow mostly the strategy of [28]. Let us describe the results of [28] and this paper in more detail.

Let $n \geq 1$ be an integer and $L/\mathbb{Q}_p$ be a finite extension with residue field k of cardinality q. Denote by $\breve{L}$ the completion of the maximal unramified extension of L. Then one has the Lubin-Tate tower $(\mathcal{M}_{\mathrm{LT},J})_{J\subseteq \mathrm{GL}_n(L)}$, indexed by compact open subgroups J of $\mathrm{GL}_n(L)$, consisting of smooth rigid-analytic varieties $\mathcal{M}_{\mathrm{LT},J}$ over $\breve{L}$ equipped with compatible actions of $D^\times$ on all $\mathcal{M}_{\mathrm{LT},J}$ where D is the central division algebra over L of invariant $1/n$. It is shown in [29] that the inverse limit

$$\mathcal{M}_{\mathrm{LT},\infty} = \varprojlim_J \mathcal{M}_{\mathrm{LT},J}$$

can be defined in a good sense in the category of perfectoid spaces. Let π be a smooth mod p representation of $\mathrm{GL}_n(L)$. The construction of Scholze involves descending the trivial sheaf π on $\mathcal{M}_{\mathrm{LT},\infty}$ along the Gross-Hopkins period map

$$\pi_{\mathrm{GH}} : \mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^{n-1}_{\breve{L}},$$

resulting in a Weil-equivariant sheaf $\mathcal{F}_\pi$ on the site $(\mathbb{P}^{n-1}_{\breve{L}}/D^\times)_{\text{ét}}$. (One may refer to Section 3 of [28] for the notations and more details.) The main theorem of [28], Theorem 1.1 in *loc.cit.*, asserts that for each $i \geq 1$, the cohomology group $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ is an admissible representation of $D^\times$ and carries an action of the Galois group $\mathrm{Gal}(\bar{L}/L)$; moreover, this cohomology group vanishes if $i > 2(n-1)$.

To state our local-global compatibility result, we need to change the notations to a global context. So fix a CM field extension K/F with K totally imaginary and F its maximal totally real subfield. Choose a place $\mathfrak{p}$ of F lying over p and an infinite place α of K. Let B be a division algebra over K of dimension n^2 with an involution of the second kind which is supposed to be positive. Assume that

$\mathfrak{p}$ is split in K and fix a place $\mathfrak{q}$ of K over $\mathfrak{p}$ where we assume moreover that B is a division algebra of invariant $1/n$. Then to these data one has a corresponding unitary similitude group $\tilde{G}$ over F with

$$\tilde{G}(F_{\mathfrak{p}}) \cong (B_{\mathfrak{q}}^{op})^{\times} \times (F_{\mathfrak{p}})^{\times},$$

and for each compact open subgroup $U \subseteq (B_{\mathfrak{q}}^{op})^{\times}$ there is a Shimura variety $\mathrm{Sh}_{UC^{\mathfrak{p}}}$ over K associated with the subgroup $U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}$ of $\tilde{G}(\mathbb{A}_{F,f})$, where $C^{\mathfrak{p}} \subseteq \tilde{G}(\mathbb{A}_{F,f}^{\mathfrak{p}})$ is a fixed sufficiently small tame level. Moreover there exists another division algebra D over K with interchanged local behaviours at $\mathfrak{q}$ and α from B, so that its associated unitary similitude group G' over F is an inner form of $\tilde{G}$, locally isomorphic to $\tilde{G}$ at all places except $\mathfrak{p}$ and $\alpha|_F$; in particular, D is split at $\mathfrak{q}$. The space of continuous functions

$$\pi = \pi_{C^{\mathfrak{p}}} := C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(\mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p)$$

is an admissible $\mathbb{Z}_p$-representation of $\mathrm{GL}_n(F_{\mathfrak{p}})$ and applying Scholze's functor, one obtains a $(\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times})$-representation on the space $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi})$. On the global side, we consider the cohomology of the above system of Shimura varieties and define

$$\rho = \rho_{C^{\mathfrak{p}}} := \varinjlim_{U} H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p)$$

which is a $(\mathrm{Gal}_K \times B_{\mathfrak{q}}^{\times})$-representation. The first result that we will prove in this article is the following weak form of local-global compatibility.

Theorem 1.1. *There is a natural isomorphism of* $(\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times})$*-representations over* $\mathbb{Z}_p$

$$H^i_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}}}}) \cong \rho_{C^{\mathfrak{p}}}.$$

As in [28], this will be proved as a consequence of p-adic uniformization of Shimura varieties. We can deduce from this theorem a more precise result using the formalism of σ-typicity in Section 5 of [28]. To state it, let $\mathbb{T}$ be the formal Hecke algebra over $\mathbb{Z}$ generated by Hecke operators at good places of K and $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ (associated with a mod p Galois representation $\bar{\sigma}$) such that

$$H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p)_{\mathfrak{m}} \neq 0$$

only when $i = n-1$; we also assume that $\mathfrak{m}$ satisfies certain "strongly irreducibile" condition (Assumption 3.5). Then there is an n-dimensional Galois representation

$$\sigma = \sigma_{\mathfrak{m}} : G_K \to \mathrm{GL}_n(\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}})$$

characterized by certain Eichler-Shimura relations. Here $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$ denotes the completed Hecke algebra at level $C^{\mathfrak{p}}$ and is a complete local Noetherian ring acting faithfully on $\pi_{\mathfrak{m}}$. The next result says that one can recover $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ from $\pi_{\mathfrak{m}}$.

Theorem 1.2. *There is a canonical* $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times}]$*-equivariant isomorphism*

$$H_{ét}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}}) \cong \sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}} \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}} \rho[\sigma].$$

for some faithful $\mathbb{T}(C^{\mathfrak{p}})$*-module* $\rho[\sigma]$ *which carries the trivial* $\mathrm{Gal}_{F_{\mathfrak{p}}}$*-action. If moreover* $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ *is absolutely irreducible, then this determines* $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ *uniquely.*

One important step in proving this theorem is the following cuspidality criterion, and its consequence on constructing congruences between automorphic forms, which will allow us to extend the Hecke action of $\mathbb{T}$ on $\pi_{\mathfrak{m}}$ to an action of $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$; see section 6 for the notations and some background unexplained here. We remark that Fintzen-Shin [12] have proved, independently and simultaneously, such results for all reductive groups over totally real fields that are compact modulo center at infinity under a mild condition on p; see Theorem 3.1.1 of their paper (and also the Appendix D therein which removes the condition on p).

Proposition 1.3. *Let L be a p-adic field and $\psi : L \to \mathbb{C}^{\times}$ be a non-trivial (additive) character of level one (that is, ψ is trivial on $\varpi\mathcal{O}_L$ but non-trivial on $\mathcal{O}_L$). Let α_m be the homomorphism*

$$\alpha_m : U^{\tilde{M}}(\mathfrak{A}) \to \varpi^{-N}\mathcal{O}_L, \quad a \mapsto tr_{A/L}(\beta_m(a-1)).$$

If π is a smooth irreducible representation of $\mathrm{GL}_n(L)$ *such that* $\pi|_{U^{\tilde{M}}(\mathfrak{A})}$ *contains the character $\psi \circ \alpha_m$, then π is cuspidal.*

Corollary 1.4. *Let* $A_m = \mathbb{Z}_p[T]/((T^{p^m}-1)/(T-1))$. *Take* $L = F_{\mathfrak{p}}$ *and let ψ be a character of L with coefficients in A_m whose restriction to* $\varpi^{-N}\mathcal{O}_L$ *is the map*

$$\varpi^{-N}\mathcal{O}_L \xrightarrow[\sim]{\times\varpi^N} \mathcal{O}_L \twoheadrightarrow \mathcal{O}_L/\varpi^{me} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z} \to A_m^{\times}$$

with the last arrow mapping $1 \in \mathbb{Z}/p^m\mathbb{Z}$ *to* $T \in A_m^{\times}$. *Define* $\psi_m = \psi \circ \alpha_m$. *Then any automorphic representation π of* $\mathrm{GL}_n(F_{\mathfrak{p}})$ *appearing in*

$$C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(U^{\tilde{M}} \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}), \psi_m)[1/p]$$

is cuspidal at $\mathfrak{p}$.

Finally we prove a torsion class version under a reasonable flatness assumption, cf. remark 7.6. It shows in particular that if $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is irreducible, then it can be read off from $H_{\text{ét}}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi[\mathfrak{m}]})$.

Theorem 1.5. *Assume that* $\pi_{\mathfrak{m}}^{\vee}$ *is flat over* $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$. *Then* $H_{ét}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi[\mathfrak{m}]})$ *is a non-zero admissible* $\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times}$*-representation, and it has the same Jordan-Hölder factors as* $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$. *In particular, if* $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ *is irreducible, then every irreducible subrepresentation of* $H_{ét}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi[\mathfrak{m}]})$ *is isomorphic to* $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$.

Remark 1.6. In [23], Le-Le Hung-Morra-Park-Qian have obtained similar mod p local-global compatibility result in the Fontaine-Lafaille cases under suitable conditions, by studying moduli stacks of Fontaine-Lafaille modules. Moreover, Zicheng Qian and the author have succeeded in giving an argument which can be used to deal with the much larger class of multiplicity-free Galois representations $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ in Theorem 1.5 above, namely:

Theorem 1.7. *Assume that* $\pi_{\mathfrak{m}}^{\vee}$ *is flat and the* $\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}}]$*-module* $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ *is multiplicity free. Then* $H_{ét}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi[\mathfrak{m}]})$ *determines* $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ *up to isomorphism.*

Here $\mathbb{F}$ is the finite field which serves as the coefficient field of $\bar{\sigma}$ above. We prove this theorem by establishing certain classification result of subrepresentations of a given σ-typic representation, cf. section 8 of the thesis.

L'existence d'une correspondance de Langlands p-adique, comme imaginé premièrement par Breuil (cf. [4]), reste encore largement ouverte au-delà du cas de $\mathrm{GL}_2(\mathbb{Q}_p)$. Le travail de Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin [6] a fourni une construction qui à des représentations p-adiques de $\mathrm{GL}_n(F)$ associe des représentations galoisiennes sur un corps p-adique F, une construction de nature globale. Dans l'autre direction, Scholze [28] a construit des représentations galoisiennes à partir de représentations mod p (et p-adiques), de manière complètement locale; puis il a prouvé la compatibilité entre sa construction et celle de Caraiani-Emerton-Gee-Geraghty-Paskunas-Shin, aussi qu'un résultat de compatibilité locale-globale, tous les deux pour GL_2.

L'objet de cette thèse est de généraliser le résultat de compatibilité locale-globale de Scholze à GL_n pour $n > 2$. Pour cela nous suivons largement le stratégie de [28]. Décrivons en détails les résultats de [28] et de celui-ci.

Soient $n \geq 1$ un entier et $L/\mathbb{Q}_p$ une extension finie de corps résiduel k de cardinalité q. Notez par $\breve{L}$ le complété de l'extension non ramifié maximale de L. On dispose alors du tour Lubin-Tate $(\mathcal{M}_{\mathrm{LT},J})_{J\subseteq \mathrm{GL}_n(L)}$, indexé par sous-groupes compacts ouverts J de $\mathrm{GL}_n(L)$, constitué de variétés lisse rigide-analytiques $\mathcal{M}_{\mathrm{LT},J}$ sur $\breve{L}$ munies d'actions compatibles de $D^\times$ où D est l'algèbre de division centrale sur L d'invariant $1/n$. Il est démontré dans [29] que la limit inverse

$$\mathcal{M}_{\mathrm{LT},\infty} = \varprojlim_J \mathcal{M}_{\mathrm{LT},J}$$

peut avoir du bon sens dans la catégorie d'espaces perfectoïdes. Soit π une représentation mod p et lisse de $\mathrm{GL}_n(L)$. La construction de Scholze consiste à descendre le faisceau trivial π sur $\mathcal{M}_{\mathrm{LT},\infty}$ le long de l'application période de Gross-Hopkins

$$\pi_{\mathrm{GH}} : \mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^{n-1}_{\breve{L}},$$

donnant un faisceau Weil-équivariant $\mathcal{F}_\pi$ sur le site $(\mathbb{P}^{n-1}_{\breve{L}}/D^\times)_{\text{ét}}$. (On peut se référer à Section 3 de [28] pour les notations et pour plus de détails.) Le théorème principal de [28], Theorem 1.1 dans *loc.cit.*, affirme que pour chaque $i \geq 1$, le groupe de cohomologie $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ est une représentation admissible de $D^\times$ et porte une action du groupe $\mathrm{Gal}(\bar{L}/L)$; par ailleurs, ce groupe de cohomologie disparaît si $i > 2(n-1)$.

Pour énoncer notre résultat de compatibilité locale-globale, il faut changer les notations à un contexte global. Donc fixons une extension CM de corps K/F avec K totalement imaginaire et F son sous-corps maximal totalement réel. Choissons une place $\mathfrak{p}$ de F au dessus de p et une place infinie α de K. Soit B une algèbre de division sur K de dimension n^2 avex une involution de seconde espèce qui est supposée être positive. Supposons que $\mathfrak{p}$ est déployé dans K et fixons une place $\mathfrak{q}$ de K au dessus de $\mathfrak{p}$ telle que B soit une algèbre de division d'invariant $1/n$ en q. Alors on a un groupe unitaire de similitude $\tilde{G}$ sur F associé à ces dononées avec

$$\tilde{G}(F_{\mathfrak{p}}) \cong (B_{\mathfrak{q}}^{op})^{\times} \times (F_{\mathfrak{p}})^{\times},$$

et pour chaque sous-groupe compact ouvert $U \subseteq (B_{\mathfrak{q}}^{op})^{\times}$ il y a une variété de Shimura $\mathrm{Sh}_{UC^{\mathfrak{p}}}$ sur K associé au sous-groupe $U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}$ de $\tilde{G}(\mathbb{A}_{F,f})$, où $C^{\mathfrak{p}} \subseteq \tilde{G}(\mathbb{A}_{F,f}^{\mathfrak{p}})$ est un niveau modéré suffisament petit et fixé. Par ailleurs, il y a une autre algèbre de division D sur K avec comportements locals échangés en $\mathfrak{q}$ et α de B, tel que le groupe unitaire de similitude associé G' sur F est une forme interieure de $\tilde{G}$, localement isomorphe à $\tilde{G}$ en toute les places sauf $\mathfrak{p}$ et $\alpha|_F$; en particulier, D est deployée en $\mathfrak{q}$. L'espace de fonctions continues

$$\pi = \pi_{C^{\mathfrak{p}}} := C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(\mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p)$$

est une $\mathbb{Z}_p$-représentation admissible de $\mathrm{GL}_n(F_{\mathfrak{p}})$ et en appliquant le foncteur de Scholze, on obtient une représentation de $(\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times})$ sur l'espace $H_{\text{ét}}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi})$. Dans le côté global, on considère la cohomologie du système de variétés de Shimura ci-dessus et definit

$$\rho = \rho_{C^{\mathfrak{p}}} := \varinjlim_{U} H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p)$$

qui est une $(\mathrm{Gal}_K \times B_{\mathfrak{q}}^{\times})$-représentation. Le premier résultat que nous allons prouver dans ce article est la forme faible de compatibilité locale-globale suivante.

Theorem 1.8. *Il y a un isomorphisme naturel de* $(\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times})$*-représentations sur* $\mathbb{Z}_p$

$$H_{\acute{e}t}^{i}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi_{C^{\mathfrak{p}}}}) \cong \rho_{C^{\mathfrak{p}}}.$$

Comme dans [28], cela sera montré comme une consequence d'uniformisation p-adic de variétés de Shimura. On peut en déduire un résultat plus précis utilisant le formalisme de σ-typicité dans Section 5 de [28].Pour le énoncer, soient $\mathbb{T}$ l'algèbre de Hecke formelle sur $\mathbb{Z}$ engendrée par des opérateurs de Hecke aux bonnes places de K et $\mathfrak{m}$ un idéal maximal de $\mathbb{T}$ (associé à une représentation galoisienne mod p, noté par $\bar{\sigma}$) tel que

$$H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p)_{\mathfrak{m}} \neq 0$$

seulement quand $i = n - 1$. Nous Supposons aussi que $\mathfrak{m}$ satisfait certaine "fort irréductible" condition (Assumption 3.5). Alors il y a une représentation galoisienne de dimension n

$$\sigma = \sigma_{\mathfrak{m}} : G_K \to \mathrm{GL}_n(\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}})$$

caractérisée par certaines relations de Eichler-Shimura. Ici $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$ désigne l'algèbre de Hecke complétée au niveau $C^{\mathfrak{p}}$ et est un anneau local noethérien complet agissant fidèlement sur $\pi_{\mathfrak{m}}$. Le résultat suivant dit que l'on peut récupérer $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ à partir de $\pi_{\mathfrak{m}}$.

Theorem 1.9. *Il y un ismorphisme canonique et $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_{F_{\mathfrak{p}}} \times B_{\mathfrak{q}}^{\times}]$-équivariant*

$$H_{\acute{e}t}^{n-1}(\mathbb{P}_{\mathbb{C}_p}^{n-1}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}}) \cong \sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}} \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}} \rho[\sigma].$$

pour certain $\mathbb{T}(C^{\mathfrak{p}})$-module $\rho[\sigma]$ qui porte l'action triviale par $\mathrm{Gal}_{F_{\mathfrak{p}}}$. Si par ailleurs $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ est absolument irréductible, alors cela determine $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ uniquement.

Une étape importante dans la démonstration de ce théorème est le critère de cuspidalité suivant, et sa conséquence sur la construction de congurences entre formes automorphes, ce qui nous permettra d'étendre l'action de Hecke de $\mathbb{T}$ sur $\pi_{\mathfrak{m}}$ à une action de $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$; voir la section 6 pour les notations et quelques arrière-plans inexpliqués ici. Nous remarquons que Fintzen-Shin [12] ont prouvé, indépendamment et simultanément, de tels résultats pour tous les groupes réductifs sur des corps totalement réels qui sont compacts modulo centre à l'infini sous une condition douce sur p ; voir le théorème 3.1.1 de leur article (aussi l'annexe D qui enlève la condition sur p).

Proposition 1.10. *Soient L un corps p-adique et $\psi : L \to \mathbb{C}^{\times}$ un caractère (additif) non trivial de niveau 1 (c'est-à-dire, ψ est trivial sur $\varpi\mathcal{O}_L$ mais non trivial sur $\mathcal{O}_L$). Soit α_m le morphism*

$$\alpha_m : U^{\tilde{M}}(\mathfrak{A}) \to \varpi^{-N}\mathcal{O}_L, \quad a \mapsto tr_{A/L}(\beta_m(a-1)).$$

Si π est une représentation lisse et irréductible de $\mathrm{GL}_n(L)$ telle que $\pi|_{U^{\tilde{M}}(\mathfrak{A})}$ contient le caractère $\psi \circ \alpha_m$, alors π est cuspidal.

Corollary 1.11. *Soit $A_m = \mathbb{Z}_p[T]/((T^{p^m}-1)/(T-1))$. Posons $L = F_{\mathfrak{p}}$ et soit ψ un caractère de L avec coefficients dans A_m dont la restriction à $\varpi^{-N}\mathcal{O}_L$ est l'application*

$$\varpi^{-N}\mathcal{O}_L \overset{\times\varpi^N}{\xrightarrow{\sim}} \mathcal{O}_L \twoheadrightarrow \mathcal{O}_L/\varpi^{me} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z} \to A_m^{\times}$$

avec la flèche dernière $1 \in \mathbb{Z}/p^m\mathbb{Z}$ à $T \in A_m^{\times}$. Define $\psi_m = \psi \circ \alpha_m$. Alors toute représentation automorphe π de $\mathrm{GL}_n(F_{\mathfrak{p}})$ qui apparaît dans

$$C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(U^{\tilde{M}} \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}), \psi_m)[1/p]$$

est cuspidale en $\mathfrak{p}$.

Enfin, nous prouvons une version de classe de torsion sous une hypothèse de platitude raisonnable, voir remarque 7.6. Il montre en particulier que si $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ est irréductible, alors il peut être lu à partir de $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$.

Theorem 1.12. *Supposons que le module* $\pi^\vee_\mathfrak{m}$ *est plat sur* $\mathbb{T}(C^\mathfrak{p})_\mathfrak{m}$*. Alors* $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ *est une représentation non zéro de* $\mathrm{Gal}_{F_\mathfrak{p}} \times B^\times_\mathfrak{q}$*, et il a les mêmes facteurs de Jordan-Hölder avec* $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$*. En particulier, si* $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ *est irréducible, alors toute sous-représentation irréductible de* $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ *est isomorphe à* $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$.

Remark 1.13. Dans [23], Le-Le Hung-Morra-Park-Qian ont obtenu un résultat similaire de compatibilité locale-globale de mod p dans les cas de Fontaine-Lafaille sous des conditions appropriées, en étudiant des champs des modules de Fontaine-Lafaille. De plus, Zicheng Qian et l'auteur ont réussi à donner un argument qui peut être utilisé pour traiter la classe beaucoup plus large des représentations galoisiennes sans multiplicité $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ dans le théorème 1.12 ci-dessus, c'est-à-dire:

Theorem 1.14. *Supposons que* $\pi^\vee_\mathfrak{m}$ *est plat sur*$\mathbb{T}(C^\mathfrak{p})_\mathfrak{m}$ *et que le* $\mathbb{F}[\mathrm{Gal}_{F_\mathfrak{p}}]$*-module* $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ *est sans multiplicité. Alors* $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ *détermine* $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ *à isomorphisme près.*

Ici $\mathbb{F}$ est le corps fini de coéfficients de la représentation $\bar{\sigma}$ ci-dessus. Nous montrons ce théorème en établissant un résultat de classification des sous-représentations d'une représentation σ-typique; voir la section 8 de cette thèse.

2. Scholze's functor

In this section we give a short review of Scholze's construction which associates p-adic Galois representations with admissible p-adic representations of $GL_n(F)$, for F a p-adic field. The proof of most statements in this section, if not provided here, can be found in [28]. We start with some background on the Gross-Hopkins period map and étale cohomology of adic spaces that are equipped with a continuous action of a profinite group.

I. Adic spaces

Definition 2.1. A Huber ring is a topological ring A which contains an open subring A_0 whose subspace topology is induced by a finitely generated ideal $I \subseteq A_0$ (i.e., the set $\{I^n\}_{n\geq 1}$ forms a basis of neighborhood of $0 \in A_0$).

A subset S of a topological ring A is said to be bounded if for every open neighborhood U of 0, there exists an open neighborhood V of 0 such that $V \cdot S \subseteq U$. An element $a \in A$ is called power-bounded if the set $\{a^n : n \geq 0\}$ is bounded. We denote the subset of bounded elements of A by A°.

Definition 2.2. Let A be a topological ring and Γ be totally ordered abelian group. A continuous valuation (with values in Γ) on A is a multiplicative map

$$|\cdot| : A \to \Gamma \cup \{0\}$$

such that $|0| = 0$, $|1| = 1$, $|a + b| \leq \max(|a|, |b|)$ and for every $\gamma \in \Gamma$ the set $\{a \in A : |a| < \gamma\}$ is open in A.

Two valuations $|\cdot|_1 : A \to \Gamma_1 \cup \{0\}$ and $|\cdot|_2 : A \to \Gamma_2 \cup \{0\}$ are equivalent if for all $a, b \in A$ we have $|a|_1 \geq |b|_1$ if and only if $|a|_2 \geq |b|_2$. Let $\mathrm{Cont}(A)$ denote the set of equivalence classes of continuous valuations of A. We endow $\mathrm{Cont}(A)$ the topology generated by subsets of the form $\{x \in \mathrm{Cont}(A) \big| \ |f(x)| \leq |g(x)| \neq 0\}$ with $f, g \in A$; here we denote by $|f(x)|$ the image of the valuation x on an element $f \in A$.

Definition 2.3. Let A be a Huber ring. A ring of integral elements of A is a subring of A which is open and integrally closed in A, and moreover satisfies $A^+ \subseteq A^\circ$. A Huber pair (A, A^+) is a pair (A, A^+) consisting of a Huber ring A and a ring of integral elements A^+ of A. Morphisms between Huber pairs (A, A^+) and (B, B^+) are continuous ring homomorphisms $A \to B$ mapping A^+ into B^+.

Just as in the construction of schemes, we can first associate a locally ringed space $\mathrm{Spa}(A, A^+)$ with a Huber pair (A, A^+), called the adic spectrum of (A, A^+), and then use these adic spectra as building blocks for general adic spaces. The underlying topological space of $\mathrm{Spa}(A, A^+)$ is the subset of $\mathrm{Cont}(A)$ given by

$$\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Cont}(A) \big| \ |f(x)| \leq 1 \text{ for all } f \in A^+\}$$

equipped with the subspace topology. To define the structure sheaf, we start with

Definition 2.4. Let $s \in A$ and $T \subseteq A$ a finite subset such that $TA \subseteq A$ is open. The the subset of $X = \mathrm{Spa}(A, A^+)$

$$U(\frac{T}{s}) := \{x \in X \big| \ |t(x)| \leq |s(x)| \text{ for all } t \in T\}$$

is called a rational subset.

It is easy to see that rational subsets of $\mathrm{Spa}(A, A^+)$ are open. The following result, which is Proposition 1.3 of [17], shows that rational subsets are themselves adic spectra.

Lemma 2.5. *Let U be a rational subset of $X = \mathrm{Spa}(A, A^+)$. Then there exists a complete Huber pair, denoted by $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$, together with a morphism of Huber pairs $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the induced map*

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$$

factors through U, and is universal for such maps. Furthermore, this map is a homeomorphism onto U.

Here we say a Huber pair (A, A^+) is complete if A is complete. Then A^+ is also complete as it is open and hence closed in A.

Definition 2.6. The structure presheaf of complete topological rings $\mathcal{O}_X$ on $\mathrm{Spa}(A, A^+)$ is defined as follows. If $\subseteq X$ is a rational subset, then $\mathcal{O}_X(U)$ is as in the above lemma. For a general open subset $W \subseteq X$, we define

$$\mathcal{O}_X(W) := \varprojlim_{U \subseteq W \text{rational}} \mathcal{O}_X(U).$$

We also define the sub-presheaf $\mathcal{O}_X^+$ similarly.

Proposition 2.7. *(Proposition 3.1.7,* [30]*) For all open subsets $U \subseteq X$, we have*

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \big| \ |f(x)| \leq 1 \text{ for all } x \in U\}.$$

Thus $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is.

Remark 2.8. The presheaf $\mathcal{O}_X$ is not necessarily a sheaf in general, cf. for instance the end of §1, [17] for such examples due to Rost. But in many important cases it is a sheaf. These cases include: (1) A is finitely generated over a notherian ring of definition (2) A is strongly noetherian (3) A is perfectoid (4) A is discrete. We call a Huber pair (A, A^+) sheafy if $\mathcal{O}_X$ is sheaf.

Definition 2.9. An adic space is a triple $(X, \mathcal{O}_X, \{|\cdot|_x\}_{x\in X})$ where $(X, \mathcal{O}_X)$ is a locally ringed space, $\mathcal{O}_X$ is a sheaf of complete topological rings, and $\{|\cdot|_x\}$ is a continuous valuation on $\mathcal{O}_{X,x}$ for each $x \in X$, such that locally on X it is isomorphic to $\mathrm{Spa}(A, A^+)$ for some sheafy Huber pair (A, A^+).

For a Huber pair (B, B^+), we write $\mathrm{CAff}^{\mathrm{op}}_{(B,B^+)}$ for the category of complete Huber pairs (A, A^+) with a morphism $(B, B^+) \to (A, A^+)$.

Remark 2.10. One may have noticed in the construction of adic space one difference with the theory of schemes, that is, the additional role of A^+. This is well motivated and explained in Section 3.3 of [30].

Remark 2.11. A slightly general notion of adic space is proposed in [29], Definition 2.1.5, to remedy the problem that a Huber pair (A, A^+) may be non-sheafy. Their approach is to enlarge the category of adic spaces from a functor of points view, analogous to the theory of algebraic spaces. In fact we will need to appeal to this more general notion later on when necessary but we do not repeat the definition here.

For a complete nonarchimedean field K with ring of integers $\mathcal{O}_K$ and a pseudo-uniformizer ϖ (that is, $|\varpi| < 1$), we let $\mathrm{Nilp}_{\mathcal{O}}$ to be the category of $\mathcal{O}_K$-algebras R such that ϖ is nilpotent on R.

Definition 2.12. Let $\mathcal{F}$ be a contravariant functor on $\mathrm{Nilp}^{\mathrm{op}}_{\mathcal{O}}$. Then $\mathcal{F}^{\mathrm{ad}}_{\eta}$ is defined to be the sheafification of the presheaf on $\mathrm{CAff}^{\mathrm{op}}_{(K,\mathcal{O})}$ given by

$$(R, R^+) \mapsto \varinjlim_{R_0 \subseteq R^+} \varprojlim_{n} \mathcal{F}(R_0/\varpi^n).$$

Remark 2.13. In the case when $\mathcal{F}$ is representable by a formal scheme $\mathcal{X}$ over $\mathcal{O}$ with locally finitely generated ideal of definition, $\mathcal{F}^{\mathrm{ad}}_{\eta}$ is exactly the adic generic fiber of the adic space $\mathcal{X}^{\mathrm{ad}}$ over $\mathrm{Spa}(\mathcal{O}, \mathcal{O})$ associated with the formal scheme $\mathcal{X}$, cf. Proposition 2.2.2 of [29].

II. Lubin-Tate spaces at infinite level

It is noticed that categorical inverse limits rarely exist in the category of adic spaces. Instead, the following well-behaved notion of inverse limit has been formulated which proves to be useful.

Definition 2.14. Let X_i be a filtered inverse system of adic spaces such that all transition maps are quasi-compact and quasi-separated, let X be an adic space, and let $f_i : X \to X_i$ be a family of compatible morphisms of adic spaces. Then we write $X \sim \varprojlim_i X_i$ if the following two properties hold:

(1) the induced map on the underlying topological spaces is a homeomorphism: $|X| \xrightarrow{\sim} \varprojlim_i |X_i|$;

(2) there exists an open cover of X by affinoid subspaces $\mathrm{Spa}(R, R^+) \subseteq X$ such that the map $\varinjlim_{\mathrm{Spa}(R_i,R_i^+)\subseteq X_i} R_i \to R$ has dense image. Here the direct limit runs over all open affinoid subspaces $\mathrm{Spa}(R_i, R_i^+) \subseteq X_i$ over which the map $\mathrm{Spa}(R, R^+) \subseteq X \to X_i$ factors.

Remark 2.15. Here we have followed the definition of [29]; it is more general than the original one, Definition 7.14 of [27].

Let H be a p-divisible group over a perfect field k of characteristic p, of height n and dimension d. We denote by $W(k)$ the Witt ring of k and $\mathrm{Nilp}_{W(k)}$ the category of $W(k)$-algebras where p is nilpotent. Then we define

Definition 2.16. A deformation of H to an object $R \in \mathrm{Nilp}_{W(k)}$ is a pair (G, ρ) where G is a p-divisible group over R and $\rho : H \otimes_k R/p \to G \otimes_R (R/p)$ is a quasi-isogeny.

Let $\mathcal{M}$ be the functor on $\mathrm{Nilp}_{W(k)}$ which sends an object R to the set of isomorphism classes of deformations of H to R. Then Rapoport-Zink [26] proved the following theorem.

Theorem 2.17. *The functor $\mathcal{M}$ is representable by a formal scheme, still denoted by $\mathcal{M}$, over* $\mathrm{Spf}\, W(k)$ *which admits locally a finitely generated ideal of definition.*

These formal schemes (or more generally, their associated adic spaces) are what we usually call Rapoport-Zink spaces. If we take the dimension of H in Definition 2.16 to be $d = 1$, then the corresponding formal scheme $\mathcal{M}$ is said to be Lubin-Tate, and it is isomorphic (non-canonically) to

$$\bigsqcup_{i\in\mathbb{Z}} \mathrm{Spf}\,(W(k)[[X_1, \ldots, X_{n-1}]])$$

when k is algebraically closed. One can add level structures to the moduli problems on the generic level and define by passing to infinity the reasonable inverse limit of the corresponding moduli spaces.

Definition 2.18. For each integer $i \geq 1$, let $\mathcal{M}_i$ be the functor on the category of complete affinoid $(W(k)[1/p], W(k))$-algebras which sends an object (R, R^+) to the set of triples (G, ρ, γ) where $(G, \rho) \in \mathcal{M}_\eta^{\mathrm{ad}}(R, R^+)$ and

$$\gamma : (\mathbb{Z}_p/p^i\mathbb{Z}_p)^n \to G[p^i]_\eta^{\mathrm{ad}}(R, R^+)$$

is a morphism of $\mathbb{Z}_p/p^i\mathbb{Z}_p$-modules. It is required that for all $x = \mathrm{Spa}(K, K^+) \in \mathrm{Spa}(R, R^+)$ the induced map

$$\alpha(x) : (\mathbb{Z}_p/p^i\mathbb{Z}_p)^n \to G[p^i]_\eta^{\mathrm{ad}}(K, K^+)$$

is an isomorphism.

Theorem 2.19. *(Rapoport-Zink) The functor $\mathcal{M}_i$ is representable by an adic space over* $\mathrm{Spa}(W(k)[1/p], W(k))$*, which is an closed and open subset of the n-fold fiber product $(G[p^i]_\eta^{\mathrm{ad}})^n$ of $G[p^i]_\eta^{\mathrm{ad}}$ over $\mathcal{M}_\eta^{\mathrm{ad}}$.*

Now we can introduce Rapoport-Zink spaces at infinite level.

Definition 2.20. Let $\mathcal{M}_\infty$ be the functor on the category of complete affinoid $(W(k)[1/p], W(k))$-algebras which sends an object (R, R^+) to the set of triples (G, ρ, γ) where $(G, \rho) \in \mathcal{M}_\eta^{\mathrm{ad}}(R, R^+)$ and

$$\gamma : \mathbb{Z}_p^n \to T(G)_\eta^{\mathrm{ad}}(R, R^+)$$

is a morphism of $\mathbb{Z}_p$-modules. It is required that for all $x = \mathrm{Spa}(K, K^+) \in \mathrm{Spa}(R, R^+)$ the induced map

$$\alpha(x) : \mathbb{Z}_p^n \to T(G)_\eta^{\mathrm{ad}}(K, K^+)$$

is an isomorphism.

Theorem 2.21. *The functor $\mathcal{M}_\infty$ is representable by an adic space over* $\mathrm{Spa}(W(k)[1/p], W(k))$ *which is in fact perfectoid. Moreover, the relation*

$$\mathcal{M}_\infty \sim \varprojlim_i \mathcal{M}_i$$

holds in the category of adic spaces.

Remark 2.22. This theorem is proved by Scholze-Weinstein, cf. Theorem 6.3.4 of [29]. There, they also give an alternative description of $\mathcal{M}_\infty$ in terms of the Dieudonné module of H and crystalline period rings from p-adic Hodge theory, thus in a way independent of deformations of H.

If the dimension of H is $d = 1$, then we will write $\mathcal{M}_{\mathrm{LT},\infty}$ for the Rapoport-Zink space at infinite level (and similarly for $\mathcal{M}_{\mathrm{LT},n}$). Note also that in these Lubin-Tate cases, $\mathcal{M}_{\mathrm{LT},\infty}$ can be described as a space cut out from $(\tilde{H}_\eta^{\mathrm{ad}})^n$ by a determinant condition where n is the height of H; see Theorem 6.4.1 of [29]. Here and in the following, $\tilde{H}$ denotes the universal cover of the p-divisble group H, cf. Page 17 of [29] for the definition.

III. Gross-Hopkins period map

Now we recall some basic facts of the Gross-Hopkins period map. It is first studied by Gross-Hopkins [14] where the authors had applications to stable homotopy theory for them. We will define it using Grothendieck-Messing theory.

Let H be as above, i.e., a p-divisible group over a perfect field k of characteristic p, of height n and dimension d. We denote by $M(H)$ the Dieudonné module of H, which is a free $W(k)$-module of rank n. Suppose that R is a $W(k)$-algebra complete with respect to the p-adic topology. Then by Grothendieck-Messing theory, a deformation (G, ρ) to R gives rise to a surjection of locally free $R[1/p]$-modules

$$M(H) \otimes_{W(k)} (R[1/p]) \twoheadrightarrow \operatorname{Lie} G[1/p].$$

Note that this map depends on (G, ρ) only up to isogeny. So it induces a map (the Grothendieck-Messing period morphism) on the generic fiber

$$\pi : \mathcal{M}_\eta^{\mathrm{ad}} \to \mathrm{Gr}(d, n)$$

where $\mathrm{Gr}(d, n)$ is the Grassmannian variety of d-dimensional quotients of the rational Dieudonné module $M(H)[1/p]$, which is considered as an adic space over $\mathrm{Spa}(W(k)[1/p], W(k))$. The following theorem is Proposition 5.17 of [26].

Theorem 2.23. *The period morphism π is étale. Each fiber consists of a single isogeny class of lifts of H.*

In the Lubin-Tate case, namely when $d = 1$, we have $\mathrm{Gr}(d, n) = \mathbb{P}^{n-1}$. In this case, Gross-Hopkins proved that π is surjective, not just on classical rigid points but on all adic points.

Remark 2.24. The surjectivity result of Gross-Hopkins shows some new features in nonarchimedean geometry, which indicates that the projective space $\mathbb{P}^{n-1}$ is not simply connected. One consequence of the above surjectivity, that is, the properness of the image of π, is crucial in the construction of Scholze's functor. In fact the Lubin-Tate case is essentially the only case of Rapoport-Zink spaces whose period maps are surjective, cf. the Appendix by Rapoport to [28].

By composing with the natural surjections $\mathcal{M}_\infty \to \mathcal{M}_n \to \mathcal{M} \xrightarrow{\pi} \mathrm{Gr}(d, n)$, we obtain period morphisms on $\mathcal{M}_\infty$ and $\mathcal{M}_n$, which will also be denoted by π.

IV. Duality between Lubin-Tate and Drinfeld towers

We can generalize the previous results to Rapoport-Zink spaces of EL type. This depends on a choice of quadruple (B, V, G, μ) where

- B is a semisimple algebra over $\mathbb{Q}_p$;
- V is a finite B-module;
- $G = \mathrm{GL}_B(V)$ considered as an algebraic group over $\mathbb{Q}_p$;
- $\mu : \mathbb{G}_m \to G_{\overline{\mathbb{Q}}_p}$ is a conjugacy class of cocharacters such that the weight decomposition of $V_{\overline{\mathbb{Q}}_p}$ has only 0 and 1 as weights:
$$V_{\overline{\mathbb{Q}}_p} = V_0 \oplus V_1$$
.

Then we set $n = \dim V$ and $d = \dim V_0$. We fix moreover a maximal order $\mathcal{O}_B$ of B and an $\mathcal{O}_B$-stable lattice Λ of V. Finally, fix a p-divisible group H over k of height n and dimension d equipped with an action of $\mathcal{O}_B$ such that
$$M(H) \otimes_{W(k)} W(k)[1/p] \cong V \otimes_{\mathbb{Q}_p} W(k)[1/p]$$
holds as $B \otimes \mathbb{Q}_p W(k)[1/p]$-modules. Then we write $\mathcal{D} = (B, V, \tilde{H}, \mu)$ for the rational EL data and $\mathcal{D}^{\mathrm{int}} = (\mathcal{O}_B, \Lambda, H, \mu)$ for the integral one. Let E be the field of definition of the conjugacy class of cocharacters μ and set $\breve{E} = E \cdot W(k)$.

To these data one can define similarly Rapoport-Zink spaces of EL type, denoted by $\mathcal{M}_{\mathcal{D}^{\mathrm{int}}}$, cf. Definition 6.5.1 and Theorem 6.5.2 of [29]. Furthermore, there is also the Grothendieck-Messing period morphism
$$\pi_{\mathrm{GM}} : (\mathcal{M}_{\mathcal{D}^{\mathrm{int}}})^{\mathrm{ad}}_\eta \to \mathrm{Gr}_{\mathrm{GM}}(d, n)$$
where $\mathrm{Gr}_{\mathrm{GM}}(d, n)$ parametrizes B-equivariant quotients of $M(H) \otimes_{W(k)} R$ which are projective finite modules over R and isomorphic to $V_0 \otimes_{\mathbb{Q}_p} R$ (as $B \otimes_{\mathbb{Q}_p} R$-modules) locally on R, for R a complete $\breve{E}$-algebra. One can also add level structures as before to obtain Rapoport-Zink spaces of EL type at finite levels $\mathcal{M}_{\mathcal{D}^{\mathrm{int}},i}$, $i \geq 1$ and at infinite level $\mathcal{M}_{\mathcal{D}^{\mathrm{int}},\infty}$. The following result is taken from pp. 56 of [29].

Theorem 2.25. *The adic space $\mathcal{M}_{\mathcal{D}^{\mathrm{int}},\infty}$ is preperfectoid and admits an alternative description as a sheaf $\mathcal{M}_{\mathcal{D},\infty}$ which depends only on the rational data $\mathcal{D}$. Here $\mathcal{M}_{\mathcal{D},\infty}$ is the sheafification of the functor which maps a complete affinoid $(\breve{E}, \mathcal{O}_{\breve{E}})$-algebra (R, R^+) to the set of B-module morphisms*
$$V \to \tilde{H}^{\mathrm{ad}}_\eta(R, R^+)$$
satisfying the following two properties.

(1) *The quotient Q of $M(H) \otimes_{W(k)} R$ by the image of $V \otimes R$ is a projective finite module over R and is isomorphic to $V_0 \otimes_{\mathbb{Q}_p} R$ (as $B \otimes_{\mathbb{Q}_p} R$-modules) locally on R.*
(2) *For all geometric point $x = \mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(R, R^+)$ the sequence*

$$0 \to V \to \tilde{H}_\eta^{\mathrm{ad}}(C, \mathcal{O}_C) \to Q \otimes_R C \to 0$$

is exact.

Moreover, the action of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$ on $\mathcal{M}_{\mathcal{D},\infty}$ is continuous, and the Grothendieck-Messing period map

$$\pi_{\mathrm{GM}} : \mathcal{M}_{\mathcal{D},\infty} \to \mathrm{Gr}_{\mathrm{GM}}(d, n)$$

is $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$-equivariant. Here J is the group of B-linear self quasi-isogenies of H.

Now we can state the duality result on Rapoport-Zink spaces. Before that, let us note that in all the constructions of EL type, if we take B to be an extension of $\mathbb{Q}_p$ of degree d and $V = F^n$, then we obtain the Lubin-Tate case which is relevant to us later. But the more general context is necessary to state the duality result. Now let $\mathrm{Gr}_{\mathrm{HT}}(d, n)$ be the adic space over $\mathrm{Spa}(\breve{E}, \mathcal{O}_{\breve{E}})$ parametrizing B-equivariant quotients of $V \otimes_{\mathbb{Q}_p} R$ which are finite projective as modules over R and are isomorphic to $V_1 \otimes_{\mathbb{Q}_p} R$ (as $B \otimes_{\mathbb{Q}_p} R$-modules) locally on R.

Proposition 2.26. *There is a Hodge-Tate period morphism*

$$\pi_{\mathrm{HT}} : \mathcal{M}_{\mathcal{D},\infty} \to \mathrm{Gr}_{\mathrm{HT}}(d, n)$$

which sends an (R, R^+)-valued points of $\mathcal{M}_{\mathcal{D},\infty}$ represented by a map $V \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)$ to the quotient of $V \otimes_{\mathbb{Q}_p} R$ given as the image of

$$V \otimes_{\mathbb{Q}_p} R \to M(H) \otimes_{W(k)} R.$$

Moreover, π_{HT} is $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p)$-equivariant.

Proof. This is Proposition 7.1 of [29]. ☐

From now on we suppose that k is algebraically closed.

Definition 2.27. For an EL data $\mathcal{D} = (B, V, \tilde{H}, \mu)$, the dual EL data $\check{\mathcal{D}}$ is defined as follows. First $\check{B} = \mathrm{End}_B(H) \otimes \mathbb{Q}$, and $\mathcal{O}_{\check{B}} = \mathrm{End}_{\mathcal{O}_B}(H)$. Then $\check{V} = \check{B}$ and $\check{\Lambda} = \mathcal{O}_{\check{B}}$ endowed with the natural left action of $\check{B}$ and $\mathcal{O}_{\check{B}}$ respectively. We set $\check{H} = \mathrm{Hom}_{\mathcal{O}_B}(\Lambda, \mathcal{O}_B) \otimes_{\mathcal{O}_B} H$. The definition of $\check{\mu}$ is a given via its induced weight decomposition on $\check{V}$ and we refer the reader to page 58 of [29].

It follows that $\check{G} \cong J$ and $g \in \check{G} = J \subseteq \check{B}$ acts on $\check{V}$ by multiplication with g^{-1} from the right. We also denote by $\check{\mathrm{Gr}}_{\mathrm{GM}}(d, n)$ and $\check{\mathrm{Gr}}_{\mathrm{HT}}(d, n)$ the corresonding Grassmannian varieties for the dual EL data.

Theorem 2.28. *There are natural action of G on $\mathrm{Gr}_{\mathrm{HT}}(d,n)$ and $\check{\mathrm{Gr}}_{\mathrm{GM}}(d,n)$, as well as a canonical G-equivariant isomorphism $\mathrm{Gr}_{\mathrm{HT}}(d,n) \cong \check{\mathrm{Gr}}_{\mathrm{GM}}(d,n)$. Similarly, there are natural action of $\check{G}$ on $\check{\mathrm{Gr}}_{\mathrm{HT}}(d,n)$ and $\mathrm{Gr}_{\mathrm{GM}}(d,n)$, as well as a canonical $\check{G}$-equivariant isomorphism $\check{\mathrm{Gr}}_{\mathrm{HT}}(d,n) \cong \mathrm{Gr}_{\mathrm{GM}}(d,n)$.*

Let $\mathcal{M}_{\check{\mathcal{D}},\infty}$ be the Rapoport-Zink space at infinite level associated with the dual EL data $\check{\mathcal{D}}$.

Theorem 2.29. *There is a natural $G(\mathbb{Q}_p) \times \check{G}(\mathbb{Q}_p)$-equivariant isomorphism*

$$\mathcal{M}_{\mathcal{D},\infty} \cong \mathcal{M}_{\check{\mathcal{D}},\infty}$$

under which $\pi_{\mathrm{GM}} : \mathcal{M}_{\mathcal{D},\infty} \to \mathrm{Gr}_{\mathrm{GM}}(d,n)$ gets identified with $\check{\pi}_{\mathrm{HT}} : \mathcal{M}_{\check{\mathcal{D}},\infty} \to \check{\mathrm{Gr}}_{\mathrm{HT}}(d,n)$, and vice versa.

For the proofs of these two theorems one may refer to Section 7 of [29].

V. Equivariant étale sites

Let X be a locally noetherian analytic adic space with a continuous action by a locally profinite group G. Let $(X/G)_{\text{ét}}$ be the site whose objects are (locally noetherian analytic) adic spaces Y equipped with a continuous action of G, and a G-equivariant étale morphism $Y \to X$. Morphisms are G-equivariant maps over X, and a family of morphisms $\{f_i : Y_i \to Y\}$ is a cover if $|Y| = \bigcup_i f_i(|Y_i|)$. We denote the associated topos of this site by $(X/G)^{\sim}_{\text{ét}}$.

One can easily verify that all finite limits exist in $(X/G)_{\text{ét}}$. Recall that a category $\mathcal{C}$ is called coherent if it satisfies the following axioms:

(A1) *The category $\mathcal{C}$ admits finite limits.*

(A2) *Every morphism $f : X \to Z$ in $\mathcal{C}$ admits a factorization $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is an effective epimorphism and h is a monomorphism.*

(A3) *For every object $X \in \mathcal{C}$, the partially ordered set $Sub(X)$ is an upper semilattice: that is, it has a least element, and every pair of subobjects of X have a least upper bound.*

(A4) *The collection of effective epimorphisms in $\mathcal{C}$ is stable under pullback.*

(A5) *For every morphism $f : X \to Y$ in $\mathcal{C}$, the map $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is a homomorphism of upper semilattices.*

One may refer to eg. Tag 00WP of [31] for the definition of effective epimorphism. Then we have the following lemma.

Lemma 2.30. *Let X be a locally noetherian analytic adic space with a continuous action by a locally profinite group G. Then $(X/G)_{\text{ét}}$ is locally coherent.*

Proof. This is (v) of [28]. □

Thus it follows that formation of cohomology of such sites commutes with direct limit; this point will be used later when we deal with the cohomology of adic projective spaces. We also need the following two propositions whose proofs can be found in Section 2 of [28].

Proposition 2.31. *Let X be a qcqs locally noetherian analytic adic space with a continuous action by a profinite group G. Then for any closed subgroup H of G, $(X/H_0)^{\sim}_{ét}$ is a projective limit of the fibred topos $(X/H)^{\sim}_{ét}$ for all open subgroups H of G containing H_0. In particular, for any sheaf $\mathcal{F} \in (X/G)^{\sim}_{ét}$, we have*

$$H^i(X/H_0)_{ét}, \mathcal{F}) = \varinjlim_{H_0 \subseteq H \subseteq G} H^i(X/H)_{ét}, \mathcal{F}).$$

Proposition 2.32. *Suppose that X is a locally noetherian adic space which is analytic and moreover equipped with a continuous action of a locally profinite group G. Let $\mathcal{F}$ be a pointed sheaf on $(X/G)_{ét}$. Then $\mathcal{F}$ is trivial if and only if its pullback to $X_{ét}$ is trivial.*

VI. Scholze's functor

With these preparations, now let us briefly recall Scholze's construction of his functor. We fix some notations first.

So let $n \geq 1$ be an integer, F be a finite extension of $\mathbb{Q}_p$ and K is a open subgroup of $\mathrm{GL}_n(F)$. Denote by $\breve{F}$ the completion of the maximal unramified extension of F and by $\mathcal{M}_{\mathrm{LT},\infty}$ the perfectoid space over $\breve{F}$ constructed in [29], so that

$$\mathcal{M}_{\mathrm{LT},\infty} \sim \varprojlim_{K} \mathcal{M}_{\mathrm{LT},K}.$$

Here $\mathcal{M}_{\mathrm{LT},K}$ is the smooth rigid-analytic Lubin-Tate space over $\breve{F}$ at finite level K, cf. [14]. Let D denote the central division algebra over F of invariant $1/n$. Then for π an admissible $\mathbb{F}_p$-representation of $\mathrm{GL}_n(F)$, one may construct a sheaf $\mathcal{F}_\pi$ on the site $(\mathbb{P}^{n-1}_{\breve{F}}/D^\times)_{\text{ét}}$ equivariant for the Weil descent datum, by descending the trivial sheaf π along the Gross-Hopkins map

$$\pi_{\mathrm{GH}} : \mathcal{M}_{\mathrm{LT},\infty} \to \mathbb{P}^{n-1}_{\breve{F}}$$

which can be considered as a $\mathrm{GL}_n(F)$-torsor.

Proposition 2.33. *The association mapping a $D^\times$-equivariant étale map $U \to \mathbb{P}^{n-1}_{\breve{F}}$ to the $\mathbb{F}_p$-vector space*

$$\mathrm{Map}_{\mathrm{cont},\mathrm{GL}_n(F)\times D^\times}(|U \times_{\mathbb{P}^{n-1}_{\breve{F}}} \mathcal{M}_{\mathrm{LT},\infty}|, \pi)$$

of continuous $\mathrm{GL}_n(F) \times D^\times$*-equivariant maps defines a Weil-equivariant sheaf* $\mathcal{F}_\pi$ *on* $(\mathbb{P}^{n-1}_{\breve{F}}/D^\times)_{\acute{e}t}$*. The association* $\pi \mapsto \mathcal{F}_\pi$ *is exact, and all geometric fibers of* $\mathcal{F}_\pi$ *are isomorphic to* π*.*

Let $C/\breve{F}$ be an algebraically closed complete extension with ring of integers $\mathcal{O}_C$.

Theorem 2.34. *(Scholze) For any* $i \geq 0$*, the* $D^\times$*-representation* $H^i_{\acute{e}t}(\mathbb{P}^{n-1}_C, \mathcal{F}_\pi)$ *is admissible, independent of* C*, and vanishes for* $i > 2(n-1)$*. Taking* $C = \mathbb{C}_p$*, the action of the Weil group* W_F *on* $H^i_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ *extends to an action of the absolute Galois group* G_F *of* F*. More generally, the same statements hold when* π *is replaced by an admissible* $A[\mathrm{GL}_n(F)]$*-module with* A *a complete Noetherian local ring with finite residue field of characteristic* p*.*

To explain the last statement of the above theorem, we recall:

Definition 2.35. Let A be a complete noetherian local ring, $\mathfrak{m}$ be its maximal ideal such that the residue field $A/\mathfrak{m}$ is finite of characteristic p. Let G be a p-adic analytic group. Then we call an $A[G]$-module V to be smooth if for every $v \in V$, there exists an open subgroup $H \subseteq G$ and an integer $j \geq 1$ such that v is invariant under the action of H, and $\mathfrak{m}^i v = 0$.

We call a smooth $A[G]$-module V to be admissible if for every integer $j \geq 1$ and H an open subgroup of G, $V^H[\mathfrak{m}^j]$ is finitely generated as a module over $A/\mathfrak{m}^j$.

Using a result of M. Strauch (cf. Theorem 4.4 of [32]), one can show that the cohomology group in degree 0 is always computable.

Theorem 2.36. *Let* V *be an admissible* $A[\mathrm{GL}_n(F)]$*-module as in the above definition, and let* $\mathcal{F}_V$ *be the associated sheaf on* $(\mathbb{P}^{n-1}_C/D^\times)_{\acute{e}t}$*. Then the map induced by the inclusion* $V^{\mathrm{SL}_n(F)} \hookrightarrow V$

$$H^0_{\acute{e}t}(\mathbb{P}^{n-1}_C, \mathcal{F}_{V^{\mathrm{SL}_n(F)}}) \to H^0_{\acute{e}t}(\mathbb{P}^{n-1}_C, \mathcal{F}_V)$$

is an isomorphism. Moreover, the action of the triple $\mathrm{GL}_n(F) \times W_F \times D^\times$ *can be described explicitly.*

Remark 2.37. The above theorem plays an important role in Scholze's proof of the complete classification in the mod p local-global compatibility in the case of $n = 2$, and its failure in higher dimensions is one of the source of difficulties to obtain a complete result as in the $n = 2$ case.

Remark 2.38. Another important topic is the vanishing property of $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ for different choices of smooth admissible representations π of $\mathrm{GL}_n(K)$. As mentioned before, Scholze has shown that it vanishes for every π whenever $i > 2(n-1)$.

In [20], Johansson-Ludwig proved that for an appropriate parabolic induction $\pi = \mathrm{Ind}_{P^*(K)}^{\mathrm{GL}_n(K)}\sigma$ of $\mathrm{GL}_n(K)$, the group $H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_\pi)$ vanishes for $i > n-1$, strengthening a previous result of Ludwig [24]. The latter was already used by Paskunas to show a non-vanishing result in degree one for (a version of) Scholze's functor for Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ corresponding via the p-adic local Langlands correspondence to reducible two-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. So it is natural to ask if there are similar consequences for $\mathrm{GL}_2(K)$ with a general $K/\mathbb{Q}_p$. We remark that in [20] the vanishing result is proved by a perfectoidness property of $\mathcal{M}_{\mathrm{LT},\infty}/P(K)$ which is the quotient of infinite level Lubin-Tate space by a parabolic subgroup P of GL_n, based on a construction of a perfectoid overconvergent anticanonical tower for certain Harris-Taylor Shimura varieties.

3. Global setup

Introduction. In this section we consider some unitary similitude Shimura varieties and their cohomology. These so-called Kottwitz-Harris-Taylor type Shimura varieties are singled out and studied in [21], and then further in [16] (on a subclass of it), hence the terminology. The class of varieties that we will consider is a variant of the KHT class, but we still call it KHT type. We remark here that we choose to work with such Shimura varieties (more precisely, the class in [2]) for the following two reasons. First, the tubular neighborhoods for them that can be p-adically uniformized by Drinfeld spaces are the whole of their associated adic spaces, which thus provides a bridge between the cohomology of these varieties and that of Drinfeld spaces; by the duality result in [29] the latter is related to cohomology of Lubin-Tate spaces which in turn is closely related to Scholze's functor. Second, the Rapoport-Zink uniformization result, cf. [26], is realized at the level of places of $\mathbb{Q}$ while we need to deal with representations of GL_n at the level of a specific place of a CM field rather than a bunch of such places over a place of $\mathbb{Q}$; Boutot-Zink provides the correct generalization of Rapoport-Zink's results in a parallel manner (see, however, Remark 4.8 and 5.5). On the other hand, in (2.10) of [11] an explicit relation between these two types of Shimura varieties (as well as their cohomology groups), is given.

Thus we will work under the context of [2]; let us first introduce some notations and establish the global setup. One can refer to chapter 0 of [2] for more details. We first change the notation to a global setup. Fix an integer $n > 2$ and a rational prime integer p. Let K be a CM field with $F \subseteq K$ the maximal totally real subfield. Let B be a division algebra over K of dimension n^2 with an involution of the second kind $'$ which we assume to be positive, i.e., for all nonzero $x \in B$ we have $tr_{B/\mathbb{Q}}(xx') > 0$. Let $W = B$ as a $B \otimes_K B^{op}$-module and $\psi : W \times W \to \mathbb{Q}$ an alternating nondegenerate pairing such that

$$\psi(bw_1, w_2) = \psi(w_1, b'w_2)$$

for all $w_1, w_2 \in W$ and $b \in B$.

Let $*$ be the unique involution on B such that

$$\psi(w_1 b, w_2) = \psi(w_1, w_2 b^*)$$

for all $w_1, w_2 \in W$ and $b \in B$. Fix an infinite place $\alpha : K \hookrightarrow \mathbb{C}$ of K and its complex conjugate $\bar{\alpha}$, as well as an embedding $v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We denote by $\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_m$ the prime ideals of $\mathcal{O}_F$ lying over p with $\mathfrak{p}_0$ induced by the embedding

v above and set $\mathfrak{p} := \mathfrak{p}_0$. We assume that they all split in K with

$$\mathfrak{p}_i\mathcal{O}_K = \mathfrak{q}_i\bar{\mathfrak{q}}_i, \quad \mathfrak{q}_i \neq \bar{\mathfrak{q}}_i, \quad i = 0, 1, \ldots, m.$$

The prime ideals $\mathfrak{q}_0$ and $\bar{\mathfrak{q}}_0$ will also be denoted by $\mathfrak{q}$ and $\bar{\mathfrak{q}}$ respectively. We assume that $B_{\mathfrak{q}} := B \otimes_K K_{\mathfrak{q}}$ is a division algebra of invariant $1/n$.

We also fix a maximal order $\mathcal{O}_B$ of B such that $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is fixed under the involution $b \to b'$. We also write Γ for $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ when we regard it as a submodule of $W \otimes \mathbb{Q}_p$.

Let $\tilde{G}$ be the algebraic group over F whose group of R-points for any F-algebra R is given by

$$\tilde{G}(R) := \{(g, \lambda) \in (B^{op} \otimes_F R)^\times \times R^\times \,|\, gg^* = \lambda\}$$

and let $G := \mathrm{Res}_{F/\mathbb{Q}}(\tilde{G})$ be the Weil restriction of scalars so that for any $\mathbb{Q}$-algebra T we have

$$G(T) := \{(g, \lambda) \in (B^{op} \otimes_{\mathbb{Q}} T)^\times \times (F \otimes_{\mathbb{Q}} T)^\times \,|\, gg^* = \lambda\}.$$

We will consider Shimura varieties associated with compact open subgroups C of

$$G(\mathbb{A}_f) = \tilde{G}(\mathbb{A}_{F,f})$$

with the form $C = C_p C^p$ where $C^p \subseteq G(\mathbb{A}_f^p)$ is compact open and

$$C_p \subseteq G(\mathbb{Q}_p) = \prod_{i=0}^{m} \tilde{G}(F_{\mathfrak{p}_i})$$

decomposes as

$$C_p = \prod_{i=0}^{m} C_{\mathfrak{p}_i}, \quad C_{\mathfrak{p}_i} \subseteq \tilde{G}(F_{\mathfrak{p}_i}).$$

As the involution $*$ induces an isomorphism

$$B_{\bar{\mathfrak{q}}_i} \xrightarrow{\sim} B_{\mathfrak{q}_i}^{op},$$

there are identifications for all i

$$\tilde{G}(F_{\mathfrak{p}_i}) \cong (B_{\mathfrak{q}_i}^{op})^\times \times F_{\mathfrak{p}_i}^\times.$$

In fact, in later parts C will usually be of the form $C = C_{\mathfrak{p}} C^{\mathfrak{p}}$ with $C^{\mathfrak{p}} \subseteq \tilde{G}(\mathbb{A}_{F,f}^{\mathfrak{p}})$ (sufficiently small) compact open and $C_{\mathfrak{p}} = U \times \mathcal{O}_{F_{\mathfrak{p}}}^\times \subseteq \tilde{G}(F_{\mathfrak{p}}) = (B_{\mathfrak{q}}^{op})^\times \times F_{\mathfrak{p}}^\times$ with $U \subseteq (B_{\mathfrak{q}}^{op})^\times$ compact open, in which case we write $\mathrm{Sh}_{UC^{\mathfrak{p}}}$ instead of $\mathrm{Sh}_{U \times \mathcal{O}_{F_{\mathfrak{p}}}^\times \times C^{\mathfrak{p}}}$ for the Shimura variety associated with C.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ be the Deligne torus and h be a morphism

$$h : \mathbb{S} \to G_{\mathbb{R}}$$

such that h defines on $W_{\mathbb{R}}$ a Hodge structure of type $(1,0),(0,1)$ and such that $\psi(w_1, h(i)w_2)$ is a symmetric positive definite bilinear form on $W_{\mathbb{R}}$. Note that h is unique up to $G(\mathbb{R})$-conjugacy and we let X denote the $G(\mathbb{R})$-conjugacy class of h. Then (G,X) defines a Shimura datum and for sufficiently small compact open subgroups $C \subseteq G(\mathbb{A}_f)$ as above we have a projective system of Shimura varieties Sh_C over its reflex field, denoted by E.

The morphism h defines a Hodge structure

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

and we require the following condition to hold: the trace of the action of an element $b \in B$ on $W^{1,0}$ is of the form

$$tr_{\mathbb{C}}(b|W^{1,0}) = \Sigma_{i:K\to\mathbb{C}} r_i(tr^0 b)$$

with r_i integers such that r_i is of type $(1, n-1)$ at α and $(0,n)$ at other infinite places. Then we have $E = \alpha(K)$ and $E_v \cong K_{\mathfrak{q}}$. Denote by κ the residue field of E_v.

We fix a tame level, i.e. a compact open subgroup $C^{\mathfrak{p}}$ of $\tilde{G}(\mathbb{A}^{\mathfrak{p}}_{F,f})$ and let $\mathcal{P}$ denote the set of finite places w of K such that

- $w|_{\mathbb{Q}} \neq p$;
- w is split over F;
- B is split at w (i.e., $\tilde{G}(F_u) \cong \mathrm{GL}_n(F_u) \times F_u^{\times}$ where $u = w|_F$) and the component C_u of $C^{\mathfrak{p}}$ at u is maximal.

Consider the abstract Hecke algebra

$$\mathbb{T} = \mathbb{T}_{\mathcal{P}} := \mathbb{Z}[T_w^{(j)} : w \in \mathcal{P}, j = 1,2,\ldots,n]$$

where $T_w^{(j)}$ is the Hecke operator corresponding to the double coset

$$\Big[\mathrm{GL}_n(\mathcal{O}_{F_w})\begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_w})\Big].$$

Here ϖ_w is a uniformizer of the local field F_w. Then the Hecke algebra $\mathbb{T}$ acts on $H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p)$ for all compact open $U \subseteq (B^{op}_{\mathfrak{q}})^{\times}$. Fix a finite field $\mathbb{F} := \mathbb{F}_q$ with q elements for q a power of p. Let

$$\bar{\sigma} : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{F}_q)$$

be an absolutely irreducible continuous representation of Gal_K. We can associate to $\bar{\sigma}$ (together with $\mathcal{P}$) a maximal ideal $\mathfrak{m}$ of $\mathbb{T}$; it is the kernel of the map

$$\mathbb{T} \to \mathbb{F}_q, \quad T_w^{(j)} \mapsto (-1)^j (\mathbf{N}w)^{-j(j-1)/2} a_w^{(j)}, \quad w \in \mathcal{P},\ j = 1,2,\ldots,n$$

where Frob_w for $w \in \mathcal{P}$ is the geometric Frobenius of Gal_{K_w}, $\mathbf{N}w$ is the cardinality of the residue field of K_w, and the $a_w^{(j)} \in \mathbb{F}_q$ are such that the characteristic polynomial of $\bar{\sigma}(\mathrm{Frob}_w)$ equals

$$X^n + \cdots + a_w^{(j)} X^{n-j} + \cdots + a_w^{(n)}.$$

Thus $\mathfrak{m}$ is a maximal ideal containing p; we assume further that

Condition 3.1. *For all compact open U*

$$H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p)_{\mathfrak{m}} \neq 0$$

only when $i = n-1$.

Remark 3.2. The above condition on vanishing of cohomology outside of the middle degree is satisfied, for instance, when we impose certain generic condition on $\mathfrak{m}$ or on its associated Galois representation, cf. eg. Théorème 4.7 of [3], and also the main results of [7].

Remark 3.3. The reason that we impose the above condition is twofold: on the one hand, it gives an easier comparison between the completed cohomology and cohomology with infinite level at $\mathfrak{p}$ in p-torsion coefficients, of the Shimura varieties; on the other hand, under the flatness assumption in Section 7 it implies an important injectivity result (cf. Lemma 7.9).

Let $\mathbb{T}(UC^{\mathfrak{p}})$ be the image of $\mathbb{T}$ in $\mathrm{End}(H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}))$ and $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$ its $\mathfrak{m}$-adic completion. Note then that $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$ is isomorphic to the localization at $\mathfrak{m}$ of the image of $\mathbb{Z}_p[T_w^{(i)} : w \in \mathcal{P}, i = 1, 2, \ldots, n]$ in $\mathrm{End}(H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p))$, which one meets more often in the literature. (This is because the localization at a maximal ideal $\mathfrak{m}$ of a finite $\mathbb{Z}_p$-algebra is automatically complete with respect to the $\mathfrak{m}$-adic topology.) Thus $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$ acts faithfully on $H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p)_{\mathfrak{m}}$ and we have an associated Galois representation:

Proposition 3.4. *Assume from now on that $\mathfrak{m}$ satisfies the following Assumption 3.5. There is a unique (up to conjugation) continuous n-dimensional Galois representation*

$$\sigma = \sigma_{\mathfrak{m}} : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}})$$

unramified at almost all places, such that for every $w \in \mathcal{P}$, $\sigma(\mathrm{Frob}_w)$ has characteristic polynomial

$$X^n + \cdots + (-1)^j (\boldsymbol{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \cdots + (-1)^n (\boldsymbol{N}w)^{n(n-1)/2} T_w^{(n)}.$$

Proof. As the $\mathrm{Sh}_{UC^{\mathfrak{p}}}$'s are projective, Matsushima's formula (cf. VII. 5.2 of [1]) gives an isomorphism

$$H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}}}(\mathbb{C}), \overline{\mathbb{Q}}_p) \cong \bigoplus_{\pi} \pi_f^{U\times\mathcal{O}_{F_{\mathfrak{p}}}^{\times}\times C^{\mathfrak{p}}} \otimes H^{n-1}(\mathrm{Lie}\, G(\mathbb{R}), K_\infty, \pi_\infty)$$

where π runs over irreducible constituents (taken with its multiplicity) of the space of automorphic forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$, and K_∞ is a maximal compact subgroup of $G(\mathbb{R})$. Then by Artin comparison theorem (cf. [15], Exposé XI, Théorème 4.4) we have $H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}}}(\mathbb{C}), \overline{\mathbb{Q}}_p) \cong H^{n-1}_{\text{ét}}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ and we may write

$$H^{n-1}_{\text{ét}}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \overline{\mathbb{Q}}_p) = \bigoplus_{\pi} \pi_f^{U\times\mathcal{O}_{F_{\mathfrak{p}}}^{\times}\times C^{\mathfrak{p}}} \otimes R^{n-1}(\pi)$$

where π runs over cuspidal automorphic representations of $G(\mathbb{A})$ over $\overline{\mathbb{Q}}_p$ (taken with its multiplicity) and where $R^{n-1}(\pi)$ is a finite dimensional continuous representation of $\mathrm{Gal}(\overline{K}/K)$. Localizing both sides at $\mathfrak{m}$, we obtain

$$H^{n-1}_{\text{ét}}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \overline{\mathbb{Q}}_p)_{\mathfrak{m}} = \bigoplus_{\pi\in\mathcal{A}(\mathfrak{m})} \pi_f^{U\times\mathcal{O}_{F_{\mathfrak{p}}}^{\times}\times C^{\mathfrak{p}}} \otimes R^{n-1}(\pi) \tag{1}$$

where $\mathcal{A}(\mathfrak{m})$ denotes the subset of cuspidal automorphic representations π of G such that the kernel of the corresponding map $\psi_\pi : \mathbb{T}(UC^{\mathfrak{p}}) \to \overline{\mathbb{Q}}_p$ induced by π is contained in $\mathfrak{m}$. As $\bar{\sigma}_{\mathfrak{m}}$ is assumed to be absolutely irreducible, each of the π belonging to $\mathcal{A}(\mathfrak{m})$ will have base change (ψ, Π) to $\mathbb{A}_E^\times \times \mathrm{GL}_n(\mathbb{A}_K)$ so that Π is cuspidal. We now make the following assumption on $\mathfrak{m}$ to apply the known cases of the Tate conjecture in the setting of Shimura varieties, see Remark 3.6 however.

Assumption 3.5. *Assume that for every $\pi \in \mathcal{A}(\mathfrak{m})$, the Galois representation $\rho_{\Pi,p}$ associated with Π by the global Langlands correspondence is strongly irreducible, i.e., $\rho_{\Pi,p}$ is irreducible and not induced from a proper open subgroup of* Gal_K.

Thus by Theorem 2.25 of [11], each $R^{n-1}(\pi)$ is a semisimple $\overline{\mathbb{Q}}_p[\mathrm{Gal}(\overline{K}/K)]$-module. Then by Proposition VII.1.8 and Proposition VI.2.7 of [16] (or similarly by Theorem 6.4 and Corollary 6.5 of [33]), for every π in $\mathcal{A}(\mathfrak{m})$, $R^{n-1}(\pi)$ is a direct sum of finite copies of an n-dimensional Gal_K representation $\tilde{R}^{n-1}(\pi)$.

We get from (1) an isomorphism

$$\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \xrightarrow{\sim} \prod_{\pi\in\mathcal{A}(\mathfrak{m})} \overline{\mathbb{Q}}_p$$

by sending $T_w^{(i)}$ on the left hand side to its corresponding Hecke eigenvalue on $\pi_w^{\mathrm{GL}_n(\mathcal{O}_w)}$. This isomorphism is defined over a finite extension of $\mathbb{Q}_p$ as $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$ is of finite type over $\mathbb{Z}_p$. Collecting the $\tilde{R}^{n-1}(\pi)$'s above we obtain a representation

$$\mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}[1/p]).$$

On the other hand, all characteristic polynomials of Frobenius elements take values in $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$, so one obtains a determinant with values in $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$. But $\bar{\sigma}$ is absolutely irreducible by assumption, so one obtains the existence of the desired representation σ, cf. Theorem 2.22 of [8]. That for every $w \in \mathcal{P}$ the Frobenius action $\sigma(\mathrm{Frob}_w)$ has the exhibited characteristic polynomial is classical and is proved in Theorem 1 of [21] (where the unitary group has similitude factor in $\mathbb{G}_{m/\mathbb{Q}}$ but it works for groups with $\mathbb{G}_{m/F}$-similitude factors, too), cf. also [36]. Finally the uniqueness follows from the fact that the set $\{\mathrm{Frob}_w | w \in \mathcal{P}\}$ is dense inside Gal_K, which is a consequence of Chebotarev's theorem. □

Remark 3.6. By a strong form of the Tate Conjecture (see eg. [25]), the semisimplicity of the $R^{n-1}(\pi)$'s above always holds true but at this moment it is not fully proved yet, so we made the assumption 3.5 on $\mathfrak{m}$ to use the results of [11].

In particular, we have $\sigma \pmod{\mathfrak{m}} = \overline{\sigma}$. Recall (Definition 5.2, [28]) that for a ring R, a group G and an n-dimensional representation $\sigma_R : G \to \mathrm{GL}_n(R)$ of G, an $R[G]$-module is said to be σ_R-typic if there exists an R-module M_0 such that $M = \sigma_R \otimes_R M_0$ with G acting trivially on M_0. The following proposition generalizes the case of $n = 2$.

Proposition 3.7. *The* $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_K]$*-module* $H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}}$ *is* σ*-typic.*

Proof. Since $H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}}$ is a submodule of $H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}}[1/p]$ (over the ring $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_K]$), by Proposition 5.4 of [28] it suffices to show that

$$H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Z}_p)_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$$

is σ-typic. But this follows from the description of the $\overline{\mathbb{Q}}_p$-cohomology of $\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}$, as in the proof of Proposition 3.4. Indeed, we may shift the multiplicity of each Galois representation $\tilde{R}^{n-1}(\pi)$ to $\pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}}$ and then use the isomorphism of Gal_K-modules $\pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}} \otimes_{\overline{\mathbb{Q}}_p} \tilde{R}^{n-1}(\pi) \cong \pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}} \otimes_{\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}} (\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}})^{\oplus n}$ obtained from the following diagram

$$\begin{array}{ccc} \pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}} \otimes_{\overline{\mathbb{Q}}_p} \tilde{R}^{n-1}(\pi) & & \pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}} \otimes_{\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}} (\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}})^{\oplus n} \\ & \searrow \quad \swarrow & \\ & \pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}} \otimes_{\mathcal{T}} (\mathcal{T})^{\oplus n} & \end{array}$$

where $\mathcal{T} = \mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ and both arrows are isomorphisms of Gal_K-modules induced by the natural inclusion on the second factor of tensor product. By summing up over $\pi \in \mathcal{A}(\mathfrak{m})$, we obtain an isomorphism

$$\bigoplus_{\pi \in \mathcal{A}(\mathfrak{m})} \left(\pi_f^{U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}} \otimes_{\overline{\mathbb{Q}}_p} R^{n-1}(\pi)\right) \cong M \otimes_{\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}} \mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}^{\oplus n}$$

as Gal_K-modules for some $\mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$-module M.

□

Now we pass to completed cohomology. It will be essentially the cohomology with torsion coefficients of Section 5 (at least under our ongoing assupmtion on $\mathfrak{m}$ that the cohomology is concentrated in middle degree when localized at $\mathfrak{m}$, cf. Condition 3.1). Let

$$\tilde{H}^{n-1}(C^{\mathfrak{p}}, \mathbb{Z}_p) := \varprojlim_k \varinjlim_U H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}}, \overline{K}}, \mathbb{Z}/p^k\mathbb{Z}),$$

and

$$\tilde{H}^{n-1}(C^{\mathfrak{p}}, \mathbb{Z}_p)_{\mathfrak{m}} := \varprojlim_k \varinjlim_U H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}}, \overline{K}}, \mathbb{Z}/p^k\mathbb{Z})_{\mathfrak{m}}$$

where U runs over all compact open subgroups of $(B_{\mathfrak{q}}^{op})^{\times}$ in each definition.

Then the inverse limit

$$\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}} := \varprojlim_U \mathbb{T}(UC^{\mathfrak{p}})_{\mathfrak{m}}$$

acts faithfully and continuously on $\tilde{H}^{n-1}(C^{\mathfrak{p}}, \mathbb{Z}_p)_{\mathfrak{m}}$. Then the same argument as in the proof of Proposition 5.7, [28] shows the following result.

Proposition 3.8. *There is a unique (up to conjugation) continuous n-dimensional Galois representation*

$$\sigma = \sigma_{\mathfrak{m}} : G_K \to \mathrm{GL}_n(\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}})$$

unramified at almost all places, such that for every $w \in \mathcal{P}$, $\sigma(\mathrm{Frob}_w)$ has characteristic polynomial

$$X^n + \cdots + (-1)^j (\boldsymbol{N}w)^{j(j-1)/2} T_w^{(j)} X^{n-j} + \cdots + (-1)^n (\boldsymbol{N}w)^{n(n-1)/2} T_w^{(n)}.$$

The ring $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$ is a complete Noetherian local ring with finite residue field.

Proposition 3.9. *The $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_K]$-module $\tilde{H}^{n-1}(C^{\mathfrak{p}}, \mathbb{Z}_p)_{\mathfrak{m}}$ is σ-typic.*

Proof. This follows from Propostion 3.7, noting that the σ's are compatible with each other and that all operations in the definition of

$$\tilde{H}^{n-1}(C^{\mathfrak{p}}, \mathbb{Z}_p)_{\mathfrak{m}} = \varprojlim_k \varinjlim_U H^{n-1}(\mathrm{Sh}_{UC^{\mathfrak{p}}, \overline{K}}, \mathbb{Z}/p^k\mathbb{Z})_{\mathfrak{m}}$$

preserve the property (of modules) of being σ-typic. $\square$

4. p-ADIC UNIFORMIZATION OF KHT TYPE SHIMURA VARIETIES

Now we describe a p-adic (in contrast to classical complex-analytic) uniformization result of the Shimura varieties defined before to compare local and global cohomology groups. We will use the results of Boutot-Zink which introduces a variant of the moduli methods of Rapoport-Zink and which obtains the same uniformization results (up to some Galois twists) with Varshavsky [34], [35].

We keep the notations from the last section on the global context. Let us consider the following moduli problem, which is a variant of the one in [26]. For C a compact open subgroup of $\tilde{G}(\mathbb{A}_{F,f})$, let $\mathcal{A}_C$ be the functor on the category of $\mathcal{O}_{E_v}$-schemes, which sends an $\mathcal{O}_{E_v}$-scheme S to the set of isomorphism classes of tuples $(A, \Lambda, \{\lambda_i\}_{i=1}^m, \bar{\eta}^p, \{\bar{\eta}_{\mathfrak{q}_i}\}_{i=1}^m)$ where

1. A is an abelian $\mathcal{O}_K$-scheme over S up to isogeny of order prime to $\mathfrak{p}$ together with an action of $\mathcal{O}_B$
$$\iota : \mathcal{O}_B \to \mathrm{End}(A);$$
2. Λ is the one dimensional vector space over F generated by an F-homogeneous polarization λ of A which is principal in $\mathfrak{p}$;
3. for each $i = 1, \dots, m$, λ_i is a generator of $\Lambda \otimes_F F_{\mathfrak{p}_i} \mod (C_{\mathfrak{p}_i} \cap F_{\mathfrak{p}_i}^\times)$;
4. $\bar{\eta}^p$ is a class of isomorphisms of $B \otimes \mathbb{A}_f^p$-modules
$$\bar{\eta}^p : V^p(A) \to W \otimes \mathbb{A}_f^p \mod C^p$$
which preserve the Riemannian form on $V^p(A)$ induced by any polarization $\lambda \in \Lambda$ and the pairing ψ on $W \otimes \mathbb{A}_f^p$ up to a constant in $(F \otimes \mathbb{A}_f^p)^\times$;
5. for each $i = 1, \dots, m$, $\bar{\eta}_{\mathfrak{q}_i}$ is class of isomorphisms of $B_{\mathfrak{q}_i}$-modules
$$\bar{\eta}_{\mathfrak{q}_i} : V_{\mathfrak{q}_i}(A) \to W_{\mathfrak{q}_i} \mod C_{\mathfrak{q}_i}.$$

such that the following properties are satisfied:

(a) The involution $b \to b'$ on $\mathcal{O}_B$ coincides with the one obtained from the Rosati involution on $\mathrm{End}(A)$ induced by Λ;
(b) there is an equality of polynomial functions on $\mathcal{O}_B$, called the determinant condition:
$$\det_{\mathcal{O}_S}(b, \mathrm{Lie} A) = \det_{\overline{\mathbb{Q}}_p}(b, W_0)$$

Remark 4.1. One can formulate condition (b) in terms of the associated p-divisible group X of A. Indeed, the determinant condition amounts to requiring: (1) $X_{\mathfrak{q}}$ is a special formal $\mathcal{O}_{B_{\mathfrak{q}}}$-module à la Drinfeld; (2) $X_{\mathfrak{q}_i}$ is étale for each $i = 1, \dots, m$.

It is proved in [2] that

Proposition 4.2. *If C is sufficiently small, then the étale sheafification of $\mathcal{A}_C$ is represented by a projective scheme over $\mathcal{O}_{E_v}$, still denoted by $\mathcal{A}_C$. For such varying C these schemes form a projective system $\{\mathcal{A}_C\}$ with finite transition maps.*

It is natural to equip the system $\{\mathcal{A}_C\}$ with a right action of $G(\mathbb{A}_f)$, for the details one may refer to [2]. Next we consider moduli problems for p-divisible formal $\mathcal{O}_{B_\mathfrak{q}}$-modules. We fix a special formal $\mathcal{O}_{B_\mathfrak{q}}$-module Φ over $\overline{\kappa}$ and denote its dual by $\hat{\Phi}$. We will often write $\mathbb{X}$ for $\Phi \times \hat{\Phi}$.

Let $\mathcal{M}$ be the functor on the category of $\mathcal{O}_{\breve{E}_v}$-schemes where p is locally nilpotent, which assigns such a scheme T the set of isomorphism classes of pairs (X, ρ) where

(1) X is a p-divisible $\mathcal{O}_{B_\mathfrak{p}}$-module of special type over T;
(2) ρ is a quasi-isogeny of $\mathcal{O}_{B_\mathfrak{p}}$-modules over T

$$\rho : \mathbb{X} \times_{\mathrm{Spec}\,\kappa} \overline{T} \to X \times_T \overline{T}.$$

Here $\overline{T} := T \times_{\mathrm{Spec}\,\mathbb{Z}_p} \mathrm{Spec}\,\mathbb{F}_p$ and it is regarded as a scheme over $\mathrm{Spec}\,\overline{\kappa}$ via $\overline{T} \to \mathrm{Spec}\,(\mathcal{O}_{\breve{E}_v}/p) \to \mathrm{Spec}\,\overline{\kappa}$. Note that by rigidity of quasi-isogenies, ρ gives rise to a quasi-isogeny $\beta : \hat{X}_2 \to X_1$; we require that Zariski locally on T one can find an element $h \in F_\mathfrak{p}^\times$ such that $h\beta$ is an isomorphism.

Proposition 4.3. *$\mathcal{M}$ is representable by a p-adic formal scheme over $\mathcal{O}_{\breve{E}_v}$, which is equipped with a Weil descent datum. Furthermore, it acquires natural actions of two groups, the group J of self quasi-isogenies of the $\mathcal{O}_{B_\mathfrak{p}}$-module $\mathbb{X}$ preserving the polarization up to a constant in $F_\mathfrak{p}^\times$, and $\tilde{G}(F_\mathfrak{p})$.*

Now we are ready to state the p-adic uniformization result. For this we fix a point $(A_s, \Lambda_s, \{\lambda_{s.i}\}_{i=1}^m, \bar{\eta}_s{}^p, \{\bar{\eta}_{s,\mathfrak{q}_i}\}_{i=1}^m) \in \mathcal{A}_C(\bar{\kappa})$ for a sufficiently small C. Using a theorem of Serre-Tate, one can define a uniformization morphism

$$\Theta : \mathcal{M} \times \tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p})/C^\mathfrak{p} \to \mathcal{A}_C \times_{\mathrm{Spec}\,\mathcal{O}_{E_v}} \mathrm{Spec}\,\mathcal{O}_{\breve{E}_v}$$

which is $\tilde{G}(\mathbb{A}_{F,f})$-equivariant. It is then natural to try to find out the fibers of this map. Define

$$\tilde{I}^\bullet(F) = \{\phi \in \mathrm{End}_B^0(A_s) \,|\, \phi \circ \phi' \in F^\times\}$$

where $\phi \mapsto \phi'$ is the Rosati involution induced by Λ on the finite dimensional $\mathbb{Q}$-algebra $\mathrm{End}_B^0(A_s)$. (Here $\mathrm{End}_B^0(A_s)$ means $\mathrm{End}_B(A_s) \otimes_\mathbb{Z} \mathbb{Q}$ which is usually called the endomorphism ring up to isogeny.) Then we regard $\tilde{I}^\bullet$ as an algebraic group over F such that its F-points are given as above.

Let G' be the inner form of G over F such that $G'(F \otimes_\mathbb{Q} \mathbb{R})$ is compact modulo center, $G'(\mathbb{A}_{F,f}^\mathfrak{p}) = \tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p})$, and $G'(F_\mathfrak{p}) \cong \mathrm{GL}_n(F_\mathfrak{p}) \times \mathcal{O}_{F_\mathfrak{p}}^\times$. Then G' is the unitary similitude group associated with (D, μ) where D is a division algebra over K

of dimension n^2 with an involution μ of the second kind over F satisfying the following conditions:

- D splits at $\mathfrak{q}$,
- (D,μ) and $(B,')$ are locally isomorphic at all finite places except $\mathfrak{q}$,
- μ is positive definite at all archimedean places of F.

Remark 4.4. The existence of such D and μ follows from the results of Kottwitz and Clozel (see [9], Prop. 2.3]).

Theorem 4.5. *(**p-adic uniformization of Shimura varieties**, Theorem 0.16,* [2]*)*
For any compact open subgroup $C \subseteq \tilde{G}(\mathbb{A}_{F,f})$, there is an isomorphism of rigid analytic spaces over $Sp\breve{E}_v$:

$$I^\bullet(\mathbb{Q})\backslash(\mathfrak{X} \times F_\mathfrak{p}^\times) \times \tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p})/C \xrightarrow{\sim} \mathrm{Sh}^{rig}_{(G,\tilde{h}),C} \times_{SpE_v} Sp\breve{E}_v$$

which is $\tilde{G}(\mathbb{A}_{F,f})$-equivariant and compatible with the Weil descent data on both sides.

We explain a bit on the notations and the proof of this theorem. It is essentially a consequence of the corresponding p-adic uniformization of Shimura varieties that are moduli spaces of abelian varieties with PEL structures. Here $\tilde{h}$ is a well chosen morphism $\mathbb{S} \to G_\mathbb{R}$ in the definition of Shimura datum to account for certain Galois twist which occurs when comparing some $\mathrm{Sh}_{(G,h)}$ and its appropriate PEL type avatar. Then one proves the following uniformization result for PEL type Shimura varieties following the method of Rapoport-Zink, [26].

Theorem 4.6. *(Page 29,* [2]*) There is an isomorphism of rigid analytic spaces over $\breve{E}_v$:*

$$I^\bullet(\mathbb{Q})\backslash(\mathfrak{X} \times F_\mathfrak{p}^\times) \times \tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p})/C \xrightarrow{\sim} \mathcal{A}_C^{rig} \times_{\mathrm{Sp}\,E_v} \mathrm{Sp}\,\breve{E}_v.$$

Here and above, $I^\bullet$ denotes $\mathrm{Res}_{F/\mathbb{Q}}\tilde{I}^\bullet$ and $\tilde{I}^\bullet$ is defined above as the algebraic group of certain self quasi-isogenies of an abelian variety over the residue field of $K_\mathfrak{q}$. It is easy to show that $\tilde{I}^\bullet$ is in fact isomorphic to our G'. Thus we can write $I^\bullet(\mathbb{Q}) = \tilde{I}^\bullet(F) = G'(F)$. Moreover, $\mathfrak{X}$ denotes the rigid analytic pro-covering space over $\breve{\mathcal{N}}^{rig}$, the rigid analytic space associated with the formal scheme $\breve{\mathcal{N}}$ classifying quasi-isogenies of some fixed special formal $O_{B_\mathfrak{q}}$-module over $\bar{\kappa}$, see e.g. 5.34 of [26] for more details. With Theorem 4.6, the p-adic uniformization of Shimura varieties Theorem 4.5 is proved by a comparison between these two types of Shimura varieties, see Lemma 0.9 of [2].

Recall that we have fixed a compact open subgroup $C^\mathfrak{p}$ of $\tilde{G}(\mathbb{A}_{F,f}^\mathfrak{p})$, that U denotes a compact open subgroup of $(B_\mathfrak{q}^{op})^\times$ and that $\mathrm{Sh}_{UC^\mathfrak{p}}$ denotes the Shimura variety associated with the subgroup $U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}$. Thus taking $C = U \times \mathcal{O}_{F_{\mathfrak{p}}}^{\times} \times C^{\mathfrak{p}}$ in Theorem 4.5 (then base change to $\mathbb{C}_p$ and take the associated adic spaces with quasi-separated rigid analytic spaces, cf. 1.1.11 of [18]) we obtain:

Theorem 4.7. *There is an isomorphism of adic spaces over* $\mathbb{C}_p$

$$(\mathrm{Sh}_{UC^{\mathfrak{p}}} \otimes_E \mathbb{C}_p)^{ad} \cong G'(F)\backslash \mathcal{M}_{\mathrm{Dr},U,\mathbb{C}_p} \times (F_{\mathfrak{p}}^{\times}/\mathcal{O}_{F_{\mathfrak{p}}}^{\times}) \times G'(\mathbb{A}_{F,f}^{\mathfrak{p}})/C^{\mathfrak{p}}$$

compatible with varying $U \subseteq (B_{\mathfrak{q}}^{op})^{\times}$ *and with the Weil descent datum to* E_v.

In this theorem, $\mathcal{M}_{\mathrm{Dr},U,\mathbb{C}_p}$ denotes the Drinfeld space of level $U \subseteq (B_{\mathfrak{q}}^{op})^{\times}$, which is a smooth adic space over $\mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p})$, cf. [29]. When U varies, these spaces form the so-called Drinfeld tower $(\mathcal{M}_{\mathrm{Dr},U,\mathbb{C}_p})_U$ with finite étale transitive maps and they are related to the above pro-space $\mathfrak{X}$ via

$$(\mathfrak{X} \otimes_{\breve{E}_v} \mathbb{C}_p)^{ad} \cong \varprojlim_U \mathcal{M}_{\mathrm{Dr},U,\mathbb{C}_p}.$$

Remark 4.8. There is a similar rigid uniformization result of Rapoport-Zink for unitary groups with similitude factor in $\mathbb{G}_{m/\mathbb{Q}}$; in our special case it has the following form

$$(\mathrm{Sh}_{K_pK^p} \otimes_E \mathbb{C}_p)^{ad} \cong G_0(\mathbb{Q})\backslash \mathcal{M}_{\mathrm{Dr},K_p,\mathbb{C}_p} \times G_0(\mathbb{A}_{\mathbb{Q},f}^{p})/K^p$$

for a group G_0 over $\mathbb{Q}$ and K_p (resp. K^p) the level group at p (resp. away from p). Then the Rapoport-Zink space $\mathcal{M}_{\mathrm{Dr},K_p,\mathbb{C}_p}$ decomposes as a product of Rapoport-Zink spaces indexed by places $\mathfrak{p}$ of F over p. Under appropriate conditions on the choice of the PEL Shimura datum, the decomposition reads:

$$\mathcal{M}_{\mathrm{Dr},K_p,\mathbb{C}_p} = \prod_{\mathfrak{p}|p} \mathcal{M}_{\mathrm{Dr},K_{\mathfrak{p}},\mathbb{C}_p} \times (\mathbb{Q}_p^{\times}/\mathbb{Z}_p^{\times})$$

where the component $\mathcal{M}_{\mathrm{Dr},K_{\mathfrak{p}},\mathbb{C}_p}$ for $\mathfrak{p}$ not equal to a fixed one (say $\mathfrak{p}_1$) can be furthermore arranged to be in the étale case, that is, isomorphic to $G_0(F_{\mathfrak{p}})/K_{\mathfrak{p}}$. Then we can make a parallel development for this type Shimura varieties on all the results in our work. The above arrangement on local components at $\mathfrak{p} \neq \mathfrak{p}_1$ is for the purpose of focusing on the fixed place $\mathfrak{p}_1$ when applying the Hodge-Tate period map and matching the fiber of the period map to automorphic forms with infinite level at $\mathfrak{p}_1$. See the proof of Proposition 5.3 and also Remark 5.5 for more explanations.

5. The case of GL_n

Now let us return to our case of $\mathrm{GL}_n, n > 2$.

Definition 5.1. Let $\rho^i_{C^{\mathfrak{p}}}$ be the admissible $\mathbb{Z}_p$-representation of $G_K \times (B^{op}_{\mathfrak{q}})^\times$ given by

$$\rho^i_{C^{\mathfrak{p}}} := \varinjlim_U H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p)$$

where U runs over compact open subgroups of $(B^{op}_{\mathfrak{q}})^\times$. We will often write ρ for $\rho^{n-1}_{C^{\mathfrak{p}}}$. Let $\pi = \pi_{C^{\mathfrak{p}}}$ be the admissible $\mathbb{Z}_p$-representation of $\mathrm{GL}_n(F_{\mathfrak{p}})$ given by the space of continuous functions

$$\pi = \pi_{C^{\mathfrak{p}}} := C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(\mathcal{O}^\times_{F_{\mathfrak{p}}} \times C^{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p).$$

Define

$$\mathrm{Sh}_{C^{\mathfrak{p}}} := G'(F)\backslash \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \times (F^\times_{\mathfrak{p}}/\mathcal{O}^\times_{F_{\mathfrak{p}}}) \times G'(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}}.$$

It is a perfectoid space over $\mathbb{C}_p$ equipped with a Weil descent datum to E_v, such that we have the following similarity relation between adic spaces (see Def. 2.4.1 of [29])

$$\mathrm{Sh}_{C^{\mathfrak{p}}} \sim \varprojlim_U (\mathrm{Sh}_{UC^{\mathfrak{p}}} \otimes_E \mathbb{C}_p)^{ad}.$$

In particular, we have

$$H^i(\mathrm{Sh}_{C^{\mathfrak{p}},\mathbb{C}_p}, \mathbb{Q}_p/\mathbb{Z}_p) = \varinjlim_U H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}_p,}, \mathbb{Q}_p/\mathbb{Z}_p) \tag{2}$$

as $(W_{F_{\mathfrak{p}}} \times B^\times_{\mathfrak{q}})$-representations where $W_{F_{\mathfrak{p}}}$ is the Weil group (here we drop the notation "op" from $(B^{op}_{\mathfrak{q}})^\times$ so that it acts from the right on $\varinjlim_U H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}_p,}, \mathbb{Q}_p/\mathbb{Z}_p)$).

Remark 5.2. It follows from the proper base change theorem (e.g., Tag 0A5I of [31]) that there is a canonical isomorphism

$$H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}_p,}, \mathbb{Q}_p/\mathbb{Z}_p) \cong H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\overline{K}}, \mathbb{Q}_p/\mathbb{Z}_p),$$

which we will use tacitly throughout.

Then we can proceed as Scholze did in [28] to obtain first a weak form of p-adic local-global compatibility. By Proposition 7.1.1 of [29] we have the Hodge-Tate period morphism

$$\pi_{\mathrm{HT}} : \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$$

compatible with Weil descent data, and it is identified with the Grothendieck-Messing period map under the duality isomorphism

$$\mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \cong \mathcal{M}_{\mathrm{LT},\infty,\mathbb{C}_p}\ ,$$

cf. Theorem 7.2.3 of [29]. Now the $\mathrm{GL}_n(F_\mathfrak{p})$-equivariance of the Hodge-Tate period map induces a map

$$\pi^{\mathrm{Sh}}_{\mathrm{HT}} : \mathrm{Sh}_{C^\mathfrak{p},\mathbb{C}_p} = G'(F)\backslash \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \times (F_\mathfrak{p}^\times/\mathcal{O}_{F_\mathfrak{p}}^\times) \times G'(\mathbb{A}^\mathfrak{p}_{F,f})/C^\mathfrak{p} \to \mathbb{P}^{n-1}_{\mathbb{C}_p}.$$

Note that π_{HT} is $(W_{F_\mathfrak{p}} \times B_\mathfrak{q}^\times)$-equivariant.

Proposition 5.3. *There is a $(W_{F_\mathfrak{p}} \times B_\mathfrak{q}^\times)$-equivariant isomorphism of sheaves on the étale site of (the adic space) $\mathbb{P}^{n-1}_{\mathbb{C}_p}$:*

$$R\pi^{\mathrm{Sh}}_{\mathrm{HT}\acute{e}t*}(\mathbb{Q}_p/\mathbb{Z}_p) \cong \mathcal{F}_{\pi_{C^\mathfrak{p}}}.$$

Proof. The proof is almost the same as in [28]. First we check that the higher direct images vanish. It is enough to check this at stalks, so let $\bar{x} = \mathrm{Spa}(C, C^+) \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$ be a geometric point, that is, $C/\breve{E}_v$ is complete algebraically closed and $C^+ \subseteq C$ is an open and bounded valuation subring. We may assume that C is the completion of the algebraic closure of the residue field of $\mathbb{P}^{n-1}_{\mathbb{C}_p}$ at the image of $\bar{x}$. Let

$$\bar{x} \to U_i \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$$

be a cofinal system of étale neighborhoods of $\bar{x}$; then we have $\bar{x} \sim \varprojlim_i U_i$. Write

$$U_i^{\mathrm{Sh}} \to \mathrm{Sh}_{C^\mathfrak{p},\mathbb{C}_p}$$

for the pullback of U_i, so that U_i^{Sh} is a perfectoid space étale over $\mathrm{Sh}_{C^\mathfrak{p},\mathbb{C}_p}$. One can form the inverse limit $U_x^{\mathrm{Sh}} = \varprojlim_i U_i^{\mathrm{Sh}}$ in the category of perfectoid spaces over $\mathbb{C}_p$. We have equalities

$$(R^j\pi^{\mathrm{Sh}}_{\mathrm{HT\acute{e}t}*}(\mathbb{Q}_p/\mathbb{Z}_p))_{\bar{x}} = \varinjlim_i H^j_{\mathrm{\acute{e}t}}(U_i^{\mathrm{Sh}}, \mathbb{Q}_p/\mathbb{Z}_p) = H^j_{\mathrm{\acute{e}t}}(U_{\bar{x}}^{\mathrm{Sh}}, \mathbb{Q}_p/\mathbb{Z}_p).$$

On the other hand, the fiber $U_{\bar{x}}^{\mathrm{Sh}}$ is given by profinitely many copies of $\bar{x}$,

$$U_{\bar{x}}^{\mathrm{Sh}} = \mathrm{Spa}(C^0(G'(F)\backslash\mathcal{G} \times \tilde{G}(\mathbb{A}^\mathfrak{p}_{F,f})/C^\mathfrak{p}, C), C^0(G'(F)\backslash\mathcal{G} \times \tilde{G}(\mathbb{A}^\mathfrak{p}_{F,f})/C^\mathfrak{p}, C^+))$$

where $\mathcal{G} = \mathrm{GL}_n(F_\mathfrak{p}) \times (F_\mathfrak{p}^\times/\mathcal{O}_{F_\mathfrak{p}}^\times)$. This implies that $H^j_{\mathrm{\acute{e}t}}(U_{\bar{x}}^{\mathrm{Sh}}, \mathbb{Q}_p/\mathbb{Z}_p)$ vanishes for $j > 0$, and equals

$$C^0(G'(F)\backslash\mathrm{GL}_n(F_\mathfrak{p}) \times F_\mathfrak{p}^\times \times \tilde{G}(\mathbb{A}^\mathfrak{p}_{F,f})/(\mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \mathbb{Q}_p/\mathbb{Z}_p)$$

in degree 0.

It remains to identify $\pi^{\mathrm{Sh}}_{\mathrm{HT\acute{e}t}*}(\mathbb{Q}_p/\mathbb{Z}_p)$. The previous computation shows that the fibers are isomorphic to $\pi_{C^\mathfrak{p}}$. Let $U \to \mathbb{P}^{n-1}_{\mathbb{C}_p}$ be an étale map. We need to construct a map

$$H^0(U \times_{\mathbb{P}^{n-1}_{\mathbb{C}_p}} (G'(F)\backslash \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \times (F_{\mathfrak{p}}^\times/\mathcal{O}_{F_\mathfrak{p}}^\times) \times G'(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}}), \mathbb{Q}_p/\mathbb{Z}_p)$$
$$\to \mathrm{Map}_{\mathrm{cont,\ GL}_n(F_\mathfrak{p})}(|U \times_{\mathbb{P}^{n-1}_{\mathbb{C}_p}} \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}|, C^0(G'(F)\backslash \mathcal{G} \times G'(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}}, \mathbb{Q}_p/\mathbb{Z}_p)).$$

The left hand side is equal to

$$C^0(|U \times_{\mathbb{P}^{n-1}_{\mathbb{C}_p}} (G'(F)\backslash \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \times (F_{\mathfrak{p}}^\times/\mathcal{O}_{F_\mathfrak{p}}^\times) \times G'(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}})|, \mathbb{Q}_p/\mathbb{Z}_p),$$

and it remains to observe that there is a natural $\mathrm{GL}_n(F_\mathfrak{p})$-equivariant map

$$(U \times_{\mathbb{P}^{n-1}_{\mathbb{C}_p}} \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p}) \times G'(F)\backslash \mathcal{G} \times G'(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}})$$
$$\to U \times_{\mathbb{P}^{n-1}_{\mathbb{C}_p}} (G'(F)\backslash \mathcal{M}_{\mathrm{Dr},\infty,\mathbb{C}_p} \times (F_{\mathfrak{p}}^\times/\mathcal{O}_{F_\mathfrak{p}}^\times) \times G'(\mathbb{A}^{\mathfrak{p}}_{F,f})/C^{\mathfrak{p}}).$$

□

It follows from Proposition 5.3 that

$$H^i(\mathrm{Sh}_{C^{\mathfrak{p}},\mathbb{C}_p}, \mathbb{Q}_p/\mathbb{Z}_p) = H^i_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}}}}).$$

Combining this with formula (2) we have proved

Theorem 5.4. *There is a natural isomorphism of* $(\mathrm{Gal}_{F_\mathfrak{p}} \times B_{\mathfrak{q}}^\times)$*-representations over* $\mathbb{Z}_p$

$$H^i_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}}}}) \cong \rho^i_{C^{\mathfrak{p}}}.$$

This is a form of local-global compatibility. We will now derive a more concrete version of it by combining it with the σ-typicity decomposition before. To this end we need to extend the Hecke action and for this we need a cuspidality criterion.

Remark 5.5. Until now we have been working with the unitary similitude group G and G' whose similitude factors lie in $\mathbb{G}_{m/F}$. As one might have noticed, this similitude factor played almost a trivial role in the sense that by our definition automorphic forms on G' have unramified character on $F_\mathfrak{p}^\times$ and all Shimura varieties associated with G have level subgroups of the form $U \times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^{\mathfrak{p}}$. Thus it is natural to imagine that one can use unitary groups $\mathbb{G}$ with $\mathbb{Q}$-similitude factors and their associated Shimura varieties to do the generalization (of local-global compatibility). Indeed, this is possible. Most of the arguments involved are formal as long as the p-adic uniformization and σ-typicity of the cohomology are available: the main change will be to replace $F_\mathfrak{p}^\times/\mathcal{O}_{F_\mathfrak{p}}^\times$ by $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$. (As mentioned in Remark 4.8, the Rapoport-Zink uniformization allows one to obtain directly a similar weak form of local-global compatibility for Shimura varieties associated with $\mathbb{G}$, while a parallel property on σ-typicity of the cohomology groups of Shimura varieties associated with $\mathbb{G}$, when combined with the corresponding weak form of local-global compatibility, will lead to again a strong form as in Section 7.)

6. Digression on type theory and a cuspidality criterion

Let $n \geq 2$ be an integer as before and $L/\mathbb{Q}_p$ a finite extension. Set $G = \mathrm{GL}_n(L)$. Just as the $n = 2$ case, for general n we also have a cuspidality criterion of admissible representations of G in terms of containment of certain characters of filtration subgroups of G. To achieve this, we will work with a special extreme case in type theory. The reference for the following and more general framework is [5].

Let E/L be the totally ramified extension of degree n obtained by adjoining an n-th root α of ω, namely $E = L[\alpha]$ with $\alpha^n = \omega$. Regarding $E =: V$ as an n-dimensional vector space over L, we have isomorphisms $\mathrm{End}_L(E) \cong M_n(L) =: A$ and $\mathrm{Aut}_L(E) \cong \mathrm{GL}_n(L) = G$. Let $\mathfrak{A}$ be the hereditary order corresponding to the chain lattice $\mathfrak{L} = \{\mathfrak{p}_E^i : i \in \mathbb{Z}\}$, that is, $\mathfrak{A} = \mathrm{End}^0_{\mathfrak{o}_L}(\mathfrak{L}) := \{x \in A \,|\, x\mathfrak{p}_E^i \subseteq \mathfrak{p}_E^i, \forall i \in \mathbb{Z}\}$. Then $\mathfrak{A}$ is the unique $\mathfrak{o}_L$-order in A such that

$$E^\times \subseteq \mathfrak{L}(\mathfrak{A}) := \{g \in G \,|\, g\mathfrak{A}g^{-1} = \mathfrak{A}\};$$

in other words, $E^\times$ normalises $\mathfrak{A}$. Moreover, $\mathfrak{A}$ is a principal order. We set $\mathfrak{P} = \mathrm{End}^1_{\mathfrak{o}_L}(\mathfrak{L}) := \{x \in A \,|\, x\mathfrak{p}_E^i \subseteq \mathfrak{p}_E^{i+1}, \forall i \in \mathbb{Z}\}$, the Jacobson radical of $\mathfrak{A}$.

Next, for m a positive integer, we can and do choose an element $\beta_m \in E$ with $\nu_E(\beta_m) = -mn - 1$ (for example, one can take $\beta_m = \varpi^{-m}\alpha^{-1}$); here ν_E is the discrete valuation on E. Then it follows that β_m is minimal in the sense of [5].

Lemma 6.1. *Set $\beta_m = \varpi^{-m}\alpha^{-1} \in E$. Then the stratum $[\mathfrak{A}, -\nu_{\mathfrak{A}}(\beta_m), 0, \beta_m]$ is simple.*

Proof. It is clear that $E = L[\beta_m]$, so $L[\beta_m]$ is a field whose non-zero elements normalise $\mathfrak{A}$. Also the inequality $M := -\nu_{\mathfrak{A}}(\beta_m) > 0$ holds as noted in the following remark. □

Remark 6.2. Note that $M := -\nu_{\mathfrak{A}}(\beta_m) > 0$ tends to infinity as m tends to infinity. Indeed, by definition $\nu_{\mathfrak{A}}(\beta_m)$ is the unique integer r such that $\beta_m \in \mathfrak{P}^r \backslash \mathfrak{P}^{r+1}$, which is easily computed to be $mn + 1$; thus $M := mn + 1$ tends to infinity when m goes to infinity. We will denote $\tilde{M} := [\frac{M}{2}] + 1 > 0$ for future use.

Recall that for the hereditary order $\mathfrak{A}$ defined above, one has a sequence of filtration subgroups $U^k(\mathfrak{A})$ given by $U^k(\mathfrak{A}) := 1 + \mathfrak{P}^k$ for $k \geq 1$ and $U^0(\mathfrak{A}) := \mathfrak{A}^\times$. Also in our case, $B := \mathrm{End}_E(V) \cong E, \mathfrak{B} := \mathfrak{A} \cap B \cong \mathcal{O}_E$ is a hereditary $\mathfrak{o}_E$-order with Jacobson radical $\mathfrak{Q} = \mathfrak{P} \cap B \cong \mathfrak{p}_E$. In particular, since β_m is minimal, we have $H^1(\beta_m, \mathfrak{A}) = (1 + \mathfrak{p}_E)U^{\tilde{M}}(\mathfrak{A})$, cf. Corollary 3.1.13 of [10], hence

$$H^1/U^{\tilde{M}}(\mathfrak{A}) \cong (1 + \mathfrak{p}_E)/(1 + \mathfrak{p}_E^{\tilde{M}}).$$

In the last step we have used the following two points: (1) The fact that $E^\times$ normalizes $U^{\tilde{M}}$ implies that H^1 normalizes $U^{\tilde{M}}$; (2) $H^1 \cap U^{\tilde{M}} = (1+\mathfrak{p}_E^{\tilde{M}})$; both follow from straightforward computations. Moreover, $\det_B : E^\times \to E^\times$ is the identity map in our case. In the following proposition, we keep the notations as above. Thus, $M = mn+1$ and we also set $N = M - \tilde{M}$ where $\tilde{M} = [\frac{M}{2}]+1$.

Proposition 6.3. *Let $\psi : L \to \mathbb{C}^\times$ be a non-trivial (additive) character of level one (that is, ψ is trivial on $\varpi\mathcal{O}_L$ but non-trivial on $\mathcal{O}_L$). Let α_m be the homomorphism*

$$\alpha_m : U^{\tilde{M}}(\mathfrak{A}) \to \varpi^{-N}\mathcal{O}_L, \quad a \mapsto tr_{A/L}(\beta_m(a-1)).$$

If π is a smooth irreducible representation of $\mathrm{GL}_n(L)$ such that $\pi|_{U^{\tilde{M}}(\mathfrak{A})}$ contains the character $\psi \circ \alpha_m$, then π is cuspidal.

Proof. We will prove that π contains a maximal simple type; hence by Theorem 6.2.2 of [5] it is cuspidal. Let us first show that π contains a simple character θ of H^1. By assumption, there is an injection

$$\psi_m := \psi \circ \alpha_m \hookrightarrow \pi|_{U^{\tilde{M}}(\mathfrak{A})}$$

which gives rises to a non-zero homomorphism

$$\text{c-Ind}_{U^{\tilde{M}}(\mathfrak{A})}^{H^1} \psi_m \to \pi|_{H^1}.$$

It then suffices to show that we have the following decomposition:

$$\text{c-Ind}_{U^{\tilde{M}}(\mathfrak{A})}^{H^1} \psi_m = \bigoplus_{\theta \in C^1(\psi_m)} \theta$$

where $C^1(\psi_m)$ is the set of simple characters of H^1 extending ψ_m. Clearly the right hand side is contained in the left; thus the equality holds if both sides have the same dimension. But by Mackey's restriction formula, the left side has dimension

$$[H^1 : U^{\tilde{M}}(\mathfrak{A})] = |(1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^{\tilde{M}})|$$

while by 1. (b), Proposition 3.1.18 of [10] the right side has dimension

$$[U^1(\mathfrak{B}) : U^{\tilde{M}}(\mathfrak{B})] = |(1+\mathfrak{p}_E)/(1+\mathfrak{p}_E^{\tilde{M}})|.$$

So they indeed have the same dimension, as claimed. We have the inclusions of compact open subgroups of G:

$$H^1 \subseteq J^1 \subseteq J$$

Since there is a unique extension $\eta(\theta)$ of the simple character θ to J^1 and J^1 is compact, it follows that $\pi|_{J^1}$ contains this unique $\eta(\theta)$. Our final task is to show that $\pi|_J$ contains a simple type (J, λ).

Note that as a result of our choice of the hereditary order and the stratum above, one has $ef = n/[E : L] = 1$, hence $e = f = 1$ in our case. Thus

$$J/J^1 \cong GL(f, k_E)^e = k_E^\times$$

and the factor σ in the definition of $\lambda = \kappa \otimes \sigma$ is nothing but a character inflated from

$$k_E^\times \cong \mathfrak{o}_E^\times/(1 + \mathfrak{o}_E).$$

So by Theorem 5.2.2 of [5] the irreducible representation λ in the definition of simple type coincides with the β-extension κ of $\eta(\theta)$. Therefore it suffices to show that $\pi|_J$ contains some β-extension of $\eta(\theta)$. We make a counting argument again. Indeed, we obtain from the containment $\eta(\theta) \hookrightarrow \pi|_{J^1}$ a non-zero map

$$\text{c-Ind}_{J^1}^J \eta(\theta) \to \pi|_J.$$

We claim that

$$\text{c-Ind}_{J^1}^J \eta(\theta) = \bigoplus \kappa$$

where κ runs over all β-extensions of $\eta(\theta)$. But again, the left side has dimension by Mackey's formula

$$[J : J^1] = |GL(f, k_E)^e| = |k_E^\times|$$

while the right side has dimension $|\mathfrak{o}_E^\times/(1 + \mathfrak{o}_E)| = |k_E^\times|$ by Theorem 5.2.2 of [5]. We conclude that π contains a simple type which is also maximal (namely $e = e(\mathfrak{B}|\mathfrak{o}_E) = 1$) and the proof is complete. □

Remark 6.4. In this proposition, we could have just dealt with such a cuspidality criterion by assuming the containment of a simple character in π, for which the proof would be a little easier. The reason that we consider characters on the smaller subgroup $U^{\tilde{M}}$ rather than on H^1 is because the latter does not form a cofinal system of subgroups of $\mathrm{GL}_n(L)$ (that is, they cannot be as small as one would want), whereas the former does. This point will be important in Corollary 6.8.

Let e be the ramification index of $[F_\mathfrak{p} : \mathbb{Q}_p]$ and fix a surjection $\mathcal{O}_L/\varpi^{me} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z}$. Taking $L = F_\mathfrak{p}$, we have the following corollary.

Corollary 6.5. *Let $A_m = \mathbb{Z}_p[T]/((T^{p^m} - 1)/(T - 1))$. Let ψ be a character of L with coefficients in A_m whose restriction to $\varpi^{-N}\mathcal{O}_L$ is the map*

$$\varpi^{-N}\mathcal{O}_L \xrightarrow[\sim]{\times\varpi^N} \mathcal{O}_L \twoheadrightarrow \mathcal{O}_L/\varpi^{me} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z} \to A_m^\times$$

with the last arrow mapping $1 \in \mathbb{Z}/p^m\mathbb{Z}$ *to* $T \in A_m^\times$*. Define* $\psi_m = \psi \circ \alpha_m$*. Then any automorphic representation* π *of* $\mathrm{GL}_n(F_\mathfrak{p})$ *appearing in*

$$C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(U^{\tilde{M}} \times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \psi_m)[1/p]$$

is cuspidal at $\mathfrak{p}$*.*

Proof. Let k be the smallest integer such that ψ is trivial on $\varpi^k\mathcal{O}_L$, so $-N < k \leq me - N$. Let ψ' be the character of L given by $x \mapsto \psi(\varpi^{k-1}x)$ so that ψ' is a character of level one. Then $\pi_\mathfrak{p}|_{U^{\tilde{M}_{m+k-1}}}$ contains the character $\psi' \circ \alpha_{m+k-1}$, where $\tilde{M}_{m+k-1} = n(m+k-1)+1$; indeed, for any $g \in U^{\tilde{M}_{m+k-1}}$ we have

$$\pi(g)v = \psi(\alpha_m(g))v = \psi'(\varpi^{-(k-1)}\alpha_m(g))v = \psi' \circ \alpha_{m+k-1}(g)v$$

where v is an eigenvector of $\pi|_{U^{\tilde{M}}}$. Therefore one concludes by the proposition above.

Remark 6.6. The existence of such an additive character ψ of L in the above corollary can be seen from the following fact: for each integer k, one can always extend a character χ on $\omega^k\mathcal{O}_L$ to $\omega^{k-1}\mathcal{O}_L$, since $\mathbb{C}^\times$ is an injective $\mathbb{Z}$-module. (Here we are regarding each $\omega^k\mathcal{O}_L$ as a $\mathbb{Z}$-module, too.)

□

Remark 6.7. Fintzen-Shin [12] have proved, independently and simultaneously, such results for all reductive groups over totally real fields that are compact modulo center at infinity under a mild condition on p; see Theorem 3.1.1 of their paper (and also the Appendix D by Raphaël Beuzart-Plessis therein which removes the condition on p). In fact they prove results on supercuspidal types to construct congruences between automorphic forms and apply such congruence results to simplify the construction of automorphic Galois representations by reducing to cases where the automorphic representation has supercuspidal component at a prime.

Let us return to our context. Note that as G and G' are isomorphic to $\mathrm{GL}_1(B)\times_{\mathbb{G}_m} \mathrm{Res}_{K/F}\mathbb{G}_m$ and $\mathrm{GL}_1(D) \times_{\mathbb{G}_m} \mathrm{Res}_{K/F}\mathbb{G}_m$ respectively, applying the results of [22] along with the classical Jacquet-Langlands correspondence between $\mathrm{GL}_1(B)$ (resp. $\mathrm{GL}_1(D)$) and GL_n, we know that an automorphic representation of G' transfers to G if and only if it is a discrete series at $\mathfrak{p}$.

Corollary 6.8. *Let* $\mathbb{T}(C^\mathfrak{p})_\mathfrak{m}$ *be defined as in Section 2, so that it acts faithfully on* $H^{n-1}(C^\mathfrak{p}, \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{m}$*. The natural action of* $\mathbb{T}$ *on*

$$\pi_\mathfrak{m} = \pi_{C^\mathfrak{p},\mathfrak{m}} = C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(\mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{m}$$

extends to a continuous action of $\mathbb{T}(C^\mathfrak{p})_\mathfrak{m}$*.*

Proof. We follow the proof of Scholze in [28]. It is enough to prove this for each group

$$C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(K' \times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \mathbb{Z}/p^m\mathbb{Z})_\mathfrak{m}$$

with $K' \subseteq \mathrm{GL}_n(F_\mathfrak{p})$ compact open subgroups. We may assume that $K' = U^{\tilde{M}}$ is of the form in Proposition 6.3 with $\tilde{M} = [\frac{mn+1}{2}] + 1$ for varying m, as these subgroups form a cofinal system. In this case, $\mathbb{Z}/p^m\mathbb{Z} \cong A_m/(T-1)$ and ψ_m mod $(T-1)$ is trivial. Hence there is a $\mathbb{T}$-equivariant surjection

$$C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(U^{\tilde{M}} \times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \psi_m)_\mathfrak{m} \\ \to C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(U^{\tilde{M}} \times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \mathbb{Z}/p^m\mathbb{Z})_\mathfrak{m}.$$

Thus it suffices to show that the action of $\mathbb{T}$ on

$$M := C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(U^{\tilde{M}} \times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \psi_m)_\mathfrak{m}$$

extends continuously to $\mathbb{T}(C^\mathfrak{p})_\mathfrak{m}$. But M is p-torsion free, so it is enough to check in characteristic 0. In that case, by Corollary 6.5 all automorphic representations of G' appearing in $M[1/p]$ are cuspidal at the place $\mathfrak{p}$ and thus transfer by Jacquet-Langlands correspondence to $\tilde{G}$, so that after transfer they show up in the cohomology group $H^{n-1}(\mathrm{Sh}_{U^{\tilde{M}}C^\mathfrak{p},\mathbb{C}}, \mathbb{Z}_p)_\mathfrak{m}$ for $U^{\tilde{M}}$ sufficiently small. Since $\mathbb{T}(C^\mathfrak{p})_\mathfrak{m} \twoheadrightarrow \mathbb{T}(U^{\tilde{M}}C^\mathfrak{p})_\mathfrak{m}$ acts continuously on

$$H^{n-1}(\mathrm{Sh}_{U^{\tilde{M}}C^\mathfrak{p},\mathbb{C}}, \mathbb{Z}_p)_\mathfrak{m},$$

the result follows.

∎

7. The local-global compatibility

With the previous preparations, let us now deduce the concrete form of local-global compatibility mentioned before. Recall that there is an n-dimensional Galois representation

$$\sigma = \sigma_{\mathfrak{m}} : G_K \to \mathrm{GL}_n(\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}})$$

associated with $\mathfrak{m}$, as well as its reduction modulo $\mathfrak{m}$

$$\bar{\sigma} = \bar{\sigma}_{\mathfrak{m}} : G_K \to \mathrm{GL}_n(\mathbb{F}_q)$$

which is by definition modular; here q is the cardinality of $\mathbb{T}(C^{\mathfrak{p}})/\mathfrak{m}$. By Proposition 3.9 and Corollary 6.8, $\rho^{n-1}_{C^{\mathfrak{p}},\mathfrak{m}}$ is a σ-typic $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_K]$-module, so we have

$$\rho^{n-1}_{C^{\mathfrak{p}},\mathfrak{m}} = \sigma \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}} \rho[\sigma]$$

for some $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[B^{\times}_{\mathfrak{q}}]$-module $\rho[\sigma]$. Combining with Theorem 5.4, we have proved the following result.

Corollary 7.1. *There is a canonical $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_{F_{\mathfrak{p}}} \times B^{\times}_{\mathfrak{q}}]$-equivariant isomorphism*

$$H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}}) \cong \sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}} \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}} \rho[\sigma].$$

The $\mathbb{T}(C^{\mathfrak{p}})$-module $\rho[\sigma]$ is faithful.

Remark 7.2. In the proof of the above corollary it is guaranteed by corollary 6.8 that there is a well defined $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$-module structure on $\pi_{\mathfrak{m}}$ (so that we can use the formalism of $A[G]$-modules). We also abused tacitly the notations $\mathrm{Gal}_{K_{\mathfrak{q}}}$ and $\mathrm{Gal}_{F_{\mathfrak{p}}}$ under the canonical isomorphism $F_{\mathfrak{p}} \cong K_{\mathfrak{q}}$.

This implies that the localization $\pi_{C^{\mathfrak{p}},\mathfrak{m}}$ determines the local Galois representation

$$\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}} : \mathrm{Gal}_{F_{\mathfrak{p}}} \to \mathrm{GL}_n(\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}})$$

at least when $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is absolutely irreducible.

Theorem 7.3. *Assume that $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is absolutely irreducible. Then $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is determined by $\pi_{C^{\mathfrak{p}},\mathfrak{m}}$. More precisely, the $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}[\mathrm{Gal}_{F_{\mathfrak{p}}}]$-module*

$$H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}})$$

is $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$-typic, and faithful as a $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$-module. It follows from Lemma 5.5 of [28] *that this module determines $\sigma|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ uniquely.*

This more concrete form of local-global compatibility shows that the local component at $\mathfrak{p}$ of the global Galois representation $\sigma_{\mathfrak{m}}$ associated with the eigensystem $\mathfrak{m}$ is determined compatibly by $\pi_{C^{\mathfrak{p}},\mathfrak{m}}$ through the cohomology group

$H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}})$. Now we want to prove a similar result for the $\mathfrak{m}$-torsion $\pi_{C^{\mathfrak{p}}}[\mathfrak{m}]$ instead of the localization $\pi_{C^{\mathfrak{p}},\mathfrak{m}}$.

Consider the following two short exact sequences of $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$-modules

$$0 \to \pi_{C^{\mathfrak{p}},\mathfrak{m}}[\mathfrak{m}] \xrightarrow{i} \pi_{C^{\mathfrak{p}},\mathfrak{m}} \to \pi_{C^{\mathfrak{p}},\mathfrak{m}}/\pi_{C^{\mathfrak{p}},\mathfrak{m}}[\mathfrak{m}] \to 0 \tag{3}$$

and

$$0 \to \pi_{C^{\mathfrak{p}},\mathfrak{m}}/\pi_{C^{\mathfrak{p}},\mathfrak{m}}[\mathfrak{m}] \xrightarrow{j} \prod_{i=1}^{r} \pi_{C^{\mathfrak{p}},\mathfrak{m}} \to Q \to 0 \tag{4}$$

where r is an integer such that there exists a sequence of generators $f_1, \dots, f_r$ of the ideal $\mathfrak{m}\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$, j induced by mapping $x \in \pi_{C^{\mathfrak{p}},\mathfrak{m}}$ to $(f_1 x, \dots, f_r x) \in \prod_{i=1}^{r} \pi_{C^{\mathfrak{p}},\mathfrak{m}}$, and Q is by definition the quotient in the second sequence. Let C denote $\pi_{C^{\mathfrak{p}},\mathfrak{m}}/\pi_{C^{\mathfrak{p}},\mathfrak{m}}[\mathfrak{m}]$. We have two corresponding long exact sequences

$$\dots \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}[\mathfrak{m}]}) \xrightarrow{i_*} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_C) \to \dots \tag{5}$$

and

$$\dots \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_C) \to \bigoplus_{i=1}^{r} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{C^{\mathfrak{p}},\mathfrak{m}}}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_Q) \to \dots \tag{6}$$

Now for simplicity, $\pi_{C^{\mathfrak{p}}}$ will be denoted by π.

Lemma 7.4. *The natural map $\pi \to \pi_{\mathfrak{m}}$ induces an isomorphism $\pi[\mathfrak{m}] \xrightarrow{\sim} \pi_{\mathfrak{m}}[\mathfrak{m}]$.*

Proof. Suppose $f \in \pi[\mathfrak{m}]$ maps to 0 in $\pi_{\mathfrak{m}}$, then there exists $t \notin \mathfrak{m}$ such that $tf = 0$. But $\mathfrak{m}$ is a maximal ideal, so there exist $r \in \mathbb{T}$ and $m \in \mathfrak{m}$ satisfying $rt + m = 1$, and we deduce that $f = rtf + mf = 0$ since f is $\mathfrak{m}$-torsion. To show surjectivity, let $f/s \in \pi_{\mathfrak{m}}[\mathfrak{m}]$. As

$$\pi = \varinjlim_{K_{\mathfrak{p}}} \varinjlim_{r} C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(K_{\mathfrak{p}} \times \mathcal{O}^{\times}_{F_{\mathfrak{p}}} \times C^{\mathfrak{p}}), \mathbb{Z}/p^r\mathbb{Z})$$

we may assume that f belongs to a member indexed by $K_{\mathfrak{p}}$ and r above, which is Hecke stable and finite as a set; denote it by $\pi_{K_{\mathfrak{p}},r}$. Then the image of $\mathbb{T}$ (resp. $\mathfrak{m}$) in $\mathrm{End}(\pi_{K_{\mathfrak{p}},r})$ is a finite ring (resp. ideal) and we choose $m_1, \dots, m_l \in \mathfrak{m}$ so that their images form a set equal to the image of $\mathfrak{m}$. By the assumption that $f/s \in \pi_{\mathfrak{m}}[\mathfrak{m}]$, there exists an element $t_i \in \mathbb{T}$ with $t_i \notin \mathfrak{m}$ for each $1 \le i \le l$ such that $t_i m_i f = 0$. Let $t = t_1 \dots t_l$, so that $tmf = 0$ for every $m \in \mathfrak{m}$. Moreover there exists $h \in \mathbb{T}$ satisfying $1 - hst \in \mathfrak{m}$. Taking $g = fht$, one verifies that $g \in \pi[\mathfrak{m}]$ and that f/s is the image of g under the localization map. □

Lemma 7.5. *Assume that* $\pi_{\mathfrak{m}}^{\vee} = \mathrm{Hom}_{\mathbb{Z}_p}(\pi_{\mathfrak{m}}, \mathbb{Q}_p/\mathbb{Z}_p)$ *is flat over* $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$ *(see the remark below). Then for all* $r \geq 1$*, we have*

$$\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r] \cong \bigoplus_s \pi[\mathfrak{m}]$$

where $s = \dim_{\mathbb{T}/\mathfrak{m}}(\mathfrak{m}^{\mathrm{r}}/\mathfrak{m}^{\mathrm{r}+1})$.

Proof. Since taking dual (with respect to $\mathbb{Q}_p/\mathbb{Z}_p$) is an exact functor, we have

$$\begin{aligned}(\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r])^{\vee} \cong & Ker(\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]^{\vee} \to \pi_{\mathfrak{m}}[\mathfrak{m}^r]^{\vee}) \\ \cong & Ker(\pi_{\mathfrak{m}}^{\vee}/\mathfrak{m}^{r+1}\pi_{\mathfrak{m}}^{\vee} \to \pi_{\mathfrak{m}}^{\vee}/\mathfrak{m}^r\pi_{\mathfrak{m}}^{\vee}) \\ \cong & \mathfrak{m}^r\pi_{\mathfrak{m}}^{\vee}/\mathfrak{m}^{r+1}\pi_{\mathfrak{m}}^{\vee}.\end{aligned}$$

But $\pi_{\mathfrak{m}}^{\vee}$ is assumed to be flat over $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$, we decuce that

$$\begin{aligned}\mathfrak{m}^r\pi_{\mathfrak{m}}^{\vee}/\mathfrak{m}^{r+1}\pi_{\mathfrak{m}}^{\vee} &\cong \pi_{\mathfrak{m}}^{\vee} \otimes (\mathfrak{m}^r/\mathfrak{m}^{r+1}) \\ &\cong (\pi_{\mathfrak{m}}^{\vee} \otimes \mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}/\mathfrak{m}) \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}/\mathfrak{m}} (\mathfrak{m}^r/\mathfrak{m}^{r+1}) \\ &\cong (\pi_{\mathfrak{m}}[\mathfrak{m}])^{\vee} \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}/\mathfrak{m}} (\mathfrak{m}^r/\mathfrak{m}^{r+1}).\end{aligned}$$

From the isomorphism $\pi[\mathfrak{m}] = \pi_{\mathfrak{m}}[\mathfrak{m}]$, the last term above can be identified with

$$(\pi_{\mathfrak{m}}[\mathfrak{m}])^{\vee} \otimes_{\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}/\mathfrak{m}} (\mathfrak{m}^r/\mathfrak{m}^{r+1}) \cong (\pi[\mathfrak{m}])^{\vee} \otimes_{\mathbb{T}/\mathfrak{m}} (\mathfrak{m}^r/\mathfrak{m}^{r+1}) \cong \bigoplus_s (\pi[\mathfrak{m}])^{\vee}$$

where $s = \dim_{\mathbb{T}/\mathfrak{m}}(\mathfrak{m}^{\mathrm{r}}/\mathfrak{m}^{\mathrm{r}+1})$. Combining all these isomorphisms and taking duals again, we have $\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r] \cong \big((\pi[\mathfrak{m}])^{\vee} \otimes_{\mathbb{T}/\mathfrak{m}} (\mathfrak{m}^r/\mathfrak{m}^{r+1})\big)^{\vee} \cong (\bigoplus_s (\pi[\mathfrak{m}])^{\vee})^{\vee} \cong \bigoplus_s \pi[\mathfrak{m}]$ where $s = \dim_{\mathbb{T}/\mathfrak{m}}(\mathfrak{m}^{\mathrm{r}}/\mathfrak{m}^{\mathrm{r}+1})$, as desired. ☐

Remark 7.6. In [13], Theorem B, Gee-Newton proved among other things that under certain assumptions on the Gelfand-Kirillov dimension of π, the corresponding flatness result for the group PGL_n indeed holds, so our flatness assumption above seems to be reasonable and one might expect a proof of it under similar conditions as in [13] or directly via the Jacquet-Langlands correspondence using Theorem B of [13].

Assume from now on that $\pi_{\mathfrak{m}}^{\vee}$ is flat over $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$. As a consequence of the above lemma, the sheaf $\mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r]}$ and hence the cohomology group $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r]})$, are determined by the torsion part $\pi[\mathfrak{m}]$. We also notice that

Lemma 7.7. *The* $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$*-module* $\pi_{\mathfrak{m}}$ *is* $\mathfrak{m}$ *power torsion:* $\pi_{\mathfrak{m}} = \varinjlim_k \pi_{\mathfrak{m}}[\mathfrak{m}^k]$. *Moreover, there exists an integer* $N > 0$ *such that* $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ *appears in* $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^N]})[\mathfrak{m}]$.

Proof. By the equality

$$\pi = \varinjlim_{K_\mathfrak{p}} C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(K_\mathfrak{p}\times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \mathbb{Q}_p/\mathbb{Z}_p)$$

and the finiteness of the cardinality of $G'(F)\backslash G'(\mathbb{A}_{F,f})/(K_\mathfrak{p}\times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p})$ it suffices to show that for each compact open subgroup $K_\mathfrak{p}\subseteq G(F_\mathfrak{p})$ and for each positive integer r, the localization at $\mathfrak{m}$ of

$$\pi_{K_\mathfrak{p},r} := C^0(G'(F)\backslash G'(\mathbb{A}_{F,f})/(K_\mathfrak{p}\times \mathcal{O}_{F_\mathfrak{p}}^\times \times C^\mathfrak{p}), \mathbb{Z}/p^r\mathbb{Z})$$

is $\mathfrak{m}$ power torsion. Let $\tilde{\mathbb{T}}$ be the image of the map $\phi:\mathbb{T}\to \mathrm{End}(\pi_{K_\mathfrak{p},r})$ and $\tilde{\mathfrak{m}}\subseteq\tilde{\mathbb{T}}$ be the image of $\mathfrak{m}$ under ϕ. We distinguish two cases:

- $\tilde{\mathfrak{m}}$ is equal to $\tilde{\mathbb{T}}$, which is equivalent to $\ker(\phi)\nsubseteq\mathfrak{m}$. In this case it is direct to check that $(\pi_{K_\mathfrak{p},r})_\mathfrak{m}=0$.
- $\ker(\phi)\subseteq\mathfrak{m}$, in which case $\tilde{\mathfrak{m}}$ is a maximal ideal of $\tilde{\mathbb{T}}$. As $\pi_{K_\mathfrak{p},r}$ is finite as a set it follows that $\tilde{\mathbb{T}}$ is a finite ring and hence an Artin ring. Thus $\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}$ is a local Artin ring and there exists an integer $M>0$ such that $\tilde{\mathfrak{m}}^M\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}=0$. Now $\mathfrak{m}^M(\pi_{K_\mathfrak{p},r})_\mathfrak{m} = \tilde{\mathfrak{m}}^M(\tilde{\mathbb{T}}_{\tilde{\mathfrak{m}}}\otimes_{\tilde{\mathbb{T}}}\pi_{K_\mathfrak{p},r})=0$.

For the second statement, we first note that from the exactness of $\pi\mapsto\mathcal{F}_\pi$ we have an injective morphism of sheaves

$$\varinjlim_k \mathcal{F}_{\pi_\mathfrak{m}[\mathfrak{m}^k]}\to\mathcal{F}_{\pi_\mathfrak{m}},$$

which is also surjective by checking stalks using the equality $\mathcal{F}_{\pi,\bar{x}}=\pi$ and the first statement of the lemma. On the other hand, by Proposition 2.8 of [28] and the coherence of the sites $(\mathbb{P}^{n-1}_{\mathbb{C}_p}/K)_{\text{ét}}$ we deduce

$$H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi_\mathfrak{m}}) = \varinjlim_k H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi_\mathfrak{m}[\mathfrak{m}^k]})$$

with injective transition maps in the direct system. As $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is finite dimensional and occurs in $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi_\mathfrak{m}})[\mathfrak{m}] = \bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}\otimes\rho_{C^\mathfrak{p}}[\mathfrak{m}]$ it must occur in some $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi_\mathfrak{m}[\mathfrak{m}^N]})[\mathfrak{m}]$ for some large N.

□

Now we can prove

Theorem 7.8. *Assume that $\pi_\mathfrak{m}^\vee$ is flat over $\mathbb{T}(C^\mathfrak{p})_\mathfrak{m}$ and that $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is semisimple with distinct irreducible factors. Then $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ appears in the composition series of $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi[\mathfrak{m}]})$.*

Proof. We may assume that $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is irreducible. Pick an $N>0$ such that $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ appears in $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p},\mathcal{F}_{\pi_\mathfrak{m}[\mathfrak{m}^N]})[\mathfrak{m}]$ whose existence is guaranteed by the above lemma. Denote $\pi_N := \pi_{\mathfrak{m}}[\mathfrak{m}^N]$ and consider the short exact sequence

$$0 \to \pi_N[\mathfrak{m}] \xrightarrow{i} \pi_N \to \pi_N/\pi_N[\mathfrak{m}] \to 0$$

and its associated long exact sequence

$$\ldots \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_N[\mathfrak{m}]}) \xrightarrow{i_*} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_N}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_1}) \to \ldots$$

where $C_1 := \pi_N/\pi_N[\mathfrak{m}]$. Since $\pi_N[\mathfrak{m}]$ is of $\mathfrak{m}$-torsion, by functoriality the map i_* factors through $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_N})[\mathfrak{m}]$ which by assumption contains $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$. If the image of i_* contains a copy of $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ in $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_N})$ then we are done, as the irreducible representation $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ occurs in the composition series of a quotient of $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_N[\mathfrak{m}]})$ (note that $\pi_N[\mathfrak{m}] = \pi[\mathfrak{m}]$). Now suppose the contrary so that $\mathrm{coker}(i_*)$, containing a copy of $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ (since $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is assumed to be irreducible), injects into $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_1})$.

Consider the further filtration

$$0 \to C_1[\mathfrak{m}] \xrightarrow{i_1} C_1 \to C_2 \to 0$$

where $C_2 := C_1/C_1[\mathfrak{m}]$ and one can check that $C_1[\mathfrak{m}] = \pi_N[\mathfrak{m}^2]/\pi_N[\mathfrak{m}]$ so that $C_2 = \pi_N/\pi_N[\mathfrak{m}^2]$. We have again the long exact sequence

$$\ldots \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_1[\mathfrak{m}]}) \xrightarrow{i_{1*}} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_1}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_2}) \to \ldots$$

By the same argument as above, either the image of i_{1*} contains a copy of $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ inside $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_1})$, or, otherwise, $\mathrm{coker}(i_{1*})$ contains a copy of $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ and injects into $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_2})$. Then one defines $C_3 := \pi_N/\pi_N[\mathfrak{m}^3]$ to continue with the above procedure.

Considering that $\pi_N = \pi_N[\mathfrak{m}^N]$ (hence $C_N := \pi_N/\pi_N[\mathfrak{m}^N] = 0$), one concludes by induction that there exists a positive integer $r < N$ such that the image of i_{r*} contains a copy of $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ inside $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_r})$, i.e., $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is an irreducible sub-representation of a homomorphic image of $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_r[\mathfrak{m}]})$ and therefore appears in the composition series of

$$H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C_r[\mathfrak{m}]}) = H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r]})$$

where we have used equalities $\pi_N[\mathfrak{m}^i] = \pi[\mathfrak{m}^i]$ for $1 \leq i \leq N$. For later reference in the following, we denote this induction process by Ω.

So we have proved that $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ appears in the composition series either of $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ or of $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r]})$ for some positive integer r, but in the latter case one has

$$H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r]}) \cong \bigoplus_s H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$$

from the equality $\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r] \cong \bigoplus_s \pi[\mathfrak{m}]$ (hence $\mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]/\pi_{\mathfrak{m}}[\mathfrak{m}^r]} \cong \bigoplus_s \mathcal{F}_{\pi[\mathfrak{m}]}$). ☐

Now we show that when $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is irreducible, it is uniquely determined by $\pi[\mathfrak{m}]$ in the sense that we can read off $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ from $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$.

Lemma 7.9. *Assume that $\pi^\vee_{\mathfrak{m}}$ is flat over $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$. Then $H^i_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]}) = 0$ for $1 \le i \le n-2$, and the map $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]}) \xrightarrow{i_*} H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}]$ is injective.*

Proof. We prove that the kernel of the map i_* in the long exact sequence (5), which is equal to the image of the map

$$H^{n-2}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_C) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]}),$$

is trivial by showing directly that $H^{n-2}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_C) = 0$. This will be a consequence of the hypothesis that $H^i(\mathrm{Sh}_{UC^{\mathfrak{p}},\mathbb{C}}, \mathbb{Z}_p)_{\mathfrak{m}}$ is concentrated in middle degree. Indeed, in degree 0 this follows from the injection

$$H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_C) \to \bigoplus_{i=1}^{r} H^0_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}}) = \bigoplus_{i=1}^{r} H^0(\mathrm{Sh}_{C^{\mathfrak{p}},\mathbb{C}_p}, \mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{m}}$$

induced by the exact sequence (4), and from the vanishing of $H^0(\mathrm{Sh}_{C^{\mathfrak{p}},\mathbb{C}_p}, \mathbb{Q}_p/\mathbb{Z}_p)_{\mathfrak{m}}$ by our assumption on $\mathfrak{m}$. Then we proceed by induction, so assume now that $H^k_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_C) = 0$ for some k with $0 \le k \le n-3$ and we want to prove the vanishing in degree $k+1$. Note that from the sequence (3) we have immediately $H^{k+1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]}) = 0$. Further, as we can write $\pi_{\mathfrak{m}}/\pi_{\mathfrak{m}}[\mathfrak{m}] = \varinjlim_N \pi_{\mathfrak{m}}[\mathfrak{m}^N]/\pi_{\mathfrak{m}}[\mathfrak{m}]$ by Lemma 7.7, it suffices to prove $H^{k+1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{C'_N}) = 0$ for all $N \ge 1$ where $C'_N := \pi_{\mathfrak{m}}[\mathfrak{m}^N]/\pi_{\mathfrak{m}}[\mathfrak{m}]$. But this follows directly by the induction process Ω in the proof of Theorem 7.8, which shows in particular that C'_N is a successive extension of copies of $\pi[\mathfrak{m}]$ (by Lemma 7.5). ☐

Theorem 7.10. *Assume that $\pi^\vee_{\mathfrak{m}}$ is flat over $\mathbb{T}(C^{\mathfrak{p}})_{\mathfrak{m}}$. Then $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ is a non-zero admissible $\mathrm{Gal}_{F_{\mathfrak{p}}} \times B^\times_{\mathfrak{q}}$-representation, and it has the same Jordan-Hölder factors with $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$. In particular, if $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is irreducible, then every irreducible subrepresentation of $H^{n-1}_{\acute{e}t}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ is isomorphic to $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$.*

Proof. By Lemma 7.9 it suffices to prove $JH(\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})$ is contained in $JH(H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]}))$. But $\bar{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is finite dimensional, so this follows again from the induction argument Ω in the proof of Theorem 7.8, using Lemma 7.5 and Lemma 7.7. ☐

8. Beyond the semisimple case (joint with Z. Qian)

Introduction. In this joint work with Zicheng Qian, we prove a stronger result on mod p local-global compatibility than that in section 7; namely, we remove the semisimple condition on $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ and instead replace it by the much weaker condtion that $\bar{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ be multiplicity free as a Galois representation of $\mathrm{Gal}_{F_\mathfrak{p}}$. We will first establish a slightly more general framework concerning submodules of a given σ-typic module before we embark on the proof of the stronger result that we mentioned.

Let G be a group, R be a commutative ring and ρ_0 be an $R[G]$-module of finite length. We have the following definition generalizing Definition 5.2 of [28].

Definition 8.1. Assume that ρ_0 is multiplicity free, namely each Jordan–Hölder factor of ρ_0 appears with multiplicity one. Then ρ_0 admits a decomposition $\rho_0 \cong \bigoplus \tilde{\rho}$ into its non-zero indecomposable direct summands. We say that an $R[G]$-module V is ρ_0*-typic* if there exists a non-zero R-module $W_{\tilde{\rho}}$ with trivial G-action for each $\tilde{\rho}$, such that $V \cong \bigoplus_{\tilde{\rho}} W_{\tilde{\rho}} \otimes_R \tilde{\rho}$.

We assume throughout this section that R is field and ρ_0 is multiplicity free. We fix from now a ρ_0-typic $R[G]$-module V equipped with an R-module $W_{\tilde{\rho}}$ for each indecomposable direct summand $\tilde{\rho}$ of ρ_0 as in Definition 8.1. The main result of this section is a criterion (see Proposition 8.6) for certain submodule of V to be ρ_0-typic.

We write Σ for the set of non-zero indecomposable $R[G]$-submodules of ρ_0, equipped with the natural partial order given by inclusion of $R[G]$-submodules. We write $\mathrm{JH}_{R[G]}(\cdot)$ for the set of Jordan–Hölder factors. As ρ_0 is multiplicity free, any $R[G]$-submodule of ρ_0 is uniquely determined by its set of Jordan–Hölder factors, and we clearly have $\#\Sigma \leq 2^\ell$ where ℓ is the length of ρ_0. Note that $V \cong \bigoplus_{\tilde{\rho}} W_{\tilde{\rho}} \otimes_R \tilde{\rho}$ forces V to be locally finite, and so is any subquotient of V.

Lemma 8.2. *Let $\rho' \subseteq \rho$ be two elements of Σ. Then the canonical map*

$$\mathrm{Hom}_{R[G]}(\rho, V) \to \mathrm{Hom}_{R[G]}(\rho', V)$$

is an isomorphism.

Proof. We first deduce from $\rho, \rho' \in \Sigma$ and $\rho' \subseteq \rho$ that there exists a unique indecomposable direct summand $\tilde{\rho} \in \Sigma$ of ρ_0 which contains ρ, ρ'. The canonical map $\mathrm{Hom}_{R[G]}(\rho, \tilde{\rho}) \to \mathrm{Hom}_{R[G]}(\rho', \tilde{\rho})$ is clearly an isomorphism of R-vector spaces of dimension one. Then the canonical map in question factors through the isomorphisms

$$\mathrm{Hom}_{R[G]}(\rho, V) \cong W_{\tilde{\rho}} \otimes_R \mathrm{Hom}_{R[G]}(\rho, \tilde{\rho}) \xrightarrow{\sim} W_{\tilde{\rho}} \otimes_R \mathrm{Hom}_{R[G]}(\rho', \tilde{\rho}) \cong \mathrm{Hom}_{R[G]}(\rho', V).$$

□

Lemma 8.3. *Let $V' \subseteq V$ be an $R[G]$-submodule with $\mathrm{cosoc}_{R[G]}(V')$ being irreducible. Then there exists $\rho \in \Sigma$ such that $V' \cong \rho$.*

Proof. We write $\tau := \mathrm{cosoc}_{R[G]}(V') \in \mathrm{JH}_{R[G]}(V) = \mathrm{JH}_{R[G]}(\rho_0)$. There exists a unique $\rho \subseteq \tilde{\rho} \subseteq \rho_0$ such that $\tilde{\rho}$ is an indecomposable direct summand of ρ_0 and $\mathrm{cosoc}_{R[G]}(\rho) \cong \tau$. As $V/W_{\tilde{\rho}} \otimes_R \rho$ does not have τ as a Jordan–Hölder factor, we may assume without loss of generality that $\rho = \tilde{\rho} = \rho_0$. Then the key observation is that

$$\mathrm{Hom}_{R[G]}(\tilde{\rho}, V) \cong W_{\tilde{\rho}} \otimes_{R\ R[G]}(\tilde{\rho}) \xrightarrow{\sim} W_{\tilde{\rho}} \otimes_{R\ R[G]}(\tau) \cong \mathrm{Hom}_{R[G]}(\tau, W_{\tilde{\rho}} \otimes_R \tau). \tag{7}$$

The $R[G]$-submodule $V' \subseteq V$ determines an embedding $\tau \hookrightarrow W_{\tilde{\rho}} \otimes_R \tau$ and thus (by (7) an embedding $f : \tilde{\rho} \hookrightarrow V$. We write $\mathrm{rad}(V')$ for the kernel of $V' \twoheadrightarrow \mathrm{cosoc}_{R[G]}(V')$. As the canonical map $V'/\mathrm{rad}(V') \to V/(\mathrm{im}(f) + \mathrm{rad}(V'))$ is zero by the choice of f, so is the map $V' \to V/\mathrm{im}(f)$, which implies that $V' \subseteq \mathrm{im}(f)$. This inclusion must be equality as both $R[G]$-modules share the same cosocle. □

Lemma 8.4. *Let $V' \subseteq V$ be an $R[G]$-submodule. If V' is multiplicity free, then there exists an embedding $V' \hookrightarrow \rho_0$.*

Proof. By writing V' as direct sum of its indecomposable direct summands, it suffices to assume that V' is indecomposable and find $\rho \in \Sigma$ such that $V' \cong \rho$. As $\mathrm{JH}_{R[G]}(V) = \mathrm{JH}_{R[G]}(\rho_0)$ is finite, we deduce that V' has finite length. By writing each $W_{\tilde{\rho}} = \varinjlim_k W_{\tilde{\rho},k}$ as direct limit of its finite dimensional subspaces and then using the fact that V' has finite length, we may assume without loss of generality that $W_{\tilde{\rho}}$ is finite dimensional for each indecomposable direct summand $\tilde{\rho}$ of ρ_0. We write $\mathrm{soc}_{R[G]} V' \cong \bigoplus_{t=1}^{s} \tau_t$, then each $\tau_t \subseteq V' \subseteq V$ determines a unique $\tilde{\rho}_t$ containing τ_t as well as an element $f_t \in \mathrm{Hom}_{R[G]}(\tau_t, V) \cong \mathrm{Hom}_{R[G]}(\tilde{\rho}_t, V) \cong W_{\tilde{\rho}_t}$. As V' is indecomposable and $W_{\tilde{\rho}} \otimes_R \tilde{\rho}$ do not share common Jordan–Hölder factor for different $\tilde{\rho}$, we deduce that all $\tilde{\rho}_t$ equal the same $\tilde{\rho}$. As it is harmless to assume that R is infinite, there exists $\ell : W_{\tilde{\rho}} \to R$ such that $\ell(f_t) \neq 0$ for each $1 \leq t \leq s$. Hence, $\ell \otimes_R \tilde{\rho} : W_{\tilde{\rho}} \otimes_R \tilde{\rho} \to \tilde{\rho}$ restricted to an injection on $\mathrm{soc}_{R[G]}(V')$, and thus an injection on V' as well. We conclude by the observation that any indecomposable $R[G]$-submodule of $\tilde{\rho}$ is in Σ. □

Lemma 8.5. *Let $V' \subseteq V$ be an $R[G]$-submodule. Assume that*

(1) $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$;

(2) *for each indecomposable direct summand* $\tilde{\rho}$ *of* ρ_0 *and each embedding* $f : \tilde{\rho} \hookrightarrow V$, *we have either* $\mathrm{im}(f) \subseteq V'$ *or* $\mathrm{im}(f) \cap V' = 0$.

Then V' *is* ρ_0*-typic.*

Proof. Recall that we have $V \cong \bigoplus_{\tilde{\rho}} W_{\tilde{\rho}} \otimes_R \tilde{\rho}$ and the identification $W_{\tilde{\rho}} \cong \mathrm{Hom}_{R[G]}(\tilde{\rho}, V)$ for each indecomposable direct summand $\tilde{\rho}$ of ρ_0. We write $W'_{\tilde{\rho}} \subseteq W_{\tilde{\rho}}$ for the subspace of all morphisms $f : \tilde{\rho} \to V$ satisfying $\mathrm{im}(f) \subseteq V'$. We claim that the natural map

$$\bigoplus_{\tilde{\rho}} W'_{\tilde{\rho}} \otimes_R \tilde{\rho} \to V' \tag{8}$$

is an isomorphism. The compatibility with $V \cong \bigoplus_{\tilde{\rho}} W_{\tilde{\rho}} \otimes_R \tilde{\rho}$ forces (8) to be injective. As V' is sum of its $R[G]$-submodules with irreducible cosocle, it suffices to prove that each such $R[G]$-submodule V'' of V' is contained in the image of (8). In fact, it follows from Lemma 8.3 that there exists $\rho \in \Sigma$ such that $V'' \cong \rho$. Hence, we deduce from Lemma 8.2 that there exists an indecomposable direct summand $\tilde{\rho}$ of ρ_0 as well as $f \in \mathrm{Hom}_{R[G]}(\tilde{\rho}, V)$ such that $\tilde{\rho} \supseteq \rho$ and $V'' \subseteq \mathrm{im}(f)$. As $\mathrm{im}(f)$ is multiplicity free, it embeds into ρ_0 by Lemma 8.4, and thus embeds into $\tilde{\rho}$ by checking Jordan–Hölder factors. This forces $\tilde{\rho} \cong \ker(f) \oplus \mathrm{im}(f)$ and thus $\ker(f) = 0$ as $\tilde{\rho}$ is indecomposable. In other words, f is an embedding with $0 \neq V'' \subseteq \mathrm{im}(f) \cap V'$, which together with our assumption implies that $\mathrm{im}(f) \subseteq V'$. Hence, $\mathrm{im}(f)$ is contained in the image of (8), and so is V''. Note finally that the equality $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$ forces $W'_{\tilde{\rho}} \neq 0$ for each indecomposable direct summand $\tilde{\rho}$ of ρ. The proof is thus completed. ☐

Proposition 8.6. *Let* ρ_0 *be a multiplicity free* $R[G]$*-module of finite length. Let* V *be a* ρ_0*-typic* $R[G]$*-module with a sequence of* $R[G]$*-submodules* $V_1 \subsetneq V_2 \subsetneq \cdots$ *satisfying the following conditions*

- $V = \bigcup_{r \geq 1} V_r$; *and*
- *for each* $r \geq 1$, *there exists an embedding* $V_{r+1}/V_r \hookrightarrow V_1^{\oplus s_r}$ *for some* $s_r \geq 1$.

Then V_1 *is* ρ_0*-typic. In particular,* V_1 *determines* ρ_0 *up to isomorphism.*

Proof. Our assumption clearly implies that $\mathrm{JH}_{R[G]}(V_1) = \mathrm{JH}_{R[G]}(\rho_0)$. Let $\tilde{\rho}$ be an indecomposable direct summand of ρ_0 and $f : \tilde{\rho} \hookrightarrow V$ be an embedding. According to Lemma 8.5, it suffices to show that either $\mathrm{im}(f) \subseteq V_1$ or $\mathrm{im}(f) \cap V_1 = 0$ holds. We set $V_{f,0} := 0 \subseteq \mathrm{im}(f)$ and $V_{f,r} := \mathrm{im}(f) \cap V_r$ for each $r \geq 1$. Our assumption on $\{V_r\}_{r \geq 1}$ implies that $\{V_{f,r}\}_{r \geq 0}$ is an increasing and exhaustive filtration on $\mathrm{im}(f)$. The inclusion $\mathrm{im}(f) \subseteq V$ induces a natural embedding

$$V_{f,r+1}/V_{f,r} \hookrightarrow V_{r+1}/V_r \hookrightarrow V_1^{\oplus s_r} \hookrightarrow V^{\oplus s_r}.$$

As $V^{\oplus s_r}$ is ρ_0-typic and $V_{f,r+1}/V_{f,r}$ is multiplicity free, we deduce from Lemma 8.4 that $V_{f,r+1}/V_{f,r}$ embeds into ρ_0, and actually embeds into $\tilde{\rho}$ by checking Jordan–Hölder factors. As $V_{f,r+1}/V_{f,r}$ embeds into $\tilde{\rho} \cong \mathrm{im}(f)$ for each $r \geq 0$, we deduce that

$$\tilde{\rho} \cong \mathrm{im}(f) \cong \bigoplus_{r\geq 0} V_{r+1,f}/V_{r,f}.$$

However, $\tilde{\rho}$ is indecomposable, and thus there exists a unique $r_f \geq 0$ such that $V_{r_f+1,f}/V_{r_f,f} \cong \tilde{\rho}$ and $V_{r+1,f} = V_{r,f}$ for all $r \neq r_f$. In particular, we have $\mathrm{im}(f) \subseteq V_1$ if $r_f = 0$, and $\mathrm{im}(f) \cap V_1 = 0$ if $r_f \geq 1$. As the ρ_0-typic $R[G]$-module V_1 determines the isomorphism class of each indecomposable direct summand $\tilde{\rho}$ of ρ_0 (by considering all possible indecomposable direct summands of V_1), it clearly determines ρ_0 up to isomorphism. The proof is thus finished. □

We also have the following more general result which captures ρ_0 from an $R[G]$-submodule $V' \subseteq V$ without knowing that V' is ρ_0-typic.

Proposition 8.7. *Let $V' \subseteq V$ be an $R[G]$-submodule. Assume that $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$. Then V' determines ρ_0 up to isomorphism.*

Proof. As ρ_0 is multiplicity free, for each $\tau \in \mathrm{JH}_{R[G]}(\rho_0)$, there exists a unique $R[G]$-submodule $\rho_\tau \subseteq \rho_0$ such that $\mathrm{cosoc}_{R[G]}(\rho_\tau) \cong \tau$. It follows from Lemma 8.3 that for each $\tau \in \mathrm{JH}_{R[G]}(\rho_0)$, any $R[G]$-submodule $V'' \subseteq V$ satisfying $\mathrm{cosoc}_{R[G]}(V') \cong \tau$ must also satisfy $V'' \cong \rho_\tau$. Consequently, we deduce from $\mathrm{JH}_{R[G]}(V') = \mathrm{JH}_{R[G]}(\rho_0)$ that V' determines the set of isomorphism classes $\{[\rho_\tau]\}_{\tau\in \mathrm{JH}_{R[G]}(\rho_0)}$. It then suffices to show that the set $\{[\rho_\tau]\}_{\tau\in \mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ_0 up to isomorphism. We prove that $\{[\rho_\tau]\}_{\tau\in \mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ up to isomorphism for each $R[G]$-submodule $\rho \subseteq \rho_0$ by induction on the length of ρ.

Let $\rho' \subseteq \rho$ be two $R[G]$-modules of ρ_0 with $\rho/\rho' \cong \tau_0$ for some $\tau_0 \in \mathrm{JH}_{R[G]}(\rho_0)$. Assume first that $\{[\rho_\tau]\}_{\tau\in \mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ' and $\rho' \cap \rho_{\tau_0}$ up to isomorphism. We choose two embeddings $f_1 : \rho' \cap \rho_{\tau_0} \to \rho'$ and $f_2 : \rho' \cap \rho_{\tau_0} \to \rho_{\tau_0}$ and note that the choice of the pair (f_1, f_2) is unique up to automorphisms of ρ', ρ_{τ_0} and $\rho' \cap \rho_{\tau_0}$. Hence, the isomoprphism class of the amalgamated sum $\rho' \oplus_{\rho'\cap\rho_{\tau_0}} \rho_{\tau_0}$ does not depend on the choice of f_1, f_2. It is obvious that $\rho \cong \rho' \oplus_{\rho'\cap\rho_{\tau_0}} \rho_{\tau_0}$ and thus $\{[\rho_\tau]\}_{\tau\in \mathrm{JH}_{R[G]}(\rho_0)}$ determines ρ up to isomorphism. The proof is then finished by induction on length. □

We can now apply this formalism to our setting in section 7. More precisely, recall that under the flatness condition on $\pi_{\mathfrak{m}}^\vee$ there exists for each $r \geq 1$ a short exact sequence (by Lemma 7.5)

$$0 \to \pi_{\mathfrak{m}}[\mathfrak{m}^r] \to \pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}] \to (\pi_{U^{\mathfrak{p}}}[\mathfrak{m}])^{\oplus s_r} \to 0 \tag{9}$$

where $s_r \geq 1$ is a positive integer. Applying Scholze's functor, we get from it an exact sequence on cohomology groups (for each $r \geq 1$)
(10)
$$0 \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^r]}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^{r+1}]}) \to \bigoplus_{s_r} H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{U^{\mathfrak{p}}}[\mathfrak{m}]}).$$

The injectivity on the left hand side follows from Lemma 7.9. Now set

$$V_r := H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}^r]})[\mathfrak{m}].$$

Taking $\mathfrak{m}$-torsion on the sequence (10) yields an exact sequence

$$0 \to V_r \to V_{r+1} \to (V_1)^{\oplus s_r}. \tag{11}$$

To apply our previous results on the classification of $R[G]$-submodules of a ρ_0-typic $R[G]$-module V, we need to assume

Condition 8.8. *The $\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}}]$-module $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ is multiplicity free; that is, each Jordan–Hölder factor of $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ appears with multiplicity one.*

Recall that here $\mathbb{F} = \mathbb{F}_q$ is the coefficient field of the representation $\bar{\sigma}$. Then we take

- $R = \mathbb{F} = \mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}}/\mathfrak{m}\mathbb{T}(U^{\mathfrak{p}})_{\mathfrak{m}}$
- $G = \mathrm{Gal}_{F_{\mathfrak{p}}}$
- $\rho_0 = \overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$
- $V = H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}]$.

Thus we have $V = \varinjlim_r V_r$; this equality together with the sequence (11) fulfills all the conditions in Proposition 8.6. Therefore we deduce

Theorem 8.9. *Assume that the flatness condition on $\pi^\vee_{\mathfrak{m}}$ and Condition 8.8 hold. Then $H^{n-1}_{ét}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ is $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$-typic. In particular, $H^{n-1}_{ét}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ up to isomorphism.*

Remark 8.10. Let us write V_1^* for the image of the homomorphism

$$H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}[\mathfrak{m}]}) \to H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi_{\mathfrak{m}}})[\mathfrak{m}]. \tag{12}$$

Assuming Condition 8.8 and moreover that

$$\mathrm{JH}_{\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}}]}(V_1^*) = \mathrm{JH}_{\mathbb{F}[\mathrm{Gal}_{F_{\mathfrak{p}}}]}(\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}), \tag{13}$$

we can show that V_1^* determines $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ up to isomorphism without showing that V_1^* is $\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$-typic. In fact, for each $\tau \in \mathrm{JH}_{[\mathrm{Gal}_{F_{\mathfrak{p}}}]}(\overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}})$, there exists a unique subrepresentation $\rho_\tau \subseteq \overline{\sigma}|_{\mathrm{Gal}_{F_{\mathfrak{p}}}}$ with cosocle τ. It follows from Lemma 8.3 that any $[\mathrm{Gal}_{F_{\mathfrak{p}}}]$-submodule V' of V_1^* with $\mathrm{cosoc}_{[\mathrm{Gal}_{F_{\mathfrak{p}}}]}(V') \cong \tau$ necessarily satisfies

$V' \cong \rho_\tau$. As $\tau \in \mathrm{JH}_{[\mathrm{Gal}_{F_\mathfrak{p}}]}(V_1^*)$ by our assumption, we deduce that V_1^* determines the isomorphism class $[\rho_\tau]$ of ρ_τ for each $\tau \in \mathrm{JH}_{[\mathrm{Gal}_{F_\mathfrak{p}}]}(\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}})$. Then we can recover the isomorphism class of $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ from

$$\{[\rho_\tau]\}_{\tau \in \mathrm{JH}_{[\mathrm{Gal}_{F_\mathfrak{p}}]}(V_1^*)}$$

by considering an amalgamate sum. However, one needs to be careful that V_1^* *a priori* depends on the structure of $\pi_\mathfrak{m}$ rather than $\pi_\mathfrak{m}[\mathfrak{m}]$, and thus the result above for V_1^* is not sufficient to imply that $\pi_\mathfrak{m}[\mathfrak{m}]$ determines $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ up to isomorphism. If (12) is an embedding, then this gives an alternative approach to Theorem 8.9 with weaker conclusion (namely without showing that V_1^* is $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$-typic). It worths to point out that there are actually examples in [19] such that

- The flatness condition fails;
- the map ((12)) is an embedding; and
- $V_1^* \cong H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ is not $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$-typic.

We also note that we do not know how to prove (13) without assuming the flatness condition when $n \geq 3$.

Remark 8.11. If $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ is not multiplicity free, then in general we cannot determine $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ uniquely from the structure of a direct summand V' of an infinite dimensional $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$-typic module, even if V' exhausts all Jordan–Hölder factors of $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$. Let $\overline{\chi}_1, \overline{\chi}_2$ be two distinct characters $\mathrm{Gal}_{F_\mathfrak{p}} \to \mathbb{F}^\times$, then we have the following simple examples for such $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$

- $r_1, r_2 \geq 2$ and $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}} \cong \overline{\chi}_1^{\oplus r_1} \oplus \overline{\chi}_2^{\oplus r_2}$;
- $[F_\mathfrak{p} : \mathbb{Q}_p] \geq 2$, $r_1, r_2 \geq 2$ and $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}} \cong \overline{\sigma}_1^{\oplus r_1} \oplus \overline{\sigma}_2^{\oplus r_2}$ with $\overline{\sigma}_1, \overline{\sigma}_2$ two non-isomorphic extensions of $\overline{\chi}_1$ by $\overline{\chi}_2$.

We expect a better understanding of the $D^\times$-action on $H^{n-1}_{\text{ét}}(\mathbb{P}^{n-1}_{\mathbb{C}_p}, \mathcal{F}_{\pi[\mathfrak{m}]})$ to be essential to generalize Theorem 8.9 to cases when $\overline{\sigma}|_{\mathrm{Gal}_{F_\mathfrak{p}}}$ has multiplicity.

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Résumé. Nous généralisons le résultat de compatibilité local-global dans [28] aux cas de dimension supérieure, en examinant la relation entre le foncteur de Scholze et la cohomologie des variétés de Shimura de type Kottwitz-Harris-Taylor. En chemin, nous prouvons un critère de cuspidalité de la théorie des types. Nous traitons également de la compatibilité des classes de torsion dans le cas des représentations semi-simples mod p Galois sans multiplicité, sous certaines hypothèses de platitude. Enfin, nous enlevons la condition sur semisimplicité et la remplaçons par la condition beaucoup plus faible d'être sans multiplicité. Ce dernier résultat est obtenu en collaboration avec Z. Qian.

Mots clés: *Compatibilité local-global, programme de Langlands, variétés de Shimura, représentations galoisiennes*

Abstract. We generalize the local-global compatibility result in [28] to higher dimensional cases, by examining the relation between Scholze's functor and cohomology of Kottwitz-Harris-Taylor type Shimura varieties. Along the way we prove a cuspidality criterion from type theory. We also deal with compatibility for torsion classes in the case of semisimple mod p Galois representations which are multiplicity free, under certain flatness hypotheses. Finally, we remove the semisimple condition and replace it by the much weaker condition of being multiplicity free. This last result is obtained in joint work with Z. Qian.

Key words: *Local-global compatibility, Langlands program, Shimura varieties, Galois representations*