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Stabilité et homologie de factorisation équivariante

Equivariant stabilization and factorization homology

THÈSE DE DOCTORAT
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Résumé

L'objectif de cette thèse est de contribuer à l'étude de la théorie de l'homotopie équivariante. Il se compose de trois parties.

Dans la première partie, nous prouvons que la stabilisation de l'∞-catégorie des G-espaces par rapport aux sphères de représentation est équivalente à l'∞-catégorie des G-spectres, où G est un groupe de Lie compact. L'∞-catégorie des G-espaces est obtenue via la structure de modèle standard sur la catégorie des G-espaces, tandis que l'∞-catégorie des G-spectres est acquise à partir de la structure du modèle stable. En fait, on prouve que ces catégories sont présentables, donc on montre l'équivalence des ∞-catégories présentables.

Dans la deuxième partie, nous utilisons la théorie paramétrée des catégories supérieures pour construire la version équivariante de l'homologie de factorisation. Il existe déjà une construction de homologie de factorisation équivariante pour les variétés avec l'action d'un groupe fini, que nous étendons aux variétés avec l'action d'un groupe de Lie compact à stabilisateurs finis.

Dans la troisième partie, nous développons la théorie des approximations des ∞-opérades paramétrées lorsque la paramétrisation est faite par rapport à l'∞-catégorie des G-espaces transitifs (i.e. les orbites) avec des stabilisateurs finis, où G est un groupe de Lie compact. Nous utilisons cette théorie pour prouver que l'∞-catégorie des G-disques est librement générée par l'∞-catégorie des H-disques avec un cadrage approprié sur les G-disques et les H-disques, où $H \leq G$ est un sous-groupe fini.

Abstract

The aim of this thesis is to contribute to the study of the equivariant homotopy theory. It consists of three parts.

In the first part we prove that the stabilization of the ∞-category of G-spaces with respect to the representation spheres is equivalent to the ∞-category of G-spectra, where G is a compact Lie group. The ∞-category of G-spaces is obtained via the standard model structure on the category of G-spaces while the ∞-category of G-spectra is acquired from the stable model structure. In fact, we prove that these categories are presentable, hence, we show the equivalence of presentable ∞-categories.

In the second part we use the parametrized higher category theory to construct the equivariant version of the factorization homology. There already exists a construction of genuine equivariant factorization homology for manifolds with an action of a finite group, which we extend to the manifolds with an action of a compact Lie group with finite stabilizers.

In the third part, we develop the theory of approximations to parametrized ∞-operads when the parametrization is done with respect to the ∞-category of transitive G-spaces (i.e. orbits) with finite stabilizers, where G is a compact Lie group. We use this theory to prove that the ∞-category of G-discs is freely generated by the ∞-category of H-discs with suitable framing on both G-discs and H-discs, where $H \leq G$ is a finite subgroup.

Contents

Chapter 0

Introduction

Objects with an action of a group G apear naturally in topology (and mathematics in general). Let X be a space with an action of a group G. Classical constructions in the algebraic topology such as homotopy groups or singular homology of X do not recognize the action of G. Therefore, we need to develop suitable machinery that will take this action into account (Bredon (co)homology etc.). Studying homotopical properties of these equivariant objects gives rise to the equivariant homotopy theory. Although many results in classical homotopy theory have their equivariant versions (Whitehead's theorem, CW-approximations just to name a few) passing to the equivariant case is not always straightforward.

This thesis consists of three topics related to equivariant homotopy theory when G is a compact Lie group:

1. Stabilization of the ∞-category of G-spaces with respect to the representation spheres.
2. Equivariant version of the factorization homology.
3. Universal property of the equivariant framed disc algebras.

Part I: Stabilization of the ∞-category of G-spaces with respect to the representation spheres

The importance of having a good stable category is evident in today's mathematics. In such category, the basic formal properties of homology and cohomology become trivialities. In the case of topological spaces, the stable homotopy category is the category obtained from the homotopy category of topological spaces by inverting the suspension functor (smash product with the circle S^1), which gives us a linear approximation of the homotopy category of spaces. The isomorphism classes in the stable homotopy category represent generalized cohomology theories.

In equivariant homotopy theory the importance of having a good stable category is even greater since much of basic equivariant algebra arises in the stable context. When talking about stabilization of the category of G-spaces, the suspension functor needs to be replaced by smash product with representation spheres, that is, one point compactifications of real G-representations. In this first part, we give a description of the stabilization of the category of G-spaces with respect to the representation spheres in the higher categorical setting.

G-spectra

In order to index G-representations, one can define a notion of a *universe* $\mathcal{U}$ as a sum of countably many copies of some set of irreducible representations of a compact Lie group G including

the trivial representation. Finite dimensional real inner-product spaces of a universe are called *indexing spaces*. We can organize them into an ∞-category $Rep^{\mathcal{U}}(G)_\infty$, with objects being the indexing spaces and morphisms G-equivariant inclusions.
We can now define a G-spectrum E as a family of pointed G-spaces EV for each indexing space $V \in \mathcal{U}$, such that for every pair of indexing spaces $V \subseteq W$ we have a structure map

$$\sigma_{V,W} : \Sigma^{W-V} EV \to EW$$

where $W - V$ is the orthogonal complement of V in W, with $\sigma_{V,V} = Id$ and such that evident diagram commutes

$$\begin{tikzcd}
\Sigma^{U-W}\Sigma^{W-V}EV \arrow[r] \arrow[d, "\simeq"] & \Sigma^{U-W}EW \arrow[d] \\
\Sigma^{U-V}EV \arrow[r] & EU
\end{tikzcd}$$

where $V \subseteq W \subseteq U$ are indexing spaces and where we denote $\Sigma^V X := S^V \wedge X$.
Note that, unlike classical spectra, G-spectra are graded by indexing spaces. The term that can be found in literature is $RO(G)$-graded. This corresponds to the fact that G-spectra encode $RO(G)$-graded (co)homology theories for equivariant spaces.

Take an indexing space V and a pointed G-space X. We can define the *free suspension G-spectrum* of X as a G-spectrum $\Sigma^\infty_V X$ such that

$$(\Sigma^\infty_V X)(W) = \Sigma^{W-V} X, \text{ when } V \subseteq W, \text{ and}$$
$$(\Sigma^\infty_V X)(W) = *, \text{ otherwise}$$

Free suspension G-spectra represent very interesting objects, since the class of free suspension G-spectra of finite G-CW-complexes generate all G-spectra under filtered homotopy colimits.

Stabilization

At the moment, let us focus our attention on the ∞-category of based, finite G-CW-complexes $Space^{finG-CW}_\infty$ and the ∞-category $(Sp^{\mathcal{U}}_{G-CW})_\infty$ of free suspension G-spectra $\Sigma^\infty_V X$ where V is an indexing space (i.e. a G-representation) and X is a based, finite G-CW-complex.
The central idea of the first part of the thesis is to look at the functor

$$\widetilde{\chi} : Rep^{\mathcal{U}}(G)_\infty \to Cat_\infty$$

which sends every indexing space to the ∞-category $Space^{finG-CW}_\infty$, and every inclusion of indexing spaces $V \hookrightarrow U$ to the smash product with the represantation sphere S^{U-V} where $U - V$ is the orthogonal complement of V in U. The stabilization of the ∞-category of based, finite G-CW-spaces with respect to the representation spheres is

$$\underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \widetilde{\chi}$$

Informally, the objects of $\underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \widetilde{\chi}$ can be represented by pairs (X, V) where X is a pointed, finite G-CW-space and V is a G-representation. There is an evident functor $F : \underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \widetilde{\chi} \to (Sp^{\mathcal{U}}_{G-CW})_\infty$ sending (X, V) to the G-spectrum $\Sigma^\infty_V X$. One of the main results of the first part is the theorem 4.2.1:

Theorem 0.1. *The functor $F : colim_{Rep^{\mathcal{U}}(G)_\infty} \widetilde{\chi} \to (Sp^{\mathcal{U}}_{G-CW})_\infty$ is an equivalence of ∞-categories.*

We can make a similar claim about the ∞-category of based G-spaces $Space_\infty^G$ and the ∞-category of G-spectra $(Sp_G^{\mathcal{U}})_\infty$, with one fundamental difference: The ∞-categories $Space_\infty^G$ and $(Sp_G^{\mathcal{U}})_\infty$ are presentable ∞-categories. In particular, we can regard them as elements of the ∞-category $\mathcal{P}r^L$, the ∞-category of presentable ∞-categories with colimit preserving functors between them. Moreover, the ∞-categories $Space_\infty^G$ and $(Sp_G^{\mathcal{U}})_\infty$ are generated under filtered colimits by the ∞-categories $Space_\infty^{finG-CW}$ and $(Sp_{G-CW}^{\mathcal{U}})_\infty$ respectively. The stabilization colimit will now be the colimit in $\mathcal{P}r^L$ which in general differs from the colimits in Cat_∞.

The universal property

Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and let $X \in \mathcal{C}^\otimes$. We say that X is an invertible object of $\mathcal{C}^\otimes$ if the map $X \otimes (-) : \mathcal{C}^\otimes \to \mathcal{C}^\otimes$ is an equivalence. To add up, given a symmetric monoidal map $\phi : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ we say that $X \in \mathcal{C}^\otimes$ is sent to an invertible object in $\mathcal{D}^\otimes$ if $\phi(X)$ is invertible in $\mathcal{D}^\otimes$.
When talking about G-spectra we can obtain a Quillen equivalence

$$S^V \wedge (-) : Sp_G^{\mathcal{U}} \rightleftarrows Sp_G^{\mathcal{U}} : \Omega^V(-)$$

on the stable model category of G-spectra. Consequently, we have that representation spheres act by equivalences on the ∞-category $(Sp_G^{\mathcal{U}})_\infty$. Moreover, the map $\Sigma^\infty : Space_\infty^G \to (Sp_G^{\mathcal{U}})_\infty$ sending a G-space to a free suspension G-spectrum now sends representation spheres to invertible objects in $(Sp_G^{\mathcal{U}})_\infty$. We want to show that $(Sp_G^{\mathcal{U}})_\infty$ is universal with respect to the inversion of representation spheres. To be more precise, we want to show that $(Sp_G^{\mathcal{U}})_\infty$ is the initial object of the ∞-category of presentable symmetric monoidal ∞-categories equipped with a map from $Space_\infty^G$ such that the representation spheres are sent to invertible objects. The main result of this part is the proposition 4.3.1:

Proposition 0.2. *The restriction functor*

$$CAlg(\mathcal{P}r^L)_{(Sp_G^{\mathcal{U}})_\infty/} \to CAlg(\mathcal{P}r^L)_{Space_\infty^G/}$$

is fully faithful, where $CAlg(\mathcal{P}r^L)$ is the ∞-category of commutative algebras in $\mathcal{P}r^L$ i.e. presentable symmetric monoidal categories, such that the essential image consists of those symmetric monoidal functors $Space_\infty^G \to \mathcal{C}^\otimes$ which send every representation sphere into an invertible object in $\mathcal{C}^\otimes$.

Organization by chapters:

1. **Preliminaries**:
Here we provide the reader with theoretical background and the machinery used in the first part of the thesis.

2. **The ∞-category of G-spaces**:
We introduce the ∞-category of G-spaces as an underlying ∞-category of the model category of G-spaces. Also, we show that this ∞-category is presentable. In particular, it is generated under filtered colimits by finite G-CW-complexes.

3. **The ∞-category of G-spectra**:
We introduce the ∞-category of G-spectra as an underlying ∞-category of the stable model category of G-spectra. Later, we give the construction of the functorial fibrant replacement in the stable model category of G-spectra. Perhaps the most important section in this chapter is 3.3, where we construct the stable homotopy category of G-spectra $\mathcal{SH}_G$. The importance of this construction lies in the fact that we can prove that every G-spectrum can be obtained as

a filtered homotopy colimit of free suspension G-spectra $\Sigma_V^\infty X$ where V is a G-representation and X is a finite G-CW-complex. Additionally, this helps us to deduce that the ∞-category of G-spectra is presentable.

4. **Stabilization of the ∞-category of G-spaces**:
In this last chapter we prove the theorem 0.1 and later deduce that the stabilization of the presentable ∞-category of G-spaces with respect to the representation spheres is equivalent to the presentable ∞-category of G-spectra. In the end, we prove proposition 0.2 i.e. we show that $(Sp_G^{\mathcal{U}})_\infty$ is universal with respect to inverting the representation spheres.

Part II: Equivariant version of the factorization homology

Factorization homology, as presented by Ayala and Francis ([AF15]), or chiral homology of Lurie ([HA]) represents homology theories for manifolds with coefficients in n-disc algebras, where n is the fixed dimension of the manifolds that we consider. It stems from the work of Beilinson and Drinfeld ([BD04]), but it also has roots in the work of Salvatore ([Sa01]) and Segal ([Se10]). There are numerous reasons to study factorization homology: for one, factorization homology defines topological quantum field teories (see [CG16]), but more importantly, it defines homology theories for manifolds and not topological spaces in general.

One of the most interesting examples for us would be the factorization homology of a circle S^1 with coefficients in an associative algebra A in some presentable ∞-category $\mathcal{C}^\otimes$, which we denote with $\int_{S^1} A$. By [AF15] *3.19* we know that $\int_{S^1} A$ is equivalent to the Hochschild complex $HC_*(A)$ of A. Additionally, we can make the connexion between the factorization homology and the topological Hochschild homology by taking A to be an associative ring spectrum. The action on the circle translates to the action on $THH(A)$ which serves as a motivation for studying factorization homology in the equivariant setting. Keeping in mind that there is an action of the group $O(2)$ on S^1 and therefore on $THH(A)$ it is reasonable to consider manifolds with an action of a compact Lie group.

Parametrized higher category theory

Let M be a n-dimensional manifold with an action of a compact Lie group G and let A be an $\mathbb{E}_n$-ring spectrum. Then, by functiorality we have an induced action of G on $\int_M A$. Unfortunately, this action is defined only up to coherent homotopy since the factorization homology is defined as a ∞-categorical colimit. Therefore $\int_M A$ is not a *genuine* G-object. More generally, we could ask ourselves: what would be the right notion of an ∞-category with an action of G, which we will simply call G-∞-categories? One obvious candidate would be an ∞-category $\mathcal{C}$ together with a coCartesian fibration $\mathcal{C} \to BG$. Sadly, as explained in [BDGNS16], this coCartesian fibration does not capture all of the information that we would like to have. This motivates the development of parametrized higher category theory, provided by Barwick and his students ([BDGNS16], [Shah18], [Nar16], [Nar17], [NS]). Similarly as one can do computations in homotopy theory in ∞-categorical setting, parametrized higher category theory provides us with a good environment in which we can work in equivariant homotopy theory. By definition, a G-∞-category is an ∞-category $\mathcal{C}$ together with a coCartesian fibration $p : \mathcal{C} \to \mathcal{O}_G^{op}$ where $\mathcal{O}_G$ is the ∞-category of orbits of G. Informally, for every orbit $G/H \in \mathcal{O}_G$, $\mathcal{C}_{[G/H]}$ can be viewed as the ∞-category of H-objects. To add up, we have a family of ∞-categories which are compatible in the sence that we can define restriction functors, conjugacy action functors between them etc.
Unfortunately, there are some theoretical limitations to this theory. Namely, when G is a compact Lie group, in general we cannot take $\mathcal{O}_G$ to be the orbit category of *all* tranzitive G-spaces, but we can always restrict our attention to those orbits with finite stabilizers (there will be more words on that in 5).

Return to factorization homology

We have already stated that factorization homology represents homology theories for manifolds, and as such needs to satisfy Eilenberg-Steenrod axioms. In view of Ayala and Francis, the construction of factorization homology as a symmetric monoidal functor satisfies an analogue of the Eilenberg-Steenrod axioms called $\otimes$-excision property: If we have a manifold M obtained as a collar gluing $M \cong M' \bigcup_{M_0 \times \mathbb{R}} M''$ then we have an equivalence:

$$\int_M A \simeq \int_{M'} A \underset{\int_{M_0 \times \mathbb{R}} A}{\otimes} \int_{M''} A$$

where the left side is the two sided bar construction. Perhaps the most important result concerning factorization homology lies in the axiomatic caracterization i.e. Ayala and Francis proved that factorization homology accounts for all homology theories on manifolds.

We want to develop factorization homology for manifolds with an action of a compact Lie group G such that the output object $\int_M A$ is a genuine G-object. In order to do so we will use the parametrized higher category theory. In this view, our G-factorization homology would need to satisfy the equivariant analogues of the ordinary factorization homology: G-$\otimes$-excision property, axiomatic caracterization etc.
Additionally, if we take A to be equivariant disc algebra with values in the G-symmetric monoidal G-∞-category of G-spectra $\underline{Sp}^G$ (see [Nar17]), the topological induction functor will be compatible with the norm maps: if $K \leq H \leq G$ are finite subgroups of G and if M is a K-manifold, then

$$\int_{H \times_K M} A \simeq N_K^H(\int_M A)$$

Framing on G-manifolds

Framing on G-manifolds provides us with G-factorization homology of G-manifolds with more general tangential structure. This way, we can, for example, obtain homology theories for equivariant oriented manifolds ([CMW01]) or manifolds with free G-action (see [Hor19] *3.3.7*). The most common example are V-framed G-manifolds, where V is a G-representation. Such framing on a G-manifold M corresponds to the trivialization of the tangent vector bundle $TM \cong M \times V$.
In general, the tangent vector bundle of a G-manifold is determined by the classifying map $\tau_M : M \to BO_n(G)$. In other words, there is a pullback diagram of G-vector bundles

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & EO_n(G) \arrow[d] \\ M \arrow[r, "\tau_M"] & BO_n(G) \end{tikzcd}$$

The data of a framing on a G-manifold M consists of a G-map $f : B \to BO_n(G)$ where B is a G-space, together with a factorization of a G-tangent bundle classifying map through f. In particular, B-framing on M corresponds to the diagram

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & EO_n(G) \arrow[d] \\ M \arrow[r, "f_M"] & B \arrow[r, "f"] & BO_n(G) \end{tikzcd}$$

where both left and right (and therefore also the outer) rectangles are pullback diagrams, with $E \to B$ being the G-vector tangent bundle corresponding to the map f. In the case when $B = *$, the $*$-framing corresponds to the diagram

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & EO_n(G) \arrow[d] \\ M \arrow[r, "f_M"] & * \arrow[r] & BO_n(G) \end{tikzcd}$$

where V is a G-representation. Hence, $*$-framing, or better said V-framing, corresponds to the trivialization of the tangent vector bundle $TM \cong V \times M$.

G-disc algebras

Generally speaking, G-factorization homology is an invariant of a geometric and an algebraic input. The geometric input is given by (framed) G-manifolds, while the algebraic input is given by (framed) G-disc algebras. Formally, (framed) G-disc algebras with coefficients in $\underline{\mathcal{C}}^{\otimes}$ are G-symmetric monoidal functors with the source being the G-∞-category of (framed) G-discs taking values in some G-symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$. Informally, G-factorization homology can be seen as gluing local data of a G-manifold M. This local data can be described by an algebraic structure given by G-discs on M. In the approach of Ayala and Francis, the algebraic input on framed n-manifolds is given by $\mathbb{E}_n$-algebra, while in the equivariant case, the algebraic input is given by V-framed G-disc algebras. To depict this structure on a G-manifold M, let $x \in M$ be a point with stabilizer $Stab(x) = H$. Then the tangent space TM_x has a linear action of H. Moreover, it is isomorphic to V as an H-representation. Then the tubular neighborhood of the orbit of x is isomorphic to $G \times_H V$. In other words, the map $G \times_H V \to G/H$ is a G-vector bundle, where G/H is regarded as the orbit of x. Therefore, intuitively, we can think of our framed discs as G-vector bundles $G \times_H V \to G/H$.

Existing work

As of the time of writing this thesis, the author is familiar with the work of Horev ([Hor19]) and Weelinck ([Wee18]) on the equivariant versions of factorization homology. While the work of Weelinck contains several good ideas, the construction presented in [Wee18] provides us with factorization homology which is not *genuine* as explained above. On the other hand, the construction of Horev, using parametrized higher category theory, gives an equivariant extension of the ordinary factorization homology given by Ayala and Francis when G is a *finite* group, which is of greater interest to us. In fact, the second part of this thesis represents the generalization of Horev's construction to the case when the group G is a compact Lie group.

Organization by chapters:

5. **Preliminaries**:
   We give the theoretical basis for the parametrized higher homotopy theory and higher algebra: we define parametrized ∞-categories, parametrized symmetric monoidal ∞-categories and parametrized ∞-operads. In addition, we give examples of the G-∞-category of G-spaces and G-∞-category of finite G-sets.

6. **G-manifolds**:
   We give the definition of G-manifolds with which we want to work with. Additionally, we give the description and construction of the G-∞-category of G-manifolds $\underline{Mfld}^G$, together with its framed variants $\underline{Mfld}^{G,B-fr}$ where the framing is given by a G-map $f : B \to BO_n(G)$. We will also see how the restriction functor, conjugacy functor and the topological induction functor are incorporated in this G-∞-structure. Finally, we will construct and describe the G-symmetric monoidal G-∞-category of G-manifolds.

7. G**-discs**:
We start by definig G-∞-category of G-discs together with its framed variants. Similar to the previous chapter, we follow this up with the section devoted to the G-symmetric monoidal structure on the G-∞-category of G-discs. In section 7.2 we will prove that the framed G-discs $\underline{Disk}^{G,B-fr}$ can be obtained as the G-symmetric monoidal envelope of G-∞-operad $\underline{Rep}_n^{B-fr,\sqcup}(G)$. Secton 7.3 serves to describe the connection between G-discs and G-configuration spaces. Finally, in the last section 7.4 we give the description of G-disc algebras taking values in the G-∞-category of G-spectra. In addition, we will see how the norm maps of Hill-Hopkins-Ravenell ([HHR16]) are incorporated in this G-parametrized structure.

8. G**-Factorization homology**:
First, we give the definition of the G-factorization homology as a parametrized colimit

$$\int_M A = \underline{G/H} - colim(\underline{Disk}^{G,B-fr}_{/\underline{M}} \to \underline{G/H} \underline{\times} \underline{Disk}^{G,B-fr} \to \underline{G/H} \underline{\times} \underline{\mathcal{C}})$$

After the definition, we proceed with the description of the G-factorization homology as a G-functor obtained as a G-left adjoint functor $i_!$

$$i_! : Fun_G(\underline{Disk}^{G,B-fr}, \underline{\mathcal{C}}) \rightleftarrows Fun_G(\underline{Mfld}^{G,B-fr}, \underline{\mathcal{C}}) : i^*$$

Finally, we extend the G-factorization homology functor to the G-symmetric monoidal functor.

9. **Properties of** G**-factorization homology**:
In the final chapter we show:

   - G-factorization homology satisfies the G-$\otimes$-excision property:
   This result can be summed by Definition 9.2.2 and Proposition 9.2.4: if M is a G-manifold with the gollar gluing decomposition i.e. a G-map $f : M \to [-1,1]$ such that $f^{-1}(-1,1) \cong M_0 \times (-1,1)$, with $M_0 = f^{-1}(0)$, $M' = f^{-1}[-1,1)$ and $M'' = f^{-1}(-1,1]$, then there is an equivalence

   $$\int_M A \simeq \int_{M'} A \underset{\int_{M_0\times\mathbb{R}} A}{\otimes} \int_{M''} A$$

   where A is a (framed) G-disc algebra taking values in a G-symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$.

   - G-factorization homology respects sequential unions:
   If there is a sequence of open G-manifolds $M_1 \subset M_2 \subset ...$ such that $M = \bigcup_{i=1}^{+\infty} M_i$ then there is an equivalence

   $$colim_i \int_{M_i} A \xrightarrow{\simeq} \int_M A$$

   - Axiomatic caracterization of G-factorization homology:
   The main result of this section is Theorem 9.4.3 which states:

   Theorem 0.3. *Let $\underline{C}^{\otimes} \to \underline{Fin}^G_*$ be a presentable G-symmetric monoidal G-∞-category and let $f : B \to BO_n(G)$ be a G-map. Then the adjunction*

   $$(i^{\otimes})_! : Fun^{\otimes}_G(\underline{Disk}^{G,B-fr,\sqcup}, \underline{C}^{\otimes}) \rightleftarrows Fun^{\otimes}_G(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes}) : (i^{\otimes})^*$$

   restricts to an equivalence

$$(i^{\otimes})_! : Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) \xrightarrow{\simeq} \mathcal{H}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$$

Where $\mathcal{H}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) \subseteq Fun_G^{\otimes}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$ is the full subcategory spanned by those functors that are *homology theories* i.e. which satisfy G-$\otimes$-excision property and respect G-sequential unions.

In other words, we show that G-factorization homology accounts for all homology theories of G-manifolds.

Part III: Universal property of the equivariant framed disc algebras

Third and final part of the thesis is dedicated to proving the universal property of the G-disc algebras. Informally, the ∞-category of G-disc algebras with coefficients in some G-symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$ is equivalent to the ∞-category of H-disc algebras taking values in the underlying H-symmetric monoidal category $\underline{\mathcal{C}}_H^{\otimes}$ of $\underline{\mathcal{C}}^{\otimes}$ with a suitable choice of framing on both G-discs and H-discs. In other words, the G-symmetric monoidal category of G-discs is freely generated by the H-symmetric monoidal category of H-discs with the suitable choice of framing on both G-discs and H-discs.

Framing on G-discs and H-discs

As we have stated in the beginning, there is a suitable choice of framing on G-discs and H-discs under which we obtain the equivalence of ∞-categories of G-disc algebras and H-disc algebras. Namely, the G-discs are framed over the orbit space G/H, while the H-discs are framed over the point. These framings are compatible in the following sense:
The $*$-framing on H-discs corresponds to the V-framing, where V is an H-representation. Then the G/H-framing corresponds to the framing with respect to the G-vector bundle $G \times_H V \to G/H$. In the case when we work with the manifolds of the same dimension as our compact Lie group G, the H-representation above corresponds to the adjoint representation of G with the restricted H-action.

Main result

The most important result of this section is the Theorem 10.2.5:

Theorem 0.4. *Let $\underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal category. Then the G-symmetric monoidal category of G/H-framed G-discs is freely generated by the H-symmetric monoidal category of $*$-framed H-discs. In other words, there is an equivalence*

$$Fun_G^{\otimes}(\underline{Disk}^{G,G/H-fr}, \underline{\mathcal{C}}^{\otimes}) \xrightarrow{\simeq} Fun_H^{\otimes}(\underline{Disk}^{H,*-fr}, \underline{\mathcal{C}}_H^{\otimes})$$

where $\underline{\mathcal{C}}_H^{\otimes}$ is the underlying H-∞-category of G-∞-category $\underline{\mathcal{C}}^{\otimes}$.

What would be the motivation for showing this universal property? The input of the G-factorization homology is a G-disc algebra together with a G-manifold which we regard as a genuine G-object in the parametrized ∞-category, and as an output we receive a genuine G-object. The universal property allows us to replace G-algebra with an H-algebra, which is in general, less complicated object. To be more precise, a V-framed (or $*$-framed) H-disc algebra A_H in $\underline{\mathcal{C}}_H^{\otimes}$ (the underlying H-∞-category of a G-∞-category $\underline{\mathcal{C}}^{\otimes}$), where V is the adjoint representation of G restricted to H-action, gives us the corresponding G/H-framed G-algebra A in $\underline{\mathcal{C}}^{\otimes}$. Moreover,

computing $\int_{G/H} A$ now gives us a genuine G-object (i.e. a coCartesian section of $\underline{\mathcal{C}}^{\otimes} \to \mathcal{O}_G^{op}$) living in $\underline{\mathcal{C}}^{\otimes}$.
In particular, taking $H = \{e\}$ to be a trivial subgroup, the $*$-framed (or better said R^n-framed) H-disc is equivalent to the $\mathbb{E}_n$-algebra object in the underlying ∞-category $\underline{\mathcal{C}}_e^{\otimes}$ of $\underline{\mathcal{C}}^{\otimes}$. Additionally, taking $n = 1$ and $G = S^1$, gives us the following:

- Taking an associative algebra object A_e in $\underline{\mathcal{C}}_e^{\otimes}$ gives us a corresponding G-framed G-disc algebra A in $\underline{\mathcal{C}}^{\otimes}$. Therefore, the associative algebra A_e produces a genuine S^1-object $\int_{S^1} A$ in $\underline{\mathcal{C}}^{\otimes}$.

This will be explained in more detail in 10.3.

To conclude, the result on the universal property of G-discs can carve a path to a new insight into the norm maps of Hill, Hopkins and Ravenel [HHR16].

Organization by chapters:

10. **G-approximations to G-∞-operads**: In the first section we introduce the reader to the theory of G-approximations to G-∞-operads. Informally, a G-approximation is an ∞-category C together with a map $f : C \to E^{\otimes}$ satisfying certain conditions, where $E^{\otimes}$ is a G-∞-operad. The ∞-category C is not a G-∞-operad, but under certain conditions does capture valuable information of the G-∞-operad it approximates (in our case $E^{\otimes}$). Moreover, given another G-∞-operad $E'^{\otimes}$ we can define the C-algebra objects in $E'^{\otimes}$ as functors $F : C \to E'^{\otimes}$ satisying some conditions. Again, informally, they are adequate replacements of maps of G-∞-operads.

    The most important result of this section is the Proposition 10.1.7, which states that if $f : C \to E^{\otimes}$ induces a categorical equivalence of the underlying ∞-categories of C and $E^{\otimes}$ then we can replace the ∞-category of $E^{\otimes}$-algebras in $E'^{\otimes}$, $Alg_G(E^{\otimes}, E'^{\otimes})$ with the ∞-category $Alg_G(C, E'^{\otimes})$ of C-algebras in $E'^{\otimes}$.

11. **The universal property**: In this section we use the theory of G-approximations to construct a map $\theta : \underline{Rep}_n^{*-fr,\sqcup}(H) \to \underline{Rep}_n^{G/H-fr,\sqcup}(G)$ and to prove that it is in fact a G-approximation satisfying the conditions of Proposition 10.1.7. Using the results $\mathcal{D}^{H,*-fr} \simeq Env_H(\underline{Rep}_n^{*-fr,\sqcup}(H))$ and $\mathcal{D}^{G,G/H-fr} \simeq Env_G(\underline{Rep}_n^{G/H-fr,\sqcup}(G))$ i.e. $\mathcal{D}^{H,*-fr}$ is equivalent to the H-symmetric monoidal envelope of $\underline{Rep}_n^{*-fr,\sqcup}(H)$ and $\mathcal{D}^{G,G/H-fr}$ is equivalent to the G-symmetric monoidal envelope of $\underline{Rep}_n^{G/H-fr,\sqcup}(G)$, we obtain the Theorem 10.2.5.

12. **Applications**: In the final section, we give two examples as applications of the universal property. In particular, we explain how:

    - Associative algebra objects with genuine involution in $\underline{Sp}^{\mathbb{Z}_2}$ correspont to the $O(2)$-genuine objects in $\underline{Sp}^{O(2)}$. In addition, we will see how to refine the $\mathbb{Z}_2$-genuine structure on the real topological Hochschild homology to $O(2)$-genuine structure;
    - Associative algebra objects in the ∞-category of spectra Sp correspond to the S^1-genuine objects in $\underline{Sp}^{S^1}$.

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Part I

Stabilization of the ∞-category of G-spaces

Chapter 1

Preliminaries I

1.1 Model categories

The theory of model categories was developed by Quillen in [Qui67] and [Qui69]. Formally, a model category is an ordinary category with three specified classes of morphisms: *weak equivalences*, *fibrations* and *cofibrations*. These classes are subject to some axioms which we will state later. The main advantage of model categories is that they provide us with decent machinery with which we can operate in homotopy theory. Some good refferences are [Hov98], [PH03], [DS95] and [IHom], but there are of course many others.

Other importance of model categories is that we can construct an underlying ∞-category of a model category, which plays a significant role in the first part.

Definition 1.1.1. Let M be an ordinary category. A morphism $q : A \to B$ is called a *retract* of $f : X \to Y$ if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "q"] \arrow[rr, bend left, "id_A"] & X \arrow[r] \arrow[d, "f"] & A \arrow[d, "q"] \\ B \arrow[r] \arrow[rr, bend right, "id_B"'] & Y \arrow[r] & B \end{tikzcd}$$

Remark 1.1.2. If we take M to be a category of topological spaces and $q = id_A$, $f = id_X$, the notion of id_A being a retract of id_X corresponds to the notion of space A being a retract of a topological space X.

Definition 1.1.3. A *model category* is a category M equipped with three classes of morphisms:

- class of *weak equivalences* $\mathcal{W}$ (which are denoted with $\xrightarrow{\sim}$)
- class of *fibrations* $\mathcal{F}$ (denoted with $\twoheadrightarrow$),
- class of *cofibrations* $\mathcal{C}$ (denoted with $\rightarrowtail$).

which satisfy the following 5 axioms:

1. M is cocomplete.
2. (2 out of 3 property) For every commutative diagram

$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & C \end{tikzcd}$$

if two of the three morphisms are weak equivalences then so is the third.

3. If q is a retract of f such that $f \in \mathcal{W}$ (resp. $f \in \mathcal{F}$, $f \in \mathcal{C}$), then $q \in \mathcal{W}$ (resp. $q \in \mathcal{F}$, $q \in \mathcal{C}$).
4. For every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & B \arrow[d, two heads, "p"] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

such that $i \in \mathcal{C}$ and $p \in \mathcal{F}$ the dashed lift exists if i or p is in addition a weak equivalence.

5. Every morphism $f : X \to Y$ admits two natural factorizations

$$X \overset{\sim}{\rightarrowtail} P_f \twoheadrightarrow Y \text{ , and}$$
$$X \rightarrowtail C_f \overset{\sim}{\twoheadrightarrow} Y$$

We will write $(M, \mathcal{W}, \mathcal{F}, \mathcal{C})$ for a model category M to indicate the classes of weak equivalences, fibrations and cofibrations. We will also write just M when the model structure is known.

Remark 1.1.4. The condition in axiom 5 that the factorizations are natural (functorial) can be omitted. The reason why it is written is of practical nature: Consider, for example, the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[d, "q"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

The functoriality of the factorization allows us to obtain the following diagrams

$$\begin{tikzcd} A \arrow[r, tail, "\sim"] \arrow[d, "p"] & P_f \arrow[r] \arrow[d, "P(p,q)"] & B \arrow[d, "q"] \\ A' \arrow[r, tail, "\sim"] & P_{f'} \arrow[r, two heads] & B' \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "p"] & C_f \arrow[r, two heads, "\sim"] \arrow[d, "C(p,q)"] & B \arrow[d, "q"] \\ A' \arrow[r, tail] & C_{f'} \arrow[r, two heads, "\sim"] & B' \end{tikzcd}$$

Remark 1.1.5. By axiom 1 every model category M has an initial and a final object since these objects correspond to the limit and colimit of empty diagrams respectively. For future reference, let us denote with 0 the initial and with $\{*\}$ the terminal object of M.

Definition 1.1.6. Let $X \in M$ where M is a model category. We say that X is

- *fibrant* if we have $X \twoheadrightarrow \{*\}$
- *cofibrant* if we have $0 \rightarrowtail X$

Definition 1.1.7. For every $X \in M$ consider the maps $0 \to X$ and $X \to \{*\}$. Axiom 5 gives us a factorization

$$X \overset{\sim}{\rightarrowtail} P_X \twoheadrightarrow \{*\} \text{ , and}$$
$$0 \rightarrowtail C_X \overset{\sim}{\twoheadrightarrow} X$$

We call P_X (resp. C_X) the *fibrant* (resp. *cofibrant*) replacement of X.

Remark 1.1.8. Given a model category M, we will denote with M_f (resp. M_c) the full subcategory spanned by fibrant (resp. cofibrant) objects.
By axiom 5 we have functors

$$R : M \to M_f$$
$$L : M \to M_c$$

such that $R(X) = P_X$ and $L(X) = C_X$.

Example 1.1.9. ***Quillen model category:*** *Quilen gave a proof that the category of topological spaces Top can be endowed with the structure of a model category with:*

- *the class of weak equivalences being* weak homotopy equivalences*,*
- *fibrations being* Serre fibrations*, and*
- *cofibrations being retracts of general cellular inclusions.*

It is worth noting that in this structure every object is fibrant.

Definition 1.1.10. Let M be a model category with $\mathcal{W}$ the class of weak equivalences. We define the *homotopy category* of M to be the localization of M at the class of maps $\mathcal{W}$ and we denote it with

$$Ho(M) := M[\mathcal{W}^{-1}]$$

Note that we also have the projection functor $\pi : M \to Ho(M)$.

Lemma 1.1.11. *Let M be a model category. The canonical inclusions $M_f \hookrightarrow M$ and $M_c \hookrightarrow M$ induce equivalences of homotopy categories*

$$Ho(M_f) \xrightarrow{\simeq} Ho(M)$$
$$Ho(M_c) \xrightarrow{\simeq} Ho(M)$$

As we said in the beginning of this section one important aspect of model categories is that they admit an underlying ∞-category

Definition 1.1.12. Let M be a model category. We define the underlying ∞-category as

$$M_\infty = N(M)[\mathcal{W}^{-1}]$$

in other words, as the simplicial localization at the class of arrows represented by weak equivalences in M.

Given a functor $F : C \to D$ between two model categories, we would like to study the conditions on F which would allow us to compare the two model categories and their underlying homotopy categories and therefore their underlying ∞-categories. To start, we can define the notion of *left* and *right Quillen* functor.

Definition 1.1.13. Let C and D be two model categories ad let $F : C \to D$ and $G : D \to C$ be two functors. We say that:

- F is a left Quillen functor if it is left adjoint and if it preserves cofibrations and trivial cofibrations,
- G is a right Quillen functor if it is right adjoint and if it preserves fibrations and trivial fibrations.

Definition 1.1.14. A *Quillen adjunction* is an adjunction $F : C \rightleftarrows D : G$ between model categories such that F is a left Quillen functor and G is a right Quillen functor.

Next thing that we will do is introduce the notion of a *derived functor*. Generally speaking, when we have a functor $F : C \to D$ between two model categories, a derived functor of F is a functor between homotopy categories of C and D induced by F (or equivalently, a functor between ∞-categories C_∞ and D_∞ induced by F).

Definition 1.1.15. Let C and D be two model categories ad let $F : C \to D$ and $G : D \to C$ be two functors. We define:

- *Left derived* fuctor of F, $\mathbb{L}F : Ho(C) \to D$ to be the Left Kan extension of F along $\pi_C : C \to Ho(C)$,
- *Right derived* fuctor of G, $\mathbb{R}G : Ho(D) \to C$ to be the Right Kan extension of G along $\pi_D : D \to Ho(D)$,
- *Total* left derived fuctor of F, $\mathbb{L}^{tot}F$ as the left derived functor of the compositionong $C \xrightarrow{F} D \xrightarrow{\pi_D} Ho(D)$,
- *Total* right derived fuctor of G, $\mathbb{R}^{tot}G$ as the right derived functor of the compositionong $D \xrightarrow{G} C \xrightarrow{\pi_C} Ho(C)$.

Proposition 1.1.16. *Let C and D be two model categories ad let $F : C \to D$ and $G : D \to C$ be two functors that form a Quillen adjunction*

$$F : C \rightleftarrows D : G$$

then the total left and right functors $\mathbb{L}^{tot}F$ and $\mathbb{R}^{tot}G$ form an adjunction

$$\mathbb{L}^{tot}F : Ho(C) \rightleftarrows Ho(D) : \mathbb{R}^{tot}G$$

Defintion 1.1.17. A *Quillen equivalence* is a Quillen adjunction $F : C \rightleftarrows D : G$ such that the induced adjunction $\mathbb{L}^{tot}F : Ho(C) \rightleftarrows Ho(D) : \mathbb{R}^{tot}G$ represents the equivalence of categories.

Remark 1.1.18. An immediate consequence of the upper definition is that given that two model categories C and D are Quillen equivalent their underlying ∞-categories C_∞ and D_∞ are also equivalent.

Example 1.1.19. *The most important examples of total derived functors are* homotopy *colimits.*

Note that, given a category K and a category C we have a category of K-indexed diagrams C^K. Moreover, we have an adjunction colim : $C^K \rightleftarrows C$: c where colim is a functor which assigns to every K-diagram $X : K \to C$ its colimit and where c is a constant functor.

If C is a model category we can equip C^K with the projective model category structure *in which the weak equivalences (resp. fibrations) are those natural transformations $\tau : X \to Y$ such that $\tau_k : X(k) \to Y(k)$ is a weak equivalence (resp. fibration) for every $k \in K$. The cofibrations would be those natural transformations that have a left lifting property (LLP) with respect to all trivial fibrations.*
Now the colim functor is a functor between two model categories and we will mark

$$hocolim := \mathbb{L}^{tot} colim$$

the homotopy colimit. We can also make an analogous definition for homotopy limits.

By [HTT] *4.2.4* we have the following:

Lemma 1.1.20. *Let $\mathcal{M}$ be a model category. Then the notion of homotopy colimit corresponds to the notion of colimit in the underlying ∞-category M_∞.*

This lemma can be used for variety of results. The one that we will be using the most in particuliar is:

Theorem 1.1.21. *Let $\mathcal{M}$ be a model category and let $\mathcal{I}$ be a filtered category with a filtered subcategory $\mathcal{J}$ together with an inclusion functor incl : $\mathcal{J} \to \mathcal{I}$ such that for every $i \in \mathcal{I}$ there exists $j \in \mathcal{J}$ such that there is a map $i \to incl(j)$. Then for an $\mathcal{I}$-diagram $X : \mathcal{I} \to \mathcal{M}$, the map*

$$hocolim_{\mathcal{J}} incl^* X \to hocolim_{\mathcal{I}} X$$

is a weak equivalence.

Proof. Let $\mathcal{M}_\infty$ be the underlying ∞-category of the model category $\mathcal{M}$, and let $N(\mathcal{I})$ and $N(\mathcal{J})$ denote the nerve of categories $\mathcal{I}$ and $\mathcal{J}$ respectively. Now we have the situation

$$N(\mathcal{J}) \hookrightarrow N(\mathcal{I}) \xrightarrow{\tilde{X}} \mathcal{M}_\infty$$

where $N(\mathcal{I}) \xrightarrow{\tilde{X}} \mathcal{M}_\infty$ is induced by $X : \mathcal{I} \to \mathcal{M}$. We would like to show that the map $N(\mathcal{J}) \hookrightarrow N(\mathcal{I})$ is cofinal in the ∞-categorical sense. For that we will use the Quillen's theorem A for ∞-categories (see [HTT] *4.1.3.1*)

Quillen's theorem A. *Let $p : X \to \mathcal{C}$ be a map of simplicial sets whose codomain is an ∞-category. Then p is cofinal if and only if for every $y \in \mathcal{C}$ the simplicial set $X \times_{\mathcal{C}} \mathcal{C}_{y/}$ is weakly contractible.*

It is easy to see that for any $i \in \mathcal{I}$ the category $\mathcal{J} \times_{\mathcal{I}} \mathcal{I}_{i/}$ is filtered. Since the nerve functor preserves pullbacks and $N(\mathcal{I}_{i/}) = N(\mathcal{I})_{i/}$, the filtered property tells us that the category $N(\mathcal{J}) \times_{N(\mathcal{I})} N(\mathcal{I})_{i/}$ is weakly contractible, hence the map $N(\mathcal{J}) \to N(\mathcal{I})$ is cofinal.

This gives us the fact that colimits in $\mathcal{M}_\infty$ computed over the category $N(\mathcal{I})$ are weakly equivalent to the colimits computed over $N(\mathcal{J})$, but the colimits in $\mathcal{M}_\infty$ corespond to the homotopy colimits in the underlying model category, hence the desired result. $\square$

Remark 1.1.22. We will finish this section with a remark on how to compute homotopy colimits in general:
Let $\mathcal{M}$ be a model category, K an ordinary category and $X : K \to \mathcal{M}$ a K-diagram. As we have defined in 1.1.19, $hocolim_K X = \mathbb{L}^{tot} colim_K X$. By the definition of derived functors $hocolim_K X = colim_K LX$ where LX is the cofibrant replacement of the K-diagram X in the projective model structure on $\mathcal{M}^K$.

For example let

$$X_1 \to X_2 \to ... \to X_n \to ...$$

be a sequence in $\mathcal{M}$. By inspection, the cofibrant replacement is the sequence

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & ... \arrow[r] & X_n \arrow[r] \arrow[d] & ... \\ CX_1 \arrow[r] & CX_2 \arrow[r] & ... \arrow[r] & CX_n \arrow[r] & ... \end{tikzcd}$$

where $CX_i \to X_i$ is a weak equivalence and $CX_i \to CX_{i+1}$ is a cofibration for every i, thus

$$hocolim_{i \in \mathbb{N}} X_i = colim_{i \in \mathbb{N}} CX_i$$

1.2 Presentable ∞-categories

Presentable ∞-categories are ∞-categories with nice formal properties. Informally speaking, a presentable ∞-category is generated under filtered colimits by some set of objects. The category of Abelian groups is one such example: even though the category is large, it is in some sense determined by the much smaller category of finitely generated abelian groups. There are many other examples of presentable categories that arise naturally in mathematics. For the theory of presentable categories in classical categorical setting one can look at [AR94], for more references one can look in [Gro15] and [HTT].

Formally, the definition of a presentable ∞-category is the following:

Definition 1.2.1. An ∞-category C is *presentable* if it is cocomplete and *accessible.*

Informally, the condition of being accessible means that our ∞-category C is generated under filtered colimits by some small set of objects in C. For that we will introduce the category of Ind-objects. Later, we will introduce the notion of left and right adjoint functor between ∞-categories in order to present the version of Freyd's adjoint functor theorem.

Ind-objects and accessible ∞-categories

Let C be a category. An Ind-object of C is a diagram $f : \mathcal{I} \to C$ where $\mathcal{I}$ is a small filtered category. We will denote with $Ind(C)$ the category of Ind-objects in C. Note that C may be recovered from $Ind(C)$ by taking diagrams indexed by the one-point category. The idea is that $Ind(C)$ is obtained from C by formally adjoining colimits of filtered diagrams. Moreover, $Ind(C)$ can be described by the following universal property: for any category D which admits filtered colimits and any functor $F : C \to D$, there exists a functor $\widetilde{F} : Ind(C) \to D$ whose restriction to C is isomorphic to F and which commutes with filtered colimits.

In higher categorical setting we will need to take some time to explain the analogues of filtered ∞-categories:

Definition 1.2.2. ([HTT] *5.3.1.7*) Let C be an ∞-category. We say that C is filtered if, for every small simplicial set K and every map $f : K \to C$, there exists a map $\widetilde{f} : K^{\triangleright} \to C$ extending f.

Remark 1.2.3. Note that in [HTT] the definition of a filtered ∞-category is given with respect to some regular cardinal κ, that is we say that C is κ-filtered if, for every κ-small simplicial set K and every map $f : K \to C$, there exists a map $\widetilde{f} : K^{\triangleright} \to C$ extending f. Then when we take $\kappa = \omega$ we arrive at our definition. Of course that we can keep this size factor throughout this section but for the sake of simplicity we will stick to the case $\kappa = \omega$ whenever it is possible.

Lemma 1.2.4. *([HTT]* 5.3.1.14*) Let C be an ∞-category. Then C is filtered if and only if it has the right extension property with respect to every inclusion*

$$\partial\Delta^n \hookrightarrow \Lambda^{n+1}_{n+1}$$

for every $n \geq 0$.

Lemma 1.2.5. *([HTT]* 5.3.1.15*) Let C be a topological category. Then C is filtered if and only if the ∞-category $N(C)$ is filtered.*

For the following part, denote with $j : C \to \mathcal{P}(C) := Fun(C^{op}, \mathcal{S})$ the Yoneda embedding where C is an ∞-category and $\mathcal{S}$ is the ∞-category of spaces.

Definition 1.2.6. ([HTT] *5.3.5.1*) Let C be a small ∞-category. We let $Ind(C)$ denote the full subcategory of $\mathcal{P}(C)$ spanned by those functors $f : C^{op} \to \mathcal{S}$ which classify right fibrations $\widetilde{C} \to C$, where the ∞-category $\widetilde{C}$ is filtered. We will refer to $Ind(C)$ as the ∞-category of Ind-objects of C.

Proposition 1.2.7. *([HTT]* 5.3.5.3*) The full subcategory $Ind(C) \hookrightarrow \mathcal{P}(C)$ is stable under filtered colimits.*

Proposition 1.2.8. *([HTT]* 5.3.5.4*) Let C be a small ∞-category and let $F : C^{op} \to \mathcal{S}$ be an object of $\mathcal{P}(C)$. The following conditions are equivalent:*

1. *There exists a (small) filtered ∞-category $\mathcal{I}$ and a diagram $p : \mathcal{I} \to C$ such that F is a colimit of the composition $j \circ p : \mathcal{I} \to \mathcal{P}(C)$.*
2. *The functor F belongs to $Ind(C)$.*

If C admits small colimits, then the upper conditions are equivalent to:

3. *The functor F preserves small limits.*

Definition 1.2.9. ([HTT] *5.3.4.5*) Let C be an ∞-category which admits small filtered colimits and let κ be a regular cardinal. We will say that a functor $f : C \to D$ is κ-continuous if it preserves κ-filtered colimits. In particuliar, we will say that f is continuous if it is ω-continuous.
Let C be an ∞-category which admits filtered colimits and let $X \in C$. Let $j_X : C \to \widehat{\mathcal{S}}$ denote the functor corepresented by X. We will say that X is compact if j_X is continuous.

Proposition 1.2.10. *([HTT]* 5.3.5.5*) Let C be a small ∞-category and let $j : C \to Ind(C)$ be the restriction of the Yoneda embedding. For each object $X \in C$, $j(X)$ is a compact object of $Ind(C)$.*

Definition 1.2.11. ([HTT] *5.4.2.1* and *5.4.2.5*) An ∞-category C is *accessible* if there exists a small category C_0 together with an equivalence

$$Ind(C_0) \to C$$

If C is an accessible ∞-category, then a functor $F : C \to D$ is *accessible* if it is κ-continuous for some regular cardinal κ.

Adjoint functors

Every reader is familiar with importance of adjunctions in classical category theory. Here, we will give a brief introduction to adjunctions in the higher categorical setting. We will start with the definition of an associated functor which we will use in the definition of adjunction:

Definition 1.2.12. ([HTT] *5.2.1.1*) Let $p : M \to \Delta^1$ be a Cartesian fibration and let C and D be ∞-categories such that we have equivalences $h_0 : C \to p^{-1}(0)$ and $h_1 : D \to p^{-1}(1)$. We will say that a functor $g : D \to C$ is associated to M if there is a commutative diagram

$$\begin{tikzcd} D \times \Delta^1 \arrow[rr, "s"] \arrow[dr] & & M \\ & \Delta^1 \arrow[ur] & \end{tikzcd}$$

such that $s|_{D\times\{1\}} = h_1$, $s|_{D\times\{0\}} = h_0 \circ g$ and $s|_{\{d\}\times\Delta^1}$ is a p-Cartesian edge of M for every $d \in D$.

We have the converse:

Proposition 1.2.13. *([HTT]* 5.2.1.3 (1)*) Let C and D be ∞-categories and let $g : D \to C$ be a functor. Then there exists a diagram*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & M \arrow[d, "p"] & D \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \Delta^1 & \{1\} \arrow[l] \end{tikzcd}$$

such that $p : M \to \Delta^1$ is a Cartesian fibration, the associated maps $C \to p^{-1}(0)$ and $D \to p^{-1}(1)$ are equivalences, and g is associated to M.

Remark 1.2.14. The proposition [HTT] *5.2.1.3* also tells us that the ∞-category M is unique up to a categorical equivalence.

Now we are ready to give the formal definition of an adjunction in the higher categorical setting:

Definition 1.2.15. ([HTT] *5.2.2.1*) An *adjunction* between ∞-categories C and D is a map $p : M \to \Delta^1$ which is a Cartesian and CoCartesian fibration.
Let $g : D \to C$ be a functor associated to $p : M \to \Delta^1$ and let $f : C \to D$ be a functor associated to $p^{op} : M^{op} \to (\Delta^1)^{op} \simeq \Delta^1$. we will say that f is *left adjoint* to g and g is *right adjoint* to f.

We will turn to some properties of adjoint functors:

Proposition 1.2.16. *([HTT]* 5.2.2.6*) Let C, D and E be ∞-categories and let $f : C \to D$ and $f' : D \to E$ be functors such that f has a right adjoint g and f' has a right adjoint g'. Then $f' \circ f$ has a right adjoint $g \circ g'$.*

Proposition 1.2.17. *([HTT]* 5.2.3.5*) Let $f : C \rightleftarrows D : g$ be an adjunction between ∞-categories C and D. Then f preserves all colimits that exist in C and g preserves all limits that exist in D.*

Proposition 1.2.18. *([HTT]* 5.5.1.4*) Let $f : C \to D$ be a functor between ∞-categories that admit small filtered colimits. Suppose that f admits a right adjoint functor $g : D \to C$ such that g is continuous. Then f carries compact objects to compact objects.*

Definition 1.2.19. ([HTT] *5.2.6.1*) Let C and D be ∞-categories. We will denote with $Fun^L(C, D)$ (resp. $Fun^R(C, D)$) the full subcategory of $Fun(C, D)$ spanned by left (resp. right) adjoint functors.

Proposition 1.2.20. *([HTT]* 5.2.6.2*) The ∞-categories $Fun^L(C, D)$ and $Fun^R(D, C)^{op}$ are equivalent.*

Presentable ∞-categories

Since we have covered enough theoretical ground, the definition of presentable ∞-categories is quite easy to formulate:

Definition 1.2.21. An ∞-category is *presentable* if it is cocomplete and accessible.

We know that the presentable ∞-categories are cocomplete by definition but we also have this important result:

Proposition 1.2.22. *([HTT]* 5.5.2.4*) A presentable ∞-category admits all (small) limits.*

One of the most important properties of presentable ∞-categories is their relation to the adjoint functors which we can express as the version of the *adjoint functor theorem*:

Theorem 1.2.23. *([HTT]* 5.5.2.9*) Let $F : C \to D$ be a functor between presentable ∞-categories. Then:*

- *F admits a right adjoint if and only if it preserves small colimits.*
- *F admits a left adjoint if and only if F is accessible and it preserves small limits.*

This description allows us to form two ∞-categories of presentable ∞-categories which would be in some sense antiequivalent.

Definition 1.2.24. ([HTT] *5.5.3.1*) Let $\widehat{Cat}_\infty$ be the ∞-category of (not necessarily small) ∞-categories. Define two subcategories $\mathcal{P}r^L$,$\mathcal{P}r^R \subseteq \widehat{Cat}_\infty$ the following way:

- The objects of both $\mathcal{P}r^L$ and $\mathcal{P}r^R$ are presentable ∞-categories.
- A functor $F : C \to D$ of presentable ∞-categories is a morphism in $\mathcal{P}r^L$ if it preserves small colimits i.e. it is a left adjoint functor.
- A functor $F : C \to D$ of presentable ∞-categories is a morphism in $\mathcal{P}r^R$ if it is accessible and if it preserves small limits i.e. it is a right adjoint functor.

Remark 1.2.25. The ∞-categories $\mathcal{P}r^L$ and $\mathcal{P}r^R$ are antiequivalent in the sense that for every simplicial set K we have a canonical bijection

$$[K, \mathcal{P}r^L] \simeq [K^{op}, \mathcal{P}r^R]$$

where $[K, C]$ denotes the collection of equivalence classes of objects of the ∞-category $Fun(K, C)$. In particular, there is a canonical isomorphism

$$\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{op}$$

in the homotopy category of ∞-categories (see [HTT] *5.5.3.4*).

Proposition 1.2.26. *([HTT]* 5.5.3.13 *and* 5.5.3.18*) The ∞-categories $\mathcal{P}r^L$ and $\mathcal{P}r^R$ admit small limits, and the inclusion functors $\mathcal{P}r^L \subseteq \widehat{Cat}_\infty$ and $\mathcal{P}r^R \subseteq \widehat{Cat}_\infty$ preserve small limits, where $\widehat{Cat}_\infty$ is the ∞-category of (not necessarily small) ∞-categories.*

Remark 1.2.27. Moreover, we have that the ∞-category $\mathcal{P}r^L$ admits small colimits: Since $\mathcal{P}r^L$ is anti-equivalent to $\mathcal{P}r^R$ by replacing each left adjoint by its right adjoint, it follows that colimits in $\mathcal{P}r^L$ are computed as limits in $\mathcal{P}r^R$, which are in turn computed as limits in $\widehat{Cat}_\infty$. This means that colimits in $\mathcal{P}r^L$ are computed as limits in $\widehat{Cat}_\infty$.

At the end of this section, we will list some properties and examples of presentable ∞-categories:

Proposition 1.2.28. *Let $\{C_\alpha\}_{\alpha \in A}$ be a family of presentable ∞-categories indexed by a small set A. Let C and D be presentable ∞-categories and let K be a small simplicial set. Then the following ∞-categories are also presentable:*

- $\prod_{\alpha \in A} C_\alpha$,
- $Fun(K, C)$,
- $Fun^L(C, D)$,
- $C_{/p}$ *and* $C_{p/}$, *for a diagram* $p : K \to C$.

Chapter 2

The ∞-category of G-spaces

The goal of this chapter is to introduce the reader with the definition and properties of compactly generated weak Hausdorff spaces with an action of a compact Lie group G (G-CGWH spaces, for short). In fact, following this chapter all spaces will be assumed to be G-CGWH (based) spaces.

We introduce the standard model structure on G-CGWH spaces and consequently the model structure on the based G-CGWH spaces. In 2.1 we study colimits and homotopy colimits in this category. Finally, in 2.2 we prove that the underlying ∞-category of the model category of based G-CGWH spaces is presentable. In particular. it is generated under filtered colimits by the full subcategory spanned by the finite G-CW-complexes.

2.1 G-CGWH spaces

In order to prove some fundamental statements that will be of use in our further work, it will be convenient to introduce the category of *compactly generated weak Hausdorff spaces* equipped with an action of a compact Lie group G, which we will denote as G-CGWH spaces.

As noted in [Sch18] (Appendix B), the category of G-CGWH spaces is (co)complete. Using [Sch18], **Proposition B.2.** (i), G/H is compactly generated, where $H \leq G$ is a closed subgroup, which gives us:

Proposition 2.1.1. *Every G-CW-complex is compactly generated.*

The category of G-CGWH spaces can be endowed with model structure, which we will call the standard model structure for G-spaces ([Sch18], **Proposition B.7.**), where:

- **weak equivalences** are weak homotopy equivalences of G-spaces, that is G-maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a weak homotopy equivalence for every closed subgroup $H \leq G$.
- **fibrations** are Serre fibrations of G-spaces, that is G-maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a Serre fibration for every closed subgroup $H \leq G$.

We also know that this model structure is cofibrantly generated, with the set of generating cofibrations

$$\mathcal{I}_G = \{G/H \times S^{k-1} \to G/H \times D^k\}_{k\geq 0, H\leq G} \tag{2.1}$$

(note that, for $k = 0$, S^{k-1} is taken to be the empty set), and with the set of generating acyclic cofibrations

$$\mathcal{J}_G = \{G/H \times D^k \to G/H \times D^k \times [0, 1]\}_{k\geq 0, H\leq G}$$

where in both cases $H \leq G$ runs through the set of closed subgroups of G. We will reffer to weak equivalences, fibrations and cofibrations in this model structure by G-weak equivalences, G-fibrations and G-cofibrations.

Imitating the proofs of [Hov98] 2.4.5 and 2.4.6 we can conclude:

Proposition 2.1.2. *Every G-cofibration is a closed inclusion.*

What we want to accomplish next is to formulate a statement similar to [Rezk] 10.14 which would take into account the action of compact Lie group G on the maping space of all continuous maps between two G-CGWH spaces X and Y.

Remark 2.1.3. When K is compact and X is compactly generated, then the mapping space $Map_{Space}(K, X)$ of all continuous maps is compactly generated. If we take spaces K and X to be equipped with an action of a compact Lie group G, we will mark $Map^G(K, X)$ the G-CGWH space of all continuous maps with G acting by conjugation. As noted in [Sch18], given a G-CGWH space X and a closed subgroup $H \leq G$, the space X^H is a closed subspace of X and hence compactly generated, meaning that the spaces $(Map^G(K, X))^H$ are all compactly generated.

Theorem 2.1.4. *Let*

$$X_0 \to X_1 \to \ldots \to X_n \to ..$$

be a sequence of G-CGWH spaces where the maps $X_i \to X_{i+1}$ are closed inclusions of G-CGWH spaces. Denote with X the colimit of the sequence. Then for every compact G-space K we have a bijection

$$colim_n Map^G(K, X_n) \to Map^G(K, colim_n X_n) \tag{2.2}$$

which is G-equivariant. Moreover, for every closed subgroup $H \leq G$, we have a bijection

$$colim_n (Map^G(K, X_n))^H \to (Map^G(K, colim_n X_n))^H$$

Proof. By first forgetting the action of G and using [Rezk] 10.14 we know that every map $f : K \to X$ factors through X_n for some n. We can look at X_n as a closed subspace of X (see for example [Sch18] $A.14(i)$). Let us denote with $i_n : X_n \hookrightarrow X$ the inclusion, which in addition is a G-map, so we can write $f = i_n \circ \widetilde{f}$. Fix an element $g \in G$. Then we again have that $g^{-1}fg$ factors through X_m for some m, and let us write that factorization as $g^{-1}fg = i_m \circ \overline{f_g}$. Since the inclusion maps are also G-maps we can assume that $n = m$ without any loss of generality. Since

$$g^{-1}\widetilde{f}(gK) \subseteq g^{-1}\widetilde{f}(K) \subseteq g^{-1}X_n \subseteq X_n$$

we have a well defined map $g^{-1}\widetilde{f}g : K \to X_n$. But now

$$i_n \circ g^{-1}\widetilde{f}g = g^{-1} \circ i_n\widetilde{f} \circ g = g^{-1}fg = i_n \circ \overline{f_g}$$

the map i_n is an inclusion and hence $g^{-1}\widetilde{f}g = \overline{f_g}$. With this we have concluded that the bijection (2.2) is G-equivariant. The second part follows automatically. $\square$

In other words, when restricting the G-action to H-action, we have a bijection

$$colim_n Map_{Space_H}(K, X_n) \to Map_{Space_H}(K, colim_n X_n)$$

Let us focus a little bit on the spaces of fixed points:

As mentioned in 2.1.3, given a G-CGWH space X and a closed subgroup $H \leq G$, the space X^H is a closed subspace of X and hence compactly generated. Then the closed inclusion of G-spaces $X \to Y$ gives us a closed inclusion $X^H \to Y^H$. This with the fact that taking H-fixed points commutes with filtered colimits along G-maps that are closed inclusions (a slight modification of [Sch18] B.1 (ii)) gives us:

Corollary 2.1.5. *Let $H \leq G$ be a closed subgroup. Then for a sequence*

$$X_0 \to X_1 \to \ldots \to X_n \to ..$$

of G-CGWH spaces where the maps $X_i \to X_{i+1}$ are closed inclusions of G-CGWH spaces and every G-fixed point $x \in X_0$, the map

$$colim_n \pi_k^H(X_n, x) \to \pi_k^H(colim_n X_n, x)$$

is an isomorphism for every $k \geq 0$.

Corrolary 2.1.6. *Let*

$$X_0 \to X_1 \to ... \to X_n \to ..$$

be a sequence of G-CGWH spaces where the maps $X_i \to X_{i+1}$ are closed inclusions of G-CGWH spaces. Then the colimit $colim_n X_n$ in the category of G-CGWH spaces is also the homotopy colimit with respect to the standard model structure on G-spaces.

Corollary 2.1.7. *Let K be a compact G-space and let*

$$X_0 \to X_1 \to ... \to X_n \to ..$$

be a sequence of G-CGWH spaces where the maps $X_i \to X_{i+1}$ are closed inclusions of G-CGWH spaces. Then the map

$$colim_n Map^G(K, X_n) \to Map^G(K, colim_n X_n)$$

is a weak homotopy equivalence of G-spaces.

Now we are ready to prove theorem which will play a significant role in the upcoming sections:

Theorem 2.1.8. *Let K be a compact G-CW-complex and let*

$$X_0 \to X_1 \to ... \to X_n \to ..$$

be a sequence of G-CGWH spaces. Then the map

$$hocolim_n Map^G(K, X_n) \to Map^G(K, hocolim_n X_n)$$

is a weak equivalence of G-spaces.

Proof. The homotopy colimit is computed by taking the colimit of the cofibrant replacement of the initial diagram with respect to the projective model structure. In our case, that would include replacing the sequence by a weakly equivalent one involving G-cofibrations. The functor $Map^G(K, \_)$ with K a compact G-CW-complex preserves weak homotopy equivalences of G-spaces and so we can assume that the maps $X_n \to X_{n+1}$ are G-cofibrations, and therefore closed inclusions of G-spaces by 2.1.2. Then each $Map^G(K, X_n) \to Map^G(K, X_{n+1})$ is also a closed inclusion, and so 2.1.6 and 2.1.7 gives us the desired result. □

We will state another theorem which will be of use further on:

Theorem 2.1.9. *Let*

$$X_0 \to X_1 \to ... \to X_n \to ..$$

be a sequence of G-CGWH spaces and let $H \leq G$ be a closed subgroup of a compact Lie group G. Then the map

$$colim_n \pi_k^H(X_n) \to \pi_k^H(hocolim_n X_n)$$

is an isomorphism.

Proof. As stated in the proof of the previous statement, the homotopy colimit is computed by replacing the initial sequence by a weakly equivalent one involving G-cofibrations, which are in particular closed inclusions. Let us denote with $\{CX_n\}$ the cofibrant replacement diagram of $\{X_n\}$. Since we are talking about weakly equivalent replacement in the projective module structure on the category of diagrams, the maps $CX_n \to X_n$ are all weak homotopy equivalences of G-spaces. By 2.1.5 we have have an isomorphism

$$colim_n \pi_k^H(X_n) = colim_n \pi_k^H(CX_n) \to \pi_k^H(colim_n CX_n) = \pi_k^H(hocolim_n X_n)$$

for every $k \geq 0$. With this the proof is finished. □

Remark 2.1.10. Note that we have been talking about G-CGWH spaces which are not pointed. But as in non-equivariant case (see [Hov98], *Corollary 2.4.24* and *Theorem 2.4.25*) that is not a big concern since all of the proofs pass when we restrict our attention to pointed G-CGWH spaces (with pointed mapping spaces).

Notation 2.1.11. *From now on, when we say space or G-space we will mean a CGWH space or G-CGWH space respectively.*

2.2 Presentable ∞-category of G-spaces

Similarly as in 2.1 the category of based G-spaces can be endowed with the model structure which we call the standard model structure on based G-spaces. In particular we have:

- **weak equivalences** are weak homotopy equivalences of based G-spaces, that is based G-maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a weak homotopy equivalence for every closed subgroup $H \leq G$.
- **fibrations** are Serre fibrations of based G-spaces, that is based G-maps $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a Serre fibration for every closed subgroup $H \leq G$.

We also know that this model structure is cofibrantly generated, with the set of generating cofibrations

$$\mathcal{I}_G = \{(G/H \times S^{k-1})_+ \to (G/H \times D^k)_+\}_{k\geq 0, H\leq G}$$

(note that, for $k = 0$, S^{k-1} is taken to be the empty set), and with the set of generating acyclic cofibrations

$$\mathcal{J}_G = \{(G/H \times D^k)_+ \to (G/H \times D^k \times [0,1])_+\}_{k\geq 0, H\leq G}$$

where in both cases $H \leq G$ runs through the set of closed subgroups of G. We will reffer to weak equivalences, fibrations and cofibrations in this model structure by G-weak equivalences, G-fibrations and G-cofibrations.

To summarize, we have the following:

Theorem 2.2.1. *([MMMF02] III 1.8) The category of based G-spaces is a compactly generated model category with respect to the based G-weak equivalences, based G-fibrations, and retracts of relative based G-cell complexes. The sets $\mathcal{I}_G$ and $\mathcal{J}_G$ are the generating G-cofibrations and the generating acyclic G-cofibrations.*

As for any model category, we have an underlying ∞-category of G-spaces $Space^G_\infty$, which is computed as the localization of the nerve

$$Space^G_\infty = N(Space_{G*})[W^{-1}]$$

where W is the collection of G-weak homotopy equivalences. We will denote by $Space^{G-CW}_\infty \subset Space^G_\infty$ the ∞-subcategory spanned by finite, based G-CW-complexes.

We want to show that the ∞-category $Space^G_\infty$ is a presentable ∞-category. In particular, we want to show that the objects of the ∞-category $Space^{G-CW}_\infty$ are compact objects in $Space^G_\infty$ and that the ∞-category $Space^G_\infty$ is, in fact, generated under filtered colimits by the objects of $Space^{G-CW}_\infty$.

Lemma 2.2.2. *Finite based G-CW-complexes are compact objects in $Space^G_\infty$*

Proof. Let X be a finite, based G-CW-complex and let $Y : \mathcal{I} \to Space^G_\infty$ be a diagram indexed by a filtered category $\mathcal{I}$. We want to show that

$$Map_{Space^G_\infty}(X, colim_{i\in\mathcal{I}} Y_i) \simeq colim_{i\in\mathcal{I}} Map_{Space^G_\infty}(X, Y_i)$$

We will prove this by induction on $k = dim(X)$.

For $k = 0$, without loss of generality, we have that $X = G/H$ where H is a closed subgroup of G. In other words, X is a G orbit space. In order to show that orbits are compact objects recall the Elmendorf theorem (see, for example [Ste10])

Elmendorf's theorem. *Let G be a topological group. Then there is a pair of Quillen equivalences*

$$\Theta : Space^{O_G^{op}} \rightleftarrows Space_G : \Phi$$

where O_G^{op} is the orbit category. The category $Space^{O_G^{op}}$ is endowed with projective model structure while the category $Space_G$ is endowed with the standard model structure.

This Elmendorf Quillen equivalence sends each orbit G/H to the corresponding representable functor $O_G^{op} \to Space$. Representable functors are always compact in the functor category, since mapping out of them is given by evaluation at the representing object, and evaluation commutes with all homotopy colimits, and in particular filtered homotopy colimits. Since the categories $Space^{O_G^{op}}$ and $Space_G$ are Quillen equivalent, their underlying ∞-categories are equivalent, and since G/H maps to a compact object, it is also compact.

For the inductive step, let X_k be a finite G-CW-complex of dimension k and X_{k-1} its $k-1$-skeleton. By the inductive hypothesis, X_{k-1} is a compact object. X_k is obtained via the pushout

$$\begin{tikzcd}
\coprod_j G/H_j \times S^{k-1} \arrow[r] \arrow[d] & X_{k-1} \arrow[d] \\
\coprod_j G/H_j \times D^k \arrow[r] & X_k
\end{tikzcd}$$

Note that since the left vertical map is in fact a cofibration we have even more, that the diagram is a homotopy pushout square. Moreover, the objects $\coprod_j G/H_j \times S^{k-1}$, X_{k-1} and $\coprod_j G/H_j \times D^k$ are all compact (fist two are clear and the third is equivalent to disjoint union of orbits and hence is compact). The statement now follows from the fact that $Map_{Space^G_\infty}(\bullet, Y)$ takes colimits to limits and that finite limits commute with filtered colimits. $\square$

Theorem 2.2.3. *The ∞-category $Space^G_\infty$ is a presentable ∞-category.*

Proof. We have that the ∞-category $Space^G_\infty$ has all small colimits, and is in fact generated under filtered colimits by $Space^{G-CW}_\infty$. By the previous lemma 2.2.2, we know that the elements of $Space^{G-CW}_\infty$ are compact objects in $Space^G_\infty$ and hence we can write $Space^G_\infty$ as the *Ind*-completion of $Space^{G-CW}_\infty$, that is

$$Space^G_\infty \simeq Ind(Space^{G-CW}_\infty)$$

$\square$

Chapter 3

The ∞-category of G-spectra

Since we have defined the ∞-category of G-spaces and concluded that it is in fact an *Ind*-completion of its subcategory spanned by finite G-spaces, it is our turn to do the same with the ∞-category of G-spectra and it's subcategory spanned by free suspension spectra of finite G-CW-complexes.

First section is dedicated to the introduction of the category of G-spectra, where we will become familiar with the model structure as well as other constructions. Secondly, we will turn our attention to the ∞-category of G-spectra where we will show that free suspension spectra of finite G-CW-complexes are compact objects in the ∞-category of G-spectra. Later, we will construct a functorial fibrant replacement QX of a G-spectrum X. This construction is important for two reasons:

- The fibrant replacement is used when computing the mapping spaces in the ∞-category of G-spectra (see [DK1], [DK2] for the overall theory and [DK3] for the case of model categories), which we will use to prove that the stabilization of the ∞-category of G-spaces with respect to the representation spheres is equivalent to the ∞-category of G-spectra.
- Secondly, the fibrant replacement will be used when defining the stable homotopy category of G-spectra $\mathcal{SH}_G$.

Finally, in the last section, we will show that the ∞-category of G-spectra is presentable. In particular, we will show that $(Sp^{\mathcal{U}}_G)_\infty = Ind((Sp^{\mathcal{U}}_{G-CW})_\infty)$ where $(Sp^{\mathcal{U}}_{G-CW})_\infty$ is the full subcategory of $(Sp^{\mathcal{U}}_G)_\infty$ spanned by free spectra $\Sigma^\infty_V X$ where $V \subset \mathcal{U}$ is an indexing space and X is a *finite* G-CW-complex.

3.1 The model category of G-spectra

Let G be a compact Lie group. Generally speaking, G-spectra are objects that encode equivariant (co)homology theories. They are generalization of spectra as one passes from stable homotopy theory to equivariant stable homotopy theory. Some good refferences include [MMMF02], [CBMS], and for a slightly different approach one can look at [Sch01].

Unlike the classical case, where (co)homology theories are indexed by numbers, equivariant (co)homology theories are indexed by G-representations i.e. real inner-product G-vector spaces in a chosen G-universe which we define:

Definition 3.1.1. ([CBMS] or [MMMF02]) A universe $\mathcal{U}$ is a sum of countably many copies of some set of irreducible representations of a compact Lie group G including the trivial representation. Finite dimensional real inner-product subspaces of $\mathcal{U}$ are called *indexing spaces*. The universe is called *complete* when we take the set of *all* irreducible representations, and is called *trivial* if we take only the trivial representation.

The work with a universe gives us more flexibility (as well as some technical advantages) as opposed to working with set of representations of G.
Where the ordinary concept of spectrum is given in terms of looping and delooping of ordinary topological spaces by ordinary spheres, a G-spectrum is instead given by looping and delooping of topological G-spaces with respect to representation spheres of G:

Definition 3.1.2. Representation sphere of the indexing space $V \in \mathcal{U}$ is the pointed G-space obtained as the one-point compactification of that indexing space, denoted S^V. Representation sphere S^V acts functorially on the category $Space_{G*}$ by the smash product. Given a pointed G-space X, we denote $S^V \wedge X$ by $\Sigma^V X$.

Definition 3.1.3. Let $Rep^{\mathcal{U}}(G)$ be the category whose objects are indexing spaces of a given universe $\mathcal{U}$, with morphisms given by inclusions. By a spectrum indexed on $\mathcal{U}$, we mean a family of based G-spaces EV, one for each indexing space $V \subset \mathcal{U}$, together with structure G-maps

$$\sigma_{V,W} : \Sigma^{W-V} EV \to EW$$

where $V \subseteq W$ are indexing spaces and $W - V$ is the orthogonal complement of V in W. We require $\sigma_{V,V} = id$, and we require evident transitivity diagram to commute for indexing spaces $V \subset W \subset U$

$$\begin{tikzcd}
\Sigma^{U-W}\Sigma^{W-V}EV \arrow[r] \arrow[d, "\simeq"] & \Sigma^{U-W}EW \arrow[d] \\
\Sigma^{U-V}EV \arrow[r] & EU
\end{tikzcd}$$

A map between two G-spectra $f : E \to F$ is given by a family of maps of based G-spaces $f(V) : EV \to FV$ for every indexing space $V \subset \mathcal{U}$ such that the diagram

$$\begin{tikzcd}
\Sigma^{W-V}EV \arrow[r] \arrow[d] & \Sigma^{W-V}FV \arrow[d] \\
EW \arrow[r] & FW
\end{tikzcd}$$

commutes for all indexing spaces $V \subseteq W$ of $\mathcal{U}$. We denote with $Sp_G^{\mathcal{U}}$ the category of G-spectra indexed on universe $\mathcal{U}$.

Remark 3.1.4. Note that we have the adjoint structure maps $\widetilde{\sigma}_{V,W} : EV \to \Omega^{W-V}EW$, where $\Omega^{W-V}EW = Map^G(S^{W-V}, EW)$.
For G-spaces X and Y $Map^G(X,Y)$ represents the internal hom object, that is, the mapping G-space of maps between bassed G-spaces X and Y, which is a space of all (not necessarily equivariant) continuous maps between based G-spaces X and Y with G acting by conjugation.

Definition 3.1.5. In the case when maps $\widetilde{\sigma}_{V,W}$ from 3.1.4 are all weak homotopy equivalences of G-spaces, we say that E is an Ω-G-spectrum.

Example 3.1.6. *Given a based G-space X and an indexing space $V \subset \mathcal{U}$, we can define a* free *G-spectrum functor, which we will denote by $\Sigma_V^\infty : Space_{G*} \to Sp_G^{\mathcal{U}}$, given by:*

$$\Sigma_V^\infty X(W) = \Sigma^{W-V} X, \textit{ for } V \subseteq W, \textit{ and}$$
$$\Sigma_V^\infty X(W) = *, \textit{ otherwise}$$

On the other hand, we have a functor $\Omega_V^\infty : Sp_G^{\mathcal{U}} \to Space_{G*}$, which assigns to a G-spectrum E a based G-space $\Omega_V^\infty E = EV$. These two functors form an adjunction (see, for example [MMMF02]):

$$\Sigma_V^\infty : Space_{G*} \rightleftarrows Sp_G^{\mathcal{U}} : \Omega_V^\infty$$

which will prove to be very useful.

The category of G-spectra $Sp_G^{\mathcal{U}}$ can be endowed with the stable model structure. We will give a quick expository here, more details can be found in [MMMF02].
First, we define homotopy groups of a G-spectrum $E \in Sp_G^{\mathcal{U}}$ ([MMMF02], definition III 3.2):

Definition 3.1.7. Let $E \in Sp_G^{\mathcal{U}}$ be a G-spectrum. The homotopy groups of E are defined the following way:

$$\pi_q^H(E) = \underset{V \in Rep^{\mathcal{U}}(G)}{colim} \pi_q^H(\Omega^V E(V)), \text{ for } q \geq 0,$$
$$\pi_{-q}^H(E) = \underset{\mathbb{R}^q \subset V \in Rep^{\mathcal{U}}(G)}{colim} \pi_0^H(\Omega^{V-\mathbb{R}^q} E(V)), \text{ for } q > 0$$

where H is the closed subgroup of G, $\pi_q^H(A) = \pi_q(A^H)$ for every based G-space A and V runs through all indexing spaces of $\mathcal{U}$. The colimit in the definition of negatively indexed homotopy groups is indexed by all indexing spaces $V \subset \mathcal{U}$ that contain $\mathbb{R}^q$.
A map $f : X \to Y$ between G-spectra will be called a π_*-isomorphism if it induces an isomomorphism between homotopy groups.

Before going to the stable model structure it is necessary to visit the level model structure on $Sp_G^{\mathcal{U}}$:

Definition 3.1.8. ([MMMF02], definition III 2.3) Given a map $f : X \to Y$ of G-spectra, we say that:

1. f is a level equivalence if $f(V) : X(V) \to Y(V)$ is a weak equivalence of G-spaces,
2. f is a level fibration if $f(V) : X(V) \to Y(V)$ is a Serre fibration of G-spaces,
3. f is a level acyclic fibration if it is both a level equivalence and level fibration,
4. f is a q-cofibration if it satisfies the LLP with respect to level acyclic fibrations.

Now we can present the stable model structure:

Theorem 3.1.9. *([MMMF02], III 4.1 and 4.2) The category $Sp_G^{\mathcal{U}}$ is a compacly generated, proper G-topological model category where:*

- *weak equivalences are the π_*-isomorphisms,*
- *cofibrations are q-cofibrations,*
- *fibrations are those maps which satisfy the RLP with respect to the acyclic cofibrations.*

Moreover, the fibrant objects are the Ω-G-spectrum objects.

Note that we have an action of $Space_{G*}$ on the category $Sp_G^{\mathcal{U}}$, where for a pointed G-space X and a G-spectrum E this action is given by $(X \wedge E)(V) = X \wedge EV$. We also have another action given by $F(X, E)(V) = Map^G(X, EV)$. In the special case when we take X to be a representation sphere S^V we define functors $\Sigma^V E(W) = \Sigma^V EW$ and $\Omega^V E(W) = Map^G(S^V, EW) = \Omega^V EW$. A result form [MMMF02] gives us:

Lemma 3.1.10. *([MMMF02], III 4.15) For every indexing space $V \subset \mathcal{U}$, the pair of endofunctors (Σ^V, Ω^V) in $Sp_G^{\mathcal{U}}$ is a Quillen equivalence.*

3.2 The ∞-category

The category of G-spectra $Sp_G^{\mathcal{U}}$ is endowed with a stable model structure which means that we have an underlying ∞-category (see also [Gro15] and [HTT]), in which, by lemma 3.1.10 representation spheres act invertibly.

Definition 3.2.1. We will denote with $(Sp_G^{\mathcal{U}})_\infty$ the underlying ∞-category of the model category of G-spectra $Sp_G^{\mathcal{U}}$ with respect to the stable model structure.

Definition 3.2.2. Let $Sp_{G-CW}^{\mathcal{U}} \subseteq Sp_G^{\mathcal{U}}$ be the full subcategory spanned by free G-spectra $\Sigma_V^\infty X$ where $V \subset \mathcal{U}$ is an indexing space and X is a finite G-CW-complex. Let us also denote with $(Sp_{G-CW}^{\mathcal{U}})_\infty$ the full subcategory of $(Sp_G^{\mathcal{U}})_\infty$ again spanned by free G-spectra $\Sigma_V^\infty X$ where $V \subset \mathcal{U}$ is an indexing space and X is a finite G-CW-complex.

Throughout the section we will heavily rely on the fact that $Rep^{\mathcal{U}}(G)$ is a filtered category, thus we will state this quick lemma which is the direct corolarry of 1.1.21:

Lemma 3.2.3. *Let $\mathcal{R}$ be a filtered subcategory of the category $Rep^{\mathcal{U}}(G)$ together with the inclusion functor $i : \mathcal{R} \hookrightarrow Rep^{\mathcal{U}}(G)$. Additionally, for every $V \in Rep^{\mathcal{U}}(G)$ there exists $W \in \mathcal{R}$ and a map $V \hookrightarrow i(W)$. Then for every functor $X : Rep^{\mathcal{U}}(G) \to Space_G$ the map*

$$hocolim_{\mathcal{R}} i^* X \to hocolim_{Rep^{\mathcal{U}}(G)} X$$

is a weak equivalence of G-spaces.

First we want to show that elements of the ∞-category $(Sp_{G-CW}^{\mathcal{U}})_\infty$ are compact objects in the ∞-category of G-spectra.

Let E be a G-spectrum, $\mathcal{I}$ a filtered category and $\{Y_i\}_{i\in\mathcal{I}}$ a diagram of G-spectra indexed by $\mathcal{I}$. We say that G-spectrum E is a compact object if the map

$$\underset{i\in\mathcal{I}}{hocolim} Map_{Sp_G^{\mathcal{U}}}(E, Y_i) \to Map_{Sp_G^{\mathcal{U}}}(E, \underset{i\in\mathcal{I}}{hocolim} Y_i)$$

is a weak equivalence. We will show the following:

Theorem 3.2.4. *Let X be a finite G-CW-complex, $V \subset \mathcal{U}$ an indexing space. Then the G-spectrum $\Sigma_V^\infty X$ is a compact object in $Sp_G^{\mathcal{U}}$.*

Proof. As before, let $\mathcal{I}$ be a filtered category and $\{Y_i\}_{i\in\mathcal{I}}$ a diagram of G-spectra indexed by $\mathcal{I}$. We want to show that the map

$$\underset{i\in\mathcal{I}}{hocolim} Map_{Sp_G^{\mathcal{U}}}(\Sigma_V^\infty X, Y_i) \to Map_{Sp_G^{\mathcal{U}}}(\Sigma_V^\infty X, \underset{i\in\mathcal{I}}{hocolim} Y_i)$$

is a weak equivalence. For that we will use the adjunction $\Sigma_V^\infty : Space_{G*} \rightleftarrows Sp_G^{\mathcal{U}} : \Omega_V^\infty$. If we could prove that the functor Ω_V^∞ commutes with filtered homotopy colimits, the lemma 2.2.2 would give us the desired conclusion.

The homotopy colimit $\underset{i\in\mathcal{I}}{hocolim} Y_i$ is obtained by computing the colimit of the diagram obtained as the cofibrant replacement of the original diagram. Note that, by [MMMF02] III *Lemma 3.3*, the level equivalence of G-spectra is a stable equivalence, and that, by definition, the cofibrations in level model structure are cofibrations in the stable model structure. That being said, the cofibrant replacement in the level model structure of the diagram $\{Y_i\}_{i\in\mathcal{I}}$ is the cofibrant replacement in the stable model structure, which we will denote as $\{CY_i\}_{i\in\mathcal{I}}$. To add up, the diagram $\{CY_i(V)\}_{i\in\mathcal{I}}$ is the cofibrant replacement of the diagram $\{Y_i(V)\}_{i\in\mathcal{I}}$, since $\{CY_i\}_{i\in\mathcal{I}}$ is obtained from the level model structure, meaning that

$$\Omega_V^\infty \underset{i\in\mathcal{I}}{hocolim} Y_i \to \underset{i\in\mathcal{I}}{hocolim} \Omega_V^\infty Y_i$$

is a weak equivalence. This finishes the proof. $\square$

3.3 The stable homotopy category of G-spectra

In this section, we give the construction and the description of the homotopy category of G-spectra which we will call *the stable homotopy category of G-spectra* and denote it by $\mathcal{SH}_G$. We will start with the introduction of the functorial fibrant replacement in the stable model category of G-spectra, since it plays a vital role in our construction of $\mathcal{SH}_G$.

The fibrant replacement

Since we know that the fibrant objects are the Ω-G-spectra we can define:

Definition 3.3.1. For a G-spectrum X we define QX to be the G-spectrum such that:

$$QX(V) = hocolim_{V\subset W}\Omega^{W-V}X(W)$$

where V is an indexing space, and the homotopy colimit is taken over all indexing spaces $W \subset \mathcal{U}$ such that $V \subseteq W$. We have the cannonical map $\mu : X \to QX$. The homotopy colimit is constructed with respect to the standard model structure on pointed G-spaces.

As mentioned before, we will restrict our attention to G-spectra X that are levelwise compactly generated, that is $X(V)$ is a G-CGWH space for every indexing space $V \subset \mathcal{U}$. Due to the work of Illman ([Ill78] and [Ill83]) we know that representation spheres admit a structure of a G-CW-complexes, therefore they are compact G-CW-complexes hence taking a G-CGWH space A, the spaces $\Sigma^V A$ and $\Omega^V A$ are again G-CGWH spaces.

In order to show that QX is indeed a fibrant replacement of X it would suffice to prove that QX is an Ω-G-spectrum and that the map μ is a weak equivalence in the model structure (that is, a π_*-isomorphism).

Lemma 3.3.2. *Let $X \in Sp_G^{\mathcal{U}}$ be a G-spectrum. Then QX is an Ω-G-spectrum.*

Proof. We want to show that the adjoint structure map

$$\widetilde{\sigma}_{W,V} : QX(V) \to \Omega^{W-V}QX(W)$$

is a weak equivalence of G-spaces, for indexing spaces $V \subset W$.

Let us take an expanding countable sequence of indexing paces U_i such that $W \subset U_i$ for all i and such that their union is the whole universe $\mathcal{U}$. Note that we automatically have $V \subset U_i$ for all i. By 3.2.3 we have a weak equivalence:

$$hocolim_{U_i}\Omega^{U_i-V}X(U_i) \to hocolim_{V\subset U}\Omega^{U-V}X(U)$$

We will stick to the computation on the sequential homotopy colimit.

$$hocolim_{U_i}\Omega^{U_i-V}X(U_i) \simeq hocolim_{U_i}\Omega^{W-V}\Omega^{U_i-W}X(U_i)$$

By 2.1.8 we have a weak equivalence of $hocolim_{U_i}\Omega^{W-V}\Omega^{U_i-W}X(U_i) \to \Omega^{W-V}hocolim_{U_i}\Omega^{U_i-W}X(U_i)$. Since the functor of mapping outside of G-CW-complexes preserves weak equivalences, we again, by using 3.2.3 have a weak equivalence

$$\Omega^{W-V}hocolim_{U_i}\Omega^{U_i-W}X(U_i) \to \Omega^{W-V}hocolim_{W\subset U}\Omega^{U-W}X(U) = \Omega^{W-V}QX(W)$$

Putting everything together we see that spaces $QX(V)$ and $\Omega^{W-V}QX(W)$ are weakly equivalent. $\square$

Lemma 3.3.3. *Let $X \in Sp_G^{\mathcal{U}}$ be a G-spectrum. Then the map $\mu : X \to QX$ is a π_*-isomorphism.*

Proof. Since the map $\mu : X \to QX$ is cannonical it will be enough to check the homotopy groups of QX. For $k \geq 0$ and $H \leq G$ a closed subgroup, we have

$$\begin{aligned}
\pi_k^H(QX) &= colim_V \pi_k^H(\Omega^V QX(V)) \\
&= colim_V \pi_k^H(\Omega^V hocolim_{V \subset W_i} \Omega^{W_i - V} X(W_i)) \\
&\to colim_V \pi_k^H(hocolim_{V \subset W_i} \Omega^V \Omega^{W_i - V} X(W_i)) \\
&\to colim_V colim_{V \subset W_i} \pi_k^H(\Omega^V \Omega^{W_i - V} X(W_i)) \\
&= colim_V colim_{V \subset W_i} \pi_k^H(\Omega^{W_i} X(W_i)) \\
&= colim_U \pi_k^H(\Omega^U X(U)) \\
&= \pi_k^H(X)
\end{aligned}$$

where W_i is an expanding countable sequence such that the union is the universe $\mathcal{U}$. Isomorophism at the 3^{rd} row is due to 2.1.8 and the isomorphism at the 4^{th} row is due to 2.1.9, while the equality at the end is due to cofinality. The case when $k < 0$ is similar. □

The stable homotopy category of G-spectra

For the sake of convenience, we will work only with G-spectra that are levelwise G-cofibrant, that is they are cofibrant in the level model structure, see 3.3.7. This is only a technical requirement and does not represent a problem since the homotopy category of the subcategory spanned by cofibrant objects is equivalent to the homotopy category of original model category.

We start with the definition of $\mathcal{SH}_G$:

Definition 3.3.4. The stable homotopy category $\mathcal{SH}_G$ has as objects G-spectra that are levelwise G-cofibrant. For two such G-spectra X and Y, the morphisms from X to Y are given by $[X, QY]_G$ the set of G-homotopy classes from X to the fibrant replacement QY of Y. If $f : X \to QY$ is a map of G-spectra, we denote by $[f] : X \to Y$ its homotopy class, considered as a morphism in $\mathcal{SH}_G$. We will also denote $\mathcal{SH}_G(X, Y)$ for the set of homotopy classes $[X, QY]_G$.

Remark 3.3.5. As we have seen in 3.3.3 for every G-spectrum X there is a stable equivalence $\mu_X : X \to QX$ to the fibrant replacement (an Ω-G-spectrum) QX. In the case when X is already an Ω-G-spectrum we will take $QX = X$ and $\mu_X = Id_X$.

In order to define the composition and to show the associativity of composition in $\mathcal{SH}_G$ we will use the following lemma:

Lemma 3.3.6. *Let $f : X \to Y$ be a map of G-spectra where X and Y are levelwise G-cofibrant. Then f is a stable equivalence if and only if*

$$f^* : [Y, E]_G \to [X, E]_G$$

is and isomorphism for every Ω-G-spectrum E.

Proof. This is basically a reformulation of [MMMF02] III *Theorem 6.1.*
By [MMMF02] III *section 2*, for any two G-spectra X and Y we have

$$Ho^l_G(X, Y) \cong [\Gamma X, Y]_G \tag{3.1}$$

where $Ho^l_G(X, Y)$ is the set of maps in the homotopy category with respect to the level model structure, and ΓX is the cofibrant replacement of X. In our case, X and Y are already cofibrant, hence the statement follows from [MMMF02] III *Theorem 6.1.* □

Remark 3.3.7. The requirement that we work only with levelwise G-cofibrant spectra lies in (3.1). Let us denote $(Sp_G^{\mathcal{U}})^{Co}$ the subcategory of $Sp_G^{\mathcal{U}}$ spanned by the G-spectra which are levelwise G-cofibrant. Since the cofibrant replacement in level model structure is the cofibrant replacement in the stable model structure this allows us to identiy the mapping space in the stable homotopy category between cofibrant G-spectra X and Y exactly with $[X, QY]_G$. Moreover, the category $(Sp_G^{\mathcal{U}})^{Co}$ admits stable model structure from the category of G-spectra, and since object of $(Sp_G^{\mathcal{U}})^{Co}$ are obtained as cofibrant replacements in the level model structure and therefore in the stable model structures, we have that the inclusion functor $(Sp_G^{\mathcal{U}})^{Co} \hookrightarrow Sp_G^{\mathcal{U}}$ induces an equivalence on homotopy categories.

First thing we want to adress is the composition in $\mathcal{SH}_G$. We define it the following way: Let $f : X \to QY$ and $g : Y \to QZ$ be morphism of G-spectra which represent morphism from X to Y and from Y to Z in $\mathcal{SH}_G$ respectively. Then, by 3.3.6 there is a morphism $\widetilde{g} : QY \to QZ$ of G-spectra, unique up to homotopy, such that $\widetilde{g} \circ \mu_Y \simeq g$. The composite of $[f] \in \mathcal{SH}_G(X, Y)$ and $[g] \in \mathcal{SH}_G(Y, Z)$ is then defined by

$$[g] \circ [f] = [\widetilde{g} \circ f] \in \mathcal{SH}_G(X, Z)$$

Secondly, we want to show that this composition is associative. For this, let us consider $X, Y, Z, T \in \mathcal{SH}_G$ with morphisms of G-spectra

$$\begin{aligned} f &: X \to QY \\ g &: Y \to QZ \\ h &: Z \to QT \end{aligned}$$

As above we can choose morphisms $\widetilde{g} : QY \to QZ$ and $\widetilde{h} : QZ \to QT$ such that $\widetilde{g} \circ \mu_Y \simeq g$ and $\widetilde{h} \circ \mu_Z \simeq h$. Then

$$([h] \circ [g]) \circ [f] = [\widetilde{h} \circ g] \circ [f] = [(\widetilde{h} \circ \widetilde{g}) \circ f] = [\widetilde{h} \circ (\widetilde{g} \circ f)] = [h] \circ [\widetilde{g} \circ f] = [h] \circ ([g] \circ [f])$$

Remark 3.3.8. Let us focus on $\mu_X : X \to QX$. From the definition of composition we see that μ_X is a two sided unit for composition, and therefore represents the identity of X in $\mathcal{SH}_G$.

Definition 3.3.9. We can define the functor

$$L : (Sp_G^{\mathcal{U}})^{Co} \to \mathcal{SH}_G$$

which is the identity on objects and which sends a map of G-spectra $f : X \to Y$ to $[\mu_Y \circ f] \in \mathcal{SH}_G(X, Y)$.

Now we prove that L indeed a functor and that it is in fact a localization functor which sends stable equivalences to isomorphisms.

Theorem 3.3.10. *The functor $L : (Sp_G^{\mathcal{U}})^{Co} \to \mathcal{SH}_G$ is a localization functor at the class of stable equivalences. In particular:*

1. *The functor L takes stable equivalences to isomorphisms. Even more, a map of G-spectra is a stable equivalence if and only if its image in $\mathcal{SH}_G$ is an isomorphism.*

2. *For every functor $F : (Sp_G^{\mathcal{U}})^{Co} \to C$ which takes stable equivalences to isomorphisms, there exists a unique functor $\widetilde{F} : \mathcal{SH}_G \to C$ such that $\widetilde{F} \circ L = F$.*

Proof. The proof is similar to the proof of [Sym] II *theorem 4.12*, with exeption to few technical differences, so we give the complete proof here.
To check functoriality, note that by 3.3.8 $[\mu_X]$ represents the identity of $X \in \mathcal{SH}_G$, hence L preserves identities. Now consider $f : X \to Y$ and $g : Y \to Z$ two morphisms in $(Sp_G^{\mathcal{U}})^{Co}$. Now we have

$$\begin{aligned} L(g) \circ L(f) =& [\mu_Z \circ g] \circ [\mu_Y \circ f] \\ =& [\widetilde{(\mu_Z \circ g)} \circ \mu_Y \circ f] \\ =& [\mu_Z \circ g \circ f] = L(g \circ f) \end{aligned}$$

where the equality at the last line is justified since $\widetilde{(\mu_Z \circ g)}$ is chosen is such way that $\widetilde{(\mu_Z \circ g)} \circ \mu_Y \simeq \mu_Z \circ g$.
Now we prove the main points of this theorem:

1. By 3.3.6 a map $f : X \to Y$ in $(Sp_G^{\mathcal{U}})^{Co}$ is a stable equivalence if $f^* : [Y, E]_G \to [X, E]_G$ is an isomorphism for every Ω-G-spectrum E. Note that we can restrict our attention to only those Ω-G-spectra that belong to $(Sp_G^{\mathcal{U}})^{Co}$ since the same proof as in [MMMF02] III *theorem 6.1* passes in this case. Now the same argument using the Yoneda lemma follows as in the proof of [MMMF02] III *theorem 6.1*:
As Z runs through the objects of $(Sp_G^{\mathcal{U}})^{Co}$, QZ runs through all Ω-G-spectra in $(Sp_G^{\mathcal{U}})^{Co}$ and we have a commutative square

$$\begin{tikzcd} \mathcal{SH}_G(Y, Z) \arrow[r, "\mathcal{SH}_G(L(f),Z)"] \arrow[d, "="] & \mathcal{SH}_G(X, Z) \arrow[d, "="] \\ {[Y, QZ]_G} \arrow[r, "f^*"] & {[X, QZ]_G} \end{tikzcd}$$

An immediate consequence is that $L(f)$ is a stable equivalence if and only if $\mathcal{SH}_G(L(f), Z)$ is an isomorphism for every G-spectrum $Z \in (Sp_G^{\mathcal{U}})^{Co}$. By the Yoneda lemma, this is the case if and only if $L(f)$ is an isomorphism.

2. Consider a functor $F : (Sp_G^{\mathcal{U}})^{Co} \to C$ which takes stable equivalences to isomorphisms. In order to show that there exists a unique functor $\widetilde{F} : \mathcal{SH}_G \to C$ such that $\widetilde{F} \circ L = F$ we have to prove two things:

   - A functor $G : \mathcal{SH}_G \to C$ is completely determined by $G \circ L : (Sp_G^{\mathcal{U}})^{Co} \to C$.
   - Existence of functor $\widetilde{F} : \mathcal{SH}_G \to C$ for $F : (Sp_G^{\mathcal{U}})^{Co} \to C$ as above.

   The first part takes care of the uniqueness requirement. Onto the proof:
   Fistly, consider a functor $G : \mathcal{SH}_G \to C$. It is ovbious to see that G is completely determined by $G \circ L$ on objects. Now, consider a morphism $[f] : X \to Y \in \mathcal{SH}_G(X, Y)$ which is represented by a map $f : X \to QY$. Note that, by 3.3.5, we have $L(\mu_Y) = [\mu_Y]$, hence for every morphism $f : X \to QY \in (Sp_G^{\mathcal{U}})^{Co}$ we have:

   $$L(\mu_Y) \circ [f] = L(f) \in \mathcal{SH}_G(X, QY)$$

   Hence, we have the following:

   $$[f] = L(\mu_Y)^{-1} \circ L(f) \in \mathcal{SH}_G$$

Now we obtain

$$G([f]) = G(L(\mu_Y)^{-1} \circ L(f)) = (G \circ L)(\mu_Y)^{-1} \circ (G \circ L)(f)$$

which finishes the first part.
For the second part, consider a functor $F : (Sp_G^{\mathcal{U}})^{Co} \to C$ which takes stable equivalences to isomorphisms. We set $\widetilde{F}(X) = F(X)$ on objects. Motivated by previous part, for a map $f : X \to QY \in (Sp_G^{\mathcal{U}})^{Co}$, we define

$$\widetilde{F}([f]) = F(\mu_Y)^{-1} \circ F(f)$$

We have to show that the functor $\widetilde{F}$ is well-defined, in particular, we need to show that $F(f)$ only depends on the homotopy class of $f : X \to QY$. For $X \in (Sp_G^{\mathcal{U}})^{Co}$ consider the G-spectrum $I \wedge X$ where I is the unit interval endowed with trivial G-action. The map $* \to I$ is a G-cofibration and therefore, by [MMMF02] III *lemma 1.22* the G-spectrum $I \wedge X$ is levelwise G-cofibrant.
The morphism $c : I \wedge X \to X$ that maps I to a single point is a stable equivalence (even more, a level equivalence). By hypothesis, $F(c) : F(I \wedge X) \to F(X)$ is an isomorphism. The composite with the two end point inclusions $i_0, i_1 : X \to I \wedge X$ satisfy $c \circ i_0 = c \circ i_1 = Id_X$, and we have

$$F(c) \circ F(i_0) = Id_X = F(c) \circ F(i_1)$$

Thus, we have $F(i_0) = F(i_1)$. Now, given two homotopic maps $f, g : X \to QY$ with homotopy given by $H : I \wedge X \to Y$ we have

$$F(f) = F(H \circ i_0) = F(H) \circ F(i_0) = F(H) \circ F(i_1) = F(H \circ i_1) = F(g)$$

Next, we show that $\widetilde{F}$ is unital which follows from

$$\widetilde{F}(Id_X) = \widetilde{F}([\mu_X]) = F(\mu_X)^{-1} \circ F(\mu_X) = Id_{\widetilde{F}(X)}$$

For composition we consider two maps $f : X \to QY$ and $g : Y \to QZ$ for $X, Y, Z \in (Sp_G^{\mathcal{U}})^{Co}$. Note that we can choose a map $\widetilde{g} : QY \to QZ$ such that $\widetilde{g} \circ \mu_Y \simeq g$. Then we have

$$\begin{aligned}
\widetilde{F}([g] \circ [f]) =& \widetilde{F}([\widetilde{g} \circ f]) \\
=& F(\mu_Z)^{-1} \circ F(\widetilde{g} \circ f) \\
=& F(\mu_Z)^{-1} \circ F(\widetilde{g}) \circ F(f) \\
=& F(\mu_Z)^{-1} \circ F(\widetilde{g}) \circ F(\mu_Y) \circ F(\mu_Y)^{-1} \circ F(f) \\
=& F(\mu_Z)^{-1} \circ F(\widetilde{g} \circ \mu_Y) \circ F(\mu_Y)^{-1} \circ F(f) \\
=& F(\mu_Z)^{-1} \circ F(g) \circ F(\mu_Y)^{-1} \circ F(f) \\
=& (F(\mu_Z)^{-1} \circ F(g)) \circ (F(\mu_Y)^{-1} \circ F(f)) \\
=& \widetilde{F}([g]) \circ \widetilde{F}([f])
\end{aligned}$$

Lastly, we have to check the relation $\widetilde{F} \circ L = F$. This is clear for objects. For a morphism $f : X \to Y$, where $X, Y \in (Sp_G^{\mathcal{U}})^{Co}$, we have

$$\begin{aligned}
\widetilde{F}(L(f)) =& \widetilde{F}([\mu_Y \circ f]) \\
=& F(\mu_Y)^{-1} \circ F(\mu_Y \circ f) \\
=& F(\mu_Y)^{-1} \circ F(\mu_Y) \circ F(f) \\
=& F(f)
\end{aligned}$$

With this the proof is finished.

$\square$

In the remainder of this section we will show that any G-spectrum X can be written as a filtered homotopy colimit indexed by free suspension G-spectra $\Sigma^\infty_V A$ where A is a finite G-CW-complex and V is an indexing space.
Since every G-spectra is stably equivalent (even more, we can choose it to be level equivalent as we have seen above) to its cofibrant replacement we can work with G-spectra in $(Sp^{\mathcal{U}}_G)^{Co}$. Without loss of generality we can furthemore assume that X has a property that $X(V)$ is a G-CW-complex for every indexing space V.

Remark 3.3.11. Now we can write X as a colimit of the sequence

$$X_0 \to X_1 \to \dots \to X_n \dots.$$

such that $X_n(V)$ is a finite G-CW-complex for every $n \geq 0$ and every indexing space V. Furthemore we can choose this sequence in such a way that $X_{n-1}(V)$ is a G-CW-subcomplex of $X_n(V)$ in which case $X_{n-1}(V) \hookrightarrow X_n(V)$ is a G-closed inclusion. Therefore, by 2.1.6 X is in fact the homotopy colimit of the above sequence.

In what follows, we will assume even more, that X is levelwise finite G-CW-complex.
For every pair of indexing spaces $V \subseteq W$ consider the map

$$\lambda^W_V : \Sigma^\infty_W(X(V) \wedge S^{W-V}) \to \Sigma^\infty_V X(V)$$

This map is a stable equivalence, and therefore admits an inverse in the stable homotopy category

$$\psi^W_V = L(\lambda^W_V)^{-1} : \Sigma^\infty_V X(V) \to \Sigma^\infty_W(X(V) \wedge S^{W-V}) \in Mor(\mathcal{SH}_G)$$

Furthemore, we can define a map

$$j^W_V : \Sigma^\infty_V X(V) \xrightarrow{\psi^W_V} \Sigma^\infty_W(X(V) \wedge S^{W-V}) \xrightarrow{L((\sigma^W_V)_*)} \Sigma^\infty_W X(W)$$

Using the adjunction $\Sigma^\infty_V : Space_G \rightleftarrows Sp^{\mathcal{U}}_G : \Omega^\infty_V$ we obtain a map:

$$i_V : \Sigma^\infty_V X(V) \to X$$

adjoint to the identity map on $X(V)$. Since the following diagram is commutative in the category of G-spectra

$$\begin{tikzcd}
\Sigma^\infty_V X(V) \arrow[rr, "i_V"] & & X \\
\Sigma^\infty_W(X(V) \wedge S^{W-V}) \arrow[u, "\lambda^W_V"] \arrow[d, "(\sigma^W_V)_*"] & & \\
\Sigma^\infty_W X(W) \arrow[uurr, "i_W"] & &
\end{tikzcd}$$

we obtain a commutative diagram in $\mathcal{SH}_G$

$$\begin{tikzcd} \Sigma^\infty_V X(V) \arrow[rr, "L(i_V)"] \arrow[d, "j_V^W"] & & X \\ \Sigma^\infty_W X(W) \arrow[urr, "L(i_W)"] & & \end{tikzcd}$$

Now we have a well-defined map in $\mathcal{SH}_G$

$$i : hocolim_V \Sigma^\infty_V X(V) \to X$$

We want to prove the following

Theorem 3.3.12. *Let X be a G-spectrum such that $X(V)$ has a structure of a finite G-CW-complex for every indexing space V. Then $X \in (Sp^{\mathcal{U}}_G)^{Co}$ and the map*

$$i : hocolim_V \Sigma^\infty_V X(V) \to X$$

is an isomorphism in the stable homotopy category $\mathcal{SH}_G$.

Before starting the proof we will show the following lemma which will be of use:

Lemma 3.3.13. *Given a sequence of G-spectra*

$$E_0 \to E_1 \to ... \to E_n \to ...$$

we have an isomorphism

$$colim_i \pi^H_k(E_i) \to \pi^H_k(hocolim_i E_i)$$

for every k and every closed subgroup $H \leq G$.

Proof. We will prove this in the case when $k \geq 0$. The case when $k < 0$ is similar.
By definition, we have

$$\pi^H_k(hocolim_i E_i) = colim_V \pi^H_k(\Omega^V(hocolim_i E_i)(V))$$

Homotopy colimits are computed by first passing to the cofibrant replacement of the original diagram in the projective model structure on diagram spaces, and then obtaining the colimit. Let $\{CE_i\}$ be the cofibrant replacement of the sequence $\{E_i\}$. In particular, the maps $CE_n \to E_n$ are stable equivalences and the maps $CE_{n-1} \to CE_n$ are G-cofibrations, hence closed G-inclusions. This with the fact that the colimits are computet levelwise gives us:

$$\begin{aligned}
\pi^H_k(hocolim_i E_i) &= colim_V \pi^H_k(\Omega^V(hocolim_i E_i)(V)) \\
&= colim_V \pi^H_k(\Omega^V(colim_i CE_i)(V)) \\
&= colim_V \pi^H_k(\Omega^V(colim_i CE_i(V))) \\
&\xrightarrow{2.1.7} colim_V \pi^H_k(colim_i \Omega^V CE_i(V)) \\
&\xrightarrow{2.1.5} colim_V colim_i \pi^H_k(\Omega^V CE_i(V)) \\
&\cong colim_i colim_V \pi^H_k(\Omega^V CE_i(V)) \\
&= colim_i \pi^H_k(CE_i) \\
&\cong colim_i \pi^H_k(E_i)
\end{aligned}$$

□

Lemma 3.3.14. *Let $\mathcal{I}$ be a filtered category that admits a countable exaustive sequence*

$$i_o \to i_1 \to ... \to i_n \to ...$$

In particular, for every $j \in \mathcal{I}$ there exists i_n in the sequence such that there is a map $j \to i_n$ in $\mathcal{I}$. Let $\{E_i\}_{i\in\mathcal{I}}$ be a diagram of G-spectra indexed by $\mathcal{I}$. Then the map

$$colim_{i\in\mathcal{I}}\pi_k^H(E_i) \to \pi_k^H(hocolim_{i\in\mathcal{I}}E_i)$$

is an isomorphism for every k and every closed subgroup $H \leq G$.

Proof. Mark the sequence with $\mathcal{J}$. The inclusion of the exaustive sequence into $\mathcal{J} \hookrightarrow \mathcal{I}$ is cofinal, and therefore we have

$$\begin{aligned} colim_{i\in\mathcal{J}}\pi_k^H(E_i) &\cong colim_{i\in\mathcal{I}}\pi_k^H(E_i) \\ \pi_k^H(hocolim_{i\in\mathcal{J}}E_i) &\cong \pi_k^H(hocolim_{i\in\mathcal{I}}E_i) \end{aligned}$$

These two parts can be connected by 3.3.13 with which the proof is finished. □

Now we are ready to prove the theorem 3.3.12:

Proof. It will be enough to show that the map i induces an isomorphism on all homotopy groups. In the ligh of this, consider a closed subgroup $H \leq G$ and $k \geq 0$. The proof when $k < 0$ is similar. For every indexing space V we can construct the map

$$\alpha_V : \pi_q^H(\Omega^V X(V)) \to \pi_q^H(\Omega^V(\Sigma_V^\infty X(V))(V)) \to \pi_q^H(\Sigma_V^\infty X(V))$$

where the first map is the identity and the second map is cannonical map to the colimit.
Next, we have that the following diagram commutes

$$\begin{tikzcd} \pi_q^H(\Omega^V X(V)) \arrow[r] \arrow[d, "\alpha_V"] & \pi_q^H(\Omega^W X(W)) \arrow[r] \arrow[d, "\alpha_W"] & \pi_q^H(X) \\ \pi_q^H(\Sigma_V^\infty X(V)) \arrow[r, "\pi_q^H(j_V^W)"] & \pi_q^H(\Sigma_W^\infty X(W)) \arrow[ur, "\pi_q^H(L(i_W))"'] & \end{tikzcd}$$

Note that, by definition $L(i_W) = [\mu_X \circ i_W]$ where $\mu_X \circ i_W : \Sigma_W^\infty X(W) \to QX$, so it is not entirely correct to write $\pi_q^H(L(i_W)) : \pi_q^H(\Sigma_W^\infty X(W)) \to \pi_q^H(X)$. Since μ_X is a stable equivalence we should have written instead $\pi_q^H(L(i_W))$ in composition with the inverse of the isomorphism induced by μ_X, but for the sake of readability we omit it in the notation.
This means that we have a well defined map

$$\alpha : \pi_q^H(X) = colim_V\pi_q^H(\Omega^V X(V)) \to colim_V\pi_q^H(\Sigma_V^\infty X(V))$$

Since the right triangle in the upper diagram commutes, the composition

$$\pi_q^K(X) \xrightarrow{\alpha} colim_V\pi_q^H(\Sigma_V^\infty X(V)) \xrightarrow{3.3.14} \pi_q^H(hocolim_V\Sigma_V^\infty X(V)) \xrightarrow{\pi_q^H(i)} \pi_q^K(X)$$

is an isomorphism, and since the middle map is an isomorphism it would suffice to show that α is also an isomorphism. Note that the middle map is an isomorphism by 3.3.14 and using the fact that there is an exaustive sequence of indexing spaces in $Rep^{\mathcal{U}}(G)$.
We do this by constructing a map

$$\xi_V : \pi_q^H(\Sigma_V^\infty X(V)) = colim_{V\le W}\pi_q^H(\Omega^W\Sigma^{W-V}X(V)) \xrightarrow{\coprod \pi_q^H((\Omega^W\sigma_V^W)_*)} \pi_q^H(\Omega^W X(W)) \to \pi_q^H(X)$$

Since the folowing diagram commutes

$$\begin{tikzcd}
\pi_q^H(\Sigma_V^\infty X(V)) \arrow[r, "\pi_q^H(j_V^W)"] \arrow[d, "\xi_V"] & \pi_q^H(\Sigma_W^\infty X(W)) \arrow[d, "\xi_W"] \\
\pi_q^H(X) \arrow[r, "Id"] & \pi_q^H(X)
\end{tikzcd}$$

we have a well defined map

$$\xi : colim_V\pi_q^H(\Sigma_V^\infty X(V)) \to \pi_q^H(X)$$

Now we have that $\alpha\circ\xi$ is identity on $colim_V\pi_q^H(\Sigma_V^\infty X(V))$ and $\xi\circ\alpha$ is identity on $\pi_q^H(X)$, meaning that α is an isomorphism.

Since α is an isomorphism, so is $\pi_q^H(i)$, hence the proof is finished. □

Corollary 3.3.15. *Every G-spectrum E can be written as a filtered homotopy colimit of free G-suspension spectra of the form $\Sigma_V^\infty A$ where V is an indexing space and A is a finite G-CW-complex.*

Proof. As we have seen before we can choose a cofibrant replacement of X of E in the stable model structure in such way that $X \in (Sp_G^{\mathcal{U}})^{Co}$. But X can be written as a homotopy colimit of a sequence $X_0 \to X_1 \to ... \to X_n \to ...$ as in 3.3.11. Furthemore, by 3.3.12 every X_n can be written as $\underset{V\in Rep^{\mathcal{U}}(G)}{hocolim}\ \Sigma_V^\infty X_n(V)$. This in total gives us a map

$$\underset{(n,V)\in\mathbb{N}\times Rep^{\mathcal{U}}(G)}{hocolim} \Sigma_V^\infty X_n(V) \to X$$

which is an isomorphism in $\mathcal{SH}_G$. Note that $\mathbb{N}\times Rep^{\mathcal{U}}(G)$ is again filtered which finishes the proof. □

Presentability

As in the case with G-spaces, we can also define the ∞-category of G-spectra as

$$(Sp_G^{\mathcal{U}})_\infty = N(Sp_G^{\mathcal{U}})[W_{st}^{-1}]$$

the localization of the nerve of the category of G-spectra with respect to the class of maps W_{st} which correspond to stable equivalences in $Sp_G^{\mathcal{U}}$.

Define with $(Sp_{G-CW}^{\mathcal{U}})_\infty$ the ∞-subcategory of $(Sp_G^{\mathcal{U}})_\infty$ spanned by elements of the form $\Sigma_V^\infty A$ where A is a finite G-CW-complex and V is an indexing space..
We close this section with the following

Proposition 3.3.16. *The ∞-category $(Sp_G^{\mathcal{U}})_\infty$ is a presentable ∞-category. In particular, it can be written as the* Ind *completion of $(Sp_{G-CW}^{\mathcal{U}})_\infty$.*

Proof. As in the case with G-spaces, the ∞-category $(Sp_G^{\mathcal{U}})_\infty$ admits all small colimits. Moreover, by 3.3.15 $(Sp_G^{\mathcal{U}})_\infty$ is generated under filtered colimits by objects of $(Sp_{G-CW}^{\mathcal{U}})_\infty$, which are, by 3.2.4 all compact objects (in all of the claims we use the fact that homotopy colimits in the model structure correspond to the colimits in the underlying ∞-categorical structure). Hence, we have the equivalence

$$(Sp_G^{\mathcal{U}})_\infty \simeq Ind((Sp_{G-CW}^{\mathcal{U}})_\infty)$$

□

Chapter 4

Stabilization of the ∞-category of G-spaces

Now that we have done most of the heavy lifting in the previous chapters, we are ready to prove the main result of the first part of this paper.
We will write $Rep^{\mathcal{U}}(G)_\infty$ for the categorical nerve $N(Rep^{\mathcal{U}}(G))$ of the category of indexing spaces of a universe $\mathcal{U}$.
The idea is to look at the following map:

$$\chi : Rep^{\mathcal{U}}(G)_\infty \to \mathcal{P}r^L$$

which sends every G-representation to the presentable ∞-category of based G-spaces, $Space^G_\infty$, and every inclusion map $V \hookrightarrow W$ to the operation of smashing with the representation sphere S^{W-V}. Our stable ∞-category would now be the colimit of χ: $\underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \chi$. The next important step would be to prove the following equivalence:

$$(Sp^{\mathcal{U}}_G)_\infty \simeq \underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \chi \tag{4.1}$$

Note that the upper equivalence is the equivalence of presentable ∞-categories. In addition, the colimit $\underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \chi$ is the colimit in the ∞-category of presentable ∞-categories $\mathcal{P}r^L$, which in general differs from the colimit in the ∞-category of ∞-categories (see 1.2.27). This problem would be hard to tackle straightforward, so it would be wise to restrict to the case of finite spaces, that is, we can look at the map

$$\widetilde{\chi} : Rep^{\mathcal{U}}(G)_\infty \to Cat_\infty$$

which sends every G-representation to the ∞-subcategory of $Space^{G-CW}_\infty$ spanned by based, finite G-CW-complexes, and every map $V \hookrightarrow W$ to the operation of smashing with the representation sphere S^{W-V}. In this case, our stable ∞-category would be the colimit of $\widetilde{\chi}$: $\underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \widetilde{\chi}$. Then we could prove the following equivalence:

$$(Sp^{\mathcal{U}}_{G-CW})_\infty \simeq \underset{Rep^{\mathcal{U}}(G)_\infty}{colim} \widetilde{\chi} \tag{4.2}$$

where $(Sp^{\mathcal{U}}_{G-CW})_\infty$ is the ∞-subcategory of the underlying ∞-category $(Sp^{\mathcal{U}}_G)_\infty$ of the model category $Sp^{\mathcal{U}}_G$ spanned by *free* G-spectra $\Sigma^\infty_V X$, where V is the indexing space of $\mathcal{U}$ and X is a finite, based G-CW-complex.

Note that (4.1) and (4.2) are similarly written, but the difference between them is fundamental:

The colimit in (4.2) is the colimit of small ∞-categories, which is in general easier to compute. Therefore, we obtain our main equivalence (4.1) from (4.2) by passing to the *Ind*-completion of ∞-categories.

In the final section, we will use the results of Marco Robalo ([Rob13]) on the inversion of an object in a symmetric monoidal ∞-category in order to show the universal property of $(Sp_G^{\mathcal{U}})_\infty$ with respect to the inversion of representation spheres.

4.1 Inversion of an object in a symmetric monoidal ∞-category

Let $X \in \mathcal{C}^\otimes$ be an object of a symmetric monoidal ∞-category $\mathcal{C}^\otimes$. The goal of this section is to give a description of a symmetric monoidal ∞-category $\mathcal{C}^\otimes[X^{-1}]$ together with a monoidal map $\mathcal{C}^\otimes \to \mathcal{C}^\otimes[X^{-1}]$ which sends the object X to the *invertible* object in $\mathcal{C}^\otimes[X^{-1}]$. In this section, we follow the notes of Marco Robalo's paper [Rob13].
By an invertible object we mean the following:

Definition 4.1.1. Let $X \in \mathcal{C}^\otimes$ be an object of a symmetric monoidal ∞-category $\mathcal{C}^\otimes$. We say that X is *invertibe* if there exists an object $Y \in \mathcal{C}^\otimes$ such that $X \otimes Y$ (and therefore $Y \otimes X$) is equivalent to the unit object of $\mathcal{C}^\otimes$. Equivalently, X is an invertible object if the map $X \otimes (-) : \mathcal{C}^\otimes \to \mathcal{C}^\otimes$ is an equivalence of ∞-categories.

The work of Marco Robalo in [Rob13] gives us the following result:

Proposition 4.1.2. *For every presentable symmetric monoidal ∞-category $\mathcal{C}^\otimes$ and for every object $X \in \mathcal{C}^\otimes$ there exists a presentable symmetric monoidal ∞-category $\mathcal{C}^\otimes[X^{-1}]$ together with a functor $F_X : \mathcal{C}^\otimes \to \mathcal{C}^\otimes[X^{-1}]$ which sends X to an invertible object. Moreover, the ∞-category $\mathcal{C}^\otimes[X^{-1}]$ admits the universal property in the sense that:*

1. *The restriction functor*

$$CAlg(\mathcal{P}r^L)_{\mathcal{C}^\otimes[X^{-1}]/} \to CAlg(\mathcal{P}r^L)_{\mathcal{C}^\otimes/}$$

is fully faithful with essential image consisting of those algebras i.e. symmetric monoidal functors $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ which send X into an invertible object in $\mathcal{D}^\otimes$, where $CAlg(\mathcal{P}r^L)$ is the ∞-category of commutative algebra objects in $\mathcal{P}r^L$ i.e. presentable symmetric monoidal ∞-categories.

2. *The forgetful functor*

$$Mod_{\mathcal{C}^\otimes[X^{-1}]}(\mathcal{P}r^L) \to Mod_{\mathcal{C}^\otimes}(\mathcal{P}r^L)$$

is fully faithful with essential image being a full subcategory spanned by those presentable ∞-categories equipped with an action of $\mathcal{C}^\otimes$ such that X acts as an equivalence, where $Mod_{\mathcal{C}^\otimes[X^{-1}]}(\mathcal{P}r^L)$ is the ∞-category of $\mathcal{C}^\otimes[X^{-1}]$-modules in $\mathcal{P}r^L$ (and similarly for $Mod_{\mathcal{C}^\otimes}(\mathcal{P}r^L)$).

Observation 4.1.3. Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category and let $X, Y \in \mathcal{C}^\otimes$. By the universal property, we have that $\mathcal{C}^\otimes[X^{-1}][Y^{-1}] \simeq \mathcal{C}^\otimes[Y^{-1}][X^{-1}]$, which we can denote with $\mathcal{C}^\otimes[X^{-1}, Y^{-1}]$. Note that, since $\mathcal{C}^\otimes$ is symmetric monoidal, the element $X \otimes Y$ is invertible if and only if the elements X and Y are invertible. Therefore, we write $\mathcal{C}^\otimes[X^{-1}, Y^{-1}] := \mathcal{C}^\otimes[(X \otimes Y)^{-1}]$.

Inversion and stabilization

Recall that, given a (ordinary) category C and an endofunctor $F : C \to C$, the stabilization of the category C with respect to F is the colimit

$$C \xrightarrow{F} C \xrightarrow{F} ...C \xrightarrow{F} ...$$

and we denote it as $Stab_F(C)$.

Similarly, given a symmetric monoidal category $\mathcal{C}^{\otimes}$ and an element $X \in \mathcal{C}^{\otimes}$, we say that the stabilization of the category $\mathcal{C}^{\otimes}$ with respect to the element X is the colimit of the sequence

$$\mathcal{C}^{\otimes} \xrightarrow{X\otimes\_} \mathcal{C}^{\otimes} \xrightarrow{X\otimes\_} ...\mathcal{C}^{\otimes} \xrightarrow{X\otimes\_} ...$$

which we denote as $Stab_X(\mathcal{C}^{\otimes})$.

It is natural to assume that $Stab_X(\mathcal{C}^{\otimes})$ is our category $\mathcal{C}^{\otimes}[X^{-1}]$ and that is correct under the condition that X is a *symmetric* object of $\mathcal{C}^{\otimes}$:

Definition 4.1.4. ([Rob13] *4.18*) The object $X \in \mathcal{C}^{\otimes}$ of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is said to be symmetric if there is a 2-equivalence in $\mathcal{C}^{\otimes}$ between the cyclic permutation $\sigma : (X \otimes X \otimes X)^{(1,2,3)}$ and the identity of $X \otimes X \otimes X$, that is, there exists a 2-cell in $\mathcal{C}^{\otimes}$

$$\begin{tikzcd}
X \otimes X \otimes X \arrow[r, "\sigma"] \arrow[d, "id"] & X \otimes X \otimes X \\
X \otimes X \otimes X \arrow[ur, "id"] &
\end{tikzcd}$$

The author of [Rob13] gives us the following results:

Proposition 4.1.5. *([Rob13]* 4.24*) Let $\mathcal{C}^{\otimes}$ be a presentable symmetric monoidal ∞-category, and let X be a symmetric object of $\mathcal{C}^{\otimes}$. Then the ∞-category of formal inversion $\mathcal{C}^{\otimes}[X^{-1}]$ is equivalent to the stabilization $Stab_X(\mathcal{C}^{\otimes})$.*

Corollary 4.1.6. *([Rob13]* 4.25*) Let $\mathcal{C}^{\otimes}$ be a stable presentable symmetric monoidal ∞-category and let X be a symmetric object in $\mathcal{C}^{\otimes}$. Then $\mathcal{C}^{\otimes}[X^{-1}]$ is again a stable presentable symmetric monoidal ∞-category.*

4.2 The ∞-category of G-spectra as the stabilization

Note that by [MMMF02], IV, 4.2, we know that the stable model category on $Sp_G^{\mathcal{U}}$ is topological. Then the topological space of maps between G-spectra X and Y is weakly equivalent to the derived mapping space if X is cofibrant and Y is fibrant (see [DK1], [DK2] and [DK3]). To add up, the set of generating cofibrations is the set $\Sigma_V^{\infty}\mathcal{I}_G$, where $\Sigma_V^{\infty}(\_)$ is the free G-spectrum functor and $\mathcal{I}_G$ is the set of maps

$$\mathcal{I}_G = \{(G/H \times S^{k-1})_+ \to (G/H \times D^k)_+\}_{k\geq 0, H\leq G}$$

In particular, the elements of $Sp_{G-CW}^{\mathcal{U}}$ are of the form $\Sigma_V^{\infty}X$ where X is a finite G-CW-complex hence they are all cofibrant. Note that they are all also levelwise G-CGWH spaces.

Proposition 4.2.1. *Define the functor $f : colim_{Rep^{\mathcal{U}}(G)_{\infty}}\widetilde{\chi} \to (Sp_{G-CW}^{\mathcal{U}})_{\infty}$ sending the object of $colim_{Rep^{\mathcal{U}}(G)_{\infty}}\widetilde{\chi}$ of the form (X, V) where X is a finite G-CW-complex and V an indexing space, to the G-spectrum $\Sigma_V^{\infty}X$. The functor f is an equivalence of ∞-categories.*

Proof. First, note that both ∞-categories have the same elements which can be written in the form of a pair (X, V), where X is a finite G-CW-complex and V is an indexing space.

In $(Sp^{\mathcal{U}}_{G-CW})_\infty$ the pair (X, V) coresponds to the free G-spectrum $\Sigma^\infty_V X$, that is, a G-spectrum such that $\Sigma^\infty_V X(W) = \Sigma^{W-V}X$ when $V \subset W$ and is a point otherwise.

In $colim_{Rep^{\mathcal{U}}(G)_\infty}\chi$ the pair (X, V) coresponds to the element X in the colimit which is present from the "V^{th} level".

Since both categories have the same elements, what is left to show is that the mapping spaces in one ∞-category are weakly equivalent to the mapping spaces in the other ∞-category. The ∞-category $(Sp^{\mathcal{U}}_{G-CW})_\infty$ is obtained from the model category $Sp^{\mathcal{U}}_{G-CW}$ meaning that the mapping space $Map_{(Sp^{\mathcal{U}}_{G-CW})_\infty}(\Sigma^\infty_V X, \Sigma^\infty_W Y)$ is computed by passing to the fibrant replacement of $\Sigma^\infty_W Y$ (see [DK3]):

$$Map_{(Sp^{\mathcal{U}}_{G-CW})_\infty}(\Sigma^\infty_V X, \Sigma^\infty_W Y) = Map_{Sp^{\mathcal{U}}_G}(\Sigma^\infty_V X, Q(\Sigma^\infty_W Y))$$

Using the adjunction $(\Sigma^\infty_V, \Omega^\infty_V)$ we have

$$\begin{aligned} Map_{Sp^{\mathcal{U}}_G}(\Sigma^\infty_V X, Q(\Sigma^\infty_W Y)) &\cong Map_{Space_{G_*}}(X, Q(\Sigma^\infty_W Y)(V)) \\ &= Map_{Space_{G_*}}(X, hocolim_{V\subset U}\Omega^{U-V}(\Sigma^\infty_W Y)(U)) \\ &= Map_{Space_{G_*}}(X, hocolim_{V,W\subset U}\Omega^{U-V}\Sigma^{U-W}Y) \end{aligned}$$

We can again take an expanding countable sequence U_i of indexing spaces such that their union is whole $\mathcal{U}$ and such that $V, W \subset U_i$ for every i. Now, by 3.2.3 we have a weak equivalence

$$Map_{Space_{G_*}}(X, hocolim_{V,W\subset U_i}\Omega^{U_i-V}\Sigma^{U_i-W}Y) \to Map_{Space_{G_*}}(X, hocolim_{V,W\subset U}\Omega^{U-V}\Sigma^{U-W}Y)$$

Furthemore, by 2.1.8 we have a weak equivalence

$$hocolim_{V,W\subset U_i}Map_{Space_{G_*}}(X, \Omega^{U_i-V}\Sigma^{U_i-W}Y) \to Map_{Space_{G_*}}(X, hocolim_{V,W\subset U_i}\Omega^{U_i-V}\Sigma^{U_i-W}Y)$$

Again, by the adjunction (Σ^V, Ω^V) and 3.2.3 we obtain

$$hocolim_{V,W\subset U_i}Map_{Space_{G_*}}(X, \Omega^{U_i-V}\Sigma^{U_i-W}Y) \simeq hocolim_{V,W\subset U_i}Map_{Space_{G_*}}(\Sigma^{U_i-V}X, \Sigma^{U_i-W}Y)$$

and a weak equivalence

$$hocolim_{V,W\subset U_i}Map_{Space_{G_*}}(\Sigma^{U_i-V}X, \Sigma^{U_i-W}Y) \to hocolim_{V,W\subset U}Map_{Space_{G_*}}(\Sigma^{U-V}X, \Sigma^{U-W}Y)$$

Finally, note that the expression $hocolim_{V,W\subset U}Map_{Space_{G_*}}(\Sigma^{U-V}X, \Sigma^{U-W}Y)$ represents the mapping space $Map_{colim_{Rep^{\mathcal{U}}(G)_\infty}\chi}((X, V), (Y, W))$ of elements (X, V) and (Y, W) in the ∞-category $colim_{Rep^{\mathcal{U}}(G)_\infty}\chi$. To summarize, we have shown that the mapping spaces $Map_{(Sp^{\mathcal{U}}_G)_\infty}(\Sigma^\infty_V X, \Sigma^\infty_W Y)$ and $Map_{colim_{Rep^{\mathcal{U}}(G)_\infty}\chi}((X, V), (Y, W))$ are weakly equivalent. With this the proof is finished. $\square$

In order to prove the equivalence for the presentable ∞-categories $colim_{Rep^{\mathcal{U}}(G)_\infty}\chi$ and $(Sp^{\mathcal{U}}_G)_\infty$, we will pass to the *Ind*-completion of the ∞-categories $colim_{Rep^{\mathcal{U}}(G)_\infty}\widetilde{\chi}$ and $(Sp^{\mathcal{U}}_{G-CW})_\infty$. There is one subtile point that we need to adress with regard to the *Ind*-completion. Namely, The functor *Ind* preserves colimits when considered as a functor from the ∞-category $Cat^{fin-colim}_\infty$ of ∞-categories with finite colimits (and finite colimit preserving functors between them) to the ∞-category $\mathcal{P}r^L$. Therefore, we do not have the commutativity of *Ind*-functor with colimits in Cat_∞. When looking at the $Ind(colim_{Rep^{\mathcal{U}}(G)_\infty}\widetilde{\chi})$ the colimit now is the colimit in the ∞-category $Cat^{fin-colim}_\infty$ and not Cat_∞. The equivalent statement of [HTT] *6.3.4.4* tells us that $Cat^{fin-colim}_\infty$ admits small colimits. The next thing that we need to prove is:

Lemma 4.2.2. *The inclusion functor*

$$Cat^{fin-colim}_\infty \to Cat_\infty$$

preserves and detects filtered colimits.

Proof. The proof is basically the reformulation and adaptation of the proof of [HTT] *5.5.7.11*. Let $\mathcal{I}$ be a (small) filtered category and let $d : \mathcal{I} \to Cat_\infty^{fin-colim}$ be a diagram in $Cat_\infty^{fin-colim}$. Since Cat_∞ is cocomplete let $\mathcal{C}$ be the colimit of the diagram $d' : \mathcal{I} \to Cat_\infty^{fin-colim} \hookrightarrow Cat_\infty$. We need to prove the following:

1. The ∞-category $\mathcal{C}$ admits finite colimits.
2. The induced functors $d(I) \to \mathcal{C}$ preserve finite colimits for every $I \in \mathcal{I}$.
3. For every functor $F : \mathcal{C} \to \mathcal{D}$ if the composite functors $d(I) \to \mathcal{C} \xrightarrow{F} \mathcal{D}$ preserve finite colimits for every $I \in \mathcal{I}$, then the functor F does as well.

Note that, since $\mathcal{I}$ is filtered every finite diagram in $\mathcal{C}$ will factor through some $d(I)$, $I \in \mathcal{I}$, therefore, it will suffice to prove point (2) from above, since in that case the points (1) and (3) follow imediately.

Arguing as in [HTT] *5.5.7.11* we can assume without the loss of generality that $\mathcal{I}$ is the nerve of a filtered partially ordered set K ([HTT] *4.2.4.4*). We can furthemore reduce to the case when d is the nerve of a functor $f : K \to Set_\Delta$, with $\mathcal{C}$ being the homotopy colimit of this diagram ([HTT] *5.3.1.18* and *4.2.4.1*). Finally, since the collection of categorical equivalences is stable under filtered colimits, we can regard $\mathcal{C}$ as the colimit of f.

Let S be a finite simplicial set and let $C_x = f(x)$ for $x \in K$. Let $\bar{g}_x : S^{\triangleright} \to C_x$ be a colimit diagram in C_x for some $x \in K$ and denote $g_x = \bar{g}_x|_S$. We would like to show that the induced map $\bar{g} : S^{\triangleright} \to \mathcal{C}$ is also a colimit diagram. It would suffice to show that the map $\theta : \mathcal{C}_{\bar{g}/} \to \mathcal{C}_{g/}$ is a trivial fibration. Note that θ is a filtered colimit of the maps $\theta_y : (C_y)_{\bar{g}_y/} \to (C_y)_{g_y/}$ with $y \geq x$, $y \in K$. Since all of the functors $C_x \to C_y$ preserve finite colimits, all of the maps θ_y are trivial fibrations, hence so is θ. $\square$

Now we are ready to prove the main equivalence.

Proposition 4.2.3. *The functor $F : colim_{Rep^{\mathcal{U}}(G)_\infty} \chi \to (Sp_G^{\mathcal{U}})_\infty$ induced by the functor f from 4.2.1 is an equivalence.*

Proof. By slight abuse of notation let us write $colim_{Rep^{\mathcal{U}}(G)_\infty} \widetilde{\chi}$ as $colim_{Rep^{\mathcal{U}}(G)_\infty} Space_\infty^{G-CW}$, and $colim_{Rep^{\mathcal{U}}(G)_\infty} \chi$ as $colim_{Rep^{\mathcal{U}}(G)_\infty} Space_\infty^{G}$. Since $Space_\infty^{G} \simeq Ind(Space_\infty^{G-CW})$ and $(Sp_G^{\mathcal{U}})_\infty \simeq Ind((Sp_{G-CW}^{\mathcal{U}})_\infty)$, and since Ind-completion commutes with our filtered colimit $colim_{Rep^{\mathcal{U}}(G)_\infty}$ we have

$$\begin{aligned}(Sp_G^{\mathcal{U}})_\infty \simeq& Ind((Sp_{G-CW}^{\mathcal{U}})_\infty)\\ \simeq& Ind(colim_{Rep^{\mathcal{U}}(G)_\infty} Space_\infty^{G-CW})\\ \simeq& colim_{Rep^{\mathcal{U}}(G)_\infty} Ind(Space_\infty^{G-CW})\\ \simeq& colim_{Rep^{\mathcal{U}}(G)_\infty} Space_\infty^{G}\end{aligned}$$

The second equivalence is given by 4.2.1, while the third is given by 4.2.2. In total, we obtain our main equivalence

$$(Sp_G^{\mathcal{U}})_\infty \simeq colim_{Rep^{\mathcal{U}}(G)_\infty} \chi$$

$\square$

4.3 The universal property

Following the work of Robalo ([Rob13]) we would like to show the analogue of the proposition *4.10*:

Proposition 4.3.1. *The restriction functor*

$$CAlg(\mathcal{P}r^L)_{(Sp^{\mathcal{U}}_G)_\infty/} \to CAlg(\mathcal{P}r^L)_{Space^G_\infty/}$$

is fully faithful, where $CAlg(\mathcal{P}r^L)$ is the ∞-category of commutative algebras in $\mathcal{P}r^L$ i.e. presentable symmetric monoidal ∞-categories, such that the essential image consists of those symmetric monoidal functors $Space^G_\infty \to \mathcal{C}^\otimes$ which send every representation sphere into an invertible object in $\mathcal{C}^\otimes$.

Let $V \subseteq W$ be two indexing spaces. Then we can write $S^W \cong S^V \wedge S^{W-V}$, hence by 4.1.2 and 4.1.3 we have a map $Space^G_\infty[(S^V)^{-1}] \to Space^G_\infty[(S^W)^{-1}]$. Moreover, we can organize the presentable ∞-categories $Space^G_\infty[(S^V)^{-1}]$ into a diagram

$$\lambda : Rep^{\mathcal{U}}(G)_\infty \to CAlg(\mathcal{P}r^L)$$

sending every indexing space V to $Space^G_\infty[(S^V)^{-1}]$. Next, note that, by 3.1.10 $S^V \wedge (-) : (Sp^{\mathcal{U}}_G)_\infty \to (Sp^{\mathcal{U}}_G)_\infty$ is an equivalence for every representation sphere, hence, for every representation V we have a map

$$\tau_V : Space^G_\infty[(S^V)^{-1}] \to (Sp^{\mathcal{U}}_G)_\infty$$

Therefore $(Sp^{\mathcal{U}}_G)_\infty$ fits into a cone of the diagram λ. We would like to show that this is in fact a colimit cone. Note that $colim_{Rep^{\mathcal{U}}(G)_\infty}\lambda$ is a colimit in $CAlg(\mathcal{P}r^L)$, but since $Rep^{\mathcal{U}}(G)_\infty$ is a filtered category the computation of this colimit is reduced to the computation in $\mathcal{P}r^L$. Additionally, we have:

Lemma 4.3.2. *The representation sphere S^V is a symmetric object of the symmetric monoidal ∞-category $Space^G_\infty$ for every indeging space $V \subset \mathcal{U}$.*

Proof. First, note that $S^V \wedge S^V \wedge S^V = S^{V\oplus V\oplus V}$, meaning that $\sigma : (S^V \wedge S^V \wedge S^V)^{(1,2,3)} = S^{(V\oplus V\oplus V)^{(1,2,3)}}$.
Basically, what we need to do is to show that the map σ, induced by the cyclic permutation, is homotopic to the identity map on $S^{V\oplus V\oplus V}$ in the underlying category of G-spaces.

For start, let $dimV = n$ and let us fix a basis $e = \{e_i\}_{o\leq i\leq n}$. Now $dim(V \oplus V \oplus V) = 3n$ and basis consists, informally speaking, of three copies of e. It will be easier if we write the maps as matrices. Let $\widetilde{\sigma} : (V \oplus V \oplus V)^{(1,2,3)}$ be the map induced by cyclic permutation on $(V \oplus V \oplus V)$. Since the representation spheres are one point compactifications of indexing spaces we see that $\widetilde{\sigma}$ extends to σ. Now we can write

$$\widetilde{\sigma} = \begin{bmatrix} 0 & 0 & E \\ E & 0 & 0 \\ 0 & E & 0 \end{bmatrix}$$

where E is the identity matrix in $GL_n(V)$ with respect to basis e. We can see that the action of G can be seen as an homomorphism $\rho : G \to GL_n(V)$, so if for $g \in G$, we have $A = \rho(g)$, the action of g on $V \oplus V \oplus V$ can be seen as

$$P_g = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}$$

It is not hard to check that $\widetilde{\sigma}P_g = P_g\widetilde{\sigma}$, that is, that $\widetilde{\sigma}$ is a well defined G-map. Now we define the homotopy between $\widetilde{\sigma}$ and the identity map Id of $V \oplus V \oplus V$ the following way:

$$\begin{aligned} H :& I \times V \oplus V \oplus V \to V \oplus V \oplus V \\ H(t, \_) &= t\sigma + (1-t)Id \end{aligned}$$

Since $H(t, \_)$ is a linear combination of two G-maps, it will also be a G-map, hence we have a well defined G-homotopy between maps $\widetilde{\sigma}$ and Id. As noted before, representation spheres are just one point compactifications, meaning that H extends to the homotopy of maps σ and the identity map of $S^{V\oplus V\oplus V}$. This finishes the proof. $\square$

As a direct corollary we have an equivalence

$$Space^G_\infty[(S^V)^{-1}] \simeq Stab_{S^V}(Space^G_\infty)$$

hence, the colimit of the diagram λ is equivalent to the colimit of the diagram $\widetilde{\lambda} : Rep^{\mathcal{U}}(G)_\infty \to \mathcal{P}r^L$ sending every G-representation V to $Stab_{S^V}(Space^G_\infty)$. Moreover, for G-representations $V \subseteq W$, the map $Stab_{S^V}(Space^G_\infty) \to Stab_{S^W}(Space^G_\infty)$ is induced by

$$\begin{tikzcd}
Space^G_\infty \arrow[r, "S^V\wedge(-)"] \arrow[d, "id"] & Space^G_\infty \arrow[r, "S^V\wedge(-)"] \arrow[d, "S^{W-V}\wedge(-)"] & Space^G_\infty \arrow[r, "S^V\wedge(-)"] \arrow[d, "S^{(W-V)\oplus(W-V)}\wedge(-)"] & ... \\
Space^G_\infty \arrow[r, "S^W\wedge(-)"] & Space^G_\infty \arrow[r, "S^W\wedge(-)"] & Space^G_\infty \arrow[r, "S^W\wedge(-)"] & ...
\end{tikzcd}$$

therefore, we have the following

Lemma 4.3.3. *The cone constructed above exibits $(Sp^{\mathcal{U}}_G)_\infty$ as a colimit of λ.*

Proof. By 4.2.3 we can write $(Sp^{\mathcal{U}}_G)_\infty$ as $colim_{Rep^{\mathcal{U}}(G)_\infty}\chi$. We can write a new universe $\widetilde{\mathcal{U}} := \mathcal{U} \oplus \mathcal{U} \oplus ...$ as an infinite sum of copies of $\mathcal{U}$. We can again construct a category $Rep^{\widetilde{\mathcal{U}}}(G)_\infty$ analogous to $Rep^{\mathcal{U}}(G)_\infty$ as well as a diagram $\chi' : Rep^{\widetilde{\mathcal{U}}}(G)_\infty \to \mathcal{P}r^L$ analogous to χ. Since $\mathcal{U}$ is a complete universe so is $\widetilde{\mathcal{U}}$ and hence the colimits $colim_{Rep^{\mathcal{U}}(G)_\infty}\chi$ and $colim_{Rep^{\widetilde{\mathcal{U}}}(G)_\infty}\chi'$ are both equivalent to $(Sp^{\mathcal{U}}_G)_\infty$. To add up, we can construct a map

$$t : Rep^{\mathcal{U}}(G)_\infty \times \mathbb{N} \to Rep^{\widetilde{\mathcal{U}}}(G)_\infty$$

which sends an object (V, n) (where V is an indexing space) to $V \oplus V \oplus ... \oplus V$ (n times). More formally, the element $V \oplus V \oplus ... \oplus V$ is viewed as taking value V in the first n-coordinates of $\mathcal{U} \oplus \mathcal{U} \oplus ... = \widetilde{\mathcal{U}}$ and nothing in the rest.
We claim that this map is cofinal:

First, note that both categories $Rep^{\mathcal{U}}(G)_\infty \times \mathbb{N}$ and $Rep^{\widetilde{\mathcal{U}}}(G)_\infty$ are filtered. Let W be an indexing space of $\widetilde{\mathcal{U}}$. Since indexing spaces are of finite dimension we can choose big enough n such that W is contained in the first n copies of $\mathcal{U}$ in $\widetilde{\mathcal{U}}$. Let W_i be a projection of W onto ith copy of $\mathcal{U}$, for $1 \leq i \leq n$. We can choose an indexing space V of $\mathcal{U}$ which contains all of the projections W_i and hence, we have a map in $Rep^{\widetilde{\mathcal{U}}}(G)_\infty$, $W \hookrightarrow t(V, n)$. Therefore, t is cofinal.

Next, we can focus our attention to the following sequence of indexing spaces in $Rep^{\widetilde{\mathcal{U}}}(G)_\infty$ given by

$$V \hookrightarrow V \oplus V \hookrightarrow V \oplus V \oplus V \hookrightarrow ...$$

where V is an indexing space of $\mathcal{U}$ regarded as an indexing space of $\widetilde{\mathcal{U}}$ that corresponds to $t(V, 1)$.
Under χ' this sequence becomes

$$(Space^G_\infty, V) \xrightarrow{S^V\wedge(-)} (Space^G_\infty, V \oplus V) \xrightarrow{S^V\wedge(-)} (Space^G_\infty, V \oplus V \oplus V) \to ...$$

where we have written $(Space^G_\infty, V \oplus ... \oplus V)$ as $\chi'(V \oplus ... \oplus V) = Space^G_\infty$ i.e. we have marked the copy of $Space^G_\infty$ to which $V \oplus ... \oplus V$ is sent. The colimit of this sequence corresponds to $Stab_{S^V}(Space^G_\infty)$. To add up, for a map of indexing spaces $V \subseteq W$, where V, W are indexing spaces of $\mathcal{U}$ regarded as indexing spaces of $\widetilde{\mathcal{U}}$, we have

$$\begin{tikzcd}
(Space^G_\infty, V) \arrow[r, "S^V \wedge(-)"] \arrow[d, "S^{W-V}\wedge(-)"] & (Space^G_\infty, V \oplus V) \arrow[r, "S^V \wedge(-)"] \arrow[d, "S^{(W-V)\oplus(W-V)}\wedge(-)"] & (Space^G_\infty, V \oplus V \oplus V) \arrow[r, "S^V \wedge(-)"] \arrow[d, "S^{(W-V)\oplus(W-V)\oplus(W-V)}\wedge(-)"] & ... \\
(Space^G_\infty, W) \arrow[r, "S^W \wedge(-)"] & (Space^G_\infty, W \oplus W) \arrow[r, "S^W \wedge(-)"] & (Space^G_\infty, W \oplus W \oplus W) \arrow[r, "S^W \wedge(-)"] & ...
\end{tikzcd}$$

which corresponds to the map $Stab_{S^V}(Space^G_\infty) \to Stab_{S^W}(Space^G_\infty)$. Note that the diagram $\chi'|_t : Rep^{\mathcal{U}}(G)_\infty \times \mathbb{N} \to \mathcal{P}r^L$ corresponds exactly to the diagram $\widetilde{\lambda}$ hence

$$\underset{Rep^{\mathcal{U}}(G)_\infty \times \mathbb{N}}{colim} \chi'|_t \simeq colim_{Rep^{\mathcal{U}}(G)_\infty} \widetilde{\lambda}$$

Moreover, since t is cofinal we have an equivalence

$$\underset{Rep^{\mathcal{U}}(G)_\infty \times \mathbb{N}}{colim} \chi'|_t \xrightarrow{\simeq} colim_{Rep^{\tilde{\mathcal{U}}}(G)_\infty} \chi'$$

Finally, by our previous remark on the completeness of universes $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ we have an equivalence $colim_{Rep^{\tilde{\mathcal{U}}}(G)_\infty} \chi' \xrightarrow{\simeq} (Sp^{\mathcal{U}}_G)_\infty$, and therefore we obtain a sequence

$$colim_{Rep^{\mathcal{U}}(G)_\infty} \widetilde{\lambda} \simeq \underset{Rep^{\mathcal{U}}(G)_\infty \times \mathbb{N}}{colim} \chi'|_t \xrightarrow{\simeq} colim_{Rep^{\tilde{\mathcal{U}}}(G)_\infty} \chi' \xrightarrow{\simeq} (Sp^{\mathcal{U}}_G)_\infty$$

With this the proof is finished.

□

Finally, we are able to give the proof of 4.3.1:

Proof. Since every map $Space^G_\infty \to Space^G_\infty[(S^V)^{-1}]$ is an epimorphism (see [Rob13] proof of *4.10*), so is the induced map $Space^G_\infty \to (Sp^{\mathcal{U}}_G)_\infty$. Therefore, the map $CAlg(\mathcal{P}r^L)_{(Sp^{\mathcal{U}}_G)_\infty/} \to CAlg(\mathcal{P}r^L)_{Space^G_\infty/}$ is fully faithful by [Rob13] *4.3* (and the proof of *4.10*). The only thing left to check is the essential image:

It is clear that if $\varphi : Space^G_\infty \to \mathcal{C}^\otimes$ is in the image of $CAlg(\mathcal{P}r^L)_{(Sp^{\mathcal{U}}_G)_\infty/} \to CAlg(\mathcal{P}r^L)_{Space^G_\infty/}$. Then φ factors through $(Sp^{\mathcal{U}}_G)_\infty$ and hence, the representation spheres are sent to invertible objects in $\mathcal{C}^\otimes$. On the other hand, if $\varphi : Space^G_\infty \to \mathcal{C}^\otimes$ is such that representation spheres are sent to invertible objects in $\mathcal{C}^\otimes$, then by [Rob13] *4.10*, for every G-representation V we have a factorization

$$\begin{tikzcd}
Space^G_\infty \arrow[r, "\varphi"] \arrow[d] & \mathcal{C}^\otimes \\
Space^G_\infty[(S^V)^{-1}] \arrow[ur, "\varphi_V"] &
\end{tikzcd}$$

Moreover, this factorization is essentially unique. Therefore, we can make a compatible choice of maps φ_V for every G-representation V and we obtain a map $\phi : colim_{Rep^{\mathcal{U}}(G)_\infty} \lambda \to \mathcal{C}^\otimes$. The statement now follows from 4.3.3. □

Part II

G-equivariant factorization homology

Chapter 5

Preliminaries II

This chapter will introduce reader to the theory of parametrized higher category theory and higher algebra. The founders of this theory are Clark Barwick and his students: Denis Nardin, Jae Shah, Emmanuel Dotto, Saul Glassman (see [BDGNS16], [Nar16], [Nar17], [Shah18]).

In general, parametrized higher category theory and higher algebra represents a framework for doing equivariant homotopy theory as we will explain below.

In order to motivate the reader and give a little bit of insight of the forthcoming theory, assume that X is a G-space where G is a finite group. Recall that we can describe X with a functor

$$F_X : \mathcal{O}_G^{op} \to Space$$
$$F_X(G/H) \mapsto X^H$$

In the setting of ∞-categories, we will replace $\mathcal{O}_G$ with the corresponding ∞-category, which we will again mark with $\mathcal{O}_G$ and we will replace *Space* with Cat_∞. The Grothendieck-Lurie correspondence (see [HTT]) gives us a functor (a coCartesian fibration)

$$p : \mathcal{X} \to \mathcal{O}_G^{op}$$

which we think of as a G-∞-category. The reason why we regard coCartesian fibrations $p : \mathcal{X} \to \mathcal{O}_G^{op}$ as equivariant objects rather than functors from $\mathcal{O}_G^{op}$ lies in the fact that coCartesian fibrations regarded as objects are much easier to manipulate than general functors.

We will start with the formal definition of coCartesian fibration, followed by technical results relating to them which will allow us to construct some important parametrized ∞-categories, such as G-∞-category of finite G-spaces (or finite G-sets). In the last two part of this introductory part we will introduce the parametrized version of symmetric monoidal ∞-categories and ∞-operads.

This chapter does not contain any original ideas or results, it is a recolection of results from beforementioned sources.

5.1 (Co)Cartesian fibrations

CoCartesian fibrations represent a type of fibrations between ∞-categories. Given that, they do admit some sort of a lifting property, but they carry much more information: The importance of coCartesian fibrations lies in the *Lurie-Grothendieck correspondence*, which tells us that the ∞-category of all coCartesian fibrations above some simplicial set S is equivalent to the ∞-category of functors from S to Cat_∞. First, we will give a definition of a coCartesian fibrations and then state the Lurie-Grothendieck correspondence. In the end, we will give a criterion on how to detect coCartesian fibrations that we will most commonly use.

Definition 5.1.1. Let $\pi : X \to S$ be a map of simplicial sets, and let $f : x \to y$ be an edge in X. We say that f is π-coCartesian if for every $n \geq 2$ and every commutative square

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma"] \arrow[d] & X \arrow[d, "\pi"] \\
\Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

such that $\sigma(\Delta^{\{0,1\}}) = f$, the dotted lift exists.

Definition 5.1.2. Let $\pi : X \to S$ be a map of simplicial sets. We say that π is a **coCartesian fibration** if:

- π is an inner fibration, that is, π satisfies the right lifting property for every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ where $n > 0$ and $0 < i < n$.
- For every $x \in X$ and every edge $\phi : \pi(x) \to s$ in S there exists a π-coCartesian edge $f : x \to y$ in X such that $\pi(f) = \phi$.

We say that π is a **Cartesian fibration** if its opposite π^{op} is a coCartesian fibration.

Remark 5.1.3. A coCartesian (resp. Cartesian) fibration $\pi : X \to S$ is a left (resp. right) fibration if the fiber X_s is a Kan complex for every $s \in S$.

Definition 5.1.4. Let $\pi : X \to S$ and $\phi : Y \to S$ be two (co)Cartesian fibrations and let $f : X \to Y$ be a map of simplicial sets. We say that f is a map of (co)Cartesian fibrations if f sends π-(co)Cartesian edges to ϕ-(co)Cartesian edges.

Remark 5.1.5. Let S be a simplicial set. Using 5.1.4 we can form an ∞-category $coCart(S)$ whose objects are coCartesian fibrations $p : X \to S$, and a map between two objects (coCartesian fibrations) $p : X \to S$ and $q : Y \to S$ is a map $f : X \to Y$ of coCartesian fibrations. This category plays a key role in the Lurie-Grothendieck correspondene.

Theorem 5.1.6. *(The Lurie-Grothendieck correspondence) Let S be a simplicial set. There exists a functor called* unstraightening *functor*

$$Un^{\infty}_S : Fun(S, Cat_\infty) \to coCart(S)$$

Moreover, this functor in an equivalence and it is compatible with base change. To add up, it reduces to the tautological identification

$$coCart(*) = Cat_\infty \xrightarrow{Id} Cat_\infty = Fun(*, Cat_\infty)$$

Remark 5.1.7. Similar statement could be made for Cartesian fibrations. Namely, given a simplicial set S we can again construct an ∞-category whose objects are Cartesian fibrations and with morphisms between two Cartesian fibrations $p : X \to S$ and $q : Y \to S$ being maps $f : X \to Y$ of Catesian fibrations, that is maps over S which send p-Cartesian arrows in X to q-Cartesian arrows in Y.

The Lurie-Grothendieck correspondance now gives us an equivalence

$$Cart(S) \xrightarrow{\simeq} Fun(S^{op}, Cat_\infty)$$

The following proposition gives us a criterion of how to detect coCartesian fibrations:

Proposition 5.1.8. *([Shah18] 2.5) Let $\pi : X \to S$ be an inner fibration and let $f : a \to b$ be an edge in X. Then f is π-coCartesian if and only if for every $c \in X$ the following diagram*

$$\begin{tikzcd} Map_X(b,c) \arrow[r, "f^*"] \arrow[d, "\pi"] & Map_X(a,c) \arrow[d, "\pi"] \\ Map_S(\pi(b),\pi(c)) \arrow[r, "\pi(f)^*"] & Map_S(\pi(a),\pi(c)) \end{tikzcd}$$

is homotopy Cartesian.

5.2 Dualizing Cartesian and CoCartesian fibrations

Let $p : X \to S$ be a coCartesian fibration. We know that p is classified by the functor $\mathcal{X} : S \to Cat_\infty$. Similarly, given a Cartesian fibration $q : Y \to S^{op}$, we have its classifying functor $\mathcal{Y} : S \to Cat_\infty$. The question now arises: Can we pass from a coCartesian fibration $p : X \to S$ to some Cartesian fibration $q : Y \to S^{op}$ which is represented by the same functor $\mathcal{X}$?
In this section we give a description of such construction. For more informations, see [BGN14].

Definition 5.2.1. Let X be an ∞-category. We define the *twisted arrow ∞-category* of X, and denote it with $\widetilde{\mathcal{O}}(X)$ the following way

$$\widetilde{\mathcal{O}}(X)_n = Mor(\Delta^{n,op} \star \Delta^n, X) \cong X_{2n+1}$$

Proposition 5.2.2. *([BGN14], 2.3) Let X be an ∞-category. The inclusion maps $\Delta^{n,op} \hookrightarrow \Delta^{n,op} \star \Delta^n$ and $\Delta^n \hookrightarrow \Delta^{n,op} \star \Delta^n$ induce the map*

$$\widetilde{\mathcal{O}}(X) \to X^{op} \times X$$

which is a left fibration. Therefore, $\widetilde{\mathcal{O}}(X)$ is an ∞-category.

Definition 5.2.3. Let X be an ∞-category and let $X_\dagger$ and $X^\dagger$ be two subcategories which contain all the equivalences. We say that $(X, X_\dagger, X^\dagger)$ is a triple of ∞-categories and we call the morphisms of $X_\dagger$ *ingressive* and the morphisms of $X^\dagger$ *egressive*.
Furthemore, we say that a triple $(X, X_\dagger, X^\dagger)$ is **adequate** if for every ingressive morphism $x_1 \rightarrowtail y_1$ and any egressive morphism $y_2 \twoheadrightarrow y_1$ there exists a pullback square

$$\begin{tikzcd} x_2 \arrow[r] \arrow[d] & y_2 \arrow[d, two heads] \\ x_1 \arrow[r, tail] & y_1 \end{tikzcd}$$

such that $x_2 \to y_2$ is ingressive and $x_2 \to x_1$ is egressive.

Definition 5.2.4. Let $(X, X_\dagger, X^\dagger)$ be an adequate triple of ∞-categories. The **effective Burnside ∞-category** $A^{eff}(X, X_\dagger, X^\dagger)$ of the triple $(X, X_\dagger, X^\dagger)$ is the simplicial set whose n-simplices are functors

$$x : \widetilde{\mathcal{O}}(\Delta^n) \to X$$

such that for every $0 \leq i \leq k \leq l \leq j \leq n$ with $x_{ij} \rightarrowtail x_{kj}$ and $x_{il} \rightarrowtail x_{kl}$ being ingressive arrows and $x_{ij} \twoheadrightarrow x_{il}$ and $x_{kj} \twoheadrightarrow x_{kl}$ being egressive arrows, the square

$$\begin{tikzcd} x_{ij} \arrow[r, tail] \arrow[d, two heads] & x_{kj} \arrow[d, two heads] \\ x_{il} \arrow[r, tail] & x_{kl} \end{tikzcd}$$

is a pullback square.

Let $\pi : X \to S$ be a cartesian fibration. Denote with $S^{eq} \subset S$ the subcategory which contains all the objects, but whose morphisms are the equivalences in S. To add on, denote with $X^{\pi-cart} \subset X$ the subcategory which contains all the objects of X but whose morphisms are π-cartesian arrows of X. The triples

$$(S, S^{eq}, S)$$
$$(X, X \times_S S^{eq}, X^{\pi-cart})$$

are adequate triples of ∞-categories. Moreover π induces a map

$$\bar{\pi} : A^{eff}(X, X \times_S S^{eq}, X^{\pi-cart}) \to A^{eff}(S, S^{eq}, S)$$

Also, the projection $\widetilde{\mathcal{O}}(\Delta^n) \to \Delta^{n,op}$ gives rise to the map

$$S^{op} \hookrightarrow A^{eff}(S, S^{eq}, S)$$

Definition 5.2.5. Let $\pi : X \to S$ be a Cartesian fibration. Then we define the dual of π denoted as $\pi^\vee$ the following way

$$\pi^\vee : A^{eff}(X, X \times_S S^{eq}, X^{\pi-cart}) \times_{A^{eff}(S,S^{eq},S)} S^{op} \to S^{op}$$

The map $\pi^\vee$ is a coCartesian fibration.

Remark 5.2.6. ([BGN14] *3.7*) The map $S^{op} \hookrightarrow A^{eff}(S, S^{eq}, S)$ is an equaivalence, hence, the projection map

$$A^{eff}(X, X \times_S S^{eq}, X^{\pi-cart}) \times_{A^{eff}(S,S^{eq},S)} S^{op} \to A^{eff}(X, X \times_S S^{eq}, X^{\pi-cart})$$

is an equivalence aswell.

Proposition 5.2.7. *([BGN14]* 3.4*) The map*

$$\bar{\pi} : A^{eff}(X, X \times_S S^{eq}, X^{\pi-cart}) \to A^{eff}(S, S^{eq}, S)$$

is a coCartesian fibration. Moreover, a morphism in $A^{eff}(X, X \times_S S^{eq}, X^{\pi-cart})$ is a span of a form

$$\begin{tikzcd} & x \arrow[dl] \arrow[dr] & \\ y & & z \end{tikzcd}$$

which is $\bar{\pi}$-coCartesian when $x \to z$ is an equivalence.

5.3 Parametrized ∞-categories

At the start of the chapter, we have given a motivation as to what the parametrized higher category should be. In this section we will give a formal definition. Additionally, we will see under which conditions we can define a G-∞-category with G being a compact Lie group.

Definition 5.3.1. Let T be an ∞-category. A T-parametrized ∞-category is a cocartesian fibration $\pi : X \to T^{op}$. We also say that X is a T-∞-category for short. A T-parametrized functor (short, T-functor) between two T-∞-categories $\pi : X \to T^{op}$ and $\phi : Y \to T^{op}$ is a map of cocartesian fibrations.
When π is a left fibration, we say that X is a T-∞-groupoid.

Example 5.3.2. *Let G be a compact Lie group and let $\mathcal{O}_G$ be a full subcategory of the category of G-spaces spanned by the transitive G-spaces (G-orbits). We call $\mathcal{O}_G$ the orbit category, and we call $\mathcal{O}_G$-category $\pi : X \to \mathcal{O}_G^{op}$ simply a G-∞-category.*
We can look at the objects $O \in \mathcal{O}_G$ as cosets G/H by choosing a basepoint $x \in O$ with $Stab_x(O) = H$.
Informally, by Lurie's straightening/unstraightening a G-∞-category $\pi : X \to \mathcal{O}_G^{op}$ is classified by a functor $\mathcal{X} : \mathcal{O}_G^{op} \to Cat_\infty$ which sends every orbit G/H to the fiber $X_{[G/H]} := \pi^{-1}(G/H)$.

We have seen that we can define a T-∞-category for any ∞-category T but in order to be able to do some non-trivial constructions, such as to define a T-symmetric monodal structure for example, the ∞-category T needs to satisfy some additional conditions. Therefore, we define:

Definition 5.3.3. ([Nar17] *1.2* and *2.2*) Let T be a small ∞-category. We define the ∞-category of finite T-sets denoted as F_T to be a subcategory of $Fun(T^{op}, Top)$ spanned by finite coproducts of representables. In the case when $T = \mathcal{O}_G$, $F_{\mathcal{O}_G} := F_G$ corresponds to the category of finite G-sets.
We say that T is *orbital* if F_T admits all pullbacks and we say that T is *cartesian orbital* if F_T admits all limits.
To add up, we say that an orbital ∞-category T is *atomic* if there are no nontrivial retracts in T i.e. every map with a left inverse is an equivalence.

Example 5.3.4. *Let G be a finite group, then $\mathcal{O}_G$ is a cartesian orbital ∞-category which is also atomic.*
If we take G to be a compact Lie group then $\mathcal{O}_G$ is not orbital in general, but the ∞-category of transitive G-spaces with finite stabilizers is atomic orbital ∞-category. It is cartesian if G is a finite group.

For more examples see [Nar17] *1.3*.

Notation 5.3.5. *From now on, we will focus on G-∞-categories $\pi : X \to \mathcal{O}_G^{op}$ where $\mathcal{O}_G$ is the category of transitive G-spaces with finite stabilizers.*

Example 5.3.6. *One of the examples of G-∞-categories that we will see throughout this paper is the G-∞-category of G-spaces, which we denote with $\underline{Top}^G$. Here, we will give the description of this category.*
Let us introduce the category $\mathcal{O}_G - Top$ whose objects are G-maps $X \to O$ where X is a G-space and $O \in \mathcal{O}_G$ and whose morphisms are G-commutative diagrams

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ O_1 \arrow[r] & O_2 \end{tikzcd}$$

where $(X_1 \to O_1), (X_2 \to O_2) \in \mathcal{O}_G - Top$. We can regard $\mathcal{O}_G - Top$ as a topological category by taking the space of morphisms to be a subspace of

$$Map_{\mathcal{O}_G - Top}(X_1 \to O_1, X_2 \to O_2) \subseteq Map_{Top^G}(X_1, X_2) \times Map_{\mathcal{O}_G}(O_1, O_2)$$

consisting of those maps such that upper diagram is commutative.
The idea behind this construction is that we can regard $X \to O$ as classifying an H-space X_H where we write $O \cong G/H$ with the choice of a basepoint of O, and where X_H is the fiber of $X \to O$ over eH.
Consider the forgetful functor $q : \mathcal{O}_G - Top \to \mathcal{O}_G$ which after applying the topological nerve functor becomes

$$N(q) : N(\mathcal{O}_G - Top) \to \mathcal{O}_G$$

where, by abuse of notation we write $\mathcal{O}_G$ for $N(\mathcal{O}_G)$. This functor is a Cartesian fibration with the Cartesian arrows given by diagrams as above which are pullback squares. Then by dualizing this Cartesian fibration we obtain our G-∞-category of G-spaces

$$((N(\mathcal{O}_G - Top))^\vee \to \mathcal{O}_G^{op}) \cong (\underline{Top}^G \to \mathcal{O}_G^{op})$$

The elements of $\underline{Top}^G$ can still be written as G-maps $X \to O$ whereas the morphisms between $X_1 \to O_1$ and $X_2 \to O_2$ are represented by diagrams

$$\begin{array}{ccccc} X_1 & \longleftarrow & X & \longrightarrow & X_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \longleftarrow & O_2 & \xrightarrow{=} & O_2 \end{array}$$

where the left square is a pullback square. This map is coCartesian just in case $X \to X_2$ is an equivalence. The fiber $\underline{Top}^G_{[G/H]}$ is equivalent to $N(Top^G_{/(G/H)})$ which is further equivalent to $N(Top^H)$ by taking the fibre over eH.
The ideas and constructions in this example are common in this paper, most notably in 6.1 where we will use almost the same construction for building our G-∞-category of G-manifolds.

Example 5.3.7. *Let $\pi : C \to \mathcal{O}_G^{op}$ be a G-∞-category and let $x \in C$. Denote with $\mathcal{O}(C)$ the ∞-category of edges of C and denote with $\mathcal{O}^{coCart}(C)$ the full subcategory spanned by the coCartesian edges. Then we define*

$$\underline{x} = \mathcal{O}^{coCart}_{x\to}(C) = \{x\} \times_C \mathcal{O}^{coCart}(C)$$

which we call the G-space associated to x (see [Shah18] 2.28*).*

5.4 G-∞-category of G-sets

In this section we will give a construction of the G-∞-category of finite pointed G-sets for G a compact Lie group, which plays a fundamental role in the definition of the symmetric monoidal G-∞-categories and more generally G-∞-algebras.
Note: Strictly speaking, this will not trully be a category of finite G-sets, but rather a finite disjoint union of tranzitive G-spaces (G-orbits). The justification for this name comes from the case when G is a finite group, which the autors of [BDGNS16] had in mind.

Let F_G be an ∞-category of finite coproduct completion of transitive G-spaces. Since F_G is orbital the target functor

$$Fun(\Delta^1, F_G) \xrightarrow{ev_1} Fun(1, F_G) \cong F_G$$

is a Cartesian fibration. Thus, by pulling back along the inclusion $\mathcal{O}_G \hookrightarrow F_G$ we obtain a Cartesian fibration

$$\pi : Fun(\Delta^1, F_G) \times_{Fun(1,F_G)} \mathcal{O}_G \to \mathcal{O}_G$$

Definition 5.4.1. (see [Nar17] *2.14*, [Hor19] *3.4.1*) Denote with

$$\mathcal{O}_G - Fin_G = Fun(\Delta^1, F_G) \times_{Fun(1,F_G)} \mathcal{O}_G$$

The elements of $\mathcal{O}_G - Fin_G$ can be seen as arrows $U \to O$ with $U \in F_G$ and $O \in \mathcal{O}_G$. Consider a morphism in $\mathcal{O}_G - Fin_G$ written as

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & O_1 \times_{O_2} U_2 \arrow[r] \arrow[d] & U_2 \arrow[d] \\ O_1 \arrow[r, "="] & O_1 \arrow[r] & O_2 \end{tikzcd}$$

We will say that this morphism is a summand inclusion just in case the map of G-sets $U_1 \to O_1 \times_{O_2} U_2$ is a summand inclusion i.e. there exists a finite G-set U such that $U_1 \coprod U \to O_1 \times_{O_2} U_2$ is an equivalence.
We will denote with $\mathcal{O}_G - Fin_G{}^{\cup} \subset \mathcal{O}_G - Fin_G$ the subcategory consisting of all objects with morphisms being the summand inclusions.

Let $\mathcal{O}_G{}^{\sim}$ denote the maximal subgroupoid of $\mathcal{O}_G$, that is $\mathcal{O}_G{}^{\sim} \subset \mathcal{O}_G$ is a subcategory consisting of all object with morphisms being the equivalences. It is not hard to show that

$$(\mathcal{O}_G - Fin_G, \mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}, \mathcal{O}_G - Fin_G{}^{\cup}) \tag{5.1}$$

is an adequate triple. Therefore we can define

Definition 5.4.2. The G-∞-category of finite pointed G-sets is defined as the pullback

$$\begin{tikzcd} \underline{Fin}_*^G \arrow[r] \arrow[d] & A^{eff}(\mathcal{O}_G - Fin_G, \mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}, \mathcal{O}_G - Fin_G{}^{\cup}) \arrow[d] \\ \mathcal{O}_G^{op} \arrow[r, hook] & A^{eff}(\mathcal{O}_G, \mathcal{O}_G{}^{\sim}, \mathcal{O}_G) \end{tikzcd}$$

It has the same objects as $\mathcal{O}_G - Fin_G$ with morphism between $(U_1 \to O_1)$ and $(U_2 \to O_2)$ being the span

$$\begin{tikzcd} U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

with the left square being a summand inclusion.
We will say that a morphism in $\underline{Fin}_*^G$ is **inert** if the map $U \to U_2$ is an equivalence, and that it is **active** if the map $U \to O_2 \times_{O_1} U_1$ is an equivalence.

Remark 5.4.3. In order to see that $\underline{Fin}_*^G$ is in fact a G-∞-category, that is that $\underline{Fin}_*^G \to \mathcal{O}_G^{op}$ is a coCartesian fibration we can consider $\underline{Fin}_*^G$ as a particular subcategory of a G-∞-category $\underline{A}^{eff}(\mathcal{O}_G)$ (see [Nar17] *2.13*) containing all of the coCartesian edges of $\underline{A}^{eff}(\mathcal{O}_G) \to \mathcal{O}_G^{op}$ making it a G-∞-subcategory (see [BDGNS16] *section 4*).

Remark 5.4.4. ([BDGNS16] *2.11*) Let $I = [U \to O] \in \underline{Fin}_*^G$ be a finite G-set and let $\widetilde{\mathcal{O}}(F_G)$ be a twisted arrow ∞-category together with a left fibration

$$\widetilde{\mathcal{O}}(F_G) \to F_G^{op} \times F_G$$

Taking a pullback of this left fibration along

$$\mathcal{O}_G^{op} \cong \mathcal{O}_G^{op} \times \{U\} \to F_G^{op} \times F_G$$

we obtain a G-∞-category (a G-space to be more precise)

$$p_U : \underline{U} \to \mathcal{O}_G^{op}$$

which is classified by the functor $\mathbf{U} : \mathcal{O}_G^{op} \to Top$, $\mathbf{U}(O) = Map_{F_G}(O, U)$. We will call $p_U : \underline{U} \to \mathcal{O}_G^{op}$ the *discrete G-space* attached to U.
Note that $\underline{U} \simeq \coprod_{W \in Orbit(U)} \underline{W}$. Also note that this G-space $\underline{U}$ is equivalent to the one of 5.3.7, hence we use the same notation.

Lemma 5.4.5. *([BDGNS16]* 2.12*) Let C be a G-∞-category and let $U \in F_G$ be a finite G-set. Then the formation of the fibers over each orbit $W \in Orbit(U)$ induces a trivial fibration*

$$Fun_G(\underline{U}, C) \overset{\simeq}{\twoheadrightarrow} \prod_{W \in Orbit(U)} C_W$$

5.5 G-symmetric monoidal G-∞-categories

In the non-parametrized setting, we define a symmetric monoidal ∞-category as a coCartesian fibration $p : \mathcal{C}^{\otimes} \to Fin_*$ satisfying some additional conditions. Naturally, we would want to imitate this definition in order to obtain G-symmetric monoidal ∞-categories by replacing Fin_* with $\underline{Fin}_*^G$.

Definition 5.5.1. (see [Nar17], *section 3.1* or [Hor19], *B.0.10*) A G-symmetric monoidal G-∞-category is a coCartesian fibration $C^{\otimes} \to \underline{Fin}_*^G$ such that for every finite pointed G-set $J = (U \to O) \in \underline{Fin}_*^G$ we have an equivalence

$$\prod_{W \in Orbit(U)} (\chi_{[W \subset U]})_! : C^{\otimes}_{[J]} \to \prod_{W \in Orbit(U)} C^{\otimes}_{[I(W)]}$$

where $I(W) = (W \xrightarrow{=} W) \in \underline{Fin}_*^G$, $C^{\otimes}_{[J]}$ and $C^{\otimes}_{[I(W)]}$ are the corresponding fibers. The functor $\chi_{[W \subset U]} : C^{\otimes}_{[J]} \to C^{\otimes}_{[I(W)]}$ is associated to the following map in $\underline{Fin}_*^G$:

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r, "="] \arrow[d, "="] & W \arrow[d, "="] \\ O & W \arrow[l] \arrow[r, "="] & W \end{tikzcd}$$

Remark 5.5.2. When defining the maps $\chi_{[W \subset U]}$ in 5.5.1 we make use of the fact that $\mathcal{O}_G$ is an atomic ∞-category to ensure that the map

$$W \to U \times_O W$$

is a summand inclusion.

Remark 5.5.3. Given a G-symmetric monoidal G-∞-category $\underline{\mathcal{C}}^{\otimes} \to \underline{Fin}_*^G$ we can define its underlying G-∞-category $\underline{\mathcal{C}}_{I(-)}$ which fits into the pullback square

$$\begin{tikzcd} \underline{\mathcal{C}}_{I(-)} \arrow[r] \arrow[d] & \underline{\mathcal{C}}^{\otimes} \arrow[d] \\ \mathcal{O}_G^{op} \arrow[r, "I(-)"] & \underline{Fin}_*^G \end{tikzcd}$$

where $I : \mathcal{O}_G \to \underline{Fin}_*^G$ is the same as in 5.5.1.

5.6 G-∞-operads

Definition 5.6.1. *([Nar17] def. 3.1)* A G-∞-operad is an inner fibration $p : O^{\otimes} \to \mathcal{O}_G^{op}$ satisfying the following conditions:

- For every inert edge $e : J_1 \to J_2$ in $\underline{Fin}_*^G$ and every $x \in O^{\otimes}_{[J_1]}$ there exists a coCartesian lift $\widetilde{e} : x \to y$ over e.
- For any $J = [U \to O] \in \underline{Fin}_*^G$ and any choice of a pushforward functors along inert edges, we have an equivalence
$$\prod_{W \in Orbit(U)} (\chi_{[W \subseteq U]})_! : O^{\otimes}_{[J]} \xrightarrow{\simeq} \prod_{W \in Orbit(U)} O^{\otimes}_{I(W)}$$
- For any choice of pushfoward functors along inert morphisms and for every map $e : J_1 = [U_1 \to O_1] \to J_2 = [U_2 \to O_2]$ in $\underline{Fin}_*^G$ and every $x \in O^{\otimes}_{[J_1]}$ and $y \in O^{\otimes}_{[J_2]}$ the map
$$Map^e_{O^{\otimes}}(x, y) \to \prod_{W \in Orbit(U_2)} Map^{\chi_{[W \subseteq U]} \circ e}_{O^{\otimes}}(x, (\chi_{[W \subseteq U]})_! y)$$
is an equivalence where $Map^e_{O^{\otimes}}(x, y)$ is the fiber over e of the map $Map_{O^{\otimes}}(x, y) \to Map_{\underline{Fin}_*^G}(J_1, J_2)$.

An arrow $e : x \to y$ of a G-∞-operad $p : O^{\otimes} \to \underline{Fin}_*^G$ is called **inert** if it is p-coCartesian and if $p \circ e$ is inert in $\underline{Fin}_*^G$. Let $p : O^{\otimes} \to \underline{Fin}_*^G$ and $q : E^{\otimes} \to \underline{Fin}_*^G$ be two G-∞-operads and let $F : O^{\otimes} \to E^{\otimes}$ be a map over $\underline{Fin}_*^G$. We say that F is a map of G-∞-operads if F sends inert edges to inert edges.

Remark 5.6.2. Every G-symmetric monoidal ∞-category is a G-∞-operad. Moreover, just as in 5.5.3 we can define the underlying G-∞-categorry of a G-∞-operad to be the pullback

$$\begin{tikzcd} {O^{\otimes}}_{I(-)} \arrow[r] \arrow[d] & O^{\otimes} \arrow[d] \\ \mathcal{O}_G^{op} \arrow[r, "I(-)"] & \underline{Fin}_*^G \end{tikzcd}$$

We will usually mark with O the underlying G-∞-category ${O^{\otimes}}_{I(-)}$.

Remark 5.6.3. Let $p : O^{\otimes} \to \underline{Fin}_*^G$ and $q : E^{\otimes} \to \underline{Fin}_*^G$ be two G-∞-operads and let $F : O^{\otimes} \to E^{\otimes}$ be a map of G-∞-operads. We will sometimes call F an $O^{\otimes}$-algebra in $E^{\otimes}$ especially when $E^{\otimes}$ is a G-symmetric monoidal ∞-category. We will mark with $Alg_{G,O^{\otimes}}(E^{\otimes})$ or $Alg_G(O^{\otimes}, E^{\otimes})$ the ∞-category of $O^{\otimes}$-algebras in $E^{\otimes}$.

Remark 5.6.4. Taking G to be a trivial group e, the notion of e-∞-operad corresponds to the notion of an ∞-operad in the sense of Lurie (see [HA]). In this case, we will write $\underline{Fin}_*^e$ simply as Fin_*.

G-coCartesian structure

Given a G-∞-category C, we will present the G-∞-operad $C^{\amalg} \to \underline{Fin}_*^G$, the parametrized version of the coCartesian structure from [HA] (see *2.4.3*) which we will naturally call the G-coCartesian structure. The construction of this G-∞-operad is given by Nardin and Shah in [NS]. The summary of this construction can be put in the following:

Proposition 5.6.5. *Let C be a G-∞-category. Then there exists an ∞-category $C^{\amalg}$ together with a map $C^{\amalg} \to \underline{Fin}_*^G$ which is a G-∞-operad such that*

- *For $I = [U \to O] \in \underline{Fin}_*^G$ the fiber $(C^{\amalg})_I$ is equivalent to $Fun_G(\underline{U}, C)$.*
- *An arrow from X_1 to X_2 lying above a map*

$$\begin{tikzcd} U_1 \arrow[d] & U \arrow[l, "f"'] \arrow[r, "g"] \arrow[d] & U_2 \arrow[d] \\ O_1 & O \arrow[l] \arrow[r] & O_2 \end{tikzcd}$$

in $\underline{Fin}_*^G$ is a natural transformation

$$f^* X_1 \to g^* X_2$$

in $Fun_G(\underline{U}, C)$. To add up, the arrow is coCartesian if it exibits X_2 as a G-left Kan extension of $f^* X_1$ along $\widetilde{g} : \underline{U} \to \underline{U_2}$ (induced by g).

Moreover, $C^{\amalg}$ is a G-symmetric monoidal G-∞-category if and only if C has all finite G-coproducts.

We have the following lemma which will prove to be useful:

Lemma 5.6.6. *([Hor19]* 3.5.1*) Let $\underline{\mathcal{C}}$ be a G-∞-category and let $\underline{\mathcal{O}}^{\otimes}$ be a unital G-∞-operad. Consider the ∞-category $Alg_G(\underline{\mathcal{O}}^{\otimes}, \underline{\mathcal{C}})$ of morphisms of G-∞-operads $\underline{\mathcal{C}}^{\amalg}$ (with G-coCartesian structure) and $\underline{\mathcal{O}}^{\otimes}$ and the ∞-category of G-functors between the underlying G-∞-categories $Fun_G(\underline{\mathcal{O}}, \underline{\mathcal{C}})$. There is an evident restriction functor*

$$Alg_G(\underline{\mathcal{O}}, \underline{\mathcal{C}}) \to Fun_G(\underline{\mathcal{O}}, \underline{\mathcal{C}})$$

which is an equivalence.

Chapter 6

G-manifolds

The second part of this project is dedicated to defining the equivariant version of factorization homology defined by Ayala and Francis (see [AF15]) by using parametrized higher category theory. Having that in mind, instead of ∞-categories we will have G-∞-categories, instead of functors we will have G-functors etc.
The first thing on the list would be to define the G-∞-category of G-manifolds, which we describe in the first section. Later, we introduce the tangent bundle classifier which we use to define the framed version of the G-∞-category of G-manifolds. Lastly, we will describe the G-symmetric monoidal structure on the G-∞-category of G-manifolds.
Note that the constructions and ideas presented here stem from the work of Horev (see [Hor19]) who gave a construction of equivariant version of factorization homology when G is a finite group. From now on we will always assume that G is a compact Lie group of dimestion l.

6.1 G-∞-category of G-manifolds

We would want to have a G-∞-category of G-manifolds, that is a coCartesian fibration $p : \underline{Mfld}^G \to \mathcal{O}_G^{op}$ such that for $G/H \in \mathcal{O}_G^{op}$, $\underline{Mfld}^G_{[G/H]} \simeq N(Mfld^H)$, the nerve of the category of H-manifolds. This coCartesian fibration is indeed classified by a functor

$$\begin{aligned} \mathcal{M}fld_G &: \mathcal{O}_G^{op} \to Cat_\infty \\ \mathcal{M}fld_G &: G/H \mapsto N(Mfld^H) \end{aligned} \tag{6.1}$$

so by the Lurie-Grothendieck correspondence we can take $\underline{Mfld}^G := Un^\infty(\mathcal{M}fld_G)$, but as Horev explained in [Hor19] *sec. 3.1*, $Un^\infty(\mathcal{M}fld_G)$ is not suitable for our work. Therefore, similar to Horev's construction, we will give an equivalent construction of $\underline{Mfld}^G$.

Definition 6.1.1. Denote with $\mathcal{O}_G - Mfld$ the topological category whose objects are G-manifold bundles $p : M \to O$ where M is a smooth G-manifold such that the fibers are of dimension n, $O \in \mathcal{O}_G$ and p is a G-equivariant manifold bundle map, which we call the $\mathcal{O}_G$-manifolds. A map between two $\mathcal{O}_G$-manifolds $p_1 : M_1 \to O_1$ and $p_2 : M_2 \to O_2$ consists of a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, "p_1"] & M_2 \arrow[d, "p_2"] \\ O_1 \arrow[r, "\varphi"] & O_2 \end{tikzcd}$$

with the following property:
For every $x \in O_1$ the induced map on the fiber $(M_1)_x \to (O_1 \times_{O_2} M_2)_x$ is a G-embedding map.
We denote with $Emb^{\mathcal{O}_G}(M_1, M_2)$ the subspace of $Map(O_1, O_2) \times C^\infty(M_1, M_2)$ consisting of maps between $\mathcal{O}_G$-manifolds endowed with the weak C^∞-topology.

Remark 6.1.2. Given an orbit $O \in \mathcal{O}_G$ and choosing a basepoint of that orbit we can write the given orbit as G/H where H is the stabilizer of the basepoint. Having this in mind, we can think of $\mathcal{O}_G$-manifolds as equivariant manifold bundle maps $M \to G/H$. The fiber M_H over the coset eH (where e is the neutral element of G) is therefore an H-manifold. Moreover, we see that in that case, the map $(M_1)_x \to (O_1 \times_{O_2} M_2)_x$ from 6.1.1 is actually an H-embedding.

Remark 6.1.3. There exists an obvious forgetful functor $\theta : \mathcal{O}_G - Mfld \to \mathcal{O}_G$ which takes an $\mathcal{O}_G$-manifold $M \to O$ to its underlying orbit O. Given a map of orbits $\varphi : O_1 \to O_2$ and two $\mathcal{O}_G$-manifolds $M_1 \to O_1$ and $M_2 \to O_2$ we will denote with $Emb_\varphi^{\mathcal{O}_G}(M_1, M_2) \subset Emb^{\mathcal{O}_G}(M_1, M_2)$ the fiber over φ of the map $Emb^{\mathcal{O}_G}(M_1, M_2) \to Map(O_1, O_2)$ induced by the forgetful functor. In the case when $\phi : O \xrightarrow{=} O$ is the identity, we will simply write $Emb_O^{\mathcal{O}_G}(M_1, M_2)$ for $Emb_\varphi^{\mathcal{O}_G}(M_1, M_2)$.
Moreover, by 6.1.2 $Emb_O^{\mathcal{O}_G}(M_1, M_2)$ is equivalent to the space of H-embeddings $Emb^H((M_1)_x, (M_2)_x)$ where $x \in O$ and $H = Stab_x(O)$.

G-isotopy maps

Up until now, we have just defined $\mathcal{O}_G$-manifolds and maps between them. In this subsection we will define G-isotopies and see their connection to G-homotpy equivalences and G-diffeomorphisms.

Definition 6.1.4. Let $M_1 \to O_1$ and $M_2 \to O_2$ be two $\mathcal{O}_G$-manifolds and let $\varphi : O_1 \to O_2$ be a map of orbits. By a G-isotopy over φ we will mean a path in $Emb_\varphi^{\mathcal{O}_G}(M_1, M_2)$. In the case when $O_1 = O_2 = O$ we will say that a path in $Emb_O^{\mathcal{O}_G}(M_1, M_2)$ is a G-isotopy over O.
A map between two $\mathcal{O}_G$-manifolds is called a G-isotopy equivalence if it induces an equivalence in $N(\mathcal{O}_G - Mfld)$.

Remark 6.1.5. Note that an equivalence in $N(\mathcal{O}_G - Mfld)$ always lies over an isomorphism of orbits.
Let $O \in \mathcal{O}_G$ be an orbit and let $x \in O$ and let $H = Stab(x)$. By taking x to be a basepoint of O we have that $O \cong G/H$. Let $M_1 \to O$ and $M_2 \to O$ be two $\mathcal{O}_G$-manifolds. A G-isotopy over O in $Emb_O^{\mathcal{O}_G}(M_1, M_2)$ is then equivalent to an H-equivariant isotopy between two H-embeddings $(M_1)_x \to (M_2)_x$. Having this in mind, a map of $\mathcal{O}_G$-manifolds $f : M_1 \to M_2$ over the orbit O is an equivalence in $N(\mathcal{O}_G - Mfld)$ if and only if the restriction $f_x : (M_1)_x \to (M_2)_x$ is invertible up to H-isotopy, which means that f does not have to induce a G-diffeomorphism.

The same thing as in 6.1.5 occurs in the work of Horev ([Hor19] *3.1*) and similarly we can deduce that the existence of a G-isotopy equivalences ensures the existence of a G-diffeomorphisms. We will show that the *Proposition 3.1.12* in [Hor19] holds in the case when G is a compact Lie group:

Proposition 6.1.6. *Let $M_1 \to O$ and $M_2 \to O$ be two $\mathcal{O}_G$-manifolds and let $f \in Emb_O^{\mathcal{O}_G}(M_1, M_2)$ and $g \in Emb_O^{\mathcal{O}_G}(M_2, M_1)$ be G-isotopy inverses over O. Then there exists G-equivariand diffeomorphism $M_1 \cong M_2$ over O*

Proof. As in 6.1.5 take $x \in O$ and denote $H = Stab(x)$. Since the space $Emb_O^{\mathcal{O}_G}(M_1, M_2)$ is homeomorphic to the space of H-equivariant embeddings $(M_1)_x \to (M_2)_x$ it is enough to consider the case when $G = H$. Since H is finite the proof follows from the proof of [Hor19] *3.1.12*. □

G-∞-category of G-manifolds

The goal of this section is to construct the G-∞-category of G-manifolds. The idea is the following:
Note that the forgetful functor

$$N(\theta) : N(\mathcal{O}_G - Mfld) \to N(\mathcal{O}_G)$$

is a Cartesian fibration (see Lemma 6.1.8 below) which is by 6.1.2 and 6.1.3 classified by the functor as in (6.1). Therefore, we will obtain our G-∞-category of G-manifolds as a dual to this Cartesian fibration. In order to do so, we will first analyse the Cartesian edges in $N(\mathcal{O}_G - Mfld)$.

Lemma 6.1.7. *The forgetful functor $N(\theta) : N(\mathcal{O}_G - Mfld) \to N(\mathcal{O}_G)$ is an inner fibration.*

Proof. By [HTT] *2.4.1.10 (i)* it will suffice to show that, for two $\mathcal{O}_G$-manifolds $p_1 : M_1 \to O_1$ and $p_2 : M_2 \to O_2$ the map

$$Map_{Sing(\mathcal{O}_G - Mfld)}(M_1, M_2) \to Map_{Sing(\mathcal{O}_G)}(O_1, O_2)$$

is a Kan fibration. We will demonstrate this by showing that the map

$$p : Emb^{\mathcal{O}_G}(M_1, M_2) \to Map_{\mathcal{O}_G}(O_1, O_2)$$

is a fiber bundle, hence in particular a Serre fibration ([Hat01] *4.48*). Take $\varphi \in Map_{\mathcal{O}_G}(O_1, O_2)$ and take $x \in O_1$. Note that since orbits are transitive G-spaces, the map φ is determined by its image $\varphi(x) = y \in O_2$. Let $H = Stab(x)$ and $K = Stab(y)$ with $H \leq K$. Furthemore, let $N_1 = (M_1)_x$ and $N_2 = (M_2)_y$ and note that N_1 is an H-manifold and N_2 a K-manifold. We have $(O_1, x) \cong G/H$, $(O_2, y) \cong G/K$ and $M_1 \cong G \times_H N_1$, $M_2 \cong G \times_K N_2$. Then, a map

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & M_2 \arrow[d] \\ O_1 \arrow[r, "\varphi"] & O_2 \end{tikzcd}$$

is determined by a G-commutative diagram

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d] & N_2 \arrow[d] \\ \{x\} \arrow[r, "\varphi"] & \{y\} \end{tikzcd}$$

Since the map $M_1 \to O_1 \times_{O_2} M_2$ has to be a fiberwise G-embedding we see that the upper horizontal map needs to be a G-embedding: a map $f : (M_1)_x \to (M_2)_y$ above φ induces a map on the pullback $\widetilde{f} : (M_1)_x \to (O_1 \times_{O_2} M_2)_x$ which is a G-embedding, but $(O_1 \times_{O_2} M_2)_x \cong (M_2)_y = N_2$ as an H-manifold (i.e. with the restriction of action of K to H). Therefore

$$Emb^{\mathcal{O}_G}_{\varphi}(M_1, M_2) \cong Emb^H(N_1, N_2)$$

Before we continue, note that, since O_1 and O_2 are G-orbits i.e. transitive G-spaces, the space of G-equivariant maps $Map_G(O_1, O_2)$ will be generated under φ by G. In other words, every $\psi \in Map_G(O_1, O_2)$ can be written as $g\varphi$ for $g \in G$. Now, consider a neighborhood V of φ in $Map_G(O_1, O_2)$. By our previous remark, we conclude that there exists an open neighborhood U of the neutral element $e \in G$ such that for every $\psi \in V$ there exists $g \in U$ such that $\psi = g\varphi$. Additionally, we can choose V small enough such that $U \cap kU = \emptyset$ for all $k \in K$ (we can do this because K is finite). This way, for every $\psi \in V$ there exists a unique $g \in U$ such that $\psi = g\varphi$. Then by letting U act on $y \in O_2$ we obtain an open neighborhood U_y of y. Again, we can choose V small enough such that $p_2^{-1}(U_y) \cong U_y \times N_2$. Then a map $(f, \psi) \in Emb^{\mathcal{O}_G}(M_1, M_2)$ with $\psi \in V$ restricts to

$$\begin{tikzcd} N_1 \arrow[r, "\widetilde{f}"] \arrow[d] & U_y \times N_2 \arrow[d] \\ \{x\} \arrow[r, "\psi"] & U_y \end{tikzcd}$$

Moreover, for the unique $g \in U$, $\psi = g\varphi$, $\psi(x) = g\varphi(x) = gy$, hence the upper diagram further restricts to

$$\begin{tikzcd} N_1 \arrow[r, "\widetilde{f}"] \arrow[d] & N_2 \arrow[d] \\ \{x\} \arrow[r, "\psi"] & \{gy\} \end{tikzcd}$$

Therefore, the map $(f, \psi) \in Emb^{\mathcal{O}_G}(M_1, M_2)$ is completely determined by the choice of ψ and $\widetilde{f} \in Emb^H(N_1, N_2)$ and we have

$$p^{-1}(V) \cong V \times Emb^H(N_1, N_2)$$

Hence, the map $p : Emb^{\mathcal{O}_G}(M_1, M_2) \to Map_{\mathcal{O}_G}(O_1, O_2)$ is a fiber bundle map, and consequently a Serre fibration, as desired.

□

Lemma 6.1.8. *$N(\theta) : N(\mathcal{O}_G - Mfld) \to N(\mathcal{O}_G)$ is a Cartesian fibration. To be more precise, let $M \to O$ be a $\mathcal{O}_G$-manifold and let $\varphi : O_1 \to O$ be a map of orbits. Then the pullback square*

$$\begin{tikzcd} O_1 \times_O M \arrow[r] \arrow[d] & M \arrow[d] \\ O_1 \arrow[r, "\varphi"] & O \end{tikzcd}$$

is a $N(\theta)$-Cartesian arrow in $N(\mathcal{O}_G - Mfld)$. Moreover, a morphism between two $\mathcal{O}_G$-manifolds $M_1 \to O_1$ and $M_2 \to O_2$ is $N(\theta)$-Cartesian if and only if the commutative diagram from 6.1.1 is equivalent to to a pullback diagram. In other words, the diagram

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & O_1 \times_{O_2} M_2 \arrow[d] \\ O_1 \arrow[r, "="] & O_1 \end{tikzcd}$$

is a G-isotopy equivalence.

Proof. Given a commutative diagram as in 6.1.1, we can compose it as

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & O_1 \times_{O_2} M_2 \arrow[r] \arrow[d] & M_2 \arrow[d] \\ O_1 \arrow[r, "="] & O_1 \arrow[r] & O_2 \end{tikzcd}$$

If we assume that the right pullback square is $N(\theta)$-Cartesian, then by [HTT] *2.4.1.7* the whole morphism is $N(\theta)$-Cartesian if and only if the left square is $N(\theta)$-Cartesian, and the left square is $N(\theta)$-Cartesian if and only if the map $M_1 \to O_1 \times_{O_2} M_2$ is an equivalence by [HTT] *2.4.1.5*. Therefore, we only need to prove the first part.

By the dual statement of 5.1.8, we need to show that for every $\mathcal{O}_G$-manifold $N \to O_2$ the diagram

$$\begin{tikzcd} Emb^{\mathcal{O}_G}(N, O_1 \times_O M) \arrow[r] \arrow[d] & Emb^{\mathcal{O}_G}(N, M) \arrow[d] \\ Map_{\mathcal{O}_G}(O_2, O_1) \arrow[r] & Map_{\mathcal{O}_G}(O_2, O) \end{tikzcd}$$

is a homotopy pullback diagram. By 6.1.7 we know that vertical arrows are fibrations hence it will suffice to show that the fibers are equivalent. Fot this note that, for every $\psi \in Map_{\mathcal{O}_G}(O_2, O_1)$ we have a continuous bijection of the fibers

$$Emb_{\psi}^{\mathcal{O}_G}(N, O_1 \times_O M) \xrightarrow{f_*} Emb_{\varphi\circ\psi}^{\mathcal{O}_G}(N, M)$$

given by the universal property of the pullback, where f_* is post composition with $f : O_1 \times_O M \to M$.

$$\begin{tikzcd} N \arrow[r] \arrow[d] & O_1 \times_O M \arrow[r, "f"] \arrow[d] & M \arrow[d] \\ O_2 \arrow[r, "\psi"] & O_1 \arrow[r, "\varphi"] & O \end{tikzcd}$$

By the choice of the basepoint of the orbits, we can write the upper diagram as

$$\begin{tikzcd} N \arrow[r] \arrow[d] & G/K_1 \times_{G/H} M \arrow[r, "f"] \arrow[d] & M \arrow[d] \\ G/K_2 \arrow[r] & G/K_1 \arrow[r] & G/H \end{tikzcd}$$

with $K_2 \leq K_1 \leq H$ all finite groups. Note that we can regard N and $G/K_1 \times_{G/H} M$ as $\mathcal{O}_G$-manifolds lying over G/H by composition $N \to G/K_2 \to G/K_1 \to G/H$ and $G/K_1 \times_{G/H} M \to G/K_1 \to G/H$. Note also that $G/K_2 \to G/K_1 \to G/H$ and $G/K_1 \to G/H$ are covering spaces with fibres corresponding to H/K_2 and H/K_1 respectively. By taking the fiber over $eH \in G/H$ the upper diagram becomes equivalent to

$$\begin{tikzcd} \bigsqcup_{H/K_2} N_1 \arrow[r] \arrow[d] & \bigsqcup_{H/K_1} M_1 \arrow[r, "f_1"] \arrow[d] & M_1 \arrow[d] \\ H/K_2 \arrow[r] & H/K_1 \arrow[r] & H/H \end{tikzcd}$$

where N_1 is the fiber of $N \to G/K_2$ and M_1 the fiber of $M \to G/H$. Note that the fibers of $M \to G/H$ and $G/K_1 \times_{G/H} M \to G/K_1$ are non-equivariantly the same (since the latter is obtained via the pullback) but with different action. The fiber of the first bundle map is the manifold M_1 equipped with H-action, while the fiber of the second is acain M_1 but with action restricted to K_1-action.

The right square is still a pullback square, hence we have that f_1 induces a continuous bijection

$$Emb^H(\bigsqcup_{H/K_2} N_1, \bigsqcup_{H/K_1} M_1) \xrightarrow{f_{1*}} Emb^H(\bigsqcup_{H/K_2} N_1, M_1)$$

Since the map f_1 is essentially given as a disjoint union of diffeomorphisms it is not hard to see that the map f_{1*} is open, hence it is a homeomorphism. By 6.1.3 we have

$$Emb_{\psi}^{\mathcal{O}_G}(N, O_1 \times_O M) \simeq Emb^H(\bigsqcup_{H/K_2} N_1, \bigsqcup_{H/K_1} M_1)$$
$$Emb_{\varphi\circ\psi}^{\mathcal{O}_G}(N, M) \simeq Emb^H(\bigsqcup_{H/K_2} N_1, M_1)$$

which ultimately gives us $Emb_{\psi}^{\mathcal{O}_G}(N, O_1 \times_O M) \simeq Emb_{\varphi\circ\psi}^{\mathcal{O}_G}(N, M)$ and the proof is finished. □

Now we are ready to define the G-∞-category of G-manifolds.

Notation 6.1.9. *For the sake of readability and easier writing we will abuse notation and write $\mathcal{O}_G$ for the nerve $N(\mathcal{O}_G)$.*

Let $\mathcal{O}_G^{\simeq}$ denote the maximal subgroupoid of $\mathcal{O}_G$. By 5.2 we have that the triples

$$(N(\mathcal{O}_G - Mfld), N(\mathcal{O}_G - Mfld) \times_{\mathcal{O}_G} \mathcal{O}_G^{\simeq}, N(\mathcal{O}_G - Mfld)^{N(\theta)-cart})$$
$$(\mathcal{O}_G, \mathcal{O}_G^{\simeq}, \mathcal{O}_G)$$

are adequate triples.

Definition 6.1.10. The G-∞-category of G-manifolds is the coCartesian fibration $\pi : \underline{Mfld}^G \to \mathcal{O}_G^{op}$ dual to the Cartesian fibration $N(\theta) : N(\mathcal{O}_G - Mfld) \to \mathcal{O}_G$. By 5.2.7 the objects of $\underline{Mfld}^G$ are $\mathcal{O}_G$-manifolds and a morphism from $M_1 \to O_1$ to $M_2 \to O_2$ is a diagram of the form

$$\begin{tikzcd} M_1 \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] & M_2 \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

such that the left square is a pullback square. This morphism is π-coCartesian just in case the right square is a G-isotopy equivalence.

Remark 6.1.11. Unwinding the definition, $\underline{Mfld}^G$ is obtained via the pullback of the effective Burnside category

$$A^{eff}(N(\mathcal{O}_G - Mfld), N(\mathcal{O}_G - Mfld) \times_{\mathcal{O}_G} \mathcal{O}_G^{\simeq}, N(\mathcal{O}_G - Mfld)^{N(\theta)-cart})$$

along the equivalence

$$\mathcal{O}_G^{op} \xrightarrow{\simeq} A^{eff}(\mathcal{O}_G, \mathcal{O}_G^{\simeq}, \mathcal{O}_G)$$

Remark 6.1.12. Since $\pi : \underline{Mfld}^G \to \mathcal{O}_G^{op}$ is the dual of $N(\theta)$ it is classified by the same functor $\mathcal{O}_G^{op} \to Cat_\infty$ sending an orbit $G/H \mapsto N(Mfld^H)$.
We have already explained how we can view orbits as quotients G/H where H is a finite subgroup of G. Then the equivalence

$$\underline{Mfld}^G_{[G/H]} \xrightarrow{\simeq} N(Mfld^H)$$

is given by taking the fiber of $M \to G/H$ over eH, with inverse given by the functor of topological induction $G \times_H (-)$ (see also [TD87] *chapter 1, 9.2*): Given an H-manifold M which we can write as $M \to *$, the topological induction gives us

$$G \times_H M \to G \times_H (*) \simeq G/H$$

which is an $\mathcal{O}_G$-manifold.

Remark 6.1.13. Apart from 6.1.12 we would like to see other advantages of using the $\underline{Mfld}^G$. Some of them include:

- **Restriction**: Given finite subgroups $K \leq H \leq G$ we would like to have a restriction functor

$$Res_K^H : \underline{Mfld}^G_{[G/H]} \to \underline{Mfld}^G_{[G/K]}$$

which under the equivalence from 6.1.12 restricts the H-action to K-action of the H-manifold M.
Given an $\mathcal{O}_G$-manifold $M \to G/H$ the restriction functor gives us $Res_K^H(M)$ the $\mathcal{O}_G$-manifold over G/K obtained as a coCartesian lift of $G/H \to G/K \in Mor(\mathcal{O}_G{}^{op})$ starting from $M \to G/H$. By the description of coCartesian edges in $\underline{Mfld}^G$ this would simply be the pullback

$$\begin{tikzcd} M & Res_K^H(M) \arrow[l] \arrow[d] \\ G/H \arrow[from=u] & G/K \arrow[l] \end{tikzcd}$$

Forgetting the action, the fibers of $M \to G/H$ and $Res_K^H(M) \to G/K$ are the same, with the ones of the former being H-manifolds and ones of the latter being K-manifolds, which is what we wanted.

- **Conjugation**: For $H \leq G$ and $g \in G$ the conjugation action equivalence

$$c_g : \underline{Mfld}^G_{[G/H]} \xrightarrow{\simeq} \underline{Mfld}^G_{[G/gHg^{-1}]}$$

is given the following way:
For $M \to G/H$, the conjugacy action on M, $c_g(M)$ is given as a coCartesian lift of the equivalence $G/H \to G/gHg^{-1} \in Mor(\mathcal{O}_G{}^{op})$ starting from M. Again, by the description of the coCartesian arrows in $\underline{Mfld}^G$, we have a pullback square

$$\begin{tikzcd} M \arrow[d] & c_g(M) \arrow[l] \arrow[d] \\ G/H & G/gHg^{-1} \arrow[l] \end{tikzcd}$$

The inverse is given by $c_{g^{-1}}$.

- **Topological induction**: Let $K \leq H \leq G$ be finite subgroups and let M_K be a K-manifold. We would like to give a description of the topological induction $H \times_K M_K$ in $\underline{Mfld}^G$.
Let $M \to G/K$ be an $\mathcal{O}_G$-manifold with fiber M_K. Then the topological induction $H \times_K M_K$ in $\underline{Mfld}^G$ is given as a fiber of

$$M \to G/K \to G/H$$

Note that $G/K \to G/H$ is a covering G-map, hence the composition $M \to G/K \to G/H$ remains a G-manifold bundle i.e. an $\mathcal{O}_G$-manifold.

6.2 The G-∞-category of framed G-manifolds

Following the non-parametrized and non-equivariant setting we would like to get to the notion of framing on G-manifolds. By "framing" we mean the G-manifolds with tangential structure. Note that this term differs in the literature. Here, we focus on the framing in the sense of Ayala and Francis [AF15] and consequently Horev [Hor19].
Similarly as in [MS74], the tangential structure on a n-dimensional G-manifold M is given by a G-map $\tau_M : M \to BO_n(G)$ (see also [Wan80]). In this section we will define the G-∞-groupoid $\underline{Rep}_n(G)$ that can be identified with the G-space $BO_n(G)$ that classifies G-vector bundles and we will define the G-tangent classifier map in the higher parametrized setting.

G-tangent classifier

In the non-equivariant (and non-parametrized) case the tangent classifier map is given by (see [AF15] *2.1*)

$$\tau : Mfld_n \to Top_{/BO(n)}$$

where $Mfld_n$ is the ∞-category of smooth n-dimensional manifolds.
Our task is to find the parametrized version of this construction. There is one difference that we need to address: we are not interested in framings on the underlying G-manifolds of a $\mathcal{O}_G$-manifolds $M \to G/H$, but we are rather interested in the framing on the fibers of $M \to G/H$. Therefore, even though the G-manifold M is of dimension $n + l$ our *fiberwise tangent classifier* map will be given by

$$\tau : Mfld^G \to Top^G_{/BO_n(G)} \tag{6.2}$$

Note that if we, by abuse of notation, write $TM \to M$ for the fiberwise tangent bundle map with respect to $M \to G/H$, then TM_H is an H-representation (a finite dimensional real vector space with H acting by linear isomorphisms), where M_H is a fiber over eH of $M \to G/H$. Therefore, $TM \to M$ is a G-vector bundle map i.e. a vector bundle map which is a G-map such that G acts on TM by bundle maps (linear actions on each fiber) and is hence classified by a map (6.2).

We define the following:

Definition 6.2.1. Let $\underline{Rep}_n(G) \subset \underline{Mfld}^G$ be the full G-∞-subcategory spanned by the $\mathcal{O}_G$-manifolds $E \to O$ which are G-vector bundles.

The G-∞-category $\underline{Rep}_n(G)$ is important since it will play a major role in the parametrized version of (6.2).
Let us denote with $Vect_n(G)_{/B}$ the category of n-dimensional G-vector bundles over the G-space B. Since G-vector bundles are stable under pullbacks we have the following:

Proposition 6.2.2. *([TD87]* I 9.2*) Let $H \subseteq G$ be a subgroup of G. For an H-space X and a G-vector bundle $p : E \to G \times_H X$ the assignment $p \mapsto p_X$, where p_X is the restriction of p to $X \cong H \times_H X$ represents an equivalence between the category of G-vector bundles over $G \times_H X$ and H-vector bundles over X. In particular, we have an equivalence $Vect_n(G)_{/(G/H)} \simeq Vect_n(H)_{/*} \simeq Rep_n(H)$ from the category of G-vector bundles over the orbit G/H to the category of H-representations. An inverse is given by sending a representation V of H to its topological induction $G \times_H V$.*

This proposition is very important since it helps us to prove that $\underline{Rep}_n(G)$ is in fact a G-∞-groupoid. But first, we will need a notion of the equivariant version of Kister-Mazur theorem (see [Kis64]).

Proposition 6.2.3. *Let $H \leq G$ be a finite subgroup of a compact lie group G, and let V be an n-dimensional H-representation. Then the spaces $Aut^H(V)$ and $Emb^H(V,V)$ endowed with the weak C^∞-topology are weakly equivalent.*

Proof. By translation, we have a weak equivalence of the spaces

$$Emb_0^H(V,V) \hookrightarrow Emb^H(V,V), \text{ and } Aut_0^H(V) \hookrightarrow Aut^H(V)$$

where $Emb_0^H(V,V)$ (resp. $Aut_0^H(V)$) are embeddings (resp. automorphisms) which preserve the origin. We will show that both these spaces are weakly equivalent to $GL_n(V)$.
In order to show that $GL_n(V) \to Emb_0^H(V,V)$ is a weak equivalence it will suffice to show the existence of the dashed lift of the diagram

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & GL_n(V) \arrow[d] \\ D^{k+1} \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & Emb_0^H(V,V) \end{tikzcd}$$

for $k \geq -1$ (with $S^{-1} = \emptyset$). We will write $f_d = f(d) \in Emb_0^H(V,V)$. Informally, we want to transform the functions f_d into linear functions. In addition, we want our transformation to send f_d into f_d in case f_d is already a linear function. We proceed with the following:

Take $\mu : [0,+\infty) \to [0,1]$ to be a decreasing smooth function which is equal to 1 near 0 and which is 0 on $[1,+\infty)$. Define

$$F_d(x,t) = (1 - t\mu(\frac{||x||}{\varepsilon}))f_d(x) + t\mu(\frac{||x||}{\varepsilon})Df_d(0)x$$

where $t \in [0,1]$, $x \in V$ and where $Df_d(0)$ is the derivative at 0 of f_d. Note that $F_d(x,t) = f_d(x)$ whenever $||x|| \geq \varepsilon$. Additionally, since D^{k+1} is compact, we can write

$$||DF_d(x,t) - Df_d(x)|| < C\varepsilon$$

for some $C > 0$, $||x|| \leq \varepsilon$ and for all $d \in D^{k+1}$. Hence, choosing ε small enough we can conclude $||DF_d(x,t)|| \neq 0$ for every $x \in V$, $t \in [0,1]$, $d \in D^{k+1}$, hence, they will all be embeddings (since they are when $t = 0$). Let us denote with $\bar{f}_d(x) = F_d(x,1)$. Note that, if f_d is linear, then $F_d(x,t) = f_d(x)$ for every $t \in [0,1]$, hence, in particular $\bar{f}_d(x) = f_d(x)$.
Next, define the function $F'_d(x,t)$ to be

$$\begin{array}{c} Df_d(0)x, \text{ for } ||x|| \leq \varepsilon, t < 1 \text{ or } t = 1 \\ (1-t)\bar{f}_d(\frac{x}{1-t}), \text{ for } ||x|| > \varepsilon, t < 1 \end{array}$$

Again, note that if $\bar{f}_d$ is linear, then $F'_d(x,t) = \bar{f}_d(x)$ for every $t \in [0,1]$.
Finally, define $\widetilde{f}_d = F'_d(x,1)$. With this, we have shown that $GL_n(V) \to Emb_0^H(V,V)$ is a weak equivalence.

The proof that $GL_n(V) \to Aut_0^H(V)$ is analogous. We consider a diagram

$$\begin{tikzcd} S^k \arrow[r] \arrow[d] & GL_n(V) \arrow[d] \\ D^{k+1} \arrow[r, "f"] & Aut_0^H(V) \end{tikzcd}$$

Since all f_d are now diffeomorphisms, then so will $F_d(-,t)$ and $F'_d(-,t)$ be for all $t \in [-1,1]$. $\square$

We are ready to present the following:

Proposition 6.2.4. *The G-∞-category $\underline{Rep}_n(G)$ is a G-∞-groupoid. In particular, for an orbit G/H, the objects of the fiber $\underline{Rep}_n(G)_{[G/H]}$ can be viewed as n-dimensional real H-representations. The mapping space between two such elements V, W is equivalent to the mapping space $Iso_{Rep_n(H)}(V,W)$ of linear, H-equivariant isomorphisms endowed with a weak C^∞-topology.*

Proof. By [HTT] *2.4.2.4* it will suffice to show that the fibers $\underline{Rep}_n(G)_{[O]}$ are ∞-groupoids, hence the coCartesian fibration $\underline{Rep}_n(G) \to \mathcal{O}_G^{op}$ is a left fibration i.e. a G-∞-groupoid. By taking a basepoint $x \in O$ and using $\underline{(O,x)} \cong G/H$ we have $\underline{Rep}_n(G)_{[G/H]} \simeq Rep_n(H)$ by taking the fibers over eH. The claim now follows from 6.2.3. $\square$

Recall that the classifying space for G-vector bundles is a G-space $BO_n(G)$ with the following property:
For every G-space X we have the following isomorphism

$$[X, BO_n(G)]_G \cong Ob(Vect_n(G)_{/X})$$

between the G-homotopy classes of maps $X \to BO_n(G)$ and the G-vector bundles over X.

Proposition 6.2.5. *The G-∞-groupoid $\underline{Rep}_n(G)$ corresponds to $BO_n(G) \in Top^G$, the classifying G-space of n-dimensional G-vector bundles.*

Proof. Space $BO_n(G)$ classifies the functor $O \mapsto Map_G(O, BO_n(G))$, but $Map_G(O, BO_n(G))$ classifies all G-vector bundles over O i.e. $Map_G(O, BO_n(G)) \simeq \underline{Rep}_n(G)_{[O]}$. Therefore, the coCartesian fibration $\underline{BO_n(G)} \to \mathcal{O}_G^{op}$ induced by $BO_n(G)$ is classified by the same functor as $\underline{Rep}_n(G) \to \mathcal{O}_G^{op}$, hence the conclusion. $\square$

Moving on with the parametrized construction of the tangent classifier map, we have to replace Top^G with a suitable parametrized invariant. Note that the model for $\underline{Top}^G$ from 5.3.6 is not a good one since we need the G-∞-category of G-∞-groupoids in the place of Top^G. We provide the following construction:

Construction 6.2.6. Let us denote with Grp^G the ∞-category obtained as the nerve of a simplicial category whose objects are given by left fibrations $X \to \mathcal{O}_G^{op}$. Note that we have a functor $Lf : Top^G \to Grp^G$ that sends a G-space X to a left fibration $Lf(X) \to \mathcal{O}_G^{op}$ classified by a functor $O \mapsto Map_G(O, X)$. We proceed with the construction of G-∞-category of G-∞-groupoids in a similar manner as in 5.3.6.
Denote with $\mathcal{O}_G - Grp$ the nerve of the simplicial category whose objects are pairs $(O, X \to Lf(O))$ where $O \in \mathcal{O}_G$ and where $X \to Lf(O)$ is a left fibration. The forgetful functor $\mathcal{O}_G - Grp \to \mathcal{O}_G$ is a Cartesian fibration and we obtain $\underline{Grp}^G \to \mathcal{O}_G^{op}$ from the construction of a dual.
The functor Lf induces a functor $\mathcal{O}_G - Top \to \mathcal{O}_G - Grp$ which further induces a functor $Lf_* : \underline{Top}^G \to \underline{Grp}^G$ sending $(X \to O)$ to the pair $(O, Lf(X) \to Lf(O))$. This functor restricts to an equivalence on a full G-∞-subcategory of $\underline{Top}^G$ spanned by elements $(X \to O)$ where X is a G-CW-complex.

Remark 6.2.7. Note that $Lf(O) \simeq \underline{O}$ since they are classified by the same functor.

With all categories in place we would like to construct a map

$$\tau : \underline{Mfld}^G \to \underline{Grp}^G_{/\underline{BO_n(G)}}$$

which corresponds to the G-tangent bundle classifying map.

We need to prepare ourselves a little bit more in order to continue our construction

Definition 6.2.8. We will denote with $\mathcal{O}_G - Rep$ the full subcategory of $\mathcal{O}_G - Mfld$ spanned by those objects $E \to O$ which are G-vector bundles.

As a direct corollary of 6.2.4 we have:

Lemma 6.2.9. *The natural map $\mathcal{O}_G - Rep^{op} \to \underline{Rep}_n(G)$ is an equivalence.*

Proof. It would be enough to verify that $(\mathcal{O}_G - Rep^{op})_{[O]} \simeq \underline{Rep}_n(G)_{[O]}$ for every $O \in \mathcal{O}_G$ which follows from the description of mapping spaces in $\underline{Rep}_n(G)$ from 6.2.4. $\square$

Construction 6.2.10. Let $(M \to O) \in \underline{Mfld}^G_{[O]}$ and let $D(M) \to \mathcal{O}_G - Rep^{op}$ be a left fibration classified by a functor $(E \to W) \mapsto Map_{\mathcal{O}_G - Mfld}(E \to W, M \to O)$. The forgetful functor $Map_{\mathcal{O}_G - Mfld}(E \to W, M \to O) \to Map_G(W, O)$ induces a diagram

$$\begin{tikzcd} D(M) \arrow[r] \arrow[d] & \mathcal{O}_G - Rep^{op} \arrow[d] \\ Lf(O) \arrow[r] & \mathcal{O}_G^{op} \end{tikzcd}$$

We first note that $Lf(O) \simeq \underline{O}$ and then, with 6.2.9 we obtain the diagram

$$\begin{tikzcd} D(M) \arrow[r] \arrow[d] & \underline{Rep}_n(G) \arrow[d] \\ \underline{O} \arrow[r] & \mathcal{O}_G^{op} \end{tikzcd}$$

Next, we see that the composition $D(M) \to \underline{O} \to \mathcal{O}_G^{op}$ makes $D(M)$ into a G-∞-groupoid since it is a composition of two left fibrations, hence a left fibration, and we have $D(M) \in \underline{Grp}^G_{[O]}$. Moreover, the structure map $D(M) \to \underline{O}$ factors through $D(M) \to \underline{O} \underline{\times} \underline{Rep}_n(G) \to \underline{O}$ hence we have constructed an object living in $(\underline{Grp}^G_{/\underline{Rep}_n(G)})_{[O]}$. This fiberwise functor $\underline{Mfld}^G_{[O]} \to (\underline{Grp}^G_{/\underline{Rep}_n(G)})_{[O]}$ assembles into a G-functor

$$\tau : \underline{Mfld}^G \to \underline{Grp}^G_{/\underline{Rep}_n(G)} \simeq \underline{Grp}^G_{/\underline{BO_n(G)}} \tag{6.3}$$

Proposition 6.2.11. *The functor τ in* (6.3) *corresponds to the G-tangent bundle classifying functor.*

Proof. For $(M \to O) \in \underline{Mfld}^G_{[O]}$ we have the corresponding left fibration $D(M) \to \underline{O}$ which factors as

$$D(M) \to \underline{O} \underline{\times} \underline{Rep}_n(G) \to \underline{O}$$

We have to show that this map corresponds to the fiberwise tangent bundle classifier of M

$$\tau : M \to BO_n(G)$$

Since the left fibration $\underline{O} \underline{\times} \underline{Rep}_n(G) \to \underline{O}$ is obtaine via the pullback of the left fibration $\underline{Rep}_n(G) \to \mathcal{O}_G^{op}$ along $\underline{O} \to \mathcal{O}_G^{op}$, they are classified by the same G-space, which is $BO_n(G)$ by 6.2.5. Therefore, we need to prove that $D(M)$ corresponds to M. In other words, we need to prove the following equivalence

$$D(M)_{[W]} \simeq Map_G(W, M)$$

where $(W \to O) \in \underline{O}$. Without the loss of generality, it will suffice to prove the statement for $(O \xrightarrow{=} O) \in \underline{O}$. Unwinding the definition of $D(M)$ we see that, by [HTT] *3.3.4.5* the fiber $D(M)_{[O]}$ is equivalent to

$$D(M)_{[O]} \simeq \underset{(E \to O) \in \underline{Rep}_n(G)}{colim} Emb^{\mathcal{O}_G}(E, M)$$

Taking a basepoint $x \in O$ and identifying $(O, x) \cong G/H$ we obtain the following equivalent expression by taking the fibers over x:

$$D(M)_{[O]} \simeq \underset{V \in Rep_n(H)}{colim} Emb^H(V, N)$$

where N is the H-manifold obtained as the fiber of $M \to O$ over x. We need to show that this colimit is equivalent to $Map_G(O, M) \simeq Map_H(*, N) \simeq N^H$. Note that the upper colimit is the colimit in the ∞-categorical sense. Equivalently, we would have to prove the equivalence

$$\underset{V \in Rep_n(H)}{hocolim} Emb^H(V, N) \xrightarrow{\simeq} Map_H(*, N)$$

in the category of spaces. Note that by the equivariant version of Kister-Mazur theorem (see 6.2.3), the colimit in our equation decomposes as a disjoint union indexed by the isomorphism classes of H-representations, hence we can focus only on one such class, and it will suffice to show $\underset{Aut^H(V)}{hocolim} Emb^H(V, N) \xrightarrow{\simeq} Map_H(*, N)$.

We can replace $Emb^H(V, N)$ with the equivalent space of frames $Fr^H(V, N)$ (see 7.3), which we can regard as the space of H-vector bundle maps

$$\begin{array}{ccc} V & \longrightarrow & TN \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & N \end{array}$$

where we have an induced map $Fr^H(V, N) \to Map_H(*, N)$ given by the downstairs map of the diagram $* \to N$, hence a map $\underset{Aut^H(V)}{hocolim} Fr^H(V, N) \to Map_H(*, N)$. We want to show that this map is in fact a weak homotopy equivalence. Note that the homotopy colimit is given by a diagram of a group action of $Aut^H(V)$ on $Fr^H(V, N)$ by precomposition, hence $\underset{Aut^H(V)}{hocolim} Fr^H(V, N)$ is given by the homotopy quotient $Fr^H(V, N)_{Aut^H(V)}$.

In order to show that the map $Fr^H(V, N)_{Aut^H(V)} \to Map_H(*, N)$ is a weak homotopy equivalence it will sufice to show that the homotopy fiber of the map $Fr^H(V, N) \to Map_H(*, N)$ is weakly equivalent to $Aut^H(V)$. Next, note that $Map_H(*, N)$ represents the equivariant configuration space, hence the map $Fr^H(V, N) \to Map_H(*, N)$ is a fibration, meaning that fibers and homotopy fibers coincide.

Let us fix $e \in Map_H(*, N)$. Note that $e(*) = y \in N^H$. Therefore $TN_y \cong V$ as an H-representation and we have that the fiber of $Fr^H(V, N) \to Map_H(*, N)$ over e is equivalent to $Fr^H(V, N)_e \simeq Aut_0^H(V)$. The claim follows from the equivalence $Aut^H(V) \simeq Aut_0^H(V)$ given by translations. $\square$

Framing on G-manifolds

Following our previous work, we are ready to define the G-∞-category of framed G-manifolds. We start with the definition of a framing:

Definition 6.2.12. Let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. Let M be a G-manifold. The B-framing on M is given by a G-map $f_M : M \to B$ together with a G-homotopy commutative diagram

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ M \arrow[ur, "f_M"] \arrow[rr, "\tau_M"] & & BO_n(G) \end{tikzcd}$$

where $M \xrightarrow{\tau_M} BO_n(G)$ is a tangent bundle classifier map.

Definition 6.2.13. Let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. The G-∞-category of B-framed $\mathcal{O}_G$-manifolds is defined to be a pullback

$$\begin{tikzcd} \underline{Mfld}^{G,B-fr} \arrow[r] \arrow[d] & \underline{Grp}^{G}_{/\underline{B}} \arrow[d, "f^*"] \\ \underline{Mfld}^{G} \arrow[r, "\tau"] & \underline{Grp}^{G}_{/\underline{BO_n(G)}} \end{tikzcd}$$

Moreover, for $O \in \mathcal{O}_G$ and for two B-framed $\mathcal{O}_G$-manifolds $M, N \in \underline{Mfld}^{G,B-fr}_{[O]}$ the mapping space $Emb^{G,B-fr}_{O}(M,N)$ is obtained via the homotopy pullback

$$\begin{tikzcd} Emb^{G,B-fr}_{O}(M,N) \arrow[r] \arrow[d] & Map^{G}_{/(B\times O)}(M \xrightarrow{\widetilde{f_M}} B\times O, N \xrightarrow{\widetilde{f_N}} B\times O) \arrow[d] \\ Emb^{G}_{O}(M,N) \arrow[r, "\tau"] & Map^{G}_{/(BO_n(G)\times O)}(M \xrightarrow{\widetilde{\tau_M}} BO_n(G)\times O, N \xrightarrow{\widetilde{\tau_N}} BO_n(G)\times O) \end{tikzcd}$$

Definition 6.2.14. Analogous to G-manifolds, we can define the G-∞-category $\underline{Rep}^{B-fr}_{n}(G)$ as a pullback

$$\begin{tikzcd} \underline{Rep}^{B-fr}_{n}(G) \arrow[r] \arrow[d] & \underline{Mfld}^{G,B-fr} \arrow[d, "f^*"] \\ \underline{Rep}_{n}(G) \arrow[r, "\tau"] & \underline{Mfld}^{G} \end{tikzcd}$$

Proposition 6.2.15. *The G-∞-category $\underline{Rep}^{B-fr}_{n}(G)$ corresponds to the G-space B.*

Proof. Left fibration corresponding to G-space B can be described as $O \mapsto Map_G(O,B)$, where $O \in \mathcal{O}_G$, which corresponds to the classification of B-framed G-vector bundles over O by composition $O \to B \to BO_n(G)$. □

Remark 6.2.16. Let D be a G-space. Then D defines a G-object in $\underline{Top}^{G}$. By definition (see [BDGNS16] *section 7*), this G-object is a coCartesian section $\underline{D} : \mathcal{O}_G^{op} \to \underline{Top}^{G}$ given by

$$\underline{D}(O) := (D \times O \to O)$$

Therefore an object of $(\underline{Top}^G_{/\underline{D}})_{[O]} \simeq \underline{Top}^G_{[O]/\underline{D}(O)} \simeq Top^G{}_{/(D\times O\to O)}$ is given by a G-space X together with G-maps $X \to O$ and $X \to D$.

Example 6.2.17. *One of the most interesting examples of framing is when the G-space B is taken to be a point $B = *$. The map $* \to BO_n(G)$ classifies a G-vector bundle over the point i.e. an n-dimensional G-representation $V \to *$. Therefore, the tangent bundle on $M \in \underline{Mfld}^{G,B-fr}$ is given by a commutative diagram*

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & EO_n(G) \arrow[d] \\ M \arrow[r] & * \arrow[r] & BO_n(G) \end{tikzcd}$$

where both inner right and left squares are pullback squares. We will call this the V-framing on $\underline{Mfld}^G$. Finally, note that V-framing corresponds to the trivialization of the fiberwise tangent bundle $TM \cong M \times V$ as a G-vector bundle.

Example 6.2.18. *By taking $B = BO_n(G)$ and $f : BO_n(G) \xrightarrow{=} BO_n(G)$ we get to the notion of G-∞-category of G-manifolds with no tangential structure in which case we have an equivalence $\underline{Mfld}^{G,B-fr} \simeq \underline{Mfld}^G$.*

Remark 6.2.19. For more example, such as orientations of G-vector bundles (see also [CMW01]) and G-∞-category of G-manifolds with free G-action, see [Hor19] *3.3.*

6.3 G-symmetric monoidal structure

In this section, we will endow the G-∞-category of G-manifolds with a G-symmetric monoidal structure. In particular, we will construct a new G-symmetric monoidal category $\underline{Mfld}^{G,\sqcup}$ whose underlying G-∞-category will be $\underline{Mfld}^G$.
After that, we will take the framing into account.

In order to construct the G-symmetric monoidal category of G-manifolds, we would like to use the unfurling construction of Barwick (see [Bar14], in particular *section 11*). In order to do so we will first introduce the category $\mathcal{O}_G - Fin - Mfld$ based on which we will build our category:

Definition 6.3.1. Let us denote with $\mathcal{O}_G - Fin - Mfld$ the topological category whose objects consist of a sequence $M \to U \to O$ where M is a G-manifold, U is a finite G-set and $O \in \mathcal{O}_G$ such that the composite map $M \to O$ is a G-manifold bundle map. We will say that $(M \to U \to O)$ is a $\mathcal{O}_G - Fin$-manifold. Let $(M_1 \to U_1 \to O_1)$ and $(M_2 \to U_2 \to O_2)$ be two $\mathcal{O}_G - Fin$-manifolds. A morphism of $\mathcal{O}_G - Fin$-manifolds consists of a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & M_2 \arrow[d] \\ U_1 \arrow[r, "\widetilde{\varphi}"] \arrow[d] & U_2 \arrow[d] \\ O_1 \arrow[r, "\varphi"] & O_2 \end{tikzcd}$$

where the lower square is a map of finite G-sets. Additionally, we will require that the induced map $M_1 \to O_1 \times_{O_2} M_2$ is fiberwise a G-embedding map.
We will denote with $Map_{\mathcal{O}_G - Fin_G}(U_1, U_2)$ the subspace of $Map_G(O_1, O_2) \times Map_G(U_1, U_2)$ such that lower square in the upper diagram commutes, with inherited weak C^∞-topology. Similarly, we will denote with $Emb^{\mathcal{O}_G - Fin}(M_1, M_2)$ the subspace of $Map_{\mathcal{O}_G - Fin}(U_1, U_2) \times C^\infty(M_1, M_2)$ forming commutative diagrams as above, with the inherited weak C^∞-topology.

Remark 6.3.2. Let $M \to U \to O$ be an $\mathcal{O}_G - Fin$-manifold. The idea of this construction is to view this $\mathcal{O}_G - Fin$-manifold as the disjoint union of $M_W \to W \to O$, where $W \in Orbit(U)$ and M_W is the restriction of $M \to U$ to W. Note that the composite map $M_W \to W \to O$ is a G-manifold bundle map. Moreover, $W \to O$ is a covering map, and as in 6.1.13, we can look at $M_W \to W \to O$ as the topological induction of $M_W \to W$ along $W \to O$. In other words, $M_W \to W$ is an $\mathcal{O}_G$-manifold. This fact will prove useful in our later work.

Again, we have a forgetful functor $q : \mathcal{O}_G - Fin - Mfld \to \mathcal{O}_G - Fin_G$ which we would like to use for our unfurling construction. The unfurling construction requires additional collection of data on $\mathcal{O}_G - Fin_G$. In particular, we need a pair of subcategories containing all the equivalences. The morphisms of the first and second subcategory will be called respectively:

- **ingressive**: In our case those will be morphisms in $\mathcal{O}_G - Fin_G$ which are isomorphisms over orbits;
- **egressive**: Morphisms in $\mathcal{O}_G - Fin_G$ containing summand inclusions.

The first step toward our construction is:

Lemma 6.3.3. *The forgetful functor $N(q) : N(\mathcal{O}_G - Fin - Mfld) \to N(\mathcal{O}_G - Fin_G)$ is an inner fibration.*

Proof. Similarly to the proof of 6.1.7 it will be enough to show that the map

$$\begin{gathered} Map_{Sing(\mathcal{O}_G - Fin - Mfld)}(M_1 \to U_1 \to O_1, M_2 \to U_2 \to O_2) \\ \downarrow \\ Map_{Sing(\mathcal{O}_G - Fin_G)}(U_1 \to O_1, U_2 \to O_2) \end{gathered}$$

is a Kan fibration. Note that this map breaks as a disjoint union of

$$\begin{gathered} (Map_{Sing(\mathcal{O}_G - Fin - Mfld)}(M_1 \to U_1 \to O_1, M_2 \to U_2 \to O_2))_{[\widetilde{\varphi}]} \\ \downarrow \\ (Map_{Sing(\mathcal{O}_G - Fin_G)}(U_1 \to O_1, U_2 \to O_2))_{[\widetilde{\varphi}]} \end{gathered}$$

indexed by $[\widetilde{\varphi}] = \pi_0(\widetilde{\varphi}, \varphi) \in \pi_0(Map_{Sing(\mathcal{O}_G - Fin_G)}(U_1 \to O_1, U_2 \to O_2))$. Suppose that these maps are all Kan fibrations. Then, for $n \geq 1$ and $0 \leq i \leq n$, the dotted lift in the following diagram exists

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & Map_{Sing(\mathcal{O}_G - Fin - Mfld)}(M_1, M_2) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Map_{Sing(\mathcal{O}_G - Fin_G)}(U_1 \to O_1, U_2 \to O_2) \end{tikzcd}$$

since the diagram factors as

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & (Map_{Sing(\mathcal{O}_G-Fin-Mfld)}(M_1, M_2))_{[\widetilde{\varphi}]} \arrow[r] \arrow[d] & Map_{Sing(\mathcal{O}_G-Fin-Mfld)}(M_1, M_2) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & (Map_{Sing(\mathcal{O}_G-Fin_G)}(U_1 \to O_1, U_2 \to O_2))_{[\widetilde{\varphi}]} \arrow[r] & Map_{Sing(\mathcal{O}_G-Fin_G)}(U_1, U_2)
\end{tikzcd}$$

for some $[\widetilde{\varphi}] \in \pi_0(Map_{Sing(\mathcal{O}_G-Fin_G)}(U_1 \to O_1, U_2 \to O_2))$. Therefore, it will suffice to show that

$$\begin{tikzcd}
(Map_{Sing(\mathcal{O}_G-Fin-Mfld)}(M_1 \to U_1 \to O_1, M_2 \to U_2 \to O_2))_{[\widetilde{\varphi}]} \arrow[d] \\
(Map_{Sing(\mathcal{O}_G-Fin_G)}(U_1 \to O_1, U_2 \to O_2))_{[\widetilde{\varphi}]}
\end{tikzcd}$$

are all Kan fibrations. We will show that

$$q_{[\widetilde{\varphi}]} : (Map_{\mathcal{O}_G-Fin-Mfld}(M_1, M_2))_{[\widetilde{\varphi}]} \to (Map_{\mathcal{O}_G-Fin_G}(U_1 \to O_1, U_2 \to O_2))_{[\widetilde{\varphi}]}$$

are all fiber bundle maps, hence Serre fibrations. Let us fix $(\widetilde{\varphi}, \varphi) \in Map_{\mathcal{O}_G-Fin_G}(U_1 \to O_1, U_2 \to O_2)$. Consider a commutative diagram

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[r, "\widetilde{\varphi}"] \arrow[d] & U_2 \arrow[d] \\
O_1 \arrow[r, "\varphi"] & O_2
\end{tikzcd}$$

As in 6.1.7 the map φ is completely determined by the image $\varphi(x) = y$ for some chosen $x \in O_1$. We will mark $H = Stab(x)$ and $K = Stab(y)$. Since $U_1 \to O_1$ and $U_2 \to O_2$ are covering maps, we can mark their fibers with S and T respectively. Since U_1 and U_2 are finite G-sets i.e. equivalent to finite disjoint union of orbits, S and T will be finite sets. Moreover, S will have induced H-action and T will be equipped with K-action. Now again, by equivariance, the map $\widetilde{\varphi}$ will be completely determined by the restriction $\xi : S \to T$ (which is an H-map) to the fibers of x and y. Then the upper diagram restricts to

$$\begin{tikzcd}
N_1 \arrow[r, "f"] \arrow[d] & N_2 \arrow[d] \\
S \arrow[r, "\xi"] \arrow[d] & T \arrow[d] \\
\{x\} \arrow[r, "\varphi"] & \{y\}
\end{tikzcd}$$

where $N_1 = (M_1)_x$ and $N_2 = (M_2)_y$. Note that the map f corresponds to an H-embedding $N_1 \to N_2$, but also note that f cannot be any H-embedding, it lies over the map $\bar{\xi} : Orbit_H(S) \to Orbit_K(T)$

$$\begin{tikzcd} N_1 \arrow[r, "f"] \arrow[d] & N_2 \arrow[d] \\ S \arrow[r, "\xi"] \arrow[d] & T \arrow[d] \\ Orbit_H(S) \arrow[r, "\bar{\xi}"] & Orbit_K(T) \end{tikzcd}$$

where $Orbit_H(S)$ is the set of H-orbits of S and $Orbit_K(T)$ is the set of all K-orbits of T. Furthemore, $\bar{\xi}$ is determined by $[\widetilde{\varphi}] \in \pi_0(Map_{\mathcal{O}_G-Fin_G}(U_1 \to O_1, U_2 \to O_2))$. We will denote with

$$Emb^H_{[\widetilde{\varphi}]}(N_1, N_2)$$

all such H-embeddings of N_1 into N_2.

Since the topology on $(Map_{\mathcal{O}_G-Fin_G}(U_1, U_2))_{[\widetilde{\varphi}]}$ is inherited from $Map_G(O_1, O_2) \times Map_G(U_1, U_2)$ we can choose a suitable neighborhood V of $(\widetilde{\varphi}, \varphi)$ such that the projection on the level of orbits represents a neighborhood W of φ as in 6.1.7 with the addition that the inverse of W_y of the covering map $U_2 \to O_2$ can be written as $W_y \times T$. To add up, the maps $(\widetilde{\psi}, \psi) \in V$ are such that $\pi_0(\widetilde{\psi}) = \pi_0(\widetilde{\varphi})$. Now the map $(f, \widetilde{\psi}, \psi) \in Emb^{\mathcal{O}_G-Fin}(M_1, M_2)$ with $(\widetilde{\psi}, \psi) \in V$ restricts to

$$\begin{tikzcd} N_1 \arrow[r, "\widetilde{f}"] \arrow[d] & W_y \times N_2 \arrow[d] \\ S \arrow[r, "\varepsilon"] \arrow[d] & W_y \times T \arrow[d] \\ \{x\} \arrow[r, "\psi"] & W_y \end{tikzcd}$$

There is a unique $g \in W$ such that $\psi = g\varphi$, hence we have

$$\begin{tikzcd} N_1 \arrow[r, "\widetilde{f}"] \arrow[d] & N_2 \arrow[d] \\ S \arrow[r, "\varepsilon"] \arrow[d] & T \arrow[d] \\ \{x\} \arrow[r, "\psi"] & \{gy\} \end{tikzcd}$$

And the map $(\widetilde{\psi}, \psi)$ is completely determined by the lower square of this diagram. Therefore, we have

$$q^{-1}_{[\widetilde{\varphi}]}(V) \cong V \times Emb^H_{[\widetilde{\varphi}]}(N_1, N_2)$$

meaning that $q_{[\widetilde{\varphi}]}$ is a fiber bundle map and consequently a Serre fibration. With this the proof is finished.

$\square$

Lemma 6.3.4. *A morphism $(f, \widetilde{\varphi}, \varphi)$ of $N(\mathcal{O}_G - Fin - Mfld)$ in which $(\widetilde{\varphi}, \varphi) \in \mathcal{O}_G - Fin_G{}^{\cup}$ is $N(q)$ - Cartesian if and only if it is equivalent to a pullback over a summand-inclusion.*

Proof. We prove this lemma in several steps:

1. We can factor a morphism between two elements of $\mathcal{O}_G - Fin - Mfld$, $(M_1 \to U_1 \to O_1)$ and $(M_2 \to U_2 \to O_2)$ as

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d] & U_1 \times_{U_2} M_2 \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[r, "="] \arrow[d] & U_1 \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O_1 \arrow[r, "="] & O_1 \arrow[r] & O_2
\end{tikzcd}$$

where the right side is a pullback over a summand inclusion. By [HTT] *2.4.1.7* and *2.4.1.5* it will suffice to prove that the right side is an $N(q)$-cartesian arrow, which we prove in the following step.

2. Consider a map in $\mathcal{O}_G - Fin - Mfld$

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[r, "="] \arrow[d] & U_2 \arrow[d] \\
O_1 \arrow[r, "="] & O_2
\end{tikzcd}$$

where the upper square is a pullback square. We can factor this map as

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d] & O_1 \times_{O_2} M_2 \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[r, hook] \arrow[d] & O_1 \times_{O_2} U_1 \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O_1 \arrow[r, "="] & O_1 \arrow[r] & O_2
\end{tikzcd}$$

Note that the right lower square and right rectangle are pulback diagrams, hence so is the right upper square. Since the upper rectangle is a pullback diagram so is the left upper square. Analogous to 6.1.8 we can show that the right side is $N(q)$-Cartesian taking into account 6.3.3, hence by [HTT] *2.4.1.7*, we need to show that the left side is an $N(q)$-Cartesian morphism.

3. Let $U_1 \coprod U_2$ be a coproduct of finite G-sets U_1 and U_2 (hence, it is a finite G-set). Note that every $\mathcal{O}_G - Fin_G$-manifold $M \to (U_1 \coprod U_2) \to O$ decomposes as $M_1 \sqcup M_2 \to U_1 \coprod U_2 \to O$ where $M_1 \to U_1 \to O$ and $M_2 \to U_2 \to O$ are $\mathcal{O}_G - Fin_G$-manifolds.
Having this in mind, consider the following diagram

$$\begin{tikzcd}
M_1 \arrow[r, hook] \arrow[d] & M_1 \sqcup M_2 \arrow[d] \\
U_1 \arrow[r, hook] \arrow[d] & U_1 \coprod U_2 \arrow[d] \\
O \arrow[r, "="] & O
\end{tikzcd}$$

Note that it is a pullback diagram. We want to show that it is a $N(q)$-Cartesian arrow. By the dual statement of 5.1.8 we have to prove that for every $N \to U \to W$ the diagram

$$\begin{array}{ccc} Map_{\mathcal{O}_G-Fin-Mfld}(N, M_1) & \longrightarrow & Map_{\mathcal{O}_G-Fin-Mfld}(N, M_1 \sqcup M_2) \\ \downarrow & & \downarrow \\ Map_{\mathcal{O}_G-Fin_G}(U \to W, U_1 \to O) & \longrightarrow & Map_{\mathcal{O}_G-Fin_G}(U \to W, U_1 \coprod U_2 \to O) \end{array}$$

is homotopy Cartesian.
By 6.3.3 the left and right vertical arrows are fibrations, hence it will suffice to show the equivalence of the fibers. First note that $W \times_O (M_1 \sqcup M_2) \cong (W \times_O M_1) \sqcup (W \times_O M_2)$ hence, for a map in $Map_{\mathcal{O}_G-Fin-Mfld}(N, M_1)$ the upper horizontal map is determined by a map

$$N \to W \times_O M_1 \hookrightarrow (W \times_O M_1) \sqcup (W \times_O M_2) \cong W \times_O (M_1 \sqcup M_2)$$

which is a fiberwise embedding map. Therefore, for $\Phi \in Map_{\mathcal{O}_G-Fin_G}(U \to W, U_1 \to O)$ the fibers $Emb_{\Phi}^{\mathcal{O}_G}(N, M_1)$ and $Emb_{\Phi \circ q(i)}^{\mathcal{O}_G}(N, M_1 \sqcup M_2)$ are the same, where $i : M_1 \hookrightarrow M_1 \sqcup M_2$ and $q(i)$ is the projection to a map in $\mathcal{O}_G - Fin_G$. With this the proof is finished.

□

Lemma 6.3.5. *A morphism $(f', \widetilde{\varphi}', \varphi')$ in which $(\widetilde{\varphi}', \varphi') \in \mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G^{\sim}$ is $N(q)$-coCartesian if and only if it is G-isotopic to an identity of manifolds over an orbit isomorphism.*

Proof. Let $(M_1 \to U_1 \to O_1)$ and $(M_2 \to U_2 \to O_2)$ be the elements of $\mathcal{O}_G - Fin - Mfld$ such that there is an isomorphism $O_1 \cong O_2$. Then every morphism of these elements lying over the isomorphism of orbits can be factored as

$$\begin{array}{ccccc} M_1 & \longrightarrow & M_1 & \longrightarrow & M_2 \\ \downarrow & & \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 & \xrightarrow{=} & U_2 \\ \downarrow & & \downarrow & & \downarrow \\ O_1 & \xrightarrow{\cong} & O_2 & \xrightarrow{=} & O_2 \end{array}$$

Then, by the dual statements of [HTT] *2.4.1.7* and *2.4.1.5* it will suffice to show that the left side is a $N(q)$-coCartesian morphism.
By 5.1.8 we have to show that for every $(M \to U \to O) \in \mathcal{O}_G - Fin - Mfld$ the commutative diagram

$$\begin{array}{ccc} Map_{\mathcal{O}_G-Fin-Mfld}(M_1, M) & \longrightarrow & Map_{\mathcal{O}_G-Fin-Mfld}(M_1, M) \\ \downarrow & & \downarrow \\ Map_{\mathcal{O}_G-Fin_G}(U_2 \to O_2, U \to O) & \longrightarrow & Map_{\mathcal{O}_G-Fin_G}(U_1 \to O_1, U \to O) \end{array}$$

is in fact a homotopy pullback diagram. By 6.3.3 we know that vertical maps are fibrations, therefore it will suffice to show that the fibers are equivalent. They will, in fact, be the same, since the induced map on fibers is given simply by precomposition $M_1 \xrightarrow{=} M_1 \to M$ on the level of manifolds i.e. we have

$$\begin{tikzcd}
M_1 \arrow[r, "="] \arrow[d] & M_1 \arrow[r] \arrow[d] & M \arrow[d] \\
U_1 \arrow[r] \arrow[d] & U_2 \arrow[r] \arrow[d] & U \arrow[d] \\
O_1 \arrow[r, "\cong"] & O_2 \arrow[r] & O
\end{tikzcd}$$

The inverse will again be the identity on the level of manifolds and is given by the commutativity of the diagram

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O_1 \arrow[r, "\cong"] & O_2
\end{tikzcd}$$

□

Proposition 6.3.6. *The inner fibration $\mathcal{O}_G - Fin - Mfld \to \mathcal{O}_G - Fin_G$ is adequate over the triple*

$$(\mathcal{O}_G - Fin_G, \mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}, \mathcal{O}_G - Fin_G{}^{\cup})$$

in the sense of [Bar14] definition 10.3.

Proof. By [Bar14] *10.3* we need to verify the following:

1. Ingressive morphisms have coCartesian lifts, which is true by 6.3.5;
2. Egressive morphisms have Cartesian lifts, which is true by 6.3.4;
3. In the terminology of Barwick [Bar14], for a given ambigressive pullback diagram in $\mathcal{O}_G - Fin_G$

$$\begin{tikzcd}
I_1 \arrow[r, "\psi_I"] \arrow[d, "\varphi_1"] & I_2 \arrow[d, "\varphi_2"] \\
J_1 \arrow[r, "\psi_J"] & J_2
\end{tikzcd}$$

where ψ_I and ψ_J are ingressive and φ_1 and φ_2 are egressive morphisms, the natural base-change operation $b : (\psi_I)_! \circ \varphi_1{}^* \to \varphi_2{}^* \circ (\psi_J)_!$ is an equivalence.

Let $I_1 = [U_{I1} \to O_{I1}]$, $I_2 = [U_{I2} \to O_{I2}]$, $J_1 = [U_{J1} \to O_{J1}]$, $J_2 = [U_{J2} \to O_{J2}]$ and let $M \to U_{J1} \to O_{J1}$ be an $\mathcal{O}_G - Fin$-manifold above J_1. Then by 6.3.4 and 6.3.5 we have:

$$\varphi_1{}^*(M \to U_{J1} \to O_{J1}) = [U_{I1} \times_{U_{J1}} M \to U_{I1} \to O_{I1}]$$
$$(\psi_I)_! \circ \varphi_1{}^*(M \to U_{J1} \to O_{J1}) = [U_{I1} \times_{U_{J1}} M \to U_{I2} \to O_{I2}]$$

given by a diagram

$$\begin{tikzcd}
M \arrow[d] & U_{I1} \times_{U_{J1}} M \arrow[l] \arrow[r, "="] \arrow[d] & U_{I1} \times_{U_{J1}} M \arrow[d] \\
U_{J1} \arrow[d] & U_{I1} \arrow[l] \arrow[r] \arrow[d] & U_{I2} \arrow[d] \\
O_{J1} & O_{I1} \arrow[l] \arrow[r, "\cong"] & O_{I2}
\end{tikzcd}$$

where the upper left square is a pullback square. For the other operation:

$$(\psi_J)_!(M \to U_{J1} \to O_{J1}) = [M \to U_{J2} \to O_{J2}]$$
$${\varphi_2}^* \circ (\psi_J)_!(M \to U_{J1} \to O_{J1}) = [U_{I2} \times_{U_{J2}} M \to U_{I2} \to O_{I2}]$$

given by a diagram

$$\begin{tikzcd}
M \arrow[r, "="] \arrow[d] & M \arrow[d] & U_{I2} \times_{U_{J2}} M \arrow[l] \arrow[d] \\
U_{J1} \arrow[r] \arrow[d] & U_{J2} \arrow[d] & U_{I2} \arrow[l] \arrow[d] \\
O_{J1} \arrow[r, "\cong"] & O_{J2} & O_{I2} \arrow[l]
\end{tikzcd}$$

where the upper right square is a pullback square.
We obtain the map b by composing the diagrams of φ_1^* and $(\psi_I)_!$ to obtain

$$\begin{tikzcd}
U_{I1} \times_{U_{J1}} M \arrow[r] \arrow[d] & M \arrow[r, "="] \arrow[d] & M \arrow[d] \\
U_{I1} \arrow[r] \arrow[d] & U_{J1} \arrow[r] \arrow[d] & U_{J2} \arrow[d] \\
O_{I1} \arrow[r] & O_{J1} \arrow[r, "\cong"] & O_{J2}
\end{tikzcd}$$

Note that by the commutativity of the ambigressive square we have that

$$U_{I1} \to U_{J1} \to U_{J2} \simeq U_{I1} \to U_{I2} \to U_{J2}$$

which induces a map $\bar{b}$ to the pullback

$$\begin{tikzcd}
U_{I1} \times_{U_{J1}} M \arrow[r, "\bar{b}"] \arrow[d] & U_{I2} \times_{U_{J2}} M \arrow[r] \arrow[d] & M \arrow[d] \\
U_{I1} \arrow[r] \arrow[d] & U_{I2} \arrow[r] \arrow[d] & U_{J2} \arrow[d] \\
O_{I1} \arrow[r, "\cong"] & O_{I2} \arrow[r] & O_{J2}
\end{tikzcd}$$

Our natural base-change operation b is thus obtained as the left side of the following commutative diagram

$$\begin{tikzcd}
U_{I1} \times_{U_{J1}} M \arrow[r, "="] \arrow[d] & U_{I1} \times_{U_{J1}} M \arrow[r, "\bar{b}"] \arrow[d] & U_{I2} \times_{U_{J2}} M \arrow[d] \\
U_{I1} \arrow[r] \arrow[d] & U_{I2} \arrow[r, "="] \arrow[d] & U_{I2} \arrow[d] \\
O_{I1} \arrow[r, "\cong"] & O_{I2} \arrow[r, "="] & O_{I2}
\end{tikzcd}$$

We still need to prove that b is an equivalence. In order to do so, we turn to the diagram

$$\begin{tikzcd}
U_{I1} \times_{U_{J1}} M \arrow[r] \arrow[d] & M \arrow[d] \\
U_{I1} \arrow[r] \arrow[d] & U_{J1} \arrow[d] \\
U_{I2} \arrow[r, "\cong"] & U_{J2}
\end{tikzcd}$$

in which the upper square is a pullback square by the construction of $U_{I1} \times_{U_{J1}} M$ and where the lower square is also a pullback square provided by the fact that that we have been given an ambigressive pullback square. Hence, the outer rectangle is a pullback diagram, which implies $U_{I1} \times_{U_{J1}} M \simeq U_{I2} \times_{U_{J2}} M$ as desired.

□

Definition 6.3.7. Let us denote with $(\mathcal{O}_G - Fin - Mfld)^+ \subset \mathcal{O}_G - Fin - Mfld$ the subcategory consisting of all the objects of $\mathcal{O}_G - Fin - Mfld$ with morphisms the $N(q)$-Cartesian edges over summand inclusions. Additionaly, denote with $(\mathcal{O}_G - Fin - Mfld)_+$ the category

$$\mathcal{O}_G - Fin - Mfld \times_{\mathcal{O}_G - Fin_G} (\mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}) \simeq \mathcal{O}_G - Fin - Mfld \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}$$

Proposition 6.3.8. *The triple $(\mathcal{O}_G - Fin - Mfld, (\mathcal{O}_G - Fin - Mfld)_+, (\mathcal{O}_G - Fin - Mfld)^+)$ is adequate. Moreover, applying the effective Burnside construction to the functor $N(q) : \mathcal{O}_G - Fin - Mfld \to \mathcal{O}_G - Fin_G$, we obtain a functor*

$$\begin{tikzcd}
A^{eff}(\mathcal{O}_G - Fin - Mfld, (\mathcal{O}_G - Fin - Mfld)_+, (\mathcal{O}_G - Fin - Mfld)^+) \arrow[d, "\gamma"] \\
A^{eff}(\mathcal{O}_G - Fin_G, \mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}, \mathcal{O}_G - Fin_G{}^{\cup})
\end{tikzcd}$$

which is a coCartesian fibration. We call γ the unfurling of $N(q)$.

Proof. By (5.1) and 6.3.6 together with [Bar14] *11.2* we have that the triple

$$(\mathcal{O}_G - Fin - Mfld, (\mathcal{O}_G - Fin - Mfld)_+, (\mathcal{O}_G - Fin - Mfld)^+)$$

is adequate, which justifies the construction of the functor γ.

By [Bar14] *11.4* the functor γ is an inner fibration. The structure of a coCartesian fibration is given by [Bar14] *11.5*, 6.3.5 and 6.3.4. □

Definition 6.3.9. Define the ∞-category $\underline{Mfld}^{G,\sqcup}$ together with a coCartesian fibration $\underline{Mfld}^{G,\sqcup} \to \underline{Fin}^G_*$ by a pullback

$$\begin{tikzcd}
\underline{Mfld}^{G,\sqcup} \arrow[r] \arrow[d] & A^{eff}(\mathcal{O}_G - Fin - Mfld, (\mathcal{O}_G - Fin - Mfld)_+, (\mathcal{O}_G - Fin - Mfld)^+) \arrow[d, "\gamma"] \\
\underline{Fin}^G_* \arrow[r, "\simeq"] & A^{eff}(\mathcal{O}_G - Fin_G, \mathcal{O}_G - Fin_G \times_{\mathcal{O}_G} \mathcal{O}_G{}^{\sim}, \mathcal{O}_G - Fin_G{}^{\cup})
\end{tikzcd}$$

As we will later see $\underline{Mfld}^{G,\sqcup}$ has a structure of a symmetric monoidal G-∞-category.

Remark 6.3.10. Unwinding the definitions we can see that the objects in $\underline{Mfld}^{G,\sqcup}$ are $\mathcal{O}_G - Fin$-manifolds, while a morphism between $(M_1 \to U_1 \to O_1)$ and $(M_2 \to U_2 \to O_2)$ is given by a diagram

$$\begin{tikzcd}
M_1 \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2
\end{tikzcd}$$

where the left side is equivalent to a pullback over a summand inclusion. This morphism is coCartesian just in case when the right side is equivalent to the identity of manifolds.

Proposition 6.3.11. *The coCartesian fibration $\underline{Mfld}^{G,\sqcup} \to \underline{Fin}^G_*$ has the structure of a G-symmetric monoidal ∞-category whose underlying G-∞-category is equivalent to $\underline{Mfld}^G$, the G-∞-category of G-manifolds.*

Proof. The underlying G-∞-category of $\underline{Mfld}^{G,\sqcup}$ is obtained via the pullback along the functor $I(-) : \mathcal{O}_G^{op} \to \underline{Fin}^G_*$ given by $I(O) = [O \xrightarrow{=} O]$. Hence, the objects of $(\underline{Mfld}^{G,\sqcup})_{I(-)}$ are given by $M \to O \xrightarrow{=} O$ and the morphisms are given by diagrams

$$\begin{tikzcd}
M_1 \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] & M_2 \arrow[d] \\
O_1 \arrow[d, "="] & O_2 \arrow[l] \arrow[r] \arrow[d, "="] & O_2 \arrow[d, "="] \\
O_1 & O_2 \arrow[l] \arrow[r] & O_2
\end{tikzcd}$$

Where the left side is a pullback diagram. Hence, $(\underline{Mfld}^{G,\sqcup})_{I(-)} \simeq \underline{Mfld}^G$ by the functor $(M \to O \xrightarrow{=} O) \to (M \to O)$.

Next, consider $I = [U \to O] \in \underline{Fin}^G_*$. An object $M \to U \to O$ decomposes as

$$\bigsqcup_{W \in Orbit(U)} M_W \to \coprod_{W \in Orbit(U)} W \to O$$

with $(M_W \to W) \in \underline{Mfld}^G_{[W]}$. Therefore, the functor

$$\prod_{W \in Orbit(U)} (\chi_{[W \subseteq U]})_! : \underline{Mfld}^{G,\sqcup}_{[I]} \to \prod_{W \in Orbit(U)} \underline{Mfld}^G_{[W]}$$

given by

$$(M \to U \to O) \simeq (\sqcup_{W \in Orbit(U)} M_W \to \coprod_{W \in Orbit(U)} W \to O) \mapsto \prod_{W \in Orbit(U)} (M_W \to W)$$

is an equivalence. Hence the statement follows. □

Framed case

For this part, let us fix a G-map $f : B \to BO_n(G)$ with B a G-space. Going further, it would be more conveniend to look at the G-∞-category of G-∞-groupoids $\underline{Grp}^G$ as a subcategory of G-∞-category of G-spaces. We have already mentioned that if we restrict our attention to the subcategory of $\underline{Top}^G$ spanned by those G-spaces of the form $X \to O$ with $O \in \mathcal{O}_G$ and X a G-CW-complex, which we mark with $\underline{Top}^{G-CW}$, we obtain an equivalence $\underline{Grp}^G \simeq \underline{Top}^{G-CW}$. Since we have already seen that the classifying G-space $BO_n(G)$ corresponds to the G-∞-groupoid $\underline{Rep}_n(G)$ we have the following:

Proposition 6.3.12. *The fibrewise tangent bundle classifying map from* (6.3) *is equivalent to the map*

$$\tau : \underline{Mfld}^G \to \underline{Top}^{G-CW}_{/\underline{BO_n(G)}}$$

sending an $\mathcal{O}_G$-manifold $M \to O$ to the G-space $(M \to O) \in \underline{Top}^{G-CW}_{[O]}$ together with the map $\tau_M : M \to BO_n(G)$ classifying the fiberwise tangent vector space of M. Equivalently, the G-∞-category of B-framed G-manifolds fits into the pullback square

$$\begin{tikzcd}
\underline{Mfld}^{G,B-fr} \arrow[r] \arrow[d] & \underline{Top}^{G-CW}_{/\underline{B}} \arrow[d, "f_*"] \\
\underline{Mfld}^G \arrow[r, "\tau"] & \underline{Top}^{G-CW}_{/\underline{BO_n(G)}}
\end{tikzcd}$$

We would like to obtain the G-symmetric monoidal G-∞-category of B-framed G-manifolds in the similar way as in 6.3.12, i.e. via the pullback square. In that case, we would naturally replace $\underline{Mfld}^G$ with $\underline{Mfld}^{G,\sqcup}$. As for $\underline{Top}^{G-CW}_{/\underline{BO_n(G)}}$ and $\underline{Top}^{G-CW}_{/\underline{B}}$ we will replace them with $(\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$ and $(\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg}$ respectively, the G-coCartesian structures on $\underline{Top}^{G-CW}_{/\underline{BO_n(G)}}$ and $\underline{Top}^{G-CW}_{/\underline{B}}$ (respectively). Note that $(\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$ and $(\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg}$ are G-∞-operads, but since $\underline{Top}^{G-CW}_{/\underline{BO_n(G)}}$ and $\underline{Top}^{G-CW}_{/\underline{B}}$ both admit G-coproducts, then by 5.6.5 we have that $(\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$ and $(\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg}$ are in fact G-symmetric monoidal ∞-categories.

We need to show that the pull-back square from 6.2.13 extends to the pull-back square of G-symmetric monoidal ∞-categories (and G-symmetric monoidal functors between them)

$$\begin{tikzcd} \underline{Mfld}^{G,B-fr,\sqcup} \arrow[r] \arrow[d] & (\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg} \arrow[d, "f_*^{\amalg}"] \\ \underline{Mfld}^{G,\sqcup} \arrow[r, "\tau"] & (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg} \end{tikzcd}$$

By 5.6.6 these G-functors extend to lax G-symmetric monoidal functors. It then remains to verify that these lax G-symmetric monoidal functors are G-symmetric monoidal.

Remark 6.3.13. In order to better understand the G-coCartesian structure, one can look at 5.6 (or [NS] to be more precise). In addition, for a more ilustrative depiction of G-coproducts in $\underline{Top}^{G-CW}$ we give the following description of Horev ([Hor19] *3.5.2*):
Let $I = [U \to O] \in \underline{Fin}^G_*$ be a finite G-set. An element of $((\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg})_I$ is given by $X \in Fun_G(\underline{U}, \underline{Top}^{G-CW})$. By definition, a G-coproduct $\coprod_I X : \underline{O} \to \underline{Top}^{G-CW}$ is given by the G-left Kan extension of X along $\underline{U} \to \underline{O}$, which we can by [Shah18] *10.9* look as a non-parametrized left Kan extension. In other words, $\coprod_I$ is left adjoint to the restriction functor along $\underline{U} \to \underline{O}$.
Since we are working with G-∞-groupoids, by Grothendieck-Lurie correspondence, $X : \underline{U} \to \underline{Top}^{G-CW}$ is given by a G-map $X_I \to U$ where X_I is a G-space. Additionally, Grothendieck-Lurie correspondence is compatible with base change, meaning that $\coprod_I$ is now left adjoint to the pull-backing along $U \to O$ and therefore given by the post composition with $U \to O$. In other words, $\coprod_I X : \underline{O} \to \underline{Top}^{G-CW}$ is given by $X_I \to U \to O$ under straightening/unstraightening.

Remark 6.3.14. Additionally, we can give the description of the objects of the G-coCartesian structure on the parametrizad ∞-category $(\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$. For $I = [U \to O] \in \underline{Fin}^G_*$ a finite G-set, an element $X \in Fun_G(\underline{U}, (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg})$ of $((\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg})_I$, is represented by a G-space $X_I \to U \times BO_n(G)$.
The G-coproduct $\coprod_I X : \underline{O} \to (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$ is then represented by a G-space $X_I \to U \times BO_n(G) \to O \times BO_n(G)$. Alternatevly, we can say that $X_I \to U \times BO_n(G) \to O \times BO_n(G)$ (the G-coproduct space) is given by $X_I \to U \to O$ together with a map $X_I \to BO_n(G)$

Lemma 6.3.15. *The G-tangent bundle classifying functor $\tau : \underline{Mfld}^G \to \underline{Top}^{G-CW}_{/\underline{BO_n(G)}}$ extends to a G-symmetric monoidal functor*

$$\tau : \underline{Mfld}^{G,\sqcup} \to (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$$

which we, by abuse of notation, again mark with τ.

Proof. Let us see how τ behaves on objects. Let $I = [U \to O] \in \underline{Fin}^G_*$ and let $(M \to U \to O) \in (\underline{Mfld}^{G,\sqcup})_{[I]}$. Then $\tau(M \to U \to O) \in ((\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg})_{[I]}$ is given by $M \to U$ together with G-tanget classifier $M \to BO_n(G)$, or simply $M \to U \times BO_n(G)$. The G-coproduct is then given simply by postcomposing $M \to U \to O$ together with the map $M \to BO_n(G)$.
Additionally, if we look at the composition map $M \to U \to O$ as an object of $\underline{Mfld}^G_{[O]}$, then $\tau(M \to U \to O)$ again maps to $M \to U \to O$ together with a map $M \to BO_n(G)$ which is compatible with the upper calculation.
Finally, let

$$\begin{tikzcd} M_1 \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] & M_2 \arrow[d] \\ U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

be a coCartesian arrow in $\underline{Mfld}^{G,\sqcup}$. By τ this arrow maps, by inspection, to an arrow

$$\begin{tikzcd} M_1 \arrow[d] \arrow[drrr] & M \arrow[l] \arrow[r] \arrow[d] \arrow[drr] & M_2 \arrow[d] \arrow[dr] & \\ U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] & BO_n(G) \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 & \end{tikzcd}$$

which is again coCartesian. $\square$

Lemma 6.3.16. *Let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. Then the induced functor $f_* : \underline{Top}^{G-CW}_{/\underline{B}} \to \underline{Top}^{G-CW}_{/\underline{BO_n(G)}}$ extends to the G-symmetric monoidal functor*

$$f_*^{\amalg} : (\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg} \to (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}$$

Proof. Let $I = [U \to O]$ be a finite G-set and let $X \in (\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg}$ be an element given by $X_I \to U \times B$. Note that we have a commutative diagram

$$\begin{tikzcd} (X_I \to U \times B) \arrow[r, "f_*"] \arrow[d] & (X_I \to U \times BO_n(G)) \arrow[d] \\ (X_I \to O \times B) \arrow[r, "f_*"] & (X_I \to O \times BO_n(G)) \end{tikzcd}$$

Finally, note that the commutativity of the upper diagram renders the following diagram

$$\begin{tikzcd} (\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg} \arrow[r, "f_*^{\amalg}"] \arrow[d] & (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg} \arrow[d] \\ \underline{Top}^{G-CW}_{/\underline{B}} \arrow[r, "f_*"] & \underline{Top}^{G-CW}_{/\underline{BO_n(G)}} \end{tikzcd}$$

comutative, which finishes the proof. $\square$

Proposition 6.3.17. *Let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. Then the G-symmetric monoidal structure on $\underline{Mfld}^{G,B-fr}$ is given by lifting the G-symmetric monoidal structure on $\underline{Mfld}^{G}$. In particular, we have a pullback diagram*

$$\begin{tikzcd}
\underline{Mfld}^{G,B-fr,\sqcup} \arrow[r] \arrow[d] & (\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg} \arrow[d, "f_*^{\amalg}"] \\
\underline{Mfld}^{G,\sqcup} \arrow[r, "\tau"] & (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}
\end{tikzcd}$$

Proof. Lemmas 6.3.15 and 6.3.16 make the upper diagram commutative. Let

$$\begin{tikzcd}
\mathcal{P}^{\otimes} \arrow[r] \arrow[d] & (\underline{Top}^{G-CW}_{/\underline{B}})^{\amalg} \arrow[d, "f_*^{\amalg}"] \\
\underline{Mfld}^{G,\sqcup} \arrow[r, "\tau"] & (\underline{Top}^{G-CW}_{/\underline{BO_n(G)}})^{\amalg}
\end{tikzcd}$$

be a pullback diagram, which provides us with a map $\underline{Mfld}^{G,B-fr,\sqcup} \to \mathcal{P}^{\otimes}$. Note that passing to the underlying G-∞-category is given by taking a pullback of the structure map along $\underline{Fin}_*^G \to \mathcal{O}_G^{op}$, hence it preserves limits. Therefore

$$\begin{tikzcd}
(\mathcal{P}^{\otimes})_{I(-)} \arrow[r] \arrow[d] & \underline{Top}^{G-CW}_{/\underline{B}} \arrow[d, "f_*"] \\
\underline{Mfld}^{G} \arrow[r, "\tau"] & \underline{Top}^{G-CW}_{/\underline{BO_n(G)}}
\end{tikzcd}$$

is a pullback diagram meaning $\mathcal{P} := (\mathcal{P}^{\otimes})_{I(-)} \simeq \underline{Mfld}^{G,B-fr} \simeq (\underline{Mfld}^{G,B-fr,\sqcup})_{I(-)}$. This is enough to conclude that the map $\underline{Mfld}^{G,B-fr,\sqcup} \to \mathcal{P}^{\otimes}$ is an equivalence since for any $I = [U \to O] \in \underline{Fin}_*^G$

$$(\mathcal{P}^{\otimes})_{[I]} \simeq \prod_{W \in Orbit(U)} \mathcal{P}_{[W]} \simeq \prod_{W \in Orbit(U)} \underline{Mfld}^{G,B-fr}_{[W]} \simeq (\underline{Mfld}^{G,B-fr,\sqcup})_{[I]}$$

$\square$

Chapter 7

G-Discs

In this chapter we will define the G-∞-category of G-discs. Equivalently to the non-equivariant case, G-discs represent the link between the algebra and the geometry of G-manifolds. The G-∞-category of G-disks is used for defining the G-disk algebras which are again used as coefficients for equivariant version of factorization homology. At the same time, G-discs provide insight into geometry of G-manifolds by capturing the local properties. Furthemore, they can be linked with equivariant configuration spaces.

In the first section, we will provide the definition of the G-∞-category of G-discs along with its framed variants and G-symmetric monoidal structure. Later, we will prove that the G-∞-category of G-discs is the G-symmetric monoidal envelope of $\underline{Rep}_n(G)$. Finally, we will finish with the section related to the G-configuration spaces, which will play a role later in this paper, and at the end a short example on G-disc algebras with coefficients in $\underline{Sp}^G$.

7.1 Definition of G-Discs and G-algebras

In the classical (non-equivariant) setting of Ayala and Francis (see [AF15] *2.2*) the discs were simply taken to be disjoint union of n-dimensional Euclidean spaces, which in line with configuration spaces lie over points. In the equivariant theory, points are replaced by orbits and so, in our new definition, G-disc ought to be something lying over an orbit in a similar matter as before.

Consider a map $U \to O$ where $O \in \mathcal{O}_G$ and where U is a finite G-set. Note that this map is a covering map. Therefore for a G-vector bundle $E \to U$ the composite map $E \to U \to O$ is a G-manifold bundle, hence it is an element of $\underline{Mfld}^G$.

Therefore, we arrive at our definition:

Definition 7.1.1. Define a G-disc to be a vector bundle $E \to O$ of rank n where $O \in \mathcal{O}_G$. Denote with $\underline{Disk}^G \subset \underline{Mfld}^G$ the full G-∞-subcategory spanned by objects equivalent to the form $E \to U \to O$ where U is a finite G-set and $E \to U$ is a G-vector bundle.

Remark 7.1.2. Let $E \to U \to O$ be a G-disc. The reason we insert U a finite G-set in the definition of the objects of $\underline{Disk}^G$ lies in the following: as in the non-equivariant case (as in [AF15]), discs are disjoiunt unions of n-dimensional Eucidean spaces. In our case, if we consider a decomposition of a finite G-set U into orbits $U = \coprod_{W \in Orbit(U)} W$, we can view $E \to U \to O$ as a G-disjoint union of G-vector bundles $E_W \to W$. The composition with $W \to O$ exhibits $E_W \to O$ as the topological induction of $E_W \to W$ along $W \to O$.

G-symmetric monoidal structure

Similar to our upper definition of $\underline{Disk}^G$ as a full G-subcategory of $\underline{Mfld}^G$ spanned by G-discs, we would hope that we could do something similar in the case of framing as well as in the case of endowing $\underline{Disk}^G$ with G-symmetric monoidal structure. Although it requires some technical work, the ideas are pretty much straightforward at this point.

Analogous to the G-manifolds, we will consider a G-symmetric monoidal category whose underlying G-∞-category is the G-∞-category of G-discs.

Definition 7.1.3. Let $\underline{Disk}^{G,\sqcup} \subset \underline{Mfld}^{G,\sqcup}$ be a full subcategory spanned by those elements equivalent to $E \to U \to V \to O$ where U, V are finite G-sets, $O \in \mathcal{O}_G$ and where $E \to U$ is a G-vector bundle.

Remark 7.1.4. Note that, given an element $E \to U \to V \to O$, the composite $E \to V \to O$ exibits $E \to U \to V \to O$ as an $\mathcal{O}_G - Fin$-manifold, hence it is truly an element of $\underline{Mfld}^{G,\sqcup}$.

Proposition 7.1.5. *The operation of G-disjoint union on $\underline{Mfld}^{G,\sqcup}$ induces a G-symmetric monoidal structure on the G-∞-subcategory $\underline{Disk}^{G,\sqcup}$.*

Proof. Since we have defined $\underline{Disk}^{G,\sqcup} \subset \underline{Mfld}^{G,\sqcup}$ as a full G-symmetric monoidal category spanned by the G-disjoint union of G-discs it will suffice to show that the underlying G-∞-category of $\underline{Disk}^{G,\sqcup}$ is equivalent to $\underline{Disk}^G$. This is evident since the G-∞-category $(\underline{Disk}^{G,\sqcup})_{I(-)}$ is spanned by elements $E \to U \to O \xrightarrow{=} O$ where $E \to U$ is a G-vector bundle. The equivalence is now induced by functor $(\underline{Disk}^{G,\sqcup})_{I(-)} \to \underline{Disk}^G$ sending $(E \to U \to O \xrightarrow{=} O) \mapsto (E \to U \to O)$. $\square$

Framed G-discs

Definition 7.1.6. Let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. We define the G-∞-categories of B-framed G-discs and disjoint union of B-framed G-discs as the pullbacks

$$\begin{tikzcd}
\underline{Disk}^{G,B-fr} \arrow[r] \arrow[d] & \underline{Mfld}^{G,B-fr} \arrow[d] \\
\underline{Disk}^{G} \arrow[r, hook] & \underline{Mfld}^{G}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{Disk}^{G,B-fr,\sqcup} \arrow[r] \arrow[d] & \underline{Mfld}^{G,B-fr,\sqcup} \arrow[d] \\
\underline{Disk}^{G,\sqcup} \arrow[r, hook] & \underline{Mfld}^{G,\sqcup}
\end{tikzcd}$$

Description 7.1.7. By definition of a G-disc $E \to U \to O$, the map $E \to U$ is a G-vector bundle, therefore, we can write this G-disc in the form

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & BO_n(G) \\
O &
\end{tikzcd}$$

where the horizontal map is a tangent bundle classifing map. Consequently, we can think of a B-framed G-disk in a form of

$$\begin{tikzcd} U \arrow[r] \arrow[d] & B \\ O & \end{tikzcd}$$

where the horizontal map is now the framing map. Note that in both cases we can write these G-discs as $U \to O \times BO_n(G)$ and $U \to O \times B$ respectively.
Reader should keep in mind that this is only the depiction of objects, as the maps between framed discs carry more information than just the map of the underlying finite G-sets.

Definition 7.1.8. Let $\underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal G-∞-category, let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. Define the ∞-category of *B-framed G-algebras with values in* $\underline{\mathcal{C}}^{\otimes}$ to be the ∞-category of G-symmetric monoidal functors $Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$.

Example 7.1.9. *Let $B = BO_n(G)$ and $f : BO_n(G) \xrightarrow{=} BO_n(G)$. Then $\underline{Disk}^{G,B-fr} \simeq \underline{Disk}^{G}$, that is, we have the category of G-discs without framing. In this case we will call the ∞-category of B-framed G-algebras with values in $\underline{\mathcal{C}}^{\otimes}$ simply by the ∞-category of G-disc algebras and mark it with $Fun_G^{\otimes}(\underline{Disk}^{G,\sqcup}, \underline{\mathcal{C}}^{\otimes})$.*
Another important example stems from the case $B = *$. In this situation $* \to BO_n(G)$ corresponds to the n-dimensional G-representation V. We will call $Fun_G^{\otimes}(\underline{Disk}^{G,V-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$ the ∞-category of V-framed G-disc algebras with coefficients in $\underline{\mathcal{C}}^{\otimes}$.

7.2 G-symmetric monoidal envelope

We have said that G-disc algebras are coefficients for the equivariant version of factorization homology. The word "algebra" suggests that an operad is involved, yet we have defined the category of (framed) G-disc algebras simply as category of G-symmetric monoidal functors from the G-∞-category of (framed) G-discs. In this section we show that this name is indeed justified.

Definition 7.2.1. Let $\underline{Rep}_n^{\sqcup}(G)$ be the full G-∞-subcategory of $\underline{Mfld}^{G,\sqcup}$ spanned by the objects of the form $E \to U \to O$ where U is a finite G-set and $E \to U$ is a G-vector bundle.

Lemma 7.2.2. *The map $\underline{Rep}_n^{\sqcup}(G) \to \underline{Fin}_*^G$ is a G-∞-operad. Moreover, the underlying G-∞-category of $\underline{Rep}_n^{\sqcup}(G)$ is equivalent to $\underline{Rep}_n(G)$.*

Proof. It is clear that the map $\underline{Rep}_n^{\sqcup}(G) \to \underline{Fin}_*^G$ is an inner fibration. By definition 5.6.1 we have to prove three points:

1. Let

$$\begin{tikzcd} U_1 \arrow[d] & U_2 \arrow[l] \arrow[r, "="] \arrow[d] & U_2 \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

be an inert arrow in $\underline{Fin}_*^G$ i.e. the right square is a summand inclusion. Let $(E \to U_1 \to O_1) \in \underline{Rep}_n^{\sqcup}(G)$ be an element above $U_1 \to O_1$. The coCartesian lift is given by

$$\begin{tikzcd} E \arrow[d] & U_2 \times_{U_1} E \arrow[l] \arrow[r, "="] \arrow[d] & U_2 \times_{U_1} E \arrow[d] \\ U_1 \arrow[d] & U_2 \arrow[l] \arrow[r, "="] \arrow[d] & U_2 \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

Indeed, the map $U_2 \times_{U_1} E \to U_2$ is obtained via pullback from $E \to U_1$ hence it is a G-vector bundle.

2. Let $I = [U \to O] \in \underline{Fin}_*^G$, and let $(E \to U \to O) \in \underline{Rep}_n^{\sqcup}(G)$. Then by the decomposition $U \simeq \coprod_{W \in Orbit(U)} W$ we have a decomposition

$$(E \to U \to O) \simeq (\bigsqcup_{W \in Orbit(U)} E_W \to \coprod_{W \in Orbit(U)} W \to O)$$

where $E_W \to W$ are all G-vector bundels. Therefore, the functor

$$\prod_{W \in Orbit(U)} (\chi_{[W \subseteq U]})_! : \underline{Rep}_n^{\sqcup}(G)_{[I]} \to \prod_{W \in Orbit(U)} (\underline{Rep}_n^{\sqcup}(G))_{[W]}$$

given by

$$(E \to U \to O) \simeq (\bigsqcup_{W \in Orbit(U)} E_W \to \coprod_{W \in Orbit(U)} W \to O) \mapsto \prod_{W \in Orbit(U)} (E_W \to W \xrightarrow{=} W)$$

is an equivalence.

3. Mark with $e := I_1 = [U_1 \to O_1] \to I_2 = [U_2 \to O_2]$ an arrow in $\underline{Fin}_*^G$ and mark with $x = E_1 \to U_1 \to O_1$ and $y = E_2 \to U_2 \to O_2$ two elements of $\underline{Rep}_n^{\sqcup}(G)$ above I_1 and I_2 respectively. A map in $Map^e_{\underline{Rep}_n^{\sqcup}(G)}(x, y)$ can be written in the form

$$\begin{tikzcd} E_1 \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d] \\ U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

We can again decompose $E_2 \to U_2$ as the disjoint union of G-vector bundles $E_W \to W$ for $W \in Orbit(U_2)$, hence the upper map decomposes as

$$\begin{tikzcd} E_1 \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d] & E_W \arrow[l] \arrow[r, "="] \arrow[d] & E_W \arrow[d] \\ U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] & W \arrow[l] \arrow[r, "="] \arrow[d] & W \arrow[d] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 & W \arrow[l] \arrow[r, "="] & W \end{tikzcd}$$

for every $W \in Orbit(U_2)$, hence we have an equivalence

$$Map^e_{O^\otimes}(x, y) \xrightarrow{\simeq} \prod_{W \in Orbit(U_2)} Map^{\chi_{[W \subseteq U]} \circ e}_{O^\otimes}(x, (\chi_{[W \subseteq U]})_! y)$$

The proof that $\underline{Rep}_n^{\sqcup}(G)$ is a G-∞-operad is finished. For its underlying G-∞-category, note that the pullback of $\underline{Rep}_n^{\sqcup}(G)$ along the map $I(-) : \mathcal{O}_G^{op} \to \underline{Fin}_*^G$, $I(O) = [O \xrightarrow{=} O]$ is a G-∞-category $(\underline{Rep}_n^{\sqcup}(G))_{I(-)}$ whose elements are $E \to O \xrightarrow{=} O$ where $E \to O$ is a G-vector bundle and whose mapping spaces consist of diagrams

$$\begin{tikzcd}
E_1 \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d] \\
O_1 \arrow[d] & O_2 \arrow[l] \arrow[r] \arrow[d] & O_2 \arrow[d] \\
O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2
\end{tikzcd}$$

where the left side is a pullback diagram. Trerefore the equivalence $(\underline{Rep}_n^{\sqcup}(G))_{I(-)} \to \underline{Rep}_n(G)$ is given by the functor $(E \to O \xrightarrow{=} O) \mapsto (E \to O)$. $\square$

Definition 7.2.3. For a G-space B and a G-map $f : B \to BO_n(G)$ define $\underline{Rep}_n^{B-fr,\sqcup}(G)$ to be a full G-∞-subcategory of $\underline{Mfld}^{G,B-fr,\sqcup}$ via the pullback

$$\begin{tikzcd}
\underline{Rep}_n^{B-fr,\sqcup}(G) \arrow[r] \arrow[d] & \underline{Mfld}^{G,B-fr,\sqcup} \arrow[d] \\
\underline{Rep}_n^{\sqcup}(G) \arrow[r] & \underline{Mfld}^{G,\sqcup}
\end{tikzcd}$$

Note that $\underline{Rep}_n^{B-fr,\sqcup}(G)$ is also a G-∞-operad.
The following definition stems from [NS]:

Definition 7.2.4. Let $p : \underline{\mathcal{O}}^{\otimes} \to \underline{Fin}_*^G$ be a G-∞-operad. The G-symmetric monoidal envelope of $\underline{\mathcal{O}}^{\otimes}$ is defined to be

$$Env_G(\underline{\mathcal{O}}^{\otimes}) = Arr_G^{act}(\underline{\mathcal{O}}^{\otimes}) \times_{\underline{Fin}_*^G} \underline{\mathcal{O}}^{\otimes}$$

where $Arr_G^{act}(\underline{\mathcal{O}}^{\otimes}) \subset Arr_G(\underline{\mathcal{O}}^{\otimes})$ is the full G-∞-subcategory of the G-∞-category of arrows in $\underline{\mathcal{O}}^{\otimes}$ spanned by fiberwise active arrows. The objects of $Env_G(\underline{\mathcal{O}}^{\otimes})$ can be represented as

$$\begin{tikzcd}
x \arrow[d] & & \\
U \arrow[d] & U \arrow[l, "="'] \arrow[r] \arrow[d] & U_1 \arrow[d] \\
O & O \arrow[l, "="'] \arrow[r, "="] & O
\end{tikzcd}$$

for $I = [U \to O]$ and $x \in \underline{\mathcal{O}}^{\otimes}_{[I]}$. Moreover, the induced map $Env_G(\underline{\mathcal{O}}^{\otimes}) \to \underline{Fin}_*^G$ exibits $Env_G(\underline{\mathcal{O}}^{\otimes})$ as a G-symmetric monoidal category whose underlying G-∞-category is equivalent to $\underline{\mathcal{O}}^{\otimes}_{act}$ the wide G-∞-subcategory of $\underline{\mathcal{O}}^{\otimes}$ on active arrows.

Proposition 7.2.5. *Let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. Then the G-symmetric monoidal ∞-category of B-framed G-disks, $\underline{Disk}^{G,B-fr,\sqcup}$ is equivalent to $Env_G(\underline{Rep}_n^{B-fr,\sqcup}(G))$, the G-symmetric monoidal envelope of $\underline{Rep}_n^{B-fr,\sqcup}(G)$.*

Proof. We will give a proof for $f : BO_n(G) \xrightarrow{=} BO_n(G)$. The general case can be proven analogously by taking into account the description of B-framed G-discs.
Since $\underline{Rep}_n^{\sqcup}(G)$ is a G-∞-operad, by definition 7.2.4 $Env_G(\underline{Rep}_n^{\sqcup}(G)) \to \underline{Fin}_*^G$ is a G-symmetric monoidal category. In order to prove that it is equivalent to $\underline{Disk}^{G,\sqcup}$ it will suffice to show that their underlying G-∞-categories are equivalent. Again, by definition, the underlying G-∞-category $(Env_G(\underline{Rep}_n^{\sqcup}(G)))_{I(-)}$ is equivalent to $(\underline{Rep}_n^{\sqcup}(G))_{act}$. It is clear that both $(\underline{Rep}_n^{\sqcup}(G))_{act}$ and $\underline{Disk}^G$ have the same objects given by $E \to U \to O$ where $E \to U$ is a G-vector bundle.
The map between $E_1 \to U_1 \to O_1$ and $E_2 \to U_2 \to O_2$ is given by active arrows

$$\begin{tikzcd}
E_1 & E \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d] \\
U_1 \arrow[d] & U \arrow[l] \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2
\arrow[from=1-1, to=2-1]
\end{tikzcd}$$

hence, the lower left square is a pullback square. By definition, the upper left square is also a pullback square, hence the whole left rectangle is a pullback diagram, which is exactly a map in $\underline{Disk}^G$. Therefore, the mapping spaces are equivalent which concludes the proof of the equivalence of the underlying G-∞-categories and consequently the equivalence of G-symmetric monoidal categories $Env_G(\underline{Rep}_n^{\sqcup}(G))$ and $\underline{Disk}^{G,\sqcup}$. $\square$

One of the main results from [NS] concerning the G-symmetric monoidal envelope is that, given a G-symmetric monoidal category $\underline{\mathcal{C}}^{\otimes}$ and a G-∞-operad $\underline{\mathcal{O}}^{\otimes}$ we can replace the ∞-category of $\underline{\mathcal{O}}$-algebras $Alg_G(\underline{\mathcal{O}}^{\otimes}, \underline{\mathcal{C}}^{\otimes})$ with the ∞-category of G-symmetric monoidal functors $Fun_G^{\otimes}(Env_G(\underline{\mathcal{O}}^{\otimes}), \underline{\mathcal{C}}^{\otimes})$. In particular, we have

Corollary 7.2.6. *Let $\underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal ∞-category, let B be a G-space and let $f : B \to BO_n(G)$ be a G-map. Then there is an equivalence of ∞-categories*

$$Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) \xrightarrow{\simeq} Alg_G(\underline{Rep}_n^{B-fr,\sqcup}(G), \underline{\mathcal{C}}^{\otimes})$$

7.3 G-configuration spaces

In this section we define G-configuration spaces and study their relation to the mapping spaces of G-discs into an $\mathcal{O}_G$-manifold. The results obtained here play a role in proving the properties of G-factorization homology (9) defined below.

Definition 7.3.1. Let $M \to O \in \underline{Mfld}^G$ be an $\mathcal{O}_G$-manifold and let $U \to O$ be a finite G-set over $O \in \mathcal{O}_G$. We will denote with $Conf_O^G(U, M)$ the space of injective G-equivariant maps $U \hookrightarrow M$ over O, with topology inherited from the weak C^{∞}-topology on $Map_O^G(U, M)$.

Let $E \to U \to O$ be a G-disc (that is $E \to U$ is a G-vector bundle) and let $M \to O$ be an $\mathcal{O}_G$-manifold. We will denote with $Emb_O^G(E, M)$ the space of equivariant smooth embeddings of E into M. Note that there is an evident map

$$c : Emb_O^G(E, M) \to Conf_O^G(U, M)$$

given by the evaluation at the zero section $s_0 : U \hookrightarrow E$. We would like to study the map c. In fact, we claim that it is a fibration. The proof is shown through several steps:

Definition 7.3.2. Let $M \to O$ be an $\mathcal{O}_G$-manifold and let $E \to U \to O$ be a G-disc, in other words, let U be a finite G-set and let $E \to U$ be a G-vector bundle. Consider the fiberwise tangent vector bundle (which we, by abuse of notation mark with TM) $TM \to M \to O$. We define the ∞-category $Fr^G_O(E, M)$ of frames E on M to be the ∞-category of G-vector bundle maps from $E \to U \to O$ to $TM \to M \to O$ covering the identity $O \xrightarrow{=} O$.

Remark 7.3.3. The motivation behind the name of the ∞-category of frames stems from the (non-equivariant) case where we can associate a frame bundle $Fr(M)$ to any manifold M as a space of pairs $(x, \mathcal{B}_x)$ where $x \in M$ and where $\mathcal{B}_x$ is the basis for TM_x. Alternatively, we can look at the frame bundle as a space of vector bundle maps $(E \to *) \to (TM \to M)$. We can make this story more general, by replacing the point $*$ with a finite set I and define the frame bundle as the ∞-category $Fr(E, M)$ of vector bundle maps $(E \to I) \to (TM \to M)$ covering an embedding $I \hookrightarrow M$ i.e. a configuration. Definition 7.3.2 is the equivariant version of this story.

Note that there is a natural map $g : Fr^G_O(E, M) \to Conf^G_O(U, M)$ which basically just forgets the map on the level of G-vector bundles. This map is in fact a fibration, which will prove to be useful in the near future.

Next, note that our map $c : Emb^G_O(E, M) \to Conf^G_O(U, M)$ actually factorizes through $Fr^G_O(E, M)$

$$\begin{tikzcd} Emb^G_O(E,M) \arrow[rr, "c"] \arrow[dr, "d"'] & & Conf^G_O(U,M) \\ & Fr^G_O(E,M) \arrow[ur, "g"'] & \end{tikzcd}$$

where the map $d : Emb^G_O(E, M) \to Fr^G_O(E, M)$ is obtained by taking the derivative at the zero section $U \hookrightarrow E$.

Lemma 7.3.4. *The map $d : Emb^G_O(E, M) \to Fr^G_O(E, M)$ is a homotopy equivalence.*

Proof. We have already stated that the map $g : Fr^G_O(E, M) \to Conf^G_O(U, M)$ is a fibration. Note that, by choosing a basepoint $x \in O$ and taking a fiber over eH of the orbit $G/H \cong (O, x)$ of $M \to O$ and $E \to U \to O$ we have that $Emb^G_O(E, M) \to Conf^G_O(U, M)$ is equivalent to $c_H : Emb^H(\bigsqcup_{i \in S} V_i, M_H) \to Cong^H(S, M_H)$ where M_H (resp. S, $\bigsqcup_{i \in S} V_i$) is the fiber of $M \to G/H$ (resp. $U \to G/H$, $E \to U \to G/H$) over eH. By a suitable version of the equivarian isotopy extension theorem (see [Las85]), we have that this map is a Serre fibration, hence so is $c : Emb^G_O(E, M) \to Conf^G_O(U, M)$. Therefore, it would be enough to check if the fibers of the maps c and g are weakly homotopy equivalent, but this now follows from 6.2.3.

□

Corollary 7.3.5. *The map $c : Emb^G_O(E, M) \to Conf^G_O(U, M)$ is a fibration.*

We conclude this section with the following proposition:

Proposition 7.3.6. *Let $(E \to U \to O) \in \underline{Disk}^G$ and let $(M \to O) \in \underline{Mfld}^G$ with $i : N \hookrightarrow M$ a submanifold-fibre bundle. Then the following diagram*

$$\begin{tikzcd} Emb^G_O(E,N) \arrow[r] \arrow[d, "c"] & Emb^G_O(E,M) \arrow[d, "c"] \\ Conf^G_O(U,N) \arrow[r] & Conf^G_O(U,M) \end{tikzcd}$$

is homotopy Cartesian.

Proof. The diagram above is equivalent to

$$\begin{tikzcd}
Emb_O^G(E,N) \arrow[r] \arrow[d, "d"] & Emb_O^G(E,M) \arrow[d, "d"] \\
Fr_O^G(E,N) \arrow[r] \arrow[d, "g"] & Fr_O^G(E,M) \arrow[d, "g"] \\
Conf_O^G(U,N) \arrow[r] & Conf_O^G(U,M)
\end{tikzcd}$$

The upper square is a homotopy pullback hence it will suffice to show that the bottom square is also homotopy pullback square. Since the lower vertical maps are fibrations it will suffice to show that the lower square induces an equivalence of the fibers. Let $f \in Conf_O^G(U,N)$. We want to show that the induced map on the fibers

$$(Fr_O^G)_f(E,N) \to (Fr_O^G)_{i\circ f}(E,M)$$

is an equivalence. But this is clear since $TN_{f(U)} \cong TM_{i\circ f(U)}$ because $N \subset M$ are of same dimension. $\square$

7.4 G-disc algebras in $\underline{Sp}^G$

In this short section we will give a description of framed G-disc algebras with coefficients in the G-∞-category of G-spectra $\underline{Sp}^G$ (see [Nar17]). We will also see the connection with the norm maps of Hopkins, Hill and Ravenel (see [HHR16]):

Let $f : B \to BO_n(G)$ be a G-map with B a G-space and let $F \in Alg_G(\underline{Disk}^{G,B-fr}, \underline{Sp}^G)$ be a B-framed G-disc algebra. By 7.1.7 we can write $(U_1 \to G/H \times B) \in \underline{Disk}^{G,B-fr}_{[G/H]}$ and $(U_2 \to G/K \times B) \in \underline{Disk}^{G,B-fr}_{[G/K]}$, where $K \leq H \leq G$ are subgroups.

- Recall $Res^H_K : \underline{Disk}^{G,B-fr}_{[G/H]} \to \underline{Disk}^{G,B-fr}_{[G/K]}$ the restriction functor from 6.1.13. We have:

$$Res^H_K : \underline{Sp}^G_{[G/H]} \to \underline{Sp}^G_{[G/K]}$$
$$Res^H_K F(U \to G/H \times B) \simeq F(Res^H_K(U \to G/H \times B)) \simeq F(G/K \times_{G/H} U \to G/K \times B)$$

- Again, recall that the topological induction functor:

$$H \times_K (-) : \underline{Disk}^{G,B-fr}_{[G/K]} \to \underline{Disk}^{G,B-fr}_{[G/H]}$$

  is given by post-composition with $G/K \to G/H$. Topological induction funcor is compatible with the norm construction of Hill, Hopkins and Ravenel

$$N^H_K(-) : Sp^K \to Sp^H$$

  which with $\underline{Sp}^G_{[G/H]} \simeq Sp^H$ and $\underline{Sp}^G_{[G/K]} \simeq Sp^K$ gives

$$F(U_2 \to G/K \times B \to G/H \times B) \simeq N^H_K F(U_2 \to G/K \times B)$$

Remark 7.4.1. For more concrete examples when G is finite group, one can look up [Hor19] *section 7*. Additionaly, note that our theory restricts to the theory of Horev when G is a finite group, therefore all the results regarding real topological Hochschild homology and twisted topological Hochschild homology of genuine C_n-ring spectra hold in our case.

In particular, for $G = S^1$ with the class of cyclic subgroups $\{C_n\}_{n\in\mathbb{N}}$ let us look at the S^1-∞-category of V-framed S^1-discs $Fun^{\otimes}_{S^1}(\underline{Disk}^{S^1, V-fr, \sqcup}, \underline{Sp}^{S^1})$, where V is some S^1 representation.

For $C_n < S^1$ we have that the total space of a V-framed disc $E \to S^1/C_n$ is equivalent to $S^1 \times_{C_n} V$ by [TD87] *I, 9.2*, hence any V-framed S^1-disc is equivalent to $S^1 \times_{C_n} V \to S^1/C_n$. Let us fix one particular $A \in Fun^{\otimes}_{G}(\underline{Disk}^{S^1, V-fr, \sqcup}, \underline{Sp}^{S^1})$.
We will denote with

$$A_{C_n} := A(S^1 \times_{C_n} V \to S^1/C_n) \in \underline{Sp}^{S^1}_{[S^1/C_n]} \simeq Sp^{C_n}$$

In other words, for every S^1-disc $S^1 \times_{C_n} V \to S^1/C_n$ we have a corresponding C_n-spectrum A_{C_n} such that for any $C_r \leq C_n$ we have

- $Res^{C_n}_{C_r} A_{C_n} = A_{C_r}$
- $A_{C_n} \simeq N^{C_n}_{C_r} A_{C_r}$

Chapter 8

G-Factorization homology

We have finally arrived to the chapter where we will define the equivariant version of factorization homology (or, simply G-factorization homology).
First, we give the definition of the G-factorization homology as parametrized colimit. Later, we give a description of a G-factorization homology as a G-functor and finally as a symmetric monoidal G-functor. Here we follow the same path as Horev ([Hor19] *section 4*):

For starters, let $A \in Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$ be a B-framed disc algebra with coefficients in $\underline{\mathcal{C}}^{\otimes}$ and let $M \in \underline{Mfld}_{[O]}^{G,B-fr}$ be a B-framed G-manifold lying over the orbit O. We can construct a diagram of $\underline{O}$-∞-categories

$$\begin{tikzcd} \underline{Disk}_{/\underline{M}}^{G,B-fr} \arrow[r] \arrow[dr] & \underline{O}\underline{\times}\underline{Disk}^{G,B-fr} \arrow[r] \arrow[d] & \underline{O}\underline{\times}\underline{\mathcal{C}} \arrow[dl] \\ & \underline{O} & \end{tikzcd}$$

where $\underline{Disk}_{/\underline{M}}^{G,B-fr}$ is the parametrized slice category. We can look at $\underline{Disk}_{/\underline{M}}^{G} \to \underline{O}$ as the coCartesian fibration dual to the Cartesian fibration $Disk_{/M}^{G} \to \mathcal{O}_{G/O}$ (see [Hor19] *4.1.1*).
The definition of the G-factorization homology would be as follows:

Definition 8.0.1. Let $M \in \underline{Mfld}_{[O]}^{G,B-fr}$ and let A be a B-framed G-disc algebra with values in $\underline{\mathcal{C}}^{\otimes}$. We define the factorization homology of M with values in A by the following parametrized colimit

$$\int_M A = \underline{O} - colim(\underline{Disk}_{/\underline{M}}^{G,B-fr} \to \underline{O}\underline{\times}\underline{Disk}^{G,B-fr} \to \underline{O}\underline{\times}\underline{\mathcal{C}})$$

Remark 8.0.2. The G-factorization homology $\int_M A$ is computed as an $\underline{O}$-colimit, therefore it can be written as a coCartesian section of $\underline{O}\underline{\times}\underline{\mathcal{C}} \to \underline{O}$. Since $\underline{O}\underline{\times}\underline{\mathcal{C}}$ is a pullback, we can also write this coCartesian section as a G-functor $\underline{O} \to \underline{\mathcal{C}}$ which corresponds to an object of $\underline{\mathcal{C}}$ in the fibre of O. Therefore, we can write $\int_M A \in \underline{\mathcal{C}}$ (or $\underline{\mathcal{C}}_{[O]}$ to be more precise).

G-factorization homology as a G-functor

Let $\underline{\mathcal{C}}$ be a G-cocomplete G-∞-category. The inclusion functor $i : \underline{Disk}^{G,B-fr} \hookrightarrow \underline{Mfld}^{G,B-fr}$ is fully faithful. Therefore, by [Shah18] *10.6*, the restriction G-functor $i^* : \underline{Fun}_G(\underline{Mfld}^{G,B-fr}, \underline{\mathcal{C}}) \to \underline{Fun}_G(\underline{Disk}^{G,B-fr}, \underline{\mathcal{C}})$ admits a fully faithful left G-adjoint

$$i_! : \underline{Fun}_G(\underline{Disk}^{G,B-fr}, \underline{\mathcal{C}}) \rightleftarrows \underline{Fun}_G(\underline{Mfld}^{G,B-fr}, \underline{\mathcal{C}}) : i^*$$

where the values of $i_!$ are given by left $\underline{O}$-Kan extensions. Taking coCartesian sections one again recovers an adjunction

$$i_! : Fun_G(\underline{Disk}^{G,B-fr}, \underline{\mathcal{C}}) \rightleftarrows Fun_G(\underline{Mfld}^{G,B-fr}, \underline{\mathcal{C}}) : i^* \tag{8.1}$$

where the values of $i_!$ are given by left G-Kan extensions.
Now we are ready to show that G-factorization homology can be expressed as a G-functor:

Proposition 8.0.3. *Let $\underline{\mathcal{C}}^{\otimes}$ be a G-cocomplete G-∞-category. Then the functor*

$$Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) \to Fun_G(\underline{Disk}^{G,B-fr}, \underline{\mathcal{C}}) \xrightarrow{i_!} Fun_G(\underline{Mfld}^{G,B-fr}, \underline{\mathcal{C}})$$

sends $A \in Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$ to $i_!(A) \in Fun_G(\underline{Mfld}^{G,B-fr}, \underline{\mathcal{C}})$, $i_!(A) : M \mapsto \int_M A$, where A in $i_!(A)$ is regarded as an ordinary (not G-symmetric monoidal) G-functor.

Proof. This is a combination of [Shah18] *10.3* and *10.4*. By the former $i_!(A)$ is given by the left G-Kan extension of A along i and by the latter, this left G-Kan extension is computated via the formula

$$i_!(A) = \underline{O} - colim(\underline{Disk}^{G,B-fr}_{/\underline{M}} \to \underline{O} \underline{\times} \underline{Disk}^{G,B-fr} \to \underline{O} \underline{\times} \underline{\mathcal{C}})$$

which is by definition equal to $\int_M A$ (see also [Hor19] *4.1.4*). $\square$

G-factorization homology as a G-symmetric monoidal functor

In order to show that G-factorization homology can be extended to a G-symmetric monoidal functor it will suffice to show the following:

Proposition 8.0.4. *Let $\underline{\mathcal{C}}^{\otimes}$ be a G-presentable G-symmetric monoidal category. Then the adjunction of* (8.1) *lifts to an adjunction*

$$\begin{tikzcd}
(i^{\otimes})_! : Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) \arrow[r, shift left] \arrow[d] & Fun_G^{\otimes}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) : (i^{\otimes})^* \arrow[l, shift left] \arrow[d] \\
i_! : Fun_G(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}) \arrow[r, shift left] & Fun_G(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}) : i^* \arrow[l, shift left]
\end{tikzcd}$$

We need to prepare for the proof. We will start with:

Lemma 8.0.5. *([Hor19]* 4.2.5*) Let $\underline{\mathcal{C}}^{\otimes} \to \underline{Fin}_*^G$ be a presentable G-symmetric monoidal category, let $\underline{\mathcal{M}}^{\otimes} \to \underline{Fin}_*^G$ be a small G-symmetric monoidal category and $i^{\otimes} : \underline{\mathcal{D}}^{\otimes} \to \underline{\mathcal{M}}^{\otimes}$ an inclusion of a full G-symmetric monoidal subcategory. Denote by $i : \underline{\mathcal{D}} \to \underline{\mathcal{M}}$ the induced G-functor on the underlying G-∞-categories. If for every active morphism $\psi : I \to J$ in the fiber ${\underline{Fin}_*^G}_{[O]}$ and every coCartesian lift $x \to y$ of ψ to $\underline{\mathcal{M}}^{\otimes}$ the $\underline{O}$-functor $\otimes_{\psi} : (\underline{\mathcal{D}}^{\otimes}{}_{<I>})_x \to (\underline{\mathcal{D}}^{\otimes}{}_{<J>})_y$ is $\underline{O}$-cofinal then the diagram*

$$\begin{tikzcd}
Fun_G^{\otimes}(\underline{\mathcal{D}}, \underline{\mathcal{C}}) \arrow[r, "(i^{\otimes})_!"] \arrow[d] & Fun_G^{\otimes}(\underline{\mathcal{M}}, \underline{\mathcal{C}}) \arrow[d] \\
Fun_G(\underline{\mathcal{D}}, \underline{\mathcal{C}}) \arrow[r, "i_!"] & Fun_G(\underline{\mathcal{M}}, \underline{\mathcal{C}})
\end{tikzcd}$$

commutes, where $(i^{\otimes})_!$ and $i_!$ the left adjoins to the restrictions along $i^{\otimes}$ and i, respectively.

To simplyfy the calculations we will state the following proposition on the equivalence of parametrized slice categories $\underline{Mfld}^{G}_{/\underline{M}}$ and $\underline{Mfld}^{G,B-fr}_{/\underline{M}}$, which allow us to work with the unframed G-manifolds. It will also prove to be very useful later on when proving the properties of G-factorization homology.

Proposition 8.0.6. *Let B be a G-space, let $f : B \to BO_n(G)$ be a G-map and let $M \in \underline{Mfld}^{G,B-fr}_{[O]}$. Then the $\underline{O}$-functor that forgets the framing structure*

$$\underline{Mfld}^{G,B-fr}_{/\underline{M}} \to \underline{Mfld}^{G}_{/\underline{M}}$$

is an equivalence of $\underline{O}$-categories.

Proof. We proceed by proving that $\underline{Mfld}^{G,B-fr}_{/\underline{M}} \to \underline{Mfld}^{G}_{/\underline{M}}$ is an equivalence. It would suffice to show that it is a levelwise equivalence. Without loss of generality it would further suffice to show that the map

$$(\underline{Mfld}^{G,B-fr}_{/\underline{M}})_{[\varphi]} \to (\underline{Mfld}^{G}_{/\underline{M}})_{[\varphi]}$$

is an equivalence only for $[\varphi : O \xrightarrow{=} O] \in \underline{O}$. By construction, theses fibers are equivalent to

$$(\underline{Mfld}^{G,B-fr}_{/\underline{M}})_{[\varphi]} \simeq (\underline{Mfld}^{G,B-fr}_{[O]})_{/M}$$
$$(\underline{Mfld}^{G}_{/\underline{M}})_{[\varphi]} \simeq (\underline{Mfld}^{G}_{[O]})_{/M}$$

To add on, by the definition of the G-∞-category of B-framed G-manifolds the fibers $\underline{Mfld}^{G}_{[O]}$ and $\underline{Mfld}^{G,B-fr}_{[O]}$ fit into the pullback square

$$\begin{tikzcd}
\underline{Mfld}^{G,B-fr}_{[O]} \arrow[r] \arrow[d] & (\underline{Top}^{G-CW}_{[O]})_{/\underline{B}(O)} \arrow[d] \\
\underline{Mfld}^{G}_{[O]} \arrow[r] & (\underline{Top}^{G-CW}_{[O]})_{/\underline{BO_n(G)}(O)}
\end{tikzcd}$$

Further on, we obtain the following diagram of slice categories

$$\begin{tikzcd}
(\underline{Mfld}^{G,B-fr}_{[O]})_{/M} \arrow[r] \arrow[d] & (Top^{G-CW}_{/B\times O})_{/M\to B\times O)} \arrow[d] \\
(\underline{Mfld}^{G}_{[O]})_{/M} \arrow[r] & (Top^{G-CW}_{/BO_n(G)\times O})_{/M\to BO_n(G)\times O)} \arrow[d] \\
 & Top^{G-CW}_{/M}
\end{tikzcd}$$

Where Top^{G-CW} is the ∞-category of G-CW-complexes and where the square part of the diagram is a pullback square. By [AF15] *2.5* the bottom vertical map is an equivalence as well as the map obtained by composing the two right vertical maps, hence

$$(Top^{G-CW}_{/B\times O})_{/(M\to B\times O)} \to (Top^{G-CW}_{/BO_n(G)\times O})_{/(M\to BO_n(G)\times O)}$$

is an equivalence as well. Since the square in the upper diagram is a pullback the left vertical map is an equivalence thus giving us

$$\underline{Mfld}^{G,B-fr}_{/\underline{M}} \xrightarrow{\simeq} \underline{Mfld}^{G}_{/\underline{M}}$$

□

Remark 8.0.7. Beside simplyfing the calculation 8.0.6 carries some more information. Namely, under the equivalence $\underline{Mfld}^{G,B-fr}_{/\underline{M}} \xrightarrow{\simeq} \underline{Mfld}^{G}_{/\underline{M}}$, the framing on M induces a framing on any $N \subseteq M$.

We are ready to present the proof of 8.0.4:

Proof. Let $I_1 = [U_1 \to O]$ and $I_2 = [U_2 \to O]$ be the elements of $\underline{Fin}^G_*$, let $\psi : I_1 \to I_2$ be an active morphism and let $f : [M_1 \to U_1 \to O] \to [M_2 \to U_2 \to O]$ be a coCartesian lift of ψ. Then we can depict f as a diagram

$$\begin{tikzcd}
M_1 \arrow[d] & M_1 \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[d] & U_1 \arrow[l, "="'] \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O & O \arrow[l, "="'] \arrow[r, "="] & O
\end{tikzcd}$$

By 8.0.5 it will suffice to prove that the induced functor

$$(\underline{Disk}^{G,B-fr,\sqcup}_{<I_1>})_{/\underline{M_1}} \to (\underline{Disk}^{G,B-fr,\sqcup}_{<I_2>})_{/\underline{M_2}}$$

is $\underline{O}$-cofinal. Note that by 8.0.6 we are reduced to showing that

$$(\underline{Disk}^{G,\sqcup}_{<I_1>})_{/\underline{M_1}} \to (\underline{Disk}^{G,\sqcup}_{<I_2>})_{/\underline{M_2}}$$

is $\underline{O}$-cofinal. We will show that it induces fiberwise equivalences and is therefore an equivalence. Without the loss of generality it will suffice to show an equivalence only for the fiber of $\varphi : O \xrightarrow{=} O \in \underline{O}$. By definition, we have

$$((\underline{Disk}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}})_{[\varphi]} \cong ((\underline{Mfld}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}})_{[\varphi]} \times_{(\underline{Mfld}^{G,\sqcup}_{<I_i>})_{[\varphi]}} (\underline{Disk}^{G,\sqcup}_{<I_i>})_{[\varphi]}$$

with

$$((\underline{Mfld}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}})_{[\varphi]} \simeq ((\underline{Mfld}^{G,\sqcup}_{<I_i>})_{[O]})_{/M_i}$$
$$(\underline{Mfld}^{G,\sqcup}_{<I_i>})_{[O]} \cong \underline{Mfld}^{G,\sqcup}_{[I_i]}$$

Unwinding the definitions, as explained by Horev, the category $((\underline{Mfld}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}})_{[\varphi]}$ can be modeled by the coherent nerve of the Moore over category $((\mathcal{O}_G - Fin - Mfld)_{I_i})^{Moore}_{/M_i}$. Therefore $((\underline{Disk}^{G,\sqcup}_{<I_i>})_{/\underline{M_i}})_{[\varphi]}$ can be regarded as a full subcategory of $((\mathcal{O}_G - Fin - Mfld)_{I_i})^{Moore}_{/M_i}$ spanned by the objects of the form

$$(E \to U \to U_i \to O) \to (M_i \to U_i \to O)$$

which can be depicted by diagrams

$$\begin{tikzcd}
E \arrow[d] & E' \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & M_i \arrow[dd] \\
U \arrow[d] & U \arrow[l, "="'] \arrow[d] & \\
U_i \arrow[d] & U_i \arrow[l, "="'] \arrow[r, "="] \arrow[d] & U_i \arrow[d] \\
O & O \arrow[l, "="'] \arrow[r, "="] & O
\end{tikzcd}$$

With this clarification, the induced functor $((\underline{Disk}^{G,\sqcup}_{<I_1>})_{/\underline{M_1}})_{[\varphi]} \to ((\underline{Disk}^{G,\sqcup}_{<I_2>})_{/\underline{M_2}})_{[\varphi]}$ is given by the composition with

$$\begin{tikzcd}
M_1 \arrow[d] & M_1 \arrow[l, "="'] \arrow[r] \arrow[d] & M_2 \arrow[d] \\
U_1 \arrow[d] & U_1 \arrow[l, "="'] \arrow[r] \arrow[d] & U_2 \arrow[d] \\
O & O \arrow[l, "="'] \arrow[r, "="] & O
\end{tikzcd}$$

which further induces an equivalence between topological subcategories of $((\mathcal{O}_G{-}Fin{-}Mfld)_{\varphi^* I_1})^{Moore}_{/\varphi^* M_1}$ and $((\mathcal{O}_G - Fin - Mfld)_{\varphi^* I_2})^{Moore}_{/\varphi^* M_2}$ corresponding to the categories $((\underline{Disk}^{G,\sqcup}_{<I_1>})_{/\underline{M_1}})_{[\varphi]}$ and $((\underline{Disk}^{G,\sqcup}_{<I_2>})_{/\underline{M_2}})_{[\varphi]}$ respectively, by sending an object

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & M_1 \arrow[dd] \\
U \arrow[dr] & \\
& U_1 \arrow[d] \\
& O
\end{tikzcd}$$

to the object

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & M_1 \arrow[dd] & M_1 \arrow[l, "="'] \arrow[r] \arrow[dd] & M_2 \arrow[dd] \\
U \arrow[dr] & & & \\
& U_1 \arrow[d] & U_1 \arrow[l, "="'] \arrow[r] \arrow[d] & U_2 \arrow[d] \\
& O & O \arrow[l, "="'] \arrow[r, "="] & O
\end{tikzcd}$$

$\square$

Definition 8.0.8. By abuse of notation we will again mark the G-symmetric monoidal functor $i_!^{\otimes}$: $Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes}) \to Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$ with $\int_{\_} A$ and call it the G-factorization homology functor with coefficients in $A \in Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$.

Chapter 9

Properties of G-factorization homology

This chapter is dedicated to the properties of G-factorization homology. These properties include:

- The G-$\otimes$-excision property;
- Respect with regard to G-sequential unions;
- Axiomatic caracterization of G-factorization homology

We start this chapter with a section dedicated to G-collar decomposition and then go on to prove before mentioned properties in the order that they are written.

9.1 G-collar decomposition

In this section we define the G-colar gluing of G-manifolds, which is used in the G-$\otimes$-excision property of G-factorization homology. The main part of this section is the construction of the inverse image functors 9.1.3 which requires attention to detail.

Definition 9.1.1. ([Hor19] *5.1.1* or [Wee18] *4.20*) Let M be a G-manifold and let $[-1,1]$ be a closed interval endowed with the trivial G-action. By G-collar decomposition we mean a surjective equivariant map $f : M \to [-1,1]$ such that the restriction $M|_{(-1,1)} \to (-1,1)$ is a manifold bundle map with a choice of trivialization $M|_{(-1,1)} \cong M_0 \times (-1,1)$ where $M_0 = f^{-1}(0)$. We will denote with $M_+ = f^{-1}(-1,1]$ and $M_- = f^{-1}[-1,1)$.

Let $M \in \underline{Mfld}^{G,B-fr}_{[O]}$ be a B-framed G-manifold over $O \in \mathcal{O}_G$. A G-collar decomposition on $M \to O$ is a G-collar decomposition on the underlying G-manifold.

Lemma 9.1.2. *Let $p : M \to O$ be an $\mathcal{O}_G$-manifold equipped with a G-collar decomposition $f : M \to [-1,1]$ of the underlying G-manifold M. Let $V \subset [-1,1]$ be an oriented embedding of a 1-dimensional manifold possibly with boundary. Then the restriction map $f^{-1}(V) \to O$ makes $f^{-1}(V)$ an $\mathcal{O}_G$-manifold.*

Proof. Let $x \in O$. By choosing x as a basepoint of O we have $(O,x) \cong G/H$ where $H = Stab(x)$. Let $U \subset G$ be a sufficiently small neighborhood of $e \in G$ such that $U \cap H = \{e\}$. Such U exists since H is a finite subgroup. By letting U act on x we obtain a neighborhood $U_x \subset O$ of x. Furthemore, we can choose U to be sufficiently small such that $p^{-1}(U_x) \cong M_x \times U_x$, where M_x is the fiber of p above $x \in O$.

Since the G-collar gluing map $f : M \to [-1,1]$ is G-equivariant, the restriction map $p^{-1}(U_x) \cong M_x \times U_x \to [-1,1]$ factors through a unique map $M_x \to [-1,1]$ given by $M_x \times \{x\} \to [-1,1]$

$$\begin{tikzcd} M_x \times U_x \arrow[r] \arrow[d] & {[-1,1]} \\ M_x \arrow[ur] & \end{tikzcd}$$

Denote with $p' : f^{-1}(V) \to O$ the restriction map $p|_{f^{-1}(V)}$. Then

$$p'^{-1}(U_x) \cong U_x \times (f^{-1}(V) \cap M_x)$$

Considering some other point $y \in O$ we can obtain the same formula $p'^{-1}(U_y) \cong U_y \times (f^{-1}(V) \cap M_y)$. It is only left to show that $f^{-1}(V) \cap M_x \cong f^{-1}(V) \cap M_y$. Since $x, y \in O$ there exists $g \in G$ such that $gx = y$. Since p and f (and therefore p') are all G-maps, the isomorphism $f^{-1}(V) \cap M_x \to f^{-1}(V) \cap M_y$ is induced by a map ξ_g, the action map of element g on M. The inverse is then induced by $\xi_{g^{-1}}$. It is clear now that $p' : f^{-1}(V) \to O$ is a G-manifold fibre bundle i. e. an $\mathcal{O}_G$-manifold. $\square$

Following the story in the classical setting, we would like to construct the map

$$Disk^{\partial,or}_{/[-1,1]} \xrightarrow{f^{-1}} (\underline{Mfld}^{G,B-fr}_{[O]})_{/M} \tag{9.1}$$

where $Disk^{\partial,or}_{/[-1,1]}$ is the ∞-category of one dimensional oriented manifolds possibly with boundary over the segment $[-1,1]$. Using 8.0.6 we can restrict to the case $Disk^{\partial,or}_{/[-1,1]} \xrightarrow{f^{-1}} (\underline{Mfld}^{G}_{[O]})_{/M}$ Both ∞-categories can be described using the coherent nerve of the corresponding Moore over categories (see for example [Hor19] *app. A*), which we will mark as $(Disk^{\partial,or})^{Moore}_{/[-1,1]}$ and $(\underline{Mfld}^{G}_{[O]})^{Moore}_{/M}$.

Construction 9.1.3. Let $(M \to O)$ be an $\mathcal{O}_G$-manifold with an underlying G-collar decomposition $f : M \to [-1,1]$. We then construct the functor $f^{-1} : (Disk^{\partial,or})^{Moore}_{/[-1,1]} \to (\underline{Mfld}^{G}_{[O]})^{Moore}_{/M}$ the following way

- An element $V \hookrightarrow [-1,1]$ of $(Disk^{\partial,or})^{Moore}_{/[-1,1]}$ is sent to the pullback $(f^{-1}(V) \to O)$ where the last arrow is given by the composition $f^{-1}(V) \to M \to O$. By 9.1.2 we see that $(f^{-1}(V) \to O)$ is indeed an element of $\underline{Mfld}^{G}_{[O]}$ and since $f : M \to [-1,1]$ is a G-map and $V \hookrightarrow [-1,1]$ is an embedding, the map $f^{-1}(V) \to M$ will be a G-embedding, hence $(f^{-1}(V) \to O) \in (\underline{Mfld}^{G}_{[O]})^{Moore}_{/M}$.

- Let $\varphi_1 : V_1 \hookrightarrow [-1,1]$ and $\varphi_2 : V_2 \hookrightarrow [-1,1]$ be two elements in the Moore over category $(Disk^{\partial,or})^{Moore}_{/[-1,1]}$. Let $(h,(r,\gamma))$ be a point in $Map_{(Mfld^{\partial,or})^{Moore}_{/[-1,1]}}(V_1, V_2)$, where $h : V_1 \to V_2$ is an oriented embedding and $(r,\gamma) \in [0,\infty) \times Emb^{\partial,or}(V_1,[-1,1])^{[0,\infty)}$ is a Moore path from φ_1 to $\varphi_2 \circ h$. We define a functor

  $$f^{-1} : Map_{(Disk^{\partial,or})^{Moore}_{/[-1,1]}}(V_1, V_2) \to Map_{(\underline{Mfld}^{G}_{[O]})^{Moore}_{/M}}(f^{-1}(V_1), f^{-1}(V_2))$$
  $$f^{-1}(h,(r,\gamma)) := (f^{-1}(h),(r,\alpha))$$

  where $f^{-1}(h)$ is obtained via the diagram

$$\begin{tikzcd} f^{-1}(V_1) \arrow[r, hook, "f^{-1}(h)"] \arrow[d] & f^{-1}(V_2) \arrow[r, hook] \arrow[d] & M \arrow[d, "f"] \\ V_1 \arrow[r, hook] & V_2 \arrow[r, hook] & {[-1,1]} \end{tikzcd}$$

where both left and right squares are pullback squares and where $\alpha \in Emb^G(V_1, M)^{[0,\infty)}$ is a Moore path of length r defined as follows. If $x \in M|_{(-1,1)} \cong M_0 \times (-1,1)$ such that x corresponds to $(y, s) \in M_0 \times (-1,1)$ define

$$\alpha_t(x) = (y, \gamma_t \circ \varphi_1^{-1}(s)) \in M_0 \times (-1,1)$$

otherwise, if $f(x) = \pm 1$ define $\alpha_t(x) = x$.

9.2 G-$\otimes$-excision

In this section we will first define what does it mean for a G-functor to satisfy the G-$\otimes$-property, and then we will prove that our G-factorization homology satisfies that property.

The main result of this section is the Proposition 9.2.4 which states that given an $\mathcal{O}_G$-manifold M together with a collar gluing map $f : M \to [-1,1]$ and a G-disc algebra A taking values in $\underline{\mathcal{C}}^{\otimes}$ satisfying certain conditions (to be specified below) we have the following equivalence

$$\int_M A \simeq \int_{M_-} A \otimes_{\int_{M_0 \times (-1,1)} A} \int_{M_+} A$$

which serves as an analogue to the Eilenberg-Steenrod axiom for the homology theories.
First, we will give the construction of the right side of the upper equivalence in general case (not only for G-factorization homology).

Construction 9.2.1. Let $F : \underline{Mfld}^{G,B-fr} \to \underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal functor. Let $(M \to O) \in \underline{Mfld}^{G,B-fr}_{[O]}$ and let $f : M \to [-1,1]$ be a G-collar decomposition map of the underlying G-manifold M.

$$Disk^{\partial,or}_{/[-1,1]} \xrightarrow{f^{-1}} (\underline{Mfld}^{G}_{[O]})_{/M} \xrightarrow{\simeq} (\underline{Mfld}^{G,B-fr}_{[O]})_{/M} \xrightarrow{F} \underline{\mathcal{C}}_{[O]/F(M)} \tag{9.2}$$

Let us assume that $\underline{\mathcal{C}}_{[O]}$ admits sifted colimits and that the tensor product functor in $\underline{\mathcal{C}}_{[O]}$ preserves sifted colimits in each variable. Then by precomposing the upper map with the cofinal map $\Delta^{op} \to Disk^{\partial,or}_{/[-1,1]}$ (see [AF15] *3.11*) and taking the colimit of 9.2 we obtain the two sided bar construction

$$(F(M_-) \otimes_{F(M_0 \times (-1,1))} F(M_+) \to F(M)) \in (\underline{\mathcal{C}}_{[O]})_{/F(M)}$$

Definition 9.2.2. Let $F : \underline{Mfld}^{G,B-fr} \to \underline{\mathcal{C}}$ be a G-symmetric monoidal functor such that for every $(M \to O) \in \underline{Mfld}^{G,B-fr}$ with a G-collar decomposition of the underlying G-manifold $f : M \to [-1,1]$ the induced map $F(M_-) \otimes_{F(M_0 \times (-1,1))} F(M_+) \to F(M)$ is an equivalence in $\underline{\mathcal{C}}_{[O]}$. In such case we say that F satisfies the G-$\otimes$-excision property.

Remark 9.2.3. By [HA] *4.4.2.8*, under the conditions imposed on $\underline{\mathcal{C}}_{[O]}$ as in 9.2.1, the relative tensor product $F(M_-) \otimes_{F(M_0 \times (-1,1))} F(M_+)$ can be identified with the two sided bar construction, hence we use the same notation in both cases.

Proposition 9.2.4. *Let $A : \underline{Disk}^{G,B-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ be a B-framed G-disc algebra with values in $\underline{\mathcal{C}}^{\otimes}$. Then the G-factorization homology functor $\int A_{-} : \underline{Mfld}^{G,B-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ satisfies the G-$\otimes$-excision property.*

The proof of the theorem can be expressed in several steps, imitating the approach of Ayala and Francis in [AF15]. We will first give one useful construction that will make the proof of 9.2.4 analogous to that of [Hor19] *5.2.3*

Construction 9.2.5. Let $(M \to O) \in \underline{Mfld}^{G,B-fr}$ be an $\mathcal{O}_G$-manifold with a G-collar decomposition of the underlying G-manifold $f : M \to [-1,1]$. We define the following $\underline{O}$-category $\underline{Disk}_f \to \underline{O}$ as the $\underline{O}$-limit of the diagram

$$\begin{tikzcd}
\underline{Disk}^{G,B-fr}_{/\underline{M}} \arrow[d] & \underline{Fun}_{\underline{O}}(\underline{O} \times \Delta^1, \underline{Mfld}^{G,B-fr}_{/\underline{M}}) \arrow[dl, "ev_0"'] \arrow[d, "ev_1"] & \underline{O} \times Disk^{\partial,or}_{/[-1,1]} \arrow[dl, "f^{-1}"] \\
\underline{Mfld}^{G,B-fr}_{/\underline{M}} & \underline{Mfld}^{G,B-fr}_{/\underline{M}} &
\end{tikzcd}$$

together with $\underline{O}$-functors $ev_0 : \underline{Disk}_f \to \underline{Disk}^{G,B-fr}_{/\underline{M}}$ and $ev_1 : \underline{Disk}_f \to \underline{O} \times Disk^{\partial,or}_{/[-1,1]}$

Description 9.2.6. *Let us give a description of the $\underline{O}$-category $\underline{Disk}_f$. Let $(\psi : W \to O) \in \underline{O}$. Then the fiber of $\underline{Disk}_f \to \underline{O}$ over ψ is the limit of the diagram*

$$\begin{tikzcd}
(\underline{Disk}^{G,B-fr}_{[W]})_{/\psi^* M} \arrow[d] & Fun(\Delta^1, (\underline{Mfld}^{G,B-fr}_{[W]})_{/\psi^* M}) \arrow[dl, "ev_0"'] \arrow[d, "ev_1"] & Disk^{\partial,or}_{/[-1,1]} \arrow[dl, "\psi^* f^{-1}"] \\
(\underline{Mfld}^{G,B-fr}_{[W]})_{/\psi^* M} & (\underline{Mfld}^{G,B-fr}_{[W]})_{/\psi^* M} &
\end{tikzcd}$$

The ∞-categories $(\underline{Disk}^{G,B-fr}_{[W]})_{/\psi^* M}$, $(\underline{Mfld}^{G,B-fr}_{[W]})_{/\psi^* M}$ and $Disk^{\partial,or}_{/[-1,1]}$ can be modeled by the adequate coherent nerves of the Moore over categories, hence an element of $(\underline{Disk}_f)_{[\psi]}$ can be written as

$$(g : E \hookrightarrow \psi^* M, \varphi : V \hookrightarrow [-1,1], h : E \hookrightarrow f^{-1}(V), \gamma)$$

where:

- *V is a finite disjoint union of 1-dimensional oriented disks with boundary, i.e oriented open intervals equivalent to $\mathbb{R}$ and oriented half open intervals equivalent to $[0,1)$ or $(0,1]$,*
- *φ is an orientation preserving embedding of V into the closed interval $[-1,1]$,*
- *$E \to U \to W$ is a finite G-disjoint union of G-disks (i.e $E \to U$ is a G-vector bundle over a finite G-set),*
- *g is a G-equivariant embedding over W of E into $\psi^* M$,*
- *h is a G-equivariant embedding over W of E into the preimage $f^{-1}(V)$*
- *γ is a Moore path in $Emb^G_W(E, \psi^* M)$ from g to $f^{-1}(\varphi) \circ h$, where $f^{-1}(\varphi) : f^{-1}(V) \hookrightarrow \psi^* M$.*

Furthemore, we have

$$(ev_0)_{[\psi]}(g : E \hookrightarrow \psi^*M, \varphi : V \hookrightarrow [-1,1], h : E \hookrightarrow f^{-1}(V), \gamma) = (g : E \hookrightarrow \psi^*M) \in (\underline{Disk}^{G,B-fr}_{[W]})_{/\psi^*M}$$
$$(ev_1)_{[\psi]}(g : E \hookrightarrow \psi^*M, \varphi : V \hookrightarrow [-1,1], h : E \hookrightarrow f^{-1}(V), \gamma) = (\varphi : V \hookrightarrow [-1,1]) \in Disc^{\partial,or}_{/[-1,1]}$$

Lemma 9.2.7. *The $\underline{O}$-functor $ev_0 : \underline{Disk}_f \to \underline{Disk}^{G,B-fr}_{/\underline{M}}$ is an $\underline{O}$-Cartesian fibration. Additionally, the $\underline{O}$-functor $ev_1 : \underline{Disk}_f \to \underline{O} \times Disk^{\partial,or}_{/[-1,1]}$ is an $\underline{O}$-coCartesian fibration.*

Proof. From [HTT] *2.4.7.12* it follows that for every $(\psi : W \to O) \in \underline{O}$ the functor

$$(ev_0)_{[\psi]} : (\underline{Disk}_f)_{[\psi]} \to (\underline{Disk}^{G,B-fr}_{[W]})_{/\psi^*M}$$

is a Cartesian fibration. Moreover, a morphism in $\underline{Disk}_f$ is (ev_0)-Cartesian if and only if its image in $\underline{Disk}^{G,B-fr}_{/\underline{M}}$ is an equivalence, hence the second condition of [Shah18] *7.1* is true by [HTT] *2.4.1.5* and *2.4.1.7*. □

Lemma 9.2.8. *The $\underline{O}$-functor $ev_0 : \underline{Disk}_f \to \underline{Disk}^{G,B-fr}_{/\underline{M}}$ is $\underline{O}$-cofinal.*

Proof. By 8.0.6 it will suffice to prove the claim in the non-framed case.
We have to show that for every $\varphi : W \to O$ the map $(ev_0)_{[\varphi]} : (\underline{Disk}_f)_{[\varphi]} \to (\underline{Disk}^G_{/\underline{M}})_{[\varphi]}$ is cofinal. Without the loss of generality it will suffice to prove the claim for $\varphi : O \xrightarrow{=} O$. In other words, we need to show that $(ev_0)_{[O]} : (\underline{Disk}_f)_{[O]} \to (\underline{Disk}^G_{/\underline{M}})_{[O]}$ is cofinal.
Note that with a choice of a basepoint $x \in O$ we have $(O, x) \cong G/H$ where $H = Stab(x)$ is a finite subgroup, and using the equivalence $\underline{G/H} \simeq \mathcal{O}_H^{op}$ we restrict to the finite group case. We have

$$(\underline{Disk}^G_{/\underline{M}})_{[G/H]} \simeq (\underline{Disk}^G_{[G/H]})_{/M} \simeq (\underline{Disk}^H_{[H/H]})_{/M_H}$$

where M_H is the fiber of $M \to G/H$ over eH where e is the neutral element of G. Moreover, $(\underline{Disk}_f)_{[O]}$ is the limit of

$$\begin{tikzcd}
(\underline{Disk}^G_{[G/H]})_{/M} \arrow[d] & Fun(\Delta^1, (\underline{Mfld}^G_{[G/H]})_{/M}) \arrow[dl, "ev_0"'] \arrow[d, "ev_1"] & Disk^{\partial,or}_{/[-1,1]} \arrow[dl, "\psi^* f^{-1}"] \\
(\underline{Mfld}^G_{[G/H]})_{/M} & (\underline{Mfld}^{G,B-fr}_{[G/H]})_{/M} &
\end{tikzcd}$$

which becomes the limit of

$$\begin{tikzcd}
(\underline{Disk}^H_{[H/H]})_{/M_H} \arrow[d] & Fun(\Delta^1, (\underline{Mfld}^H_{[H/H]})_{/M_H}) \arrow[dl, "ev_0"'] \arrow[d, "ev_1"] & Disk^{\partial,or}_{/[-1,1]} \arrow[dl] \\
(\underline{Mfld}^H_{[H/H]})_{/M_H} & (\underline{Mfld}^H_{[H/H]})_{/M_H} &
\end{tikzcd}$$

The proof now follows from [Hor19] *5.2.7*.

□

We are ready to give the proof of 9.2.4:

Proof. By 9.2.8 we can write

$$\int_M A \simeq \underline{O} - colim(F) \tag{9.3}$$

where F is an $\underline{O}$-functor

$$F : \underline{Disk}_f \to \underline{Disk}^{G,B-fr}_{/\underline{M}} \to \underline{O \times Disk}^{G,B-fr} \to \underline{O \times \mathcal{C}}$$

Consider the following diagram of $\underline{O}$-categories

$$\begin{tikzcd}
\underline{Disk}_f \arrow[r, "ev_0"] \arrow[d, "ev_1"] & \underline{Disk}^{G,B-fr}_{/\underline{M}} \arrow[r] & \underline{O \times Disk}^{G,B-fr} \arrow[r] & \underline{O \times \mathcal{C}} \\
\underline{O} \times Disk^{\partial,or}_{/[-1,1]} \arrow[urrr, "L"] & & &
\end{tikzcd}$$

where L is the parametrized left Kan extension of F along ev_0. Note that L is obtained via the $\underline{O}$-left adjoint functor $(ev_1)_!$ i.e. $L = (ev_1)_!(F)$ where

$$(ev_1)_! : \underline{Fun}_{\underline{O}}(\underline{Disk}_f, \underline{O \times \mathcal{C}}) \leftrightarrows \underline{Fun}_{\underline{O}}(\underline{O} \times Disk^{\partial,or}_{/[-1,1]}, \underline{O \times \mathcal{C}}) : (ev_1)^*$$

is an $\underline{O}$-adjoint pair. Since parametrized left adjoint functors preserve parametrized colimits (see [Shah18] *8.7*) we have

$$\underline{O} - colim(F_1) \simeq \underline{O} - colim((ev_1)_!(F)) \simeq \underline{O} - colim(F) \simeq \int_M A$$

Additionally, since the $\underline{O}$-colimit over $\underline{O} \times Disk^{\partial,or}_{/[-1,1]}$ is equivalent to the (non-parametrized) colimit over $Disk^{\partial,or}_{/[-1,1]}$ by the equivalence

$$\begin{aligned}
Fun_{\underline{O}}(\underline{O} \times Disk^{\partial,or}_{/[-1,1]}, \underline{O \times \mathcal{C}}) &\xrightarrow{\simeq} Fun(Disk^{\partial,or}_{/[-1,1]}, \underline{\mathcal{C}}_{[O]}) \\
L &\mapsto L|_{(O \xrightarrow{=} O) \times Disk^{\partial,or}_{/[-1,1]}} = L_1
\end{aligned}$$

we obtain the following

$$\int_M A \simeq colim(L_1) = \underset{V \in Disk^{\partial,or}_{/[-1,1]}}{colim} L_1(V)$$

Using the same arguments as in [Hor19] *proof of 5.2.3*, again by passing to the finite group case by writing $(O, x) \cong G/H$, $\underline{G/H} \simeq \mathcal{O}_H^{op}$ with the choice of basepoint $x \in O$ and $H = Stab(x)$ a finite group, we can conclude

$$L_1(V) \simeq \int_{f^{-1}(V)} A$$

We finish the proof by using the result [AF15] *3.11* the same way as in 9.2.1 which gives us the desired result. ∎

9.3 G-sequential unions

Definition 9.3.1. Let $M \to O$ be an $\mathcal{O}_G$-manifold. A G-sequential union of (the underlying G-manifold) M is a sequence of open $\mathcal{O}_G$-submanifolds $M_1 \subset M_2 \subset ... \subset M$ with $M = \cup_{i=1}^{\infty} M_i$. A G-sequential union of a B-framed $\mathcal{O}_G$-manifold $M \in \underline{Mfld}_{[O]}^{G,B-fr}$ is a G-sequential union of its underlying G-manifold.

Let $F : \underline{Mfld}^{G,B-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal functor and $M = \cup_{i=1}^{\infty} M_i$ a G-sequential union of $M \in \underline{Mfld}_{[O]}^{G,B-fr}$, then F induces a map

$$colim_i F(M_i) \to F(M) \tag{9.4}$$

in $\underline{\mathcal{C}}_{[O]}^{\otimes}$.

Definition 9.3.2. Let $F : \underline{Mfld}^{G,B-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal functor and $M = \cup_{i=1}^{\infty} M_i$ a G-sequential union of $M \in \underline{Mfld}_{[O]}^{G,B-fr}$. We say that F respects G-sequential unions if the map (9.4) is an equivalence in $\underline{\mathcal{C}}_{[O]}^{\otimes}$.

Proposition 9.3.3. *Let $A \in Fun_G^{\otimes}(\underline{Disk}^{G,B-fr,\sqcup}, \underline{\mathcal{C}}^{\otimes})$ be a B-framed G-disc algebra with values in $\underline{\mathcal{C}}$. Then the G-factorization homology functor $\int A : \underline{Mfld}^{G,B-fr} \to \underline{\mathcal{C}}$ respects G-sequential unions.*

First we prove the following useful lemmas:

Lemma 9.3.4. *Let $M \in \underline{Mfld}_{[O]}^{G,B-fr}$ be a B-framed $\mathcal{O}_G$-manifold over $O \in \mathcal{O}_G$, and $M = \cup_{i=1}^{\infty} M_i$ a G-sequential union of M and let $(E \to U \to O) \in \underline{Disk}_{[O]}^{G,B-fr}$ be a finite disjoint union of G-discs. Then the induced map*

$$hocolim_i Emb_O^G(E, M_i) \to Emb_O^G(E, M)$$

is a weak equivalence.

Proof. By 7.3.6 the following square

$$\begin{tikzcd} Emb_O^G(E, M_i) \arrow[r] \arrow[d] & Emb_O^G(E, M) \arrow[d] \\ Conf_O^G(U, M_i) \arrow[r] & Conf_O^G(U, M) \end{tikzcd}$$

is a homotopy pullback square. Since homotopy colimits preserve homotopy pullbacks we obtain the homotopy pullback square

$$\begin{tikzcd} hocolim_i Emb_O^G(E, M_i) \arrow[r] \arrow[d] & Emb_O^G(E, M) \arrow[d] \\ hocolim_i Conf_O^G(E, M_i) \arrow[r] & Conf_O^G(E, M) \end{tikzcd}$$

Fortunately, since M_i are all open in M and since $M = \cup_{i=1}^{\infty} M_i$, the collection of $Conf_O^G(E, M_i)$ represents the complete open cover of $Conf_O^G(E, M)$ hence the lower horizontal line is an equivalence by [DI04] *1.6.* $\square$

Lemma 9.3.5. *Let $M \in \underline{Mfld}^{G,B-fr}_{[O]}$ be a B-framed $\mathcal{O}_G$-manifold over $O \in \mathcal{O}_G$, and $M = \cup_{i=1}^{\infty} M_i$ a G-sequential union of M. Then the $\underline{O}$-functor*

$$colim_i \underline{Disk}^{G,B-fr}_{/\underline{M_i}} \to \underline{Disk}^{G,B-fr}_{/\underline{M}}$$

is an equivalence of $\underline{O}$-categories.

Proof. By 8.0.6 it will suffice to show that the $\underline{O}$-functor $colim_i \underline{Disk}^{G,B-fr}_{/\underline{M_i}} \to \underline{Disk}^{G,B-fr}_{/\underline{M}}$ is a fiberwise equivalence. Without loss of generality, it will further suffice to prove that $colim_i(\underline{Disk}^{G,B-fr}_{/\underline{M_i}})_{[\varphi]} \to (\underline{Disk}^{G,B-fr}_{/\underline{M}})_{[\varphi]}$ is an equivalence for $\varphi : O \xrightarrow{=} O$. Therefore, we need to show that the functor

$$colim_i(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}} \to (\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M}}$$

is fully faithful and essentially surjective.
For the former, let $E_1 \to U_1 \to O$ and $E_2 \to U_2 \to O$ be the two elements of $colim_i(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}$. There is big enough $i \in \mathbb{N}$ such that $(E_1 \to U_1 \to O), (E_2 \to U_2 \to O) \in (\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}$ given by G-embeddings $e_1 : E_1 \hookrightarrow M_i$ and $e_2 : E_2 \hookrightarrow M_i$. Then the mapping space $Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}}(E_1, E_2)$ is given as the homotopy fiber

$$Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}}(E_1, E_2) \to Emb^G_O(E_1, E_2) \xrightarrow{(e_2)_*} Emb^G_O(E_1, M_i)$$

over e_1. Since the homotopy fibers are preserved by filtered homotopy colimits, we obtain the following sequence

$$hocolim_i Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}}(E_1, E_2) \to Emb^G_O(E_1, E_2) \to hocolim_i Emb^G_O(E_1, M_i)$$

Recall that by 9.3.4 we have an equivalence $hocolim_i Emb^G_O(E_1, M_i) \simeq Emb^G_O(E_1, M)$ giving us

$$hocolim_i Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}}(E_1, E_2) \to Emb^G_O(E_1, E_2) \to Emb^G_O(E_1, M)$$

In particular, the upper sequence expreses $hocolim_i Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}}(E_1, E_2)$ as the homotopy fiber of $Emb^G_O(E_1, E_2) \to Emb^G_O(E_1, M)$ over the map $E_1 \xrightarrow{e_1} M_i \hookrightarrow M$. On the other hand, this homotopy fiber is exactly equivalent to $Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M}}}(E_1, E_2)$, hence giving us a homotopy equivalence

$$hocolim_i Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}}(E_1, E_2) \simeq Map_{(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M}}}(E_1, E_2)$$

hence proving the fully faithful condition.

For essential surjectivity, consider $(E \to U \to O) \in (\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M}}$. We can chose $t > 0$ small enough such that $B_t(E) \hookrightarrow E \hookrightarrow M$ factors through some M_i. By radial dilation $E \to U \to O$ and $B_t(E) \to U \to O$ are equivalent objects in $(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M}}$. On the other hand, $B_t(E) \to U \to O$ represents an object of $colim_i(\underline{Disk}^{G,B-fr}_{[O]})_{/\underline{M_i}}$ thus proving the essential surjectivity condition. $\square$

The proof of 9.3.3 goes as follows:

Proof. Let $M \in \underline{Mfld}^{G,B-fr}_{[O]}$ be a B-framed G-manifold such that $M = \cup_i M_i$ is a sequential union of G-manifolds. Then

$$\begin{aligned}colim_i \int_{M_i} A =& colim_i(\underline{O} - colim(\underline{Disk}^{G,B-fr}_{/\underline{M_i}} \to \underline{O}\underline{\times}\underline{Disk}^{G,B-fr} \to \underline{O}\underline{\times}\underline{\mathcal{C}}))\\ \simeq& \underline{O} - colim(colim_i(\underline{Disk}^{G,B-fr}_{/\underline{M_i}} \to \underline{O}\underline{\times}\underline{Disk}^{G,B-fr} \to \underline{O}\underline{\times}\underline{\mathcal{C}}))\end{aligned}$$

Then, by using 9.3.5 we obtain

$$colim_i \int_{M_i} A \simeq \underline{O} - colim(\underline{Disk}^{G,B-fr}_{/\underline{M}} \to \underline{O}\underline{\times}\underline{Disk}^{G,B-fr} \to \underline{O}\underline{\times}\underline{\mathcal{C}}) = \int_M A$$

□

9.4 Axiomatic caracterization of G-factorization homology

In this final section we give the axiomatic caracterization of the G-factorization homology. In other words, we will prove that G-factorization homology represents all of the G-homology theories, that is, G-symmetric monoidal functors whose domain is $\underline{Mfld}^{G,B-fr}$ and which take values in presentable G-symmetric monoidal categories who, in addition, satisfy the G-$\otimes$-excision property and who respect G-sequential unions.

Definition 9.4.1. Let $\underline{C}^{\otimes} \to \underline{Fin}^G_*$ be a G-symmetric monoidal G-∞-category, $f : B \to BO_n(G)$ be a G-map and let $F \in Fun^{\otimes}_G(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes})$ be a G-symmetric monoidal functor. We will call F an *equivariant homology theory* if F satisfies G-$\otimes$-excision property and is compatible with G-sequential unions. We will denote with $\mathcal{H}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes})$ the full subcategory of $Fun^{\otimes}_G(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes})$ consisting of equivariant homology theories.

Construction 9.4.2. Let N, M with $N \subseteq M$ be compact G-manifolds. Additionally, let $f : M \to \mathbb{R}$ be an equivariant Morse function (which exists by [Was69] *4.10*) such that $M \smallsetminus N$ contains a single critical orbit, which we denote with W. Furthemore, $T_W M \to W$ decomposes as the sum of G-vector bundles $T_W M \cong P \oplus B$ such that the Hessian of f at W is positive on P and negative on B. Denote with $\overline{\mathbb{D}}(P) \to W$ and $\overline{\mathbb{D}}(B) \to W$ the closed unit disc bundles and let $\mathbb{S}(B) \to W$ be the unit sphere bundle. Then

$$M \cong N \bigcup_{\overline{\mathbb{D}}(P)\times_W \mathbb{S}(B)} \overline{\mathbb{D}}(P) \times_W \overline{\mathbb{D}}(B)$$

In other words, we say that M is obtained from N by attaching a handle-bundle of type $(P, B)_{W,f}$.

Now assume that M is a G-manifold, not necessarily compact. We can again choose a Morse function on $f : M \to \mathbb{R}$ such that $f^{-1}(-\infty, r]$ is a compact G-submanifold for every $r \in \mathbb{R}$. Moreover, we can choose the increasing sequence of regular values $r_0, r_1, ...$ such that $lim_{i\to+\infty} r_i = +\infty$, $f^{-1}(-\infty, r_0) = \emptyset$ and (r_i, r_{i+1}) contains a critical point. With the notation $M_i = f^{-1}(-\infty, r_i)$, M becomes the G-sequential union $\bigcup_{i=0}^{+\infty} M_i$. Notice that $M_{i+1} \smallsetminus \overline{M_i}$ contains only a finite number of critical orbits (since $\overline{M_i}$ is compact for all i), which we can, without the loss of generality reduce to one critical orbit because the orbits are disjoint. Therefore, $\overline{M_{i+1}}$ is obtained from $\overline{M_i}$ by attaching a single handle bundle of type $(P, B)_{W,f}$ described above. Therefore

$$\overline{M}_{i+1} \cong \overline{M}_i \bigcup_{\overline{\mathbb{D}}(P)\times_W \mathbb{S}(B)} \overline{\mathbb{D}}(P) \times_W \overline{\mathbb{D}}(B)$$

Taking $\mathbb{A}(B)$ to be a unit annulus bundle (which is a G-tubular neighborhood of the unit sphere bundle) and discarding the boundary points we have

$$M_{i+1} \cong M_i \bigcup_{\mathbb{D}(P)\times_W \mathbb{A}(B)} \mathbb{D}(P) \times_W \mathbb{D}(B)$$

Therefore, M_{i+1} is obtained from M_i by attaching an "open" handle bundle of type $(P, B)_{W,f}$. Since M is a sequential union of G-manifolds M_i we have a handle bundle decomposition of M.

Theorem 9.4.3. *Let $\underline{C}^{\otimes} \to \underline{Fin}^G_*$ be a presentable G-symmetric monoidal ∞-category and let $f : B \to BO_n(G)$ be a G-map. Then the adjunction*

$$(i^{\otimes})_! : Fun^{\otimes}_G(\underline{Disk}^{G,B-fr,\sqcup}, \underline{C}^{\otimes}) \rightleftarrows Fun^{\otimes}_G(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes}) : (i^{\otimes})^*$$

restricts to an equivalence

$$(i^{\otimes})_! : Fun^{\otimes}_G(\underline{Disk}^{G,B-fr,\sqcup}, \underline{C}^{\otimes}) \xrightarrow{\simeq} \mathcal{H}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes})$$

Proof. Let $H \in \mathcal{H}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes})$ and let A_H be the restriction of H along $i^{\otimes}$. Since the G-symmetric monoidal functor $(i^{\otimes})_!$ factors through the full ∞-subcategory $\mathcal{H}(\underline{Mfld}^{G,B-fr,\sqcup}, \underline{C}^{\otimes})$ (by 9.2.4 and 9.3.3) it would be enough to show that the counit map

$$\int_{-} A_H \to H$$

is an equivalence. We will prove this by induction:

Let $\mathcal{F}_k$ for $k = 0, 1, ..., n$ be the full G-∞-subcategory of $\underline{Mfld}^{G,B-fr}$ such that the underlying $\mathcal{O}_G$-manifolds can be written in the form $M \times_O E \to O'$, where $O \to O'$ is a map in $\mathcal{O}_G$, $M \to O$ is a $(l+k)$-dimensional $\mathcal{O}_G$-manifold and $E \to O$ is a finite disjoint union of $(n+l-k)$-dimensional G-discs. Note that the fibers of $M \to O$ are k-dimensional smooth manifolds. We prove our claim by induction on k:

For $k = 0$, $M \to O$ a finite G-set. Since $\mathcal{O}_G$ is an orbital ∞-category $M \times_O E \to O$ is just a disjoint union of G-discs and hence

$$\int_{M \times_O E} A_H \simeq A_H(M \times_O E) = H(M \times_O E)$$

since $(i^{\otimes})_!$ is fully faithful.
When $k > 0$ note that M can be written as a G-sequential union $M = \bigcup_{i=1}^{+\infty} M_i$ such that each M_i admits a finite handle bundle decomposition. Since both H and $\int_{-} A_H$ respect G-sequential unions it will suffice to provide proof when M admits a finite handle bundle decomposition. We proceed with the induction on the handle decomposition. The base case of induction is assured. For the inductive step, assume that M is obtained from N by attaching a handle-bundle of type $(P, B)_{W,f}$. Then we have

$$M \cong N \bigcup_{\mathbb{D}(P)\times_W \mathbb{A}(B)} \mathbb{D}(P) \times_W \mathbb{D}(B), \text{ and consequently}$$

$$M \times_O E \cong N \times_O E \bigcup_{(\mathbb{D}(P)\times_W \mathbb{A}(B))\times_O E} (\mathbb{D}(P) \times_W \mathbb{D}(B)) \times_O E$$

The decomposition of $M \times_O E$ presented as above is a G-collar gluing map. Intuitively, this collar gluing map sends $N \times_O E$ to the point -1 in the segment $[-1, 1]$, $(\mathbb{D}(P) \times_W \mathbb{D}(N)) \times_O E$ sends to 1, and the manifold bundle structure of the restriction $(\mathbb{D}(P) \times_W \mathbb{A}(B)) \times_O E \to (-1, 1)$ is given by the decomposition $\mathbb{A}(B) \cong \mathbb{S}(B) \times (-1, 1)$ i.e. the fibers of the manifold bundle are given by $(\mathbb{D}(P) \times_W \mathbb{S}(B)) \times_O E$.
Proceeding further with the proof, we distinguish between two cases:

1. $dim(B) = l$: In which case $N \to W$ is just a finite G-set and therefore $\mathbb{A}(B) = \emptyset$. Moreover, M decomposes as

$$M \times_O E \cong (N \times_O E) \sqcup (\mathbb{D}(P) \times_W B) \times_O E$$

which gives us

$$\int_{M \times_O E} A_H \simeq \int_{N \times_O E} A_H \otimes \int_{(\mathbb{D}(P) \times_W B) \times_O E} A_H$$

By induction hypotesis $\int_{N \times_O E} A_H \simeq H(N \times_O E)$, and since $(\mathbb{D}(P) \times_W B) \times_O E$ is disjoint union of G-discs, we have $\int_{(\mathbb{D}(P) \times_W B) \times_O E} A_H \simeq A_H((\mathbb{D}(P) \times_W B) \times_O E) = H((\mathbb{D}(P) \times_W B) \times_O E)$, which in total gives us

$$\int_M A_H \simeq H(N) \otimes H((\mathbb{D}(P) \times_W B) \times_O E) \simeq H((N \times_O E) \sqcup (\mathbb{D}(P) \times_W B) \times_O E) \simeq H(M)$$

2. $dim(B) > l$ in which case $\mathbb{A}(B) \cong \mathbb{S}(B) \times (-1, 1)$ where G acts trivially on the interval (see 9.4.4). Therefore

$$(\mathbb{A}(B) \times_W \mathbb{D}(P)) \times_O E \cong (\mathbb{S}(B) \times (-1, 1)) \times_W (P \times_O E) \cong \mathbb{S}(B) \times_W ((-1, 1) \times (P \times_O E))$$

which leads us to the conclusion $dim(\mathbb{S}(B)) = dim(B) - 1 \leq M - 1 = l + k - 1$. Hence, $(\mathbb{A}(B) \times_W \mathbb{D}(P)) \times_O E \in \mathcal{F}_{k-1}$.
By the induction hypothesis (on k) we have

$$\int_{(\mathbb{A}(B) \times_W \mathbb{D}(P)) \times_O E} A_H \simeq H((\mathbb{A}(B) \times_W \mathbb{D}(P)) \times_O E)$$

and by the same reasoning as before

$$\int_{N \times_O E} A_H \simeq H(N \times_O E)$$
$$\int_{(\mathbb{D}(P) \times_W \mathbb{D}(B)) \times_O E} A_H \simeq H((\mathbb{D}(P) \times_W \mathbb{D}(B)) \times_O E)$$

which in the end gives us

$$\int_M A_H \simeq \int_{N \times_O E} A_H \otimes_{\int_{(\mathbb{A}(B) \times_W \mathbb{D}(P)) \times_O E} A_H} \int_{(\mathbb{D}(P) \times_W \mathbb{D}(B)) \times_O E} A_H$$
$$H(N \times_O E) \otimes_{H((\mathbb{A}(B) \times_W \mathbb{D}(P)) \times_O E)} H((\mathbb{D}(P) \times_W \mathbb{D}(B)) \times_O E) \simeq H(M)$$

which finishes the proof.

$\square$

Remark 9.4.4. In [Was69], Wasserman works with *Riemannian* G-manifolds, meaning that the equivariant form given on the tangent bundle is orthogonal i.e. G acts by linear isometries. In our work, we have only recquired the form to be equivariantly linear. Luckily, this does not represent a problem since the general linear and orthogonal group are homotopy equivalent, meaning that we can always replace our linear form with an orthogonal one. Therefore we can claim that the annulus bundle $\mathbb{A}(N)$ is a tubular neighborhood of the unit sphere bundle $\mathbb{S}(N)$ and hence we can write $\mathbb{A}(N) \cong \mathbb{S}(N) \times (-1, 1)$.

Remark 9.4.5. In the introductory part of [Hor19], Horev stated some dificulties when trying to expand to the case of compact Lie groups. Namely, the handle bundles over critical orbits can be non-trivial, since the orbits are of possibly positive dimension, but in our case they are still G-discs, since we have defined them as (not necessarily trivial) G-vector bundles over orbits. Secondly, all of the orbits are of the same dimension, which makes the proof almost analogous to that of Horev. And finally, our ∞-category $\mathcal{O}_G$ is taken to be the ∞-category of orbits with finite stabilizers, which is *orbital* i.e. it is closed under fiber products, which is also important in the proof.

Part III

Universal property of framed G-disc algebras

Chapter 10

Universal property of framed G-disc algebras

10.1 G-approximations to G-∞-operads

In this section we will present the theory of G-approximations to G-∞-operads. It is rather technical and serves to develop the machinery used in the following section for proving the universal property of framed G-disc algebras. Informally, a G-approximation to a G-∞-operad is an ∞-category that somewhat behaves as a G-∞-operad. It is a more general object that can capture information about the original G-∞-operad.

Definition 10.1.1. Let $p : E^{\otimes} \to \underline{Fin}_*^G$ be a G-∞-operad and let $f : C \to E^{\otimes}$ be a categorical fibration. We say that f is a *G-approximation* to $E^{\otimes}$ if it satisfies the following conditions:

1. Let $p' = p \circ f$. The ∞-category $C_{I(-)}$ obtained as the pullback

$$\begin{tikzcd} C_{I(-)} \arrow[r] \arrow[d] & C \arrow[d, "p'"] \\ \mathcal{O}_G^{op} \arrow[r] & \underline{Fin}_*^G \end{tikzcd}$$

   together with the map $C_{I(-)} \to \mathcal{O}_G^{op}$ is a G-∞-category i.e. the map is a coCartesian fibration.

2. Let $c \in C$ and let $p'(c) = I = [U \to O]$. Then there exists p'-coCartesian morphism $c \to c_W$ in C lifting $\chi_{W \subseteq U} : I \to I(W)$ for every $W \in Orbit(U)$. Additionally, the map $f(c \to c_W)$ is an inert map in $E^{\otimes}$.

3. Let $c \in C$ and let $\alpha : x \to f(c)$ be an active morphism in $E^{\otimes}$. Then there exists a Cartesian lift $\bar{\alpha} : \bar{x} \to c$ of α in C.

Remark 10.1.2. Recall that, given a G-∞-operad $E^{\otimes}$ we can set up a factorization system on the arrows of $E^{\otimes}$ the following way

- the *inert* arrows are the inert arrows of G-∞-operads.
- the *active* arrows are the fiberwise active arrows, that is, active arrows that lie over an equivalence in $\mathcal{O}_G^{op}$.

Let $f : C \to E^{\otimes}$ be a G-approximation. Then by [HA] *2.1.2.5* we have an induced factorization system on C by taking

- f*-inert* arrows to be arrows α such that $f(\alpha)$ is an inert arrow in $E^{\otimes}$.
- f*-active* arrows to be the arrows α such that they are f-Cartesian and $f(\alpha)$ is active in $E^{\otimes}$.

Definition 10.1.3. Let $p : E^{\otimes} \to \underline{Fin}^G_*$ and $q : E'^{\otimes} \to \underline{Fin}^G_*$ be two G-∞-operads and let $f : C \to E^{\otimes}$ be a G-approximation. Denote with $p' = p \circ f$ and consider a functor $F : C \to E'^{\otimes}$. We will say that F is a C*-algebra object* in $E'^{\otimes}$ if the following conditions are satisfied:

- F induces a G-functor between underlying G-∞-categories $C_{I(-)} \to E'$.
- The following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "f"] & E'^{\otimes} \arrow[d, "q"] \\ E^{\otimes} \arrow[r, "p"] & \underline{Fin}^G_* \end{tikzcd}$$

- Let $x \in C$ and let $p'(x) = [U \to O]$. Let $\alpha_W : x \to x_W$ be a p'-coCartesian arrow lifting $\chi : [U \to O] \to [W \xrightarrow{=} W]$ for $W \in Orbit(U)$. Then the map $F(\alpha_W)$ is an inert map in $E'^{\otimes}$.

We will denote with $Alg_G(C, E')$ the ∞-category of C-algebra objects in $E'^{\otimes}$.

We would like to prove the proposition 10.1.7 (the equivariant version of [HA] *2.3.3.23*). The following lemmas will be helpful:

Lemma 10.1.4. *Let $p : O^{\otimes} \to \underline{Fin}^G_*$ and $q : E^{\otimes} \to \underline{Fin}^G_*$ be two G-∞-operads and let $F : O^{\otimes} \to E^{\otimes}$ be a map over $\underline{Fin}^G_*$. Let $\mathcal{F}$ be a class of arrows in $\underline{Fin}^G_*$ of type*

$$\begin{tikzcd} U \arrow[d] & O_2 \arrow[l] \arrow[r, "="] \arrow[d, "="] & O_2 \arrow[d, "="] \\ O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 \end{tikzcd}$$

Then F is a map of G-∞-operads if and only if it sends an inert arrow in $O^{\otimes}$ over $\mathcal{F}$ to an inert arrow in $E^{\otimes}$.

Proof. For a finite G-set $I = [U \to O]$ note that the maps $\chi_{[W \subseteq U]}$ all belong to $\mathcal{F}$. The proof is now analogous to the [HA] *2.1.2.9*. $\square$

Lemma 10.1.5. *Let $p : E^{\otimes} \to \underline{Fin}^G_*$ be a G-∞-operad and let $f : C \to E^{\otimes}$ be a G-approximation. Let $c \in C$ and let $\alpha : x \to f(c)$ be any morphism in $E^{\otimes}$. Consider*

$$\Sigma \subseteq C_{/c} \times_{E^{\otimes}_{/f(c)}} E^{\otimes}_{x//f(c)}$$

the full subcategory spanned by the objects corresponding to pairs $\{\beta : d \to c, \gamma : x \to f(d)\}$ such that γ is inert. The ∞-category Σ is contractible.

Proof. We can factorize α as in the following commutative diagram

$$\begin{tikzcd} & y \arrow[dr, "\alpha_2"] & \\ x \arrow[ur, "\alpha_1"] \arrow[rr, "\alpha"] & & f(c) \end{tikzcd}$$

such that α_1 is inert and α_2 is active. By definition of a G-approximation, we have a Cartesian lift $\bar{\alpha}_2 : \bar{y} \to c$ of γ in C. Now, we claim that the pair $\sigma = (\bar{\alpha}_2 : \bar{y} \to c, \alpha_1 : x \to y = f(\bar{y}))$ is a terminal object of Σ. We will show this, by proving that the map

$$\Sigma_{/\sigma} \to \Sigma$$

is a trivial Kan fibration. In other words, we need to prove the existence of the dashed lift

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "s_0"] \arrow[d, hook] & \Sigma_{/\sigma} \arrow[d] \\
\Delta^n \arrow[r, "s"] \arrow[ur, dashed] & \Sigma
\end{tikzcd}$$

The map $s_0 : \partial\Delta^n \to \Sigma_{/\sigma}$ corresponds to the map $s_0' : \Lambda^{n+1}_{n+1} \to \Sigma$ sending $\{n+1\} \mapsto \sigma$. Consider the restriction map

$$t_0 : \Lambda^{n+1}_{n+1} \xrightarrow{s_0'} \Sigma \to E^{\otimes}_{x/ /f(c)} \to E^{\otimes}_{x/}$$

Next, note that this map corresponds to

$$t_0' : \Lambda^{n+2}_0 \to E^{\otimes}$$

such that $t_0' : \{0,1\} \mapsto \alpha_1 : x \to y$. Furthermore, we have that

$$\Delta^{n+1} \cong \Delta^{\{0,2,\ldots,n+2\}} \xrightarrow{t_0'|_{\Delta^{\{0,2,\ldots,n+2\}}}} E^{\otimes}$$

corresponds to

$$\Delta^{n+1} \cong \{0\} \star \Delta^n \to E^{\otimes}$$

sending $\{0\}$ to x, which is induced by the map $\Delta^n \xrightarrow{s} \Sigma \to E^{\otimes}_{x/ /f(c)} \to E^{\otimes}_{x/}$. Since $t_0' : \Delta^{\{0,1\}} \mapsto (x \to y)$ is inert and hence coCartesian, we have a lift

$$\begin{tikzcd}
\Lambda^{n+2}_{n+2} \arrow[r, "t_0'"] \arrow[d, hook] & E^{\otimes} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \underline{Fin}^G_*
\end{tikzcd}$$

Therefore, we also have a lift $\Delta^{n+1} \to E^{\otimes}_{x/}$ given by

$$\begin{tikzcd}
\Lambda^{n+1}_{n+1} \arrow[r, "s_0'"] \arrow[d, hook] & \Sigma \arrow[r] & E^{\otimes}_{x/ /f(c)} \arrow[r] & E^{\otimes}_{x/} \\
\Delta^{n+1} \arrow[urr, dashed] \arrow[urrr] & & &
\end{tikzcd}$$

Finally, since the map $E^{\otimes}_{x/ /f(c)} \to E^{\otimes}_{x/}$ is a right fibration, we have the existence of a dashed lift in the upper diagram. Let us denote the restriction of that lift $\Delta^{n+1} \to E^{\otimes}_{x/ /f(c)} \to E^{\otimes}_{/f(c)}$ with χ, and the corresponding map $\Delta^{n+2} \to E^{\otimes}$ with χ'.

Similarly, if we look at the map

$$t : \Lambda^{n+1}_{n+1} \xrightarrow{s_0'} \Sigma \to C_{/c}$$

we see that it corresponds to the map

$$t' : \Lambda^{n+2}_{n+2} \to C$$

such that $t' : \Delta^{\{n+1,n+2\}} \mapsto (\bar{\alpha}_2 : \bar{y} \to c)$. Again, $\Delta^{n+1} \cong \Delta^{\{0,1,\ldots,n,n+2\}} \subset \Lambda^{n+2}_{n+2} \to C$ is induced by the map $\Delta^n \xrightarrow{s} \Sigma \to C_{/c} \to C$. Since $\bar{\alpha}_2$ is Cartesian, we have the following lift

$$\begin{tikzcd}
\Lambda^{n+2}_{n+2} \arrow[r, "t'"] \arrow[d, hook] & E^{\otimes} \\
\Delta^{n+2} \arrow[ur, dashed] &
\end{tikzcd}$$

which in total gives us the lift

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "s_0"] \arrow[d, hook] & \Sigma_{/\sigma} \arrow[d] \\
\Delta^n \arrow[r, "s"] \arrow[ur, dashed] & \Sigma
\end{tikzcd}$$

Therefore $\Sigma_{/\sigma} \to \Sigma$ is a trivial Kan fibration, and hence, by [HTT] *1.2.12.3* and *1.2.12.5* σ is a terminal object.

$\square$

Remark 10.1.6. In Lurie's book [HA], one can find a definition of a *weak approximation* to be a map $C \to E^{\otimes}$ satisfying the second point from the definition 10.1.1 and the condition from 10.1.5 (in the non-equivariant case). In this paper, we will stick to the work with (strong) G-approximation altough one could add the definition of the weak one.

Proposition 10.1.7. *Let $p : E^{\otimes} \to \underline{Fin}^G_*$ and $q : E'^{\otimes} \to \underline{Fin}^G_*$ be two G-∞-operads and let $f : C \to E^{\otimes}$ be a G-approximation. Furthemore, assume that the induced map $C_{I(-)} \to E$ is an equivalence of G-∞-categories. Then the induced map*

$$\theta : Alg_G(E, E') \to Alg_G(C, E')$$

is an equivalence of ∞-categories.

Proof. Firsty, we can choose a Cartesian fibration $u : M \to \Delta^1$ associated to the functor f, together with isomorphisms $M \times_{\Delta^1} \{0\} \simeq E^{\otimes}$, $M \times_{\Delta^1} \{1\} \simeq C$ and a retraction $r : M \to E^{\otimes}$ such that $r|_C = f$. Denote with $\Upsilon \subseteq Fun_{\underline{Fin}^G_*}(M, E')$ the full subcategory spanned by the functors $F : M \to E'$ such that:

1. The restriction $F|_{E^{\otimes}}$ belongs to $Alg_G(E, E')$,
2. For every u-Cartesian morphism α of M, the image $F(\alpha)$ is an equivalence in E' (or, equivalently, F is a q-left Kan extension of $F|_{E^{\otimes}}$).

We will continue the proof and complete it in several steps:

Step 1. *The restriction map $\Upsilon \to Alg_G(E, E')$ is a trivial Kan fibration.*

Let $\mathcal{X} \subseteq Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ be the full subcategory spanned by those functors $F : M \to E'^{\otimes}$ which are q-left Kan extensions of $F|_{E^{\otimes}}$, and let $\mathcal{Y} \subseteq Fun_{\underline{Fin}^G_*}(E^{\otimes}, E'^{\otimes})$ be the full subcategory spanned by those functors $F : E^{\otimes} \to E'^{\otimes}$ such that for every $x \in M$ the induced functor $E^{\otimes}_{/x} \to E'^{\otimes}$ has a q-colimit, where $E^{\otimes}_{/x} := E^{\otimes} \times_M M_{/x}$. By [HTT] *4.3.2.15* the restriction functor $\mathcal{X} \to \mathcal{Y}$ is a trivial Kan fibration. Moreover, we claim that $\mathcal{Y} \simeq Fun_{\underline{Fin}^G_*}(E^{\otimes}, E'^{\otimes})$. This follows from the fact that $E^{\otimes}_{/x}$ has a terminal object, hence every functor in $Fun_{\underline{Fin}^G_*}(E^{\otimes}, E'^{\otimes})$ has a q-colimit. The terminal object in $E^{\otimes}_{/x}$ is given by $x \to x$ if $x \in M|_{\{0\}} \simeq E^{\otimes}$ and by a Cartesian lift $y \to x$ over x if $x \in M|_{\{1\}} \simeq C$. By the description of the ∞-category Υ the restriction functor $\Upsilon \to Alg_G(E, E')$ fits in the commutative diagram

$$\begin{tikzcd}
\Upsilon \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d] & Fun_{\underline{Fin}^G_*}(M, E') \arrow[d] \\
Alg_G(E, E') \arrow[r, hook] & \mathcal{Y} \arrow[r, "\simeq"] & Fun_{\underline{Fin}^G_*}(E, E')
\end{tikzcd}$$

In fact, the left square is a pullback square, again by the description of Υ, hence $\Upsilon \to Alg_G(E, E')$ is a trivial Kan fibration.

Going toward our next step, note that precomposition with r gives a section of this trivial fibration, which we will traditionally mark with s. Let $\epsilon : \Upsilon \to Fun_{\underline{Fin}^G_*}(C, E'^{\otimes})$ be the other restriction. The functor θ is given by the composition $\epsilon \circ s$, which means that it will suffice to prove that ϵ induces an equivalence of ∞-categories Υ and $Alg_G(C, E'^{\otimes})$.
Again, by [HTT] *4.3.2.15* it will suffice to show the following:

1. For every $F_0 \in Alg_G(C, E')$ there exists $F \in Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ such that F is a q-right Kan extension of F_0.
2. A functor $F \in Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ belongs to Υ if and only if it is a q-right Kan extension of $F_0 = F|_C$, with $F_0 \in Alg_G(C, E')$.

Naturally, our next step would be to prove:

Step 2. *For every $F_0 \in Alg_G(C, E')$ there exists $F \in Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ such that F is a q-right Kan extension of F_0.*

Take $x \in E^{\otimes}$, let $C_{x/} = M_{x/} \times_M C$ and let $F_x = F_0|_{C_{x/}}$. By [HTT] *4.3.2.13* it will suffice to show that F_x can be extended to a q-limit diagram $C^{\triangleleft}_{x/} \to E'^{\otimes}$ (covering the map $C^{\triangleleft}_{x/} \to M \to \underline{Fin}^G_*$). Denote with $C'_{x/}$ the full subcategory of $C_{x/}$ spanned by those morphisms $x \to y$ in M such that $x \to f(y)$ is an inert arrow in $E^{\otimes}$. By 10.1.5 the inclusion functor $C'_{x/} \hookrightarrow C_{x/}$ is final, thus by [HTT] *4.3.1.7* it will suffice to show that $F'_x = F_x|_{C'_{x/}}$ can be extended to a q-limit $C'^{\triangleleft}_{x/} \to E'^{\otimes}$.

Let $p(x) = [U \to O]$ and let $C''_{x/}$ be the full subcategory of $C'_{x/}$ spanned by inert morphisms $x \to f(y)$ such that their underlying map in $\underline{Fin}^G_*$ can be written as a span

$$\begin{tikzcd}
U \arrow[d] & V \arrow[l] \arrow[r, "="] \arrow[d, "="] & V \arrow[d, "="] \\
O & V \arrow[l] \arrow[r, "="] & V
\end{tikzcd}$$

where $V \in \mathcal{O}_G$. At the moment, we wish to prove that F'_x is a q-right Kan extension of $F''_x = F'_x|_{C''_{x/}}$ so that we can use [HTT] *4.3.2.7*:
Let $\alpha : x \to y$ be a map in M which is an object of $C'_{x/}$ and denote $p \circ f(y) = [U_1 \to O_1]$. The ∞-category $C''_{x/} \times_{C'_{x/}} (C'_{x/})_{\alpha/}$ can be identified as the full subcategory of $M_{\alpha/}$ spanned by diagrams $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ such that $p \circ f(\beta)$ can be written as the span

$$\begin{tikzcd} U_1 \arrow[d] & V \arrow[l] \arrow[r, "="] \arrow[d, "="] & V \arrow[d, "="] \\ O_1 & V \arrow[l] \arrow[r, "="] & V \end{tikzcd}$$

with $V \in \mathcal{O}_G$. Note that the upper left G-map $V \to U_1$ factors through some $W \in Orbit(U_1)$. Let, $\mathcal{F}_{W \subseteq U_1}$ be a full subcategory of $\underline{Fin}^G_{*\ \chi_{[W \subseteq U_1]}/}$ spanned by those object (or better said, morpshisms) of the form

$$\begin{tikzcd} U_1 \arrow[d] & V \arrow[l] \arrow[r, "="] \arrow[d, "="] & V \arrow[d, "="] \\ O_1 & V \arrow[l] \arrow[r, "="] & V \end{tikzcd}$$

with $V \in \mathcal{O}_G$. In particular, one such morphism can be factored as composition of maps

$$\begin{tikzcd} U_1 \arrow[d] & W \arrow[l] \arrow[r, "="] \arrow[d, "="] & W \arrow[d, "="] & V \arrow[l] \arrow[r, "="] \arrow[d, "="] & V \arrow[d, "="] \\ O_1 & W \arrow[l] \arrow[r, "="] & W & V \arrow[l] \arrow[r, "="] & V \end{tikzcd}$$

Now we can write our ∞-category $C''_{x/} \times_{C'_{x/}} (C'_{x/})_{\alpha/}$ as the disjoint union of ∞-categories $C''(W)_{y/}$, for $W \in Orbit(U_1)$, where each $C''(W)_{y/}$ is equivalent to the full subcategory of $C_{y/}$ spanned by objects (that is, morphisms) covering a map from $\mathcal{F}_{W \subseteq U_1}$. Since f is a G-approximation all of these ∞-categories $C''(W)_{y/}$ have an initial object given by the $p \circ f$-coCartesian lift $y \to y_W$ covering $\chi_{[W \subseteq U_1]}$. Hence, it will suffice to show that $F_0(y)$ is a q-product of the objects $\{F_0(y_W)\}_{W \in Orbit(U_1)}$. Since $E'^{\otimes}$ is a G-∞-operad it will sufice to show that the maps $F_0(y) \to F_0(y_W)$ are inert, which is true since $F_0 \in Alg_G(C, E)$.

We have shown that F'_x is a q-right Kan extension of $F''_x = F'_x|_{C''_{x/}}$. By [HTT] *4.3.2.7* it will suffice to prove that F''_x can be extended to a q-limit diagram $C''^{\triangleleft}_{x/} \to E'^{\otimes}$ covering the map $C''^{\triangleleft}_{x/} \to M \to \underline{Fin}^G_*$.
Similarly as before, let $C''(W)_{x/}$ (for $W \in Orbit(U)$ with $p(x) = [U \to O]$) be the full subcategory of $C''_{x/}$ spanned by objects covering a map from $\mathcal{F}_{W \subseteq U}$. Now the ∞-category $C''_{x/}$ can be written as the disjoint union of ∞-categories $C''(W)_{x/}$. Denote with $E(W)$ the full subcategory of $E^{\otimes}_{x/} \times_{\underline{Fin}^G_{*\ [U \to O]/}} \mathcal{F}_{W \subseteq U}$ such that we have an equivalence $C''(W)_{x/} \simeq E(W) \times_E C_{I(-)}$. Since f induces an equivalence on the underlying G-∞-categories $C_{I(-)} \simeq E$ this is possible to do. We can choose the inert morphisms $x \to x_W$ as coCartesian lifts of $\chi_{[W \subseteq U]}$ which in turn represent the initial objects of $E(W)$. Since f induces a categorical equivalence $C_{I(-)} \xrightarrow{\simeq} E$ we can write $x_W \simeq f(c_W)$ with $\alpha_W : x \to c_W$ an initial object of $C''(W)_{x/}$. Similarly to the previous part, we are required to prove the existence of a q-product of objects $F_0(c_W)$, $W \in Orbit(U)$ which follows from the fact that $E'^{\otimes}$ is a G-∞-operad. This finishes the proof of the second step.

The arguments as above give us the following equivalent condition for point 2):

Help lemma. *Let $F \in Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ be such that $F_0 = F|_C \in Alg_G(C, E')$. Let $x \in E^{\otimes}$ such that $p(x) = [U \to O]$, and let $c_W \in C$ such that $f \circ p(c_W) = [W \xrightarrow{=} W]$ with $W \in Orbit(U)$. Additionaly, let $\alpha_W : x \to c_W$ be maps in M covering $\chi_{[W \subseteq U]}$ constructed as above. Then F is a q-right Kan extension of F_0 if and only if $F(\alpha_W)$ are inert arrows in $E'^{\otimes}$ for $W \in Orbit(U)$.*

This help lemma will turn fundamental in the proof of our third and final step:

Step 3. *A functor $F \in Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ belongs to Υ if and only if it is a q-right Kan extension of $F_0 = F|_C$, with $F_0 \in Alg_G(C, E')$.*

Let $F \in \Upsilon$. The functor $F_0 = F|_C$ is equivalent to the functor $F|_{E^{\otimes}} \circ f$ hence $F_0 \in Alg_G(C, E')$. Our help lemma now implies that F is a q-right Kan extension of F_0.

For the other direction, let $F \in Fun_{\underline{Fin}^G_*}(M, E'^{\otimes})$ and let $F_0 = F|_C \in Alg_G(C, E')$. Assume that F is a q-right Kan extension of F_0. Let $c \in C$ with $f \circ p(c) = [U \to O]$. Let $x = f(c)$. We would like to show that $F(x) \to F(c)$ in an equivalence in $E'^{\otimes}$. For that, let us choose p'-coCartesian (with $p' = f \circ p$) lifts $c \to c_W$ over $\chi_{[W \subseteq U]}$. Since $F_0 \in Alg_G(C, E')$ by assumption, the maps $F_0(c) \to F_0(c_W)$ are all inert. Since $E'^{\otimes}$ is an G-∞-operad, it will suffice to show that $F(x) \to F(c_W)$ are all inert, which is true by help lemma above.

What is left to show is that $F|_E \in Alg_G(E, E')$. By 10.1.4 it would only suffice to check that that the inert map $x \to x_V$ lying over

$$\begin{tikzcd} U \arrow[d] & V \arrow[l] \arrow[r, "="] \arrow[d, "="] & V \arrow[d, "="] \\ O & V \arrow[l] \arrow[r, "="] & V \end{tikzcd}$$

maps to an inert map. This map factors as $x \to x_W \to x_V$ lying over

$$\begin{tikzcd} U \arrow[d] & W \arrow[l] \arrow[r, "="] \arrow[d, "="] & W \arrow[d, "="] & V \arrow[l] \arrow[r, "="] \arrow[d, "="] & V \arrow[d, "="] \\ O & W \arrow[l] \arrow[r, "="] & W & V \arrow[l] \arrow[r, "="] & V \end{tikzcd}$$

with $x \to x_W$ being inert and $W \in Orbit(U)$. By the dual of [HTT] *2.4.1.7* the map $x_W \to x_V$ is also inert. It is enough to show that $F(x) \to F(x_W)$ and $F(x_W) \to F(x_V)$ are inert.
Similarly as before, we can assume that $x_W = f(c_W)$ and $x_V = f(c_V)$. Since the maps $F(x_W) \to F(c_W)$ and $F(x_V) \to F(c_V)$ are equivalences in $E'^{\otimes}$, for the map $F(x) \to F(x_W)$ to be inert it would suffice to show that the composite map $F(x) \to F(x_W) \to F(c_W)$ is inert, which follows from our new criterion. As for $F(x_W) \to F(x_V)$, note that it is equivalent to $F(c_W) \to F(c_V)$ which is further equivalent to $F_0(c_W) \to F_0(c_V)$. Finally, since $F_0 \in Alg_G(C, E')$ this map is inert and so is $F(x_W) \to F(x_V)$. With this the proof is finished. □

As an immediate consequence we have the following:

Corollary 10.1.8. *Let $f : E^{\otimes} \to E'^{\otimes}$ be a map of G-∞-operads. Furhemore, assume f is a G-approximation map that induces an equivalence on the underlying G-∞-categories. Then f is an equivalence of G-∞-operads.*

10.2 The universal property

In this section we will prove the universal property of G-disc algebras:
Let $H \leq G$ be a finite subgroup. The statement that we want to prove is the following: The G-symmetric monoidal category of G/H-framed G-discs is freely generated by the H-symmetric monoidal category of $*$-framed H-discs. In other words, for $\underline{\mathcal{C}}^{\otimes}$ a G-symmetric monoidal category we have the following equivalence:

$$Fun_G^{\otimes}(\underline{Disk}^{G,G/H-fr}, \underline{\mathcal{C}}^{\otimes}) \simeq Fun_H^{\otimes}(\underline{Disk}^{H,*-fr}, \underline{G/H}\underline{\times}\underline{\mathcal{C}}^{\otimes})$$

where $\underline{G/H}\underline{\times}\underline{\mathcal{C}}^{\otimes}$ is the underlying H-∞-category of $\underline{\mathcal{C}}^{\otimes}$.

Note that by 7.2 $\underline{Disk}^{G,G/H-fr}$ and $\underline{Disk}^{H,*-fr}$ are equivalent to the G-symmetric monoidal envelopes $Env_G(\underline{Rep}_n^{G/H-fr,\sqcup}(G))$ and $Env_H(\underline{Rep}_n^{*-fr,\sqcup}(H))$ respectively. For simplicity, let us denote with $\mathcal{D}^{G,G/H-fr} := \underline{Rep}_n^{G/H-fr,\sqcup}(G)$ and $\mathcal{D}^{H,*-fr} := \underline{Rep}_n^{*-fr,\sqcup}(H)$. Now the equivalence above can be written as

$$Alg_G(\mathcal{D}^{G,G/H-fr}, \underline{\mathcal{C}}) \simeq Alg_H(\mathcal{D}^{H,*-fr}, \underline{G/H}\underline{\times}\underline{\mathcal{C}}) \tag{10.1}$$

Description 10.2.1. *Before continuing, let us give more insight into the objects of* $\underline{Rep}_n^{*-fr,\sqcup}(H)$ *and* $\underline{Rep}_n^{G/H-fr,\sqcup}(G)$. *Since the objects of* $(\underline{Rep}_n^{*-fr,\sqcup}(H))_{I(-)}$ *are framed over point, they correspond to the* V*-framed* G*-discs where* V *is an* H*-representation. Informally, using 7.1.7 we can depict these objects as*

$$\begin{tikzcd} H \times_K V \arrow[r] \arrow[d] & V \arrow[d] \\ H/K \arrow[r] \arrow[d, "="] & * \\ H/K & \end{tikzcd}$$

where the lower horizontal arrow is the framing map and the square is the pullback square. Using 6.1.12 and 6.1.13 we can write these elements as

$$\begin{tikzcd} G \times_K V \arrow[d] & \\ G/K \arrow[r] \arrow[d, "="] & G/H \\ G/K \arrow[d] & \\ G/H & \end{tikzcd}$$

Note that the framing map is the same as the map inducing topological induction. Therefore we can write this element as $G \times_K V \to G/K \xrightarrow{=} G/K \to G/H$ *or even simpler as* $G/K \xrightarrow{=} G/K \to G/H$ *(since all the information on the bundle over* G/K *is carried by the framing map). With this depiction the general object of* $\underline{Rep}_n^{*-fr,\sqcup}(H)$ *can be written as*

$$U \to O \to G/H$$

where U is a finite G-set and $O \in \mathcal{O}_G$ is an orbit.
Similarly, an object of $(\underline{Rep}_n^{G/H-fr,\sqcup}(G))_{I(-)}$ can be written in the form

$$\begin{tikzcd} G \times_K V \arrow[r] \arrow[d] & G \times_H V \arrow[d] \\ G/K \arrow[r] \arrow[d, "="] & G/H \\ G/K & \end{tikzcd}$$

where the lower horizontal map is the framing map and the square is a pullback square. Again, by 7.1.7 we can write this element as

$$\begin{tikzcd} G/K \arrow[r] \arrow[d, "="] & G/H \\ G/K & \end{tikzcd}$$

Therefore, the elements of $\underline{Rep}_n^{G/H-fr,\sqcup}(G)$ can be written in the form

$$O \leftarrow U \rightarrow G/H$$

where U is a finite G-set and $O \in \mathcal{O}_G$ is an orbit, the left arrow is the structure map while the right arrow is the framing map.

In order prove the equivalence (10.1) we would like to be able to use 10.1.7.

Construction 10.2.2. *Let us construct a map $\theta : \mathcal{D}^{H,*-fr} \to \mathcal{D}^{G,G/H-fr}$ as part of the commutative diagram*

$$\begin{tikzcd} \mathcal{D}^{H,*-fr} \arrow[r, "\theta"] \arrow[d] & \mathcal{D}^{G,G/H-fr} \arrow[d] \\ \underline{Fin}_*^H \arrow[r] & \underline{Fin}_*^G \end{tikzcd}$$

Note that $\underline{Fin}_*^H$ is equivalent to $\underline{G/H} \underline{\times} \underline{Fin}_*^G$. Therefore the map $\underline{Fin}_*^H \to \underline{Fin}_*^G$ is a left fibration. By 10.2.1 we have the description of the objects of $\mathcal{D}^{H,*-fr}$ and $\mathcal{D}^{G,G/H-fr}$. The object $(U \to O \to G/H) \in \mathcal{D}^{H,*-fr}$ is then sent by the map θ to the object $(O \leftarrow U \to O \to G/H) \in \mathcal{D}^{G,G/H-fr}$.

Proposition 10.2.3. *The map $\theta : \mathcal{D}^{H,*-fr} \to \mathcal{D}^{G,G/H-fr}$ is a G-approximation map. Moreover, θ induces an equivalence on the underlying G-∞-categories.*

Proof. In order to prove that θ is a G-approximation map we need to prove the three points from 10.1.1:

1. Consider the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}^{H,*-fr} \arrow[rr, "\theta"] \arrow[dd] & & \mathcal{D}^{G,G/H-fr} \arrow[dd] & \\
& \underline{Rep}_n^{*-fr}(H) \arrow[ul] \arrow[dd] & & \underline{Rep}_n^{G/H-fr}(G) \arrow[ul] \arrow[dd] \\
\underline{Fin}_*^H \arrow[rr] & & \underline{Fin}_*^G & \\
& \mathcal{O}_H^{op} \arrow[ul, "I(-)"] \arrow[rr] & & \mathcal{O}_G^{op} \arrow[ul, "I(-)"]
\end{tikzcd}$$

where the right rhombus is a pullback diagram. If we restrict our attention to the part of the diagram

$$\begin{tikzcd}
\underline{Rep}_n^{*-fr}(H) \arrow[r] \arrow[d] & \mathcal{D}^{H,*-fr} \simeq \underline{Rep}_n^{*-fr,\sqcup}(H) \arrow[d] \\
\mathcal{O}_H^{op} \arrow[r, "I(-)"] \arrow[d] & \underline{Fin}_*^H \arrow[d] \\
\mathcal{O}_G^{op} \arrow[r, "I(-)"] & \underline{Fin}_*^G
\end{tikzcd}$$

we have that the upper square is a pullback square by definition. Also, by inspection the lower square is a pullback square meaning that the whole square is a pullback square. Therefore, the pullback of $\mathcal{D}^{H,*-fr} \to \underline{Fin}_*^H \to \underline{Fin}_*^G$ along $I(-) : \mathcal{O}_G^{op} \to \underline{Fin}_*^G$ is $\underline{Rep}_n^{*-fr}(H) \to \mathcal{O}_H^{op} \to \mathcal{O}_G^{op}$. Additionally, the first map in the composition is a coCartesian fibration since $\underline{Rep}_n^{*-fr}(H)$ is an H-∞-category and $\mathcal{O}_H^{op} \simeq \underline{G/H} \to \mathcal{O}_G^{op}$ is a left fibration, meaning that the composition is a coCartesian fibration i.e. $\underline{Rep}_n^{*-fr}(H) \to \mathcal{O}_G^{op}$ is a G-∞-category. Additionally, since the right rhombus is a pullback diagram θ induces a map that fits into the following diagram

$$\begin{tikzcd}
\underline{Rep}_n^{*-fr}(H) \arrow[r] \arrow[d] & \underline{Rep}_n^{G/H-fr}(G) \arrow[d] \\
\underline{G/H} \simeq \mathcal{O}_H^{op} \arrow[r] & \mathcal{O}_G^{op}
\end{tikzcd}$$

2. Let $x \in \mathcal{D}^{H,*-fr}$ be such that $\theta(x)$ lies over $I : U \to O \in \underline{Fin}_*^G$. Note that I lies in an image of $\underline{Fin}_*^H \to \underline{Fin}_*^G$ hence we can consider I as an element of $\underline{Fin}_*^H$. Since $\underline{Fin}_*^H \to \underline{Fin}_*^G$ is a left fibration, the maps $\chi_{[W \subseteq U]}$ in $\underline{Fin}_*^G$ have coCartesian lifts which are exactly maps $\chi_{[W \subseteq U]}$ in $\underline{Fin}_*^H$. Now, considering that $\mathcal{D}^{H,*-fr} \to \underline{Fin}_*^H$ is an H-∞-operad, the coCartesian lifts of $\chi_{[W \subseteq U]}$ exist with the source x and therefore the coCartesian lifts of $\chi_{[W \subseteq U]}$ in $\underline{Fin}_*^G$ also exist.

3. Recall that, by 10.2.1, objects of $\mathcal{D}^{G,G/H-fr}$ can be seen as

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & G/H \\
O &
\end{tikzcd}$$

with $U \in \underline{Fin}_*^G$ and $O \in \mathcal{O}_G$ where the horizontal map corresponds to the framing map. Now let $x \in \mathcal{D}^{G,G/H-fr}$ lying over $U_1 \to O$, $c \in \mathcal{D}^{H,*-fr}$ lying over $U_2 \to O$ and let $x \to \theta(c)$ be an active arrow lying over diagram

$$\begin{tikzcd}
U_1 \arrow[r] \arrow[d] \arrow[rr, bend left] & U_2 \arrow[d] & G/H \arrow[d, "="] \\
O \arrow[r, "="] & O \arrow[r] & G/H
\end{tikzcd}$$

Note that a map in $\mathcal{D}^{G,G/H-fr}$ should be represented by a span, but since the underlying arrow is fiberwise active, the left side of the span would be the identity, so we omit it in the diagram.
The element $\bar{x}$ of the Cartesian lift corresponds to the element $U_1 \to O \xrightarrow{=} O \to G/H$.

What is left is to show is that θ induces equivalence on the underlying G-∞-categories. Let us denote with $\theta_{I(-)} : \mathcal{D}_{I(-)}^{H,*-fr} \to \mathcal{D}_{I(-)}^{G,G/H-fr}$ the induced functor.
Note that the element of $\mathcal{D}_{I(-)}^{G,G/H-fr}$ can be written as

$$\begin{tikzcd}
O \arrow[r] \arrow[d, "="] & G/H \\
O
\end{tikzcd}$$

which is the same as $O \xrightarrow{=} O \to G/H$, an element of $\mathcal{D}_{I(-)}^{H,*-fr}$, hence $\theta_{I(-)}$ is essentially surjective. As for the mapping spaces, recall taht the mapping space in $\mathcal{D}_{I(-)}^{H,*-fr}$ between $E_1 \to O_1 \xrightarrow{=} O_1 \to G/H$ and $E_2 \to O_2 \xrightarrow{=} O_2 \to G/H$ consists of the spans

$$\begin{tikzcd}
E_1 \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d] \\
O_1 \arrow[d, "="] & O_2 \arrow[l] \arrow[r, "="] \arrow[d, "="] & O_2 \arrow[d, "="] \\
O_1 \arrow[dr] & O_2 \arrow[l] \arrow[r, "="] \arrow[d] & O_2 \arrow[dl] \\
& G/H &
\end{tikzcd}$$

which is the same as the span

$$\begin{tikzcd}
E_1 \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d] & \\
O_1 \arrow[d, "="] \arrow[drrr] & O_2 \arrow[l] \arrow[r, "="] \arrow[d, "="] \arrow[drr] & O_2 \arrow[d, "="] \arrow[dr] & \\
O_1 & O_2 \arrow[l] \arrow[r, "="] & O_2 & G/H
\end{tikzcd}$$

representing the map of the image of $\theta_{I(-)}$. We conclude that $\theta_{I(-)}$ is fully faithful and hence an equivalence which finishes the proof. □

Propositions 10.2.3 and 10.1.7 give us the following result:

Corollary 10.2.4. *Let $E^{\otimes}$ be a G-∞-operad. The map θ induces an equivalence*

$$Alg_G(\mathcal{D}^{G,G/H-fr}, E) \xrightarrow{\simeq} Alg_G(\mathcal{D}^{H,*-fr}, E)$$

We are one step away from proving our universal property:

Theorem 10.2.5. *Let $\underline{\mathcal{C}}^{\otimes}$ be a G-symmetric monoidal category. Then the G-symmetric monoidal category of G/H-framed G-discs is freely generated by the H-symmetric monoidal category of $*$-framed H-discs. In other words, there is an equivalence*

$$Fun_G^{\otimes}(\underline{Disk}^{G,G/H-fr}, \underline{\mathcal{C}}^{\otimes}) \xrightarrow{\simeq} Fun_H^{\otimes}(\underline{Disk}^{H,*-fr}, \underline{\mathcal{C}}_H^{\otimes})$$

where $\underline{\mathcal{C}}_H^{\otimes}$ is the underlying H-∞-category of G-∞-category $\underline{\mathcal{C}}^{\otimes}$.

Proof. As discussed in the beginning of this section, the above statement is equivalent to

$$Alg_G(\mathcal{D}^{G,G/H-fr}, \underline{\mathcal{C}}) \xrightarrow{\simeq} Alg_H(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}_H)$$

By 10.2.4 it will suffice to show an equivalence

$$Alg_G(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}) \simeq Alg_H(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}_H)$$

Recall that the H-symmetric monoidal category $\underline{\mathcal{C}}_H^{\otimes}$ is obtained via the commutative diagram

$$\begin{tikzcd}
\underline{\mathcal{C}}_H^{\otimes} \arrow[r] \arrow[d] & \underline{\mathcal{C}}^{\otimes} \arrow[d] \\
\underline{Fin}_*^H \arrow[r] \arrow[d] & \underline{Fin}_*^G \arrow[d] \\
\underline{G/H} \arrow[r] & \mathcal{O}_G^{op}
\end{tikzcd}$$

where both inner rectangles (and consequently also the outer rectangle) are pullbacks.
Let F be a $\mathcal{D}^{H,*-fr}$-algebra object in $\underline{\mathcal{C}}^{\otimes}$ i.e. $F \in Alg_G(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}})$, and consider the following diagram

$$\begin{tikzcd}
 & & \underline{\mathcal{C}}_H^{\otimes} \arrow[dr] \arrow[dd] & \\
\mathcal{D}^{H,*-fr} \arrow[urr, dashed, "F'"] \arrow[rrr, "F"] \arrow[drr] \arrow[ddd] & & & \underline{\mathcal{C}}^{\otimes} \arrow[ddd] \\
 & & \underline{Fin}_*^H \arrow[ddr] & \\
 & & & \\
\mathcal{D}^{G,G/H-fr} \arrow[rrr] & & & \underline{Fin}_*^G
\end{tikzcd}$$

By the universal property of the pullback F induces a functor $F' : \mathcal{D}^{H,*-fr} \to \underline{\mathcal{C}}^{\otimes}_{H}$. We would like to show that $F' \in Alg_H(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}^{\otimes}_{H})$.
Let α_W be an inert map in $\mathcal{D}^{H,*-fr}$ covering the map $\chi_{[W\subseteq U]} : [U \to O] \to [W \xrightarrow{=} W]$ in $\underline{Fin}^{H}_{*}$. Since $\underline{Fin}^{H}_{*} \to \underline{Fin}^{G}_{*}$ is a left fibration and $F \in Alg_G(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}^{\otimes})$, $F(\alpha_W)$ is inert in $\underline{\mathcal{C}}^{\otimes}$. Again, since $\underline{Fin}^{H}_{*} \to \underline{Fin}^{G}_{*}$ is a left fibration, and consequently $\underline{\mathcal{C}}^{\otimes}_{H} \to \underline{\mathcal{C}}^{\otimes}$, the pullback of $F(\alpha_W)$ is inert in $\underline{\mathcal{C}}^{\otimes}_{H}$, which is exactly $F'(\alpha_W)$, meaning $F' \in Alg_H(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}^{\otimes}_{H})$. Similarly, if we would have taken $F' \in Alg_H(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}^{\otimes}_{H})$ the induced functor $F : \mathcal{D}^{H,*-fr} \to \underline{\mathcal{C}}^{\otimes}_{H} \to \underline{\mathcal{C}}^{\otimes}$ would lie in $Alg_G(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}^{\otimes})$. By the universality of the maps to the pullback we have

$$Alg_G(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}) \simeq Alg_H(\mathcal{D}^{H,*-fr}, \underline{\mathcal{C}}_H)$$

and the proof is finished. □

10.3 Applications of the universal property

Algebras with genuine involution and $O(2)$-genuine objects

Let $G = O(2)$ and $H = \mathbb{Z}_2$ and let $\underline{Sp}^{O(2)}$ be a $O(2)$-∞-category of spectra. Let V be the adjoint representation of $O(2)$ at the identity element. It is evident that V is endowed with $O(2)$-action, which, when we restrict to $\mathbb{Z}_2$-action becomes $\mathbb{R}^{\sigma}$ where σ is an one dimensional sign representation.

Let A_H be the $\mathbb{R}^{\sigma}$-framed $\mathbb{Z}_2$-disc algebra object in $\underline{Sp}^{\mathbb{Z}_2}$. Let us give more insight into the structure of A_H. Using 6.1.13 we conclude that the objects of $\underline{Disk}^{\mathbb{Z}_2,\mathbb{R}^{\sigma}-fr}$ are given by the finite disjoint unions of

- $\mathbb{R}^{\sigma}$, which corresponds to the element $\mathbb{R}^{\sigma} \to \mathbb{Z}_2/\mathbb{Z}_2$;
- The restriction $Res^{\mathbb{Z}_2}_{\{e\}}(\mathbb{R}^{\sigma}) = \mathbb{R}^1$ which corresponds to the element $\sqcup_{\mathbb{Z}_2}\mathbb{R}^1 \to \mathbb{Z}_2/\{e\}$;
- The topological induction $\sqcup_{\mathbb{Z}_2}\mathbb{R}^1$ obtained as the element $\sqcup_{\mathbb{Z}_2}\mathbb{R}^1 \to \mathbb{Z}_2/\{e\} \to \mathbb{Z}_2/\mathbb{Z}_2$

Therefore, by 7.4 we can write

- $Res^{\mathbb{Z}_2}_{\{e\}} A_H(\mathbb{R}^{\sigma}) = A_H(\mathbb{R}^1)$;
- $A(\sqcup_{\mathbb{Z}_2}\mathbb{R}^1) \simeq N^{\mathbb{Z}_2}_{e}(A_H(\mathbb{R}^1))$.

Additionally, the equivariant embedding

$$(\sqcup_{\mathbb{Z}_2}\mathbb{R}^1) \sqcup \mathbb{R}^{\sigma} \hookrightarrow \mathbb{R}^{\sigma}$$

induces a map $N^{\mathbb{Z}_2}_{e}(A_H(\mathbb{R}^1)) \otimes A(\mathbb{R}^{\sigma}) \to A(\mathbb{R}^{\sigma})$. Meaning that the $\mathbb{Z}_2$-spectrum $A(\mathbb{R}^{\sigma})$ has a structure of a $N^{\mathbb{Z}_2}_{e}(A_H(\mathbb{R}^1))$-module.
Next, note that $A_H(\mathbb{R}^1)$ has a structure of an $\mathbb{E}_1$-algebra object in $\underline{Sp}^{\mathbb{Z}_2}_{[\mathbb{Z}_2/\{e\}]} \simeq Sp$, since

$$Fun^{\otimes}(\underline{Disk}^{\mathbb{Z}_2,\mathbb{R}^{\sigma}-fr}_{[\mathbb{Z}_2/\{e\}]}, \underline{Sp}^{O(2)}_{[\mathbb{Z}_2/\{e\}]}) \simeq Fun^{\otimes}(Disk^{1,fr}, Sp) \simeq Alg_{\mathbb{E}_1}(Sp)$$

Moreover, the $\mathbb{Z}_2$-action makes $A_H(\mathbb{R}^1)$ into an associative ($\mathbb{E}_1$) algebra with involution i.e. $A_H(\mathbb{R}^1)$ is a $A_H(\mathbb{R}^1) \otimes A_H(\mathbb{R}^1)$-module. To add up, we have seen that this $A_H(\mathbb{R}^1) \otimes A_H(\mathbb{R}^1)$-module structure lifts to $N^{\mathbb{Z}_2}_{e}(A_H(\mathbb{R}^1))$-module structure on $A(\mathbb{R}^{\sigma})$, hence we say that A_H is an associative algebra with genuine involution.

The univesal property of equivariant disc algebras, Theorem 10.2.5, gives us the equivalence

$$Fun^{\otimes}_{\mathbb{Z}_2}(\underline{Disk}^{\mathbb{Z}_2,\mathbb{R}^\sigma-fr}, \underline{Sp}^{\mathbb{Z}_2}) \simeq Fun^{\otimes}_{O(2)}(\underline{Disk}^{O(2),S^1-fr}, \underline{Sp}^{O(2)})$$

where we have used the fact that $O(2)/\mathbb{Z}_2 \cong S^1$.
Let $A \in Fun^{\otimes}_{O(2)}(\underline{Disk}^{O(2),S^1-fr}, \underline{Sp}^{O(2)})$ be an $O(2)$-disc algebra corresponding to A_H. We can regard S^1 as a genuine $O(2)$-object in the $O(2)$-∞-category of manifolds $\underline{Mfld}^{O(2),S^1-fr}$ given by

$$\begin{aligned}\underline{S}^1 :& \mathcal{O}^{op}_{O(2)} \to \underline{Mfld}^{O(2),S^1-fr}\\ O &\mapsto O \times S^1\end{aligned}$$

Therefore, we can build the following map

$$\begin{gathered}\{\text{Associative algebras with genuine involution in } \underline{Sp}^{\mathbb{Z}_2}\}\\ \downarrow\\ \{\text{Genuine } O(2)\text{-objects in } \underline{Sp}^{O(2)}\}\end{gathered}$$

given by

$$A_H \mapsto \int_{S^1} A$$

The result of Horev, [Hor19] *7.1.1* and *7.1.2* tells us that the underlying $\mathbb{Z}_2$-genuine object of $\int_{S^1} A$ is equivalent to the real topological Hochschild homology $THR(A)$ (see also [DMPR17]). In other words, we have

$$\left(\int_{S^1} A\right)_{[O(2)/\mathbb{Z}_2]} \simeq THR(A)$$

Therefore, we have obtained the refinement of the $\mathbb{Z}_2$-genuine structure on $THR(A)$ to the $O(2)$-genuine structure.

Associative algebras and S^1-genuine objects

For the second example, let us take $G = S^1$ and $H = \{e\}$ the trivial subgroup. The adjoint representation is equal to $\mathbb{R}^1$ when forgeting the action, hence the ∞-category $Fun^{\otimes}(\underline{Disk}^{e,*-fr}, \underline{Sp}^{S^1}_e)$ is equivalent to the ∞-category of $\mathbb{E}_1$-algebra objects in $\underline{Sp}^{S^1}_e \simeq Sp$. Therefore, the Theorem 10.2.5 gives us

$$Alg_{\mathbb{E}_1}(Sp) \simeq Fun^{\otimes}_{S^1}(\underline{Disk}^{S^1,S^1-fr}, \underline{Sp}^{S^1})$$

Let A^{ass}_e be an associative algebra object in the ∞-category of spectra Sp which corresponds to the S^1-framed S^1-algebra object A in $\underline{Sp}^{S^1}$. Then we can construct a map

$$\begin{gathered}\{\text{Associative algebras in } Sp\}\\ \downarrow\\ \{\text{Genuine } S^1\text{-objects in } \underline{Sp}^{S^1}\}\end{gathered}$$

given by

$$A_e^{ass} \mapsto \int_{S^1} A$$

Where the S^1-factorization homology is taken with respect to S^1 as an S^1-genuine object given by

$$\begin{aligned} \underline{S}^1 : & \mathcal{O}_{S^1}^{op} \to \underline{Mfld}^{S^1, S^1-fr} \\ & O \mapsto O \times S^1 \end{aligned}$$

By [Hor19] *7.2.2* we have

$$\left(\int_{S^1} A \right)_{[S^1/C_n]}^{\Phi C_n} \simeq THH(A; A^\tau)$$

where A^τ is a A-A^{op}-bimodule given by the formula

$$\begin{aligned} A \otimes A^\tau \otimes A \to A^\tau \\ x \otimes a \otimes y \to \tau x \otimes a \otimes y \end{aligned}$$

with $\tau \in C_n$ being the group generator.

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