UNIVERSITÉ **SORBONNE PARIS NORD**

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Étude des instabilités non linéaires autour de solutions laminaires de systèmes d'EDP de la mécanique des fluides ou des modèles de propagation en biologie

Thèse de doctorat présentée par

Tiến-Tài NGUYỄN

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Prof. Isabelle GALLAGHER	Rapportrice
Prof. David LANNES	Rapporteur
Prof. Francis NIER	Examinateur
Prof. David GÉRARD-VARET	Examinateur
MdC-HDR. Emmanuel AUDUSSE	Examinateur
MdC-HDR. Anna ROZANOVA-PIERRAT	Examinateure
Prof. Olivier LAFITTE	Directeur de thèse
Prof. Jean-Marc DELORT	Co-directeur de thèse
Assoc. Prof. Quốc Anh NGÔ	Membre invité

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Study of nonlinear instabilities around laminar solutions of PDE systems in fluid mechanics or propagation models in biology

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Tiến-Tài NGUYỄN

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Prof. Isabelle GALLAGHER	Reporter
Prof. David LANNES	Reporter
Prof. Francis NIER	Examinator
Prof. David Gérard-Varet	Examinator
MdC-HDR. Emmanuel AUDUSSE	Examinator
MdC-HDR. Anna Rozanova-Pierrat	Examinator
Prof. Olivier LAFITTE	Advisor
Prof. Jean-Marc DELORT	Co-advisor
Assoc. Prof. Quốc Anh NGÔ	Invited member

Résumé

L'étude de la stabilité des écoulements laminaires satisfaisant un système d'équations hyperboliques a beaucoup attiré l'attention des physiciens et des mathématiciens en raison de son apparition dans de nombreux modèles en mécanique des fluides, par ex. Rayleigh-Taylor, Kelvin-Hemholtz, détonation de Zeldovitch-von Neumann-Döring. En particulier, l'existence de modes normaux pour les problèmes linéarisés autour de ces écoulements stationnaires a conduit à de nombreux résultats théoriques et numériques. Pendant ce temps, l'utilisation des résultats d'instabilité linéaire pour obtenir des résultats d'instabilité non linéaire est un défi mathématique en raison du manque d'informations sur le spectre complet du problème linéarisé. Différents cadres abstraits sont développés pour prouver l'instabilité non linéaire avec des applications à divers systèmes non linéaires en mécanique des fluides. Dans cette thèse, nous nous intéressons à un modèle particulier, l'instabilité visqueuse de Rayleigh–Taylor pour un profil lisse de densité croissante. Le premier objectif est de décrire et d'utiliser une nouvelle méthode, basée sur la théorie des opérateurs, pour prouver l'existence de multiples modes normaux au problème linéarisé. Ces multiples modes normaux, ainsi que les estimations d'énergie non linéaires, nous aident à prouver une instabilité non linéaire générale, étendant un cadre célèbre de Guo et Strauss et de Grenier. Il semble pertinent d'appliquer cette méthode à d'autres systèmes non linéaires, notamment le model de bi-fluid ou les équations de détonation de Zeldovitch-von Neumann-Döring (généralisant les résultats semi-classiques du problème linéarisé déjà connus).

Abstract

The study of the stability of laminar flows satisfying a system of hyperbolic equations has attracted a lot of attention of physicians and mathematicians due to its appearance in numerous models in fluid mechanics, e.g. Rayleigh-Taylor, Kelvin-Hemholtz, Zeldovitch-von Neumann-Döring detonation. In particular, the existence of normal modes for the linearized problems around this steady flows led to many theoretical and numerical results. Meanwhile, using the linear instability results to obtain nonlinear instability results is a mathematical challenge because of the lack of information about the complete spectrum of the linearized problem. Different abstract frameworks are developed to prove the nonlinear instability with applications to various nonlinear systems in fluid mechanics. In this thesis, we are interested in a particular model, the viscous Rayleigh-Taylor instability for a smooth increasing density profile. The first goal is to describe and use a new method, based on the operator theory, to prove the existence of multiple normal modes to the linearized problem. These multiple normal modes, along with nonlinear energy estimates, help us to prove a general nonlinear instability, extending the previous framework of Guo-Strauss and of Grenier. It seems relevant to apply that method to other nonlinear systems, including the two-fluid model or the Zeldovitch-von Neumann-Döring detonation equations (generalizing the semi-classical results of the linearized problem already known).

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Contents

PRINCIPALES NOTATIONS ADOPTÉES

$\mathbf{R}$	the real line
$\mathbb{C}$	the set of complex numbers
$\mathbf{Z}$	the set of integers
$\mathbb{T}$	the 1D torus
ρ	the density
u	the velocity
P	the pressure
g	the gravity constant
μ	the viscosity coefficient
H^s	the Sobolev space
$a \lesssim b$	$a \leqslant Cb$ with a generic constant C
$a \gtrsim b$	$a \geqslant Cb$ with a generic constant C

Chapter 0

Introduction

Many decades ago, the mathematical theory of partial differential equations of nonlinear evolution began with the study of the local/global existence in time of solutions in some suitable functional spaces. Beyond the local/global existence in time of regular solutions to the evolution PDEs in fluid mechanics, the stability study of a steady flow or of a travelling wave solution, including the theoretical and numerical investigation, is also a vast subject that has attracted both mathematicians and physicians.

Let us quote the definition from Lyapunov in 1892 [56] and from Chandrasekhar in 1961 [7]. We assume that there is a local existence theorem for the nonlinear and we consider its perturbed form departing from the equilibrium. From the initial data, either the perturbation terms measured in a Sobolev norm will moderately slow down, or at least one of the perturbation terms measured in a Sobolev norm will blow up and never revert to its initial value. If the former case happens, we say that the equilibrium is stable with respect to the particular perturbation terms and if the latter case happens, we say that it is unstable.

There is still another viewpoint on instability, initially formulated by Hadamard in 1902 [39], and is called the instability in the sense of Hadamard. It states that a problem is Hadamard stable or well-posed if the solution is unique on a time interval $(0, T)$ and the solution's behaviour depends continuously on the initial conditions. Otherwise, the problem is called Hadamard unstable or ill-posed.

The aim of this thesis is the mathematical study of instability properties of a viscous system of conservation laws, described by a nonlinear model. For this purpose, we follow the definition of nonlinear instability of Grenier [35] and refer to (0.1) for technical details.

In the first step of studying the stability of the fully nonlinear equations, we intend to analyze the linearized equations, which are obtained by omitting all nonlinear terms. This corresponds to a spectral problem depending on various physical parameters. Once this spectral problem is solved, one looks at the behavior of the linear solutions using the spectral results. If the linear problem is unstable, the second step is to obtain the nonlinear instability from the linear one. The nonlinear results are harder to obtain and less frequent in the literature, because of the unboundedness of spectral radius and the effect of nonlinear terms on the control of the exponential growth of any solution to the linearized equations.

In this introductory chapter, we will present the problem in an abstract setting and then describe the structure of this thesis. The rest of the manuscript is to present the three main problems completed during this thesis, which resulted in one paper published and two papers submitted.

0.1 Abstract problem

We consider a viscous system of conservation laws,

$$\partial_t U + \sum_{j=1}^{d} \partial_{x_j} F_j(U) = \sum_{j,l=1}^{d} \partial_{x_j}(B_{jl}(U)\partial_{x_l} U) + F_{ext}(U), \quad x \in \Omega \subset \mathbf{R}^d, t > 0, \tag{0.1}$$

where $U \in \mathbf{R}^n$, $F_j, \in \mathbf{R}^n$, B_{jk} is a square matrice of order n, F_{ext} is the external force, with initial data $U(x,0) = U_0(x)$, which admits a steady-state solution depending only on x_d,

$$\bar{U} = \bar{U}(x_d), \qquad \lim_{x_d \to \pm\infty} \bar{U}(x_d) = \bar{U}_\pm. \tag{0.2}$$

The domain Ω is of type $\mathbf{R}^d, \mathbf{R}^{d-1} \times \mathbf{R}_+, \mathbf{R} \times [-1,1], \mathbb{T}^{d-1} \times \mathbf{R}, \mathbb{T}^{d-1} \times \mathbf{R}^+$ or $\mathbb{T}^{d-1} \times [-1,1]$. That type of hyperbolic PDE appears in numerous models of hydrodynamics, for example, Rayleigh-Taylor [69, 75], Kelvin-Hemholtz [40, 48], Zeldovich-Von Neuman-Doring detonation [17, 18, 19, 20, 21], Hodgkin-Huxley [43], etc. A fundamental question connected to the physical motivations is to study the stability of such a solution $\bar{U}$ in the sense of PDE. That is the goal of this thesis.

Writing the perturbation $V = U - \bar{U}$, we obtain another formulation of (0.1) around $\bar{U}$ written in (0.2), by writing

$$\partial_t V + \sum_{j=1}^{d} \partial_{x_j} F_j(\bar{U}+V) = \sum_{j,l=1}^{d} \partial_{x_j}(B_{jl}(\bar{U}+V)\partial_{x_l}(\bar{U}+V)) + F_{ext}(\bar{U}+V), \quad x \in \Omega, t > 0,$$

which rewrites

$$\partial_t V + L_{\bar{U}} V = \mathcal{N}_{\bar{U}}(V), \tag{0.3}$$

where $L_{\bar{U}} V$ is the linearized part

$$L_{\bar{U}} V := \sum_{j=1}^{d} \partial_{x_j}(\partial_{\bar{U}} F_j(\bar{U})V) - \sum_{j=1}^{d} \partial_{x_j}\Big(B_{jl}(\bar{U})\partial_{x_l} V + \Big(\partial_{\bar{U}_d} B_{jd}(\bar{U})\partial_{x_d}\bar{U}\Big)V\Big) + \partial_{\bar{U}} F_{ext}(\bar{U})V,$$

and $\mathcal{N}_{\bar{U}}(V)$ is a nonlinear term on V.

We aim at studying the stability of the solution $V \equiv 0$ to (0.3). As we said before, the general strategy contains two steps:

Step 1. Prove the linear instability, by performing the spectral analysis of the linearized equations.

Step 2. Investigate the nonlinear instability by using the spectral results obtained for the linear equation and by exploiting some energy estimates.

We discuss first the linear instability. Note that the linearized system

$$\partial_t V + L_{\bar{U}} V = 0 \tag{0.4}$$

can be rewritten as

$$\mathcal{L}(\partial_t, \partial_{x_1}, \ldots, \partial_{x_d}, \partial_{x_1^2}, \ldots, \partial_{x_d^2})V = 0,$$

where $\mathcal{L}$ is a linear operator from $\mathbf{R}^{2d+1}$ to $\mathbf{R}^n$, which coefficients depend only on x_d through the stationary solution. As the coefficients of $\mathcal{L}$ do not depend on t or $x_j (1 \leqslant j \leqslant d-1)$, the linear study focuses on solutions of (0.4) of the form

$$V(t, x) = e^{\lambda t} Q(\mathbf{k}, x) = e^{\lambda t + ik_1 x_1 + \cdots + ik_{d-1} x_{d-1}} W(x_d). \tag{0.5}$$

Such functions $V(t, x)$, as well as $\mathrm{Re}V(t, x)$, are called normal mode solutions to the linearized equation. Substituting (0.5) into (0.4), we obtain a system of ODEs of W in the variable x_d,

$$\mathcal{L}\Big(\lambda, ik_1, \ldots, ik_{d-1}, \frac{d}{dx_d}, -k_1^2, \ldots, -k_{d-1}^2, \frac{d^2}{dx_d^2}\Big) W = 0. \tag{0.6}$$

The system (0.6) might also be seen as the system obtained on the Fourier transform in $x_1, \ldots, x_{d-1}$ and the Laplace transform in t of a solution V of (0.4). The vector $\mathbf{k} = (k_1, \ldots, k_{d-1})$ is called the wave number and throughout this thesis, we write

$$k = |\mathbf{k}| = \sqrt{k_1^2 + \cdots + k_{d-1}^2}.$$

We call *linear growth rate* or *characteristic value* of system (0.6) a value of $\lambda \in \mathbb{C}$ (depending on $\mathbf{k}$) such that $\mathrm{Re}\lambda > 0$ and there exists a non-trivial and bounded solution $W(x, \mathbf{k}, \lambda)$ to (0.6).

In aforementioned cases, some methods have been developed to solve system (0.6). For the Rayleigh-Taylor instability, we refer to Lafitte [50], Guo-Hwang [29], Helffer-Lafitte [42] and Guo-Tice [32], where the authors exploit the variational structure of the linearized equation due to its self-adjoint setting. In case of non self-adjoint problems, there is a method due to Evans [22, 23, 24, 25], defining a particular function of λ, called now the Evans function, which roots in $\mathrm{Re}\lambda > 0$ are the desired characteristic values. The Evans function is related to the determinant at an arbitrary point x_d^0 of the family containing the independent decaying solutions at $+\infty$ and the similar one at $-\infty$. That method is useful in numerical analysis and has been frequently used by Zumbrun and his collaborators, see e.g. [28, 63, 2, 57], to study the stability of shock waves. Mention also the application of the linear turning point theory for a system of ODEs to study the Zeldovitch-Von Neumann-Döring detonation by Erpenbeck first [17] and then by Lafitte, Williams and Zumbrun [52, 53] completely.

For the fully nonlinear equation (0.3) in this thesis, we use the following definition of nonlinear instability of Grenier [35, Definition 2.1].

Definition 0.1. *We say that the trivial solution $V = 0$ of (0.3) is nonlinearly unstable if there exist positive constants ε_0 and C_s such that for every s arbitrarily large and every $\delta > 0$ sufficiently small, there exists a solution V of (0.3) satisfying*

$$\|V(0,x)\|_{H^s(\Omega)} \leqslant \delta, \quad \text{and } \|V(T^\delta, x)\|_{L^2(\Omega)} \geqslant \varepsilon_0$$

for some times T^δ, where

$$T^\delta \leqslant C_s(1 + \ln(1 + \delta^{-1})).$$

Based on the existence of normal modes to the linearized equation (0.4), different frameworks are constructed to prove the nonlinear instability in the sense of Definition 0.1, see e.g. Guo-Strauss [30], Grenier [35], Desjardins-Grenier [12, 13]. These frameworks were used to study the nonlinear instability in numerous models, e.g. Rayleigh-Taylor instability by Guo-Hwang [29], Tice-Wang [71], instability of the Lane–Emden steady star configurations by Jang-Tice [36], instability of solitary waves by Rousset-Tzvetkov [66, 67, 68], instability of Euler-Korteweg solitons by Paddick [64].

0.2 Plan of this thesis

In this thesis, we study an incompressible model related to the viscous Rayleigh-Taylor instability, which can be stated as in (0.1)-(0.2). The domain Ω is $\Sigma^{d-1} \times I$, where $\Sigma = \mathbf{R}$ or $2\pi L\mathbb{T}$ $(L > 0)$ and $I = \mathbf{R}, \mathbf{R}_- = (-\infty, 0)$ or $(-1, 1)$. The governing equations are

$$\begin{cases} \partial_t \rho + \sum_{j=1}^d \partial_{x_j}(\rho u_j) = 0, \\ \partial_t(\rho u_m) + \sum_{j=1}^d \partial_{x_j}(\rho u_m u_j + P) = \sum_{j=1}^d \partial_{x_j}(\mu \partial_{x_j} u_m) - g\rho e_d, \\ \sum_{j=1}^d \partial_j u_j = 0, \end{cases} \tag{0.7}$$

where $e_d = (0, \dots, 0, 1)^T$. We are interested in the instability of a hydrostatic equilibrium to (0.7),

$$(\rho_0(x_d), 0, P_0(x_d)), \quad \text{satisfying} \quad \frac{dP_0}{dx_d} = -g\rho_0. \tag{0.8}$$

For this function ρ_0, we define the characteristic length L_0 such that $L_0^{-1} := \max_{x_d \in I} |\rho_0'/\rho_0|$.

We recall the classical (inviscid) problem and well-known results in mathematics in the next part, Section 0.3. In this thesis:

1. The first chapter presents the results of

[51]: *Spectral analysis of the incompressible viscous Rayleigh-Taylor system in* $\mathbf{R}^3$,

which appeared in *Water Waves* (to be summarized in Section 0.4). This chapter studies the spectral analysis of the viscous Rayleigh-Taylor instability around an increasing density profile $\rho_0(x_d)$. For this, we develop an operator method, using the spectral theory of self-adjoint and compact operators, to prove the existence of multiple characteristic values $\lambda(\mathbf{k})$, i.e.

$$\lambda_1(\mathbf{k}) > \lambda_2(\mathbf{k}) > \dots,$$

such that

$$\Lambda := \sup_{\mathbf{k}} \lambda_1(\mathbf{k}) \leqslant \sqrt{\frac{g}{L_0}}. \tag{0.9}$$

This implies the existence of multiple normal modes $\{e^{\lambda_j(\mathbf{k})t}Q_j(\mathbf{k},x)\}_{j\geqslant 1}$ of type (0.5) to the linearized equations. That result is inspired by the inviscid study of Helffer-Lafitte [42] and gives more information on the discrete spectrum of the linearized equations than previous results of Guo-Hwang [29] and Guo-Tice [32] where only the largest characteristic value $\lambda_1(\mathbf{k})$ is found via the variational approach.

2. In Chapter 2 (to be detailed in Section 0.5) and Chapter 3 (to be detailed in Section 0.6), we continue the investigation of Rayleigh-Taylor instability in other settings, respectively,

1. [61]: Linear and Nonlinear analysis of the viscous Rayleigh-Taylor system with Navier-slip boundary conditions, preprint, `https://arxiv.org/abs/2204.09857`,

2. [62]: Nonlinear Rayleigh-Taylor instability of the viscous surface waves in an infinitely deep ocean, preprint.

We apply the operator method initiated in [51] to prove the existence of multiple normal modes to the linearized equations. Furthermore, we prove the nonlinear viscous Rayleigh-Taylor instability, i.e. the steady state solution (0.8) if the viscous RT system is nonlinearly unstable. Since the characteristic values $\lambda_j(\mathbf{k})$ $(j \geqslant 1)$ are bounded by $\sqrt{\frac{g}{L_0}}$ for any $\mathbf{k}$, we use the following procedure:

Step 1. Establish the *a priori* energy estimate for the local exact solution to the nonlinear equation of type (0.3) with any initial data,

Step 2. Formulate a linear combination of normal modes to the linearized equation of type (0.4) to make it to be an approximate solution V^{app}, i.e.

$$V^{\text{app}}(t,x) = \sum_{j=1}^{M} \mathsf{c}_j e^{\lambda_j(\mathbf{k})t} Q_j(\mathbf{k},x) \quad \text{for any } M \in \mathbb{N}^\star,$$

to the nonlinear equation (0.3). We then set $\delta V^{\text{app}}(0,x)$ $(0 < \delta \ll 1)$ as the initial data to the nonlinear equation (0.3). Eq. (0.3) with those initial data has a unique local exact solution V^{exact} on $[0, T^{\max})$, Step 3. We then define the difference $V^{\text{diff}} = V^{\text{exact}} - \delta V^{\text{app}}$, which satisfies

$$\partial_t V^{\text{diff}} + L_{\bar{U}} V^{\text{diff}} = \mathcal{N}_{\bar{U}}(V^{\text{exact}}). \tag{0.10}$$

We deduce the bound in time of $\|V^{\text{diff}}(t)\|_{L^2(\Omega)}$ by exploiting energy estimates to (0.10) and the bound in time of $\|V^{\text{exact}}(t)\|_{H^s(\Omega)}$, which is obtained thanks to the *a priori* energy estimates in Step 1,

Step 4. Conclude on the nonlinear instability by combining these above estimates.

Note that, in the results of Guo-Strauss [30] and of Grenier [35], only the maximal normal mode $e^{\lambda_1(\mathbf{k})t}Q_1(\mathbf{k}, x)$ was used in Step 2 to derive a solution of the nonlinear equation whose initial datum $\delta Q_1(\mathbf{k}, x)$ $(0 < \delta \ll 1)$. Let us emphasize that, our nonlinear results show that a wide class of initial data to the nonlinear problem departing from the equilibrium give rise to the nonlinear instability. These initial data are deduced from a linear combination of normal modes. Beyond the viscous Rayleigh-Taylor instability, we intend to use our arguments for other models, such as a two-fluid model initiated by Bresch et. al. [5]. Further discussions are left to the last chapter of this thesis, Chapter 4.

0.3 The classical Rayleigh-Taylor instability

In 1883, Lord Rayleigh [69] studied the linear stability of the eigenvalue problem for two layers of gravity-driven incompressible and inviscid fluids, the heavy one is on the top of the light one and addressed the general stability criterion. Rayleigh's work was taken up by Taylor [75] in 1950, in a more general set-up considering the effect of any accelerating field. This Rayleigh-Taylor (RT) instability appears and plays a key role in many physical phenomena, e.g., interstellar medium and galaxy clusters [60], inertial confinement fusion [55], oceanography [11], etc. For a pedagogical presentation of the mathematical study, we refer to the book of Chandrasekhar [7] in 1961 or a report of Kull [49] in 1991 and we refer to the physical reports of Zhou [77, 78] for the summary of applications of RT instability in physics.

We now describe the mathematical problem. Following Rayleigh's paper [69], we state the governing equations, which are the incompressible Euler equations in $\mathbf{R}^3$ as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = -g\rho e_3, \\ \operatorname{div} u = 0. \end{cases} \tag{0.11}$$

We recall that system (0.11) admits a hydrostatic equilibrium state

$$(\rho_0(x_3), 0, P_0(x_3)), \quad \text{satisfying } P_0' = -g\rho_0 \text{ with } ' = \frac{d}{dx_3}. \tag{0.12}$$

Denote by

$$\sigma = \rho - \rho_0, \quad u = u - 0, \quad q = P - P_0,$$

the nonlinear equations read as

$$\begin{cases} \partial_t \sigma + \rho_0' u_3 = -u \cdot \nabla \sigma, \\ (\rho_0 + \sigma)(\partial_t u + u \cdot \nabla u) + \nabla q = -g\sigma e_3, \\ \mathrm{div} u = 0. \end{cases} \tag{0.13}$$

We thus obtain the linearized equations

$$\begin{cases} \partial_t \sigma + \rho_0' u_3 = 0, \\ \rho_0 \partial_t u + \nabla q = -g\sigma e_3, \\ \mathrm{div} u = 0. \end{cases} \tag{0.14}$$

Studying the linear instability amounts to finding normal modes, which increase in time, of the form

$$\begin{cases} \sigma(t,x) = e^{\lambda t} \cos(k_1 x_1 + k_2 x_2)\omega(x_3), \\ u_1(t,x) = e^{\lambda t} \sin(k_1 x_1 + k_2 x_2)\psi(x_3), \\ u_2(t,x) = e^{\lambda t} \sin(k_1 x_1 + k_2 x_2)\theta(x_3), \\ u_3(t,x) = e^{\lambda t} \cos(k_1 x_1 + k_2 x_2)\phi(x_3), \\ q(t,x) = e^{\lambda t} \cos(k_1 x_1 + k_2 x_2) r(x_3). \end{cases} \tag{0.15}$$

Substituting (0.15) into (0.14), we have

$$\begin{cases} \lambda\omega + \rho_0'\phi = 0, \\ \lambda\rho_0\psi - k_1 r = 0, \\ \lambda\rho_0\theta - k_2 r = 0, \\ \lambda\rho_0\phi + r' = -g\omega, \\ k_1\psi + k_2\theta + \phi' = 0. \end{cases} \tag{0.16}$$

That implies

$$\omega = -\frac{\rho_0'\phi}{\lambda}, \quad r = -\frac{\lambda\rho_0\phi'}{k^2}$$

and

$$\psi = \frac{\lambda\rho_0 r}{k_1}, \quad \theta = \frac{\lambda\rho_0 r}{k_2}.$$

Eq. (0.16) reduces to the following second-order ODE on ϕ,

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') = gk^2\rho_0'\phi. \tag{0.17}$$

These solutions decay to zero at $\pm\infty$, i.e. ϕ satisfies $\lim_{x_3\to\infty}\phi(x_3) = 0$.

Let $\rho_\pm > 0$, Rayleigh considered a discontinuous density profile $\rho_0(x_3)$ such that

$$\rho_0(x_3) = \rho_- \mathbf{1}_{\{x_3<0\}} + \rho_+ \mathbf{1}_{\{x_3>0\}}. \tag{0.18}$$

It has been shown first by Rayleigh, then by Taylor that if $\rho_+ > \rho_-$, then $(\rho_0(x_3), 0, P_0(x_3))$ is linearly unstable, i.e. there are exponentially normal mode solutions of the linearized Euler equations (0.11) around that profile. Precisely, the authors proved that there is a unique growth rate

$$\lambda_0 = \sqrt{gk\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}} \tag{0.19}$$

such that (0.17) has a unique family of solutions, spanned by the function $\phi_0(x_3) = 2\sqrt{k}e^{-k|x_3|} \in H^1(\mathbf{R})$ (being normalized in $L^2(\mathbf{R})$) with $\lambda = \lambda_0$. The existence of a linearly unstable mode for Rayleigh–Taylor instability gives rise to the nonlinear instability for the fully nonlinear system (0.13). This was shown by a rigorous framework thanks to Desjardins and Grenier [13].

Let us consider now the case, where ρ_0 is smooth and satisfies

$$+\infty > \lim_{x_3\to+\infty} \rho_0(x_3) = \rho_+ > \rho_- = \lim_{x_3\to-\infty} \rho_0(x_3) > 0. \tag{0.20}$$

For some particular profiles ρ_0 satisfying (0.20), the linear instability is proven, see e.g. [7, Chapter X], [8]. For profiles satisfying (0.20) and the additional condition

$$\Big(\int_{-\infty}^0 |\rho_0(x_3) - \rho_-|^2 dx_3 \Big)^{1/2} + \Big(\int_0^{+\infty} |\rho_0(x_3) - \rho_+|^2 dx_3 \Big)^{1/2} < +\infty,$$

we refer to [8], [9], [50]. Note that, in [8], a quantization of λ (multiple countable normal modes) was shown explicitly and in [9], an asymptotic expansion of the largest value of λ was found under a small perturbation of wave number k. In 2003, the paper of Hwang and Guo [29] exploited the natural variational structure of the inviscid problem pointed out in [7] to describe the square of the maximum growth rate λ_1^2 as the maximum of the Rayleigh quotient

$$\frac{gk^2 \int_{\mathbf{R}} \rho_0' \phi^2 dx_3}{\int_{\mathbf{R}} \rho_0((\phi')^2 + k^2\phi^2) dx_3}.$$

Furthermore, based on its associated normal mode for the linear Euler equations, Guo and Hwang used the method initiated by Grenier in his celebrated paper [35] to construct an approximate solution to the nonlinear perturbation equations. Combining with some classical energy estimates, the nonlinear instability follows.

Let us add additional comments on the linear RT result. It can be seen that (0.17) is a generalized eigenvalue problem. Multiple growth rates of the inviscid problem are first mentioned by Cherfils and Lafitte [8] through the precise study of the inviscid ODE for an affine profile. Mention also the paper of Mikaelian [58] for the connection between the Rayleigh (0.17) and the Schrödinger equations. Lafitte [50] then observed that possible growth rates λ's for the classical Rayleigh-Taylor problem (not only the largest one) are such that λ^2 is an eigenvalue of a suitable self-adjoint operator and described this spectrum, for large values of k, as the eigenvalues of a 1D Schrodinger operator. This approach is extended by Helffer and Lafitte [42]. It can be noticed that the multiple growth rates are given by $\{\lambda_2, \dots, \lambda_j, \dots\}$, where λ_{j+1}^2 is equal to

$$\max_{\phi \in H_j^1} \frac{gk^2 \int_{\mathbf{R}} \rho_0' \phi^2 dx_3}{\int_{\mathbf{R}} \rho_0((\phi')^2 + k^2\phi^2) dx_3}$$

where $H_j^1 = \{\phi \in H^1(\mathbf{R}), \int_{\mathbf{R}} \phi\phi_l dx_3 = 0, \forall l \leqslant j\}$, ϕ_l is a nontrivial eigenfunction associated with λ_l.

We use in this thesis the self-adjoint operator approach for the spectral analysis.

0.4 Spectral analysis of the viscous Rayleigh-Taylor instability

The first result of this thesis is to obtain the spectral analysis when there is a viscosity coefficient in the system, using the approach of Helffer-Lafitte [42]. The RT instability with the presence of viscosity dates back at least to Chandrasekhar [7, Chapter X]. Let us consider the domain $\Omega = \mathbf{R}^3$, i.e. $d = 3, \Sigma = \mathbf{R}$ and $I = \mathbf{R}$. We are concerned with the following Navier–Stokes equations describing the motion of a nonhomogeneous incompressible viscous fluid in the presence of a uniform gravity field in $\mathbf{R}_+ \times \Omega$,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P = \mu \Delta u - g\rho e_3, \\ \mathrm{div} u = 0, \end{cases} \tag{0.21}$$

Let $' = \frac{d}{dx_3}$, note that $(\rho, u, P) = (\rho_0, 0, P_0)$ with $P_0' = -g\rho_0$ is still an equilibrium state of (0.21). The quantities

$$\sigma = \rho - \rho_0, \quad u = u - 0, \quad p = P - P_0$$

satisfy the following nonlinear equations

$$\begin{cases} \partial_t \sigma + u \cdot \nabla(\rho_0 + \sigma) = 0, \\ (\rho_0 + \sigma)\partial_t u + (\rho_0 + \sigma) u \cdot \nabla u + \nabla p = \mu \Delta u - \sigma g, \\ \mathrm{div} u = 0. \end{cases} \tag{0.22}$$

That implies the following linearized system

$$\begin{cases} \partial_t \sigma + \rho_0' u_3 = 0, \\ \rho_0 \partial_t u + \nabla p = \mu \Delta u - g\sigma e_3, \\ \mathrm{div} u = 0. \end{cases} \tag{0.23}$$

Since ρ_0 depends only on x_3, we perform the analysis into normal modes as in [7, Chapter X, Section 91]. Precisely, we seek the perturbations under the form as (0.15),

$$\begin{cases} \sigma(t,x) = e^{\lambda t}\cos(k_1x_1 + k_2x_2)\omega(x_3), \\ u_1(t,x) = e^{\lambda t}\sin(k_1x_1 + k_2x_2)\psi(x_3), \\ u_2(t,x) = e^{\lambda t}\sin(k_1x_1 + k_2x_2)\theta(x_3), \\ u_3(t,x) = e^{\lambda t}\cos(k_1x_1 + k_2x_2)\phi(x_3), \\ p(t,x) = e^{\lambda t}\cos(k_1x_1 + k_2x_2)r(x_3), \end{cases} \tag{0.24}$$

where $\mathbf{k} = (k_1, k_2) \in \mathbf{R}^2$, $\lambda \in \mathbb{C}\backslash\{0\}$ and $\mathrm{Re}\lambda \geqslant 0$. We deduce

$$\begin{cases} \lambda\zeta + \rho_0'\phi = 0, \\ \lambda\rho_0\psi - k_1 q + \mu(k^2\psi - \psi'') = 0, \\ \lambda\rho_0\theta - k_2 q + \mu(k^2\theta - \theta'') = 0, \\ \lambda\rho_0\phi + q' + \mu(k^2\phi - \phi'') + g\zeta = 0, \\ k_1\psi + k_2\theta + \phi' = 0. \end{cases} \tag{0.25}$$

Note that $\zeta = -\frac{\rho_0'\phi}{\lambda}$. Hence, $(0.25)_4$ becomes

$$\lambda^2\rho_0\phi + \lambda q' + \lambda\mu(k^2\phi - \phi') = g\rho_0'\phi. \tag{0.26}$$

We multiply $(0.25)_2$ by k_1 and $(0.25)_3$ by k_2, then use $(0.25)_4$ to obtain the equality

$$\lambda^2\rho_0\phi' + k^2\lambda q + \lambda\mu(k^2\phi' - \phi''') = 0.$$

Deriving this equation, and replacing $\lambda q'$ through (0.26), we get the following fourth-order ODE

$$-\lambda^2(\rho_0 k^2\phi - (\rho_0\phi')') = \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) - gk^2\rho_0'\phi. \tag{0.27}$$

The investigation of normal modes (0.24) amounts to finding regular solutions $\phi \in H^4(\mathbf{R})$ of (0.27). These solutions decay to zero at $\pm\infty$, i.e. ϕ satisfies

$$\lim_{x_3 \to \pm\infty} \phi(x_3) = 0. \tag{0.28}$$

Note that, $\phi \in L^\infty(\mathbf{R})$ is enough to fulfil the condition (0.28).

In the book of Chandrasekhar [7, Chapter X], the author considered two uniform viscous fluid separated by a horizontal boundary and generalized the classical result of Rayleigh and Taylor. Precisely, in the toy model case

$$\rho_0 = \rho_- 1_{\{x_3<0\}} + \rho_+ 1_{\{x_3>0\}} \quad (0 < \rho_- < \rho_+),$$

there is a unique strictly positive value of λ_μ such that Eq. (0.27) with $\lambda = \lambda_\mu$ has a non trivial solution. The equation satisfied by λ_μ is

$$\lambda^2 = \lambda_0^2 \times \frac{\frac{\sqrt{k^2+\lambda\rho_+/\mu}}{\rho_+} + \frac{\sqrt{k^2+\lambda\rho_-/\mu}}{\rho_-} - k\frac{\rho_++\rho_-}{\rho_+\rho_-}}{(\rho_++\rho_-)\Big(\frac{\sqrt{k^2+\lambda\rho_+/\mu}}{\rho_+} + \frac{\sqrt{k^2+\lambda\rho_-/\mu}}{\rho_-}\Big) - k\frac{(\rho_+-\rho_-)^2}{\rho_+\rho_-(\rho_++\rho_-)}}, \tag{0.29}$$

where λ_0 is the classical RT growth rate (see (0.19)). Hence, we have the relation between λ_μ and λ_0 in the inviscid limit, i.e. $\mu \to 0$,

$$\lambda_\mu = \lambda_0 + O(\sqrt{\mu}) \quad \text{as } \mu \to 0, \tag{0.30}$$

For the viscous problem with a smooth profile ρ_0, Jiang, Jiang and Ni [37] used a modified variational approach, described by Guo and Tice [32], where a bootstrap argument yields the largest characteristic value and an associated solution being regular under the assumption $\rho_0' \in C_0^\infty(\mathbf{R}), \inf_{\mathbf{R}} \rho_0 > 0$ and $\rho_0'(x_3^0) > 0$ for some points $x_3^0 \in \mathbf{R}$.

As far as we know, no other authors performed studies of the discrete spectrum of the viscous linearized RT instability. We then illustrate our spectral analysis.

Let us consider an increasing C^1 function ρ_0. For such a density profile ρ_0, we show that:

Lemma 0.1. *All characteristic values λ for Eq. (0.27)-(0.28) in $H^4(\mathbf{R})$ are real and satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

In view of Lemma 0.1, a characteristic value satisfies $\lambda > 0$ and we are left to look for $\lambda > 0$. We seek functions ϕ being real and only consider the vector spaces of real-valued functions in what follows in the linear analysis.

In the case $\rho_0' \geqslant 0$ compactly supported, our assumption is

$$\rho_0' \text{ is a nonnegative function of class } C_0^0(\mathbf{R}), \quad \operatorname{supp}(\rho_0') = [-a, a], \tag{0.31}$$

Outside $(-a, a)$, we denote

$$\rho_0(x_3) = \begin{cases} \rho_- & \text{as } x_3 \in (-\infty, -a], \\ \rho_+ & \text{as } x_3 \in [a, +\infty), \end{cases} \tag{0.32}$$

with $0 < \rho_- < \rho_+$ are two positive constants. This can be seen, physically speaking, as the situation of the toy model with a layer, of size $2a$, in which there is a mixture of the two fluids of density ρ_- and ρ_+.

Theorem 0.1. *Let ρ_0 satisfy (0.31) and (0.32). There exists an infinite sequence $(\lambda_n, \phi_n)_{n\geqslant 1}$ with $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ and nontrivial $\phi_n \in H^4(\mathbf{R})$ satisfying Eq. (0.27). In addition, λ_n decreases towards 0 as n goes to ∞.*

In the next part, we consider ρ_0' no longer compactly supported. The assumptions on ρ_0 are

$$\rho_0 \in C^1(\mathbf{R}), \qquad \lim_{x_3 \to \pm\infty} \rho_0(x_3) = \rho_\pm \in (0, +\infty) \tag{0.33}$$

and

$$0 < \rho_0'(x_3) < \rho_m < +\infty \text{ for all } x_3 \in \mathbf{R}. \tag{0.34}$$

The second theorem is as follows.

Theorem 0.2. *Let ρ_0 satisfy (0.33) and (0.34). For $0 < \epsilon_\star \ll 1$, there exists $N(\epsilon_\star) \in \mathbb{N}^\star$ such that there exists a finite sequence $(\lambda_n, \phi_n)_{1\leqslant n\leqslant N(\epsilon_\star)}$ with $\lambda_n \in [\epsilon_\star, \sqrt{\frac{g}{L_0}}]$ and $\phi_n \in H^4(\mathbf{R})$ satisfying Eq. (0.27).*

The proof of Theorem 0.1 and Theorem 0.2 shares the same strategy, illustrated below.

We rewrite (0.27) as a linear system of ODEs on $(\phi, \phi', \phi'', \phi''')^T$. As the solution of this system must tend to 0 when $x_3 \to \pm\infty$, using the fact that the profile ρ_0 goes to $\rho_\pm$ at $\pm\infty$, we are able to deduce the stable linear space S_+ at $+\infty$ and the unstable linear space S_- at $-\infty$, which are vector spaces of dimension two. Hence, we transform the problem for the normal modes on $\mathbf{R}$ into an ODE problem stated on a compact interval (x_-, x_+) with appropriate boundary conditions deduced from the outer solutions. Note that, if ρ_0' is compactly supported, the natural choice of $x_\pm$ is to choose the ends of suppρ_0'. However, if ρ_0' non compactly supported, $x_\pm$ are deduced from the behavior at $\pm\infty$ of the outer solutions and depend also on k and λ. In the case of a compactly supported ρ_0' (the simplest case of a convergence), they are described by

$$\begin{cases} k\tau_-\phi(-a) - (k+\tau_-)\phi'(-a) + \phi''(-a) = 0, \\ k\tau_-(k+\tau_-)\phi(-a) - (k^2 + k\tau_- + \tau_-^2)\phi'(-a) + \phi'''(-a) = 0, \end{cases} \tag{0.35}$$

and

$$\begin{cases} k\tau_+\phi(a) + (k+\tau_+)\phi'(a) + \phi''(a) = 0, \\ -k\tau_+(k+\tau_+)\phi(a) - (k^2 + k\tau_+ + \tau_+^2)\phi'(a) + \phi'''(a) = 0, \end{cases} \tag{0.36}$$

where $\tau_\pm = \sqrt{k^2 + \lambda\rho_\pm/\mu}$. In the case of a non compactly supported ρ_0', they are described by

$$\begin{cases} n_{11}^-\phi(x_-) + n_{12}^-\phi'(x_-) + \phi''(x_-) = 0, \\ n_{21}^-\phi(x_-) + n_{22}^-\phi'(x_-) + \phi'''(x_-) = 0 \end{cases} \tag{0.37}$$

and

$$\begin{cases} n_{11}^+\phi(x_+) + n_{12}^+\phi'(x_+) + \phi''(x_+) = 0, \\ n_{21}^+\phi(x_+) + n_{22}^+\phi'(x_+) + \phi'''(x_+) = 0, \end{cases} \tag{0.38}$$

Constants $n_{ij}^{\pm}$ depend on $x_{\pm}, k$ and λ.

In order to solve (0.27) on (x_-, x_+), the crucial tool in our study is to construct two bilinear forms on $H^2((x_-, x_+))$, which are continuous and coercive, $\mathcal{B}_\lambda$, denoted respectively by $\mathcal{B}_{a,\lambda}$ (see (0.42)) for ρ_0' being compactly supported and by $\mathcal{B}_{x_-,x_+,\lambda}$ (see (0.45)) for ρ_0' being strictly positive everywhere. So that the finding of a solution $\phi \in H^4((x_-, x_+))$ of Eq. (0.27) with the boundary conditions (0.35)-(0.36) or (0.37)-(0.38) is equivalent to finding a weak solution $\phi \in H^2((x_-, x_+))$ to the variational problem

$$\lambda \mathcal{B}_\lambda(\phi, \omega) = gk^2 \int_{x_-}^{x_+} \rho_0' \phi \omega dx_3 \quad \text{for all } \omega \in H^2((x_-, x_+)) \tag{0.39}$$

and thus improving the regularity of that weak solution ϕ.

The expressions of $\mathcal{B}_{a,\lambda}$ and $\mathcal{B}_{x_-,x_+,\lambda}$ are given as follows. Let us denote by

$$BV_{-a,\lambda}(\vartheta, \varrho) := \mu \begin{pmatrix} k\tau_-(k+\tau_-)\vartheta(-a)\varrho(-a) - k\tau_-\vartheta'(-a)\varrho(-a) \\ \qquad - k\tau_-\vartheta(-a)\varrho'(-a) + (k+\tau_-)\vartheta'(-a)\varrho'(-a) \end{pmatrix} \tag{0.40}$$

and

$$BV_{a,\lambda}(\vartheta, \varrho) := \mu \begin{pmatrix} k\tau_+(k+\tau_+)\vartheta(a)\varrho(a) - k\tau_+\vartheta'(a)\varrho(a) \\ \qquad - k\tau_+\vartheta(a)\varrho'(a) + (k+\tau_+)\vartheta'(a)\varrho'(a) \end{pmatrix}. \tag{0.41}$$

Define then

$$\begin{aligned} \mathcal{B}_{a,\lambda}(\vartheta, \varrho) := BV_{a,\lambda}(\vartheta, \varrho) + BV_{-a,\lambda}(\vartheta, \varrho) + \lambda \int_{-a}^{a} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 \\ + \mu \int_{-a}^{a} (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3. \end{aligned} \tag{0.42}$$

Similarly, let

$$\begin{aligned} BV_{x_+,\lambda}(\vartheta, \varrho) := -\lambda(\rho_0\vartheta'\varrho)(x_+) - \mu(n_{21}^+(x_+,\lambda)\vartheta(x_+) + n_{22}^+(x_+,\lambda)\vartheta'(x_+))\varrho(x_+) \\ + \mu(n_{11}^+(x_+,\lambda)\vartheta(x_+) + n_{12}^+(x_+,\lambda)\vartheta'(x_+))\varrho'(x_+) - 2k^2\mu(\vartheta'\varrho)(x_+) \end{aligned} \tag{0.43}$$

and

$$\begin{aligned} BV_{x_-,\lambda}(\vartheta, \varrho) := \lambda(\rho_0\vartheta'\varrho)(x_-) + \mu(n_{21}^-(x_-,\lambda)\vartheta(x_-) + n_{22}^-(x_-,\lambda)\vartheta'(x_-))\varrho(x_-) \\ - \mu(n_{11}^-(x_-,\lambda)\vartheta(x_-) + n_{12}^-(x_-,\lambda)\vartheta'(x_-))\varrho'(x_-) + 2k^2\mu(\vartheta'\varrho)(x_-). \end{aligned} \tag{0.44}$$

We have

$$\begin{aligned} \mathcal{B}_{x_-,x_+,\lambda}(\vartheta, \varrho) :=& BV_{x_-,\lambda}(\vartheta, \varrho) + BV_{x_+,\lambda}(\vartheta, \varrho) + \lambda \int_{x_-}^{x_+} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 \\ &+ \mu \int_{x_-}^{x_+} (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3, \end{aligned} \tag{0.45}$$

As we can choose $x_\pm$ depending on k, λ, μ and the profile ρ_0 such that the bilinear form $\mathcal{B}_\lambda$ is coercive on $H^2((x_-,x_+))$, one thus has $\sqrt{\mathcal{B}_\lambda(\cdot,\cdot)}$ is a norm on $H^2((x_-,x_+))$. One denotes by $(H^2((x_-,x_+)))'$ the dual space of $H^2((x_-,x_+))$ induced by the norm $\sqrt{\mathcal{B}_\lambda(\cdot,\cdot)}$. In view of Riesz's representation theorem, we thus define an abstract operator

$$Y_\lambda \in \mathcal{L}(H^2((x_-,x_+)),(H^2((x_-,x_+))'),$$

such that

$$\mathcal{B}_\lambda(\vartheta,\varrho) = \langle Y_\lambda\vartheta,\varrho\rangle, \quad \text{for all } \vartheta,\varrho \in H^2((x_-,x_+)) \tag{0.46}$$

From (0.39) and (0.46), we see that the existence of a solution $\phi \in H^4((x_-,x_+))$ of Eq. (0.27) on (x_-,x_+) with the boundary conditions (0.35)-(0.36) or (0.37)-(0.38) is thus reduced to the finding of a weak solution $\phi \in H^2((x_-,x_+))$ of

$$\lambda Y_\lambda\phi = gk^2\rho_0'\phi \quad \text{in } (H^2((x_-,x_+)))'. \tag{0.47}$$

Restricting $\varrho \in C_0^\infty((x_-,x_+))$ in (0.46), we find the precise expression of Y_λ in both cases, i.e. for all $\vartheta \in H^2((x_-,x_+))$,

$$Y_\lambda\vartheta = \lambda(\rho_0k^2\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta) \tag{0.48}$$

in $\mathcal{D}'((x_-,x_+))$ (see $Y_{a,\lambda}$ in Proposition 1.3 for $\rho_0' \geqslant 0$ being compactly supported and $Y_{x_-,x_+,\lambda}$ in Proposition 1.9 for ρ_0' being positive everywhere). Note that $\rho_0 \in C^1$ implies $(\rho_0\vartheta')'$ is well-defined. We further use a bootstrap argument to define the inverse operator Y_λ^{-1} of Y_λ, from $L^2((x_-,x_+))$ to a subspace of $H^4((x_-,x_+))$ requiring all elements satisfy (0.35)-(0.36) or (0.37)-(0.38). Note that, since $\phi \in H^4((x_-,x_+))$, these boundary conditions (involving the derivatives ϕ'',ϕ''' of ϕ at $x_3 = x_-$ and at $x_3 = x_+$) are well defined. Composing the above operator Y_λ^{-1} with the continuous injection from $H^4((x_-,x_+))$ to $L^2((x_-,x_+))$, we obtain that Y_λ^{-1} is a compact and self-adjoint operator from $L^2((x_-,x_+))$ to itself.

We introduce $\mathcal{M}$ the operator of multiplication by $\sqrt{\rho_0'}$ in $L^2((x_-,x_+))$. Note from (0.47) that, we will find v satisfying

$$\mathcal{M}Y_\lambda^{-1}\mathcal{M}v = \frac{\lambda}{gk^2}v. \tag{0.49}$$

We show that the operator $\mathcal{M}Y_\lambda^{-1}\mathcal{M}$ from $L^2((x_-,x_+))$ to itself is compact and self-adjoint, which enables us to use the spectral theory of compact and self-adjoint operators. Indeed, we obtain

the discrete spectrum of the operator $\mathcal{M}Y_\lambda^{-1}\mathcal{M}$ is an infinite sequence of eigenvalues (denoted by $\{\gamma_n(\lambda)\}_{n\geqslant 1}$).

Let $v_{n,\lambda}$ be an eigenfunction associated with the eigenvalue $\gamma_n(\lambda)$ and let $\phi_{n,\lambda} = Y_\lambda^{-1}\mathcal{M}v_{n,\lambda} \in H^4((x_-,x_+))$, we have

$$\gamma_n(\lambda)Y_\lambda\phi_{n,\lambda} = \mathcal{M}^2\phi_{n,\lambda}. \tag{0.50}$$

From (0.47) and (0.50), it can be seen that, for each n, let us solve the equation

$$\gamma_n(\lambda) = \frac{\lambda}{gk^2}. \tag{0.51}$$

When $\rho_0' \geqslant 0$ is compactly supported, for each n, we will show the existence and uniqueness of a solution λ_n to Eq. (0.51) owing first to the differentiability in λ of $\gamma_n(\lambda)$ (see Lemma 1.2), which is an extension of Kato's perturbation theory described in [45], and to the fact that $\lambda \to \gamma_n(\lambda)$ is decreasing in λ, through the derivative $\frac{d}{d\lambda}(\frac{1}{\gamma_n(\lambda)})$ (see Lemma 1.3) which exists also thanks to a similar argument of [45]. Furthermore, λ_n decreases towards 0 as $n \to \infty$. For each λ_n, we have that $\phi_{n,\lambda_n} = Y_{\lambda_n}^{-1}\mathcal{M}v_{n,\lambda_n} \in H^4((x_-, x_+))$ thanks to Propositions 1.3, 1.9 again. That function ϕ_{n,λ_n} is glued with the decaying solutions of Eq. (0.27) in the outer regions $(-\infty, x_-)$ and $(x_+, +\infty)$ by the boundary conditions at $x_\pm$, which yields a solution of Eq. (0.27) in $H^4(\mathbf{R})$ associated with $\lambda = \lambda_n$.

When $\rho_0' > 0$ everywhere, unlike in the first case, we lack an easy-to-use expression of the boundary conditions and we also do not have a uniform control of $n_{ij}^{\pm}$. We thus do not have the decrease of $\gamma_n(\lambda)$ or any control of $\gamma_n(\lambda)$ when λ goes to 0. Consequently, for $\rho_0' > 0$ everywhere, our arguments only lead to a possibly multiple existence of positive characteristic values λ such that $\lambda \geqslant \epsilon_\star > 0$.

0.5 Nonlinear Rayleigh-Taylor instability of the incompressible viscous fluid with Navier-slip boundary conditions

In [61], we consider the domain $\Omega = 2\pi L\mathbb{T} \times (-1, 1)$ with $L > 0$ (i.e. $d = 2, \Sigma = 2\pi L\mathbb{T}$ and $I = (-1, 1)$) and we are concerned with the viscous RT instability of the gravity-driven incompressible Navier-Stokes equations, which read as

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P = \mu\Delta u - \rho g e_2, \\ \mathrm{div} u = 0. \end{cases} \tag{0.52}$$

Let $\Sigma_\pm = 2\pi L\mathbb{T} \times \{\pm 1\}$, the Navier-slip boundary conditions proposed by Navier (see [59]) are given on $\Sigma_\pm$ as follows

$$\begin{aligned} u \cdot n = 0 \quad \text{on } \Sigma_+ \cup \Sigma_-, \\ (\mu(\nabla u + \nabla u^T) \cdot n)_\tau = \xi(x)u \quad \text{on } \Sigma_+ \cup \Sigma_-, . \end{aligned} \tag{0.53}$$

Here, n is the outward normal vector of the boundary, $(\mu(\nabla u + \nabla u^T) \cdot n)_\tau$ is the tangential component of $\mu(\nabla u + \nabla u^T) \cdot n$ and $\xi(x)$ is a scalar function describing the slip effect on the boundary. We assume that $\xi(x) = \xi_+$ on Σ_+ and $\xi(x) = \xi_-$ on Σ_-, where $\xi_\pm$ are two nonnegative constants.

Let us recall the steady state $(\rho_0(x_2), 0, P_0(x_2))$ of Eq. (0.52), where ρ_0 satisfies

$$\rho_0 \in C^1([-1,1]), \quad \rho_0' > 0 \text{ on } [-1,1], \quad \rho_0(\pm 1) = \rho_\pm \in (0,+\infty). \tag{0.54}$$

We now derive the linearization of Eq. (0.52) around the steady state $(\rho_0(x_2), 0, P_0(x_2))$. The perturbations

$$\sigma = \rho - \rho_0, \quad u = u - 0, \quad p = P - P_0$$

thus satisfy

$$\begin{cases} \partial_t \sigma + u \cdot \nabla(\rho_0 + \sigma) = 0, \\ (\rho_0 + \sigma)\partial_t u + (\rho_0 + \sigma) u \cdot \nabla u + \nabla p = \mu \Delta u - \sigma g, \\ \mathrm{div} v = 0. \end{cases} \tag{0.55}$$

Note that $(\mu(\nabla u + \nabla u^T) \cdot n)_\tau = n \times (\mu(\nabla u + \nabla u^T) \cdot n) \times n$ and that $n = (0, \pm 1)^T$. Hence, the boundary conditions (0.53) rewrite

$$\begin{cases} u_2 = 0, & \text{on } \Sigma_\pm, \\ \mu \partial_{x_2} u_1 = \xi_+ u_1 & \text{on } \Sigma_+, \\ \mu \partial_{x_2} u_1 = -\xi_- u_1 & \text{on } \Sigma_-. \end{cases} \tag{0.56}$$

The linearized equations read

$$\begin{cases} \partial_t \sigma + \rho_0' u_2 = 0, \\ \rho_0 \partial_t u + \nabla p = \mu \Delta u - \sigma g, \\ \mathrm{div} u = 0, \end{cases} \tag{0.57}$$

with the corresponding boundary conditions remaining (0.56).

The linear RT instability problem is still to seek a normal mode of the form

$$\begin{cases} \sigma(t,x) = e^{\lambda t} \cos(k x_1) \omega(x_2), \\ u_1(t,x) = e^{\lambda t} \sin(k x_1) \theta(x_2), \\ u_2(t,x) = e^{\lambda t} \cos(k x_1) \phi(x_2), \\ q(t,x) = e^{\lambda t} \cos(k x_1) q(x_2). \end{cases} \tag{0.58}$$

where $k \in L^{-1}\mathbf{Z} \backslash \{0\}$, $\lambda \in \mathbb{C} \backslash \{0\}$ and $\mathrm{Re}\lambda \geqslant 0$. It follows from (0.57) that

$$\begin{cases} \lambda \omega + \rho_0' \phi = 0, \\ \lambda \rho_0 \theta - k q + \mu(k^2 \theta - \theta'') = 0, \\ \lambda \rho_0 \phi + q' + \mu(k^2 \phi - \phi'') = -g\omega, \\ k\theta + \phi' = 0 \end{cases} \tag{0.59}$$

and from (0.56) that

$$\phi(\pm 1) = 0, \quad \mu\theta'(1) = \xi_+\theta(1), \quad \mu\theta'(-1) = -\xi_-\theta(-1). \tag{0.60}$$

We obtain

$$\omega = -\frac{\rho_0'}{\lambda}\phi, \quad \theta = -\frac{1}{k}\phi', \quad q = -\frac{1}{k^2}(\lambda\rho_0\phi' + \mu(k^2\phi' - \phi''')). \tag{0.61}$$

Then, we substitute q and ω into $(0.59)_3$ to get a fourth-order ODE (0.27). The boundary conditions deduced from (0.56) are obtained by assuming the solution to be in $C^2([-1,1])$,

$$\begin{cases} \phi(-1) = \phi(1) = 0, \\ \mu\phi''(1) = \xi_+\phi'(1), \\ \mu\phi''(-1) = -\xi_-\phi'(-1). \end{cases} \tag{0.62}$$

Note that $H^4((-1,1)) \hookrightarrow C^3((-1,1))$ allows us to write (0.62).

When the density profile ρ_0 is increasing, we first show that all characteristic values λ are real. We look for positive characteristic values and further obtain the uniform upper bound $\sqrt{\frac{g}{L_0}}$ of λ.

We now study the linearized problem, i.e. Eq. (0.27)-(0.62). Note that, it suffices to seek functions ϕ being real and consider the vector spaces of real-valued functions in what follows in the linear analysis. Of importance is to construct a continuous and coercive bilinear form $\mathcal{B}_{k,\lambda,\mu}$ as $\lambda \geqslant 0$ and $k \in \mathbf{R}\backslash\{0\}$ (i.e. we do not restrict $\lambda \in (0, \sqrt{\frac{g}{L_0}})$ and $k \in L^{-1}\mathbf{Z}\backslash\{0\}$ at this step) on the functional space

$$\tilde{H}^2((-1,1)) := \{\varphi \in H^2((-1,1)), \varphi(\pm 1) = 0\},$$

so that the finding of a solution $\phi \in H^4((-1,1))$ of Eq. (0.27)-(0.62) on $(-1,1)$ is equivalent to finding a weak solution $\phi \in \tilde{H}^2((-1,1))$ to the variational problem

$$\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\theta) = gk^2\int_{-1}^{1}\rho_0'\phi\theta dx_2 \quad \text{for all } \theta \in \tilde{H}^2((-1,1)), \tag{0.63}$$

and thus improving the regularity of that weak solution ϕ.

The desired bilinear form $\mathcal{B}_{k,\lambda,\mu}$ is

$$\begin{aligned} \mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) :=& \lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 \\ & - \xi_-\vartheta'(-1)\varrho'(-1) - \xi_+\vartheta'(1)\varrho'(1), \end{aligned} \tag{0.64}$$

For all $\lambda \geqslant 0$ and $k \in \mathbf{R}\backslash\{0\}$, we will consider a k-supercritical regime of the viscosity coefficient $\mu > \mu_c(k,\Xi)$ with $\Xi = (\xi_-,\xi_+)$ (see the precise formula $\mu_c(k,\Xi)$ in Proposition 2.1) such that

$$\mathcal{B}_{k,0,\mu} \text{ is coercive if and only if } \mu > \mu_c(k,\Xi). \tag{0.65}$$

It yields that $\mathcal{B}_{k,\lambda,\mu}$ is coercive for all $\lambda \geqslant 0$ and $\mu > \mu_c(k,\Xi)$. As $\mathcal{B}_{k,\lambda,\mu}$ is a coercive form on $H^2((-1,1))$, we have that $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$ is a norm on $H^2((-1,1))$. Let $(\tilde{H}^2((-1,1)))'$ be the dual space of $\tilde{H}^2((-1,1))$ associated with the norm $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$. In view of Riesz's representation theorem, we thus obtain an abstract operator $Y_{k,\lambda,\mu}$ from $\tilde{H}^2((-1,1))$ to $(\tilde{H}^2((-1,1)))'$) such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) = \langle Y_{k,\lambda,\mu}\vartheta,\varrho\rangle \quad \text{for all } \vartheta,\varrho \in \tilde{H}^2((-1,1)). \tag{0.66}$$

Owing to (0.63) and (0.66), it turns out that the existence of a solution $\phi \in H^4((-1,1))$ of Eq. (0.27)-(0.62) is reduced to the existence of a weak solution $\phi \in \tilde{H}^2((-1,1)$ of

$$\lambda Y_{k,\lambda,\mu}\phi = gk^2\rho_0'\phi \quad \text{in } (\tilde{H}^2((-1,1)))'. \tag{0.67}$$

Restricting $\varrho \in C_0^\infty((-1,1))$ in (0.66), we find that, for all $\vartheta \in H^2((-1,1))$ (see Proposition 2.3),

$$Y_{k,\lambda,\mu}\vartheta = \lambda(\rho_0k^2\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta) \quad \text{in } \mathcal{D}'((-1,1)).$$

In view of a bootstrap argument, we are able to define the inverse operator $Y_{k,\lambda,\mu}^{-1}$ of $Y_{k,\lambda,\mu}$, from $L^2((-1,1))$ to a subspace of $H^4((-1,1))$ requiring all elements satisfy (0.62). Composing the above operator $Y_{k,\lambda,\mu}^{-1}$ with the continuous injection from $H^4((-1,1))$ to $L^2((-1,1))$ (see Proposition 2.4), we obtain that $Y_{k,\lambda,\mu}^{-1}$ is a compact and self-adjoint operator from $L^2((-1,1))$ to itself.

Denoting by $\mathcal{M}$ the operator of multiplication by $\sqrt{\rho_0'}$ in $L^2((-1,1))$. Note from (0.67) that, we thus find (λ, v) such that

$$\frac{\lambda}{gk^2}v = \mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}v.$$

Once it is proven that the operator $\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$ is compact and self-adjoint from $L^2((-1,1))$ to itself, then

the discrete spectrum of the operator $\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$ is an infinite sequence of eigenvalues (denoted by $\{\gamma_n(k,\lambda,\mu)\}_{n\geqslant 1}$).

Let $v_{n,k,\lambda,\mu}$ be an eigenfunction of $\mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$ associated with the eigenvalue $\gamma_n(k,\lambda,\mu)$ and let $\phi_{n,k,\lambda,\mu} = Y_{k,\lambda,\mu}^{-1}\mathcal{M}v_{n,k,\lambda,\mu} \in H^4((-1,1))$, we have

$$\gamma_n(k,\lambda,\mu)Y_{k,\lambda,\mu}\phi_{n,k,\lambda,\mu} = \mathcal{M}^2\phi_{n,k,\lambda,\mu} = \rho_0'\phi_{n,k,\lambda,\mu}. \tag{0.68}$$

From (0.68), it can be seen that, for each n, we have to solve the equation

$$\gamma_n(k,\lambda,\mu) = \frac{\lambda}{gk^2}. \tag{0.69}$$

We will show that Eq. (0.69) has a unique root $\lambda_n(k,\mu) \in \mathbf{R}_+$ because of the decrease of γ_n in λ (see Lemma 2.3), which is an extension of Kato's perturbation theory of the spectrum of operators [45]. In addition, when λ_n is a characteristic value, we have $\lambda_n \leqslant \sqrt{\frac{g}{L_0}}$ for all $n \geqslant 1$.

This yields that for any horizontal spatial frequency $k \in L^{-1}\mathbf{Z}\backslash\{0\}$, there exists a sequence of characteristic values $(\lambda_n(k,\mu))_{n\geqslant 1}$, that is uniformly bounded and we further obtain that λ_n decreases towards 0 as $n \to \infty$. For each λ_n, we have that $\phi_{n,k,\lambda_n,\mu} = Y_{k,\lambda_n,\mu}^{-1} v_{n,k,\lambda_n,\mu}$ is a solution in $H^4((-1,1))$ of (0.27)-(0.62) associated with $\lambda = \lambda_n$.

We sum up the above arguments.

Theorem 0.3. *Let $k \in L^{-1}\mathbf{Z}\backslash\{0\}$ be fixed and let ρ_0 satisfy that* (0.54)*, i.e.*

$$\rho_0 \in C^1([-1,1]), \quad \rho_0(\pm 1) = \rho_\pm \in (0,\infty), \quad \rho_0' > 0 \text{ everywhere on } [-1,1].$$

For all $\mu > \mu_c(k,\Xi)$, there exists an infinite sequence $(\lambda_n, \phi_n)_{n\geqslant 1}$ with $\lambda_n > 0$ decreasing towards 0 as $n \to \infty$ and $\phi_n \in H^4((-1,1))$, ϕ_n nontrivial, satisfying (0.27)-(0.62).

Once Eq. (0.27)-(0.62) is solved, we go back to the linearized equations (0.57). For a fixed $k \in L^{-1}\mathbf{Z}\backslash\{0\}$, we obtain a sequence of solutions to the linearized equations (0.57)-(0.56) as indicated in Proposition 2.7, which are $(e^{\lambda_j(k,\mu)t} U_j(k,\mu,x))_{j\geqslant 1}$, with

$$U_j(k,\mu,x) = (\sigma_j, u_j, p_j)^T(k,\mu,x).$$

We now prove the nonlinear instability in the regime

$$\mu > 3\mu_c(\Xi), \quad \text{with } \mu_c(\Xi) := \sup_{k\in L^{-1}\mathbf{Z}\backslash\{0\}} \mu_c(k,\Xi). \tag{0.70}$$

The first important things are the local existence of strong solutions to the nonlinear equations and the *a priori* energy estimates to those solutions (see [14, Proposition 4.1]).

Proposition 0.1. *Suppose that the steady state $(\rho_0(x_2), 0, P_0(x_2))$ satisfies* (0.54)*. Then for any given initial data $(\sigma_0, u_0) \in (H^1(\Omega) \cap L^\infty(\Omega)) \times (H^2(\Omega))^2$ satisfying $\mathrm{div} u_0 = 0$, and also being compatible with the boundary conditions* (0.53)*, the nonlinear equations* (0.55) *has a local strong solution*

$$(\sigma, u, \nabla q) \in C([0, T^{\max}), H^1(\Omega) \times (H^2(\Omega))^2 \times (L^2(\Omega))^2). \tag{0.71}$$

Let $\mathcal{E}(t) := \sqrt{\|\sigma(t)\|_{H^1(\Omega)}^2 + \|u(t)\|_{H^2(\Omega)}^2}$ and $\delta > 0$ be sufficiently small, we further get that if $\sup_{0\leqslant s\leqslant t} \mathcal{E}(s) \leqslant \delta$, there holds

$$\begin{aligned} \mathcal{E}^2(t) + \|(\nabla q, \partial_t u)\|_{L^2(\Omega)}^2 + \int_0^t (\|\partial_t u(s)\|_{H^1(\Omega)}^2 + \|u(s)\|_{H^2(\Omega)}^2) ds \\ \lesssim \mathcal{E}^2(0) + \int_0^t \|(\sigma,u)(s)\|_{L^2(\Omega)}^2 ds. \end{aligned} \tag{0.72}$$

Since $\mu > 3\mu_c(\Xi)$, we can choose a constant $\varpi_0 > 0$ such that

$$\mu > (3 + \varpi_0)\mu_c(\Xi). \tag{0.73}$$

Hence, $\nu_0 = \frac{3+\varpi_0}{2+\varpi_0} \in (1, \frac{3}{2})$. Since all characteristic values are bounded by $\sqrt{\frac{g}{L_0}}$, we define

$$0 < \Lambda := \sup_{k \in L^{-1}\mathbf{Z}\backslash\{0\}} \lambda_1(k, \mu) \leqslant \sqrt{\frac{g}{L_0}} \tag{0.74}$$

and set $B_\Lambda := \{k \in L^{-1}\mathbf{Z}\backslash\{0\}, \lambda_1(k, \mu) > \frac{2\nu_0}{3}\Lambda\}$. For a $k_0 \in B_\Lambda$, we define $N \geqslant 1$ such that

$$\Lambda \geqslant \lambda_1(k_0, \mu) > \lambda_2(k_0, \mu) > \cdots > \lambda_N(k_0, \mu) > \frac{2\nu_0}{3}\Lambda > \lambda_{N+1}(k_0, \mu) > \ldots. \tag{0.75}$$

We introduce a linear combination of normal modes

$$U^M(t, x) = \sum_{j=1}^{M} \mathsf{c}_j e^{\lambda_j(k_0,\mu)t} U_j(k_0, \mu, x) \tag{0.76}$$

to construct an approximate solution to the nonlinear problem (0.55)-(0.56), with constants c_j being chosen such that

$$\text{at least one of } \mathsf{c}_j \ (1 \leqslant j \leqslant N) \text{ is non-zero} \tag{0.77}$$

and let $j_m := \min\{j : 1 \leqslant j \leqslant N, \mathsf{c}_j \neq 0\}$,

$$\frac{1}{2}|\mathsf{c}_{j_m}| \|u_{j_m}\|_{L^2(\Omega)} > \sum_{j \geqslant j_m+1} |\mathsf{c}_j| \|u_j\|_{L^2(\Omega)}. \tag{0.78}$$

Eq. (0.55)–(0.56), supplemented with the initial data $\delta U^M(0, x)$ $(0 < \delta \ll 1)$, admits a unique local strong solution $(\sigma^\delta, u^\delta)$ with an associated pressure q^δ on $[0, T_{\max})$ (see Proposition 0.1). We define the differences

$$\sigma^d = \sigma^\delta - \delta\sigma^M, \quad u^d = u^\delta - \delta u^M, \quad q^d = q^\delta - \delta q^M,$$

They satisfy

$$\begin{cases} \partial_t \sigma^d + \rho_0' u_2^d = -u^\delta \cdot \nabla \sigma^\delta, \\ \rho_0 \partial_t u^d - \mu \Delta u^d + \nabla q^d + g\sigma^d e_2 = -\sigma^\delta \partial_t u^\delta - (\rho_0 + \sigma^\delta) u^\delta \cdot \nabla u^\delta, \\ \mathrm{div} u^d = 0. \end{cases} \tag{0.79}$$

The initial condition of (0.79) is

$$(\sigma^d, u^d)(0) = 0 \tag{0.80}$$

and the boundary conditions of (0.79) are

$$\begin{cases} u_2^d = 0, & \text{on } \Sigma_\pm, \\ \mu \partial_{x_2} u_1^d = \xi_+ u_1^d & \text{on } \Sigma_+, \\ \mu \partial_{x_2} u_1^d = -\xi_- u_1^d & \text{on } \Sigma_-. \end{cases} \tag{0.81}$$

For t small enough, we then estimate the bound in time of $\|(\sigma^d, u^d)(t)\|_{L^2(\Omega)}$ (see Proposition 2.8), that is

$$\|(\sigma^d, u^d)(t)\|^2_{L^2(\Omega)} \lesssim \delta^3 \Big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \Big(\max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| \Big) e^{\frac{2}{3}\nu_0 \Lambda t} \Big)^3.$$

by exploiting some energy estimates of (0.79)-(0.80)-(0.81) and by using the bound in time $\|\sigma^\delta(t)\|_{H^1(\Omega)}$ and $\|u^\delta(t)\|_{H^2(\Omega)}$, which are deduced from the *a priori* energy estimate (0.72). The nonlinear result then follows.

Theorem 0.4. *Let $\mu_c(\Xi)$ be defined as in (0.70) and $\mu > 3\mu_c(\Xi)$. Let ρ_0 satisfies (0.54), i.e.*

$$\rho_0 \in C^1([-1,1]), \quad \rho_0(\pm 1) = \rho_\pm \in (0, \infty), \quad \rho_0' > 0 \text{ everywhere on } [-1,1].$$

Let $M \in \mathbb{N}^\star$, there exist a constant $m_0 > 0$ and positive constants ε_0 and δ_0 sufficiently small such that for any $\delta \in (0, \delta_0)$, the nonlinear equations (0.55) with the boundary conditions (0.56) and the initial data

$$\delta \sum_{j=1}^{M} \mathsf{c}_j U_j(x)$$

satisfying (0.77)-(0.78), has a unique local strong solution $(\sigma^\delta, u^\delta)$ with an associated pressure q^δ such that

$$\|u^\delta(T^\delta)\|_{L^2(\Omega)} \geqslant m_0 \epsilon_0, \tag{0.82}$$

where $T^\delta \in (0, T_{\max})$ is given by $\delta \sum_{j=j_m}^{M} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \epsilon_0$.

We end this section with the following remark.

Remark 0.1. *In the linear analysis, we revise the formula of the critical viscosity coefficient $\mu_c(\Xi) = \sup_{k>0} \mu_c(k, \Xi)$ of Ding, Li and Xin in [15, Proposition 2.2]. After we uploaded this paper on Arxiv, our computations lead to a corrigendum posted by the above authors recently (see [16]).*

0.6 Nonlinear Rayleigh-Taylor instability of the viscous surface wave in an infinitely deep ocean

We study, in Chapter 3 of this thesis, summarized in the present section, the nonlinear RT instability in our last setting, the viscous surface wave in an infinitely deep ocean. Let $\mathbf{T}^2 = 2\pi L_1 \mathbb{T} \times 2\pi L_2 \mathbb{T}$, the domain of the fluid is

$$\Omega(t) = \{x = (x_h, x_3) = (x_1, x_2, x_3) \in \mathbf{T}^2 \times \mathbf{R}, x_3 < \eta(t, x_1, x_2)\}, \tag{0.83}$$

hence, $\Omega(t)$ is bounded above by the free surface $\Gamma(t) = \{x_3 = \eta(t, x_1, x_2)\}$, where η is an unknown of the problem. We are concerned with the viscous RT instability of the nonhomogeneous incompressible Navier-Stokes equations without any effects of surface tension, which read

$$\begin{cases} \partial_t \tilde{\rho} + \mathrm{div}(\tilde{\rho}\tilde{u}) = 0 & \text{in } \Omega(t), \\ \partial_t(\tilde{\rho}\tilde{u}) + \mathrm{div}(\tilde{\rho}\tilde{u} \otimes \tilde{u}) + \nabla \tilde{p} = \mu\Delta\tilde{u} - g\tilde{\rho}e_3 & \text{in } \Omega(t), \\ \mathrm{div}\tilde{u} = 0 & \text{in } \Omega(t), \\ (\tilde{p}\mathrm{Id} - \mu\mathbb{S}\tilde{u}) \cdot n = p_{atm} n & \text{in } \Gamma(t), \\ \partial_t \eta = \tilde{u}_3 - \tilde{u}_1\partial_1\eta - \tilde{u}_2\partial_2\eta & \text{in } \Gamma(t). \end{cases} \tag{0.84}$$

The unknowns $\tilde{\rho} := \tilde{\rho}(t,x)$, $\tilde{u} := \tilde{u}(t,x)$ and $\tilde{p} := \tilde{p}(t,x)$ denote the density, the velocity and the pressure of the fluid, respectively. The stress tensor is $\mathbb{S}\tilde{u} = \nabla\tilde{u} + \nabla\tilde{u}^T$. The outward normal vector n of the boundary $\Gamma(t)$ is given by

$$n = \frac{(-\partial_1\eta, -\partial_2\eta, 1)^T}{\sqrt{1 + |\partial_1\eta|^2 + |\partial_2\eta|^2}}. \tag{0.85}$$

The constant p_{atm} is the atmospheric pressure. For a more physical description of the equations (0.84) and of the boundary conditions in (0.84), we refer to [54, Sect. 1.8].

To complete the statement of the problem, we must specify the initial conditions. We suppose that the initial surface $\Gamma(0)$ is given by the graph of the function $\eta(0) = \eta_0$, which yields the open set $\Omega(0)$ on which we specify the initial data for the velocity, $u(0) = u_0 : \Omega(0) \to \mathbf{R}^3$. We assume that the initial surface function satisfies the "zero-average" condition

$$\int_{\mathbf{T}^2} \eta_0 = 0 \tag{0.86}$$

and $\eta(0), u(0)$ satisfy certain compatibility conditions, which we will present in detail later (see Proposition 0.2). Note that, for sufficiently regular solutions of the problem, the condition (0.86) persists in time, that is

$$\int_{\mathbf{T}^2} \eta(t) = 0 \quad \text{for all } t \geqslant 0. \tag{0.87}$$

Indeed,

$$\frac{d}{dt}\int_{\mathbf{T}^2} \eta = \int_{\mathbf{T}^2} \partial_t \eta = \int_{\Gamma(t)} \tilde{u} \cdot n = \int_{\Omega(t)} \mathrm{div}\tilde{u} = 0.$$

The movement of the free boundary $\Gamma(t)$ and of the domain $\Omega(t)$ create numerous mathematical difficulties. To handle that, following Beale [3], we use the function η to transform the free boundary problem (0.84) into the equivalent problem (0.95) in a fixed domain $\Omega = \mathbf{T}^2 \times \mathbf{R}_-$ (i.e. $d = 3, \Sigma = \mathbf{T}^2$ and $I = \mathbf{R}_-$), which the fixed upper boundary is $\Gamma = \mathbf{T}^2 \times \{0\}$.

We now define the appropriate Poisson sum that allows us to extend η, to be determined on the surface Ω. For any $\mathbf{k} \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}$, we write

$$\hat{f}(\mathbf{k}) = \int_{\mathbf{T}^2} f(x_h) \frac{e^{-i\mathbf{k}\cdot x_h}}{2\pi\sqrt{L_1 L_2}} dx_h$$

and define the Poisson sum on Ω by

$$(\mathsf{p}f)(x_h, x_3) := \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}} \frac{e^{i\mathbf{k}\cdot x_h}}{2\pi\sqrt{L_1 L_2}} e^{|\mathbf{k}|x_3} \hat{f}(\mathbf{k}) \tag{0.88}$$

We then have $\mathsf{p} : H^s(\Gamma) \to H^{s+1/2}(\Omega)$ is a bounded linear operator for $s > 0$.

Lemma 0.2. *For $q \in \mathbb{N}$, let H_h^q be the usual homogeneous Sobolev space of order q and $\mathsf{p}f$ be the Poisson sum of a function f in $H_h^{q-1/2}(\Gamma)$. There holds*

$$\|\nabla^q \mathsf{p}f\|_{L^2(\Omega)}^2 \lesssim \|f\|_{H_h^{q-1/2}(\Gamma)}^2. \tag{0.89}$$

We extend η to be defined on Ω by

$$\theta(t, x) := (\mathsf{p}\eta)(t, x_h, x_3) \tag{0.90}$$

for all $x_h \in \mathbf{T}^2, x_3 \leqslant 0$. Lemma 0.2 implies in particular that if $\eta \in H^{q-1/2}(\Gamma)$, then $\theta \in H^q(\Omega)$ for $q \geqslant 0$. We introduce the following coordinate transformation:

$$\Omega \ni x = (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + \theta(t, x)) =: \Theta(t, x) = (y_1, y_2, y_3) \in \Omega(t). \tag{0.91}$$

If the function η is sufficiently small (in an appropriate Sobolev norm), then the mapping Θ is a diffeomorphism.

From the definition of Θ (0.91), we first compute

$$\nabla\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_1\theta & \partial_2\theta & 1 + \partial_3\theta \end{pmatrix}. \tag{0.92}$$

Following [3] again, we denote

$$A = \partial_1\theta, \quad B = \partial_2\theta, \quad J = 1 + \partial_3\theta, \quad K = J^{-1} \tag{0.93}$$

and

$$\mathcal{A} := ((\nabla\Theta)^{-1})^T = \begin{pmatrix} 1 & 0 & -AK \\ 0 & 1 & -BK \\ 0 & 0 & K \end{pmatrix}. \tag{0.94}$$

We write the differential operators $\nabla_{\mathcal{A}}, \mathrm{div}_{\mathcal{A}}, \Delta_{\mathcal{A}}$ with their actions given by

$$(\nabla_{\mathcal{A}} f)_i := \sum_{j=1}^{3} \mathcal{A}_{ij}\partial_j f, \quad \mathrm{div}_{\mathcal{A}} X := \sum_{1\leqslant i,j\leqslant 3} \mathcal{A}_{ij}\partial_j X_i, \quad \Delta_{\mathcal{A}} f = \mathrm{div}_{\mathcal{A}}\nabla_{\mathcal{A}} f.$$

We write

$$\mathcal{N} := (-\partial_1\eta, -\partial_2\eta, 1)^T$$

for the non-unit normal vector to $\Gamma(t)$, and we also write the stress tensor $\mathbb{S}_{\mathcal{A}}(u)$ as

$$(\mathbb{S}_{\mathcal{A}}u)_{ij} = \mathcal{A}_{ik}\partial_k u_j + \mathcal{A}_{jk}\partial_k u_i.$$

We now define the density ρ, the velocity u and the pressure p on the domain Ω by the composition

$$(\rho, u, p)(t, x) = (\tilde{\rho}, \tilde{u}, \tilde{p})(t, \Theta(t, x)).$$

We transform (0.84) into the following system in the new coordinates

$$\begin{cases} \partial_t\rho - K\partial_t\theta\partial_3\rho + \mathrm{div}_{\mathcal{A}}(\rho u) = 0 & \text{in } \Omega, \\ \rho(\partial_t u - K\partial_t\theta\partial_3 u + u\cdot\nabla_{\mathcal{A}}u) + \nabla_{\mathcal{A}}p - \mu\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}u = -g\rho e_3 & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}}u = 0 & \text{in } \Omega, \\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Gamma, \\ (p\mathrm{Id} - \mu\mathbb{S}_{\mathcal{A}}u)\mathcal{N} = p_{atm}\mathcal{N} & \text{on } \Gamma. \end{cases} \tag{0.95}$$

Let

$$C_0^\infty(\mathbf{R}_-) \ni \rho_0' \geqslant 0 \quad \text{such that } \mathrm{supp}\rho_0' = [-a, 0] \text{ with } a > 0. \tag{0.96}$$

We denote by

$$0 < \rho_- = \rho_0(x_3) \text{ for all } x_3 \leqslant -a, \quad \rho_0(0) = \rho_+. \tag{0.97}$$

This means that a layer of finite depth models the heavier fluid before the perturbation. We will now rewrite (0.95) around the steady-state solution

$$(\rho(t,x), u(t,x), p(t,x), \eta(t,x_h)) = (\rho_0(x_3), 0, P_0(x_3), 0),$$

recalling $P_0' = -g\rho_0$ and adding the condition $P_0(0) = p_{atm}$. We define a particular density and pressure perturbation by

$$\zeta = \rho - \rho_0 - \rho_0'\theta, \quad q = p - P_0 + g\rho_0\theta. \tag{0.98}$$

We still call the perturbations of the velocity and of the characterization of surface as (u, η) respectively. The equations for the perturbation $U = (\zeta, u, q, \eta)$ write

$$\begin{cases} \partial_t\zeta + \rho_0' u_3 = \mathcal{Q}^1(U) & \text{in } \Omega, \\ \rho_0\partial_t u + \nabla q - \mu\Delta u + g\zeta e_3 = \mathcal{Q}^2(U) & \text{in } \Omega, \\ \mathrm{div}u = \mathcal{Q}^3(U) & \text{in } \Omega, \\ \partial_t\eta - u_3 = \mathcal{Q}^4(U) & \text{on } \Gamma, \\ ((q - g\rho_+\eta)\mathrm{Id} - \mu\mathbb{S}u)e_3 = \mathcal{Q}^5(U) & \text{on } \Gamma. \end{cases} \tag{0.99}$$

The nonlinear terms $\mathcal{Q}^i(U)$ $(1 \leqslant i \leqslant 5)$ (for short $\mathcal{Q}^i$) are given by

$$
\begin{aligned}
\mathcal{Q}^1 &= \rho_0' u_3 - \rho_0' \partial_t \theta + K \partial_t \theta (\partial_3 \zeta + \rho_0' + \rho_0'' \theta + \rho_0' \partial_3 \theta), \\
&\qquad - \mathrm{div}_{\mathcal{A}}((\rho_0 + \rho_0' \theta + \zeta) u) \\
\mathcal{Q}^2 &= -(\zeta + \rho_0' \theta) \partial_t u - (\zeta + \rho_0 + \rho_0' \theta) K \partial_t \theta \partial_3 u - (\nabla_{\mathcal{A}} p - \nabla q - g \zeta e_3) \\
&\quad - (\zeta + \rho_0 + \rho_0' \theta) u \cdot \nabla_{\mathcal{A}} u - \mu (\Delta u - \mathrm{div}_{\mathcal{A}}(\mathbb{S}_{\mathcal{A}} u)), \\
\mathcal{Q}^3 &= \mathrm{div} u - \mathrm{div}_{\mathcal{A}} u, \\
\mathcal{Q}^4 &= -u_1 \partial_1 \eta - u_2 \partial_2 \eta, \\
\mathcal{Q}^5 &= (q - g\rho_+ \eta) \mathrm{Id} \cdot (e_3 - \mathcal{N}) - \mu \mathbb{S} u e_3 + \mu (\mathbb{S}_{\mathcal{A}} u) \mathcal{N}.
\end{aligned} \tag{0.100}
$$

The linearized equations are

$$
\begin{cases}
\partial_t \zeta + \rho_0' u_3 = 0 & \text{in } \Omega, \\
\rho_0 \partial_t u + \nabla q - \mu \Delta u + g \zeta e_3 = 0, & \text{in } \Omega, \\
\mathrm{div} u = 0 & \text{in } \Omega, \\
\partial_t \eta = u_3 & \text{on } \Gamma, \\
((q - g\rho_+ \eta) \mathrm{Id} - \mu \mathbb{S} u) e_3 = 0 & \text{on } \Gamma.
\end{cases} \tag{0.101}
$$

Following again [7, Chapter XI], we look for normal modes $U(t,x) = e^{\lambda t} V(x)$ of Eq. (0.101), which are

$$
(\zeta, u, q)(t, x) = e^{\lambda t} (\omega, v, r)(x), \quad \eta(t, x_h) = e^{\lambda t} \nu(x_h). \tag{0.102}
$$

The system on (w, v, r, ν) is

$$
\begin{cases}
\lambda \omega + \rho_0' v_3 = 0 & \text{in } \Omega, \\
\lambda \rho_0 v + \nabla r - \mu \Delta v + g \omega e_3 = 0 & \text{in } \Omega, \\
\mathrm{div} v = 0 & \text{in } \Omega, \\
\lambda \nu = v_3 & \text{on } \Gamma, \\
((r - g\rho_+ \nu) \mathrm{Id} - \mu (\nabla v + \nabla v^T)) e_3 = 0 & \text{on } \Gamma.
\end{cases} \tag{0.103}
$$

That implies

$$
\omega = -\frac{1}{\lambda} \rho_0' v_3, \quad \nu = \frac{1}{\lambda} v_3|_\Gamma \tag{0.104}
$$

and

$$
\begin{cases}
\lambda^2 \rho_0 v + \lambda \nabla r - \lambda \mu \Delta v - g \rho_0' v_3 e_3 = 0 & \text{in } \Omega, \\
\mathrm{div} v = 0 & \text{in } \Omega, \\
((\lambda r - g\rho_+ v_3) \mathrm{Id} - \lambda \mu (\nabla v + \nabla v^T)) e_3 = 0 & \text{on } \Gamma.
\end{cases} \tag{0.105}
$$

Let $\mathbf{k} = (k_1, k_2) \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}\backslash\{0\}$, we assume further that

$$\begin{cases} v_1(x) = \sin(k_1x_1 + k_2x_2)\psi(\mathbf{k}, x_3), \\ v_2(x) = \sin(k_1x_1 + k_2x_2)\varphi(\mathbf{k}, x_3), \\ v_3(x) = \cos(k_1x_1 + k_2x_2)\phi(\mathbf{k}, x_3), \\ r(x) = \cos(k_1x_1 + k_2x_2)\pi(\mathbf{k}, x_3). \end{cases} \tag{0.106}$$

Substituting (0.106) into (0.105), we deduce that

$$\begin{cases} \lambda^2\rho_0\psi - \lambda k_1\pi + \lambda\mu(k^2\psi - \psi'') = 0 & \text{in } \mathbf{R}_-, \\ \lambda^2\rho_0\varphi - \lambda k_2\pi + \lambda\mu(k^2\varphi - \varphi'') = 0 & \text{in } \mathbf{R}_-, \\ \lambda^2\rho_0\phi + \lambda\pi' + \lambda\mu(k^2\phi - \phi'') = g\rho_0'\phi & \text{in } \mathbf{R}_-, \\ k_1\psi + k_2\varphi + \phi' = 0 & \text{in } \mathbf{R}_-, \end{cases} \tag{0.107}$$

At $x_3 = 0$, we have the boundary conditions

$$\begin{cases} \mu(k_1\phi(0) - \psi'(0)) = 0, \\ \mu(k_2\phi(0) - \varphi'(0)) = 0, \\ \lambda\pi(0) - g\rho_+\phi(0) - 2\lambda\mu\phi'(0) = 0. \end{cases} \tag{0.108}$$

We also have the decaying condition at $-\infty$,

$$\lim_{x_3\to-\infty}(\psi, \varphi, \phi, \pi)(x_3) = 0. \tag{0.109}$$

Note that, due to $(0.107)_{1,2,4}$

$$\pi = -\frac{1}{k^2}(\lambda\rho_0\phi' + \mu(k^2\phi' - \phi''')) \quad \text{in } \mathbf{R}_-. \tag{0.110}$$

Hence, from (0.110) and $(0.107)_3$, we get a fourth-order ODE for ϕ, (0.27), i.e.

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi.$$

The boundary conditions at $x_3 = 0$ deduced from $(0.107)_4$, (0.108) and (0.110) are

$$\mu(k^2\phi(0) + \phi''(0)) = 0, \quad -\lambda\mu\phi'''(0) + (3\lambda\mu k^2 + \lambda^2\rho_+)\phi'(0) + gk^2\rho_+\phi(0) = 0 \tag{0.111}$$

and from (0.109), the function ϕ decays at $-\infty$, i.e.

$$\lim_{x_3\to-\infty}\phi(x_3) = 0. \tag{0.112}$$

Finding normal modes of the form (0.102) to Eq. (0.101) relies on the investigation of the characteristic value $\lambda \in \mathbb{C}$ ($\mathrm{Re}\lambda > 0$) such that Eq. (0.27)-(0.111)-(0.112) has a nontrivial solution ϕ living at least in $H^4(\mathbf{R}_-)$.

We again show that all characteristic values λ are real. Consequently, we look for positive characteristic values and obtain the uniform upper bound $\sqrt{\frac{g}{L_0}}$ of λ. We still consider functions ϕ being real.

We state our first theorem solving the ODE (0.27)–(0.111)–(0.112) .

Theorem 0.5. *Let ρ_0 satisfy (0.96)–(0.97), there exist an infinite sequence $(\lambda_n, \phi_n)_{n\geqslant 1}$ with $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ and nontrivial $\phi_n \in H^\infty(\mathbf{R}_-)$ satisfying Eq. (0.27)-(0.111)-(0.112).*

Proving Theorem 0.5 is in the same spirit as the proof of Theorem 0.1. We first look for a solution $\phi \in H^4(\mathbf{R}_-)$.

Indeed, on $(-\infty, -a)$, the ODE (0.27) is an ODE with constant coefficients, for which we can find explicit solutions in Proposition 3.1 which decay to 0 at $-\infty$. We thus transform the problem for the normal modes on $\mathbf{R}_-$ into an ODE problem stated on the compact interval $(-a, 0)$ with appropriate boundary conditions deduced from the outer solutions. The boundary conditions at $x_3 = -a$ are

$$\begin{cases} k\tau_-\phi(-a) - (k+\tau_-)\phi'(-a) + \phi''(-a) = 0, \\ k\tau_+(k+\tau_-)\phi(-a) - (k^2 + k\tau_- + \tau_-^2)\phi'(-a) + \phi'''(-a) = 0, \end{cases} \tag{0.113}$$

with $\tau_- = \sqrt{k^2 + \lambda\rho_-/\mu}$, and the boundary conditions at $x_3 = 0$ are (0.111).

In order to solve the fourth-order ODE (0.27) with the boundary conditions (0.113) and (0.111), the crucial step is to construct a continuous and coercive bilinear form $\mathcal{B}_{a,k,\lambda}$ on $H^2((-a,0))$, such that the finding of a solution $\phi \in H^4((-a,0))$ of Eq. (0.27)-(0.111)-(0.113) is equivalent to finding a weak solution $\phi \in H^2((-a,0))$ to the variational problem

$$\lambda\mathcal{B}_{a,k,\lambda}(\phi,\omega) = gk^2\int_{-a}^0 \rho_0'\phi\omega dx_3 \quad \text{for all } \omega \in H^2((-a,0)), \tag{0.114}$$

and thus improving the regularity of that weak solution ϕ.

The expression of $\mathcal{B}_{a,k,\lambda}$ is given as follows. Let us denote by

$$\begin{aligned} BV_{-a,k,\lambda}(\vartheta,\varrho) &:= \mu\begin{pmatrix} k\tau_-(k+\tau_-)\vartheta(-a)\varrho(-a) - k\tau_-\vartheta'(-a)\varrho(-a) \\ \quad - k\tau_-\vartheta(-a)\varrho'(-a) + (k+\tau_-)\vartheta'(-a)\varrho'(-a)\end{pmatrix}, \\ BV_{0,k,\lambda}(\vartheta,\varrho) &:= \mu k^2(\vartheta'(0)\varrho(0) + \vartheta(0)\varrho'(0)) + \frac{gk^2\rho_+}{\lambda}\vartheta(0)\varrho(0), \end{aligned} \tag{0.115}$$

The bilinear form $\mathcal{B}_{a,k,\lambda}$ is

$$\begin{aligned} \mathcal{B}_{a,k,\lambda}(\vartheta,\varrho) := BV_{0,k,\lambda}(\vartheta,\varrho) + BV_{-a,k,\lambda}(\vartheta,\varrho) + \lambda\int_{-a}^0 \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 \\ + \mu\int_{-a}^0(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3. \end{aligned} \tag{0.116}$$

As one proves that $\mathcal{B}_{a,k,\lambda}$ is coercive for all values of the parameters $\lambda > 0, \mu > 0$ and $k > 0$, we have that $\sqrt{\mathcal{B}_{a,k,\lambda}(\cdot,\cdot)}$ is a norm on $H^2((-a,0))$. Let $(H^2((-a,0)))'$ be the dual space of the functional space $H^2((-a,0))$, associated with the norm $\sqrt{\mathcal{B}_{a,k,\lambda}(\cdot,\cdot)}$. In view of Riesz's representation theorem, we obtain an abstract operator $Y_{a,k,\lambda}$ from $H^2((-a,0))$ to $(H^2((-a,0)))'$, such that

$$\mathcal{B}_{a,k,\lambda}(\vartheta,\varrho) = \langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \quad \text{for all } \vartheta,\varrho \in H^2((-a,0)). \tag{0.117}$$

Hence, from (0.114) and (0.117), we have that the existence of a solution $\phi \in H^4((-a,0))$ of Eq. (0.27)-(0.111)-(0.113) is thus reduced to the finding of a weak solution $\phi \in H^2((-a,0))$ of

$$\lambda Y_{a,k,\lambda}\phi = gk^2\rho_0'\phi \quad \text{in } (H^2((-a,0)))'. \tag{0.118}$$

Restricting $\varrho \in C_0^\infty((-a,0))$ in (0.117), we find the precise expression of $Y_{a,k,\lambda}$ (see Proposition 3.3(1)), i.e. for all $\vartheta \in H^2((-a,0))$,

$$Y_{a,k,\lambda}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta) \quad \text{in } \mathcal{D}'((-a,0)). \tag{0.119}$$

Furthermore, a classical bootstrap argument (see Proposition 3.3(2)) shows that we are able to define the inverse operator $Y_{a,k,\lambda}^{-1}$ of $Y_{a,k,\lambda}$, from $L^2((-a,0))$ to a subspace of $H^4((-a,0))$ requiring all elements satisfy (0.113)-(0.111). Note that, because ϕ belongs to $H^4((-a,0))$, these boundary conditions (involving the derivatives ϕ'', ϕ''' of ϕ at $x_3 = -a$ and at $x_3 = 0$) are well defined. Composing the above operator $Y_{a,k,\lambda}^{-1}$ with the continuous injection from $H^4((-a,0))$ to $L^2((-a,0))$ (see Proposition 3.4), we obtain that $Y_{a,k,\lambda}^{-1}$ is a compact and self-adjoint operator from $L^2((-a,0))$ to itself.

We introduce $\mathcal{M}$ the operator of multiplication by $\sqrt{\rho_0'}$ in $L^2((-a,0))$. Note from (0.118) that, we thus find v satisfying

$$\frac{\lambda}{gk^2}v = \mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}v.$$

We show that the operator $\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ is compact and self-adjoint from $L^2((-a,0))$ to itself (see Proposition 3.5), which enables to use the spectral theory of self-adjoint and compact operators to obtain that

the discrete spectrum of the operator $\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ is thus an infinite sequence of eigenvalues (denoted by $\{\gamma_n(\lambda,k)\}_{n\geqslant 1}$).

Let $v_{n,k,\lambda}$ be an eigenfunction of $\mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}$ associated with the eigenvalue $\gamma_n(\lambda,k)$ and let $\phi_{n,k,\lambda} = Y_{a,k,\lambda}^{-1}\mathcal{M}v_{n,k,\lambda} \in H^4((-a,0))$, we obtain

$$\gamma_n(\lambda,k)Y_{a,k,\lambda}\phi_{n,k,\lambda} = \mathcal{M}^2\phi_{n,k,\lambda_n} = \rho_0'\phi_{n,k,\lambda_n}. \tag{0.120}$$

From (0.118) and (0.120), we see that the problem of finding characteristic values of (0.27) amounts to solving all the equations

$$\gamma_n(\lambda,k) = \frac{\lambda}{gk^2}. \tag{0.121}$$

In Proposition 3.6, for each n, we will show the existence and uniqueness of a solution λ_n to (0.121) owing first to the differentiability in λ of $\gamma_n(\lambda,k)$ (see Lemma 3.5), which is an extension of Kato's perturbation theory of the spectrum of operators (see [45]), and to the fact that $\lambda \to \gamma_n(\lambda, k)$ is decreasing in λ (see Lemma 3.6), through the derivative $\frac{d}{d\lambda}(\frac{1}{\gamma_n(\lambda,k)})$ which exists also thanks to a similar argument (see [45]). Furthermore, we have that $\{\lambda_n\}_{n\geqslant 1}$ is a decreasing sequence towards 0.

For each λ_n, we have that $\phi_{n,k,\lambda_n} = Y^{-1}_{a,k,\lambda_n}\mathcal{M}v_{n,k,\lambda_n} \in H^4((-a,0))$ satisfies Eq. (0.27) with the boundary conditions (0.113)-(0.111) thanks to Proposition 3.3(2) again. Hence, ϕ_{n,k,λ_n} is glued with the decaying solutions of (0.27) in the outer region $(-\infty, -a)$ by the boundary conditions at $x_3 = -a$ to become a solution of (0.27)-(0.111)-(0.112) in $H^4(\mathbf{R}_-)$ associated with $\lambda = \lambda_n$. Theorem 0.5 is proven.

Once Eq. (0.27)-(0.111)-(0.112) is solved, we go back to the linearized equations (0.101). For a fixed $\mathbf{k} \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}\backslash\{0\}$, we obtain a sequence of real solutions to the linearized equations (0.101) (see Proposition 3.7), which are

$$e^{\lambda_j(\mathbf{k})t}V_j(\mathbf{k},x) = (e^{\lambda_j(\mathbf{k})t}(\zeta_j(\mathbf{k},x), u_j(\mathbf{k},x), q_j(\mathbf{k},x), \eta_j(\mathbf{k},x_h))^T.$$

Since all the characteristic values are bounded by $\sqrt{\frac{g}{L_0}}$, we set

$$0 < \Lambda := \sup_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}\backslash\{0\}} \lambda_1(\mathbf{k}) \leqslant \sqrt{\frac{g}{L_0}}, \tag{0.122}$$

and we show that Λ is the maximal growth rate of the linearized equations, see Proposition 3.9.

We move to show the nonlinear instability.

The local well-posedness of (0.99) in our functional framework can be established similarly as in [33, Theorem 6.2] for the incompressible viscous surface wave problem, that is used in [71] for the incompressible viscous surface-internal wave problem and [70] for the incompressible viscous fluid with magnetic field. Thus we refer to [33, 71, 70] for the construction of local solutions to (0.99) with some specific compatibility conditions. We restate it below and then derive the *a priori* energy estimate to the nonlinear equations (0.99) in Proposition 0.3 (see (0.128)).

Let us define the full energy functional $\mathcal{E}_f(U(t)) > 0$ such that

$$\begin{aligned}\mathcal{E}_f^2(U(t)) := \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \sum_{l=0}^{2}\|\partial_t^l\eta\|^2_{H^{4-2l}(\Gamma)} + \sum_{l=0}^{2}\|\partial_t^l(\zeta,u)(t)\|^2_{H^{4-2l}(\Omega)} \\ + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)}\end{aligned} \tag{0.123}$$

and its corresponding dissipation $\mathcal{D}_f((u,q)(t)) > 0$,

$$\mathcal{D}_f^2((u,q)(t)) := \sum_{l=0}^{2}\|\partial_t^l u(t)\|^2_{H^{5-2l}(\Omega)} + \|\partial_t q\|^2_{H^2(\Omega)} + \|q\|^2_{H^4(\Omega)}. \tag{0.124}$$

For notational convenience, we only write $\mathcal{E}_f(t)$ and $\mathcal{D}_f(t)$ in this section.

Let us recall the definition of K from (0.93) and $\mathcal{A}$ from (0.94) and define $\mathbb{M} = K\mathcal{A}$, $R = \partial_t\mathbb{M}\mathbb{M}^{-1}$ and $D_t u = \partial_t u - Ru$. We also define an orthogonal projection onto the tangent space of the surface $\{x_3 = \eta_0(x_1, x_2)\}$ according to

$$\Pi_0 v = v - \frac{v \cdot \mathcal{N}_0}{|\mathcal{N}_0|^2}\mathcal{N}_0 \quad \text{for } \mathcal{N}_0 = (-\partial_1\eta_0, \partial_2\eta_0, 1)^T. \tag{0.125}$$

Let us write

$$\begin{aligned} G^{2,0} &= g\rho_+\eta\mathcal{N} \text{ on } \Gamma, \\ G^{2,1} &= D_t G^{2,0} + \mu\mathbb{S}_{\mathcal{A}}(Ru)\mathcal{N} + (\mu\mathbb{S}_{\mathcal{A}}u - q\mathrm{Id})\partial_t\mathcal{N} + \mu\mathbb{S}_{\partial_t\mathcal{A}}u\mathcal{N} \text{ on } \Gamma. \end{aligned}$$

Proposition 0.2. *Suppose that there is a sufficiently small constant $\nu_1 \in (0,1)$ such that $(\zeta_0, u_0, q_0, \eta_0)$ satisfying*

$$\|\zeta_0\|^2_{H^4(\Omega)} + \|u_0\|^2_{H^4(\Omega)} + \|q_0\|^2_{H^3(\Omega)} + \|\eta_0\|^2_{H^{9/2}(\Gamma)} \leqslant \nu_1.$$

Suppose also that the following compatibility conditions hold for $j = 0$ and 1,

$$\begin{cases} div_{\mathcal{A}_0} D_t^j u_0 = 0 & \text{in } \Omega, \\ \Pi_0(G^{2,j}(0) + \mu\mathbb{S}_{\mathcal{A}_0} D_t^j u_0 \mathcal{N}_0) = 0 & \text{on } \Gamma. \end{cases} \tag{0.126}$$

Then, there exist $\nu_2 > 0$ and $T_{\max} > 0$ such that if $\mathcal{E}_f(0) \leqslant \nu_2$, Eq. (0.99) with the initial data $(\zeta_0, u_0, q_0, \eta_0)$ satisfying the compatibility conditions (0.126) has a unique solution (ζ, u, q, η) on the time interval $[0, T_{\max})$. Moreover, we have

$$\mathcal{E}_f(t) \lesssim (1 + T_{\max})\mathcal{E}_f(0),$$

and η is such that the mapping $\Theta(\cdot, t)$ defined by (0.91) is a C^2-diffeomorphism for each $t \in [0, T_{\max})$.

With that regular solution (ζ, u, q, η) of (0.99) on a finite time interval $[0, T_{\max})$, we aim at showing *a priori* energy estimates for the nonlinear equations (0.99).

Proposition 0.3. *Let C_{emb} be the optimal constant of the Sobolev embedding*

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega)$$

and let $\delta_0 > 0$ be sufficiently small such that

$$0 < \delta_0 \leqslant \frac{\rho_-}{2C_{emb}\max(1, \max_{\mathbf{R}_-} \rho_0'(x_3))}, \tag{0.127}$$

and (3.279) holds later. Hence, there exists $\varepsilon > 0$ sufficiently small such that for all $\delta \in (0, \delta_0)$ if $\sup_{0 \leqslant s \leqslant t} \mathcal{E}_f(s) \leqslant \delta$, we have

$$\begin{aligned} &\mathcal{E}_f^2(t) + \int_0^t \mathcal{D}_f^2(s)ds \\ &\lesssim \varepsilon^{-5}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-5}\int_0^t \mathcal{E}_f(s)(\mathcal{E}_f^2(s) + \mathcal{D}_f^2(s))ds + \varepsilon^{-5}\mathcal{E}_f^3(t) \\ &\quad + \varepsilon^{-59}\int_0^t (\|(\zeta, u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds. \end{aligned} \tag{0.128}$$

Thanks to (0.122), we define the non-empty set

$$\tilde{B}_\Lambda := \Big\{\mathbf{k} \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}\backslash\{0\} : \lambda_1(\mathbf{k}) > \frac{2\Lambda}{3}\Big\}.$$

We further fix a $\mathbf{k}_0 \in \tilde{B}_\Lambda$. There is a unique $N = N(\mathbf{k}_0) \in \mathbb{N}^\star$ such that

$$\Lambda \geqslant \lambda_1(\mathbf{k}_0) > \lambda_2(\mathbf{k}_0) > \cdots > \lambda_N(\mathbf{k}_0) > \frac{2\Lambda}{3} > \lambda_{N+1}(\mathbf{k}_0) > \dots . \tag{0.129}$$

Let $M \in \mathbb{N}^\star$ be arbitrary. In view of getting infinitely many characteristic values of the linearized problem, we consider a linear combination of normal modes

$$U^M(t,x) = \sum_{j=1}^{M} \mathsf{c}_j e^{\lambda_j(\mathbf{k}_0)t} V_j(\mathbf{k}_0, x) \tag{0.130}$$

to be an approximate solution to the nonlinear problem (0.99), with constants c_j being chosen such that

$$\text{at least one of } \mathsf{c}_j \ (1 \leqslant j \leqslant N) \text{ is non-zero} \tag{0.131}$$

and let $j_m := \min\{j : 1 \leqslant j \leqslant N, \mathsf{c}_j \neq 0\}$,

$$\frac{1}{2}|\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)} > \sum_{j \geqslant j_m+1} |\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}. \tag{0.132}$$

In order to prove the nonlinear instability result, we would like to use $U^M(0,x)$ as the initial data for the nonlinear equations (0.99). However, the initial data for the nonlinear equations (0.99) must satisfy the compatibility conditions (0.126) stated in Proposition 0.2 to ensure the local existence. In this case, the normal modes $V_j(\mathbf{k}_0, x)$ do not enjoy (0.126). Thanks to an abstract argument from [36, Section 5C], which was used in [71, 70], we obtain the modified initial data $U_0^{\delta,M}(x)$.

Proposition 0.4. *There exist a number $\delta_0 > 0$ and a family of initial data*

$$U_0^{\delta,M}(x) = \delta U^M(0,x) + \delta^2 U_\star^{\delta,M}(x) \tag{0.133}$$

for $\delta \in (0, \delta_0)$ such that

1. *$\mathcal{E}_f(U_\star^{\delta,M}) \leqslant C_M^\star$ with $C_M^\star$ being independent of δ and $U_0^{\delta,M}$ satisfies the compatibility conditions (0.126),*
2. *the nonlinear equations (0.99) with the above initial data $U_0^{\delta,M}$ has a unique solution $U^{\delta,M}$ on $[0, T^{\max})$ satisfying that $\sup_{0\leqslant t<T^{\max}} \mathcal{E}_f(U^{\delta,M}(t)) < \infty$.*

Note that $U^d(t) = U^{\delta,M}(t) - \delta U^M(t)$ solves (0.99) with the initial data $U^d(0) = \delta^2 U_\star^{\delta,M}$ and the same nonlinear terms $\mathcal{Q}^i (1 \leqslant i \leqslant 5)$. Precisely, U^d satisfies

$$\begin{cases} \partial_t \zeta^d + \rho_0' u_3^d = \mathcal{Q}^1(U^{\delta,M}) & \text{in } \Omega, \\ \rho_0 \partial_t u^d - \mu \Delta u^d + \nabla q^d + g\zeta^d e_3 = \mathcal{Q}^2(U^{\delta,M}) & \text{in } \Omega, \\ \operatorname{div} u^d = \mathcal{Q}^3(U^{\delta,M}) & \text{in } \Omega, \\ \partial_t \eta^d = u_3^d + \mathcal{Q}^4(U^{\delta,M}) & \text{on } \Gamma, \\ ((q^d - g\rho_+\eta^d)\mathrm{Id} - \mu \mathbb{S}u^d)e_3 = \mathcal{Q}^5(U^{\delta,M}) & \text{on } \Gamma. \end{cases} \tag{0.134}$$

along with the initial condition

$$U^d(0) = (\zeta^d, u^d, \eta^d, q^d)(0) = \delta^2 U_\star^{\delta,M}. \tag{0.135}$$

For t small enough, we deduce the following bound in time (see Proposition 3.18)

$$\begin{aligned} &\|(\zeta^d, u^d)(t)\|_{L^2(\Omega)}^2 + \|\eta^d(t)\|_{L^2(\Gamma)}^2 \\ &\quad\lesssim \delta^3 \Big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| e^{\frac{2}{3}\Lambda t} \Big)^3, \end{aligned} \tag{0.136}$$

by exploiting some energy estimates of Eq. (0.134) and by using the bound in time of $\mathcal{E}_f(U^{\delta,M}(t))$ (see Proposition 3.19), which is obtained thanks to the *a priori* energy estimate established in Proposition 0.3. Combining those estimates, we obtain the nonlinear result.

Theorem 0.6. *Assume that ρ_0 satisfies (0.96)-(0.97). Let $M \in \mathbb{N}^\star$, there exist two constants $\epsilon_0, \delta_0 > 0$ sufficiently small and another constant $m_0 > 0$ such that for any $\delta \in (0, \delta_0)$, the nonlinear equations (0.99) with the initial data (0.133), i.e.*

$$\delta \sum_{j=1}^{M} \mathsf{c}_j V_j(x) + \delta^2 U_\star^{\delta,M}(x),$$

satisfying (0.131) and (0.132) has a unique local strong solution $U^{\delta,M}$ such that

$$\|u^{\delta,M}(T^\delta)\|_{L^2(\Omega)} \geqslant m_0 \epsilon_0, \tag{0.137}$$

where $T^\delta \in (0, T_{\max})$ is given by $\delta \sum_{j=j_m}^{M} |\mathsf{c}_j| e^{\lambda_j T^\delta} = \epsilon_0$.

Chapter 1

Spectral analysis of the viscous Rayleigh-Taylor instability

This chapter is presented in the paper [51], joint work with Prof. Olivier Lafitte. We perform the spectral analysis of the viscous Rayleigh-Taylor instability (as mentioned in Section 0.4). It consists in finding a sequence of characteristic values λ_n such that for these λ_n, the normal modes problem for the gravity-driven incompressible Navier-Stokes equations has a non trivial, bounded solution. This problem will be stated as a generalized eigenvalue problem for a self-adjoint operator. In the case where the variations of the profile considered ρ_0' are compactly supported, we get, whatever the viscosity is, a countable infinite sequence of characteristic values, which decrease towards zero. In the case ρ_0' not compactly supported, we are only able to provide a count of the values of λ_n greater than any small constant $\epsilon_\star > 0$. These two results are deduced, in the case $\mu > 0$, from a variational formulation of the characteristic problem on a compact set $[x_-, x_+]$, using adapted boundary conditions.

1.1 Preliminaries

Throughout this chapter, we write x instead of x_3 for notational convenience.

We begin with some crucial material for the spectral study. The first one is the differentiability of eigenvalues of self-adjoint and compact operators.

The classical perturbation theory from [45, Chapter VII, \$3] has shown the continuous property of the eigenvalues for a family of holomorphic self-adjoint operators in an infinite-dimensional Hilbert space. If the operators are only differentiable, we will present a proof of the differentiability of the eigenvalues for compact and self-adjoint operators in an infinite-dimensional Hilbert space deduced from that one for matrix functions in a finite-dimensional space (see [45, Chapter II, \$5]).

Theorem 1.1. *Let I be a closed interval and H be an infinite-dimensional Hilbert* *space and $(A(\lambda))_{\lambda\in I}$ be a family of self-adjoint and compact operators in H depending continuously differentiable on λ. Then, all eigenvalues and all eigenvectors of $A(\lambda)$ are differentiable functions on λ.*

Proof. Let $\lambda_0 \in I$ be fixed. Since $A(\lambda_0)$ is a self-adjoint and compact operator in H, the spectrum of $A(\lambda_0)$ is discrete. Let γ_0 be an arbitrary eigenvalue of $A(\lambda_0)$ and $E = \mathrm{Ker}(A(\lambda_0) - \gamma_0 \mathrm{Id}_H)$, we have the decomposition $H = E \oplus E^\perp$. Consequently, for all $\lambda \in I$,

$$A(\lambda) = \begin{pmatrix} \mathrm{Proj}_E(A(\lambda)\mathrm{Proj}_E) & \mathrm{Proj}_E(A(\lambda)\mathrm{Proj}_{E^\perp}) \\ \mathrm{Proj}_{E^\perp}(A(\lambda)\mathrm{Proj}_E) & \mathrm{Proj}_{E^\perp}(A(\lambda)\mathrm{Proj}_{E^\perp}) \end{pmatrix}$$

that we will denote by $(A_{ij}(\lambda))_{1\leqslant i,j\leqslant 2}$ for brevity. Notice that

$$A(\lambda_0) = \begin{pmatrix} \gamma_0 \mathrm{Id}_E & 0 \\ 0 & A_{22}(\lambda_0) \end{pmatrix},$$

and $A_{22}(\lambda_0) - \gamma_0 \mathrm{Id}_{E^\perp}$ is invertible.

Let $0 < \varepsilon \ll 1$ and γ be an eigenvalue of $A(\lambda)$ being close to γ_0, i.e. $|\gamma - \gamma_0| < \varepsilon$. We write that

$$A(\lambda) - \gamma \mathrm{Id} = \begin{pmatrix} A_{11}(\lambda) - \gamma \mathrm{Id}_E & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) - \gamma \mathrm{Id}_{E^\perp} \end{pmatrix}. \tag{1.1}$$

If $x = (y, z)^T$ is a corresponding eigenvector, we obtain

$$\begin{cases} A_{11}(\lambda)y + A_{12}(\lambda)z = \gamma y, \\ A_{21}(\lambda)y + A_{22}(\lambda)z = \gamma z. \end{cases}$$

Consequently,

$$\begin{cases} (A_{11}(\lambda) - \gamma \mathrm{Id}_E)y + A_{12}(\lambda)z = 0, \\ A_{21}(\lambda)y + (A_{22}(\lambda) - \gamma \mathrm{Id}_{E^\perp})z = 0. \end{cases} \tag{1.2}$$

Since $A_{22}(\lambda_0) - \gamma_0 \mathrm{Id}_{E^\perp}$ is invertible, we have that $A_{22}(\lambda) - \gamma_0 \mathrm{Id}_{E^\perp}$ is invertible for $|\lambda - \lambda_0| < \delta \ll 1$. We further get that $A_{22}(\lambda) - \gamma \mathrm{Id}_{E^\perp}$ is also invertible for $|\lambda - \lambda_0| < \delta$ and $|\gamma - \gamma_0| < \varepsilon$. Hence, we deduce that (1.2) is equivalent to

$$\begin{cases} z = -(A_{22}(\lambda) - \gamma \mathrm{Id}_{E^\perp})^{-1} A_{21}(\lambda) y, \\ (A_{11}(\lambda) - A_{12}(\lambda)(A_{22}(\lambda) - \gamma \mathrm{Id}_{E^\perp})^{-1} A_{21}(\lambda))y = \gamma y. \end{cases} \tag{1.3}$$

If $y = 0$, Eq. $(1.3)_1$ implies $z = 0$, which is impossible. We have $y \neq 0$, this means that if $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$ is an eigenvalue of $A(\lambda)$, Eq. $(1.3)_2$ tells us that γ is also an eigenvalue of $B(\lambda, \gamma)$ defined by

$$B(\lambda, \gamma) := A_{11}(\lambda) - A_{12}(\lambda)(A_{22}(\lambda) - \gamma \mathrm{Id}_{E^\perp})^{-1} A_{21}(\lambda) : E \to E$$

Notice that E is finite-dimensional thanks to Riesz's theorem. $B(\lambda,\gamma)$ turns out to be a matrix, having eigenvalues $\gamma_j(\lambda,\gamma)(1 \leqslant j \leqslant \dim E)$. Then, there exists j such that $\gamma_j(\lambda,\gamma) = \gamma$. It follows from [45, Chapter II, \$5] that $\gamma_j(\lambda,\gamma)$ and its associated eigenvector are differentiable at λ_0, so is γ. □

We then state here the key ingredient for our analysis in Section 1.3 due to E. A. Coddington and N. Levinson [10, Theorem 8.1, Chapter 3].

Theorem 1.2. *We consider a linear system*

$$W'(y) = (A + L(y) + R(y))W(y). \tag{1.4}$$

Let A be a constant matrix with characteristic roots $\mu_j, j = 1, \ldots, n$, all of which are distinct. Let the matrix L be differentiable and satisfy

$$\int_0^\infty \|L'(y)\|dy < \infty \tag{1.5}$$

and let $L(y) \to 0$ as $y \to \infty$. Let the matrix R be integrable and let

$$\int_0^\infty \|R(y)\|dy < \infty. \tag{1.6}$$

Let the roots of $det(A + L(y) - \lambda I_n) = 0$ be denoted by $\lambda_j(y), j = 1, \ldots, n$. Clearly, by reordering the μ_j if necessary, $\lim_{y\to\infty} \lambda_j(y) = \mu_j$. For a given h, let

$$d_{hj}(y) = Re(\lambda_h(y) - \lambda_j(y)).$$

Suppose all $\lambda_j(1 \leqslant j \leqslant n)$ fall into one of two classes H_1 and H_2, where

$$\lambda_j \in H_1 \quad if \int_0^y d_{hj}(s)ds \to \infty \text{ as } y \to \infty \text{ and } \int_{y_1}^{y_2} d_{hj}(s)ds \geqslant -K \quad (y_2 \geqslant y_1 \geqslant 0),$$

and

$$\lambda_j \in H_2 \quad if \int_{y_1}^{y_2} d_{hj}(s)ds \leqslant K \quad (y_2 \geqslant y_1 \geqslant 0),$$

where h is fixed and K is a constant. Let p_h be the eigenvector corresponding to μ_h, i.e. $Ap_h = \mu_h p_h$. Hence, there is a solution ϕ_h of (1.4) and a $y_0 \in (0,\infty)$ such that

$$\lim_{y\to\infty} \phi_h(y) exp\Big[-\int_{y_0}^y \lambda_h(s)ds\Big] = p_h.$$

We end this section by the proof of Lemma 0.1.

Proof of Lemma 0.1. Multiplying by $\overline{\phi}$ on both sides of (0.27) and then integrating by parts, we obtain that

$$\begin{aligned} -\lambda^2 \int_{\mathbf{R}} \Big(k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2\Big)dx = \lambda\mu \int_{\mathbf{R}} \Big(|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2\Big)dx \\ - gk^2 \int_{\mathbf{R}} \rho_0'\phi^2 dx. \end{aligned} \tag{1.7}$$

Suppose that $\lambda = \lambda_1 + i\lambda_2$, then one deduces from (1.7) that

$$\begin{aligned} -(\lambda_1^2 - \lambda_2^2)\int_{\mathbf{R}}\Big(k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2\Big)dx = \lambda_1\mu\int_{\mathbf{R}}\Big(|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2\Big)dx \\ - gk^2\int_{\mathbf{R}}\rho_0'|\phi|^2 dx \end{aligned} \tag{1.8}$$

and that

$$-2\lambda_1\lambda_2\int_{\mathbf{R}}\Big(k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2\Big)dx = \lambda_2\mu\int_{\mathbf{R}}\Big(|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2\Big)dx. \tag{1.9}$$

If $\lambda_2 \neq 0$, (1.9) leads us to

$$-2\lambda_1\int_{\mathbf{R}}\Big(k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2\Big)dx = \mu\int_{\mathbf{R}}\Big(|\phi''|^2 + 2k^2|\phi'|^2 + k^4|\phi|^2\Big)dx > 0,$$

which yields a contradiction that $\lambda_1 < 0$. Hence, we have that $\lambda_2 = 0$, i.e. λ is real. Using (1.7) again, we further get that

$$\lambda^2\int_{\mathbf{R}}\rho_0(k^2|\phi|^2 + |\phi'|^2)dx \leqslant gk^2\int_{\mathbf{R}}\rho_0'|\phi|^2 dx.$$

It tells us that λ is bounded by $\sqrt{\frac{g}{L_0}}$. This finishes the proof of Lemma 0.1. □

1.2 The compactly supported profile

In this section, we consider ρ_0 satisfying (0.31) and (0.32). We remark that in this section, we use the notations $\nu_\pm = \frac{\rho_\pm}{\mu}$ and $\tau_\pm = (k^2 + \lambda\nu_\pm)^{1/2}$.

1.2.1 The solution in outer regions and reduction to a problem on a finite interval

We derive, in this subsection, the precise expression of $\phi(x)$ as $|x| \geqslant a$.

Proposition 1.1. *There are two linearly independent solutions of (0.27) decaying to 0 at $+\infty$ as $x \in [a, +\infty)$, i.e.*

$$\phi_1^+(x) = e^{-kx} \quad \text{and} \quad \phi_2^+(x) = e^{-\tau_+ x}. \tag{1.10}$$

and two linearly independent solutions of (0.27) decaying to 0 at $-\infty$ as $x \in (-\infty, -a]$, i.e.

$$\phi_1^-(x) = e^{kx} \quad \text{and} \quad \phi_2^-(x) = e^{\tau_- x}. \tag{1.11}$$

All solutions decaying to 0 at $+\infty$ (respectively at $-\infty$) are spanned by (ϕ_1^+, ϕ_2^+) (respectively by (ϕ_1^-, ϕ_2^-)).

Proof. For $x \in [a, +\infty)$, Eq. (0.27) reduces to

$$-\lambda\nu_+(k^2\phi - \phi'') = \phi^{(4)} - 2k^2\phi'' + k^4\phi.$$

We seek ϕ as $\phi(x) = e^{rx}$. Hence,

$$-\lambda\nu_+(k^2 - r^2) = r^4 - 2k^2r^2 + k^4,$$

which yields $r = \pm k$ or $r = \pm(k^2 + \lambda\nu_+)^{1/2}$. Since ϕ tends to 0 at $+\infty$, we get two linearly independent solutions, which are (1.10),

$$\phi_1^+(x) = e^{-kx} \quad \text{and} \quad \phi_2^+(x) = e^{-\tau_+ x}.$$

Hence, all solutions ϕ decaying to 0 at $+\infty$ are of the form

$$\phi(x) = A_1^+ e^{-k(x-a)} + A_2^+ e^{-\tau_+(x-a)} \tag{1.12}$$

for all $x \in [a, +\infty)$ and for some real constants A_1^+ and A_2^+.

If $x \in (-\infty, -a]$, the same calculation implies (1.11). Then, all solutions ϕ decaying to 0 at $-\infty$ are of the form

$$\phi(x) = A_1^- e^{k(x+a)} + A_2^- e^{\tau_-(x+a)} \tag{1.13}$$

for all $x \in (-\infty, -a]$ and for some real constants A_1^- and A_2^-. □

Once it is proven that $\phi(x)$ outside $(-a, a)$ is of the form (1.12) or (1.13), we look for ϕ on $(-a, a)$. That solution has to match with (1.12) and (1.13) well, i.e. there are some conditions on $(\phi, \phi', \phi'', \phi''')$ at $x = \pm a$. We will show the conditions in the following lemma.

Lemma 1.1. *The boundary conditions of (0.27) at $x = -a$, for $\phi \in H^4(\mathbf{R})$, are (0.35)*

$$\begin{cases} k\tau_-\phi(-a) - (k + \tau_-)\phi'(-a) + \phi''(-a) = 0, \\ k\tau_-(k + \tau_-)\phi(-a) - (k^2 + k\tau_- + \tau_-^2)\phi'(-a) + \phi'''(-a) = 0. \end{cases}$$

and at $x = a$ are (0.36)

$$\begin{cases} k\tau_+\phi(a) + (k + \tau_+)\phi'(a) + \phi''(a) = 0, \\ -k\tau_+(k + \tau_+)\phi(a) - (k^2 + k\tau_+ + \tau_+^2)\phi'(a) + \phi'''(a) = 0. \end{cases}$$

Proof. The boundary conditions of a solution ϕ of (0.27) at $x = \pm a$ are equivalent to the fact that ϕ belongs to the space of decaying solutions at $\pm\infty$. On the one hand, it can be seen from (1.12) and (1.13) that

$$\begin{pmatrix} \phi(x) \\ \phi'(x) \\ \phi''(x) \\ \phi'''(x) \end{pmatrix} = A_1^- e^{k(x+a)} \begin{pmatrix} 1 \\ k \\ k^2 \\ k^3 \end{pmatrix} + A_2^- e^{\tau_-(x+a)} \begin{pmatrix} 1 \\ \tau_- \\ \tau_-^2 \\ \tau_-^3 \end{pmatrix} \quad \text{for } x \leqslant -a$$

and that

$$\begin{pmatrix} \phi(x) \\ \phi'(x) \\ \phi''(x) \\ \phi'''(x) \end{pmatrix} = A_1^+ e^{-k(x-a)} \begin{pmatrix} 1 \\ -k \\ k^2 \\ -k^3 \end{pmatrix} + A_2^+ e^{-\tau_+(x-a)} \begin{pmatrix} 1 \\ -\tau_+ \\ \tau_+^2 \\ -\tau_+^3 \end{pmatrix}, \quad \text{for } x \geqslant a.$$

On the other hand, the orthogonal complement of the subspace of $\mathbf{R}^4$ spanned by two vectors $(1, k, k^2, k^3)^T$ and $(1, \tau_-, \tau_-^2, \tau_-^3)^T$ is spanned by

$$(k\tau_-, -(k+\tau_-), 1, 0)^T \text{ and } (k\tau_-(k+\tau_-), -(k^2+k\tau_-+\tau_-^2), 0, 1)^T.$$

Similarly, the orthogonal complement of the subspace of $\mathbf{R}^4$ spanned by two vectors $(1, -k, k^2, -k^3)^T$ and $(1, -\tau_+, \tau_+^2, -\tau_+^3)$ is spanned by two vectors

$$(k\tau_+, k+\tau_+, 1, 0)^T \quad \text{and} \quad (-k\tau_+(k+\tau_+), -(k^2+k\tau_++\tau_+^2), 0, 1)^T.$$

The above arguments allow us to set (0.35) and (0.36) as boundary conditions of Eq. (0.27) on $(-a, a)$. □

We aim at solving (0.27) on $(-a, a)$ with the boundary conditions (0.35)-(0.36).

1.2.2 A bilinear form and a self-adjoint invertible operator

We study the bilinear form $\mathcal{B}_{a,\lambda}$ (0.42) in the following proposition.

Proposition 1.2. *We have that*

$$\mathcal{B}_{a,\lambda}(\vartheta, \varrho) := BV_{a,\lambda}(\vartheta, \varrho) + BV_{-a,\lambda}(\vartheta, \varrho) + \lambda \int_{-a}^{a} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx$$
$$+ \mu \int_{-a}^{a} (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx.$$

is a continuous and coercive bilinear form on $H^2((-a, a))$.

Furthermore, let $(H^2((-a, a)))'$ be the dual space of $H^2((-a, a))$ associated with the norm $\sqrt{\mathcal{B}_{a,\lambda}(\cdot, \cdot)}$, there exists a unique operator

$$Y_{a,\lambda} \in \mathcal{L}(H^2((-a, a)), (H^2((-a, a)))'),$$

that is also bijective, such that

$$\mathcal{B}_{a,\lambda}(\vartheta, \varrho) = \langle Y_{a,\lambda}\vartheta, \varrho \rangle \tag{1.14}$$

for all $\vartheta, \varrho \in H^2((-a, a))$.

Proof. Clearly, $\mathcal{B}_{a,\lambda}$ is a bilinear form on $H^2((-a,a))$ since the terms $BV_{\pm a,\lambda}(\vartheta,\varrho)$ are well defined. We then establish the boundedness of $\mathcal{B}_{a,\lambda}$. The integral terms of $\mathcal{B}_{a,\lambda}$ are $\lesssim \|\vartheta\|_{H^2((-a,a))}\|\varrho\|_{H^2((-a,a))}$. About the two first terms $BV_{\pm a,\lambda}(\vartheta,\varrho)$, it follows from the general Sobolev inequality that

$$(\vartheta(a))^2 + (\vartheta(-a))^2 \lesssim \|\vartheta\|^2_{H^1((-a,a))}.$$

Similarly,

$$(\vartheta'(a))^2 + (\vartheta'(-a))^2 \lesssim \|\vartheta'\|^2_{H^1((-a,a))}.$$

Consequently, we get

$$\begin{aligned}|BV_{\pm a,\lambda}(\vartheta,\varrho)| &\lesssim (|\vartheta(\pm a)| + \vartheta'(\pm a)|)(|\varrho(\pm a)| + |\varrho'(\pm a)|)\\ &\lesssim \|\vartheta\|_{H^2((-a,a))}\|\varrho\|_{H^2((-a,a))}.\end{aligned}$$

We find that

$$|\mathcal{B}_{a,\lambda}(\vartheta,\varrho)| \lesssim \|\vartheta\|_{H^2((-a,a))}\|\varrho\|_{H^2((-a,a))}, \tag{1.15}$$

i.e. $\mathcal{B}_{a,\lambda}$ is bounded.

We move to show the coercivity of $\mathcal{B}_{a,\lambda}$. We have that

$$\begin{aligned}\mathcal{B}_{a,\lambda}(\vartheta,\vartheta) = BV_{a,\lambda}(\vartheta,\vartheta) + BV_{-a,\lambda}(\vartheta,\vartheta) + \lambda\int_{-a}^{a}\rho_0(k^2\vartheta^2 + (\vartheta')^2)dx\\ + \mu\int_{-a}^{a}((\vartheta'')^2 + 2k^2(\vartheta')^2 + k^4\vartheta^2)dx.\end{aligned}$$

We have that $BV_{a,\lambda}(\vartheta,\vartheta) \geqslant 0$ follows from the following equality

$$\begin{aligned}\frac{1}{\mu}BV_{a,\lambda}(\vartheta,\vartheta) &= k\tau_+(k+\tau_+)(\vartheta(a))^2 - 2k\tau_+\vartheta(a)\vartheta'(a) + (k+\tau_+)(\vartheta'(a))^2\\ &= k\tau_+(k+\tau_+)\Big(\vartheta(a) - \frac{\vartheta'(a)}{k+\tau_+}\Big)^2 + \frac{k^2+k\tau_+ + \tau_+^2}{k+\tau_+}(\vartheta'(a))^2.\end{aligned}$$

We also obtain that $BV_{-a,\lambda}(\vartheta,\vartheta) \geqslant 0$. Therefore, we deduce that

$$\mathcal{B}_{a,\lambda}(\vartheta,\vartheta) \geqslant \mu\min(k^4, 2k^2, 1)\|\vartheta\|^2_{H^2((-a,a))}. \tag{1.16}$$

Two inequalities (3.22) and (3.23) tell us that $\mathcal{B}_{a,\lambda}$ is a continuous and coercive bilinear form on $H^2((-a,a))$. It follows from Reisz's representation theorem that there is a unique operator $Y_{a,\lambda} \in \mathcal{L}(H^2((-a,a)),(H^2((-a,a)))')$, that is also bijective, satisfying (1.14) for all $\vartheta,\varrho \in H^2((-a,a))$. Proof of Proposition 1.2 is complete. □

The next proposition is to devoted to studying the properties of $Y_{a,\lambda}$.

Proposition 1.3. *We have the following results.*

1. *For all* $\vartheta \in H^2((-a,a))$,

$$Y_{a,\lambda}\vartheta = \lambda(\rho_0k^2\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta)$$

in $\mathcal{D}'((-a,a))$.

2. *Let $f \in L^2((-a,a))$ be given, there exists a unique solution $\vartheta \in H^2((-a,a))$ of*

$$Y_{a,\lambda}\vartheta = f \text{ in } (H^2((-a,a)))'. \tag{1.17}$$

Moreover, we have that $\vartheta \in H^4((-a,a))$ and satisfies the boundary conditions (0.35)–(0.36).

Proof. Let $\varrho \in C_0^\infty((-a,a))$, it follows from Proposition 1.2 that, for $\vartheta \in H^2((-a,a))$

$$\lambda \int_{-a}^{a} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx + \mu \int_{-a}^{a} (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx = \langle Y_{a,\lambda}\vartheta, \varrho\rangle. \tag{1.18}$$

We respectively define $(\vartheta'')'$ and $(\vartheta'')''$ in the distributional sense as the first and second derivative of ϑ'', which is in $L^2((-a,a))$. Hence, (1.18) is equivalent to

$$\lambda \int_{-a}^{a} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx + \mu \int_{-a}^{a} (2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx + \mu\langle(\vartheta'')'', \varrho\rangle = \langle Y_{a,\lambda}\vartheta, \varrho\rangle. \tag{1.19}$$

for all $\varrho \in C_0^\infty((-a,a))$. We deduce from (1.19) that

$$\int_{-a}^{a} (\lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(-2k^2\vartheta'' + k^4\vartheta))\varrho dx + \mu\langle(\vartheta'')'', \varrho\rangle = \langle Y_{a,\lambda}\vartheta, \varrho\rangle \tag{1.20}$$

for all $\varrho \in C_0^\infty((-a,a))$. The first assertion follows.

Let $f \in L^2((-a,a))$ and $\vartheta \in H^2((-a,a))$ be the solution of (1.17), we improve the regularity of the weak solution ϑ of (1.20). Indeed, we rewrite (1.20) as

$$\mu\langle(\vartheta'')'', \varrho\rangle = \int_{-a}^{a} (f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')\varrho dx \tag{1.21}$$

for all $\varrho \in C_0^\infty((-a,a))$. Since $(f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')$ belongs to $L^2((-a,a))$, it then follows from (1.21) that $(\vartheta'')'' \in L^2((-a,a))$.

Furthermore, let $\chi \in C_0^\infty((-a,a))$ satisfy $\int_{-a}^{a}\chi(y)dy = 1$. From the distribution theory, we define $\Psi \in \mathcal{D}'((-a,a))$ such that

$$\langle\Psi, \varrho\rangle = \langle(\vartheta'')'', \zeta_\rho\rangle \tag{1.22}$$

for all $\varrho \in C_0^\infty((-a,a))$, where

$$\zeta_\rho(x) = \int_{-a}^{x} \Big(\varrho(y) - \chi(y)\int_{-a}^{a}\varrho(s)ds\Big)dy \quad \text{for } -a < x < a.$$

Hence, it can be seen that

$$\langle\Psi', \varrho\rangle = -\langle\Psi, \varrho'\rangle = -\langle(\vartheta'')'', \zeta_{\rho'}\rangle = -\langle(\vartheta'')', \varrho\rangle$$

that implies $(\vartheta'')' + \Psi \equiv$ constant. In view of $(\vartheta'')'' \in L^2((-a,a))$ and (1.22), we know that $(\vartheta'')' \in L^2((-a,a))$. Since $\vartheta \in H^2((-a,a))$ and $(\vartheta'')', (\vartheta'')'' \in L^2((-a,a))$, it tells us that ϑ belongs to $H^4((-a,a))$.

By exploiting (1.21), we then show that ϑ satisfies (0.35) and (0.36). Indeed, consider now $\varrho \in C^\infty((-a,a))$, one has, using the integration by parts

$$\int_{-a}^{a} (\vartheta'')''(x)\varrho(x)dx = (\vartheta'')'(a)\varrho(a) - (\vartheta'')'(-a)\varrho(-a) - (\vartheta'')(a)\varrho'(a) + (\vartheta'')(-a)\varrho'(-a) + \int_{-a}^{a} \vartheta''(x)\varrho''(x)dx.$$

We perform on the other terms of (1.20) the integration by parts, which yields

$$\lambda \int_{-a}^{a} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx + \mu \int_{-a}^{a} (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx - \lambda\rho_0\vartheta'\varrho\Big|_{-a}^{a} + \mu\Big(\vartheta'''\varrho\Big|_{-a}^{a} - \vartheta''\varrho'\Big|_{-a}^{a} - 2k^2\vartheta'\varrho\Big|_{-a}^{a}\Big) = \int_{-a}^{a} (Y_{a,\lambda}\vartheta)\varrho dx.$$

It then follows from the definition of the bilinear form $\mathcal{B}_{a,\lambda}$ that

$$BV_{a,\lambda}(\vartheta,\varrho) + BV_{-a,\lambda}(\vartheta,\varrho) = -\lambda\rho_0\vartheta'\varrho\Big|_{-a}^{a} + \mu\Big(\vartheta'''\varrho\Big|_{-a}^{a} - \vartheta''\varrho'\Big|_{-a}^{a} - 2k^2\vartheta'\varrho\Big|_{-a}^{a}\Big) \quad (1.23)$$

for all $\varrho \in C^\infty((-a,a))$.

By collecting all terms corresponding to $\varrho(-a)$ in (1.23), we deduce that

$$\mu(k\tau_-(k+\tau_-)\vartheta(-a) - k\tau_-\vartheta'(-a)) = \lambda\rho_0(-a)\vartheta'(-a) - \mu(\vartheta'''(-a) - 2k^2\vartheta'(-a)).$$

It yields

$$\vartheta''(-a) - (k^2 + k\tau_- + \tau_-^2)\vartheta'(-a) + k\tau_-(k+\tau_-)\vartheta(-a) = 0$$

owing to the definition of τ_-. Then, we collect all terms corresponding to $\varrho(a)$ or to $\varrho'(\pm a)$ in (1.23) to conclude that ϑ satisfies (0.35) and (0.36). This ends the proof of Proposition 1.2. □

We have the following proposition on $Y_{a,\lambda}^{-1}$.

Proposition 1.4. *The operator $Y_{a,\lambda}^{-1} : L^2((-a,a)) \to L^2((-a,a))$ is compact and self-adjoint.*

Proof. It follows from Proposition 1.3 that $Y_{a,\lambda}$ admits an inverse operator $Y_{a,\lambda}^{-1}$ from $L^2((-a,a))$ to a subspace of $H^4((-a,a))$ requiring all elements satisfy (0.35)–(0.36), which is symmetric due to Proposition 1.2. We compose $Y_{a,\lambda}^{-1}$ with the continuous injection from $H^4((-a,a))$ to $L^2((-a,a))$. Notice that the embedding $H^p((-a,a)) \hookrightarrow H^q((-a,a))$ for $p > q \geqslant 0$ is compact. Therefore, $Y_{a,\lambda}^{-1}$ is compact and self-adjoint from $L^2((-a,a))$ to $L^2((-a,a))$. Proposition 1.4 is shown. □

Remark 1.1. *In this paper, we choose to define the operator*

$$\phi \mapsto \lambda(\rho_0 k^2\phi - (\rho_0\phi')') + \mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = Y_{a,\lambda}\phi$$

with boundary conditions (0.35)-(0.36) *through Riesz's representation theorem. We can also define that by the following way.*

The operator $Y_{a,\lambda}$ *is well defined on*

$$D(Y_{a,\lambda}) = \{\phi \in C^4((-a,a)), \phi \text{ verifies } (0.35) - (0.36)\}$$

and that we can extend $Y_{a,\lambda}$ *over the closure of* $D(Y_{a,\lambda})$*. Furthermore,* $Y_{a,\lambda}$ *with the domain* $H^4((-a,a))$ *containing functions that satisfy* (0.35)-(0.36) *is symmetric and positive. It follows from Friedrichs extension (see [41, Theorem 4.3.1] e.g.) that* $Y_{a,\lambda}$ *admits a self-adjoint extension.*

1.2.3 A sequence of characteristic values

We continue considering $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$ and study the operator $S_{a,\lambda} := \mathcal{M} Y_{a,\lambda}^{-1} \mathcal{M}$, where $\mathcal{M}$ is the operator of multiplication by $\sqrt{\rho_0'}$. Note that this choice prevents to consider a case where ρ_0' could be negative.

Proposition 1.5. *The operator* $S_{a,\lambda} : L^2((-a,a)) \to L^2((-a,a))$ *is compact and self-adjoint, under the hypothesis* (0.31)*.*

Proof. Due to the boundedness of ρ_0', the operator $S_{a,\lambda}$ is well-defined from $L^2((-a,a))$ to itself. $Y_{a,\lambda}^{-1}$ is compact, so is $S_{a,\lambda}$. Moreover, because both the inverse $Y_{a,\lambda}^{-1}$ and $\mathcal{M}$ are self-adjoint, the self-adjointness of $S_{a,\lambda}$ follows. □

As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{a,\lambda}$ is discrete, i.e. is a sequence $\{\gamma_n(\lambda)\}_{n\geqslant 1}$ of eigenvalues of $S_{a,\lambda}$, associated with normalized orthogonal eigenfunctions $\{\varpi_n\}_{n\geqslant 1}$ in $L^2((-a,a))$. That means

$$\gamma_n(\lambda)\varpi_n = \mathcal{M} Y_{a,\lambda}^{-1} \mathcal{M} \varpi_n.$$

so that with $\phi_n = Y_{a,\lambda}^{-1} \mathcal{M} \varpi_n \in H^4((-a,a))$, one has

$$\gamma_n(\lambda) Y_{a,\lambda}\phi_n = \rho_0'\phi_n \tag{1.24}$$

and ϕ_n satisfies (0.35)-(0.36). Eq. (1.24) also tells us that $\gamma_n(\lambda) > 0$ for all n. Indeed, we obtain

$$\gamma_n(\lambda) \int_{-a}^{a} (Y_{a,\lambda}\phi_n)\phi_n dx = \int_{-a}^{a} \rho_0'\phi_n^2 dx.$$

That implies

$$\gamma_n(\lambda) \mathcal{B}_{a,\lambda}(\phi_n, \phi_n) = \int_{-a}^{a} \rho_0'\phi_n^2 dx. \tag{1.25}$$

Since $\mathcal{B}_{a,\lambda}(\phi_n, \phi_n) > 0$ and $\rho_0' > 0$ on $(-a,a)$, we know that $\gamma_n(\lambda)$ is positive. Hence, by reordering and using the spectral theory of compact and self-adjoint operators again, we have that $\{\gamma_n(\lambda)\}_{n\geqslant 1}$ is a positive sequence decreasing towards 0 as $n \to \infty$.

For each n, in order to verify that ϕ_n is a solution of (0.27), we are left to look for real values of λ_n such that (0.51). To solve (0.51), we have to prove that $\gamma_n(\lambda)$ is differentiable and decreasing in terms of λ, respectively in two next lemmas.

Lemma 1.2. *For each n, the functions $\gamma_n(\lambda)$ and ϕ_n are differentiable in terms of $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$.*

Proof. The family $(Y_{a,\lambda})_{\lambda \in (0,\sqrt{\frac{g}{L_0}}]}$ is a family of bounded operators owing to Proposition 1.4. It can be seen that the boundary conditions (0.35)-(0.36) differentiable in the parameter λ will tell us that $(Y_{a,\lambda})_{\lambda \in (0,\sqrt{\frac{g}{L_0}}]}$ is also a family of differentiable operators on λ by following a generalized treatment of [45, Example 1.15, Chapter VII, \$1.6]. Since $Y_{a,\lambda}^{-1}$ exists for all $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$, it follows from [45, Theorem 2.23, Chapter IV, \$2.6] that $Y_{a,\lambda}^{-1}$ is differentiable for all $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$, so is $S_{a,\lambda}$. We then apply the differentiable property of eigenvalues of self-adjoint and compact operators, demonstrated in Theorem 1.1 to deduce that $\gamma_n(\lambda)$ and ϕ_n are differentiable functions. □

Lemma 1.3. *For each n, the function $\gamma_n(\lambda)$ is decreasing in $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$.*

Proof. Let $z_n = \frac{d\phi_n}{d\lambda}$. It follows from (1.24) that

$$k^2\rho_0\phi_n - (\rho_0\phi_n')' + Y_{a,\lambda}z_n = \frac{1}{\gamma_n(\lambda)}\rho_0' z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda)}\Big)\rho_0'\phi_n \tag{1.26}$$

on $(-a, a)$. In addition, we have that at $x = -a$,

$$\begin{cases} z_n''(-a) - (k+\tau_-)z_n'(-a) + k\tau_- z_n(-a) \\ \qquad = \frac{\nu_-}{2\tau_-}\phi_n'(-a) - \frac{k\nu_-}{2\tau_-}\phi_n(-a), \\ z_n'''(-a) - (k^2 + k\tau_- + \tau_-^2)z_n'(-a) + k\tau_-(k+\tau_-)z_n(-a) \\ \qquad = \Big(\frac{k\nu_-}{2\tau_-} + \nu_-\Big)\phi_n'(-a) - \Big(\frac{k^2\nu_-}{2\tau_-} + k\nu_-\Big)\phi_n(-a) \end{cases} \tag{1.27}$$

and that at $x = a$,

$$\begin{cases} z_n''(a) + (k+\tau_+)z_n'(a) + k\tau_+ z_n(a) \\ \qquad = -\frac{\nu_+}{2\tau_+}\phi_n'(a) - \frac{k\nu_+}{2\tau_+}\phi_n(a), \\ z_n'''(a) - (k^2 + k\tau_+ + \tau_+^2)z_n'(a) - k\tau_+(k+\tau_+)z_n(a) \\ \qquad = \Big(\frac{k\nu_+}{2\tau_+} + \nu_+\Big)\phi_n'(a) + \Big(\frac{k^2\nu_+}{2\tau_+} + k\nu_+\Big)\phi_n(a). \end{cases} \tag{1.28}$$

Multiplying by ϕ_n on both sides of (1.26) and then integrating by parts to obtain that

$$\begin{aligned} &\int_{-a}^{a}(k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx + \int_{-a}^{a}(Y_{a,\lambda}z_n)\phi_n dx \\ &\qquad = \frac{1}{\gamma_n(\lambda)}\int_{-a}^{a}\rho_0' z_n\phi_n dx + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda)}\Big)\int_{-a}^{a}\rho_0'\phi_n^2 dx. \end{aligned} \tag{1.29}$$

Thanks to the integration by parts, we have

$$\int_{-a}^{a} (k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx = \int_{-a}^{a} \rho_0(k^2(\phi_n)^2 + (\phi_n')^2)dx - (\rho_0\phi_n'\phi_n)\Big|_{-a}^{a} \tag{1.30}$$

and

$$\begin{aligned}\int_{-a}^{a} (Y_{a,\lambda}z_n)\phi_n dx = \int_{-a}^{a} (Y_{a,\lambda}\phi_n)z_n dx + \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)\Big|_{-a}^{a} \\ - \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)\Big|_{-a}^{a}.\end{aligned} \tag{1.31}$$

Owing to (1.30), (1.31) and also (1.24), Eq. (1.29) becomes

$$\begin{aligned}&\int_{-a}^{a} \rho_0(k^2(\phi_n)^2 + (\phi_n')^2)dx + \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)\Big|_{-a}^{a} \\ &\qquad - \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)\Big|_{-a}^{a} - (\rho_0\phi_n'\phi_n)\Big|_{-a}^{a} \\ &= \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda)}\Big)\int_{-a}^{a} \rho_0'\phi_n^2 dx.\end{aligned} \tag{1.32}$$

Using (1.27), we obtain

$$\begin{aligned}&-\Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(-a) \\ &\qquad + \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)(-a) + \rho_-\phi_n'(-a)\phi_n(-a) \\ &= \mu\Big(\frac{k^2\nu_-}{2\tau_-} + k\nu_-\Big)(\phi_n(-a))^2 - \mu\Big(\frac{k\nu_-}{2\tau_-} + \nu_-\Big)\phi_n'(-a)\phi_n(-a) \\ &\qquad - \mu\frac{k\nu_-}{2\tau_-}\phi_n(-a)\phi_n'(-a) + \mu\frac{\nu_-}{2\tau_-}(\phi_n'(-a))^2 + \rho_-\phi_n'(-a)\phi_n(-a).\end{aligned}$$

Keep in mind that $\mu\nu_- = \rho_-$, one has

$$\begin{aligned}&-\Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(-a) \\ &\qquad + \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)(-a) + \rho_-\phi_n'(-a)\phi_n(-a) \\ &= k\rho_-(\phi_n(-a))^2 + \frac{\rho_-}{2\tau_-}(\phi_n'(-a) - k\phi_n(-a))^2.\end{aligned} \tag{1.33}$$

Similarly, using (1.28), we obtain

$$\begin{aligned}&\Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(a) \\ &\qquad - \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)(a) - \rho_-\phi_n'(a)\phi_n(a) \\ &= k\rho_+(\phi_n(a))^2 + \frac{\rho_+}{2\tau_+}(\phi_n'(a) - k\phi_n(a))^2.\end{aligned} \tag{1.34}$$

Combining (1.32), (1.33) and (1.34), we deduce that

$$\begin{aligned}&\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda)}\Big)\int_{-a}^{a} \rho_0'\phi_n^2 dx \\ &= \int_{-a}^{a} \rho_0(k^2\phi_n^2 + (\phi_n')^2)dx + k\rho_-(\phi_n(-a))^2 + \frac{\rho_-}{2\tau_-}(\phi_n'(-a) - k\phi_n(-a))^2 \\ &\qquad + k\rho_+(\phi_n(a))^2 + \frac{\rho_+}{2\tau_+}(\phi_n'(a) - k\phi_n(a))^2.\end{aligned} \tag{1.35}$$

It yields that $\frac{1}{\gamma_n(\lambda)}$ is increasing in λ, i.e. $\gamma_n(\lambda)$ is decreasing in λ. This ends the proof of Lemma 1.3. □

Now we are in proposition to solve (0.51).

Proposition 1.6. *For each n, there exists only one positive λ_n satisfying (0.51). In addition, λ_n decreases towards 0 as n goes to ∞.*

Proof. Using (1.25), we know that

$$\frac{1}{\gamma_n(\lambda)}\int_{-a}^{a}\rho_0'\phi_n^2dx = \int_{-a}^{a}(Y_{a,\lambda}\phi_n)\phi_n dx = \mathcal{B}_{a,\lambda}(\phi_n,\phi_n),$$

that implies

$$\frac{1}{L_0\gamma_n(\lambda)} \geqslant \lambda k^2 + \frac{\mu k^4}{\rho_+}.$$

Consequently, for all $n \geqslant 1$,

$$\lim_{\lambda\to\sqrt{\frac{g}{L_0}}}\frac{\lambda}{\gamma_n(\lambda)} > gk^2. \tag{1.36}$$

As $0 < \frac{1}{\gamma_n(\lambda)}$ and it is a increasing function, $\frac{1}{\gamma_n(\lambda)} \leqslant \frac{1}{\gamma_n(\frac{1}{2}\sqrt{\frac{g}{L_0}})}$ for all $\lambda \leqslant \frac{1}{2}\sqrt{\frac{g}{L_0}}$. That implies

$$\lim_{\lambda\to 0}\frac{\lambda}{\gamma_n(\lambda)} \leqslant \lim_{\lambda\to 0}\frac{\lambda}{\gamma_n(\frac{1}{2}\sqrt{\frac{g}{L_0}})} = 0 \quad \text{for all } n \geqslant 1. \tag{1.37}$$

Combining (1.36), (1.37) and using Lemma 1.3, we deduce that there is only one $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ satisfying (0.51) for each $n \geqslant 1$.

We then prove that the sequence $(\lambda_n)_{n\geqslant 1}$ is decreasing. Indeed, if $\lambda_m < \lambda_{m+1}$ for some $m \geqslant 1$, we have

$$\gamma_m(\lambda_m) > \gamma_m(\lambda_{m+1}).$$

Meanwhile, we also have

$$\gamma_m(\lambda_{m+1}) > \gamma_{m+1}(\lambda_{m+1}).$$

That implies

$$\frac{\lambda_m}{gk^2} = \gamma_m(\lambda_m) > \gamma_{m+1}(\lambda_{m+1}) = \frac{\lambda_{m+1}}{gk^2}.$$

That contradiction tells us that $(\lambda_n)_{n\geqslant 1}$ is a decreasing sequence. Suppose that

$$\lim_{n\to\infty}\lambda_n = c_0 > 0.$$

Note that

$$\gamma_n(c_0) \geqslant \gamma_n(\lambda_n) = \frac{\lambda_n}{gk^2}.$$

Let $n \to \infty$, we get a contradiction that $0 \geqslant c_0$. Hence, λ_n decreases towards 0 as $n \to \infty$. We conclude Proposition 1.6. □

1.2.4 Proof of Theorem 0.1

Let λ_n be found from Proposition 1.6 and $\phi_n(x) = Y^{-1}_{a,\lambda_n}\mathcal{M}\varpi_n(x)$ on $(-a,a)$. Keep in mind our computations in Section 1.2.1, we extend ϕ_n to the whole line by requiring ϕ_n satisfies (1.12) and (1.13) for some constants $A^{\pm}_{n,1}$ and $A^{\pm}_{n,2}$ as $\lambda = \lambda_n$. Those constants $A^{\pm}_{n,1}$ and $A^{\pm}_{n,2}$ are defined by

$$\begin{cases} \phi_n(a) = A^{+}_{n,1} + A^{+}_{n,2}, \\ \phi_n'(a) = kA^{+}_{n,1} + \sqrt{k^2+\lambda_n\nu_+}A^{+}_{n,2}. \end{cases} \tag{1.38}$$

and by

$$\begin{cases} \phi_n(-a) = A^{-}_{n,1} + A^{-}_{n,2}, \\ \phi_n'(-a) = kA^{-}_{n,1} + \sqrt{k^2+\lambda_n\nu_-}A^{-}_{n,2}. \end{cases} \tag{1.39}$$

Solving (1.38) and (1.39), we get that

$$A^{+}_{n,1} = \frac{\sqrt{k^2+\lambda_n\nu_+}\phi_n(a) - \phi_n'(a)}{\sqrt{k^2+\lambda_n\nu_+}-k}, \quad A^{+}_{n,2} = \frac{\phi_n'(a) - k\phi_n(a)}{\sqrt{k^2+\lambda_n\nu_+}-k}. \tag{1.40}$$

and that

$$A^{-}_{n,1} = \frac{\sqrt{k^2+\lambda_n\nu_-}\phi_n(-a) - \phi_n'(-a)}{\sqrt{k^2+\lambda_n\nu_-}-k}, \quad A^{-}_{n,2} = \frac{\phi_n'(-a) - k\phi_n(-a)}{\sqrt{k^2+\lambda_n\nu_-}-k}. \tag{1.41}$$

Therefore, the function $\phi_n \in H^4(\mathbf{R})$ is a regular solution of (0.27) as $\lambda = \lambda_n$ for each $n \geqslant 1$. Proof of Theorem 0.1 is complete.

1.3 The strictly increasing profile case

The proof of Theorem 0.2 remains the same to that one of the first case, but more complicated. We point out the main differences as follows.

Questions concerning the existence of solutions of Eq. (0.27) being bounded at ∞ are not straightforward as in the first case. In Section 1.3, we transform Eq. (0.27) into a system of ODEs (1.44). The matrix $L(x,\lambda)$ has 4 eigenvalues $\pm k$ and $\pm\sqrt{k^2+\lambda\rho_0(x)/\mu}$, which are different for all $\lambda > 0$. We then follow [10, Theorem 8.1, Chapter 3], whose statement is Theorem 1.2, to deduce that Eq. (0.27) admits two linearly independent solutions decaying to 0 at $+\infty$ (respectively $-\infty$). A suitable interval (x_-, x_+) is thus determined through a precise calculation of the family of solutions decaying to 0 at $\pm\infty$, which yields appropriate boundary conditions (0.37) at x_- and (0.38) at x_+ in Proposition 1.7.

We solve Eq. (0.27) on the finite interval (x_-, x_+) with the boundary conditions (0.37)–(0.38). To do that, in Section 1.3.2, we construct the bilinear form $\mathcal{B}_{x_-,x_+,\lambda}$ (0.45) in Proposition 1.8 and continue the same arguments as in Section 1.2 to obtain the solution in the inner region (x_-, x_+). Note that, the coercivity of $\mathcal{B}_{x_-,x_+,\lambda}$ relies on the positivity of the terms $BV_{x_+,\lambda}$ (0.43) and $BV_{x_-,\lambda}$ (0.44) stated in Lemma 1.4. Due to the lack of an easy-to-use expression of boundary conditions in this case, it turns out that the positivity of $BV_{x_+,\lambda}$ and $BV_{x_-,\lambda}$ are derived, in Proposition 1.8, by deducing the behavior at $\pm\infty$ of coefficients $n_{ij}^{\pm}$ $(i, j = 1, 2)$ depending on $(x_\pm, \lambda)$ and appearing in the boundary conditions (0.37)-(0.38). Having the bilinear form $\mathcal{B}_{x_-,x_+,\lambda}$, we continue our arguments in Propositions 1.9 and 1.10, that follows the same line of Section 1.2, to prove Theorem 0.2.

Note that Lemma 1.4 does not give any control on x_- and x_+. It is interesting for computational purposes as well as for a study of particular profiles (for example profiles decaying to ρ_+ at $+\infty$ at rate $e^{-\alpha_+ x}(\alpha_+ > 0)$ and to ρ_- at $-\infty$ at rate $e^{\alpha_- x}(\alpha_- > 0)$), to be able to derive an explicit interval on which this is true. Notice that the restriction $\lambda \geqslant \epsilon_\star > 0$ of Theorem 0.2 implies that the matrix $R(\lambda)$ (see (1.43)) in Eq. (1.44) is non singular in the region $[\varepsilon_\star, \sqrt{\frac{g}{L_0}}]$. Hence, for a profile such that $\rho_0' > 0$ everywhere, we devote Section 1.3.4 to establish a control of x_- and x_+ independent of λ. In Propositions 1.13 and 1.14, through a careful construction of Volterra series, we can obtain refined estimates of bounded solutions of Eq. (1.44) at ∞ uniformly in $\lambda \in [\epsilon_\star, \sqrt{\frac{g}{L_0}}]$. They allow us to have refined estimates on the coefficients $n_{ij}^{\pm}$ $(i, j = 1, 2)$ uniformly in $\lambda \in [\epsilon_\star, \sqrt{\frac{g}{L_0}}]$. Hence, we obtain a criterion for x_- and x_+ in Proposition 1.11 to fulfill the conditions of Lemma 1.4 and extend Proposition 1.8.

In this section, let (e_1, e_2, e_3, e_4) be the canonical basis of $\mathbf{R}^4$ and we pay attention to an increasing profile ρ_0 satisfying (0.33) and (0.34). We will use the Frobenius matrix norm $\|\cdot\|_F$ and the Euclidean vector norm $\|\cdot\|_2$.

1.3.1 Solutions decaying to 0 at infinity and reduction to a problem on a finite interval

Let

$$L(x,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda k^2 \rho_0(x)}{\mu} - k^4 & 0 & \frac{\lambda \rho_0(x)}{\mu} + 2k^2 & 0 \end{pmatrix} \tag{1.42}$$

and

$$R(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{gk^2}{\lambda\mu} & \frac{\lambda}{\mu} & 0 & 0 \end{pmatrix} \tag{1.43}$$

We set $U = (\phi, \phi', \phi'', \phi''')^T$ and then rewrite (0.27) as

$$U'(x) = (L(x,\lambda) + \rho_0'(x) R(\lambda)) U(x), \tag{1.44}$$

The eigenvalues of $L(x,\lambda)$, $\pm k$ and $\pm\sigma_0(x,\lambda)$, with $\sigma_0(x,\lambda) = \sqrt{k^2 + \frac{\lambda}{\mu}\rho_0(x)}$ are different for all $\lambda > 0$ and for all $x \in \mathbf{R}$. Furthermore,

$$\int_{-\infty}^{+\infty} |L'(x,\lambda)|dx < +\infty, \qquad \int_{-\infty}^{+\infty} |\rho_0'(x)R(\lambda)|dx < +\infty. \tag{1.45}$$

The inequalities (1.45) allow us to use [10, Theorem 8.1, Chapter 3] (see Section 1.2 for the statement) to find bounded solutions of (1.44) near $\pm\infty$. We denote $\sigma_\pm(\lambda) = \lim_{x\to\pm\infty}\sigma_0(x,\lambda)$. Then, there exist $\bar{x}_- < \bar{x}_+$ such that we have the existence of bounded solutions $U_{1,2}^+$ on $(\bar{x}_+,+\infty)$ and $U_{3,4}^-$ on $(-\infty,\bar{x}_-)$ such that in a neighborhood of $+\infty$,

$$\begin{aligned} &e^{kx}U_1^+(x,\lambda) \to (-k^{-3},k^{-2},-k^{-1},1)^T, \\ &\exp\Big(\textstyle\int_{\bar{x}_+}^x \sigma_0(y,\lambda)dy\Big)U_2^+(x,\lambda) \to (-\sigma_+^{-3}(\lambda),\sigma_+^{-2}(\lambda),-\sigma_+^{-1}(\lambda),1)^T, \end{aligned} \tag{1.46}$$

and in a neighborhood of $-\infty$,

$$\begin{aligned} &e^{-kx}U_3^-(x,\lambda) \to (k^{-3},k^{-2},k^{-1},1)^T, \\ &\exp\Big(-\textstyle\int_{\bar{x}_-}^x \sigma_0(y,\lambda)dy\Big)U_4^-(x,\lambda) \to (\sigma_-^{-3}(\lambda),\sigma_-^{-2}(\lambda),\sigma_-^{-1}(\lambda),1)^T. \end{aligned} \tag{1.47}$$

Let us prove that $U_1^+(x,\lambda)$ and $U_2^+(x,\lambda)$ are linearly independent. Through Cauchy-Lipschitz's theorem, if they are linearly dependent at a particular x_0 then they are linearly dependent for all x, that is there exists $T(\lambda)$ such that

$$U_1^+(x,\lambda) = T(\lambda)U_2^+(x,\lambda).$$

In particular, as the limit of $kx - \int_{\bar{x}_+}^x \sigma_0(y,\lambda)dy$ when $x \to +\infty$ is equal to $-\infty$, one observes that

$$e^{kx}U_1^+(x,\lambda) = T(\lambda)\exp\Big(kx - \int_{\bar{x}_+}^x \sigma_0(y,\lambda)dy\Big)\Big(U_2^+(x,\lambda)\exp\Big(\int_{\bar{x}_+}^x \sigma_0(y,\lambda)dy\Big)\Big).$$

The r.h.s. of this identity converges to 0 when $x \to +\infty$, while the l.h.s. converges to $(-k^{-3},k^{-2},-k^{-1},1)^T$ when $x \to +\infty$, contradiction. Similarly, $U_3^-(x,\lambda)$ and $U_4^-(x,\lambda)$ are linearly independent.

The aim of the next proposition is to reduce the study of Eq. (0.27) on the real line to its study on a finite interval as in the previous section. This is really the key of our result for a smooth general profile because it entitles us to use the compact injection of $H^i((a,b))$ into $H^j((a,b))$ if $i > j$.

Proposition 1.7. *There exist $x_-^0 \leqslant \bar{x}_-$ and $x_+^0 \geqslant \bar{x}_+$ such that, for all $x_- \leqslant x_-^0$ and $x_+ \geqslant x_+^0$, there are constants $n_{ij}^\pm$ $(i,j = 1,2)$, depending on $x_\pm$ and λ such that equation*

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi,$$

where $\phi \in H^4((x_-,x_+))$, supplemented with the boundary conditions at x_- are (0.37)

$$\begin{cases} n_{11}^-\phi(x_-) + n_{12}^-\phi'(x_-) + \phi''(x_-) = 0, \\ n_{21}^-\phi(x_-) + n_{22}^-\phi'(x_-) + \phi'''(x_-) = 0 \end{cases}$$

and at x_+*, are* (0.38)

$$\begin{cases} n_{11}^+\phi(x_+) + n_{12}^+\phi'(x_+) + \phi''(x_+) = 0, \\ n_{21}^+\phi(x_+) + n_{22}^+\phi'(x_+) + \phi'''(x_+) = 0, \end{cases}$$

is equivalent to equation

$$\lambda^2(k^2\rho_0\Phi - (\rho_0\Phi')') + \lambda\mu(\Phi^{(4)} - 2k^2\Phi'' + k^4\Phi) = gk^2\rho_0'\Phi,$$

where $\Phi \in H^4(\mathbf{R})$*.*

In this case, on (x_-, x_+) *we have that* $\Phi = \phi$ *and on* $(x_+, +\infty)$ *(respectively* $(-\infty, x_-)$*),* Φ *is the first component of a linear combination of* $U_{1,2}^+(x,\lambda)$ *(see* (1.46)*) (respectively* $U_{3,4}^-(x,\lambda)$*, see* (1.47)*).*

Proof. Notice that if Φ is a bounded solution of

$$\lambda^2(k^2\rho_0\Phi - (\rho_0\Phi')') + \lambda\mu(\Phi^{(4)} - 2k^2\Phi'' + k^4\Phi) = gk^2\rho_0'\Phi,$$

$\Theta = (\Phi, \Phi', \Phi'', \Phi''')^T$ is a bounded solution of (1.44). Any decaying solution Θ of (1.44) on $(\bar{x}_+, +\infty)$ belongs to the space spanned by $U_1^+(x,\lambda)$ and $U_2^+(x,\lambda)$, which is of dimension 2 because they are linearly independent. It is equivalent to say that, for any $x_+ \geqslant \bar{x}_+$,

$$U_1^+(x_+,\lambda) \wedge U_2^+(x_+,\lambda) \wedge \Theta(x_+,\lambda) = 0 \tag{1.48}$$

and

$\Theta(x_+,\lambda)$ belongs to the space spanned by $U_1^+(x_+,\lambda)$ and $U_2^+(x_+,\lambda)$.

Let us write $U_i^+ = (U_{i1}^+, U_{i2}^+, U_{i3}^+, U_{i4}^+)^T$ for $i = 1,2$ and $U_i^- = (U_{i1}^-, U_{i2}^-, U_{i3}^-, U_{i4}^-)^T$ for $i = 3,4$. System (1.48) is a system of four equations on the components of $\Theta(x_+,\lambda)$, hence there exists a couple of equations which are linearly independent (the system being of rank 2). Let us notice that two of these four equations contain, in Φ'' and Φ''', respectively the term

$$(U_{11}^+U_{22}^+ - U_{12}^+U_{21}^+)(x_+,\lambda)\Phi''(x_+,\lambda)$$

and

$$(U_{11}^+U_{22}^+ - U_{12}^+U_{21}^+)(x_+,\lambda)\Phi'''(x_+,\lambda).$$

As the limit, when $x_+ \to +\infty$, of

$$\exp\Big(kx_+ + \int_{\bar{x}_+}^{x_+} \sigma_0(y,\lambda)dy\Big)(U_{11}^+U_{22}^+ - U_{12}^+U_{21}^+)(x_+,\lambda)$$

is

$$-\frac{1}{k^3\sigma_+^2(\lambda)} + \frac{1}{k^2\sigma_+^3(\lambda)} = -\frac{\lambda\rho_+}{\mu k^3\sigma_+^3(\lambda)(k+\sigma_+)} < 0,$$

by continuity there exists a $x_+^0 \geqslant \bar{x}_+$ such that, for all $x_+ \geqslant x_+^0$,

$$(U_{11}^+U_{22}^+ - U_{12}^+U_{21}^+)(x_+,\lambda) < 0$$

hence the equations for (1.48) on the components $e_1 \wedge e_2 \wedge e_3$ and $e_1 \wedge e_2 \wedge e_4$ write as $N^+\Theta(x_+, \lambda) = 0$ with N^+ is a 4×2 matrix of the form

$$N^+ = \begin{pmatrix} n_{11}^+ & n_{12}^+ & 1 & 0 \\ n_{21}^+ & n_{22}^+ & 0 & 1 \end{pmatrix}.$$

We are now able to write the couple $N^+\Theta(x_+, \lambda) = 0$ as

$$\begin{cases} n_{11}^+\Phi(x_+, \lambda) + n_{12}^+\Phi'(x_+, \lambda) + \Phi''(x_+, \lambda) = 0, \\ n_{21}^+\Phi(x_+, \lambda) + n_{22}^+\Phi'(x_+, \lambda) + \Phi'''(x_+, \lambda) = 0, \end{cases}$$

In a similar way, there exist n_{ij}^- $(i, j = 1, 2)$ depending on x_- and λ such that

$$\begin{cases} n_{11}^-\Phi(x_-, \lambda) + n_{12}^-\Phi'(x_-, \lambda) + \Phi''(x_-, \lambda) = 0, \\ n_{21}^-\Phi(x_-, \lambda) + n_{22}^-\Phi'(x_-, \lambda) + \Phi'''(x_-, \lambda) = 0, \end{cases}$$

Hence Φ is a solution of the ODE on (x_-, x_+):

$$\lambda^2(k^2\rho_0\Phi - (\rho_0\Phi')') + \lambda\mu(\Phi^{(4)} - 2k^2\Phi'' + k^4\Phi) = gk^2\rho_0'\Phi,$$

with the boundary conditions (0.37) and (0.38).

Conversely, assume that $\phi \in H^4((x_-, x_+))$ is a solution of equation

$$\lambda^2(k^2\rho_0\phi - (\rho_0\phi')') + \lambda\mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi) = gk^2\rho_0'\phi,$$

with boundary conditions (0.37)-(0.38). From the boundary conditions, we deduce that there exist $C_j^+(j = 1, 2), D_k^-(k = 3, 4)$ such that

$$U(x_+, \lambda) = C_1^+U_1^+(x_+, \lambda) + C_2^+U_2^+(x_+, \lambda)$$

and

$$U(x_-, \lambda) = D_3^-U_3^-(x_-, \lambda) + D_4^-U_4^-(x_-, \lambda).$$

Then, through Cauchy-Lipschitz's theorem,

$$U(x, \lambda) = C_1^+U_1^+(x, \lambda) + C_2^+U_2^+(x, \lambda) \quad \text{for all } x \geqslant x_+$$

and

$$U(x, \lambda) = D_3^-U_3^-(x, \lambda) + D_4^-U_4^-(x, \lambda) \quad \text{for all } x \leqslant x_-.$$

As these are decaying solutions at $\pm\infty$ respectively, and as there is no jump at x_+ (or x_-) for $\phi, \phi', \phi'', \phi'''$ (which have a meaning as ϕ is assumed to be in $H^4((x_-, x_+))$), the function

$$\Phi(x) = \begin{cases} C_1^+U_{11}^+(x, \lambda) + C_2^+U_{21}^+(x, \lambda), & \text{as } x \geqslant x_+ \\ \phi(x), & \text{as } x_- < x < x_+ \\ D_3^-U_{31}^-(x, \lambda) + D_4^-U_{41}^-(x, \lambda), & \text{as } x \leqslant x_- \end{cases}$$

belongs to $H^4(\mathbf{R})$ and solves equation

$$\lambda^2(k^2\rho_0\Phi - (\rho_0\Phi')') + \lambda\mu(\Phi^{(4)} - 2k^2\Phi'' + k^4\Phi) = gk^2\rho_0'\Phi \quad \text{on } \mathbf{R}.$$

□

Remark that in this proof we have no additional information on the values of x_-^0 and of x_+^0. This will be the aim of Section 1.3.4.

1.3.2 A bilinear form and a self-adjoint invertible operator

The aim of Section 1.3.2 is to study the $BV_{x_\pm,\lambda}$ terms (0.43), (0.44) which come from the bilinear form (0.45).

Lemma 1.4. *Necessary and sufficient conditions to get*

$$BV_{x_+,\lambda}(\vartheta,\vartheta) \geqslant 0 \text{ and } BV_{x_-,\lambda}(\vartheta,\vartheta) \geqslant 0 \tag{1.49}$$

for all $\vartheta \in H^2((x_-,x_+))$ *are*

$$\begin{aligned} &n_{12}^+(x_+,\lambda) \geqslant 0, \quad n_{21}^+(x_+,\lambda) \leqslant 0, \\ &(n_{11}^+(x_+,\lambda) - n_{22}^+(x_+,\lambda) - k^2 - \sigma_0^2(x_+,\lambda))^2 + 4n_{12}^+n_{21}^+(x_+,\lambda) \leqslant 0, \end{aligned} \tag{1.50}$$

and

$$\begin{aligned} &n_{12}^-(x_-,\lambda) \geqslant 0, \quad n_{21}^-(x_-,\lambda) \leqslant 0, \\ &(n_{11}^-(x_-,\lambda) - n_{22}^-(x_-,\lambda) - k^2 - \sigma_0^2(x_-,\lambda))^2 + 4n_{12}^-n_{21}^-(x_-,\lambda) \leqslant 0. \end{aligned} \tag{1.51}$$

Proof of Lemma 1.4. We treat only the case $BV_{x_+,\lambda}(\vartheta,\vartheta) \geqslant 0$. Since $\mu\sigma_0^2(x,\lambda) = \mu k^2 + \lambda\rho_0(x)$, we rewrite

$$\begin{aligned} \frac{1}{\mu}BV_{x_+,\lambda}(\vartheta,\vartheta) = &-n_{21}^+(x_+,\lambda)(\vartheta(x_+))^2 + n_{12}^+(\vartheta'(x_+))^2 \\ &+ (n_{11}^+(x_+,\lambda) - n_{22}^+(x_+,\lambda) - k^2 - \sigma_0^2(x_+,\lambda))\vartheta(x_+)\vartheta'(x_+). \end{aligned}$$

We observe that it is a quadratic polynomial in $\vartheta(x_+), \vartheta'(x_+)$. The first case is the case where $n_{12}^+ = n_{21}^+ = 0$. The inequalities (1.50) imply that $BV_{x_+,\lambda}(\vartheta,\vartheta) = 0$ for all ϑ. The second case is the case where at least one of these two real numbers is not zero. For example, if $n_{12}^+ \neq 0$, we have that the polynomial

$$n_{12}^+t^2 + (n_{11}^+ - n_{22}^+ - k^2 - \sigma_0^2)t - n_{21}^+$$

is always of the sign of n_{12}^+ hence positive, hence $BV_{x_+,\lambda}(\vartheta,\vartheta) \geqslant 0$ for all $\vartheta \in H^2((x_-,x_+))$. Lemma 1.4 is proven. □

Proposition 1.8. *There exists $x_-^1 \leqslant x_-^0$, $x_+^1 \geqslant x_+^0$ such that, for any $x_- \leqslant x_-^1$ and $x_+ \geqslant x_+^1$ chosen arbitrarily, $\mathcal{B}_{x_-,x_+,\lambda}$ (0.45) is a bilinear form on $H^2((x_-,x_+))$, that is continuous and coercive.*

Proof. Recall $n_{ij}^{\pm}$ $(i,j=1,2)$ are given in Proposition 1.7 and that $BV_{x_-,\lambda}$, $BV_{x_+,\lambda}$ are given in (0.43), (0.44). One observes that one needs to prove that

$$|\mathcal{B}_{x_-,x_+,\lambda}(\vartheta,\varrho)| \lesssim \|\vartheta\|_{H^2((x_-,x_+))}\|\varrho\|_{H^2((x_-,x_+))} \tag{1.52}$$

and that

$$\mathcal{B}_{x_-,x_+,\lambda}(\vartheta,\vartheta) \gtrsim \|\vartheta\|^2_{H^2((x_-,x_+))}. \tag{1.53}$$

For positive λ,μ and k, we have

$$\lambda\int_{x_-}^{x_+}\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx+\mu\int_{x_-}^{x_+}(\vartheta''\varrho''+2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx \lesssim \|\vartheta\|_{H^2((x_-,x_+))}\|\varrho\|_{H^2((x_-,x_+))}$$

and that

$$\lambda\int_{x_-}^{x_+}\rho_0(k^2\vartheta^2+(\vartheta')^2)dx+\mu\int_{x_-}^{x_+}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx \gtrsim \|\vartheta\|^2_{H^2((x_-,x_+))}.$$

Note that the condition $\mu>0$ is necessary to obtain the coercivity on $H^2((x_-,x_+))$. Note also that the case $\mu=0$ amounts to the inviscid Rayleigh-Taylor instability, for which similar results are known (and the corresponding problem needs only to be defined in H^1). In addition

$$|BV_{x_+,\lambda}(\vartheta,\varrho)| \lesssim (|\vartheta(x_+)|+|\varrho(x_+)|)(|\vartheta'(x_+)|+|\varrho'(x_+)|),$$
$$|BV_{x_-,\lambda}(\vartheta,\varrho)| \lesssim (|\vartheta(x_-)|+|\varrho(x_-)|)(|\vartheta'(x_-)|+|\varrho'(x_-)|)$$

and the Sobolev embedding yields (1.52). The continuity of $\mathcal{B}_{x_-,x_+,\lambda}$ on $H^2((x_-,x_+))$ follows.

To show (1.53), it suffices to prove that (1.49) holds. In view of Lemma 1.4, we verify (1.50). Since $N^+U_1^+(x_+,\lambda)=N^+U_2^+(x_+,\lambda)=0$, we have that n_{ij}^+ $(i,j=1,2)$ depend on $x_+$and λ and satisfy

$$\begin{cases} n_{11}^+U_{11}^+(x_+,\lambda)+n_{12}^+U_{12}^+(x_+,\lambda)+U_{13}^+(x_+,\lambda)=0,\\ n_{11}^+U_{21}^+(x_+,\lambda)+n_{12}^+U_{22}^+(x_+,\lambda)+U_{23}^+(x_+,\lambda)=0,\end{cases} \tag{1.54}$$

and

$$\begin{cases} n_{21}^+U_{11}^+(x_+,\lambda)+n_{22}^+U_{12}^+(x_+,\lambda)+U_{14}^+(x_+,\lambda)=0,\\ n_{21}^+U_{21}^+(x_+,\lambda)+n_{22}^+U_{22}^+(x_+,\lambda)+U_{24}^+(x_+,\lambda)=0.\end{cases} \tag{1.55}$$

Let $\mathsf{n}_{ij}^+(\lambda)$ be the limit of $n_{ij}^+(x_+,\lambda)$ as $x\to+\infty$. When $x_+\to+\infty$, two systems (1.54)-(1.55) converge to

$$\begin{cases} -\mathsf{n}_{11}^+(\lambda)k^{-3}+\mathsf{n}_{12}^+(\lambda)k^{-2}-k^{-1}=0,\\ -\mathsf{n}_{11}^+(\lambda)\sigma_+^{-3}(\lambda)+\mathsf{n}_{12}^+(\lambda)\sigma_+^{-2}(\lambda)-\sigma_+^{-1}(\lambda)=0,\end{cases} \tag{1.56}$$

and

$$\begin{cases} -\mathsf{n}_{21}^{+}(\lambda)k^{-3} + \mathsf{n}_{22}^{+}(\lambda)k^{-2} + 1 = 0, \\ -\mathsf{n}_{21}^{+}(\lambda)\sigma_{+}^{-3}(\lambda) + \mathsf{n}_{22}^{+}(\lambda)\sigma_{+}^{-2}(\lambda) + 1 = 0, \end{cases} \tag{1.57}$$

hence

$$\mathsf{n}_{11}^{+}(\lambda) = k\sigma_{+}(\lambda), \quad \mathsf{n}_{12}^{+}(\lambda) = k + \sigma_{+}(\lambda)$$

and

$$\mathsf{n}_{21}^{+}(\lambda) = -k\sigma_{+}(\lambda)(k + \sigma_{+}(\lambda)), \quad \mathsf{n}_{22}^{+}(\lambda) = -(k^2 + k\sigma_{+}(\lambda) + \sigma_{+}^2(\lambda)).$$

One thus has

$$\begin{aligned} &(\mathsf{n}_{11}^{+}(\lambda) - \mathsf{n}_{22}^{+}(\lambda) - k^2 - \sigma_{+}^2(\lambda))^2 + 4\mathsf{n}_{12}^{+}\mathsf{n}_{21}^{+}(\lambda) \\ &= ((k + \sigma_{+}(\lambda))^2 - k^2 - \sigma_{+}^2(\lambda))^2 - 4k\sigma_{+}(\lambda)(k + \sigma_{+}(\lambda))^2 \\ &= -4k\sigma_{+}(\lambda)(k^2 + k\sigma_{+}(\lambda) + \sigma_{+}^2(\lambda)) < 0. \end{aligned}$$

Hence, by continuity, there exists $x_{+}^{1} \geqslant x_{+}^{0}$ such that, for all $x_{+} \geqslant x_{+}^{1}$,

$$(n_{11}^{+}(x_{+},\lambda) - n_{22}^{+}(x_{+},\lambda) - k^2 - \sigma_0^2(x_{+},\lambda))^2 + 4n_{12}^{+}n_{21}^{+}(x_{+},\lambda) < 0.$$

The proof of the existence of $x_{-}^{1} \leqslant x_{-}^{0}$ such that, for all $x_{-} \leqslant x_{-}^{1}$ such that

$$(n_{11}^{-}(x_{-},\lambda) - n_{22}^{-}(x_{-},\lambda) - k^2 - \sigma_0^2(x_{-},\lambda))^2 + 4n_{12}^{-}n_{21}^{-}(x_{-},\lambda) < 0$$

follows the same pattern. Hence, by application of Lemma 1.4, the inequality (1.53) follows, which ends the proof of Proposition 1.8. □

Mimicking the arguments in Propositions 1.2, 1.3, 1.4 we obtain the following proposition.

Proposition 1.9. *Let $(H^2((x_{-},x_{+})))'$ be the dual space of $H^2((x_{-},x_{+}))$ associated with the norm $\sqrt{\mathcal{B}_{x_{-},x_{+},\lambda}(\cdot,\cdot)}$, there exists a unique operator*

$$Y_{x_{-},x_{+},\lambda} \in \mathcal{L}(H^2((x_{-},x_{+})),(H^2((x_{-},x_{+}))'),$$

that is also bijective, such that

$$\mathcal{B}_{x_{-},x_{+},\lambda}(\vartheta,\varrho) = \langle Y_{x_{-},x_{+},\lambda}\vartheta,\varrho\rangle \tag{1.58}$$

for all $\vartheta,\varrho \in H^2((x_{-},x_{+}))$. Furthermore, we have

1. *For all $\vartheta \in H^2((x_{-},x_{+}))$,*

$$Y_{x_{-},x_{+},\lambda}\vartheta = \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') + \mu(\vartheta^{(4)} - 2k^2\vartheta'' + k^4\vartheta)$$

in $\mathcal{D}'((x_{-},x_{+}))$.

2. *Let $f \in L^2((x_-,x_+))$ be given, there exists a unique $\vartheta \in H^4((x_-,x_+))$ satisfying the boundary conditions (0.37)–(0.38).*

$$Y_{x_-,x_+,\lambda}\vartheta = f \text{ in } (H^2((x_-,x_+)))', \tag{1.59}$$

3. *The operator $Y^{-1}_{x_-,x_+,\lambda} : L^2((x_-,x_+)) \to L^2((x_-,x_+))$ is compact and self-adjoint.*

It is then straightforwad to obtain the following spectral result. The discrete spectrum of $\mathcal{M}Y^{-1}_{x_-,x_+,\lambda}\mathcal{M}$ is a sequence of eigenvalues $(\gamma_n(\lambda))_{n\geqslant 1}$. The function $\gamma_n(\lambda)$ is a continuous function of λ for all n from the arguments of Lemma 1.2. The problem of finding a characteristic value amounts to solving the equality (0.51) as before. However, no control is possible on $\gamma_n(\lambda)$ when λ goes to 0. In addition, no control of $x^0_\pm$ and $x^1_\pm$ (because it may depend on λ) is available to have a possibility of having estimates of $\gamma_n(\lambda)$ as well. Having an explicit (even if not optimal) criterion on $x^1_\pm$ such that the inequalities of coercivity (1.50)-(1.51) are true is the aim of the refined estimates of the solutions $U^+_{1,2}$ and $U^-_{3,4}$ (deducing in Propositions 1.13, 1.14 below) which follow. That will be postponed to Section 1.3.4 below.

1.3.3 The finding of characteristic value λ_n and Proof of Theorem 0.2

Let $\epsilon_\star > 0$ be given, we look for $\lambda_n \in (\epsilon_\star, \sqrt{\frac{g}{L_0}})$ satisfying (0.51). However, unlike the previous case with $\rho_0' \geqslant 0$ being compactly supported, we do not have here the decrease of γ_n on λ to obtain the uniqueness of λ_n.

Proposition 1.10. *For $0 < \epsilon_\star \ll 1$, there exists $N(\epsilon_\star) \in \mathbb{N}$ such that there is at least one positive $\lambda_n \in (\epsilon_\star, \sqrt{\frac{g}{L_0}})$ satisfying (0.51) for each $1 \leqslant n \leqslant N(\epsilon_\star)$.*

Proof. We still have

$$\lim_{\lambda\to\sqrt{\frac{g}{L_0}}} \frac{\lambda}{\gamma_n(\lambda)} > gk^2. \tag{1.60}$$

and $b_n(\epsilon_\star) := \inf_{\lambda\geqslant\epsilon_\star}\gamma_n(\lambda) > 0$. Notice that $\{b_n(\epsilon_\star)\}_{n\geqslant 1}$ is a sequence decreasing to 0 as $n \to \infty$. Set

$$N(\epsilon_\star) := \sup\left\{n | b_n(\epsilon_\star) > \frac{\epsilon_\star}{gk^2}\right\} \in [1, +\infty).$$

For $1 \leqslant n \leqslant N(\epsilon_\star)$,

$$\lim_{\lambda\to\epsilon_\star} \frac{\lambda}{\gamma_n(\lambda)} \leqslant \lim_{\lambda\to\epsilon_\star} \frac{\lambda}{b_n(\epsilon_\star)} = \frac{\epsilon_\star}{b_n(\epsilon_\star)} < gk^2. \tag{1.61}$$

It then follows from (1.60) and (1.61) that we have at least one desired $\lambda_n \in (\epsilon_\star, \sqrt{\frac{g}{L_0}})$ for $1 \leqslant n \leqslant N(\epsilon_\star)$. □

Now, we are able to prove Theorem 0.2.

Proof. Let ϖ_n be an eigenfunction associated with γ_n of $\mathcal{M}Y^{-1}_{x_+,x_-,\lambda}\mathcal{M}$. That means

$$\mathcal{M}Y^{-1}_{x_-,x_+,\lambda_n}\mathcal{M}\varpi_n = \gamma_n(\lambda_n,k)\varpi_n = \frac{\lambda_n}{gk^2}\varpi_n.$$

Hence, $\phi_n = Y^{-1}_{x_-,x_+,\lambda_n}\mathcal{M}\varpi_n \in H^4((x_-,x_+))$ satisfies

$$\lambda_n Y_{x_-,x_+,\lambda_n}\phi_n = gk^2\rho_0'\phi_n$$

on (x_-,x_+). In order to conclude that ϕ_n is a solution of (0.27), we then extend ϕ_n on $\mathbf{R}$ by continuity.

Let us take $\lambda = \lambda_n$ in the formulas of $U^+_{1,2}$ from (1.46) and in the formulas of $U^-_{3,4}$ from (1.47). Hence, ϕ_n is of the form

$$\phi_n(x) = B^+_{n,1}U^+_{11}(x,\lambda_n) + B^+_{n,2}U^+_{21}(x,\lambda_n)$$

as $x \geqslant x_+$ and

$$\phi_n(x) = B^-_{n,3}U^-_{31}(x,\lambda_n) + B^-_{n,4}U^-_{41}(x,\lambda_n)$$

as $x \leqslant x_-$ for some real constants $B^+_{n,1}, B^+_{n,2}, B^-_{n,3}$ and $B^-_{n,4}$. The constants $B^+_{n,1}, B^+_{n,2}$ are defined by

$$\begin{cases} \phi_n(x_+) = B^+_{n,1}U^+_{11}(x_+,\lambda_n) + B^+_{n,2}U^+_{21}(x_+,\lambda_n) \\ \phi_n'(x_+) = B^+_{n,1}U^+_{12}(x_+,\lambda_n) + B^+_{n,2}U^+_{22}(x_+,\lambda_n). \end{cases} \tag{1.62}$$

Similarly, we have the system for $B^-_{n,3}$ and $B^-_{n,4}$ is

$$\begin{cases} \phi_n(x_-) = B^-_{n,3}U^-_{31}(x_-,\lambda_n) + B^-_{n,4}U^-_{41}(x_-,\lambda_n), \\ \phi_n'(x_-) = B^-_{n,3}U^-_{32}(x_-,\lambda_n) + B^-_{n,4}U^-_{42}(x_-,\lambda_n). \end{cases} \tag{1.63}$$

Solving (1.62) and (1.63), we obtain that

$$B^+_{n,1} = \frac{U^+_{22}(x_+,\lambda_n)\phi_n(x_+) - U^+_{21}(x_+,\lambda_n)\phi_n'(x_+)}{(U^+_{11}U^+_{22} - U^+_{12}U^+_{21})(x_+,\lambda_n)},$$

that

$$B^+_{n,2} = \frac{-U^+_{12}(x_+,\lambda_n)\phi_n(x_+) + U^+_{11}(x_+,\lambda_n)\phi_n'(x_+)}{(U^+_{11}U^+_{22} - U^+_{12}U^+_{21})(x_+,\lambda_n)},$$

that

$$B^-_{n,3} = \frac{U^-_{42}(x_-,\lambda_n)\phi_n(x_-) - U^-_{41}(x_-,\lambda_n)\phi_n'(x_-)}{(U^-_{31}U^-_{42} - U^-_{41}U^-_{32})(x_-,\lambda_n)},$$

and that

$$B^-_{n,4} = \frac{-U^-_{32}(x_-,\lambda_n)\phi_n(x_-) + U^-_{31}(x_-,\lambda_n)\phi_n'(x_-)}{(U^-_{31}U^-_{42} - U^-_{41}U^-_{32})(x_-,\lambda_n)}.$$

Therefore, we get that ϕ_n is a regular solution of (0.27) as $\lambda = \lambda_n$ for each $1 \leqslant n \leqslant N(\epsilon_\star)$. This ends the proof of Theorem 0.2. ☐

Remark 1.2. *1. For $|x_\pm|$ large enough, the investigation of regular solutions to Eq. (0.27) on the real line is equivalent to that one on (x_-, x_+) with boundary conditions (0.37) and (0.38) at $x_\pm$. More computations are required in the second case due to the lack of compact assumption on ρ_0'.*

2. For $|x_\pm|$ large enough, the problem (0.27) on (x_-, x_+) with boundary conditions (0.37) and (0.38) is equivalent to a weaker version of (0.27) that can be solved by applying Riesz's representation theorem on the bilinear form $\mathcal{B}_{x_-,x_+,\lambda}$ and then improving the regularity.

3. Generally speaking, a problem on the real line with decaying solutions at infinity and transversality hypotheses is equivalent to a problem with the compact setting when we have enough decays on the solutions at $\pm\infty$.

1.3.4 Explicit construction and refined estimates of the decaying solutions at infinity

In the regime $0 < \epsilon_\star \leqslant \lambda \leqslant \sqrt{\frac{g}{L_0}}$, we notice again that $R(\lambda)$ is uniformly bounded. So that, as $0 < \epsilon_\star \leqslant \lambda \leqslant \sqrt{\frac{g}{L_0}}$,, we further derive a control of the inner region (x_-, x_+) independent of λ in the following proposition, extending the result of Lemma 1.4 and Proposition 1.8.

Proposition 1.11. *Let $\epsilon_\star > 0$ given and let $\lambda \geqslant \epsilon_\star$. Let $z_{+,\epsilon_\star}$ (respectively, $z_{-,\epsilon_\star}$) be the sum of two upper bounds, which are functions decreasing towards 0 at $+\infty$ (respectively, at $-\infty$), on the r.h.s. of (1.84) and (1.85) (respectively, (1.88) and (1.89)). There exist positive constants $\Gamma_\pm(\epsilon_\star)$ such that, for all x_+, x_- satisfy*

$$z_{+,\epsilon_\star}(x_+) \leqslant \Gamma_+(\epsilon_\star) \quad \textit{and} \quad z_{-,\epsilon_\star}(x_-) \leqslant \Gamma_-(\epsilon_\star), \tag{1.64}$$

we have $\mathcal{B}_{x_-,x_+,\lambda}$ is coercive.

The proof of Proposition 1.11 relies on the refined estimates of the bounded solutions of (1.44) near ∞, presented in Propositions 1.13, 1.14. Before going to the proof of Propositions 1.13, 1.14, thus Proposition 1.11, we present some materials. Notice from (1.42) that one has $L(x,\lambda) = P(x,\lambda)D(x,\lambda)P(x,\lambda)^{-1}$, where

$$D(x,\lambda) = \operatorname{diag}(-k, -\sigma_0(x,\lambda), k, \sigma_0(x,\lambda)), \tag{1.65}$$

$$P(x,\lambda) = \begin{pmatrix} -k^{-3} & -\sigma_0^{-3}(x,\lambda) & k^{-3} & \sigma_0^{-3}(x,\lambda) \\ k^{-2} & \sigma_0^{-2}(x,\lambda) & k^{-2} & \sigma_0^{-2}(x,\lambda) \\ -k^{-1} & -\sigma_0^{-1}(x,\lambda) & k^{-1} & \sigma_0^{-1}(x,\lambda) \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{1.66}$$

and

$$P(x,\lambda)^{-1} = \frac{\mu}{2\lambda\rho_0(x)}\begin{pmatrix} -k^3\sigma_0^2(x,\lambda) & k^2\sigma_0^2(x,\lambda) & k^3 & -k^2 \\ k^2\sigma_0^3(x,\lambda) & -k^2\sigma_0^2(x,\lambda) & -\sigma_0^3(x,\lambda) & \sigma_0^2(x,\lambda) \\ k^3\sigma_0^2(x,\lambda) & k^2\sigma_0^2(x,\lambda) & -k^3 & -k^2 \\ -k^2\sigma_0^3(x,\lambda) & -k^2\sigma_0^2(x,\lambda) & \sigma_0^3(x,\lambda) & \sigma_0^2(x,\lambda) \end{pmatrix} \tag{1.67}$$

The columns of matrix P are denoted by P_1, P_2, P_3, P_4 and P_2, P_4 depend on (x,λ). Note that for every positive k,λ and μ, P and P^{-1} are bounded uniformly in $\mathbf{R}$ and P^{-1} becomes singular when $\lambda \to 0$ and x is fixed.

Then, we set $U(x) = P(x,\lambda)V(x)$, Eq. (1.44) becomes

$$V'(x) = (D(x,\lambda) + \rho_0'(x)M(x,\lambda))V(x), \tag{1.68}$$

where

$$M(x,\lambda) = P(x,\lambda)^{-1}R(\lambda)P(x,\lambda) - \frac{d\sigma_0(x,\lambda)}{d\rho_0(x)}P(x,\lambda)^{-1}\frac{dP(x,\lambda)}{d\sigma_0(x,\lambda)}.$$

Lemma 1.5. *Let*

$$\delta(\epsilon_\star) := \sqrt{k^2 + \frac{\epsilon_\star\rho_-}{\mu}} \quad \textit{and} \quad \delta_s := \sqrt{k^2 + \sqrt{\frac{g}{L_0}}\frac{\rho_+}{\mu}}.$$

For any $\lambda \in [\epsilon_\star, \sqrt{\frac{g}{L_0}}]$, *there hold*

$$\sup_{x\in\mathbf{R}} \|P(x,\lambda)\|_F \leqslant \Gamma_p := \max\Big(1, \frac{1}{k}, \frac{1}{k^2}, \frac{1}{k^3}\Big) \tag{1.69}$$

and

$$\begin{aligned}\sup_{x\in\mathbf{R}} \|M(x,\lambda)\|_F \leqslant \Gamma_m(\epsilon_\star) := &\frac{1}{\rho_-\epsilon_\star^2}\max\Big(g\Big(k+\frac{1}{L_0}\Big), g\Big(\frac{k^2}{\delta(\epsilon_\star)}+\frac{1}{L_0}\Big)\Big) \\ &+ \frac{1}{4\delta(\epsilon_\star)}\sqrt{\frac{g}{L_0}}\max\Big(\frac{2k^2}{\delta^2(\epsilon_\star)}(k+\delta_s), \frac{5k^2}{\delta(\epsilon_\star)}+\delta_s, \frac{k^2}{\delta(\epsilon_\star)}+\delta_s\Big).\end{aligned} \tag{1.70}$$

Proof. Due to $\epsilon_\star \leqslant \lambda \leqslant \sqrt{\frac{g}{L_0}}$, we get that $\delta(\epsilon_\star) < \sigma_0(x,\lambda) \leqslant \delta_s$ for all $x \in \mathbf{R}$, it yields (1.69). We move to demonstrate (1.70). Let

$$a_\pm(\lambda) = gk \pm \lambda^2, \quad b_\pm(x,\lambda) = \frac{gk^2}{\sigma_0(x,\lambda)} \pm \lambda^2$$

and

$$c_\pm(x,\lambda) = k \pm \sigma_0(x,\lambda), \quad d(x,\lambda) = \frac{5k^2}{\sigma_0(x,\lambda)} - \sigma_0(x,\lambda).$$

Direct computations show that

$$\frac{d\sigma_0(x,\lambda)}{d\rho_0(x)} = \frac{\lambda}{2\mu}\frac{\rho_0'(x)}{\sigma_0(x,\lambda)},$$

that

$$P(x,\lambda)^{-1}R(\lambda)P(x,\lambda) = \frac{1}{2\lambda^2\rho_0(x)}\begin{pmatrix} a_- & \frac{k^2b_-}{\sigma_0^2} & -a_+ & -\frac{k^2b_+}{\sigma_0^2} \\ -\frac{a_-\sigma_0^2}{k^2} & -b_- & \frac{a_+\sigma_0^2}{k^2} & -b_+ \\ a_- & \frac{k^2b_-}{\sigma_0^2} & -a_+ & -\frac{k^2b_+}{\sigma_0^2} \\ -\frac{a_-\sigma_0^2}{k^2} & -b_- & \frac{a_+\sigma_0^2}{k^2} & -b_+ \end{pmatrix}(x,\lambda)$$

and that

$$P(x,\lambda)^{-1}\frac{dP(x,\lambda)}{d\sigma_0(x,\lambda)} = \frac{\mu}{2\lambda\rho_0(x)}\begin{pmatrix} 0 & -\frac{2k^2c_+}{\sigma_0^2} & 0 & \frac{2k^2c_-}{\sigma_0^2} \\ 0 & d & 0 & -\frac{k^2}{\sigma_0}+\sigma_0 \\ 0 & \frac{2k^2c_-}{\sigma_0^2} & 0 & -\frac{2k^2c_+}{\sigma_0^2} \\ 0 & -\frac{k^2}{\sigma_0}+\sigma_0 & 0 & d \end{pmatrix}(x,\lambda).$$

For all $x \in \mathbf{R}$, it is clear that

$$|a_\pm(\lambda)| \leqslant g\Big(k + \frac{1}{L_0}\Big), \quad |b_\pm(x,\lambda)| \leqslant g\Big(\frac{k^2}{\delta(\epsilon_\star)} + \frac{1}{L_0}\Big)$$

and that

$$|c_\pm(x,\lambda)| \leqslant k + \delta_s, \quad |d(x,\lambda)| \leqslant \frac{5k^2}{\delta(\epsilon_\star)} + \delta_s.$$

Therefore, (1.70) follows. Proof of Lemma 1.5 is complete. □

Let $\tilde{x}_+$ be chosen such that

$$\int_{\tilde{x}_+}^{+\infty} \rho_0'(\tau)\|M(\tau,\lambda)\|_F d\tau \leqslant \Gamma_m(\epsilon_\star)(\rho_+ - \rho_0(\tilde{x}_+)) < \frac{1}{2} \tag{1.71}$$

and $\tilde{x}_-$ be chosen such that

$$\int_{-\infty}^{\tilde{x}_-} \rho_0'(\tau)\|M(\tau,\lambda)\|_F d\tau \leqslant \Gamma_m(\epsilon_\star)(\rho_0(\tilde{x}_-) - \rho_-) < \frac{1}{2}. \tag{1.72}$$

Let $\alpha_\pm(x)$ and $\beta_\pm(x,\lambda)$ be defined

$$\alpha_\pm(x) = \pm k(x - \tilde{x}_\pm), \quad \beta_\pm(x,\lambda) = \pm\int_{\tilde{x}_\pm}^{x} \sigma_0(y,\lambda)dy. \tag{1.73}$$

We then study the solutions of (1.68) decaying to 0 at $+\infty$.

Proposition 1.12. *Eq. (1.68) on $(\tilde{x}_+, +\infty)$ admits a unique solution $V_1(x,\lambda)$ such that $e^{-\alpha_+(x)}V_1(x,\lambda)$ converges to e_1 as $x \to +\infty$ and a unique solution $V_2(x,\lambda)$ such that $e^{-\beta_+(x,\lambda)}V_2(x,\lambda)$ converges to e_2 as $x \to +\infty$. Furthermore, we have the following estimates*

$$\|e^{\alpha_+(x)}V_1(x,\lambda) - e_1\|_2 \leqslant 2\Gamma_m(\epsilon_\star)\left(\begin{aligned}&(\rho_+ - \rho_0(x)) + \rho_0(\tilde{x}_+)e^{-(\delta(\epsilon_\star)-k)(x-\tilde{x}_+)} \\ &+ \Big|\rho_0(x) - (\delta(\epsilon_\star) - k)\int_{\tilde{x}_+}^{x}\rho_0(\tau)e^{-(\delta(\epsilon_\star)-k)(x-\tau)}d\tau\Big|\end{aligned}\right) \tag{1.74}$$

and

$$\|e^{\beta_+(x,\lambda)}V_2(x,\lambda) - e_2\|_2 \leqslant 2\Gamma_m(\epsilon_\star)(\rho_+ - \rho_0(x)) \tag{1.75}$$

for all $x \geqslant \tilde{x}_+$.

Proof. We define the matrices

$$\Psi(x,\lambda) = \operatorname{diag}(e^{-\alpha_+(x)}, e^{-\beta_+(x,\lambda)}, e^{\alpha_+(x)}, e^{\beta_+(x,\lambda)}),$$

$$\Psi_1(x,\lambda) = \operatorname{diag}(0, e^{-\beta_+(x,\lambda)}, 0, 0)$$

and

$$\Psi_2(x,\lambda) = \operatorname{diag}(e^{-\alpha_+(x)}, 0, e^{\alpha_+(x)}, e^{\beta_+(x,\lambda)}).$$

Then, we consider the equation

$$\begin{aligned} V_1(x,\lambda) = e^{-\alpha_+(x)}e_1 &+ \int_{\tilde{x}_+}^{x} \Psi_1(x,\lambda)\Psi^{-1}(\tau,\lambda)\rho_0'(\tau)M(\tau,\lambda)V_1(\tau,\lambda)d\tau \\ &- \int_{x}^{+\infty} \Psi_2(x,\lambda)\Psi(\tau,\lambda)^{-1}\rho_0'(\tau)M(\tau,\lambda)V_1(\tau,\lambda)d\tau. \end{aligned} \tag{1.76}$$

It can be seen that a solution V_1 of (1.76) satisfies (1.68). We solve (1.76) by the Picard iteration method. Indeed, let $V_1^{(0)}(x,\lambda) = 0$ and

$$\begin{aligned} V_1^{(j+1)}(x,\lambda) = e^{-\alpha_+(x)}e_1 &+ \int_{\tilde{x}_+}^{x} \Psi_1(x,\lambda)\Psi^{-1}(\tau,\lambda)\rho_0'(\tau)M(\tau,\lambda)V_1^{(j)}(\tau,\lambda)d\tau \\ &- \int_{x}^{+\infty} \Psi_2(x,\lambda)\Psi^{-1}(\tau,\lambda)\rho_0'(\tau)M(\tau,\lambda)V_1^{(j)}(\tau,\lambda)d\tau. \end{aligned}$$

We have that

$$\Psi_1(x,\lambda)\Psi^{-1}(\tau,\lambda) = \operatorname{diag}(0, e^{-(\beta_+(x,\lambda)-\beta_+(\tau,\lambda))}, 0, 0)$$

and that

$$\Psi_2(x,\lambda)\Psi^{-1}(\tau,\lambda) = \operatorname{diag}(e^{-(\alpha_+(\tau)-\alpha_+(x))}, 0, e^{\alpha_+(\tau)-\alpha_+(x)}, e^{-(\beta_+(x,\lambda)-\beta_+(\tau,\lambda))}).$$

Hence, we can estimate for $\tilde{x}_+ \leqslant \tau \leqslant x$,

$$\begin{aligned} \|\Psi_1(x,\lambda)\Psi^{-1}(\tau,\lambda)\|_F &\leqslant e^{-(\alpha_+(x)-\alpha_+(\tau))+\int_\tau^x (k-\sigma_0(s))ds} \\ &\leqslant e^{-(\alpha_+(x)-\alpha_+(\tau))-(\delta(\epsilon_\star)-k)(x-\tau)} \end{aligned} \tag{1.77}$$

and for $\tau \geqslant x$,

$$\|\Psi_2(x,\lambda)\Psi^{-1}(\tau,\lambda)\|_F \leqslant e^{-(\alpha_+(x)-\alpha_+(\tau))}. \tag{1.78}$$

Using (1.77) and (1.78), we get

$$\begin{aligned} &e^{\alpha_+(x)}\|V_1^{(j+1)}(x,\lambda) - V_1^{(j)}(x,\lambda)\|_2 \\ &\leqslant \Gamma_m(\epsilon_\star)\int_{\tilde{x}_+}^{\infty} e^{\alpha_+(\tau)}\rho_0'(\tau)\|V_1^{(j)}(\tau,\lambda) - V_1^{(j-1)}(\tau,\lambda)\|_F d\tau. \end{aligned}$$

Thanks to the induction, we get for all $x \geqslant \tilde{x}_+$ and for all $j \geqslant 0$,

$$e^{\alpha_+(x)}\|V_1^{(j+1)}(x,\lambda) - V_1^{(j)}(x,\lambda)\|_2 \leqslant \left(\frac{1}{2}\right)^j, \tag{1.79}$$

yielding the uniform convergence of $\{V_1^{(j)}(x,\lambda)\}_{j\geqslant 0}$ on any subinterval of $(\tilde{x}_+,+\infty)$. Let $V_1(x,\lambda)$ be the limit function. $V_1^{(j)}(x,\lambda)$ is continuous, so is $V_1(x,\lambda)$. Moreover, (1.79) implies that

$$e^{\alpha_+(x)}\|V_1^{(j+1)}(x,\lambda)\|_2 \leqslant \sum_{i=0}^{j} e^{\alpha_+(x)}\|V_1^{(i+1)}(x,\lambda)-V_1^{(i)}(x,\lambda)\| \leqslant \sum_{i=0}^{j}\Big(\frac{1}{2}\Big)^i.$$

That tells us for $x\geqslant \tilde{x}_+$,

$$\|V_1(x,\lambda)\|_2 \leqslant 2e^{-\alpha_+(x)}. \tag{1.80}$$

Once we have (1.80), we then prove (1.74). Indeed,

$$\begin{aligned} e^{\alpha_+(x)}V_1(x,\lambda)-e_1 &= e^{\alpha_+(x)}\int_{\tilde{x}_+}^{x}\Psi_1(x,\lambda)\Psi^{-1}(\tau,\lambda)\rho_0'(\tau)M(\tau,\lambda)V_1(\tau,\lambda)d\tau \\ &\quad - e^{\alpha_+(x)}\int_x^{+\infty}\Psi_2(x,\lambda)\Psi(\tau,\lambda)^{-1}\rho_0'(\tau)M(\tau,\lambda)V_1(\tau,\lambda)d\tau. \end{aligned}$$

We make use of (1.78) and (1.80) to have that

$$e^{\alpha_+(x)}\int_x^{+\infty}\|\Psi_2(x,\lambda)\Psi^{-1}(\tau,\lambda)\rho_0'(\tau)M(\tau,\lambda)V_1(\tau,\lambda)\|_2 d\tau \leqslant 2\Gamma_m(\epsilon_\star)(\rho_+-\rho_0(x)). \tag{1.81}$$

From (1.77) and (1.80), we obtain that

$$\begin{aligned} &\|e^{\alpha_+(x)}\int_{\tilde{x}_+}^{x}\Psi_1(x,\lambda)\Psi^{-1}(\tau,\lambda)\rho_0'(\tau)M(\tau,\lambda)V_1(\tau,\lambda)d\tau\|_2 \\ &\qquad\leqslant 2\Gamma_m(\epsilon_\star)\int_{\tilde{x}_+}^{x}\rho_0'(\tau)e^{-(\delta(\epsilon_\star)-k)(x-\tau)}d\tau. \end{aligned} \tag{1.82}$$

After integrating by parts, we get

$$\begin{aligned} \int_{\tilde{x}_+}^{x}\rho_0'(\tau)e^{-(\delta(\epsilon_\star)-k)(x-\tau)}d\tau &= -\rho_0(\tilde{x}_+)e^{-(\delta(\epsilon_\star)-k)(x-\tilde{x}_+)}+\rho_0(x) \\ &\quad -(\delta(\epsilon_\star)-k)\int_{\tilde{x}_+}^{x}\rho_0(\tau)e^{-(\delta(\epsilon_\star)-k)(x-\tau)}d\tau. \end{aligned} \tag{1.83}$$

Combining (1.81), (1.82) and (1.83) gives (1.74).

By considering the eigenvalue $-\sigma_0(x,\lambda)$ of $L(x,\lambda)$, we continue the idea in Theorem 1.2 and mimic the above arguments to the solution $V_2(x,\lambda)$ such that $e^{\beta_+(x,\lambda)}V_2(x,\lambda)$ converges to e_2 at $+\infty$ and $V_2(x,\lambda)$ enjoys (1.75). This ends the proof of Lemma 1.12. □

Now, we get back to (1.44) to find solutions that are bounded near $+\infty$.

Proposition 1.13. *Eq.* (1.44) *on* $(\tilde{x}_+,+\infty)$ *admits*

1. *a unique solution* $U_1^+(x,\lambda)$ *satisfying that as* $x \to +\infty$, $e^{\alpha_+(x)}U_1^+(x,\lambda)$ *converges to* $(-k^{-3}, k^{-2}, -k^{-1}, 1)^T$ *and that for all* $x \geqslant \tilde{x}_+$,

$$\begin{aligned} &\|e^{\alpha_+(x)}U_1^+(x,\lambda) - (-k^{-3}, k^{-2}, -k^{-1}, 1)^T\|_2 \\ &\leqslant 2\Gamma_p\Gamma_m(\epsilon_\star)\left(\begin{aligned} &(\rho_+ - \rho_0(x)) + \rho_0(\tilde{x}_+)e^{-(\delta(\epsilon_\star)-k)(x-\tilde{x}_+)} \\ &+ \left|\rho_0(x) - (\delta(\epsilon_\star) - k)\int_{\tilde{x}_+}^{x} \rho_0(\tau)e^{-(\delta(\epsilon_\star)-k)(x-\tau)}d\tau\right| \end{aligned}\right), \end{aligned} \tag{1.84}$$

2. *a unique solution* $U_2^+(x,\lambda)$ *satisfying that as* $x \to +\infty$, $e^{\beta_+(x,\lambda)}U_2^+(x,\lambda)$ *converges to* $(-\sigma_+^{-3}(\lambda), \sigma_+^{-2}(\lambda), -\sigma_+^{-1}(\lambda), 1)^T$ *and that for all* $x \geqslant \tilde{x}_+$,

$$\begin{aligned} &\|e^{\beta_+(x,\lambda)}U_2^+(x,\lambda) - (-\sigma_+^{-3}(\lambda), \sigma_+^{-2}(\lambda), -\sigma_+^{-1}(\lambda), 1)^T\|_2 \\ &\leqslant \Big(\sqrt{\frac{g(4\delta^{10}(\epsilon_\star) + 16\delta^{12}(\epsilon_\star) + 9\delta_s^4)}{16L_0\mu^2\delta^{16}(\epsilon_\star)}} + 2\Gamma_p\Gamma_m(\epsilon_\star)\Big)(\rho_+ - \rho_0(x)). \end{aligned} \tag{1.85}$$

Proof. We define $U_j^+(x,\lambda) = P(x,\lambda)V_j(x,\lambda)(j = 1,2)$, with V_1 and V_2 are two solutions of (1.68) satisfying (1.74) and (1.75) respectively. It can be seen that $U_1^+(x,\lambda)$ and $U_2^+(x,\lambda)$ are two solutions of (1.44).

Note that

$$\begin{aligned} e^{\alpha_+(x)}U_1^+(x,\lambda) &= P_1 + P(x,\lambda)(e^{\alpha_+(x)}V_1(x,\lambda) - e_1) \\ &= (-k^{-3}, k^{-2}, -k^{-1}, 1)^T + P(x,\lambda)(e^{\alpha_+(x)}V_1(x,\lambda) - e_1). \end{aligned}$$

The inequality (1.84) is then clear due to the estimate (1.74). According to l'Hôpital's rule, we have

$$\lim_{x\to+\infty} \frac{\int_{\tilde{x}_+}^{x} \rho_0(\tau)e^{(\delta(\epsilon_\star)-k)\tau}d\tau}{e^{(\delta(\epsilon_\star)-k)x}} = \lim_{x\to+\infty} \frac{\rho_0(x)e^{(\delta(\epsilon_\star)-k)x}}{(\delta(\epsilon_\star)-k)e^{(\delta(\epsilon_\star)-k)x}} = \frac{\rho_+}{\delta(\epsilon_\star)-k},$$

that implies

$$\lim_{x\to+\infty} \left|\rho_0(x) - (\delta(\epsilon_\star) - k)\int_{\tilde{x}_+}^{x} \rho_0(\tau)e^{-(\delta(\epsilon_\star)-k)(x-\tau)}d\tau\right| = 0.$$

The behavior of $U_1^+(x,\lambda)$ at $+\infty$ follows.

To prove (1.85), we write

$$\begin{aligned} &e^{\beta_+(x,\lambda)}U_2^+(x,\lambda) - (-\sigma_+^{-3}(\lambda), \sigma_+^{-2}(\lambda), -\sigma_+^{-1}(\lambda), 1)^T \\ &= P_2(x,\lambda) - (-\sigma_+^{-3}(\lambda), \sigma_+^{-2}(\lambda), -\sigma_+^{-1}(\lambda), 1)^T + P(x,\lambda)(e^{\beta_+(x,\lambda)}V_2(x,\lambda) - e_2). \end{aligned}$$

Since $\delta(\epsilon_\star) < \sigma_+(\lambda) < \delta_s$ for all $\lambda \in [\epsilon_\star, \sqrt{\frac{g}{L_0}}]$, we bound that

$$\begin{aligned} &\|P_2(x,\lambda) - (-\sigma_+^{-3}(\lambda), \sigma_+^{-2}(\lambda), -\sigma_+^{-1}(\lambda), 1)^T\|_2^2 \\ &= \frac{\lambda^2}{\mu^2}(\rho_0(x) - \rho_+)^2 \left[\begin{aligned} &\frac{1}{\sigma_0^2(x,\lambda)\sigma_+^2(\lambda)(\sigma_0(x,\lambda)+\sigma_+(\lambda))^2} + \frac{1}{\sigma_0^2(x,\lambda)\sigma_+^2(\lambda)} \\ &+ \Big(\frac{\sigma_0^2(x,\lambda) + \sigma_0(x,\lambda)\sigma_+(\lambda) + \sigma_+^2(\lambda)}{\sigma_0^3(x,\lambda)\sigma_+^3(\lambda)(\sigma_0(x,\lambda)+\sigma_+(\lambda))}\Big)^2 \end{aligned}\right] \\ &\leqslant \frac{g(4\delta^{10}(\epsilon_\star) + 16\delta^{12}(\epsilon_\star) + 9\delta_s^4)}{16L_0\mu^2\delta^{16}(\epsilon_\star)}(\rho_0(x) - \rho_+)^2. \end{aligned} \tag{1.86}$$

Meanwhile, as a result of (1.75),

$$\|P(x,\lambda)(e^{\beta_+(x,\lambda)}V_2(x,\lambda)-e_2)\|_2 \leqslant 2\Gamma_p\Gamma_m(\epsilon_\star)(\rho_+-\rho_0(x)). \tag{1.87}$$

Thanks to (1.86) and (1.87), we obtain (1.85) hence the behavior of $U_2^+(x,\lambda)$ at $+\infty$. Proof of Proposition 1.13 is complete. □

We now fix two positive eigenvalues of $L(x,\lambda)$, k and $\sigma_0(x,\lambda)$ and thus follow Theorem 1.2 again. We are able to construct solutions of (1.44), which are bounded near $-\infty$ as in Proposition 1.13.

Proposition 1.14. *Eq. (1.44) on $(-\infty,\tilde{x}_-)$ admits*

1. *a unique solution $U_3^-(x,\lambda)$ satisfying that as $x\to-\infty$, $e^{\alpha_-(x)}U_3^-(x,\lambda)$ converges to $(k^{-3},k^{-2},k^{-1},1)^T$ and that, for all $x\leqslant\tilde{x}_-$*

$$\begin{aligned}&\|e^{\alpha_-(x)}U_3^-(x,\lambda)-(k^{-3},k^{-2},k^{-1},1)^T\|_2\\ &\leqslant 2\Gamma_p\Gamma_m(\epsilon_\star)\left(\begin{aligned}&(\rho_0(x)-\rho_-)+\rho_0(\tilde{x}_-)e^{-(\delta(\epsilon_\star)-k)(\tilde{x}_--x)}\\ &+\left|\rho_0(x)-(\delta(\epsilon_\star)-k)\int_x^{\tilde{x}_-}\rho_0(\tau)e^{-(\delta(\epsilon_\star)-k)(\tau-x)}d\tau\right|\end{aligned}\right),\end{aligned} \tag{1.88}$$

2. *a unique solution $U_4^-(x,\lambda)$ satisfying that as $x\to-\infty$, $e^{\beta_-(x,\lambda)}U_4^-(x,\lambda)$ converges to $(\sigma_-^{-3}(\lambda),\sigma_-^{-2}(\lambda),\sigma_-^{-1}(\lambda),1)^T$ and that for all $x\leqslant\tilde{x}_-$,*

$$\begin{aligned}&\|e^{\beta_-(x,\lambda)}U_4^-(x,\lambda)-(\sigma_-^{-3}(\lambda),\sigma_-^{-2}(\lambda),\sigma_-^{-1}(\lambda),1)^T\|_2\\ &\leqslant\left(\sqrt{\frac{g(4\delta^{10}(\epsilon_\star)+16\delta^{12}(\epsilon_\star)+9\delta_s^4)}{16L_0\mu^2\delta^{16}(\epsilon_\star)}}+2\Gamma_p\Gamma_m(\epsilon_\star)\right)(\rho_0(x)-\rho_-).\end{aligned} \tag{1.89}$$

We are now in position to prove Proposition 1.11.

Proof. We recall n_{ij}^+ $(i,j=1,2)$ from (1.54) and (1.55) to have that

$$\begin{pmatrix}n_{11}^+\\ n_{12}^+\end{pmatrix}(x_+,\lambda)=-\frac{1}{U_{11}^+U_{22}^+-U_{21}^+U_{12}^+}\begin{pmatrix}U_{22}^+ & -U_{12}^+\\ -U_{21}^+ & U_{11}^+\end{pmatrix}\begin{pmatrix}U_{13}^+\\ U_{23}^+\end{pmatrix}(x_+,\lambda) \tag{1.90}$$

and

$$\begin{pmatrix}n_{21}^+\\ n_{22}^+\end{pmatrix}(x_+,\lambda)=-\frac{1}{U_{11}^+U_{22}^+-U_{21}^+U_{12}^+}\begin{pmatrix}U_{22}^+ & -U_{12}^+\\ -U_{21}^+ & U_{11}^+\end{pmatrix}\begin{pmatrix}U_{14}^+\\ U_{24}^+\end{pmatrix}(x_+,\lambda). \tag{1.91}$$

We are ready to prove the estimates (1.50) needed for Lemma 1.4. Now using (1.84) and (1.85) into (1.90) yields that

$$n_{12}^+(x_+,\lambda)=\frac{U_{21}^+U_{13}^+-U_{11}^+U_{23}^+}{U_{11}^+U_{22}^+-U_{21}^+U_{12}^+}(x_+,\lambda)=\frac{k+\sigma_+(\lambda)+f_1(x_+,\lambda)}{1+f_2(x_+,\lambda)},$$

where $|f_j(x,\lambda)| = O(z_{+,\epsilon_\star}(x))$ $(j = 1, 2)$ uniformly in $\lambda \in [\epsilon_\star, \sqrt{\frac{g}{L_0}}]$ as $x \to \infty$. Hence, there exists a constant $\xi_+(\epsilon_\star) > 0$ such that

$$\begin{aligned} n_{12}^+(x_+,\lambda) &\geqslant k + \sigma_+(\lambda) - \xi_+(\epsilon_\star) z_{+,\epsilon_\star}(x_+) \\ &\geqslant k + \delta(\epsilon_\star) - \xi_+(\epsilon_\star) z_{+,\epsilon_\star}(x_+). \end{aligned}$$

That implies $n_{12}^+(x_+,\lambda) > 0$ if

$$z_{+,\epsilon_\star}(x_+) < \frac{k + \delta(\epsilon_\star)}{\xi_+(\epsilon_\star)}. \tag{1.92}$$

We estimate

$$\Delta_+(x_+,\lambda) := (n_{11}^+(x_+,\lambda) - n_{22}^+(x_+,\lambda) - k^2 - \sigma_0^2(x_+,\lambda))^2 + 4n_{12}^+ n_{21}^+(x_+,\lambda)$$

From (1.84) and (1.85) again, we obtain from (1.90) and (1.91) that

$$\begin{aligned} n_{11}^+(x_+,\lambda) &= k\sigma_+(\lambda) + O(z_{+,\epsilon_\star}(x_+)), \\ n_{22}^+(x_+,\lambda) + k^2 + \sigma_0^2(x_+,\lambda) &= -k\sigma_+(\lambda) + O(z_{+,\epsilon_\star}(x_+)), \\ n_{21}^+(x_+,\lambda) &= -k\sigma_+(\lambda)(k + \sigma_+(\lambda)) + O(z_{+,\epsilon_\star}(x_+)). \end{aligned}$$

Hence, there exists a constant $w_+(\epsilon_\star) > 0$ such that

$$\begin{aligned} \Delta_+(x_+,\lambda) &\leqslant -4k\sigma_+(\lambda)(k^2 + k\sigma_+(\lambda) + \sigma_+^2(\lambda)) + w_+(\epsilon_\star) z_{+,\epsilon_\star}(x_+) \\ &\leqslant -4k\delta(\epsilon_\star)(k^2 + k\delta(\epsilon_\star) + \delta^2(\epsilon_\star)) + w_+(\epsilon_\star) z_{+,\epsilon_\star}(x_+). \end{aligned}$$

The inequality $\Delta_+(x_+,\lambda) \leqslant 0$ is equivalent to

$$z_{+,\epsilon_\star}(x_+) \leqslant \frac{4k\delta(\epsilon_\star)(k^2 + k\delta(\epsilon_\star) + \delta^2(\epsilon_\star))}{w_+(\epsilon_\star)}. \tag{1.93}$$

Combining (1.92) and (1.93), we take

$$\Gamma_+(\epsilon_\star) = \min\Big(\frac{k + \delta(\epsilon_\star)}{\xi_+(\epsilon_\star)}, \frac{4k\delta(\epsilon_\star)(k^2 + k\delta(\epsilon_\star) + \delta^2(\epsilon_\star))}{w_+(\epsilon_\star)}\Big).$$

If x_+ satisfies $z_{+,\epsilon_\star}(x_+) \leqslant \Gamma_+(\epsilon_\star)$, then one has $n_{12}^+(x_+,\lambda) > 0 \geqslant \Delta_+(x_+,\lambda)$, i.e. (1.50). That implies $BV_{x_+,\lambda}(\vartheta,\vartheta) \geqslant 0$.

Similarly, we get that, from (1.55), we follow the above arguments to show that there exists $\Gamma_-(\epsilon_\star) > 0$ such that for $z_{-,\epsilon_\star}(x_-) \leqslant \Gamma_-(\epsilon_\star)$, (1.51) holds. It yields $BV_{x_-,\lambda}(\vartheta,\vartheta) \geqslant 0$. Proposition 1.11 is proven. □

1.4 A remark on the relation between the formulation on $[x_-, x_+]$ and the formulation on R of the viscous RT problem

We end this chapter by the following remark on the bilinear form $\mathcal{B}_{x_-,x_+,\lambda}$.

Proposition 1.15. *For all bounded solutions ϕ of (0.27) on $\mathbf{R}$ and for all $\theta \in H^2(\mathbf{R})$, there holds*

$$\begin{aligned} \lambda \int_{-\infty}^{+\infty} \rho_0(k^2\phi\theta + \phi'\theta')dx + \mu \int_{-\infty}^{+\infty} (\phi''\theta'' + 2k^2\phi\theta' + k^4\phi\theta)dx \\ = \mathcal{B}_{x_-,x_+,\lambda}(\phi,\theta) + \int_{\mathbf{R}\setminus[x_-,x_+]} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx. \end{aligned} \tag{1.94}$$

We immediately have two remarks from (1.94).

1. In the case of ρ_0' being compactly supported ($\text{supp}\rho_0' = [-a,a]$), we have
$$\lambda \int_{-\infty}^{+\infty} \rho_0(k^2\phi\theta + \phi'\theta')dx + \mu \int_{-\infty}^{+\infty} (\phi''\theta'' + 2k^2\phi\theta' + k^4\phi\theta)dx = \mathcal{B}_{a,\lambda}(\phi,\theta). \tag{1.95}$$
That means $\mathcal{B}_{x_-,x_+,\lambda}$ is independent of $x_\pm$ if and only if $x_- \leqslant -a < a \leqslant x_+$.
2. In the case $\rho_0' > 0$ everywhere, for each (x_-, x_+), a penalization of $\mathcal{B}_{x_-,x_+,\lambda}$ by the term $\int_{x_-}^{x_+} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx$ is necessary to obtain the ODE (0.27) on the whole space.

Proof of Proposition 1.15. To prove (1.94), we show two following identities

$$\begin{aligned} \int_{x_+}^{+\infty} \lambda\rho_0(k^2\phi\theta + \phi'\theta')dx + \mu \int_{x_+}^{+\infty} (k^4\phi\theta + 2k^2\phi'\theta' + \phi''\theta'')dx \\ = \int_{x_+}^{+\infty} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + BV_{x_+,\lambda}(\phi,\theta). \end{aligned} \tag{1.96}$$

and

$$\begin{aligned} \int_{-\infty}^{x_-} \lambda\rho_0(k^2\phi\theta + \phi'\theta')dx + \mu \int_{-\infty}^{x_-} (k^4\phi\theta + 2k^2\phi'\theta' + \phi''\theta'')dx \\ = \int_{-\infty}^{x_-} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + BV_{x_+,\lambda}(\phi,\theta). \end{aligned} \tag{1.97}$$

We have the following remark that we state on $(x_+, +\infty)$ (a similar expression holds true for $(-\infty, x_-)$) that if ϕ is a bounded solution of (0.27) on $(x_+, +\infty)$, we then have from Proposition 1.7 that ϕ satisfies (0.38) at x_+. We integrate by parts to have that

$$\begin{aligned} \int_{x_+}^{+\infty} \phi''\theta'' dx &= \phi''\theta'\Big|_{x_+}^{+\infty} - \phi'''\theta\Big|_{x_+}^{+\infty} + \int_{x_+}^{+\infty} \phi^{(4)}\theta dx \\ &= -(\phi''\theta')(x_+) + (\phi'''\theta)(x_+) + \int_{x_+}^{+\infty} \phi^{(4)}\theta dx. \end{aligned} \tag{1.98}$$

Because of (0.38), we have

$$\phi''(x_+) = -(n_{11}^+\phi(x_+) + n_{12}^+\phi'(x_+)) \tag{1.99}$$

and

$$\phi'''(x_+) = -(n_{21}^+\phi(x_+) + n_{22}^+\phi'(x_+)). \tag{1.100}$$

Substituting (1.99) and (1.100) into (1.98), we obtain

$$\begin{aligned}\int_{x_+}^{+\infty} \phi''\theta'' dx = (n_{11}^+\phi(x_+) + n_{12}^+\phi'(x_+))\theta'(x_+) - (n_{21}^+\phi(x_+) + n_{22}^+\phi'(x_+))\theta(x_+) \\ + \int_{x_+}^{+\infty} \phi^{(4)}\theta dx.\end{aligned} \tag{1.101}$$

Using the integration by parts again, we deduce that

$$\int_{x_+}^{+\infty} \phi'\theta' dx = \phi'\theta\Big|_{x_+}^{+\infty} - \int_{x_+}^{+\infty} \phi''\theta dx = -(\phi'\theta)(x_+) - \int_{x_+}^{+\infty} \phi''\theta dx \tag{1.102}$$

and that

$$\int_{x_+}^{+\infty} \rho_0\phi'\theta' dx = -(\rho_0\phi'\theta)(x_+) - \int_{x_+}^{+\infty} (\rho_0\phi')'\theta dx. \tag{1.103}$$

In view of (1.101), (1.102) and (1.103), we obtain

$$\begin{aligned}&\int_{x_+}^{+\infty} \lambda\rho_0(k^2\phi\theta + \phi'\theta')dx + \mu\int_{x_+}^{+\infty} (k^4\phi\theta + 2k^2\phi'\theta' + \phi''\theta'')dx \\ &= \int_{x_+}^{+\infty} (\lambda(k^2\rho_0\phi - (\rho_0\phi')') + \mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi))\theta dx \\ &\quad + \mu(n_{11}^+\phi(x_+) + n_{12}^+\phi'(x_+))\theta'(x_+) - \mu(n_{21}^+\phi(x_+) + n_{22}^+\phi'(x_+))\theta(x_+) \\ &\quad - 2k^2\mu(\phi'\theta)(x_+) - \lambda(\rho_0\phi'\theta)(x_+) \\ &= \int_{x_+}^{+\infty} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + BV_{x_+,\lambda}(\phi,\theta).\end{aligned} \tag{1.104}$$

Hence, (1.96) follows from (1.104). Similarly, we obtain (1.97). It follows from (1.96) and (1.97) that

$$\begin{aligned}&\lambda\int_{-\infty}^{+\infty} \rho_0(k^2\phi\theta + \phi'\theta')dx + \mu\int_{-\infty}^{+\infty} (\phi''\theta'' + 2k^2\phi\theta' + k^4\phi\theta)dx \\ &= \int_{-\infty}^{x_-} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + \int_{x_+}^{+\infty} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + BV_{x_-,\lambda}(\phi,\theta) + BV_{x_+,\lambda}(\phi,\theta) \\ &\quad + \int_{x_-}^{x_+} \lambda\rho_0(k^2\phi\theta + \phi'\theta')dx + \mu\int_{x_-}^{x_+} (k^4\phi\theta + 2k^2\phi'\theta' + \phi''\theta'')dx \\ &= \int_{-\infty}^{x_-} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + \int_{x_+}^{+\infty} \frac{gk^2\rho_0'}{\lambda}\phi\theta dx + \mathcal{B}_{x_-,x_+,\lambda}(\phi,\theta).\end{aligned} \tag{1.105}$$

It yields (1.94). □

Chapter 2

Nonlinear Rayleigh-Taylor instability of the incompressible viscous fluid with Navier-slip boundary conditions

This chapter is presented in the preprint [61]. For the Rayleigh-Taylor instability for the incompressible viscous fluid with Navier-slip boundary conditions, the search of normal modes is once again equivalent to the investigation of solutions of a fourth-order ODE on a compact interval $(-1,1)$. A point where it differs from the first paper is that, due to the presence of slip coefficients, denoted by Ξ, the spectral analysis is performed in a *supercritical* regime of μ ($\mu > \mu_c(k,\Xi)$). In this regime, we apply the operator method of the previous chapter to prove the existence of infinitely many characteristic values to the linearized equations. As the spectral analysis is proven, we construct a wide class of initial data, based on the existence of infinitely many normal modes of the linearized equations, to prove the nonlinear Rayleigh-Taylor instability. We are not able to prove the nonlinear instability in the supercritical regime $\mu > \sup_{k\in L^{-1}\mathbf{Z}\setminus\{0\}} \mu_c(k,\Xi)$. We have to add a sharper condition, $\mu > 3\sup_{k\in L^{-1}\mathbf{Z}\setminus\{0\}} \mu_c(k,\Xi)$, to prove the nonlinear instability.

2.1 Preliminaries

The first aim is to prove Lemma 2.1, showing that all characteristic values λ are real for increasing density profile ρ_0. Secondly, we find the exact formula of the k-critical viscosity coefficient $\mu_c(k,\Xi)$ (see (0.65) above) for all $k > 0$. The last goal is to study the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ in Section 2.1.3 to prepare for our linear study.

2.1.1 Positivity of characteristic values λ

Lemma 2.1. *For any $k \in L^{-1}\mathbf{Z}\backslash\{0\}$,*

- *all characteristic values λ are always real,*
- *all characteristic values λ satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

Proof. Let $\phi \in H^4((-1,1))$ satisfy (0.27)-(0.62). Multiplying by $\overline{\phi}$ on both sides of (0.27) and then using the integration by parts, we get that

$$-\int_{-1}^{1}(\rho_0\phi')'\overline{\phi}dx_2 = -\rho_0\phi'\overline{\phi}\Big|_{-1}^{1} + \int_{-1}^{1}\rho_0|\phi'|^2dx_2$$

that

$$-\int_{-1}^{1}\phi''\overline{\phi}dx_2 = -\phi'\overline{\phi}\Big|_{-1}^{1} + \int_{-1}^{1}|\phi'|^2dx_2$$

and that

$$\int_{-1}^{1}\phi^{(4)}\overline{\phi}dx_2 = \phi'''\overline{\phi}\Big|_{-1}^{1} - \phi''\overline{\phi}'\Big|_{-1}^{1} + \int_{-1}^{1}|\phi''|^2dx_2,$$

we obtain that

$$\begin{aligned}\lambda\Big(\mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2 - \xi_-|\phi'(-1)|^2 - \xi_+|\phi'(1)|^2\Big)\\ + \lambda^2\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2 = gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.\end{aligned} \tag{2.1}$$

Suppose that $\lambda = \lambda_1 + i\lambda_2$, then one deduces from (2.1) that

$$\begin{aligned}\lambda_1\Big(\mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2 - \xi_-|\phi'(-1)|^2 - \xi_+|\phi'(1)|^2\Big)\\ + (\lambda_1^2-\lambda_2^2)\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2 = gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2\end{aligned} \tag{2.2}$$

and that

$$\begin{aligned}\lambda_2\Big(\mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2 - \xi_-|\phi'(-1)|^2 - \xi_+|\phi'(1)|^2\Big)\\ = -2\lambda_1\lambda_2\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2.\end{aligned} \tag{2.3}$$

If $\lambda_2 \neq 0$, Eq. (2.3) leads us to

$$\begin{aligned}-2\lambda_1\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2 = \mu\int_{-1}^{1}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_2\\ - \xi_-|\phi'(-1)|^2 - \xi_+|\phi'(1)|^2,\end{aligned}$$

which yields

$$-(\lambda_1^2-\lambda_2^2)\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2=-2\lambda_1^2\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2 - gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

Equivalently,

$$(\lambda_1^2+\lambda_2^2)\int_{-1}^{1}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_2=-gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2. \tag{2.4}$$

That implies

$$(\lambda_1^2+\lambda_2^2)k^2\rho_-\int_{-1}^{1}|\phi|^2dx_2\leqslant -gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

The positivity of ρ_0' yields a contradiction, then λ is real. Due to (2.2) again, we further get that

$$\lambda^2\int_{-1}^{1}\rho_0(k^2|\phi|^2+|\phi'|^2)dx_2\leqslant gk^2\int_{-1}^{1}\rho_0'|\phi|^2dx_2.$$

It tells us that λ is bounded by $\sqrt{\frac{g}{L_0}}$. This finishes the proof of Lemma 2.1. □

Note again that, thanks to Lemma 2.1, in what follows in this section, we only use real-valued functions for the linear analysis.

2.1.2 The threshold of viscosity coefficient

We obtain the precise formula of the critical viscosity coefficient $\mu_c(k,\Xi)$ for all $k\in\mathbf{R}\backslash\{0\}$. Note that $\mu_c(k,\Xi)=\mu_c(-k,\Xi)$ for all $k\in\mathbf{R}\backslash\{0\}$, it suffices to find $\mu_c(k,\Xi)$ for $k\in\mathbf{R}_+$.

Proposition 2.1. *The following results hold.*

1. *For all $k\in\mathbf{R}_+$, we have*

$$\mu_c(k,\Xi)=\max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-(\phi'(-1))^2+\xi_+(\phi'(1))^2}{\int_{-1}^{1}((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_2}. \tag{2.5}$$

Moreover,

$$\mu_c(k,\Xi)=\frac{1}{4k\sinh^2(2k)}\left(\begin{aligned}&(\sinh(2k)\cosh(2k)-2k)(\xi_++\xi_-)\\&+\left(\begin{aligned}&(\sinh(2k)-2k\cosh(2k))^2(\xi_++\xi_-)^2\\&+\sinh^2(2k)(\sinh^2(2k)-4k^2)(\xi_+-\xi_-)^2\end{aligned}\right)^{\frac{1}{2}}\end{aligned}\right). \tag{2.6}$$

2. *$\mu_c(k,\Xi)$ is a decreasing function in $k \in \mathbf{R}_+$ and*

$$\lim_{k\to 0}\mu_c(k,\Xi) = \sup_{k\in\mathbf{R}\setminus\{0\}} \mu_c(k,\Xi) =: \mu_c^s(\Xi). \tag{2.7}$$

We have the asymptotic expansion of $\mu_c(k,\Xi)$ as $k \to 0^+$,

$$\begin{aligned}\mu_c(k,\Xi) = &\frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big)\\ &- \frac{2}{15}\Big(4(\xi_+ + \xi_-) + \frac{4\xi_+^2 - \xi_+\xi_- + 4\xi_-^2}{\sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}}\Big)k^2 + O(k^3).\end{aligned} \tag{2.8}$$

That implies

$$\mu_c^s(\Xi) = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big). \tag{2.9}$$

As $k \gg 1$, we obtain the limit

$$\mu_c(k,\Xi) \leqslant \frac{\sqrt{2(\xi_+^2 + \xi_-^2)}}{k} \to 0. \tag{2.10}$$

3. *We have*

$$\begin{aligned}\mu_c^s(\Xi) &= \max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1 (\phi'')^2 dx_2}\\ &= \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).\end{aligned} \tag{2.11}$$

The proof is postponed to Section 2.4.

2.1.3 A bilinear form and a self-adjoint invertible operator

In what follows in this section we have $\lambda \geqslant 0$ and $k \in \mathbf{R}_+$ being fixed. Let us recall the definition of $\mathcal{B}_{k,\lambda,\mu}$ from (0.64),

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) := \lambda\int_{-1}^1 \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^1 (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2\\ - \xi_-\vartheta'(-1)\varrho'(-1) - \xi_+\vartheta'(1)\varrho'(1).\end{aligned}$$

Lemma 2.2. *We have the followings.*

- *For all $\mu > 0$, $\mathcal{B}_{k,\lambda,\mu}$ is a continuous bilinear form on $\tilde{H}^2((-1,1))$.*
- *For all $\mu > \mu_c(k,\Xi)$, $\mathcal{B}_{k,\lambda,\mu}$ is coercive.*

Proof of Lemma 2.2. Clearly, $\mathcal{B}_{k,\lambda,\mu}$ is a bilinear form on $\tilde{H}^2((-1,1))$. We then establish the boundedness of $\mathcal{B}_{k,\lambda,\mu}$. The integral terms of $\mathcal{B}_{k,\lambda,\mu}$ are $\lesssim (\lambda + 1)\|\vartheta\|_{\tilde{H}^2((-1,1))}\|\varrho\|_{\tilde{H}^2((-1,1))}$. Meanwhile, it follows from the general Sobolev inequality that

$$(\vartheta'(-1))^2 + (\vartheta'(1))^2 \lesssim \|\vartheta'\|_{H^1((-1,1))}^2.$$

Consequently, we get

$$|\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho)| \lesssim (1+\lambda)\|\vartheta\|_{\tilde{H}^2((-1,1))}\|\varrho\|_{\tilde{H}^2((-1,1))}, \tag{2.12}$$

i.e. $\mathcal{B}_{k,\lambda,\mu}$ is bounded.

We show the coercivity of $\mathcal{B}_{k,\lambda,\mu}$. We have that

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\vartheta) = \lambda\int_{-1}^{1}\rho_0(k^2\vartheta^2+(\vartheta')^2)dx_2 + \mu\int_{-1}^{1}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_2 \\ - \xi_-(\vartheta'(-1))^2 - \xi_+(\vartheta'(1))^2.\end{aligned}$$

As $\lambda \geqslant 0$ and $\mu > \mu_c(k,\Xi)$, we have

$$\begin{aligned}\mathcal{B}_{k,\lambda,\mu}(\vartheta,\vartheta) &\geqslant \lambda\int_{-1}^{1}\rho_0(k^2\vartheta^2+(\vartheta')^2)dx_2 \\ &\quad + (\mu-\mu_c(k,\Xi))\int_{-1}^{1}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_2 \\ &\geqslant (\mu-\mu_c(k,\Xi))\int_{-1}^{1}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_2.\end{aligned} \tag{2.13}$$

It then follows from (2.12) and (2.13) that $\mathcal{B}_{k,\lambda,\mu}$ is a continuous and coercive bilinear form on $\tilde{H}^2((-1,1))$. □

With the above property of $\mathcal{B}_{k,\lambda,\mu}$, we then establish:

Proposition 2.2. *Let $\mu > \mu_c(k,\Xi)$ and $(\tilde{H}^2((-1,1)))'$ be the dual space of $\tilde{H}^2((-1,1))$, associated with the norm $\sqrt{\mathcal{B}_{k,\lambda,\mu}(\cdot,\cdot)}$. There is a unique operator*

$$Y_{k,\lambda,\mu} \in \mathcal{L}(H^2((-1,1)),(\tilde{H}^2((-1,1)))'),$$

which is also bijective, such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) = \langle Y_{k,\lambda,\mu}\vartheta,\varrho\rangle \tag{2.14}$$

for all $\vartheta,\varrho \in \tilde{H}^2((-1,1))$.

Proof. It follows from Riesz's representation theorem that there exists an operator $Y_{k,\lambda,\mu} \in \mathcal{L}(\tilde{H}^2((-1,1)),(\tilde{H}^2((-1,1)))')$ such that

$$\mathcal{B}_{k,\lambda,\mu}(\vartheta,\varrho) = \langle Y_{k,\lambda,\mu}\vartheta,\varrho\rangle$$

for all $\varrho \in \tilde{H}^2((-1,1))$. Proof of Proposition 2.2 is complete. □

Proposition 2.3. *We have the following results.*

1. *For all $\vartheta \in \tilde{H}^2((-1,1))$,*

$$Y_{k,\lambda,\mu}\vartheta = \lambda(k^2\rho_0\vartheta-(\rho_0\vartheta')')+\mu(\vartheta^{(4)}-2k^2\vartheta''+k^4\vartheta)$$

   in $\mathcal{D}'((-1,1))$.

2. *Let $f \in L^2((-1,1))$ be given, there exists a unique solution $\vartheta \in \tilde{H}^2((-1,1))$ of*

$$Y_{k,\lambda,\mu}\vartheta = f \text{ in } (\tilde{H}^2((-1,1)))'. \tag{2.15}$$

Moreover, we have that $\vartheta \in H^4((-1,1))$ satisfies the boundary conditions (0.62).

Proof. It follows from Proposition 2.2 that there is a unique $\vartheta \in \tilde{H}^2((-1,1))$ such that

$$\lambda \int_{-1}^{1} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu \int_{-1}^{1} (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{2.16}$$

for all $\varrho \in C_0^\infty((-1,1))$. We respectively define $(\vartheta'')'$ and $(\vartheta'')''$ in the distributional sense as the first and second derivative of ϑ'' which is in $L^2((-1,1))$. Hence, (2.16) is equivalent to

$$\lambda \int_{-1}^{1} \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\langle(\vartheta'')'', \varrho\rangle + \mu \int_{-1}^{1} (2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2 = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{2.17}$$

for all $\varrho \in C_0^\infty((-1,1))$. We deduce from (2.17) that

$$\lambda \int_{-1}^{1} (k^2\rho_0\vartheta - (\rho_0\vartheta')')\varrho dx_2 + \mu\langle(\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta, \varrho\rangle = \langle Y_{k,\lambda,\mu}\vartheta, \varrho\rangle \tag{2.18}$$

for all $\varrho \in C_0^\infty((-1,1))$. The resulting equation implies that

$$\mu((\vartheta'')'' - 2k^2\vartheta'' + k^4\vartheta) + \lambda(k^2\rho_0\vartheta - (\rho_0\vartheta')') = Y_{k,\lambda,\mu}\vartheta \quad \text{in } \mathcal{D}'((-1,1)). \tag{2.19}$$

The first assertion holds.

Under the assumption $f \in L^2((-1,1))$, we improve the regularity of the weak solution $\vartheta \in \tilde{H}^2((-1,1))$ of (2.19). Indeed, we rewrite (2.19) as

$$\mu\langle(\vartheta'')'', \varrho\rangle = \int_{-1}^{1} (Y_{k,\lambda,\mu}\vartheta + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')\varrho dx_2$$

for all $\varrho \in C_0^\infty((-1,1))$. Since $(f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')$ belongs to $L^2((-1,1))$, it then follows from (2.18) that $(\vartheta'')'' \in L^2((-1,1))$. Let $\chi \in C_0^\infty((-1,1))$ satisfy $\int_{-1}^1 \chi(y)dy = 1$. Using the distribution theory, we define $\Sigma \in \mathcal{D}'((-1,1))$ such that

$$\langle\Sigma, \theta\rangle = \langle(\vartheta'')'', \zeta_\theta\rangle \tag{2.20}$$

for all $\theta \in C_0^\infty((-1,1))$, where

$$\zeta_\theta(x_2) = \int_{-1}^{x_2} \Big(\theta(y) - \chi(y)\int_{-1}^{1}\theta(s)ds\Big)dy$$

for all $-1 < x_2 < 1$. We obtain

$$\langle\Sigma', \theta\rangle = -\langle\Sigma, \theta'\rangle = -\langle(\vartheta'')'', \zeta_{\theta'}\rangle.$$

Note that

$$\langle(\vartheta'')'', \zeta_{\theta'}\rangle = \langle(\vartheta'')'', \theta(x_2) - \int_{-1}^{x_2}\chi(y)\int_{-1}^{1}\theta'(s)dsdy\rangle = \langle(\vartheta'')'', \theta\rangle,$$

this yields $\langle\Sigma', \theta\rangle = -\langle(\vartheta'')'', \theta\rangle$. Hence, we have that $(\vartheta'')' + \Sigma \equiv$ constant. In view of $(\vartheta'')'' \in L^2((-1,1))$ and (2.20), we know that $(\vartheta'')' \in L^2((-1,1))$. Since $\vartheta \in \tilde{H}^2((-1,1))$ and $(\vartheta'')', (\vartheta'')'' \in L^2((-1,1))$, it tells us that ϑ belongs to $H^4((-1,1))$ and we can take their traces of derivatives of ϑ up to order 3.

By performing (2.18), we then show that ϑ satisfies (0.62). Indeed, for all $\varrho \in \tilde{H}^2((-1,1))$, we perform the integration by parts to obtain from (2.18) that

$$\begin{aligned}\lambda\int_{-1}^{1}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_2 + \mu\int_{-1}^{1}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_2\\ - \lambda\rho_0\vartheta'\varrho\Big|_{-1}^{1} + \mu\Big(\vartheta'''\varrho\Big|_{-1}^{1} - \vartheta''\varrho'\Big|_{-1}^{1} - 2k^2\vartheta'\varrho\Big|_{-1}^{1}\Big) = \int_{-1}^{1}(Y_{k,\lambda,\mu}\vartheta)\varrho dx_2.\end{aligned}$$

It then follows from the definition of the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ that

$$\lambda\rho_0\vartheta'\varrho\Big|_{-1}^{1} - \mu\Big(\vartheta'''\varrho\Big|_{-1}^{1} - \vartheta''\varrho'\Big|_{-1}^{1} - 2k^2\vartheta'\varrho\Big|_{-1}^{1}\Big) = \xi_-\vartheta'(-1)\varrho'(-1) + \xi_+\vartheta'(1)\varrho'(1), \quad (2.21)$$

for all $\varrho \in \tilde{H}^2((-1,1))$. By collecting all terms corresponding to $\varrho'(\pm1)$ in (2.21), we deduce that

$$\mu\vartheta''(\pm1) = \pm\xi_\pm\vartheta'(\pm1).$$

This yields that ϑ satisfies (0.62). The proof of Proposition 2.3 is complete. □

We obtain more information on the inverse operator $Y_{k,\lambda,\mu}^{-1}$.

Proposition 2.4. *The operator* $Y_{k,\lambda,\mu}^{-1} : L^2((-1,1)) \to L^2((-1,1))$ *is compact and self-adjoint.*

Proof. It follows from Proposition 2.3 that $Y_{k,\lambda,\mu}$, being supplemented with (0.62), admits an inverse operator $Y_{k,\lambda,\mu}^{-1}$ from $L^2((-1,1))$ to a subspace of $H^4((-1,1))$ requiring all elements satisfy (0.62), which is symmetric due to Proposition 2.2. We compose $Y_{k,\lambda,\mu}^{-1}$ with the continuous injection from $H^4((-1,1))$ to $L^2((-1,1))$. Notice that the embedding $H^p((-1,1)) \hookrightarrow H^q((-1,1))$ for $p > q \geqslant 0$ is compact. Therefore, the operator $Y_{k,\lambda,\mu}^{-1}$ is compact and self-adjoint from $L^2((-1,1))$ to $L^2((-1,1))$. □

2.2 Linear instability

2.2.1 A sequence of characteristic values

We continue considering $\lambda \geqslant 0$ and $k \in L^{-1}\mathbf{Z}\backslash\{0\}$ being fixed. We study the operator $S_{k,\lambda,\mu} := \mathcal{M}Y_{k,\lambda,\mu}^{-1}\mathcal{M}$, where $\mathcal{M}$ is the operator of multiplication by $\sqrt{\rho_0'}$.

Proposition 2.5. *Under the hypothesis (0.54), the operator $S_{k,\lambda,\mu} : L^2((-1,1)) \to L^2((-1,1))$ is compact and self-adjoint.*

Proof. Due to the assumption on ρ_0 (0.54), the operator $S_{k,\lambda,\mu}$ is well-defined and bounded from $L^2((-1,1))$ to itself. $Y^{-1}_{k,\lambda,\mu}$ is compact, so is $S_{k,\lambda,\mu}$. Moreover, because both the inverse $Y^{-1}_{k,\lambda,\mu}$ and $\mathcal{M}$ are self-adjoint, the self-adjointness of $S_{k,\lambda,\mu}$ follows. □

As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{k,\lambda,\mu}$ is discrete, i.e. is a sequence $\{\gamma_n(k,\lambda,\mu)\}_{n\geqslant 1}$ of eigenvalues of $S_{k,\lambda,\mu}$, associated with normalized orthogonal eigenfunctions $\{\varpi_n\}_{n\geqslant 1}$ in $L^2((-1,1))$. That means

$$S_{k,\lambda,\mu}\varpi_n = \mathcal{M}Y^{-1}_{k,\lambda,\mu}\mathcal{M}\varpi_n = \gamma_n(k,\lambda,\mu)\varpi_n.$$

So that $\phi_n = Y^{-1}_{k,\lambda,\mu}\mathcal{M}\varpi_n$ belongs to $H^4((-1,1))$ and satisfies (0.62). One thus has

$$\gamma_n(k,\lambda,\mu)Y_{k,\lambda,\mu}\phi_n = \rho_0'\phi_n \tag{2.22}$$

and ϕ_n satisfies (0.62). Eq. (2.22) also tells us that $\gamma_n(k,\lambda,\mu) > 0$ for all n. Indeed, we obtain

$$\gamma_n(k,\lambda,\mu)\int_{-1}^{1}(Y_{k,\lambda,\mu}\phi_n)\phi_n dx_2 = \int_{-1}^{1}\rho_0'\phi_n^2 dx_2.$$

That implies

$$\gamma_n(k,\lambda,\mu)\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n) = \int_{-1}^{1}\rho_0'\phi_n^2 dx_2. \tag{2.23}$$

Since $\mathcal{B}_{k,\lambda,\mu}(\phi_n,\phi_n) > 0$ and $\rho_0' > 0$ on $(-1,1)$, we know that $\gamma_n(k,\lambda,\mu)$ is positive. Hence, by reordering and using the spectral theory of compact and self-adjoint operators again, we have that $\{\gamma_n(k,\lambda,\mu)\}_{n\geqslant 1}$ is a positive sequence decreasing towards 0 as $n \to \infty$.

For each n, ϕ_n is a solution of (0.27)-(0.62) if and only if there are positive λ_n such that (0.69) holds. To solve (0.69), we use the two following lemmas.

Lemma 2.3. *For each n,*

- *$\gamma_n(k,\lambda,\mu)$ and ϕ_n are differentiable in λ.*
- *$\gamma_n(k,\lambda,\mu)$ is decreasing in λ.*

Proof. The proof of Lemma 2.3(1) is the same as Lemma 1.2, we omit the details here. We now prove that $\gamma_n(k,\lambda,\mu)$ is decreasing in λ.

Let $z_n = \frac{d\phi_n}{d\lambda}$, it follows from (2.22) that

$$k^2\rho_0\phi_n - (\rho_0\phi_n')' + Y_{k,\lambda,\mu}z_n = \frac{1}{\gamma_n(k,\lambda,\mu)}\rho_0' z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big)\rho_0'\phi_n \tag{2.24}$$

on $(-1, 1)$. At $x_2 = \pm 1$, we have

$$\begin{cases} z_n(-1) = z_n(1) = 0, \\ \mu z_n''(1) = \xi_+ z_n'(1), \\ \mu z_n''(-1) = -\xi_- z_n'(-1). \end{cases} \tag{2.25}$$

Multiplying by ϕ_n on both sides of (2.24), we obtain that

$$\begin{aligned} &\int_{-1}^{1} (k^2 \rho_0 \phi_n - (\rho_0 \phi_n')')\phi_n dx_2 + \int_{-1}^{1} (Y_{k,\lambda,\mu} z_n)\phi_n dx_2 \\ &\qquad = \frac{1}{\gamma_n(k,\lambda,\mu)} \int_{-1}^{1} \rho_0' z_n \phi_n dx_2 + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big) \int_{-1}^{1} \rho_0' \phi_n^2 dx_2. \end{aligned} \tag{2.26}$$

Note that z_n enjoys (2.25), then

$$\int_{-1}^{1} (Y_{k,\lambda,\mu} z_n)\phi_n dx_2 = \int_{-1}^{1} (Y_{k,\lambda,\mu} \phi_n) z_n dx_2 = \frac{1}{\gamma_n(k,\lambda,\mu)} \int_{-1}^{1} \rho_0' z_n \phi_n dx_2.$$

That implies

$$\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big) \int_{-1}^{1} \rho_0' \phi_n^2 dx_2 = \int_{-1}^{1} (k^2 \rho_0 \phi_n - (\rho_0 \phi_n')')\phi_n dx_2. \tag{2.27}$$

Using the integration by parts, we obtain from (2.27) that

$$\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(k,\lambda,\mu)}\Big) \int_{-1}^{1} \rho_0' \phi_n^2 dx_2 = \int_{-1}^{1} \rho_0 (k^2 \phi_n^2 + (\phi_n')^2) dx_2 > 0.$$

Consequently, $\gamma_n(k,\lambda,\mu)$ is decreasing in $\lambda > 0$. □

2.2.2 Proof of Theorem 0.3 and normal modes of the linearized equations

In view of Lemma 2.3, we are able to prove Theorem 0.3.

Proof of Theorem 0.3. For each n, there is only one solution λ_n of (0.69). Indeed, using (2.23), we know that

$$\frac{1}{\gamma_n(k,\lambda,\mu)} \int_{-1}^{1} \rho_0' \phi_n^2 dx_2 = \int_{-1}^{1} (Y_{k,\lambda,\mu} \phi_n)\phi_n dx_2 = \mathcal{B}_{k,\lambda,\mu}(\phi_n, \phi_n).$$

Hence, it follows from (2.13) that

$$\begin{aligned} \frac{1}{\gamma_n(k,\lambda,\mu)} \int_{-1}^{1} \rho_0' \phi_n^2 dx_2 &\geqslant \lambda \int_{-1}^{1} \rho_0 (k^2 \phi_n^2 + (\phi_n')^2) dx_2 \\ &\qquad + (\mu - \mu_c(k,\Xi)) \int_{-1}^{1} ((\phi_n'')^2 + 2k^2 (\phi_n')^2 + k^4 \phi_n^2) dx_2 \\ &\geqslant \lambda k^2 \int_{-1}^{1} \rho_0 \phi_n^2 dx_2 + (\mu - \mu_c(k,\Xi)) k^4 \int_{-1}^{1} \phi_n^2 dx_2. \end{aligned}$$

That implies
$$\frac{1}{L_0\gamma_n(k,\lambda,\mu)} \geqslant \lambda k^2 + \frac{(\mu-\mu_c(k,\Xi))k^4}{\rho_+}.$$
Consequently, for all $n \geqslant 1$,
$$\frac{\lambda}{\gamma_n(k,\lambda,\mu)} > gk^2 \text{ for } \lambda \text{ large.} \tag{2.28}$$
Meanwhile, for all $n \geqslant 1$ and $\lambda \leqslant \frac{1}{2}\sqrt{\frac{g}{L_0}}$,
$$\frac{\lambda}{\gamma_n(k,\lambda,\mu)} \leqslant \frac{\lambda}{\gamma_n(k,\frac{1}{2}\sqrt{\frac{g}{L_0}},\mu)} \to 0 \text{ as } \lambda \to 0. \tag{2.29}$$

In view of (2.28), (2.29) and Lemma 2.3, we obtain only one solution λ_n of (0.69) and (λ_n, ϕ_n) satisfies (0.27)-(0.62). That means for all n, λ_n is a characteristic value, hence it is bounded by $\sqrt{\frac{g}{L_0}}$.

We now prove that $(\lambda_n)_{n\geqslant 1}$ decreases towards 0 as $n \to \infty$. If $\lambda_m < \lambda_{m+1}$ for some $m \geqslant 1$, we have
$$\gamma_m(k,\lambda_m,\mu) > \gamma_m(k,\lambda_{m+1},\mu).$$
Meanwhile, we also have
$$\gamma_m(k,\lambda_{m+1},\mu) > \gamma_{m+1}(k,\lambda_{m+1},\mu).$$
That implies
$$\frac{\lambda_m}{gk^2} = \gamma_m(k,\lambda_m,\mu) > \gamma_{m+1}(k,\lambda_{m+1},\mu) = \frac{\lambda_{m+1}}{gk^2}.$$
That contradiction tells us that $(\lambda_n)_{n\geqslant 1}$ is a decreasing sequence. Suppose that
$$\lim_{n\to\infty} \lambda_n = d_0 > 0.$$
Note that for all n, $\gamma_n(k,\lambda_n,\mu) = \frac{\lambda_n}{gk^2}$, then
$$\gamma_n(k,d_0,\mu) \geqslant \gamma_n(k,\lambda_n,\mu) = \frac{\lambda_n}{gk^2}.$$
Let $n \to \infty$, we get that $0 \geqslant d_0$, a contradiction. Hence λ_n decreases towards 0 as $n \to \infty$. The proof of Theorem 0.3 is complete. □

We derive the following property for the largest characteristic value λ_1 found in Theorem 0.3.

Proposition 2.6. *Let us recall the bilinear form $\mathcal{B}_{k,\lambda,\mu}$ on $H^2((-1,1))$ (0.64) and (λ_1,ϕ_1) from Theorem 0.3. We have that*
$$\frac{1}{gk^2} = \max_{\phi\in H^2((-1,1))} \frac{\int_{-1}^1 \rho_0'\phi^2 dx_2}{\lambda_1 \mathcal{B}_{k,\lambda_1,\mu}(\phi,\phi)}, \tag{2.30}$$
and the variational problem (2.30) is attained by the function ϕ_1.

Proof. For all $\lambda > 0$, we solve the variational problem

$$\beta(k,\lambda,\mu) = \max\Big(\int_{-1}^{1} \rho_0'\phi^2 dx_2 \Big| \phi \in \tilde{H}^2((-1,1)), \quad \lambda \mathcal{B}_{k,\lambda,\mu}(\phi,\phi) = 1\Big).$$

Let us define the Lagrangian functional

$$\mathcal{L}_{\mathcal{B}}(\phi,\beta) = \int_{-1}^{1} \rho_0'\phi^2 dx_2 - \beta(\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi) - 1).$$

Thanks to the Lagrange multiplier theorem, the extrema of the quotient

$$\frac{\int_{-1}^{1} \rho_0'\phi^2 dx_2}{\lambda\mathcal{B}_{k,\lambda,\mu}(\phi,\phi)}$$

are necessarily the stationary points $(\beta_\star, \phi_\star)$ of $\mathcal{L}_{\mathcal{B}}$, which satisfy

$$\lambda\mathcal{B}_{k,\lambda,\mu}(\phi_\star,\phi_\star) = 1 \tag{2.31}$$

and

$$\int_{-1}^{1} \rho_0'\phi_\star\theta dx_2 - \beta_\star\lambda\mathcal{B}_{k,\lambda,\mu}(\phi_\star,\theta) = 0, \tag{2.32}$$

for all $\theta \in \tilde{H}^2((-1,1))$. Restricting $\theta \in C_0^\infty((-1,1))$ and following the line of the proof of Proposition 2.3, one deduces from (2.32) that $\phi_\star$ has to satisfy

$$\beta_\star\lambda Y_{k,\lambda,\mu}\phi_\star = \rho_0'\phi_\star \tag{2.33}$$

in a weak sense. We further get that $\phi_\star \in H^4((-1,1))$ and satisfies (2.31) and the boundary conditions (0.62). Hence, $\lambda\beta_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{k,\lambda,\mu}$ from $L^2((-1,1))$ to itself, with $\mathcal{M}^{-1}Y_{k,\lambda,\mu}\phi_\star \in L^2((-1,1))$ being an associated eigenfunction. That implies

$$\beta(k,\lambda,\mu) \leqslant \lambda^{-1}\gamma_1(k,\lambda,\mu). \tag{2.34}$$

Meanwhile, since the operator $S_{k,\lambda,\mu}$ is self-adjoint and positive (see again Proposition 2.5), we thus obtain that

$$\gamma_1(k,\lambda,\mu) = \sup_{\omega\in L^2((-1,1))} \frac{\langle S_{k,\lambda,\mu}\omega,\omega\rangle}{\|\omega\|^2_{L^2((-1,1))}}.$$

Hence, for all $\omega \in L^2((-1,1))$ and for $\phi = Y^{-1}_{k,\lambda,\mu}\mathcal{M}\omega \in H^4((-1,1))$, we have

$$\gamma_1(k,\lambda,\mu)\langle Y_{k,\lambda,\mu}\phi,\phi\rangle \leqslant \frac{\langle S_{k,\lambda,\mu}\omega,\omega\rangle^2}{\|\omega\|^2_{L^2((-1,1))}} \leqslant \|S_{k,\lambda,\mu}\omega\|^2_{L^2((-1,1))}.$$

Equivalently,

$$\gamma_1(k,\lambda,\mu) \leqslant \sup\Big\{\frac{\|\mathcal{M}\phi\|^2_{L^2((-1,1))}}{\langle Y_{k,\lambda,\mu}\phi,\phi\rangle} | \phi \in H^4((-1,1)) \text{ and } \mathcal{M}^{-1}Y_{k,\lambda,\mu}\phi \in L^2((-1,1))\Big\},$$

it yields

$$\lambda^{-1}\gamma_1(k,\lambda,\mu) \leqslant \beta(k,\lambda,\mu). \tag{2.35}$$

Two inequalities (2.34) and (2.35) tell us that $\beta(k,\lambda,\mu) = \lambda^{-1}\gamma_1(k,\lambda,\mu)$ for all $\lambda > 0$. We thus obtain $\beta(k,\lambda_1,\mu) = \frac{1}{gk^2}$ and the variational problem (2.30) is attained by the function ϕ_1. Proof of Proposition 2.6 is complete. □

We now solve the linearized equations (0.57) to prepare for our nonlinear part.

Proposition 2.7. *For each $k \in L^{-1}\mathbf{Z}\backslash\{0\}$ and for all $\mu > \mu_c(k,\Xi)$, there exists an infinite sequence of solutions ($n \geqslant 1$)*

$$\begin{aligned} e^{\lambda_n(k,\mu)t}U_n(k,\mu,x) &= e^{\lambda_n(k,\mu)t}(\sigma_n,u_n,p_n)^T(k,\mu,x) \\ &= e^{\lambda_n(k,\mu)t}\begin{pmatrix} \cos(kx_1)\omega_n(k,\mu,x_2) \\ \sin(kx_1)\theta_n(k,\mu,x_2) \\ \cos(kx_1)\phi_n(k,\mu,x_2) \\ \cos(kx_1)q_n(k,\mu,x_2) \end{pmatrix} \end{aligned}$$

to the linearized equations (0.57)-(0.56), such that

$$\sigma_n \in H^2(\Omega), u_n \in (H^3(\Omega))^2 \text{ and } p_n \in H^1(\Omega).$$

Proof. For each solution $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ of (0.51), we have a solution $\phi_n = Y^{-1}_{k,\lambda_n,\mu}\mathcal{M}\varpi_n \in H^4((-1,1))$ of (0.27)-(0.62) as $\lambda = \lambda_n$. We now find a solution to the system (0.59) as $\lambda = \lambda_n$. First, we obtain $\theta_n = -\frac{\phi_n'}{k}$ and $\omega_n = -\frac{\rho_0'\phi_n}{\lambda_n}$. Due to (0.61), we get

$$q_n = -\frac{1}{k^2}(\lambda_n\rho_0\phi_n' + \mu(k^2\phi_n' - \phi_n''')) \in H^1((-1,1)).$$

With a solution $(\omega_n,\theta_n,\phi_n,q_n)$ of (0.59), we then conclude that

$$e^{\lambda_n(k,\mu)t}(\sigma_n,u_{n,1},u_{n,2},p_n)^T(k,\mu,x) = e^{\lambda_n(k,\mu)t}\begin{pmatrix} \cos(kx_1)\omega_n(k,\mu,x_2) \\ \sin(kx_1)\theta_n(k,\mu,x_2) \\ \cos(kx_1)\phi_n(k,\mu,x_2) \\ \cos(kx_1)q_n(k,\mu,x_2) \end{pmatrix}$$

is a solution to the linearized equations (0.57)-(0.56). □

2.3 Nonlinear instability

2.3.1 The local existence

Thanks to Proposition 2.7, we will formulate a sequence of approximate solutions $e^{\lambda_n(k,\mu)}U_n(k,\mu,x)$ to the nonlinear equations (0.55)-(0.56), which are solutions to the linearized equations (0.57). Let us fix a $k = k_0 \in L^{-1}\mathbf{Z}\backslash\{0\}$ such that (0.75) holds and $\mu > 3\mu_c(\Xi)$. We recall (0.76),

$$(\sigma^M, u^M, q^M)(t,x) := \sum_{j=j_m}^{M} \mathsf{c}_j e^{\lambda_j(k,\mu)t} U_j(k,\mu,x),$$

where the constants c_j $(j \geqslant 1)$ satisfy (0.77)-(0.78).

Keeping in mind that $\min_{[-1,1]} \rho_0 > 0$, then due to the embedding from $H^2(\Omega)$ to $L^\infty(\Omega)$, there exists a constant $\delta_0 > 0$ such that

$$\delta_0 \| \sum_{j\geqslant 1} \sigma_j(0,x)\|_{L^\infty(\Omega)} > \frac{1}{2} \min_{[-1,1]} \rho_0(x_2). \tag{2.36}$$

Hence, for $\delta \leqslant \delta_0$,

$$\frac{1}{2} \min_{[-1,1]} \rho_0(x_2) \leqslant \min_{\Omega}(\rho_0(x_2) + \delta\sigma^M(0,x))$$

By virtue of Proposition 0.1, the nonlinear equations (0.55)-(0.56) with initial data $\delta(\sigma^M, u^M)(0)$ admits a local solution

$$(\sigma^\delta, u^\delta) \in C^0([0,T^{\max}), H^1(\Omega) \times (H^2(\Omega))^2)$$

with an associated pressure $q^\delta \in C^0([0,T^{\max}), L^2(\Omega))$. Furthermore, we have for all $t \in [0,T^{\max})$,

$$\frac{1}{2} \min_{[-1,1]} \rho_0(x_2) \leqslant \inf_{\Omega}(\rho_0(x_2) + \sigma^\delta(t,x)).$$

In what follows in this chapter, the constants C_i $(i \geqslant 1)$ are universal ones depending only on physical parameters, M and $\mathsf{c}_j (j \geqslant 1)$.

Let $F_M(t) = \sum_{j=j_m}^M |\mathsf{c}_j| e^{\lambda_j t}$ and $0 < \epsilon_0 \ll 1$ be fixed later (2.71). There exists a unique T^δ such that $\delta F_M(T^\delta) = \epsilon_0$. Let

$$C_1 = \sqrt{\|\sigma^M(0)\|^2_{H^1(\Omega)} + \|u^M(0)\|^2_{H^2(\Omega)}}, \quad C_2 = \sqrt{\|\sigma^M(0)\|^2_{L^2(\Omega)} + \|u^M(0)\|^2_{L^2(\Omega)}}.$$

We define

$$\begin{aligned} T^\star &:= \sup\Big\{t \in (0,T^{\max}) | \mathcal{E}(\sigma^\delta(t), u^\delta(t)) \leqslant C_1\delta_0\} > 0, \\ T^{\star\star} &:= \sup\{t \in (0,T^{\max}) | \|(\sigma^\delta, u^\delta)(t)\|_{L^2(\Omega)} \leqslant 2C_2\delta F_M(t)\Big\} > 0. \end{aligned} \tag{2.37}$$

Note that $\mathcal{E}(\sigma^\delta(0), u^\delta(0)) = C_1\delta < C_1\delta_0$, we have $T^\star > 0$. Similarly, we have $T^{\star\star} > 0$. Then for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, it follows from the *a priori* energy estimate (0.72) of Proposition 0.1 that

$$\mathcal{E}^2(\sigma^\delta(t), u^\delta(t)) + \|\partial_t u^\delta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla\partial_t u^\delta(\tau)\|^2_{L^2(\Omega)} d\tau \leqslant C_3\delta^2 F_M^2(t). \tag{2.38}$$

2.3.2 The difference functions

Let

$$\sigma^d = \sigma^\delta - \delta\sigma^M, \quad u^d = u^\delta - \delta u^M, \quad q^d = q^\delta - \delta q^M.$$

Since $(\sigma^\delta, u^\delta, q^\delta)$ solves the nonlinear equations (0.55) and (σ^M, u^M, q^M) solves the linearized equations (0.57), we have that (σ^d, u^d, q^d) satisfies (0.79)

$$\begin{cases} \partial_t\sigma^d + \rho_0' u_2^d = -u^\delta\cdot\nabla\sigma^\delta, \\ \rho_0\partial_t u^d - \mu\Delta u^d + \nabla q^d = -\sigma^\delta\partial_t u^\delta - (\rho_0+\sigma^\delta)u^\delta\cdot\nabla u^\delta - g\sigma^d e_2, \\ \mathrm{div} u^d = 0. \end{cases}$$

along with the initial condition (0.80),

$$(\sigma^d, u^d)(0) = 0$$

and the boundary conditions (0.81).

$$\begin{cases} u_2^d = 0, & \text{on } \Sigma_\pm, \\ \mu\partial_{x_2}u_1^d = \xi_+ u_1^d & \text{on } \Sigma_+, \\ \mu\partial_{x_2}u_1^d = -\xi_- u_1^d & \text{on } \Sigma_-. \end{cases}$$

The compatibility conditions read as

$$u_1^d(0,x_1,-1) = u_1^d(0,x_1,1), \quad \mathrm{div} u^d(0) = 0. \tag{2.39}$$

We now establish the error estimate for $\|(\sigma^d, u^d)\|_{L^2(\Omega)}$.

Proposition 2.8. *For all $t \leqslant \min(T^\delta, T^\star, T^{\star\star})$, there holds*

$$\|(\sigma^d, u^d)(t)\|^2_{L^2(\Omega)} \leqslant C_4\delta^3\Big(\sum_{j=1}^N |\mathsf{c}_j|e^{\lambda_j t} + \max(0, M-N)\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|e^{\frac{2}{3}\nu_0\Lambda t}\Big)^3. \tag{2.40}$$

The proof of Proposition 2.8 relies on Lemmas 2.4, 2.5, 2.6, 2.7 below.

Lemma 2.4. *We have the following inequalities*

$$\sum_{0\leqslant s\leqslant 2, 0\leqslant\tau\leqslant 1} \|\partial_t^\tau u^d(t)\|_{H^s(\Omega)} \leqslant C_5\delta F_M(t), \tag{2.41}$$

and

$$\|\sigma^d(t)\|_{H^1(\Omega)} + \|\partial_t\sigma^d(t)\|_{L^2(\Omega)} \leqslant C_6\delta F_M(t). \tag{2.42}$$

Proof. For $\tau \in \{0,1\}$,

$$\partial_t^\tau u^M(t) = \sum_{j=1}^M \lambda_j^\tau \mathsf{c}_j e^{\lambda_j t} U_j(k_0, x),$$

it yields, for all $s \in \{0, 1, 2\}$,

$$\|\partial_t^\tau u^M(t)\|_{H^s(\Omega)} \leqslant C_7 F_M(t).$$

In view of (2.38), we then obtain that for $s \in \{0, 1, 2\}$ and $\tau \in \{0, 1\}$,

$$\|\partial_t^\tau u^d(t)\|_{H^s(\Omega)} \leqslant \delta \|\partial_t^\tau u^M(t)\|_{H^s(\Omega)} + \|\partial_t^\tau u^\delta(t)\|_{H^s(\Omega)} \leqslant C_8 \delta F_M(t).$$

To prove (2.42), we use $(0.79)_1$ and (2.38) again,

$$\begin{aligned}\|\sigma^d(t)\|_{H^1(\Omega)} + \|\partial_t \sigma^d(t)\|_{L^2(\Omega)} &\leqslant \|\sigma^\delta(t)\|_{H^1(\Omega)} + \delta\|\sigma^M(t)\|_{H^1(\Omega)} \\ &\quad + C_9 \|u_2^d(t)\|_{L^2(\Omega)} + \|u^\delta(t)\|_{L^2(\Omega)} \|\nabla \sigma^\delta\|_{L^2(\Omega)} \\ &\leqslant C_{10} \delta F_M(t).\end{aligned}$$

Lemma 2.4 is proven. □

Lemma 2.5. *There holds*

$$\|\partial_t u^d(0)\|_{L^2(\Omega)}^2 \leqslant C_{11} \delta^3. \tag{2.43}$$

Proof. From $(0.79)_{2,3}$ and the boundary conditions (0.81), we use the integration by parts to obtain that

$$\begin{aligned}\int_\Omega \rho_0 |\partial_t u^d|^2 dx = \int_\Omega \mu \Delta u^d \cdot \partial_t u^d dx - \int_\Omega (\sigma^\delta \partial_t u^\delta + (\rho_0 + \sigma^\delta) u^\delta \cdot \nabla u^\delta) \cdot \partial_t u^d dx \\ - \int_\Omega g \sigma^d \partial_t u_2^d dx.\end{aligned}$$

Thanks to Lemma 2.4, one has

$$-\int_\Omega (\sigma^\delta \partial_t u^\delta + (\rho_0 + \sigma^\delta) u^\delta \cdot \nabla u^\delta) \cdot \partial_t u^d dx \leqslant C_{12} \delta^3 F_M^3(t). \tag{2.44}$$

That implies

$$\|\partial_t u^d(t)\|_{L^2(\Omega)}^2 \leqslant C_{13} \Big((\|u^d(t)\|_{H^2(\Omega)} + \|\sigma^d(t)\|_{L^2(\Omega)}) \|\partial_t u^d(t)\|_{L^2(\Omega)} + \delta^3 F_M^3(t) \Big).$$

Using Young's inequality, we further get

$$\|\partial_t u^d(t)\|_{L^2(\Omega)}^2 \leqslant \frac{1}{2} \|\partial_t u^d(t)\|_{L^2(\Omega)}^2 + C_{14} (\|u^d(t)\|_{H^2(\Omega)}^2 + \|\sigma^d(t)\|_{L^2(\Omega)}^2) + C_{13} \delta^3 F_M^3(t).$$

That implies

$$\|\partial_t u^d(t)\|_{L^2(\Omega)}^2 \leqslant C_{15} \Big(\|u^d(t)\|_{H^2(\Omega)}^2 + \|\sigma^d(t)\|_{L^2(\Omega)}^2 + \delta^3 F_M^3(t) \Big). \tag{2.45}$$

Letting $t \to 0$ in (2.45), we complete the proof Lemma 2.5. □

Lemma 2.6. *Let*

$$\mathsf{X} := \{w \in (H^2(\Omega))^2, w \text{ satisfies } (0.53) \text{ and } divw = 0\}.$$

There holds for all $w \in H^2_\star(\Omega)$,

$$\begin{aligned}\int_\Omega g\rho_0'|w_2|^2 dx + \Lambda \int_{(2\pi L\mathbb{T})^2} (\xi_+|w_1(x_1,1)|^2 + \xi_-|w_1(x_1,-1)|^2) dx_1 \\ \leqslant \Lambda^2 \int_\Omega \rho_0|w|^2 dx + \Lambda\mu \int_\Omega |\nabla w|^2 dx.\end{aligned} \tag{2.46}$$

The proof of Lemma 2.6 is due to the definition of Λ (0.74) and Proposition 2.6, that is similar to [14, Lemma 5.1], hence we omit the details here.

Lemma 2.7. *There holds for all* $w \in \mathsf{X}\backslash\{0\}$,

$$\sup_{w\in\mathsf{X}} \frac{\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2 + \xi_-|w_1(x_1,-1)|^2)dx_1}{\|\nabla w\|^2_{L^2(\Omega)}} \leqslant \mu_c(\Xi). \tag{2.47}$$

Proof. Let us fix a horizontal frequency $k \in L^{-1}\mathbf{Z}$ and introduce the horizontal Fourier transform

$$\hat{f}(k,x_2) = \int_{2\pi L\mathbb{T}} f(x)e^{-ikx_1}dx_1.$$

For $w \in \mathsf{X}$, we write

$$\hat{w}_1(k,x_2) = -i\theta(k,x_2), \quad \hat{w}_2(k,x_2) = \phi(k,x_2).$$

Then, $k\theta + \phi' = 0$ and (θ,ϕ) enjoys (0.60). Following Fubini's and Parseval's theorem, one thus deduces

$$\begin{aligned}\int_{2\pi L\mathbb{T}} (\xi_+|w_1(x_1,1)|^2 + \xi_-|w_1(x_1,-1)|^2)dx_1 \\ = \frac{1}{2\pi L}\sum_{k\in L^{-1}\mathbb{Z}} (\xi_+(|\theta(k,1)|^2 + \xi_-|\theta(k,-1)|^2)\end{aligned} \tag{2.48}$$

and

$$\|\nabla w\|^2_{L^2(\Omega)} = \frac{1}{2\pi L}\sum_{k\in L^{-1}\mathbb{Z}} \int_{-1}^1 (k^2(|\theta|^2 + |\phi|^2) + |\theta'|^2 + |\phi'|^2)(k,x_2)dx_2. \tag{2.49}$$

We may reduce to estimate (2.47) when θ and ϕ are real-valued and continue the estimate to the real and imaginary parts of θ and ϕ. For any $k \in L^{-1}\mathbf{Z}\backslash\{0\}$, we have from $k\theta + \phi' = 0$ that

$$\xi_+(\theta(k,1))^2 + \xi_-(\theta(k,-1))^2 = \frac{1}{k^2}(\xi_+((\phi'(k,1))^2 + \xi_-(\phi'(k,-1))^2) \tag{2.50}$$

and that

$$\begin{aligned}\int_{-1}^1 \Big(k^2(\theta^2 + \phi^2) + (\theta')^2 + (\phi')^2\Big)(k,x_2)dx_2 \\ = \frac{1}{k^2}\int_{-1}^1 (k^4\phi^2 + 2k^2(\phi')^2 + (\phi'')^2)(k,x_2)dx_2.\end{aligned} \tag{2.51}$$

Owing to (2.48), (2.50) and the definition of $\mu_c(k,\Xi)$, we get

$$\begin{aligned}&\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ &\quad\leqslant\frac{1}{2\pi L}\left(\begin{aligned}&\limsup_{k\to0}\frac{1}{k^2}(\xi_+(\phi'(k,1))^2+\xi_-(\phi'(k,-1))^2)\\ &+\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{1}{k^2}(\xi_+(\phi'(k,1))^2+\xi_-(\phi'(k,-1))^2)\end{aligned}\right)\\ &\quad\leqslant\frac{1}{2\pi L}\left(\begin{aligned}&\limsup_{k\to0}\frac{\mu_c(k,\Xi)}{k^2}\int_{-1}^1(k^4\phi^2+2k^2\phi'^2+\phi''^2)(k,x_2)dx_2\\ &+\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{\mu_c(k,\Xi)}{k^2}\int_{-1}^1(k^4\phi^2+2k^2\phi'^2+\phi''^2)(k,x_2)dx_2\end{aligned}\right).\end{aligned}$$

Thanks to Proposition 2.1, we obtain

$$\begin{aligned}&\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\\ &\leqslant\frac{\mu_c(\Xi)}{2\pi L}\left(\begin{aligned}&\limsup_{k\to0}\frac{1}{k^2}\int_{-1}^1(k^4\phi^2+2k^2(\phi')^2+(\phi'')^2)(k,x_2)dx_2\\ &+\sum_{k\in L^{-1}\mathbb{Z}\backslash\{0\}}\frac{1}{k^2}\int_{-1}^1(k^4\phi^2+2k^2(\phi')^2+(\phi'')^2)(k,x_2)dx_2\end{aligned}\right).\end{aligned}\tag{2.52}$$

Combining (2.49), (2.51) and (2.52), it gives

$$\int_{2\pi L\mathbb{T}}(\xi_+|w_1(x_1,1)|^2+\xi_-|w_1(x_1,-1)|^2)dx_1\leqslant\mu_c(\Xi)\|\nabla w\|^2_{L^2(\Omega)}.$$

Lemma 2.7 is proven. □

We now prove Proposition 2.8.

Proof of Proposition 2.8. We rewrite $(0.79)_2$ as

$$(\rho_0+\sigma^\delta)\partial_t u^d-\mu\Delta u^d+\nabla q^d=f^\delta-g\sigma^d e_2,$$

where $f^\delta=-\sigma^\delta\partial_t u^M-(\rho_0+\sigma^\delta)u^\delta\cdot\nabla u^\delta$. Differentiate the resulting equation with respect to t and then multiply by $\partial_t u^d$, we obtain after integration that

$$\begin{aligned}&\int_\Omega\partial_t\sigma^\delta|\partial_t u^d|^2dx+\int_\Omega(\rho_0+\sigma^\delta)\partial_t^2u^d\cdot\partial_t u^d dx\\ &=\int_\Omega\mu\Delta\partial_t u^d\cdot\partial_t u^d dx-\int_\Omega\nabla\partial_t q^d\cdot\partial_t u^d dx+\int_\Omega(\partial_t f^\delta-g\partial_t\sigma^d e_2)\cdot\partial_t u^d dx.\end{aligned}$$

Since $\text{div}\partial_t u^d=0$, we use the integration by parts to further obtain

$$\begin{aligned}&\int_\Omega\partial_t\sigma^\delta(t)|\partial_t u^d(t)|^2dx+\int_\Omega(\rho_0+\sigma^\delta(t))\partial_t^2u^d(t)\cdot\partial_t u^d(t)dx\\ &=\int_\Omega(\partial_t f^\delta(t)-g\partial_t\sigma^d(t)e_2)\cdot\partial_t u^d(t)dx-\mu\int_\Omega|\nabla\partial_t u^d(t)|^2dx\\ &\qquad+\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(t,x_1,1)|^2+\xi_-|\partial_t u_1^d(t,x_1,-1)|^2)dx_1.\end{aligned}$$

That means,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_0+\sigma^\delta(t))|\partial_t u^d(t)|^2 dx \\
&= -\frac{1}{2}\int_\Omega \partial_t\sigma^\delta(t)|\partial_t u^d(t)|^2 dx + \int_\Omega (\partial_t f^\delta(t) - g\partial_t \sigma^d(t) e_2)\cdot \partial_t u^d(t) dx - \mu\int_\Omega |\nabla \partial_t u^d(t)|^2 dx \\
&\quad + \int_{2\pi L\mathbb{T}} (\xi_+|\partial_t u_1^d(t,x_1,1)|^2 + \xi_-|\partial_t u_1^d(t,x_1,-1)|^2) dx_1.
\end{aligned}$$

Using $(0.79)_1$, we then get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big((\rho_0+\sigma^\delta(t))|\partial_t u^d(t)|^2 - g\rho_0'|u_2^d(t)|^2\Big) dx \\
&\qquad + 2\mu\int_\Omega |\nabla\partial_t u^d(t)|^2 dx - 2\int_{2\pi L\mathbb{T}} (\xi_+|\partial_t u_1^d(t,x_1,1)|^2 + \xi_-|\partial_t u_1^d(t,x_1,-1)|^2) dx_1 \\
&= -\int_\Omega \partial_t \sigma^\delta(t)|\partial_t u^d(t)|^2 dx + 2\int_\Omega (\partial_t f^\delta(t) + g u^\delta(t)\cdot\nabla\sigma^\delta(t) e_2)\cdot\partial_t u^d(t) dx.
\end{aligned}$$

Integrating in time variable, we get

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|_{L^2(\Omega)}^2 + 2\mu\int_0^t \|\nabla\partial_t u^d(s)\|_{L^2(\Omega)}^2 ds \\
&\qquad - 2\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(s,x_1,1)|^2 + \xi_-|u_1^d(s,x_1,-1)|^2) dx_1 ds \\
&= \int_\Omega g\rho_0'|u_2^d(t)|^2 dx + \Big(\int_\Omega (\rho_0+\sigma^\delta(t))|\partial_t u^d(t)|^2 dx\Big)\Big|_{t=0} \\
&\qquad + \int_0^t\int_\Omega (2\partial_t f^\delta(s) + 2g u^\delta(s)\cdot\nabla\sigma^\delta(s) e_2 - \partial_t\sigma^\delta(s)\partial_t u^d(s))\cdot\partial_t u^d(s) ds.
\end{aligned} \tag{2.53}$$

We continue using (2.41), (2.42) and (2.43) to estimate each term of the r.h.s of (2.53). This yields

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|_{L^2(\Omega)}^2 + 2\mu\int_0^t \|\partial_t u^d(s)\|_{L^2(\Omega)}^2 ds \\
&\qquad - 2\int_{2\pi L\mathbb{T}} (\xi_+|\partial_t u_1^d(t,x_1,1)|^2 + \xi_-|\partial_t u_1^d(t,x_1,-1)|^2) dx_1 \\
&\leqslant \int_\Omega g\rho_0'|u_2^d(t)|^2 dx + C_{16}\delta^3 F_M^3(t).
\end{aligned} \tag{2.54}$$

Due to (2.46), we further get that

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|_{L^2(\Omega)}^2 + 2\mu\int_0^t \|\nabla\partial_t u^d(s)\|_{L^2(\Omega)}^2 ds \\
&\qquad - 2\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|\partial_t u_1^d(s,x_1,1)|^2 + \xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1 ds \\
&\leqslant \Lambda^2\int_\Omega \rho_0|u^d(t)|^2dx + \Lambda\mu\int_\Omega |\nabla u^d(t)|^2 dx \\
&\qquad - \Lambda\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(t,x_1,1)|^2 + \xi_-|u_1^d(t,x_1,-1)|^2)dx_1 + C_{16}\delta^3F_M^3(t) \\
&\leqslant \Lambda^2\int_\Omega (\rho_0+\sigma^\delta(t))|u^d(t)|^2dx + \Lambda\mu\int_\Omega |\nabla u^d(t)|^2 dx \\
&\qquad - \Lambda\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(t,x_1,1)|^2 + \xi_-|u_1^d(t,x_1,-1)|^2)dx_1 + C_{17}\delta^3F_M^3(t).
\end{aligned} \tag{2.55}$$

On the other hand, we have

$$\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|_{L^2(\Omega)}^2 = 2\int_\Omega (\rho_0+\sigma^\delta(t))u^d(t)\cdot\partial_t u^d(t)dx + \int_\Omega \partial_t\sigma^\delta(t)|u^d(t)|^2dx.$$

Let us recall ϖ_0 from (0.73) and $\nu_0 = \frac{3+\varpi_0}{2+\varpi_0} \in (1,\frac{3}{2})$. We fix two positive constants $m_{1,2}$ such that

$$m_1 = \nu_0 + \sqrt{\nu_0^2-1} \tag{2.56}$$

and that

$$m_2 = \mu(m_1^2-m_1+1) - \mu_c(\Xi)(m_1^2+1) + \sqrt{(\mu(m_1^2-m_1+1)-\mu_c(\Xi)(m_1^2+1))^2 - \mu^2 m_1^2}. \tag{2.57}$$

With $m_1 > 0$ from (2.56), we use Young's inequality to observe

$$\begin{aligned}
&2\int_\Omega (\rho_0+\sigma^\delta(t))u^d(t)\cdot\partial_t u^d(t)dx \\
&\quad\leqslant \frac{1}{\Lambda m_1}\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|_{L^2(\Omega)}^2 + \Lambda m_1\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|_{L^2(\Omega)}^2.
\end{aligned}$$

That will imply

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|_{L^2(\Omega)}^2 &\leqslant \frac{1}{\Lambda m_1}\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|_{L^2(\Omega)}^2 \\
&\qquad + \Lambda m_1\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|_{L^2(\Omega)}^2 + C_{18}\delta^3F_M^3(t).
\end{aligned} \tag{2.58}$$

With $m_2 > 0$ defined as in (2.57), we obtain from (2.55) and (2.58) that

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|_{L^2(\Omega)}^2 + m_2\|\nabla u^d(t)\|_{L^2(\Omega)}^2 \\
&\leqslant \Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|_{L^2(\Omega)}^2 + \Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla u^d\|_{L^2(\Omega)}^2 \\
&\qquad + \frac{2}{\Lambda m_1}\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|\partial_t u_1^d(s,x_1,1)|^2 + \xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1 ds \\
&\qquad - \frac{2\mu}{\Lambda m_1}\int_0^t \|\nabla\partial_t u^d(s)\|_{L^2(\Omega)}^2 ds + C_{19}\delta^3F_M^3(t).
\end{aligned}$$

Together with (2.47), we deduce

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla u^d(t)\|^2_{L^2(\Omega)}\\&\leqslant\Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}+\Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla u^d(t)\|^2_{L^2(\Omega)}\\&\qquad-\frac{2(\mu-\mu_c(\Xi))}{\Lambda m_1}\int_0^t\|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds+C_{19}\delta^3F_M^3(t).\end{aligned}\tag{2.59}$$

We use Young's inequality to get that

$$\begin{aligned}\Big(\frac{\mu}{m_1}+m_2\Big)\|\nabla u^d(t)\|^2_{L^2(\Omega)}&=2\Big(\frac{\mu}{m_1}+m_2\Big)\int_0^t\int_\Omega\nabla u^d(s)\cdot\nabla\partial_t u^d(s)dxds\\&\leqslant\frac{2(\mu-\mu_c(\Xi))}{\Lambda m_1}\int_0^t\|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds\\&\qquad+\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}\int_0^t\|\nabla u^d(s)\|^2_{L^2(\Omega)}ds.\end{aligned}\tag{2.60}$$

Combining (2.59) and (2.60) gives us

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla u^d(t)\|^2_{L^2(\Omega)}\\&\leqslant\Big(m_1+\frac{1}{m_1}\Big)\Lambda\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}\\&\qquad+\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}\int_0^t\|\nabla u^d(s)\|^2_{L^2(\Omega)}ds+C_{19}\delta^3F_M^3(t).\end{aligned}\tag{2.61}$$

It follows from (2.56) and (2.57) that

$$\frac{\Lambda m_1\Big(\frac{\mu}{m_1}+m_2\Big)^2}{2(\mu-\mu_c(\Xi))}=\Lambda\Big(m_1+\frac{1}{m_1}\Big)m_2=2\nu_0\Lambda m_2.$$

Therefore, (2.61) becomes

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}+m_2\|\nabla u^d(t)\|^2_{L^2(\Omega)}\\&\leqslant 2\nu_0\Lambda\Big(\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}+m_2\int_0^t\|\nabla u^d(s)\|^2_{L^2(\Omega)}ds\Big)+C_{19}\delta^3F_M^3(t).\end{aligned}\tag{2.62}$$

Recalling that $u^d(0)=0$, thus, applying Gronwall's inequality to (2.62), one obtains

$$\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)}+m_2\int_0^t\|\nabla u^d(s)\|^2_{L^2(\Omega)}ds\leqslant C_{19}\delta^3e^{2\nu_0\Lambda t}\int_0^t e^{-2\nu_0\Lambda s}F_M^3(s)ds.\tag{2.63}$$

Since $F_M^3(t)\leqslant M^2\max_{j_m\leqslant j\leqslant M}|\mathsf{c}_j|^2F_M(3t)$, we then have from (2.63) that

$$\|u^d(t)\|^2_{L^2(\Omega)}\leqslant C_{20}\delta^3e^{2\nu_0\Lambda t}\sum_{j=j_m}^M\int_0^t|\mathsf{c}_j|e^{(3\lambda_j-2\nu_0\Lambda)s}ds.\tag{2.64}$$

Because of (0.75), we have $\lambda_j > \frac{2}{3}\nu_0\Lambda$ for $j_m \leqslant j \leqslant N$ and $\lambda_j < \frac{2}{3}\nu_0\Lambda$ for $j \geqslant N+1$. It yields that for $j_m \leqslant j \leqslant N$,

$$\int_0^t e^{(3\lambda_j - 2\nu_0\Lambda)s} ds = \frac{1}{3\lambda_j - 2\nu_0\Lambda}(e^{(3\lambda_j-2\nu_0\Lambda)t} - 1) \leqslant \frac{1}{3\lambda_j - 2\nu_0\Lambda} e^{(3\lambda_j-2\nu_0\Lambda)t} \tag{2.65}$$

and that for $j \geqslant N+1$,

$$\int_0^t e^{(3\lambda_j - 2\nu_0\Lambda)s} ds = \frac{1}{3\lambda_j - 2\nu_0\Lambda}(e^{(3\lambda_j-2\nu_0\Lambda)t} - 1) \leqslant \frac{1}{2\nu_0\Lambda - 3\lambda_j}. \tag{2.66}$$

In view of (2.65) and (2.66), we obtain from (2.64) that if $M \leqslant N$,

$$\|u^d(t)\|^2_{L^2(\Omega)} \leqslant C_{20}\delta^3 \Big(\sum_{j=j_m}^{M} \frac{|\mathsf{c}_j|}{3\lambda_j - 2\nu_0\Lambda} e^{3\lambda_j t} \Big)$$

and if $M \geqslant N+1$,

$$\|u^d(t)\|^2_{L^2(\Omega)} \leqslant C_{20}\delta^3 \Big(\sum_{j=j_m}^{M} \frac{|\mathsf{c}_j|}{3\lambda_j - 2\nu_0\Lambda} e^{3\lambda_j t} + \sum_{j=N+1}^{M} \frac{|\mathsf{c}_j|}{2\nu_0\Lambda - 3\lambda_j} e^{2\nu_0\Lambda t} \Big).$$

That means

$$\|u^d(t)\|^2_{L^2(\Omega)} \leqslant C_{21}\delta^3 \Big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \Big(\max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| \Big) e^{\frac{2}{3}\nu_0\Lambda t} \Big)^3. \tag{2.67}$$

To show the bound of $\|\sigma^d(t)\|_{L^2(\Omega)}$, we use Cauchy-Schwarz's inequality to deduce from $(0.79)_1$ that

$$\frac{d}{dt}\|\sigma^d(t)\|_{L^2(\Omega)} \leqslant \|\sigma^d(t)\|_{L^2(\Omega)} \leqslant (\max \rho_0')\|u_2^d(t)\|_{L^2(\Omega)} + \|u^\delta(t)\|_{L^2(\Omega)}\|\sigma^\delta(t)\|_{H^1(\Omega)}.$$

Using (2.38), we obtain

$$\frac{d}{dt}\|\sigma^d(t)\|_{L^2(\Omega)} \leqslant C_{22}(\|u_2^d(t)\|_{L^2(\Omega)} + \delta^2 F_M^2(t)).$$

Note that $\sigma^d(0) = 0$, integrating the resulting inequality in time, we have

$$\|\sigma^d(t)\|_{L^2(\Omega)} \leqslant C_{22} \int_0^t (\|u_2^d(s)\|_{L^2(\Omega)} + \delta^2 F_M^2(s)) ds.$$

Together with (2.67), we have

$$\|\sigma^d(t)\|^2_{L^2(\Omega)} \leqslant C_{23}\delta^3 \Big(\sum_{j=j_m}^{N} |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \Big(\max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| \Big) e^{\frac{2}{3}\nu_0\Lambda t} \Big)^3. \tag{2.68}$$

The inequality (2.40) follows from (2.67) and (2.68). Proof of Proposition 2.8 is complete. □

2.3.3 Proof of Theorem 0.6

Note that

$$\|u^M(t)\|^2_{L^2(\Omega)} = \sum_{i=j_m}^{M} \mathsf{c}_i^2 e^{2\lambda_i t}\|u_i\|^2_{L^2(\Omega)} + 2\sum_{j_m\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i(x)\cdot u_j(x)dx. \quad (2.69)$$

It can be seen that

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant& \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} + 2\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i(x)\cdot u_j(x)dx \\ &- |\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)e^{(\lambda_{j_m}+\lambda_{j_m+1})t}.\end{aligned}$$

By Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}2\sum_{j_m+1\leqslant i<j\leqslant M} &\mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i(x)\cdot u_j(x)dx \\ &\geqslant -2\sum_{j_m+1\leqslant i<j\leqslant M}|\mathsf{c}_i||\mathsf{c}_j|e^{(\lambda_i+\lambda_j)t}\|u_i\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} \\ &\geqslant -e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2.\end{aligned}$$

This yields

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant& \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2 \\ &- |\mathsf{c}_{j_m}|e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big).\end{aligned}$$

Due to the assumption (0.78), we deduce that

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant& \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} - \frac{1}{2}\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|^2_{L^2(\Omega)} \\ &- \frac{1}{4}\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_{j_m}\|^2_{L^2(\Omega)}.\end{aligned}$$

This yields

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant& \mathsf{c}_{j_m}^2\Big(e^{2\lambda_{j_m}t} - \frac{1}{2}e^{(\lambda_{j_m}+\lambda_{j_m+1})t} - \frac{1}{4}e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big)\|u_{j_m}\|^2_{L^2(\Omega)} \\ &+ \sum_{j=j_m+1}^{M}\mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)}.\end{aligned}$$

Notice that for all $t \geqslant 0$,

$$e^{2\lambda_{j_m}t} - e^{(\lambda_{j_m}+\lambda_{j_m+1})t} - e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} \geqslant \frac{1}{4}e^{2\lambda_{j_m}t}.$$

Hence, we have

$$\|u^M(t)\|_{L^2(\Omega)} \geqslant C_{24}F_M(t), \tag{2.70}$$

for all $t \leqslant \min(T^\delta, T^\star, T^{\star\star})$.

Let

$$\tilde{\mathsf{c}}(M) = \max_{N+1\leqslant j\leqslant M} \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|} \geqslant 0.$$

We recall the definition of $T^\star$ and $T^{\star\star}$ from (2.37) and the fact that T^δ satisfies uniquely $\delta F_M(T^\delta) = \epsilon_0$, provided that ϵ_0 is taken to be

$$\epsilon_0 < \min\Big(\frac{C_2\delta_0}{C_3}, \frac{C_2^2}{2C_4(1+M\tilde{\mathsf{c}}(M))^3}, \frac{C_{24}^2}{4C_4(1+M\tilde{\mathsf{c}}(M))^3}\Big). \tag{2.71}$$

We then prove that

$$T^\delta \leqslant \min\{T^\star, T^{\star\star}\}. \tag{2.72}$$

In fact, if $T^\star < T^\delta$, we have from (2.38) that

$$\mathcal{E}((\sigma^\delta, u^\delta)(T^\star)) \leqslant C_3\delta F_M(T^\star) \leqslant C_3\delta F_M(T^\delta) = C_3\epsilon_0 < C_2\delta_0.$$

And if $T^{\star\star} < T^\delta$, we have by (2.40) and the definition of T^δ that

$$\begin{aligned}
&\|(\sigma^\delta, u^\delta)(T^\delta)\|_{L^2(\Omega)} \\
&\leqslant \delta\|(\sigma^M, u^M)(T^\delta)\|_{L^2(\Omega)} + \|(\sigma^d, u^d)(T^\delta)\|_{L^2(\Omega)} \\
&\leqslant C_2\delta F_M(T^\delta) + \sqrt{C_4}\delta^{\frac{3}{2}}\Big(\sum_{j=1}^N |\mathsf{c}_j|e^{\lambda_j T^\delta} + \max(0, M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac{2}{3}\nu_0\Lambda T^\delta}\Big)^{\frac{3}{2}}.
\end{aligned} \tag{2.73}$$

Notice from (0.75) that for $N+1 \leqslant j \leqslant M$,

$$|\mathsf{c}_j|\delta e^{\frac{2}{3}\nu_0\Lambda T^\delta} < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}(\delta|\mathsf{c}_{j_m}|e^{\lambda_1 T^\delta}) < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\delta F_M(T^\delta) = \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\epsilon_0.$$

Then, it follows from (2.73) that

$$\begin{aligned}
\|(\sigma^\delta, u^\delta)(T^\delta)\|_{L^2(\Omega)} &\leqslant C_2\delta F_M(T^\delta) + \sqrt{C_4}\delta^{\frac{3}{2}}(1+M\tilde{\mathsf{c}}(M))^{\frac{3}{2}}F_M^{\frac{3}{2}}(T^\delta) \\
&\leqslant C_2\epsilon_0 + \sqrt{C_4}(1+M\tilde{\mathsf{c}}(M))^{\frac{3}{2}}\epsilon_0^{\frac{3}{2}}.
\end{aligned}$$

Using (2.71) again, we deduce

$$\|(\sigma^\delta, u^\delta)(T^\delta)\|_{L^2(\Omega)} < 2C_2\epsilon_0 = 2C_2\delta F_M(T^\delta).$$

which also contradicts the definition of $T^{\star\star}$.

Once we have (2.72), we then get from (2.40) and (2.70) that

$$\begin{aligned}
&\|u^\delta(T^\delta)\|_{L^2(\Omega)} \\
&\geqslant \delta\|u^M(T^\delta)\|_{L^2(\Omega)} - \|u^d(T^\delta)\|_{L^2(\Omega)} \\
&\geqslant C_{24}\delta F_M(T^\delta) - \sqrt{C_4}\delta^{\frac{3}{2}}\Big(\sum_{j=1}^N |\mathsf{c}_j|e^{\lambda_j T^\delta} + \max(0, M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{\frac{2}{3}\nu_0\Lambda T^\delta}\Big)^{\frac{3}{2}}.
\end{aligned}$$

Therefore,

$$\|u^\delta(T^\delta)\|_{L^2(\Omega)} \geqslant C_{24}\epsilon_0 - \sqrt{C_4}(1 + M\tilde{\mathsf{c}}(M))^{\frac{3}{2}}\epsilon_0^{\frac{3}{2}} \geqslant \frac{C_{24}\epsilon_0}{2} > 0. \tag{2.74}$$

The inequality (0.82) is proven by taking δ_0 satisfying (2.36), ϵ_0 satisfying (2.71) and $m_0 = \frac{C_{24}}{2}$. This ends the proof of Theorem 0.4.

2.4 Proof of Proposition 2.1

2.4.1 The precise value of $\mu_c(k,\Xi)$

In this part, we prove Proposition 2.1(1). The equality (2.5) can be seen immediately from the definition of $\mathcal{B}_{k,0,\mu}$.

Note that the quotient

$$\frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_2}$$

is bounded because of the embedding $H^2((-1,1)) \hookrightarrow C^1((-1,1))$. To prove (2.6), let us consider the Lagrangian functional

$$\mathcal{L}_k(\phi,\beta) = \beta\Big(\int_{-1}^1((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_2 - 1\Big) - (\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2) \tag{2.75}$$

for any $\phi \in \tilde{H}^2((-1,1))$ and $\beta \neq 0$. Using Lagrange multiplier theorem, the extrema of the quotient

$$\frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_2}$$

are necessarily the stationary points of (ϕ_k,β_k) of $\mathcal{L}_k$, which satisfy

$$\int_{-1}^1((\phi_k'')^2 + 2k^2(\phi_k')^2 + k^4\phi_k^2)dx_2 = 1, \tag{2.76}$$

and

$$\beta_k\int_{-1}^1(\phi_k''\omega'' + 2k^2\phi_k'\omega' + k^4\phi_k\omega)dx_2 = \xi_-\phi_k'(-1)\omega'(-1) + \xi_+\phi_k'(1)\omega'(1) \tag{2.77}$$

for all $\omega \in \tilde{H}^2((-1,1))$. We obtain from (2.77) after taking integration by parts that

$$\phi_k^{(4)} - 2k^2\phi_k'' + k^4\phi_k = 0,$$

and

$$\begin{cases} \beta_k(\phi_k'''(1) + 2k^2\phi_k'(1))\omega(1) = 0, \\ (\beta_k\phi_k''(1) - \xi_+\phi_k'(1))\omega'(1) = 0, \\ \beta_k(\phi_k'''(-1) + 2k^2\phi_k'(-1))\omega(-1) = 0, \\ (\beta_k\phi_k''(-1) + \xi_-\phi_k'(-1))\omega'(-1) = 0, \end{cases}$$

for all $\omega \in \tilde{H}^2((-1,1))$. This yields

$$\begin{cases} \beta_k \phi_k''(1) - \xi_+ \phi_k'(1) = 0, \\ \beta_k \phi_k''(-1) + \xi_- \phi_k'(-1) = 0. \end{cases} \tag{2.78}$$

Hence, ϕ_k is of the form

$$\phi_k(x_2) = (Ax_2 + B)\sinh(kx_2) + (Cx_2 + D)\cosh(kx_2),$$

with A, B, C, D are four constants such that $A^2 + B^2 + C^2 + D^2 > 0$. Since $\phi_k \in \tilde{H}^2((-1,1))$, we get

$$\begin{cases} (A + B)\sinh k + (C + D)\cosh k = 0, \\ (-A + B)\sinh(-k) + (-C + D)\cosh(-k) = 0. \end{cases}$$

It yields

$$C = -B\tanh k \quad \text{and} \quad D = -A\tanh k. \tag{2.79}$$

We then compute

$$\phi_k'(x_2) = (A + kD + kCx_2)\sinh(kx_2) + (C + kB + kAx_2)\cosh(kx_2)$$

and

$$\phi_k''(x_2) = (2kC + k^2B + k^2Ax_2)\sinh(kx_2) + (2kA + k^2D + k^2Cx_2)\cosh(kx_2).$$

Substituting these formulas into (2.78), we have

$$\begin{cases} \beta_k\Big((2kC + k^2(B + A))\sinh k + (2kA + k^2(D + C))\cosh k\Big) \\ \qquad = \xi_+\Big((A + k(D + C))\sinh k + (C + k(B + A))\cosh k\Big), \\ \beta_k\Big((2kC + k^2(B - A))\sinh(-k) + (2kA + k^2(D - C))\cosh(-k)\Big) \\ \qquad = -\xi_-\Big((A + k(D - C))\sinh(-k) + (C + k(B - A))\cosh(-k)). \end{cases}$$

Thanks to (2.79), that reduces to

$$\begin{cases} 2k\beta_k(C\sinh k + A\cosh k) \\ \qquad = \xi_+\Big(A + k(D + C)\sinh k + (C + k(B + A))\cosh k\Big), \\ 2k\beta_k(-C\sinh k + A\cosh k) \\ \qquad = \xi_-\Big((A + k(D - C))\sinh(k) - (C + k(B - A))\cosh(k)\Big). \end{cases}$$

Equivalently,

$$\begin{cases} 2k\beta_k\Big(A + B + (A - B)\cosh(2k)\Big) = \xi_+\Big((A - B)\sinh(2k) + 2k(A + B)\Big), \\ 2k\beta_k\Big(A - B + (A + B)\cosh(2k)\Big) = \xi_-\Big((A + B)\sinh(2k) + 2k(A - B)\Big). \end{cases} \tag{2.80}$$

Then, (A, B) is a solution of the following system

$$\begin{cases} A\Big(2k(1+\cosh(2k))\beta_k - \xi_+(2k+\sinh(2k))\Big) \\ \qquad = B\Big(2k(\cosh(2k)-1)\beta_k - \xi_+(\sinh(2k)-2k)\Big), \\ A\Big(2k(1+\cosh(2k))\beta_k - \xi_-(2k+\sinh(2k))\Big) \\ \qquad = -B\Big(2k(\cosh(2k)-1)\beta_k - \xi_-(\sinh(2k)-2k)\Big). \end{cases} \tag{2.81}$$

System (2.81) admits a nontrivial solution if and only

$$\begin{aligned} &\Big(2k(1+\cosh(2k))\beta_k - \xi_+(2k+\sinh(2k))\Big) \\ &\qquad \times \Big(2k(\cosh(2k)-1)\beta_k - \xi_-(\sinh(2k)-2k)\Big) \\ &= -\Big(2k(1+\cosh(2k))\beta_k - \xi_-(2k+\sinh(2k))\Big) \\ &\qquad \times \Big(2k(\cosh(2k)-1)\beta_k - \xi_+(\sinh(2k)-2k)\Big). \end{aligned} \tag{2.82}$$

We rewrite Eq. (2.82) as a quadratic equation of β_k, that is

$$\begin{aligned} 4k^2(\cosh^2(2k)-1)\beta_k^2 - 2k(\sinh(2k)\cosh(2k)-2k)(\xi_+ + \xi_-)\beta_k \\ + (\sinh^2(2k)-4k^2)\xi_+\xi_- = 0. \end{aligned} \tag{2.83}$$

The discriminant is

$$\begin{aligned} \Delta_{k,\Xi} &= k^2(\sinh(2k)\cosh(2k)-2k)^2(\xi_+ + \xi_-)^2 \\ &\quad - 4k^2(\cosh^2(2k)-1)(\sinh^2(2k)-4k^2)\xi_+\xi_- \\ &= k^2(\sinh(2k)-2k\cosh(2k))^2(\xi_+ + \xi_-)^2 \\ &\quad + k^2\sinh^2(2k)(\sinh^2(2k)-4k^2)(\xi_+ - \xi_-)^2. \end{aligned}$$

Because $\tanh(2k) < 2k$ for all $k > 0$ and $\xi_+^2 + \xi_-^2 > 0$, we have $\Delta_{k,\xi}$ is always positive. Hence, we have that (2.83) has two roots

$$\beta_{k,\pm} = \frac{k(\sinh(2k)\cosh(2k)-2k)(\xi_+ + \xi_-) \pm \sqrt{\Delta_{k,\Xi}}}{4k^2\sinh^2(2k)}.$$

We take the higher value $\beta_{k,+} > 0$ and then solve the system (2.81) as $\beta_k = \beta_{k,+}$.

If $\xi_- \geqslant \xi_+$, we have

$$\begin{aligned} &4\sinh^2 k\Big(2k(\cosh(2k)-1)\beta_{k,+} - \xi_+(\sinh(2k)-2k)\Big) \\ &= (\sinh(2k)\cosh(2k)-2k)(\xi_- - \xi_+) - 2(\sinh(2k)-2k\cosh(2k))\xi_+ + \frac{1}{k}\sqrt{\Delta_{k,\Xi}} \\ &> 0. \end{aligned}$$

Then, we obtain from $(2.81)_1$ that

$$B = A\frac{2k(1+\cosh(2k))\beta_{k,+} - \xi_+(2k+\sinh(2k))}{2k(\cosh(2k)-1)\beta_{k,+} - \xi_+(\sinh(2k)-2k)} =: Aa_{k,\Xi}. \tag{2.84}$$

So that

$$(A, B, C, D) = A(1, a_{k,\Xi}, -a_{k,\Xi} \tanh k, -\tanh k)$$

with $A \neq 0$ and $\phi_k(x_2) = Az_k(x_2)$, with

$$z_k(x_2) = (x_2 + a_{k,\Xi}) \sinh(kx_2) - \tanh k(a_{k,\Xi}x_2 + 1) \cosh(kx_2).$$

We find A from (2.76), such that

$$A^2 \int_{-1}^{1} ((z_k'')^2 + 2k^2(z_k')^2 + k^4 z_k^2) dx_2 = 1. \tag{2.85}$$

If $0 < \xi_- < \xi_+$, that will imply

$$2k(\cosh(2k) - 1)\beta_{k,+} - \xi_-(\sinh(2k) - 2k) > 0.$$

We further get from $(2.81)_2$ that

$$B = -A \frac{2k(1 + \cosh(2k))\beta_{k,+} - \xi_-(2k + \sinh(2k))}{2k(\cosh(2k) - 1)\beta_{k,+} - \xi_-(\sinh(2k) - 2k)} =: -Ab_{k,\Xi}.$$

So that, we have

$$(A, B, C, D) = A(1, -b_{k,\Xi}, b_{k,\Xi} \tanh k, -\tanh k)$$

with $A \neq 0$ and $\phi_k(x_2) = Aw_k(x_2)$, with

$$w_k(x_2) = (x_2 - b_{k,\Xi}) \sinh(kx_2) + (b_{k,\Xi}x_2 - 1) \tanh k \cosh(kx_2).$$

We still find A from (2.76),

$$A^2 \int_{-1}^{1} ((w_k'')^2 + 2k^2(w_k')^2 + k^4 w_k^2) dx_2 = 1. \tag{2.86}$$

We obtain that

$$\begin{aligned}\mu_c(k, \Xi) &= \max_{\phi \in \tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^{1}((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_2} \\ &= \frac{1}{4k \sinh^2(2k)} \left(\begin{array}{l} (\sinh(2k)\cosh(2k) - 2k)(\xi_+ + \xi_-) \\ + \left(\begin{array}{l} (\sinh(2k) - 2k\cosh(2k))^2(\xi_+ + \xi_-)^2 \\ + \sinh^2(2k)(\sinh^2(2k) - 4k^2)(\xi_+ - \xi_-)^2 \end{array} \right)^{\frac{1}{2}} \end{array} \right).\end{aligned}$$

That variational problem is attained by the function $\phi_k(x_2) = Az_k(x_2)$, where A satisfies (2.85) or $\phi_k(x_2) = Aw_k(x_2)$, where A satisfies (2.86). The equality (2.6) is shown and the first part of Proposition 2.1 then follows.

2.4.2 Asymptotic behavior of $\mu_c(k,\Xi)$ in low/high regime of wave number

Let us prove Proposition 2.1(2). Clearly, we have that $\mu_c(k,\Xi)$ is a decreasing function in $k>0$. It yields (2.7).

We first consider $k\to 0$. Let us recall the Taylor's expansion of $\sinh(2k)$ and $\cosh(2k)$. We have

$$\sinh(2k) = 2k + \frac{4}{3}k^3 + \frac{4}{15}k^5 + O(k^6), \quad \text{and} \quad \cosh(2k) = 1 + 2k^2 + \frac{2}{3}k^4 + O(k^5).$$

We deduce that

$$\frac{\sinh(2k)\cosh(2k) - 2k}{4k\sinh^2(2k)} = \frac{\frac{1}{6} + \frac{2}{15}k^2 + O(k^3)}{\frac{1}{2} + \frac{2}{3}k^2 + O(k^3)} = \frac{1}{3} - \frac{8}{15}k^2 + O(k^3),$$

that

$$\frac{\sinh(2k) - 2k\cosh(2k)}{4k\sinh^2(2k)} = \frac{-\frac{8}{3} - \frac{16}{15}k^2 + O(k^3)}{16 + \frac{64}{3}k^2 + O(k^3)} = -\frac{1}{6} + \frac{7}{45}k^2 + O(k^3)$$

and that

$$\frac{\sinh^2(2k)(\sinh^2(2k) - 4k^2)}{16k^2\sinh^4(2k)} = \frac{\frac{16}{3} + \frac{128}{45}k^2 + O(k^2)}{64 + \frac{256}{3}k^2 + O(k^2)} = \frac{1}{12} - \frac{1}{15}k^2 + O(k^3).$$

We deduce that

$$\lim_{k\to 0}\mu_c(k,\Xi) = \frac{1}{3}(\xi_+ + \xi_-) + \sqrt{\frac{1}{36}(\xi_+ + \xi_-)^2 + \frac{1}{12}(\xi_+ - \xi_-)^2}. \tag{2.87}$$

That will imply (2.9), i.e.

$$\mu_c^s(\Xi) = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

Furthermore, we have that

$$\lim_{k\to 0}\frac{\mu_c(k,\Xi) - \mu_c^s(\Xi)}{k^2} = -\frac{2}{15}\Big(4(\xi_+ + \xi_-) + \frac{4\xi_+^2 - \xi_+\xi_- + 4\xi_-^2}{\sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-}}\Big). \tag{2.88}$$

Two limits (2.87) and (2.88) help us to get (2.8).

For high wave number, i.e. $k\to+\infty$, we can see that

$$\frac{\sinh(2k)\cosh(2k) - 2k}{\sinh^2(2k)} = \frac{1 - e^{-8k} - 8ke^{-4k}}{1 + e^{-8k} - 2e^{-4k}} \leqslant 2,$$

that

$$\frac{\sinh(2k) - 2k\cosh(2k)}{\sinh^2(2k)} = \frac{1}{2}\frac{1 - 2k - (1+2k)e^{-4k}}{e^{2k} + e^{-6k} - 2e^{-2k}} \leqslant 1.$$

Hence,

$$\mu_c(k,\Xi) = \frac{1}{4k}\left(\begin{array}{l}\frac{\sinh(2k)\cosh(2k)-2k}{\sinh^2(2k)}(\xi_+ + \xi_-) \\ + \left(\begin{array}{l}\left(\frac{\sinh(2k)-2k\cosh(2k)}{\sinh^2(2k)}\right)^2(\xi_+ + \xi_-)^2 \\ + \left(1 - \frac{4k^2}{\sinh^2(2k)}\right)(\xi_+ - \xi_-)^2\end{array}\right)^{\frac{1}{2}}\end{array}\right)$$
$$\leqslant \frac{1}{4k}\Big(2(\xi_+ + \xi_-) + \sqrt{2(\xi_+^2 + \xi_-^2)}\Big).$$

That implies (2.10). The proof of the second assertion of Proposition 2.1 is complete.

2.4.3 Proof of Proposition 2.1(3)

In this appendix, we prove Proposition 2.1(3). We first show that

$$\mu_c^s(\Xi) = \sup_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2} \tag{2.89}$$

Indeed, we write

$$\tilde{\mu}_c(\Xi) = \sup_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2}$$

and then prove that $\mu_c^s(\Xi) = \tilde{\mu}_c(\Xi)$. Clearly, we have $\mu_c(k,\Xi) \leqslant \tilde{\mu}_c(\Xi)$ for all $k \in \mathbf{R}\backslash\{0\}$. It yields $\mu_c^s(\Xi) \leqslant \tilde{\mu}_c(\Xi)$. It suffices to show that $\tilde{\mu}_c(\Xi) \geqslant \mu_c^s(\Xi)$. For any $\varepsilon > 0$, we fix a function $\phi_\varepsilon \in \tilde{H}^2((-1,1))$ such that

$$\frac{\xi_-(\phi_\varepsilon'(-1))^2 + \xi_+(\phi_\varepsilon'(1))^2}{\int_{-1}^1(\phi_\varepsilon'')^2 dx_2} \geqslant \tilde{\mu}_c(\Xi) - \varepsilon.$$

Let $k \neq 0$ be small enough, we then obtain

$$\frac{\xi_-(\phi_\varepsilon'(-1))^2 + \xi_+(\phi_\varepsilon'(1))^2}{\int_{-1}^1((\phi_\varepsilon'')^2 + 2k^2(\phi_\varepsilon')^2 + k^4\phi_\varepsilon^2)dx_2} > \frac{\xi_-(\phi_\varepsilon'(-1))^2 + \xi_+(\phi_\varepsilon'(1))^2}{\int_{-1}^1(\phi_\varepsilon'')^2 dx_2} - \varepsilon.$$

That implies

$$\mu_c(k,\Xi) > \tilde{\mu}_c(\Xi) - 2\varepsilon.$$

We deduce that $\tilde{\mu}_c(\Xi) = \sup_{k\in\mathbf{R}\backslash\{0\}} \mu_c(k,\Xi)$, i.e. (2.89).

Then, we show that

$$\max_{\phi\in\tilde{H}^2((-1,1))} \frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2 dx_2} = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big). \tag{2.90}$$

Let us consider the Lagrangian functional

$$\mathcal{L}_0(\phi,\beta) = \beta\Big(\int_{-1}^1 (\phi'')^2 dx_2 - 1\Big) - \xi_-(\phi'(-1))^2 - \xi_+(\phi'(1))^2. \tag{2.91}$$

for any $\phi \in \tilde{H}^2((-1,1))$ and $\beta \neq 0$. Owing to Lagrange multiplier theorem, we find that the extrema of the quotient

$$\frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^{1}(\phi'')^2 dx_2}$$

are necessarily the stationary points (ϕ_0, β_0) of $\mathcal{L}_0$, which satisfy

$$\int_{-1}^{1}(\phi_0'')^2 dx_2 = 1, \tag{2.92}$$

and

$$\beta_0 \int_{-1}^{1} \phi_0''\omega'' dx_2 - (\xi_-\phi_0'(-1)\omega'(-1) + \xi_+\phi_0'(1)\omega'(1)) = 0 \tag{2.93}$$

for all $\omega \in \tilde{H}^2((-1,1))$. We obtain from (2.93) after taking integration by parts that

$$\phi_0^{(4)} = 0 \quad \text{on } (-1,1).$$

and

$$\begin{cases} \phi_0''(1) = \xi_+\phi_0'(1), \\ \phi_0''(-1) = -\xi_-\phi_0'(-1). \end{cases} \tag{2.94}$$

Hence, ϕ_0 is of the form

$$\phi_0(x_2) = (x_2^2 - 1)(Ax_2 + B).$$

Substituting this form of ϕ_0 into (2.94), we have that

$$\begin{cases} \beta_0(3A + B) = \xi_+(A + B), \\ \beta_0(3A - B) = \xi_-(A - B). \end{cases}$$

Hence,

$$\begin{cases} A(3\beta_0 - \xi_+) + B(\beta_0 - \xi_+) = 0, \\ A(3\beta_0 - \xi_-) - B(\beta_0 - \xi_-) = 0. \end{cases} \tag{2.95}$$

System (2.95) admits a nontrivial solution (A, B) if and only if

$$(3\beta_0 - \xi_+)(\beta_0 - \xi_-) + (3\beta_0 - \xi_-)(\beta_0 - \xi_+) = 0.$$

It yields

$$3\beta_0^2 - 2(\xi_+ + \xi_-)\beta_0 + \xi_-\xi_+ = 0. \tag{2.96}$$

The discriminant of (2.96) is

$$\Delta_{0,\xi} = (\xi_+ + \xi_-)^2 - 3\xi_-\xi_+ = \xi_+^2 - \xi_+\xi_- + \xi_-^2 > 0.$$

Then, Eq. (2.96) has two roots

$$\beta_{0,\pm} = \frac{1}{3}\Big(\xi_+ + \xi_- \pm \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

We take the higher value $\beta_{0,+}$. As $\beta_0 = \beta_{0,+}$, we have from $(2.95)_2$ that

$$A(3\beta_{0,+} - \xi_-) = B(\beta_{0,+} - \xi_-).$$

It is obvious that

$$3\beta_{0,+} - \xi_- = \xi_+ + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2} > 0.$$

Then we have

$$A = B\frac{\beta_{0,+} - \xi_-}{3\beta_{0,+} - \xi_-}$$

and

$$\phi_0(x_2) = Bz_0(x_2), \quad \text{with } z_0(x_2) = \Big(\frac{\beta_{0,+} - \xi_-}{3\beta_{0,+} - \xi_-}x_2 + 1\Big)(x_2^2 - 1).$$

We continue using (2.92) to find a non-zero B. This yields

$$B^2\int_{-1}^{1}(z_2''(x_2))^2dx_2 = 1.$$

That is equivalent to

$$8B^2\Big(3\frac{(\beta_{0,+} - \xi_-)^2}{(3\beta_{0,+} - \xi_-)^2} + 4\Big) = 1.$$

this yields

$$B = \pm\frac{1}{2\sqrt{2}}\frac{3\beta_{0,+} - \xi_-}{\sqrt{39\beta_{0,+}^2 - 30\beta_{0,+}\xi_- + 7\xi_-^2}}.$$

That means, we observe

$$\max_{\phi\in\tilde{H}^2((-1,1))}\frac{\xi_-(\phi'(-1))^2 + \xi_+(\phi'(1))^2}{\int_{-1}^1(\phi'')^2dx_2} = \frac{1}{3}\Big(\xi_+ + \xi_- + \sqrt{\xi_+^2 - \xi_+\xi_- + \xi_-^2}\Big).$$

That variational problem is attained by the function

$$\phi_0(x_2) = \pm\frac{1}{2\sqrt{2}}\frac{3\beta_{0,+} - \xi_-}{\sqrt{39\beta_{0,+}^2 - 30\beta_{0,+}\xi_- + 7\xi_-^2}}\Big(\frac{\beta_{0,+} - \xi_-}{3\beta_{0,+} - \xi_-}x_2 + 1\Big)(x_2^2 - 1).$$

Combining (2.89) and (2.90), we obtain Proposition 2.1(3).

2.5 Comments on the paper of Ding, Zi and Li

In [14], the authors Ding, Zi and Li construct an approximate solution generated by the maximal normal mode, $(\sigma^a, u^a, q^a)(t,x) = \delta e^{\lambda_1(k)t}U_1(x)$ with k being fixed such that $\frac{2\Lambda}{3} < \lambda_1(k) < \Lambda$. Applying Proposition 0.1, the nonlinear equations (0.55)-(0.56) with the initial data

$$(\sigma^\delta, u^\delta, q^\delta)(0) = (\sigma^a, u^a, q^a)(0).$$

admits a strong solution $(\sigma^\delta, u^\delta) \in C^0([0, T^{\max}), H^1 \times H^2)$ with an associated pressure $q^\delta \in C^0([0, T^{\max}), L^2)$. Let T^δ such that $\delta e^{\lambda_1 T^\delta} = \epsilon_0 \ll 1$. We define

$$\begin{aligned} T^\star &:= \sup \Big\{ t \in (0, T^{\max}) | \mathcal{E}(\sigma^\delta(t), u^\delta(t)) \leqslant C\delta_0 \} > 0, \\ T^{\star\star} &:= \sup \{ t \in (0, T^{\max}) | \|(\sigma^\delta, u^\delta)(t)\|_{L^2(\Omega)} \leqslant C\delta e^{\lambda_1 t} \Big\} > 0. \end{aligned}$$

Then for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, we have

$$\mathcal{E}^2(\sigma^\delta(t), u^\delta(t)) + \|\partial_t u^\delta(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla \partial_t u^\delta(\tau)\|^2_{L^2(\Omega)} d\tau \leqslant C\delta^2 e^{2\lambda_1 t}.$$

In [14, Proposition 5.2], they claim that the difference functions

$$(\sigma^d, u^d, q^d) = (\sigma^\delta, u^\delta, q^\delta) - (\sigma^a, u^a, q^a)$$

enjoy

$$\|(\sigma^d, u^d)\|^2_{L^2(\Omega)} \leqslant C\delta^3 e^{3\lambda_1 t} \tag{2.97}$$

for all $\mu > 0$. We believe that (2.97) needs to be corrected, not for all $\mu > 0$. Precisely, we are in doubt about inequality **(137)** in that paper, that is for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$,

$$\begin{aligned} &\|\sqrt{\rho_0 + \sigma^\delta(t)} \partial_t u^d(t)\|^2_{L^2(\Omega)} + \Lambda\mu \|\nabla u^d(t)\|^2_{L^2(\Omega)} + \mu \int_0^t \|\nabla \partial_t u^d(s)\|^2_{L^2(\Omega)} ds \\ &\leqslant \Lambda \Big(\int_0^t \|\sqrt{\rho_0 + \sigma^\delta(s)} u^d(s)\|^2_{L^2(\Omega)} + \Lambda\mu \|\nabla u^d(s)\|^2_{L^2(\Omega)} \Big) \\ &\qquad + \Lambda \|\sqrt{\rho_0 + \sigma^\delta(t)} u^d(t)\|^2_{L^2(\Omega)} + C\delta^3 e^{3\lambda_1 t}. \end{aligned} \tag{2.98}$$

Due to (2.98) and the following inequality

$$\begin{aligned} &\frac{d}{dt} \|\sqrt{\rho_0 + \sigma^\delta(t)} u^d(t)\|^2_{L^2(\Omega)} \\ &= 2 \int_\Omega (\rho_0 + \sigma^\delta(t)) u^d(t) \cdot \partial_t u^d(t) dx + \int_\Omega \partial_t \sigma^\delta(t) |u^d(t)|^2 dx \\ &\leqslant \frac{1}{\Lambda} \|\sqrt{\rho_0 + \sigma^\delta(t)} \partial_t u^d(t)\|^2_{L^2(\Omega)} + \Lambda \|\sqrt{\rho_0 + \sigma^\delta(t)} u^d(t)\|^2_{L^2(\Omega)} + C\delta^3 e^{3\lambda_1 t}, \end{aligned} \tag{2.99}$$

it is claimed in [14, **(138)**] that

$$\begin{aligned} &\frac{d}{dt} \|\sqrt{\rho_0 + \sigma^\delta(t)} u^d(t)\|^2_{L^2(\Omega)} + \Big(\|\sqrt{\rho_0 + \sigma^\delta(t)} \partial_t u^d(t)\|^2_{L^2(\Omega)} + \Lambda\mu \|\nabla u^d(t)\|^2_{L^2(\Omega)} \Big) \\ &\leqslant \Lambda \int_0^t \Big(\|\sqrt{\rho_0 + \sigma^\delta(s)} \partial_t u^d(s)\|^2_{L^2(\Omega)} + \Lambda\mu \|\nabla u^d(s)\|^2_{L^2(\Omega)} \Big) ds \\ &\qquad + \Lambda \|\sqrt{\rho_0 + \sigma^\delta(t)} u^d(t)\|^2_{L^2(\Omega)} + C\delta^3 e^{3\lambda_1 t}. \end{aligned} \tag{2.100}$$

The inequality (2.97) is followed by applying Gronwall's inequality to (2.100).

We shall explain the arguments of (2.98) in [14]. First, we still have

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|^2_{L^2(\Omega)} + 2\mu\int_0^t \|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds \\
&\qquad - 2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds \\
&= \int_\Omega g\rho_0'|u_2^d(t)|^2dx + \Big(\int_\Omega(\rho_0+\sigma^\delta(t))|\partial_t u^d(t)|^2dx\Big)\Big|_{t=0} \\
&\qquad + \int_0^t\int_\Omega(2\partial_t f^\delta(s)+2gu^\delta(s)\cdot\nabla\sigma^\delta(s)e_2-\partial_t\sigma^\delta(s)\partial_t u^d(s))\cdot\partial_t u^d(s)ds.
\end{aligned} \tag{2.101}$$

We estimate

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|^2_{L^2(\Omega)} + 2\mu\int_0^t \|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds \\
&\qquad - 2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds \\
&\leqslant \int_\Omega g\rho_0'|u_2^d(t)|^2dx + C\delta^3e^{3\lambda_1 t}.
\end{aligned} \tag{2.102}$$

That implies

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|^2_{L^2(\Omega)} + 2\mu\int_0^t \|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds \\
&\qquad - 2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2+\xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1ds \\
&\leqslant \Lambda^2\int_\Omega(\rho_0+\sigma^\delta(t))|u^d(t)|^2dx + \Lambda\mu\int_\Omega|\nabla u^d(t)|^2dx \\
&\qquad - \Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1 + C\delta^3e^{3\lambda_1 t}.
\end{aligned} \tag{2.103}$$

By using the inequality

$$\Lambda\mu\|\nabla u^d\|^2_{L^2(\Omega)} \leqslant \Lambda^2\mu\int_0^t\|\nabla u^d(s)\|^2_{L^2(\Omega)}ds + \mu\int_0^t\|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds$$

and the identity

$$\begin{aligned}
&\Lambda\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(t,x_1,1)|^2+\xi_-|u_1^d(t,x_1,-1)|^2)dx_1 \\
&= \Lambda^2\int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|u_1^d(s,x_1,1)|^2+\xi_-|u_1^d(s,x_1,-1)|^2)dx_1ds \\
&\qquad + \int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\partial_t u_1^d(s,x_1,1)|^2+\xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1ds \\
&\qquad - \int_0^t\int_{2\pi L\mathbb{T}}(\xi_+|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,1)+\xi_-|\Lambda u_1^d-\partial_t u_1^d|^2(s,x_1,-1))dx_1ds,
\end{aligned}$$

it is obtained from (2.103) that (see **(134)** in [14])

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\Lambda\Big(\mu\|\nabla u^d(t)\|^2_{L^2(\Omega)} - \int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(t,x_1,1)|^2 + \xi_-|u_1^d(t,x_1,-1)|^2)dx_1\Big) \\
&\quad + \frac{1}{2}\int_0^t \Big(\mu\|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)} - \int_{2\pi L\mathbb{T}} (\xi_+|\partial_t u_1^d(s,x_1,1)|^2 + \xi_-|\partial_t u_1^d(s,x_1,-1)|^2)dx_1\Big)ds \\
&\leqslant \Lambda^2\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)} + C\delta^3 e^{\lambda_1 t} \\
&\quad + \frac{3}{2}\Lambda^2\int_0^t \Big(\mu\|\nabla u^d(s)\|^2_{L^2(\Omega)} - \int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(s,x_1,1)|^2 + \xi_-|u_1^d(s,x_1,-1)|^2)dx_1\Big)ds \\
&\quad + \frac{3}{2}\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|\Lambda u_1^d - \partial_t u_1^d|^2(s,x_1,1) + \xi_-|\Lambda u_1^d - \partial_t u_1^d|^2(s,x_1,-1))dx_1 ds.
\end{aligned} \tag{2.104}$$

Integrating (2.99) in time from 0 to t and using (2.104) and Young's inequality, the authors deduce (2.98) without providing any detailed explanations.

However, we observe by integrating (2.99) in time that

$$\|\sqrt{\rho_0+\sigma^\delta(t)}u^d(t)\|^2_{L^2(\Omega)} \leqslant \frac{1}{\Lambda}\int_0^t e^{\Lambda(t-s)}\|\partial_t u^d(s)\|^2_{L^2(\Omega)}ds + Ce^{3\lambda_1 t}.$$

Then the l.h.s of (2.98) will be bounded by

$$\begin{aligned}
&\|\sqrt{\rho_0+\sigma^\delta(t)}\partial_t u^d(t)\|^2_{L^2(\Omega)} + \Lambda\mu\|\nabla u^d(t)\|^2_{L^2(\Omega)} + \mu\int_0^t \|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds \\
&\leqslant \Lambda\int_0^t e^{\Lambda(t-s)}\|\partial_t u^d(s)\|^2_{L^2(\Omega)}ds + \frac{1}{2}\Lambda\mu\|\nabla u^d(t)\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2}\mu\int_0^t \|\nabla\partial_t u^d(s)\|^2_{L^2(\Omega)}ds + \frac{3}{2}\Lambda^2\mu\int_0^t \|\nabla u^d(s)\|^2_{L^2(\Omega)}ds \\
&\quad + \Lambda\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(t,x_1,1)|^2 + \xi_-|u_1^d(t,x_1,-1)|^2)dx_1 \\
&\quad - \Lambda^2\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|u_1^d(s,x_1,1)|^2 + \xi_-|u_1^d(s,x_1,-1)|^2)dx_1 ds \\
&\quad + \int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|\Lambda u_1^d - \partial_t u_1^d|^2(s,x_1,1) + \xi_-|\Lambda u_1^d - \partial_t u_1^d|^2(s,x_1,-1))dx_1 + C\delta^3 e^{3\lambda_1 t}.
\end{aligned} \tag{2.105}$$

We are not clear about the way in [14] to remove all integral terms over $2\pi L\mathbb{T}$ in the r.h.s of (2.105) to get (2.98) for all $\mu > 0$, especially the following term

$$\int_0^t\int_{2\pi L\mathbb{T}} (\xi_+|\Lambda u_1^d - \partial_t u_1^d|^2(s,x_1,1) + \xi_-|\Lambda u_1^d - \partial_t u_1^d|^2(s,x_1,-1))dx_1.$$

Chapter 3

Nonlinear Rayleigh-Taylor instability of the viscous surface wave in an infinitely deep ocean

This chapter is presented in the paper [62], mentioned in Section 0.6. We consider the boundary value problem on a moving domain and with a free surface. No surface tension is accounted for. Hence, the first step is to use a Lagrangian transformation to transform the original problem into another problem on the fixed domain $\Omega = \mathbf{T}^2 \times \mathbf{R}_-$. Let us consider an increasing profile ρ_0 such that ρ_0' is compactly supported. The spectral analysis in this setting is in the same spirit as the one in the first paper [51] and we obtain infinitely many characteristic values to the linearized equations.

In this case, there are two striking differences to show the nonlinear instability. The first reason is due to the compatibility conditions for the initial data of the nonlinear equations, that are not satisfied in general by the normal modes of the linearized equations. A modification of the normal modes is needed to fulfil the compatibility conditions. The second one is the presence of more nonlinear terms due to the Lagrangian transformation, that requires more efforts to construct the *a priori* energy estimates, such as using Gagliardo-Nirenberg's inequality. The linear and nonlinear instability occur for all positive viscosity.

3.1 Preliminaries

We will employ the Einstein convention of summing over repeated indices. We present here some material.

Proof of Lemma 0.2. Thanks to Fubini's theorem and Parseval's formula, we obtain

$$\begin{aligned}\|\nabla^q \mathsf{p} f\|^2_{L^2(\Omega)} &\lesssim \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}} \int_{-\infty}^0 |\mathbf{k}|^{2q}|\hat{f}(\mathbf{k})|^2 e^{2|\mathbf{k}|x_3}dx_3 \\ &\lesssim \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}} |\mathbf{k}|^{2q-1}|\hat{f}(\mathbf{k})|^2.\end{aligned}$$

The inequality (0.89) then follows.

Product estimate. Suppose that $\Sigma = \Omega$ or Γ, let $f \in H^{s_1}(\Sigma), g \in H^{s_2}(\Sigma)$,

1. if $0 \leqslant r \leqslant s_1 \leqslant s_2$ and $s_2 > r + 3/2$, then $fg \in H^r(\Sigma)$,
2. if $0 \leqslant r \leqslant s_1 \leqslant s_2$ and $s_1 > 3/2$, then $fg \in H^r(\Sigma)$.

In both cases, we have

$$\|fg\|_{H^r(\Sigma)} \lesssim \|f\|_{H^{s_1}(\Sigma)}\|g\|_{H^{s_2}(\Sigma)}, \tag{3.1}$$

We refer to [33, Lemma 10.1] for the proof of (3.1).

Gagliardo-Nirenberg's inequality. Let $s \geqslant 0$, $\Sigma = \Omega$ or Γ and $f, g \in H^s(\Sigma) \cap L^\infty(\Sigma)$, we have

$$\|fg\|_{H^s(\Sigma)} \lesssim \|f\|_{H^s(\Sigma)}\|g\|_{L^\infty(\Sigma)} + \|f\|_{L^\infty(\Sigma)}\|g\|_{H^s(\Sigma)}. \tag{3.2}$$

Elliptic estimates. Let $r \geqslant 2$ and $\phi \in H^{r-2}(\Omega), \psi \in H^{r-1}(\Omega)$ and $\alpha \in H^{r-1/2}(\Gamma)$. There exist unique $u \in H^r(\Omega)$ and $q \in H^{r-1}(\Omega)$ solving

$$\begin{cases} -\Delta u + \nabla q = \phi & \text{in } \Omega, \\ \mathrm{div} u = \psi & \text{in } \Omega, \\ (q\mathrm{Id} - \mu \mathbb{S}u)e_3 = \alpha & \text{on } \Gamma. \end{cases}$$

Moreover, we have

$$\|u\|^2_{H^r(\Omega)} + \|q\|^2_{H^{r-1}(\Omega)} \lesssim \|\phi\|^2_{H^{r-2}(\Omega)} + \|\psi\|^2_{H^{r-1}(\Omega)} + \|\alpha\|^2_{H^{r-3/2}(\Gamma)}. \tag{3.3}$$

thanks to [33, Lemma A.15] for example.

We also recall the classical regularity theory for the Stokes problem with Dirichlet boundary conditions (see [76, Theorem 2.4] after using the domain expansion technique). Let $r \geqslant 2$ and $f \in H^{r-2}(\Omega), g \in H^{r-1}(\Omega)$ and $h \in H^{r-1/2}(\Gamma)$ such that

$$\int_\Omega g = \int_\Gamma h \cdot \nu, \text{ where } \nu \text{ is the outward unit normal vector to the boundary.}$$

There exist uniquely $u \in H^r(\Omega)$ and $q \in H^{r-1}(\Omega)$ solving

$$\begin{cases} -\Delta u + \nabla q = f & \text{in } \Omega, \\ \mathrm{div} u = g & \text{in } \Omega, \\ u = h & \text{on } \Gamma. \end{cases}$$

There also holds

$$\|u\|^2_{H^r(\Omega)} + \|q\|^2_{H^{r-1}(\Omega)} \lesssim \|f\|^2_{H^{r-2}(\Omega)} + \|g\|^2_{H^{r-1}(\Omega)} + \|h\|^2_{H^{r-1/2}(\Gamma)}. \tag{3.4}$$

Korn's inequality. The following Korn's inequality is proven in [47, Theorem 5.12],

$$\|\nabla u\|^2_{L^2(\Omega)} \lesssim \|\mathbb{S}u\|^2_{L^2(\Omega)}. \tag{3.5}$$

Commutator estimates.

Let $\mathcal{J} = \sqrt{1 - \partial_1^2 - \partial_2^2}$ and let us define the commutator

$$[\mathcal{J}^s, f]g = \mathcal{J}^s(fg) - f\mathcal{J}^s g.$$

We have

$$\|[\mathcal{J}^s, f]g\|_{L^2(\Gamma)} \lesssim \|\nabla f\|_{L^\infty(\Gamma)}\|\mathcal{J}^{s-1}g\|_{L^2(\Gamma)} + \|\mathcal{J}^s f\|_{L^2(\Gamma)}\|g\|_{L^\infty(\Gamma)}. \tag{3.6}$$

The proof of (3.6) is similar to that one of [46, Lemma X1].

Interpolation inequality. It can be found in [1, Chapter 5] that

$$\|u\|_{H^j(\Omega)} \lesssim \|u\|^{1/(j+1)}_{L^2(\Omega)}\|u\|^{j/(j+1)}_{H^{j+1}(\Omega)}.$$

That implies for $\varepsilon > 0$, there is a universal constant $C(j)$ such that

$$\|u\|_{H^j(\Omega)} \leqslant \varepsilon\|u\|_{H^{j+1}(\Omega)} + C(j)\varepsilon^{-j}\|u\|_{L^2(\Omega)}. \tag{3.7}$$

Coefficient estimates. If $\|\eta\|_{H^{5/2}(\Gamma)} \lesssim 1$, we have

$$\|J - 1\|_{L^\infty(\Omega)} + \|\mathcal{N} - 1\|_{L^\infty(\Gamma)} + \|K - 1\|_{L^\infty(\Gamma)} \lesssim \|\eta\|_{H^{5/2}(\Gamma)}. \tag{3.8}$$

Also, the map Θ defined by (0.91) is a diffeomorphism. We refer to [34, Lemma 2.4] for the proof of (3.8). In the following lemma, we provide some additional estimates.

Lemma 3.1. *Under the assumption $\|\eta\|_{H^{9/2}(\Gamma)} \lesssim 1$, the following inequalities hold*

$$\|\partial_t^l(A, B)\|_{H^s(\Omega)} \lesssim \|\partial_t^l\eta\|_{H^{s+1/2}(\Gamma)} \quad \text{for any } 0 \leqslant l \leqslant 2 \text{ and } 0 \leqslant s \leqslant 4, \tag{3.9}$$

and

$$\begin{cases} \|K - 1\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)} & \text{for } 0 \leqslant s \leqslant 4, \\ \|\partial_t K\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)} & \text{for } 0 \leqslant s \leqslant 2, \\ \|\partial_t^2 K\|_{L^2(\Omega)} \lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}, \\ \|\partial_t^3 K\|_{L^2(\Omega)} \lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \|(AK, BK)\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)} \quad \textit{for } 0 \leqslant s \leqslant 4, \\ \|\partial_t(AK, BK)\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)} \quad \textit{for } 0 \leqslant s \leqslant 2, \\ \|\partial_t^2(AK, BK)\|_{L^2(\Omega)} \lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}, \\ \|\partial_t^3(AK, BK)\|_{L^2(\Omega)} \lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}, \end{cases} \tag{3.11}$$

and

$$\begin{cases} \|\mathcal{A} - Id\|_{H^s(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)} \quad \textit{for } 0 \leqslant s \leqslant 4, \\ \|\partial_t\mathcal{A}\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)} \quad \textit{for } 0 \leqslant s \leqslant 2, \\ \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)} \lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}, \\ \|\partial_t^3\mathcal{A}\|_{L^2(\Omega)} \lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}. \end{cases} \tag{3.12}$$

Proof. To prove (3.9), we use Lemma 0.2 to obtain

$$\|\partial_t^l(A, B)\|_{H^s(\Omega)} = \|\partial_t^l(\partial_1\theta, \partial_2\theta)\|_{H^s(\Omega)} \lesssim \|\partial_t^l\theta\|_{H^{s+1}(\Omega)} \lesssim \|\partial_t^l\eta\|_{H^{s+1/2}(\Gamma)}.$$

We then claim (3.10). Since $K - 1 = J^{-1}(1 - J) = -J^{-1}\partial_3\theta$, we have

$$\|K - 1\|_{H^s(\Omega)} \lesssim \|J^{-1}\partial_3\theta\|_{H^s(\Omega)} \lesssim \|\theta\|_{H^{s+1}(\Omega)} \lesssim \|\eta\|_{H^{s+1/2}(\Gamma)}.$$

Note that $K = J^{-1}$, we have $\partial_t K = -J^{-2}\partial_t J$. Owing to the product estimate (3.1), Lemma 0.2 and the fact that $\|J - 1\|_{L^\infty(\Omega)} \lesssim 1$ (3.8), we get

$$\|\partial_t K\|_{H^s(\Omega)} \lesssim \|J^{-2}\partial_t\partial_3\theta\|_{H^s(\Omega)} \lesssim \|\partial_t\partial_3\theta\|_{H^s(\Omega)} \lesssim \|\partial_t\eta\|_{H^{s+1/2}(\Gamma)}.$$

Since $\partial_t^2 K = -J^{-2}\partial_t^2 J + 2J^{-3}(\partial_t J)^2$, we continue applying Sobolev embedding, Lemma 0.2 and (3.8) to obtain

$$\begin{aligned} \|\partial_t^2 K\|_{L^2(\Omega)} &\lesssim \|J^{-2}\partial_t^2\partial_3\theta\|_{L^2(\Omega)} + \|J^{-3}(\partial_t\partial_3\theta)^2\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^2\partial_3\theta\|_{L^2(\Omega)}(1 + \|\partial_3\theta\|_{H^2(\Omega)}) + \|\partial_t\partial_3\theta\|_{L^2(\Omega)}\|\partial_t\partial_3\theta\|_{H^2(\Omega)} \\ &\lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}) + \|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{5/2}(\Gamma)} \\ &\lesssim \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)}. \end{aligned}$$

Similarly, we have

$$\partial_t^3 K = -J^{-2}\partial_t^3 J + 6J^{-3}\partial_t J\partial_t^2 J - 6J^{-4}(\partial_t J)^3.$$

This yields

$$\begin{aligned} \|\partial_t^3 K\|_{L^2(\Omega)} &\lesssim \|J^{-2}\partial_t^3\partial_3\theta\|_{L^2(\Omega)} + \|J^{-3}\partial_t\partial_3\theta\partial_t^2\partial_3\theta\|_{L^2(\Omega)} + \|J^{-4}(\partial_t\partial_3\theta)^3\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^3\partial_3\theta\|_{L^2(\Omega)} + \|\partial_t\partial_3\theta\|_{H^2(\Omega)}\|\partial_t^2\partial_3\theta\|_{L^2(\Omega)} + \|\partial_t\partial_3\theta\|^2_{H^2(\Omega)}\|\partial_t\partial_3\theta\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}. \end{aligned}$$

Hence, (3.10) is proven.

We combine (3.9) and (3.10) to prove (3.11). Note that $XK = X(K-1) + X$ for $X = A$ or B, we use Sobolev embedding and (3.10) to obtain that

$$\|XK\|_{L^2(\Omega)} \lesssim \|X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) \lesssim \|\eta\|_{H^{1/2}(\Gamma)}.$$

We make use (3.1) and (3.9), (3.10) to obtain

$$\|XK\|_{H^1(\Omega)} \lesssim \|X\|_{H^1(\Omega)}(1 + \|K-1\|_{H^3(\Omega)}) \lesssim \|\eta\|_{H^{3/2}(\Gamma)}$$

and if $s = 2, 3$ or 4, we use also Gagliardo-Nirenberg's inequality to have

$$\begin{aligned}\|XK\|_{H^s(\Omega)} &\lesssim \|X\|_{H^s(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|X\|_{H^2(\Omega)}(1 + \|K-1\|_{H^s(\Omega)}) \\ &\lesssim \|\eta\|_{H^{s+1/2}(\Gamma)}.\end{aligned}$$

We further obtain

$$\|\partial_t(XK)\|_{H^s(\Omega)} \lesssim \|\partial_t X\|_{H^s(\Omega)} + \|\partial_t X(K-1)\|_{H^s(\Omega)} + \|X\partial_t(K-1)\|_{H^s(\Omega)}.$$

If $s = 0$, we use Sobolev embedding and (3.9), (3.10) again to have

$$\begin{aligned}\|\partial_t(XK)\|_{L^2(\Omega)} &\lesssim \|\partial_t X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|X\|_{H^2(\Omega)}\|\partial_t K\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t \eta\|_{H^{1/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}).\end{aligned}$$

If $s = 1$ or 2, we use (3.1) and also (3.9), (3.10) to obtain

$$\begin{aligned}\|\partial_t(XK)\|_{H^1(\Omega)} &\lesssim \|\partial_t X\|_{H^1(\Omega)}(1 + \|K-1\|_{H^3(\Omega)}) + \|X\|_{H^3(\Omega)}\|\partial_t K\|_{H^1(\Omega)} \\ &\lesssim \|\partial_t \eta\|_{H^{3/2}(\Gamma)}(1 + \|\eta\|_{H^{7/2}(\Gamma)}).\end{aligned}$$

or

$$\begin{aligned}\|\partial_t(XK)\|_{H^2(\Omega)} &\lesssim \|\partial_t X\|_{H^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|X\|_{H^2(\Omega)}\|\partial_t K\|_{H^2(\Omega)} \\ &\lesssim \|\partial_t \eta\|_{H^{5/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}).\end{aligned}$$

Similarly, it can be seen that

$$\begin{aligned}\|\partial_t^2(XK)\|_{L^2(\Omega)} &\lesssim \|\partial_t^2 X\|_{L^2(\Omega)} + \|\partial_t^2 X(K-1)\|_{L^2(\Omega)} + \|\partial_t X\partial_t(K-1)\|_{L^2(\Omega)} \\ &\quad + \|X\partial_t^2(K-1)\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^2 X\|_{L^2(\Omega)}(1 + \|K-1\|_{H^2(\Omega)}) + \|\partial_t X\|_{L^2(\Omega)}\|\partial_t K\|_{H^2(\Omega)} \\ &\quad + \|X\|_{H^2(\Omega)}\|\partial_t^2 K\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^2 \eta\|_{H^{1/2}(\Gamma)}(1 + \|\eta\|_{H^{5/2}(\Gamma)}) + \|\partial_t \eta\|_{H^{1/2}(\Gamma)}\|\partial_t \eta\|_{H^{5/2}(\Gamma)} \\ &\quad + \|\eta\|_{H^{5/2}(\Gamma)}(\|\partial_t^2 \eta\|_{H^{1/2}(\Gamma)} + \|\partial_t \eta\|_{H^{5/2}(\Gamma)}^2) \\ &\lesssim \|\partial_t^2 \eta\|_{H^{1/2}(\Gamma)} + \|\partial_t \eta\|_{H^{5/2}(\Gamma)}^2.\end{aligned}$$

In a same way, we have

$$\begin{aligned}\|\partial_t^3(XK)\|_{L^2(\Omega)} &\lesssim \|\partial_t^3 X\|_{L^2(\Omega)}(1+\|K-1\|_{H^2(\Omega)}) + \|\partial_t^2 X\|_{L^2(\Omega)}\|\partial_t K\|_{H^2(\Omega)} \\ &\quad + \|\partial_t X\|_{H^2(\Omega)}\|\partial_t^2 K\|_{L^2(\Omega)} + \|X\|_{H^2(\Omega)}\|\partial_t^3 K\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)}.\end{aligned}$$

Thus, the proof of (3.11) is complete.

Note that

$$\|\partial_t^l(\mathcal{A}-\mathrm{Id})\|_{H^s(\Omega)} \leqslant \|\partial_t^l(K-1)\|_{H^s(\Omega)} + \|\partial_t^l(AK)\|_{H^s(\Omega)} + \|\partial_t^l(BK)\|_{H^s(\Omega)}.$$

Hence, (3.12) follows from (3.9), (3.10) and (3.11). □

3.2 The linear analysis

We begin with the following lemma.

Lemma 3.2. *For any $k>0$,*

- *all characteristic values λ are always real,*
- *all characteristic values λ satisfy that $\lambda \leqslant \sqrt{\frac{g}{L_0}}$.*

Proof. Multiplying by $\overline{\phi}$ on both sides of (0.27) and then integrating by parts, we obtain that

$$\begin{aligned}&\lambda^2\Big(\int_{\mathbf{R}_-}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_3 - \rho_0\phi'\overline{\phi}\Big|_{-\infty}^0\Big) \\ &\qquad + \lambda\mu\Big(\int_{\mathbf{R}_-}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_3 + (\phi'''\overline{\phi}-\phi''\overline{\phi}'-2k^2\phi'\overline{\phi})\Big|_{-\infty}^0\Big) \\ &= gk^2\int_{\mathbf{R}_-}\rho_0'|\phi|^2dx_3.\end{aligned}$$

Using (0.111) and (0.112), we get

$$\begin{aligned}&\lambda^2\Big(\int_{\mathbf{R}_-}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_3 - \rho_+\phi(0)\overline{\phi}'(0)\Big) + \lambda\mu\int_{\mathbf{R}_-}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_3 \\ &\quad + (3\lambda\mu k^2+\lambda^2\rho_+)\phi'(0)\overline{\phi}(0) + gk^2\rho_+|\phi(0)|^2 + \lambda\mu k^2\overline{\phi}'(0)\phi(0) - 2\lambda\mu k^2\phi'(0)\overline{\phi}(0) \\ &= gk^2\int_{\mathbf{R}_-}\rho_0'|\phi|^2dx_3.\end{aligned}$$

This yields

$$\begin{aligned}&\lambda^2\int_{\mathbf{R}_-}(k^2\rho_0|\phi|^2+\rho_0|\phi'|^2)dx_3 + \lambda\mu\int_{\mathbf{R}_-}(|\phi''|^2+2k^2|\phi'|^2+k^4|\phi|^2)dx_3 \\ &\qquad + \lambda\mu k^2(\phi'(0)\overline{\phi}(0)+\overline{\phi}'(0)\phi(0)) + gk^2\rho_+|\phi(0)|^2 = gk^2\int_{\mathbf{R}_-}\rho_0'|\phi|^2dx_3.\end{aligned}$$

Using the integration by parts and (0.111) again, we have

$$\lambda^2 \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 + \lambda\mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3 \\ = -gk^2\rho_+|\phi(0)|^2 + gk^2 \int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3. \tag{3.13}$$

Suppose that $\lambda = \lambda_1 + i\lambda_2$, then one deduces from (3.13) that

$$(\lambda_1^2 - \lambda_2^2) \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 + \lambda_1\mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3 \\ = -gk^2\rho_+|\phi(0)|^2 + gk^2 \int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3 \tag{3.14}$$

and that

$$-2\lambda_1\lambda_2 \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 = \lambda_2\mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3. \tag{3.15}$$

If $\lambda_2 \neq 0$, (3.15) leads us to

$$-2\lambda_1 \int_{\mathbf{R}_-} (k^2\rho_0|\phi|^2 + \rho_0|\phi'|^2)dx_3 = \mu \int_{\mathbf{R}_-} (|\phi'' + k^2\phi|^2 + 4k^2|\phi'|^2)dx_3 < 0,$$

that contradiction yields $\lambda_2 = 0$, i.e. λ is real. Using (3.13) again, we further get that

$$\lambda^2 \int_{\mathbf{R}_-} \rho_0(k^2|\phi|^2 + |\phi'|^2)dx_3 \leqslant gk^2 \int_{\mathbf{R}_-} \rho_0'|\phi|^2 dx_3.$$

It tells us that λ is bounded by $\sqrt{\frac{g}{L_0}}$. This finishes the proof of Lemma 3.2. □

Note again that, thanks to Lemma 3.2, in what follows in this section, we only use real-valued functions.

3.2.1 Solutions on the outer region $(-\infty, -a)$ and reduction to an ODE on the finite interval $(-a, 0)$

Proposition 3.1. *Let $\tau_- = \sqrt{k^2 + \lambda\rho_-/\mu}$. There are two linearly independent solutions of (0.27) decaying to 0 at $-\infty$ as $x_3 \in (-\infty, -a]$, i.e.*

$$\phi_1^-(x_3) = e^{kx_3} \quad \text{and} \quad \phi_2^-(x_3) = e^{\tau_- x_3}. \tag{3.16}$$

All solutions decaying to 0 at $-\infty$ are spanned by (ϕ_1^-, ϕ_2^-).

Proof. On the interval $(-\infty, -a)$, Eq. (0.27) is an ODE with constant coefficients,

$$-\lambda\rho_-(k^2\phi - \phi'') = \mu(\phi^{(4)} - 2k^2\phi'' + k^4\phi). \tag{3.17}$$

We seek ϕ as $\phi(x_3) = e^{rx_3}$. Hence,

$$-\lambda\rho_-(k^2 - r^2) = \mu(r^4 - 2k^2r^2 + k^4),$$

which yields $r = \pm k$ or $r = \pm(k^2 + \lambda\rho_-/\mu)^{1/2}$. Since ϕ tends to 0 at $-\infty$, we get two independent solutions of (3.17),

$$\phi_1^-(x_3) = e^{kx_3} \quad \text{and} \quad \phi_2^-(x_3) = e^{(k^2+\lambda\rho_-/\mu)^{1/2}x_3}.$$

Hence, we can find all bounded solutions of (3.17) of the form

$$\phi(x_3) = A_1 e^{k(x_3+a)} + A_2 e^{\tau_-(x_3+a)}. \tag{3.18}$$

Proof of Proposition 3.1 is finished. □

Once it is proven that $\phi(x_3)$ outside $(-a, 0)$ is precise, we look for ϕ on $(-a, 0)$. That solution has to match with (3.18) well, i.e. there is a condition on $(\phi, \phi', \phi'', \phi''')$ at $x_3 = -a$. We will show that in the following lemma.

Lemma 3.3. *The boundary conditions of* (0.27) *at* $x_3 = -a$, *for* $\phi \in H^4(\mathbf{R}_-)$, *are* (0.113), *i.e.*

$$\begin{cases} k\tau_-\phi(-a) - (k + \tau_-)\phi'(-a) + \phi''(-a) = 0, \\ k\tau_-(k + \tau_-)\phi(-a) - (k^2 + k\tau_- + \tau_-^2)\phi'(-a) + \phi'''(-a) = 0, \end{cases}$$

and at $x_3 = 0$, *are* (0.111).

Proof. For a solution ϕ of Eq. (0.27) on $(-a, 0)$, the boundary conditions at $x_3 = -a$ are equivalent to the fact that ϕ belongs to the space of decaying solutions at ∞. On the one hand, it can be seen from (3.18) that

$$\begin{pmatrix} \phi(x_3) \\ \phi'(x_3) \\ \phi''(x_3) \\ \phi'''(x_3) \end{pmatrix} = A_1^- e^{k(x_3+a)} \begin{pmatrix} 1 \\ k \\ k^2 \\ k^3 \end{pmatrix} + A_2^- e^{\tau_-(x_3+a)} \begin{pmatrix} 1 \\ \tau_- \\ \tau_-^2 \\ \tau_-^3 \end{pmatrix} \quad \text{for } x_3 \leqslant -a.$$

On the other hand, direct computations show that the orthogonal complement of the subspace of $\mathbf{R}^4$ spanned by two vectors $(1, k, k^2, k^3)^T$ and $(1, \tau_-, \tau_-^2, \tau_-^3)^T$ is spanned by

$$(k\tau_-, -(k + \tau_-), 1, 0)^T \quad \text{and} \quad (k\tau_-(k + \tau_-), -(k^2 + k\tau_- + \tau_-^2), 0, 1)^T.$$

The above arguments allow us to set (0.113) as boundary conditions of Eq. (0.27) at $x_3 = -a$. □

3.2.2 A bilinear form and a self-adjoint invertible operator

Proposition 3.2. *Let us recall* (0.115),

$$BV_{-a,k,\lambda}(\vartheta,\varrho) := \mu\begin{pmatrix} k\tau_-(k+\tau_-)\vartheta(-a)\varrho(-a) - k\tau_-\vartheta'(-a)\varrho(-a) \\ \qquad - k\tau_-\vartheta(-a)\varrho'(-a) + (k+\tau_-)\vartheta'(-a)\varrho'(-a)\end{pmatrix},$$

$$BV_{0,k,\lambda}(\vartheta,\varrho) := \mu k^2(\vartheta'(0)\varrho(0) + \vartheta(0)\varrho'(0)) + \frac{gk^2\rho_+}{\lambda}\vartheta(0)\varrho(0),$$

and (0.116),

$$\mathcal{B}_{a,k,\lambda}(\vartheta,\varrho) := BV_{0,k,\lambda}(\vartheta,\varrho) + BV_{-a,k,\lambda}(\vartheta,\varrho) + \lambda\int_{-a}^{0}\rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3$$
$$+ \mu\int_{-a}^{0}(\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3.$$

We have that $\mathcal{B}_{a,k,\lambda}$ is a continuous and coercive bilinear form on $H^2((-a,0))$.

Furthermore, let $(H^2((-a,0)))'$ be the dual space of $H^2((-a,0))$ associated with the norm $\sqrt{\mathcal{B}_{a,k,\lambda}(\cdot,\cdot)}$, there exists a unique operator

$$Y_{a,k,\lambda} \in \mathcal{L}(H^2((-a,0)),(H^2((-a,0)))'),$$

that is also bijective, such that

$$\mathcal{B}_{a,k,\lambda}(\vartheta,\varrho) = \langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \tag{3.19}$$

for all $\vartheta,\varrho \in H^2((-a,0))$.

Before proving Proposition 3.2, we state our key lemma, whose proof is postponed to Section 3.2.6. This yields the coercivity of $\mathcal{B}_{a,k,\lambda}$ as it will appear in the proof of Proposition 3.2.

Lemma 3.4. *We have*

$$\min_{\vartheta\in H^2((-a,0))}\begin{pmatrix} 2k^2(\vartheta'(0)\vartheta(0) - \vartheta'(-a)\vartheta(-a)), \vartheta \text{ satisfies the constraint} \\ \int_{-a}^{0}((\vartheta'')^2 + 2k^2(\vartheta')^2 + k^4\vartheta^2)dx_3 = 1.\end{pmatrix}$$
$$= -\frac{\sinh(ka)+ka}{3\sinh(ka)-ka},$$

and

$$\max_{\vartheta\in H^2((-a,0))}\begin{pmatrix} 2k^2(\vartheta'(0)\vartheta(0) - \vartheta'(-a)\vartheta(-a)), \vartheta \text{ satisfies the constraint} \\ \int_{-a}^{0}((\vartheta'')^2 + 2k^2(\vartheta')^2 + k^4\vartheta^2)dx_3 = 1.\end{pmatrix} = 1.$$

Proof of Proposition 3.2. Clearly, $\mathcal{B}_{a,k,\lambda}$ is a bilinear form on $H^2((-a,0))$ since the terms $BV_{a,k,\lambda}(\vartheta,\varrho)$ and $BV_{0,k,\lambda}(\vartheta,\varrho)$ are well defined. We then establish the boundedness of $\mathcal{B}_{a,k,\lambda}$. The integral terms of $\mathcal{B}_{a,k,\lambda}$ are clearly $\lesssim \|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}$. About the two boundary value terms, it follows from the general Sobolev inequality that

$$\max(\vartheta^2(0),\vartheta^2(-a)) \lesssim \|\vartheta\|^2_{H^1((-a,0))}$$

and that

$$\max((\vartheta'(0))^2,(\vartheta'(-a))^2) \lesssim \|\vartheta'\|^2_{H^1((-a,0))}.$$

Consequently, we get

$$\begin{aligned}|BV_{-a,k,\lambda}(\vartheta,\varrho)| &\lesssim (|\vartheta(-a)|+|\vartheta'(-a)|)(|\varrho(-a)|+|\varrho'(-a)|)\\ &\lesssim \|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}.\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}|BV_{0,k,\lambda}(\vartheta,\varrho)| &\lesssim \frac{\lambda+1}{\lambda}(|\vartheta(0)|+|\vartheta'(0)|)(|\varrho(0)|+|\varrho'(0)|)\\ &\lesssim \frac{\lambda+1}{\lambda}\|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}.\end{aligned} \tag{3.21}$$

In view of (3.20) and (3.21), we find that

$$|\mathcal{B}_{a,k,\lambda}(\vartheta,\varrho)| \lesssim \frac{\lambda+1}{\lambda}\|\vartheta\|_{H^2((-a,0))}\|\varrho\|_{H^2((-a,0))}, \tag{3.22}$$

i.e. $\mathcal{B}_{a,k,\lambda}$ is bounded.

We move to show the coercivity of $\mathcal{B}_{a,k,\lambda}$. We have that

$$\begin{aligned}\mathcal{B}_{a,k,\lambda}(\vartheta,\vartheta) &= BV_{0,k,\lambda}(\vartheta,\vartheta) + BV_{-a,k,\lambda}(\vartheta,\vartheta) + \lambda\int_{-a}^0 \rho_0(k^2\vartheta^2+(\vartheta')^2)dx_3\\ &\quad + \mu\int_{-a}^0 ((\vartheta'')^2+2k^2(\vartheta')^2+k^4(\vartheta)^2)dx_3.\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{\mu}BV_{-a,k,\lambda}(\vartheta,\vartheta) &= k\tau_-(k+\tau_-)(\vartheta(-a))^2 - 2k\tau_-\vartheta(-a)\vartheta'(-a) + (k+\tau_-)(\vartheta'(-a))^2\\ &= (k+\tau_-)\Big(\vartheta'(-a)+\frac{k(k-\tau_-)}{k+\tau_-}\vartheta(-a)\Big)^2\\ &\quad + \frac{k(\tau_-(k+\tau_-)^2-k(k-\tau_-)^2)}{k+\tau_-}(\vartheta(-a))^2 - 2k^2\vartheta(-a)\vartheta'(-a)\\ &\geqslant -2k^2\vartheta(-a)\vartheta'(-a).\end{aligned}$$

Therefore, we deduce that

$$\frac{1}{\mu}\mathcal{B}_{a,k,\lambda}(\vartheta,\vartheta) \geqslant 2k^2(\vartheta(0)\vartheta'(0)-\vartheta(-a)\vartheta'(-a)) + \int_{-a}^0 ((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_3.$$

Notice from Lemma 3.4 that

$$\frac{1}{\mu}\mathcal{B}_{a,k,\lambda}(\vartheta,\vartheta) \geqslant \frac{2(\sinh(ka)-ka)}{3\sinh(ka)-ka}\int_{-a}^{0}((\vartheta'')^2+2k^2(\vartheta')^2+k^4\vartheta^2)dx_3. \tag{3.23}$$

The inequalities (3.22) and (3.23) tell us that $\mathcal{B}_{a,k,\lambda}$ is a continuous and coercive bilinear form on $H^2((-a,0))$. It follows from Riesz's representation theorem that there is a unique operator $Y_{a,k,\lambda} \in \mathcal{L}(H^2((-a,0)),(H^2((-a,0)))')$, that is also bijective, satisfying (3.19) for all $\vartheta,\varrho \in H^2((-a,0))$. Proof of Proposition 1.2 is complete. ☐

The next proposition is to devoted to studying the properties of $Y_{a,k,\lambda}$.

Proposition 3.3. *We have the following results.*

1. *For all $\vartheta \in H^2((-a,0))$,*

$$Y_{a,k,\lambda}\vartheta = \lambda(k^2\rho_0\vartheta-(\rho_0\vartheta')')+\lambda\mu(\vartheta^{(4)}-2k^2\vartheta''+k^4\vartheta)$$

   in $\mathcal{D}'((-a,0))$.

2. *Let $f \in L^2((-a,0))$ be given, there exists a unique solution $\vartheta \in H^2((-a,0))$ of*

$$Y_{a,k,\lambda}\vartheta = f \ \text{ in } (H^2((-a,0)))'. \tag{3.24}$$

   Moreover, $\vartheta \in H^4((-a,0))$ and satisfies the boundary conditions (0.113)–(0.111).

Proof. It follows from Proposition 3.2 that there is a unique $\vartheta \in H^2((-a,0))$ such that

$$\lambda\int_{-a}^{0}\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx_3+\mu\int_{-a}^{0}(\vartheta''\varrho''+2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx_3 = \langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \tag{3.25}$$

for all $\varrho \in C_0^\infty((-a,0))$. We respectively define $(\vartheta'')'$ and $(\vartheta'')''$ in the distributional sense as the first and second derivative of ϑ'' which is in $L^2((-a,0))$. Hence, Eq. (3.25) is equivalent to

$$\lambda\int_{-a}^{0}\rho_0(k^2\vartheta\varrho+\vartheta'\varrho')dx_3+\mu\langle(\vartheta'')'',\varrho\rangle+\mu\int_{-a}^{0}(2k^2\vartheta'\varrho'+k^4\vartheta\varrho)dx_3 = \langle Y_{a,k,\lambda}\vartheta,\varrho\rangle \tag{3.26}$$

for all $\varrho \in C_0^\infty((-a,0))$. Eq. (3.26) implies that

$$\mu((\vartheta'')''-2k^2\vartheta''+k^4\vartheta)+\lambda(k^2\rho_0\vartheta-(\rho_0\vartheta')') = Y_{a,k,\lambda}\vartheta \quad \text{in } \mathcal{D}'((-a,0)). \tag{3.27}$$

The first assertion holds.

Under the assumption $f \in L^2((-a,0))$, we improve the regularity of the weak solution $\vartheta \in H^2((-a,0))$ of (3.27). Indeed, we rewrite (3.27) as

$$\mu\langle(\vartheta'')'',\varrho\rangle = \int_{-a}^{0}(Y_{a,k,\lambda}\vartheta+2\mu k^2\vartheta''-\mu k^4\vartheta-\lambda k^2\rho_0\vartheta+\lambda(\rho_0\vartheta')')\varrho dx_3$$

for all $\varrho \in C_0^\infty((-a,0))$. Since $(f + 2\mu k^2\vartheta'' - \mu k^4\vartheta - \lambda k^2\rho_0\vartheta + \lambda(\rho_0\vartheta')')$ belongs to $L^2((-a,0))$, it follows from (3.26) that $(\vartheta'')'' \in L^2((-a,0))$.

Let $\chi_1 \in C_0^\infty((-a,0))$ satisfy $\int_{-a}^0 \chi_1(y)dy = 1$. Using the distribution theory, we define $\Sigma \in \mathcal{D}'((-a,0))$ such that

$$\langle \Sigma, \varrho \rangle = \langle (\vartheta'')'', \zeta_\varrho \rangle \tag{3.28}$$

for all $\varrho \in C_0^\infty((-a,0))$, where

$$\zeta_\varrho(x_3) = \int_{-a}^{x_3} \Big(\varrho(y) - \chi_1(y) \int_{-a}^0 \varrho(s)ds \Big) dy$$

for all $-a < x_3 < 0$. We obtain

$$\langle \Sigma', \varrho \rangle = -\langle \Sigma, \varrho' \rangle = -\langle (\varrho'')'', \zeta_{\varrho'} \rangle.$$

Note that

$$\langle (\vartheta'')'', \zeta_{\varrho'} \rangle = \langle (\vartheta'')'', \varrho(x_3) - \int_{-a}^{x_3} \chi_1(y) \int_{-a}^0 \varrho'(s)dsdy \rangle = \langle (\vartheta'')'', \varrho \rangle,$$

this yields $\langle \Sigma', \varrho \rangle = -\langle (\vartheta'')'', \varrho \rangle$. Hence, we have that $(\vartheta'')' + \Sigma \equiv \text{constant}$. In view of $(\vartheta'')'' \in L^2((-a,0))$ and (3.28), we know that $(\vartheta'')' \in L^2((-a,0))$. Since $\vartheta \in H^2((-a,0))$ and $(\vartheta'')', (\vartheta'')'' \in L^2((-a,0))$, it tells us that ϑ belongs to $H^4((-a,0))$ and we can take their traces up to order 3.

By exploiting (3.25), we then show that ϑ satisfies (0.113)-(0.111). Indeed, for all $\varrho \in H^2((-a,0))$, we use the integration by parts to obtain from (3.25) that

$$\begin{aligned} \lambda \int_{-a}^0 \rho_0(k^2\vartheta\varrho + \vartheta'\varrho')dx_3 + \mu \int_{-a}^0 (\vartheta''\varrho'' + 2k^2\vartheta'\varrho' + k^4\vartheta\varrho)dx_3 \\ - \lambda\rho_0\vartheta'\varrho\Big|_{-a}^0 + \mu\Big(\vartheta'''\varrho\Big|_{-a}^0 - \vartheta''\varrho'\Big|_{-a}^0 - 2k^2\vartheta'\varrho\Big|_{-a}^0 \Big) = \int_{-a}^0 (Y_{a,k,\lambda}\vartheta)\varrho dx_3. \end{aligned}$$

It then follows from the definition of the bilinear form $\mathcal{B}_{a,k,\lambda}$ that

$$-\lambda\rho_0\vartheta'\varrho\Big|_{-a}^0 + \mu\Big(\vartheta'''\varrho\Big|_{-a}^0 - \vartheta''\varrho'\Big|_{-a}^0 - 2k^2\vartheta'\varrho\Big|_{-a}^0 \Big) = BV_0(\vartheta,\varrho) + BV_{-a}(\vartheta,\varrho), \tag{3.29}$$

for all $\varrho \in H^2((-a,0))$. By collecting all terms corresponding to $\varrho(-a)$ in (3.29), we deduce that

$$\lambda\rho_-\vartheta'(-a) - \mu\vartheta'''(-a) + 2\mu k^2\vartheta'(-a) = \mu k\tau_-(k+\tau_-)\vartheta(-a) - \mu k\tau_-\vartheta'(-a).$$

This yields,

$$\begin{aligned} \vartheta'''(-a) &= (\tau_-^2 - k^2)\vartheta'(-a) + 2k^2\vartheta'(-a) + k\tau_-\vartheta'(-a) - k\tau_-(k+\tau_-)\vartheta(-a) \\ &= (k^2 + k\tau_- + \tau_-^2)\vartheta'(-a) - k\tau_-(k+\tau_-)\vartheta(-a). \end{aligned}$$

We just proved that ϑ satisfies $(0.113)_2$. Similarly, ϑ also fulfils $(0.113)_1$ and (0.111). This ends the proof of Proposition 3.3. □

We have the following proposition on $Y_{a,k,\lambda}^{-1}$.

Proposition 3.4. *The operator* $Y_{a,k,\lambda}^{-1} : L^2((-a,0)) \to L^2((-a,0))$ *is compact and self-adjoint.*

Proof. It follows from Proposition 3.3 that $Y_{a,k,\lambda}$ admits an inverse operator $Y_{a,k,\lambda}^{-1}$ from $L^2((-a,0))$ to a subspace of $H^4((-a,0))$ requiring all elements satisfy (0.111)-(0.113), which is symmetric. We compose $Y_{a,k,\lambda}^{-1}$ with the continuous injection from $H^4((-a,0))$ to $L^2((-a,0))$. Notice that the embedding $H^p((-a,0)) \hookrightarrow H^q((-a,0))$ for $p > q \geqslant 0$ is compact. Therefore, the opearator $Y_{a,k,\lambda}^{-1}$ is compact and self-adjoint from $L^2((-a,0))$ to $L^2((-a,0))$. Proposition 3.4 is shown. □

3.2.3 A sequence of characteristic values

We continue considering $\lambda \in (0, \sqrt{\frac{g}{L_0}}]$ and we study the operator $S_{a,k,\lambda} := \mathcal{M} Y_{a,k,\lambda}^{-1} \mathcal{M}$, where $\mathcal{M}$ is the operator of multiplication by $\sqrt{\rho_0'}$.

Proposition 3.5. *The operator* $S_{a,k,\lambda} : L^2((-a,0)) \to L^2((-a,0))$ *is compact and self-adjoint.*

Proof. Due to the assumptions on ρ_0, the operator $S_{a,k,\lambda}$ is well-defined from $L^2((-a,0))$ to itself. The operator $Y_{a,k,\lambda}^{-1}$ is compact, so is $S_{a,k,\lambda}$. Moreover, because both the inverse $Y_{a,k,\lambda}^{-1}$ and $\mathcal{M}$ are self-adjoint, the self-adjointness of $S_{a,k,\lambda}$ follows. □

As a result of the spectral theory of compact and self-adjoint operators, the point spectrum of $S_{a,k,\lambda}$ is discrete, i.e. is a sequence $\{\gamma_n(k,\lambda)\}_{n\geqslant 1}$ of eigenvalues of $S_{a,k,\lambda}$, associated with normalized orthogonal eigenfunctions $\{\varpi_n\}_{n\geqslant 1}$ in $L^2((-a,0))$. That means

$$\gamma_n(\lambda,k)\varpi_n = S_{a,k,\lambda}\varpi_n = \mathcal{M} Y_{a,k,\lambda}^{-1} \mathcal{M}\varpi_n.$$

So that with $\phi_n = Y_{a,k,\lambda}^{-1}\mathcal{M}\varpi_n \in H^4((-a,0))$, one has

$$\gamma_n(\lambda,k) Y_{a,k,\lambda}\phi_n = \rho_0'\phi_n \tag{3.30}$$

and ϕ_n satisfies (0.113)-(0.111). Eq. (3.30) also tells us that $\gamma_n(\lambda,k) > 0$ for all n. Indeed, we obtain

$$\gamma_n(\lambda,k)\int_{-a}^0 (Y_{a,k,\lambda}\phi_n)\phi_n dx_3 = \int_{-a}^0 \rho_0'\phi_n^2 dx_3.$$

That implies

$$\gamma_n(\lambda,k)\mathcal{B}_{a,k,\lambda}(\phi_n,\phi_n) = \int_{-a}^0 \rho_0'\phi_n^2 dx_3. \tag{3.31}$$

Since $\mathcal{B}_{a,k,\lambda}(\phi_n,\phi_n) > 0$ and $\rho_0' > 0$ on $(-a,0)$, we know that $\gamma_n(\lambda,k)$ is positive for all n. Hence, by reordering and using the spectral theory of compact and self-adjoint operators again, we obtain that $\gamma_n(\lambda, k)$ is a positive sequence decreasing towards 0 as $n \to \infty$.

For each n, in order to verify that ϕ_n is a solution of (0.27), we are left to look for real values of λ_n such that (0.121) holds, i.e.

$$\gamma_n(\lambda_n, k) = \frac{\lambda_n}{gk^2}.$$

To solve (0.121), we need the two following lemmas.

Lemma 3.5. *For each n, $\gamma_n(\lambda, k)$ and ϕ_n are differentiable in λ.*

Proof. The proof of Lemma 3.5 is the same as Lemma 1.2, we omit the details here. □

Lemma 3.6. *For each n, $\gamma_n(\lambda, k)$ is strictly decreasing in λ.*

Proof. Let $z_n = \frac{d\phi_n}{d\lambda}$, it follows from (3.30) that

$$k^2\rho_0\phi_n - (\rho_0\phi_n')' + Y_{a,k,\lambda}z_n = \frac{1}{\gamma_n(\lambda,k)}\rho_0' z_n + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\rho_0'\phi_n \tag{3.32}$$

on $(-a, 0)$. At $x_3 = -a$, we have

$$\begin{cases} z_n''(-a) - (k+\tau_-)z_n'(-a) + k\tau_- z_n(-a) = \frac{\rho_-}{2\mu\tau_-}\phi_n'(-a) - \frac{k\rho_-}{2\mu\tau_-}\phi_n(-a), \\ z_n'''(-a) - (k^2 + k\tau_- + \tau_-^2)z_n'(-a) + k\tau_-(k+\tau_-)z_n(-a) \\ \qquad = \Big(\frac{k\rho_-}{2\mu\tau_-} + \frac{\rho_-}{\mu}\Big)\phi_n'(-a) - \Big(\frac{k^2\rho_-}{2\mu\tau_-} + \frac{k\rho_-}{\mu}\Big)\phi_n(-a), \end{cases} \tag{3.33}$$

and at $x_3 = 0$, we also have

$$\begin{cases} z_n''(0) + k^2 z_n(0) = 0, \\ z_n'''(0) - (3k^2 + \frac{\lambda\rho_+}{\mu})z_n'(0) + \frac{gk^2\rho_+}{\lambda\mu}z_n(0) = \frac{\rho_+}{\mu}\phi_n'(0) + \frac{gk^2\rho_+}{\lambda^2\mu}\phi_n(0). \end{cases} \tag{3.34}$$

Multiplying by ϕ_n on both sides of (3.32), we obtain that

$$\begin{aligned} &\int_{-a}^0 (k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_3 + \int_{-a}^0 (Y_{a,k,\lambda}z_n)\phi_n dx_3 \\ &\quad = \frac{1}{\gamma_n(\lambda,k)}\int_{-a}^0 \rho_0' z_n\phi_n dx_3 + \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\int_{-a}^0 \rho_0'\phi_n^2 dx_3. \end{aligned} \tag{3.35}$$

Thanks to the integration by parts, we have

$$\int_{-a}^0 (k^2\rho_0\phi_n - (\rho_0\phi_n')')\phi_n dx_3 = \int_{-a}^0 \rho_0(k^2\phi_n^2 + (\phi_n')^2)dx_3 - (\rho_0\phi_n'\phi_n)\Big|_{-a}^0 \tag{3.36}$$

and

$$\begin{aligned} \int_{-a}^0 (Y_{a,k,\lambda}z_n)\phi_n dx_3 = \int_{-a}^0 &(Y_{a,k,\lambda}\phi_n)z_n dx_3 + \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)\Big|_{-a}^0 \\ &- \Big(\mu(\phi_n'''z_n - \phi_n''z_n' - 2k^2\phi_n'z_n) - \lambda\rho_0\phi_n'z_n\Big)\Big|_{-a}^0. \end{aligned} \tag{3.37}$$

Owing to (3.36), (3.37) and (3.30), Eq. (3.35) becomes

$$\begin{aligned}\int_{-a}^{0} \rho_0(k^2\phi_n^2 + (\phi_n')^2)dx_3 &+ \Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)\Big|_{-a}^{0} \\ &- \Big(\mu(\phi_n''' z_n - \phi_n'' z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)\Big|_{-a}^{0} - (\rho_0\phi_n'\phi_n)\Big|_{-a}^{0} \\ = \frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\int_{-a}^{0}\rho_0'\phi_n^2 dx_3.\end{aligned} \tag{3.38}$$

Using (3.33), we obtain

$$\begin{aligned} &-\Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(-a) \\ &\quad + \Big(\mu(\phi_n''' z_n - \phi_n'' z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)(-a) + \rho_-\phi_n'(-a)\phi_n(-a) \\ &= \Big(\frac{k^2\rho_-}{2\tau_-} + k\rho_-\Big)(\phi_n(-a))^2 - \Big(\frac{k\rho_-}{2\tau_-} + \rho_-\Big)\phi_n'(-a)\phi_n(-a) \\ &\quad - \frac{k\rho_-}{2\tau_-}\phi_n(-a)\phi_n'(-a) + \frac{\rho_-}{2\tau_-}(\phi_n'(-a))^2 + \rho_-\phi_n'(-a)\phi_n(-a) \\ &= k\rho_-(\phi_n(-a))^2 + \frac{\rho_-}{2\tau_-}(\phi_n'(-a) - k\phi_n(-a))^2.\end{aligned} \tag{3.39}$$

Using (3.34) and (0.111), we also have

$$\begin{aligned} &-\Big(\mu(z_n'''\phi_n - z_n''\phi_n' - 2k^2 z_n'\phi_n) - \lambda\rho_0 z_n'\phi_n\Big)(0) + \rho_+\phi_n'(0)\phi_n(0) \\ &\quad + \Big(\mu(\phi_n''' z_n - \phi_n'' z_n' - 2k^2\phi_n' z_n) - \lambda\rho_0\phi_n' z_n\Big)(0) = \frac{gk^2\rho_+}{\lambda^2}(\phi_n(0))^2.\end{aligned} \tag{3.40}$$

Combining (3.38), (3.39) and (3.40), we deduce that

$$\begin{aligned}\frac{d}{d\lambda}\Big(\frac{1}{\gamma_n(\lambda,k)}\Big)\int_{-a}^{0}\rho_0'\phi_n^2 dx_3 &= \int_{-a}^{0}\rho_0(k^2\phi_n^2 + (\phi_n')^2)dx_3 + k\rho_-(\phi_n(-a))^2 \\ &\quad + \frac{\rho_-}{2\tau_-}(\phi_n'(-a) - k\phi_n(-a))^2 + \frac{gk^2\rho_+}{\lambda^2}(\phi_n(0))^2.\end{aligned} \tag{3.41}$$

This yields that $\frac{1}{\gamma_n(\lambda,k)}$ is strictly increasing in λ, i.e. $\gamma_n(\lambda,k)$ is strictly decreasing in λ. This ends the proof of Lemma 3.6. $\square$

Now we are in position to solve (0.121).

Proposition 3.6. *For each $n \geqslant 1$, there exists a unique $\lambda_n > 0$ solving (0.121). In addition, λ_n decreases towards 0 as n goes to ∞.*

Proof. Using (3.31), we know that

$$\frac{1}{\gamma_n(\lambda,k)}\int_{-a}^{0}\rho_0'\phi_n^2 dx_3 = \int_{-a}^{0}(Y_{a,k,\lambda}\phi_n)\phi_n dx_3 = \mathcal{B}_{a,k,\lambda}(\phi_n,\phi_n),$$

Keep in mind the definition of $\mathcal{B}_{a,k,\lambda}$ (0.116), we deduce that

$$\frac{1}{\gamma_n(\lambda,k)}\int_{-a}^{0}\rho_0'\phi_n^2 dx_3 \geqslant \lambda\int_{-a}^{0}k^2\rho_0\phi_n^2 dx_3 + \mu\int_{-a}^{0}k^4\phi_n^2 dx_3,$$

that implies
$$\frac{1}{L_0\gamma_n(\lambda,k)} \geqslant \lambda k^2 + \frac{\mu k^4}{\rho_+}.$$
Consequently, for all $n \geqslant 1$,
$$\lim_{\lambda\to\sqrt{\frac{g}{L_0}}} \frac{\lambda}{\gamma_n(\lambda,k)} > gk^2. \tag{3.42}$$

Since $\gamma_n(\lambda,k)$ is a decreasing function, we have $\frac{1}{\gamma_n(\lambda,k)} \leqslant \frac{1}{\gamma_n(\frac{1}{2}\sqrt{\frac{g}{L_0}},k)}$ for all $\lambda \leqslant \frac{1}{2}\sqrt{\frac{g}{L_0}}$. Hence,
$$\lim_{\lambda\to 0} \frac{\lambda}{\gamma_n(\lambda,k)} = 0. \tag{3.43}$$
Combining (3.42), (3.43) and the fact that γ_n is decreasing in λ, we obtain a unique λ_n solving (0.121).

We prove that the sequence $(\lambda_n)_{n\geqslant 1}$ is decreasing. Indeed, if $\lambda_m < \lambda_{m+1}$ for some $m \geqslant 1$, we have
$$\gamma_m(\lambda_m,k) > \gamma_m(\lambda_{m+1},k).$$
Meanwhile, we also have
$$\gamma_m(\lambda_{m+1},k) > \gamma_{m+1}(\lambda_{m+1},k).$$
That implies
$$\frac{\lambda_m}{gk^2} = \gamma_m(\lambda_m,k) > \gamma_{m+1}(\lambda_{m+1},k) = \frac{\lambda_{m+1}}{gk^2}.$$
That contradiction tells us that $(\lambda_n)_{n\geqslant 1}$ is a decreasing sequence.

To conclude Proposition 3.6, we prove that $\lim_{n\to\infty}\lambda_n = 0$. Indeed, suppose that $\lim_{n\to\infty}\lambda_n = c_0 > 0$, one has that $\lambda_n \geqslant c_0$ for all $n \geqslant 1$. This yields
$$\gamma_n(c_0,k) \geqslant \gamma_n(\lambda_n,k) = \frac{\lambda_n}{gk^2} \geqslant \frac{c_0}{gk^2}.$$
Letting $n \to \infty$, we obtain that $0 \geqslant \frac{c_0}{gk^2}$, which is a contradiction. Hence, $\lim_{n\to\infty}\lambda_n = 0$. Proposition 3.6 is proven. □

3.2.4 Proof of Theorem 0.5 and normal modes to the linearized equations

We are in position to prove Theorem 0.5.

Proof of Theorem 0.5. For each $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ being found from Proposition 3.6, let $\phi_n(x_3) = Y_{a,k,\lambda_n}^{-1}\mathcal{M}\varpi_n(x_3)$ in $(-a,0)$. Keep in mind our computations in Section 3.2.1, we extend ϕ_n to $\mathbf{R}_-$ by requiring ϕ_n satisfies (3.18) for some constants $A_{n,1}, A_{n,2}$ as $\lambda = \lambda_n$. Those constants $A_{n,1}, A_{n,2}$ are defined by

$$\begin{cases} \phi_n(-a) = A_{n,1} + A_{n,2}, \\ \phi_n'(-a) = kA_{n,1} + A_{n,2}\sqrt{k^2 + \frac{\lambda_n \rho_-}{\mu}}. \end{cases} \tag{3.44}$$

Solving (3.44), we get that

$$A_{n,1} = \frac{\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}}\phi_n(-a) - \phi_n'(-a)}{\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}} - k}, \quad A_{n,2} = \frac{\phi_n'(-a) - k\phi_n(-a)}{\sqrt{k^2 + \frac{\lambda_n\rho_-}{\mu}} - k}. \tag{3.45}$$

Therefore, the function $\phi_n \in H^4(\mathbf{R}_-)$ is a solution of (0.27) satisfying (0.111) and (0.112) as $\lambda = \lambda_n$ for each $n \geqslant 1$. Using a bootstrap argument, we have $\phi_n \in H^\infty(\mathbf{R}_-)$. Proof of Theorem 0.5 is complete. □

Once we have solutions of (0.27)-(0.111)-(0.112), we go back to the linearized equations (0.101).

Proposition 3.7. *For each* $\mathbf{k} = (k_1, k_2) \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}\backslash\{0\}$, *there exists an infinite sequence of normal modes*

$$e^{\lambda_n(\mathbf{k})t} V_n(\mathbf{k}, x) = e^{\lambda_n(\mathbf{k})t}(\zeta_n(\mathbf{k}, x), u_n(\mathbf{k}, x), q_n(\mathbf{k}, x), \eta_n(\mathbf{k}, x_h)) \tag{3.46}$$

to the linearized equations (0.101), *such that*

$$\zeta_n \in H^\infty(\Omega), u_n \in (H^\infty(\Omega))^3 \text{ and } q_n \in H^\infty(\Omega). \tag{3.47}$$

Proof. For each solution $\lambda_n \in (0, \sqrt{\frac{g}{L_0}})$ of (0.121), we have a solution ϕ_n in $H^4(\mathbf{R}_-)$ of (0.27) as $\lambda = \lambda_n$, being found in Theorem 0.5. Furthermore, $\phi_n \in H^\infty(\mathbf{R}_-)$. We find uniquely $\pi_n \in H^\infty(\mathbf{R}_-)$ from (0.110) such that

$$\pi_n(\mathbf{k}, x_3) = -\frac{1}{k^2}(\lambda_n\rho_0\phi_n' + \mu(k^2\phi_n' - \phi_n'''))(\mathbf{k}, x_3).$$

To look for ψ_n, we rewrite (0.107) as a second order ODE,

$$-\mu\psi_n'' + (\lambda_n\rho_0\psi_n + \mu k^2\psi_n - k_1\pi_n) = 0.$$

Note from (0.108) and (0.109) that ψ_n satisfies that $\psi_n'(0) = k_1\phi_n(0)$ and that $\lim_{x_3\to-\infty}\psi_n(x_3) = 0$. By the ODE theory on a bounded interval and the domain expansion technique, we obtain a unique solution $\psi_n \in H^\infty(\mathbf{R}_-)$, where the solution ψ_n depends on the known functions ϕ_n and π_n. We get φ_n in a similar way. Hence, $(\psi_n, \varphi_n, \phi_n, \pi_n) \in (H^\infty(\mathbf{R}_-))^4$ is a solution of (0.107)-(0.108).

Following (0.106), we then construct the functions

$$\begin{aligned} v_{1,n}(\mathbf{k}, x) &= \sin(k_1x_1 + k_2x_2)\psi_n(\mathbf{k}, x_3), \\ v_{2,n}(\mathbf{k}, x) &= \sin(k_1x_1 + k_2x_2)\varphi_n(\mathbf{k}, x_3), \\ v_{3,n}(\mathbf{k}, x) &= \cos(k_1x_1 + k_2x_2)\phi_n(\mathbf{k}, x_3), \\ r_n(\mathbf{k}, x_3) &= \cos(k_1x_1 + k_2x_2)\pi_n(\mathbf{k}, x_3). \end{aligned}$$

Keep in mind (0.104), let us define also

$$\omega_n(\mathbf{k}, x) = -\frac{1}{\lambda_n(\mathbf{k})}\rho_0'(x_3)v_{3,n}(\mathbf{k}, x_3) \quad \text{and} \quad \nu_n(\mathbf{k}, x_h) = \frac{1}{\lambda_n(\mathbf{k})}v_{3,n}(\mathbf{k}, x_h, 0).$$

Hence

$$(\zeta_n(t, \mathbf{k}, x), u_n(t, \mathbf{k}, x), q_n(t, \mathbf{k}, x), \eta_n(t, \mathbf{k}, x_h)) = e^{\lambda_n(\mathbf{k})t}(\omega_n, v_n, r_n, \nu_n)(\mathbf{k}, x)$$

is a real-valued solution of the linearized equations (0.101). We claim (3.47) by virtue of $(\psi_n, \varphi_n, \phi_n, \pi_n) \in (H^\infty(\mathbf{R}_-))^4$. □

3.2.5 Maximal growth rate

We derive the following proposition on the largest characteristic value λ_1 found in Theorem 0.5.

Proposition 3.8. *Let us recall the bilinear form $\mathcal{B}_{a,k,\lambda}$ on $H^2((-a, 0))$ (0.116) and (λ_1, ϕ_1) from Theorem 0.5. We have that*

$$\frac{1}{gk^2} = \max_{\phi \in H^2((-a,0))} \frac{\int_{-a}^0 \rho_0'\phi^2 dx_3}{\lambda_1 \mathcal{B}_{a,k,\lambda_1}(\phi, \phi)}, \tag{3.48}$$

and the variational problem (3.48) is attained by the function ϕ_1 restricted on $(-a, 0)$.

Furthermore, let us define the following bilinear form on $H^2(\mathbf{R}_-)$,

$$\mathbf{B}_{k,\lambda}(\phi, \theta) := \lambda \int_{\mathbf{R}_-} \rho_0(k^2\phi\theta + \phi'\theta')dx_3 + \mu \int_{\mathbf{R}_-} ((\phi'' + k^2\phi)(\theta'' + k^2\theta) + 4k^2\phi'\theta')dx_3 + \frac{gk^2\rho_+}{\lambda}\phi(0)\theta(0).$$

Hence, we have

$$\frac{1}{gk^2} = \max_{\phi \in H^2(\mathbf{R}_-)} \frac{\int_{\mathbf{R}_-} \rho_0'\phi^2 dx_3}{\lambda_1 \mathbf{B}_{k,\lambda_1}(\phi, \phi)}. \tag{3.49}$$

The variational problem (3.49) is attained by the function ϕ_1.

Proof of Proposition 3.8. We divide the proof into two parts, proving (3.48) and (3.49), respectively.

Part 1. We show that (3.48) holds. For all $\lambda > 0$, we solve the variational problem

$$\alpha_1(\lambda, k) = \max\Big(\int_{-a}^0 \rho_0'\phi^2 dx_3 \Big| \phi \in H^2((-a, 0)), \quad \lambda\mathcal{B}_{a,k,\lambda}(\phi, \phi) = 1\Big). \tag{3.50}$$

Let us define the Lagrangian function

$$\mathcal{L}_{\mathcal{B}}(\nu, \phi) = \int_{-a}^0 \rho_0'\phi^2 dx_3 - \nu(\lambda\mathcal{B}_{a,k,\lambda}(\phi, \phi) - 1). \tag{3.51}$$

It follows from the Lagrange multiplier theorem that the extrema of the quotient

$$\frac{\int_{-a}^{0} \rho_0' \phi^2 dx_3}{\lambda \mathcal{B}_{a,k,\lambda}(\phi, \phi)}$$

are necessarily the stationary points $(\nu_\star, \phi_\star)$ of $\mathcal{L}_\mathcal{B}$, which satisfy

$$\lambda \mathcal{B}_{a,k,\lambda}(\phi_\star, \phi_\star) = 1 \tag{3.52}$$

and

$$\int_{-a}^{0} \rho_0' \phi_\star \theta dx_3 - \lambda \nu_\star \mathcal{B}_{a,k,\lambda}(\phi_\star, \theta) = 0, \tag{3.53}$$

for all $\theta \in H^2((-a,0))$. Restricting $\theta \in C_0^\infty((-a,0))$ and following the line of the proof of Proposition 3.3, one deduces from (3.53) that $\phi_\star$ has to satisfy

$$\lambda \nu_\star Y_{a,k,\lambda} \phi_\star = \rho_0' \phi_\star \tag{3.54}$$

in a weak sense. We further get that $\phi_\star \in H^4((-a,0))$ and satisfies (3.52) and the boundary conditions (0.113)-(0.111). Hence, all stationary points $(\nu_\star, \phi_\star)$ of $\mathcal{L}_\mathcal{B}$ satisfy that, $\lambda\nu_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{a,k,\lambda} = \mathcal{M} Y_{a,k,\lambda}^{-1} \mathcal{M}$ from $L^2((-a,0))$ to itself, with

$$\mathcal{M}^{-1} Y_{a,k,\lambda} \phi_\star = \frac{1}{\lambda \nu_\star} \mathcal{M} \phi_\star \in L^2((-a,0))$$

being an associated eigenfunction. That implies

$$\alpha_1(\lambda, k) \leqslant \lambda^{-1} \gamma_1(\lambda, k). \tag{3.55}$$

Meanwhile, since the operator $S_{a,k,\lambda}$ is self-adjoint and positive, we thus obtain that

$$\gamma_1(\lambda, k) = \sup_{\phi \in L^2((-a,0))} \frac{\langle S_{a,k,\lambda} \phi, \phi \rangle}{\|\phi\|^2_{L^2((-a,0))}}.$$

Hence, for all $\phi \in L^2((-a,0))$ and for $\psi = Y_{a,k,\lambda}^{-1} \mathcal{M} \phi \in H^4((-a,0))$, we have

$$\langle Y_{a,k,\lambda} \psi, \psi \rangle = \langle S_{a,k,\lambda} \phi, \phi \rangle,$$

which yields

$$\gamma_1(\lambda, k) \langle Y_{a,k,\lambda} \psi, \psi \rangle \leqslant \frac{\langle S_{a,k,\lambda} \phi, \phi \rangle^2}{\|\phi\|^2_{L^2((-a,0)}} \leqslant \|S_{a,k,\lambda} \phi\|^2_{L^2((-a,0))}.$$

This yields

$$\gamma_1(\lambda, k) \leqslant \sup \Big\{ \frac{\|\mathcal{M}\psi\|^2_{L^2((-a,0))}}{\langle Y_{a,k,\lambda} \psi, \psi \rangle} | \psi \in H^4((-a,0)) \text{ and } \mathcal{M}^{-1} Y_{a,k,\lambda} \psi \in L^2((-a,0)) \Big\}.$$

Owing to (3.19), we have that

$$\gamma_1(\lambda, k) \leqslant \sup \Big\{ \frac{\int_{-a}^{0} \rho_0' \psi^2 dx_3}{\mathcal{B}_{a,k,\lambda}(\psi, \psi)} | \psi \in H^4((-a,0)) \text{ and } \mathcal{M}^{-1} Y_{a,k,\lambda} \psi \in L^2((-a,0)) \Big\}.$$

We thus obtain

$$\lambda^{-1}\gamma_1(\lambda,k) \leqslant \alpha_1(\lambda,k) \tag{3.56}$$

The two inequalities (3.55) and (3.56) tell us that $\alpha_1(k,\lambda) = \lambda^{-1}\gamma_1(k,\lambda)$ for all $\lambda > 0$, from which we deduce $\alpha_1(\lambda_1,k) = \frac{1}{gk^2}$ and the variational problem (3.48) is attained by the function ϕ_1.

Part 2. We prove that (3.49) holds. We set

$$\alpha_2(\lambda,k) = \max_{\phi\in H^2(\mathbf{R}_-)}\Big(\int_{\mathbf{R}_-}\rho_0'\phi^2 dx_3\Big|\lambda\mathbf{B}_{k,\lambda}(\phi,\phi) = 1\Big).$$

and consider the Lagrangian function

$$\mathbf{L_B}(\omega,\phi) = \int_{\mathbf{R}_-}\rho_0'\phi^2 dx_3 - \omega(\mathbf{B}_{k,\lambda}(\phi,\phi) - 1).$$

Thanks to Lagrange multiplier theorem again, the extrema of the quotient

$$\frac{\int_{\mathbf{R}_-}\rho_0'\phi^2 dx_3}{\lambda\mathbf{B}_{k,\lambda}(\phi,\phi)}$$

are necessarily the stationary points $(\omega_\star,\Phi_\star) \in \mathbf{R}_+ \times H^2(\mathbf{R}_-)$ of $\mathbf{L_B}$, which satisfy

$$\lambda\mathbf{B}_{k,\lambda}(\Phi_\star,\Phi_\star) = 1 \tag{3.57}$$

and

$$\int_{\mathbf{R}_-}\rho_0'\Phi_\star\theta dx_3 - \lambda\omega_\star\mathbf{B}_{k,\lambda}(\Phi_\star,\theta) = 0 \tag{3.58}$$

for all $\theta \in H^2(\mathbf{R}_-)$.

We now improve the regularity of $\Phi_\star$. We respectively define $(\Phi_\star'')'$ and $(\Phi_\star'')''$ in the distributional sense as the first and second derivative of $\Phi_\star''$ which is in $L^2(\mathbf{R}_-)$. Hence, (3.58) will imply that

$$\begin{aligned}\lambda\int_{\mathbf{R}_-}\rho_0(k^2\Phi_\star\theta + \Phi_\star'\theta')dx_3 + \mu\langle(\Phi_\star'')'',\theta\rangle + \mu\int_{\mathbf{R}_-}(2k^2\Phi_\star''\theta + 4k^2\Phi_\star'\theta' + k^4\Phi_\star\theta)dx_3 \\ = \frac{1}{\lambda\omega_\star}\int_{\mathbf{R}_-}\rho_0'\Phi_\star\theta dx_3\end{aligned} \tag{3.59}$$

for all $\theta \in C_0^\infty(\mathbf{R}_-)$. We deduce from (3.59) that

$$\mu((\Phi_\star'')'' - 2k^2\Phi_\star'' + k^4\Phi_\star) + \lambda(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')') = \frac{1}{\lambda\omega_\star}\rho_0'\Phi_\star \quad \text{in } \mathcal{D}'(\mathbf{R}_-). \tag{3.60}$$

Thanks to (3.59) again, we obtain $(\Phi_\star'')'' \in L^2(\mathbf{R}_-)$. Let $b > 0$ be fixed and arbitrary, we have that $\Phi \in H^2((-b,0))$. Let $\chi_2 \in C_0^\infty((-b,0))$ satisfy $\int_{-b}^0\chi_2(y)dy = 1$. Using the distribution theory, we define $\Sigma_b \in \mathcal{D}'((-b,0))$ such that

$$\langle\Sigma_b,\theta\rangle = \langle(\Phi_\star'')'',\zeta_{\theta,b}\rangle \tag{3.61}$$

for all $\theta \in C_0^\infty((-b,0))$, where

$$\zeta_{\theta,b}(x_3) = \int_{-b}^{x_3} \Big(\theta(y) - \chi_2(y)\int_{-b}^{0}\theta(s)ds\Big)dy$$

for all $-b < x_3 < 0$. We obtain

$$\langle \Sigma_b', \theta\rangle = -\langle \Sigma_b, \theta'\rangle = -\langle (\Phi_\star'')'', \zeta_{\theta',b}\rangle.$$

Note that

$$\langle (\Phi_\star'')'', \zeta_{\theta',b}\rangle = \langle (\Phi_\star'')'', \theta(x_3) - \int_{-b}^{x_3}\chi_2(y)\int_{-b}^{0}\theta'(s)dsdy\rangle = \langle (\Phi_\star'')'', \theta\rangle,$$

this yields $\langle \Sigma_b', \theta\rangle = -\langle (\Phi_\star'')'', \theta\rangle$. Hence, we have that $(\Phi_\star'')' + \Sigma_b \equiv$ constant. In view of $(\Phi_\star'')'' \in L^2((-b,0))$ and (3.61), we know that $(\Phi_\star'')' \in L^2((-b,0)))$. Since $\Phi_\star \in H^2(\mathbf{R}_-)$ and $(\Phi_\star'')', (\Phi_\star'')'' \in L^2((-b,0))$, it tells us that Φ belongs to $H^4((-b,0))$.

Next, let us take $\theta \in C_0^\infty((-\infty,-b))$ with $b > a$. Due to (0.96) and (0.97), one has

$$\mu((\Phi_\star'')'' - 2k^2\Phi_\star'' + k^4\Phi_\star) + \lambda\rho_+(k^2\Phi_\star - \Phi_\star'') = 0 \quad \text{in } \mathcal{D}'((-\infty,-b)).$$

As Φ is bounded at $-\infty$, hence we have

$$\Phi_\star(x_3) = a_1 e^{kx_3} + a_2 e^{(k^2+\lambda\rho_-/\mu)^{1/2}x_3}.$$

Since $\Phi_\star$ is explicit, we see that $\Phi_\star \in H^4((-\infty,-b))$. Consequently, $\Phi_\star \in H^4(\mathbf{R}_-)$ and $\Phi_\star$ decays to 0 at infinity.

By exploiting (3.58), we show that $\Phi_\star$ satisfies (0.111). Indeed, for all $\theta \in H^2(\mathbf{R}_-)$, we use the integration by parts to obtain from (3.58) that

$$\begin{aligned}
&\lambda\int_{\mathbf{R}_-}(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')')\theta dx_3 + \lambda\rho_0\Phi_\star'\theta\Big|_{-\infty}^{0} \\
&\quad + \mu\int_{\mathbf{R}_-}((\Phi_\star'' + k^2\Phi_\star)(\theta_\star'' + k^2\theta_\star) + 4k^2\Phi_\star'\theta')dx_3 \\
&\quad + \mu\Big(\Phi_\star''\theta'\Big|_{-\infty}^{0} - \Phi_\star'''\theta\Big|_{-\infty}^{0} + k^2\Phi_\star\theta'\Big|_{-\infty}^{0} + 3k^2\Phi_\star'\theta\Big|_{-\infty}^{0}\Big) + \frac{gk^2\rho_+}{\lambda}\Phi_\star(0)\theta(0) \\
&= \frac{1}{\lambda\omega_\star}\int_{\mathbf{R}_-}\rho_0'\Phi_\star\theta dx_3.
\end{aligned}$$

By collecting all terms corresponding to $\theta'(0)$ and $\theta'(0)$ respectively, we obtain that $\Phi_\star''(0) + k^2\Phi_\star(0) = 0$ and that

$$\lambda\rho_+\Phi_\star'(0) - \mu\Phi_\star'''(0) + 3k^2\Phi_\star'(0) + \frac{gk^2\rho_+}{\lambda}\Phi_\star(0) = 0.$$

This yields that $\Phi_\star$ satisfies (0.111).

We have just shown that $\Phi_\star \in H^4(\mathbf{R}_-)$ is a solution to

$$\mu(\Phi_\star^{(4)} - 2k^2\Phi_\star'' + k^4\Phi_\star) + \lambda(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')') = \frac{1}{\lambda\omega_\star}\rho_0'\Phi_\star \quad \text{on } \mathbf{R}_- \tag{3.62}$$

satisfying (0.111)-(0.112). Since supp$\rho_0' = [-a, 0]$, we see that $\Phi_\star$ is a solution of

$$\mu(\Phi_\star^{(4)} - 2k^2\Phi_\star'' + k^4\Phi_\star) + \lambda(k^2\rho_0\Phi_\star - (\rho_0\Phi_\star')') = 0 \quad \text{on } (-\infty, -a).$$

Then, $\Phi_\star$ on $(-\infty, -a)$ is of the form (3.18). Mimicking the computations in the proof of Lemma 3.3, we deduce $\Phi_\star$ on $(-a, 0)$ is a solution of

$$\lambda\omega_\star Y_{a,k,\lambda}(\Phi_\star|_{(-a,0)}) = \rho_0'\Phi_\star|_{(-a,0)} = \mathcal{M}^2\Phi_\star|_{(-a,0)}$$

with the boundary conditions (0.113)-(0.111). Set

$$\tilde{\Phi} = \mathcal{M}^{-1}Y_{a,k,\lambda}(\Phi_\star|_{(-a,0)}) = \frac{1}{\lambda\omega_\star}\mathcal{M}\Phi_\star|_{(-a,0)} \in L^2((-a,0)), \tag{3.63}$$

it yields

$$\lambda\omega_\star\tilde{\Phi} = \mathcal{M}Y_{a,k,\lambda}^{-1}\mathcal{M}\tilde{\Phi} = S_{a,k,\lambda}\tilde{\Phi}.$$

That means $\lambda\omega_\star$ is an eigenvalue of the compact and self-adjoint operator $S_{a,k,\lambda}$ from $L^2((-a,0))$ to itself, with $\tilde{\Phi} \in L^2((-a,0))$ (defined as in (3.63)) being an associated eigenfunction. Hence, we get

$$\lambda\alpha_2(\lambda, k) \leqslant \gamma_1(\lambda, k). \tag{3.64}$$

Let us recall the function ϕ_1 from Theorem 0.5. One thus has

$$\alpha_2(\lambda, k) \geqslant \frac{\int_{\mathbf{R}_-} \rho_0'\phi_1^2 dx_3}{\lambda\mathbf{B}_{k,\lambda}(\phi_1, \phi_1)}. \tag{3.65}$$

Note that from the proof of Theorem 0.5,

$$\phi_1(x_3) = A_1 e^{k(x_3+a)} + A_2 e^{\sqrt{k^2+\frac{\lambda_1\rho_-}{\mu}}(x_3+a)} \quad \text{as } -\infty < x_3 < -a.$$

Let us write $\phi_1|_{(-a,0)}$ as the function ϕ_1 being restricted on $(-a, 0)$. Hence, the direct computations show that

$$\mathbf{B}_{k,\lambda}(\phi_1, \phi_1) = \mathcal{B}_{a,k,\lambda}(\phi_1|_{(-a,0)}, \phi_1|_{(-a,0)}), \tag{3.66}$$

and we keep in mind the assumption supp$\rho_0' = [-a, 0]$. Then, from (3.65) and (3.66), we have

$$\alpha_2(\lambda, k) \geqslant \frac{\int_{-a}^0 \rho_0'\phi_1^2 dx_3}{\lambda\mathcal{B}_{a,k,\lambda}(\phi_1|_{(-a,0)}, \phi_1|_{(-a,0)})}.$$

It then follows

$$\alpha_2(\lambda_1, k) \geqslant \frac{\int_{-a}^0 \rho_0'\phi_1^2 dx_3}{\lambda_1\mathcal{B}_{a,k,\lambda_1}(\phi_1|_{(-a,0)}, \phi_1|_{(-a,0)})} = \frac{1}{gk^2}. \tag{3.67}$$

Combining (3.64) and (3.67) gives us that $\alpha_2(\lambda_1, k) = \frac{1}{gk^2}$ and the variational problem (3.49) is attained by the function ϕ_1. We finish the proof of Proposition 3.8. □

Recall the definition of Λ from (0.122), we prove that Λ is the maximal growth rate of the linearized equations (0.101) in the following sense:

Proposition 3.9. *For all $t \geqslant 0$, the following inequalities hold*

$$\begin{aligned}\|\zeta(t)\|^2_{H^1(\Omega)} + \|u(t)\|^2_{H^1(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \int_0^t \|u(s)\|^2_{H^1(\Omega)} ds \\ \lesssim (\|\eta(0)\|^2_{H^{1/2}(\Gamma)} + \|u(0)\|^2_{H^2(\Omega)} + \|\zeta(0)\|_{H^1(\Omega)})e^{2\Lambda t},\end{aligned} \tag{3.68}$$

and

$$\begin{aligned}\|\eta(t)\|^2_{H^{1/2}(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)} \\ \lesssim (\|\eta(0)\|^2_{H^{1/2}(\Gamma)} + \|u(0)\|^2_{H^2(\Omega)} + \|\zeta(0)\|^2_{L^2(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.69}$$

The proof of Proposition 3.9 relies on the three lemmas below.

Lemma 3.7. *There holds*

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u|^2 - \int_\Omega g\rho_0'|u_3|^2 + \int_\Gamma g\rho_+|u_3|^2\Big) + \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t u|^2 = 0. \tag{3.70}$$

Proof. We differentiate $(0.101)_2$ in time, multiply the resulting equation by $\partial_t u$ and then use $(0.101)_1$ to obtain

$$\int_\Omega \rho_0\partial_t^2 u\cdot\partial_t u + \int_\Omega \nabla\partial_t q\cdot\partial_t u - \mu\int_\Omega \Delta\partial_t u\cdot\partial_t u - \int_\Omega g\rho_0' u_3\partial_t u_3 = 0.$$

That is equivalent to

$$\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u|^2 - \int_\Omega g\rho_0'|u_3|^2\Big) + \int_\Omega \nabla\partial_t q\cdot\partial_t u - \mu\int_\Omega \Delta\partial_t u\cdot\partial_t u = 0. \tag{3.71}$$

We use the integration by parts over Ω to have

$$\begin{aligned}\int_\Omega \nabla\partial_t q\cdot\partial_t u - \mu\int_\Omega \Delta\partial_t u\cdot\partial_t u = \int_\Gamma (\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u)e_3\cdot\partial_t u - \int_\Omega \partial_t q\mathrm{div}\partial_t u \\ + \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t u|^2\end{aligned}$$

Thanks to $(0.101)_{3,4,5}$, we obtain

$$\begin{aligned}\int_\Omega \nabla\partial_t q\cdot\partial_t u - \mu\int_\Omega \Delta\partial_t u\cdot\partial_t u &= \int_\Gamma g\rho_+\partial_t\eta\partial_t u_3 + \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t u|^2 \\ &= \int_\Gamma g\rho_+ u_3\partial_t u_3 + \frac{\mu}{2}\int_\Omega |\mathbb{S}\partial_t u|^2.\end{aligned} \tag{3.72}$$

Substituting (3.72) into (3.71), we conclude (3.70). □

Lemma 3.8. *There holds*

$$\int_\Omega g\rho_0'|u_3|^2 \leqslant \int_\Gamma g\rho_+|u_3|^2 + \Lambda^2\int_\Omega \rho_0|u|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}u|^2. \tag{3.73}$$

Proof. Let $\mathbf{k} = (k_1, k_2) \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}$ be fixed and $\hat{f}$ be the horizontal Fourier transform of f, i.e.

$$\hat{f}(\mathbf{k}, x_3) = \int_{\mathbf{T}} f(x_h, x_3) e^{-i\mathbf{k}\cdot x_h} dx_h.$$

We write

$$\hat{u}_1(\mathbf{k}, x) = -i\psi(\mathbf{k}, x_3), \quad \hat{u}_2(\mathbf{k}, x) = -i\varphi(\mathbf{k}, x_3), \quad \hat{u}_3(\mathbf{k}, x) = \phi(\mathbf{k}, x_3).$$

Notice that for $\mathbf{k} = 0$,

$$\phi(0,0) = \int_\Gamma u_3 = \int_\Omega \mathrm{div} u = 0.$$

Hence, together with Parseval's theorem, we have

$$\int_\Gamma g\rho_+ |u_3|^2 = \frac{1}{4\pi^2 L_1 L_2} \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}\setminus\{0\}} g\rho_+ |\phi(\mathbf{k}, 0)|^2. \tag{3.74}$$

We may reduce to estimate (3.74) when ψ, φ and ϕ are real-valued and then continue the estimate to the real and imaginary parts of ψ, φ and ϕ

For each $\mathbf{k} \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}\setminus\{0\}$, we deduce from Proposition 3.8 that

$$\begin{aligned}\int_{\mathbf{R}_-} g\rho_0' \phi^2(\mathbf{k}, x_3) dx_3 \leqslant g\rho_+(\phi(k,0))^2 + \lambda_1^2 \int_{\mathbf{R}_-} \rho_0 \Big(\phi^2 + \frac{(\phi')^2}{k^2}\Big)(\mathbf{k}, x_3) dx_3 \\ + \lambda_1 \mu \int_{\mathbf{R}_-} \Big(\Big(\frac{\phi''}{k} + k\phi\Big)^2 + 4(\phi')^2\Big)(\mathbf{k}, x_3) dx_3.\end{aligned}$$

It thus follows from the definition of Λ (0.122) that

$$\begin{aligned}\int_{\mathbf{R}_-} g\rho_0' \phi^2(\mathbf{k}, x_3) dx_3 \leqslant g\rho_+(\phi(\mathbf{k},0))^2 + \Lambda^2 \int_{\mathbf{R}_-} \rho_0 \Big(\phi^2 + \frac{(\phi')^2}{k^2}\Big)(\mathbf{k}, x_3) dx_3 \\ + \Lambda \mu \int_{\mathbf{R}_-} \Big(\Big(\frac{\phi''}{k} + k\phi\Big)^2 + 4(\phi')^2\Big)(\mathbf{k}, x_3) dx_3\end{aligned} \tag{3.75}$$

for all $\mathbf{k} \in L_1^{-1}\mathbf{Z} \times L_2^{-1}\mathbf{Z}\setminus\{0\}$.

Meanwhile, for $\mathbf{k} \neq 0$, notice that $k_1\psi + k_2\varphi + \phi' = 0$. One thus has

$$(\phi')^2 \leqslant (k_1\psi + k_2\varphi)^2 + (k_1\varphi - k_2\psi)^2 = k^2(\psi^2 + \varphi^2), \tag{3.76}$$

and

$$2(\phi')^2 = 2k_1^2\psi^2 + 2k_2^2\varphi^2 + 4k_1k_2\psi\varphi \leqslant 2k_1^2\psi^2 + 2k_2^2\varphi^2 + (k_1\varphi + k_2\psi)^2. \tag{3.77}$$

Furthermore, we obtain that

$$(\phi'')^2 \leqslant (k_1\psi' + k_2\varphi')^2 + (k_1\varphi' - k_2\psi')^2 = k^2((\psi')^2 + (\varphi')^2).$$

This yields

$$\Big(\frac{1}{k}\phi'' + k\phi\Big)^2 = \frac{1}{k^2}(\phi'')^2 + 2\phi\phi'' + k^2\phi^2 \leqslant (\psi')^2 + (\varphi')^2 - 2\phi(k_1\psi' + k_2\varphi') + k^2\phi^2,$$

so that

$$\Big(\frac{1}{k}\phi'' + k\phi\Big)^2 \leqslant (k_1\phi - \psi')^2 + (k_2\phi - \varphi')^2. \tag{3.78}$$

Then, in view of Fubini's and Parseval's theorem again, we find that due to (3.76),

$$\begin{aligned} \int_\Omega \rho_0|u|^2 &= \frac{1}{4\pi^2 L_1 L_2} \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}} \int_{\mathbf{R}_-} \rho_0(\psi^2 + \varphi^2 + \phi^2)(\mathbf{k}, x_3)dx_3 \\ &\geqslant \frac{1}{4\pi^2 L_1 L_2} \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}\setminus\{0\}} \int_{\mathbf{R}_-} \rho_0\Big(\phi^2 + \frac{(\phi')^2}{k^2}\Big)(\mathbf{k}, x_3)dx_3 \end{aligned} \tag{3.79}$$

and that due to (3.77) and (3.78),

$$\begin{aligned} &\frac{1}{2}\int_\Omega \mu|\mathbb{S}u|^2 \\ &= \frac{\mu}{4\pi^2 L_1 L_2} \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}} \int_{\mathbf{R}_-} \begin{pmatrix} 2(\phi')^2 + 2k_1^2\psi^2 + 2k_2^2\varphi^2 + (k_1\varphi + k_2\psi)^2 \\ + (k_1\phi - \psi')^2 + (k_2\phi - \varphi')^2 \end{pmatrix} dx_3 \\ &\geqslant \frac{\mu}{4\pi^2 L_1 L_2} \sum_{\mathbf{k}\in L_1^{-1}\mathbf{Z}\times L_2^{-1}\mathbf{Z}\setminus\{0\}} \int_{\mathbf{R}_-} \Big(\Big(\frac{\phi''}{k} + k\phi\Big)^2 + 4(\phi')^2\Big)dx_3. \end{aligned} \tag{3.80}$$

Combining (3.74), (3.75), (3.79) and (3.80), the inequality (3.73) follows, we end the proof here. □

We are in position to prove Proposition 3.9.

Proof of Proposition 3.9. Owing to (3.70) and (3.73), we have that

$$\begin{aligned} \int_\Omega \rho_0|\partial_t u(t)|^2 + \int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u(s)|^2 ds &= y_1 + \int_\Omega g\rho_0'|u_3(t)|^2 - \int_\Gamma g\rho_+|u_3(t)|^2 \\ &\leqslant y_1 + \Lambda^2\int_\Omega \rho_0|u(t)|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}u(t)|^2, \end{aligned} \tag{3.81}$$

where

$$y_1 = \int_\Omega \rho_0|\partial_t u(0)|^2 - \int_\Omega g\rho_0'|u_3(0)|^2 + \int_\Gamma g\rho_+|u_3(0)|^2.$$

Using Cauchy-Schwarz's inequality, we have that

$$\begin{aligned} \int_\Omega \mu|\mathbb{S}u(t)|^2 &= \int_\Omega \mu|\mathbb{S}u(0)|^2 + 2\int_0^t\int_\Omega \mu\mathbb{S}u(s) : \mathbb{S}\partial_t u(s)ds \\ &\leqslant \int_\Omega \mu|\mathbb{S}u(0)|^2 + \frac{1}{\Lambda}\int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u(s)|^2 ds + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u(s)|^2 ds \end{aligned} \tag{3.82}$$

and that

$$\frac{d}{dt}\int_\Omega \rho_0|u|^2 \leqslant \frac{1}{\Lambda}\int_\Omega \rho_0|\partial_t u|^2 + \Lambda\int_\Omega \rho_0|u|^2. \tag{3.83}$$

Three inequalities (3.81), (3.82) and (3.83) imply that

$$\frac{d}{dt}\int_\Omega \rho_0|u(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u(t)|^2 \leqslant y_2 + 2\Lambda\int_\Omega \rho_0|u(t)|^2 + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u(s)|^2 ds. \tag{3.84}$$

where
$$y_2 = \frac{y_1}{\Lambda} + \int_\Omega \mu|\mathbb{S}u(0)|^2.$$
In view of Gronwall's inequality, we obtain from (3.84) that
$$\int_\Omega \rho_0|u(t)|^2 + \frac{1}{2}\int_0^t\int_\Omega \mu|\mathbb{S}u(s)|^2 ds \leqslant e^{2\Lambda t}\int_\Omega \rho_0|u(0)|^2 + \frac{y_2}{2\Lambda}(e^{2\Lambda t}-1). \tag{3.85}$$
Hence,
$$\begin{aligned}\frac{1}{\Lambda}\int_\Omega \rho_0|\partial_t u(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u(t)|^2 &\leqslant y_2 + \Lambda\int_\Omega \rho_0|u(t)|^2 + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u(s)|^2 ds\\ &\leqslant \Big(y_2 + 2\Lambda\int_\Omega \rho_0|u(0)|^2\Big)e^{2\Lambda t}\end{aligned} \tag{3.86}$$
Using the trace theorem, we have
$$y_1 + y_2 \lesssim \|u(0)\|^2_{H^1(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)}. \tag{3.87}$$
Because of (3.85), (3.86) and (3.87), we observe
$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)} + \|\mathbb{S}u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\mathbb{S}u(s)\|^2_{L^2(\Omega)}ds\\ &\lesssim (\|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.88}$$
In view of Korn's inequality (see (3.5)), that implies
$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\partial_t u(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}ds\\ &\lesssim (\|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.89}$$
Using $(0.101)_1$ and (3.89) also, we get
$$\begin{aligned}\|\zeta(t)\|^2_{H^1(\Omega)} &\lesssim \|\zeta(0)\|^2_{H^1(\Omega)} + \int_0^t \|u(s)\|^2_{H^1(\Omega)}ds\\ &\lesssim (\|\zeta(0)\|^2_{H^1(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.90}$$
The inequality (3.68) follows from (3.89) and (3.90).

To prove (3.69), we use the trace theorem to obtain that
$$\begin{aligned}\|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)}ds &= \|u_3(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|u_3(s)\|^2_{H^{1/2}(\Gamma)}ds\\ &\leqslant \|u_3(t)\|^2_{H^1(\Omega)} + \int_0^t \|u_3(s)\|^2_{H^1(\Omega)}ds.\end{aligned}$$
Together with (3.85), (3.87) and (3.89), we deduce that
$$\|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)}ds \lesssim (\|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}. \tag{3.91}$$
The resulting inequality tells us that
$$\begin{aligned}\|\eta(t)\|^2_{H^{1/2}(\Gamma)} &\leqslant \|\eta(0)\|^2_{H^{1/2}(\Gamma)} + \int_0^t \|\partial_t\eta(s)\|^2_{H^{1/2}(\Gamma)}ds\\ &\lesssim (\|\eta(0)\|^2_{H^{1/2}(\Gamma)} + \|\partial_t u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{H^1(\Omega)})e^{2\Lambda t}.\end{aligned} \tag{3.92}$$
The inequality (3.69) follows from (3.91) and (3.92). Proposition 3.9 is proven. □

3.2.6 Proof of Lemma 3.4

Note that the quotient

$$\frac{2k^2(\phi'(0)\phi(0) - \phi'(-a)\phi(-a))}{\int_{-a}^0((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_3} \tag{3.93}$$

is bounded because of the embedding $H^2((-a,0)) \hookrightarrow C^1((-a,0))$. To prove Lemma 3.4, let us consider the Lagrangian functional

$$\mathcal{L}_k(\phi,\beta) = \beta\Big(\int_{-a}^0((\phi'')^2 + 2k^2(\phi')^2 + k^4\phi^2)dx_3 - 1\Big) - 2k^2(\phi'(0)\phi(0) - \phi'(-a)\phi(-a)),$$

for any $\phi \in H^2((-a,0))$. Using Lagrange multiplier theorem again, we find that the extrema of the quotient (3.93) are necessarily the stationary points (ϕ_k,β_k) of $\mathcal{L}_k$, which satisfy

$$\int_{-a}^0((\phi_k'')^2 + 2k^2(\phi_k')^2 + k^4\phi_k^2)dx_3 = 1 \tag{3.94}$$

and

$$\begin{aligned}\beta_k &\int_{-a}^0(\phi_k''\theta'' + 2k^2\phi_k'\theta' + k^4\phi_k\theta)dx_3 \\ &= k^2(\phi_k'(0)\theta(0) + \phi_k(0)\theta'(0) - \phi_k'(-a)\theta(-a) - \phi_k(-a)\theta'(-a)).\end{aligned} \tag{3.95}$$

for all $\theta \in H^2((-a,0))$.

Taking the integration by parts, we obtain that

$$\begin{aligned}\beta_k &\int_{-a}^0(\phi_k^{(4)} - 2k^2\phi_k'' + k^4\phi_k)\theta dx_3 + \beta_k(\phi_k''\theta' - \phi_k'''\theta + 2k^2\phi_k'\theta)\Big|_{-a}^0 \\ &= k^2(\phi_k'(0)\theta(0) + \phi_k(0)\theta'(0) - \phi_k'(-a)\theta(-a) - \phi_k(-a)\theta'(-a)).\end{aligned} \tag{3.96}$$

Restricting $\theta \in C_0^\infty((-a,0))$, the resulting equality yields

$$\phi_k^{(4)} - 2k^2\phi_k'' + k^4\phi_k = 0 \tag{3.97}$$

on $(-a,0)$. Hence, (3.96) tells us that

$$\begin{cases}\beta_k\phi_k''(0) = k^2\phi_k(0), \\ \beta_k(-\phi_k'''(0) + 2k^2\phi_k'(0)) = k^2\phi_k'(0), \\ \beta_k\phi_k''(-a) = k^2\phi_k(-a), \\ \beta_k(-\phi_k'''(-a) + 2k^2\phi_k'(-a)) = k^2\phi_k'(-a).\end{cases} \tag{3.98}$$

Any solution ϕ_k of (3.97) is of the form

$$\phi_k(x_3) = (Ax_3 + B)\sinh(kx_3) + (Cx_3 + D)\cosh(kx_3), \tag{3.99}$$

with A, B, C, D are four constants such that $A^2 + B^2 + C^2 + D^2 > 0$. Let us compute from (3.99) that

$$\phi_k'(x_3) = (A + kD + kCx_3)\sinh(kx_3) + (C + kB + kAx_3)\cosh(kx_3),$$
$$\phi_k''(x_3) = (2kC + k^2B + k^2Ax_3)\sinh(kx_3) + (2kA + k^2D + k^2Cx_3)\cosh(kx_3).$$

and

$$\phi_k'''(x_3) = (3k^2A + k^3D + k^3C)\sinh(kx_3) + (3k^2C + k^3B + k^3Ax_3)\cosh(kx_3).$$

Substituting these formulas into (3.98), we obtain

$$\begin{cases} \beta_k(2kA + k^2D) = k^2D, \\ \beta_k(-k^2C + k^3B) = k^2(C + kB), \end{cases} \tag{3.100}$$

and

$$\begin{cases} \beta_k\Big(-(2kC + k^2(B - Aa))\sinh(ka) + (2kA + k^2(D - Ca))\cosh(ka)\Big) \\ \qquad = k^2(-(B - Aa)\sinh(ka) + (D - Ca)\cosh(ka)), \\ \beta_k\Big(-(3k^2A + k^3(D - Ca))\sinh(ka) + (3k^2C + k^3(B - Aa))\cosh(ka)\Big) \\ \qquad = k^2(2\beta_k - 1)\Big(-(A + k(D - Ca))\sinh(ka) + (C + k(B - Aa))\cosh(ka)\Big). \end{cases} \tag{3.101}$$

System (3.100) is equivalent to

$$\begin{cases} k(\beta_k - 1)B = (\beta_k + 1)C, \\ k(\beta_k - 1)D = -2\beta_kA. \end{cases} \tag{3.102}$$

We also obtain that (3.101) is equivalent to

$$\begin{cases} \Big((-\beta_k(ka\sinh(ka) + 2\cosh(ka)) + ka\sinh(ka)\Big)A + (\beta_k - 1)k\sinh(ka)B \\ \quad + \Big((2\sinh(ka) + ka\cosh(ka))\beta_k - ka\cosh(ka)\Big)C + (-\beta_k + 1)k\cosh(ka)D = 0, \\ \Big(-(\beta_k + 1)\sinh(ka) + (\beta_k - 1)ka\cosh(ka)\Big)A + (-\beta_k + 1)k\cosh(ka)B \\ \quad + \Big((-\beta_k + 1)ka\sinh(ka) + (\beta_k + 1)\cosh(ka)\Big)C + (\beta_k - 1)k\sinh(ka)D = 0. \end{cases} \tag{3.103}$$

Substituting (3.102) into (3.103), we deduce

$$\begin{cases} ka\tanh(ka)(-\beta_k + 1)A + ((3\beta_k + 1)\tanh(ka) + ka(\beta_k - 1))C = 0, \\ (-(3\beta_k + 1)\tanh(ka) + ka(\beta_k - 1))A + (-\beta_k + 1)ka\tanh(ka)C = 0. \end{cases}$$

Hence, system (3.98) is equivalent to

$$\begin{cases} (\beta_k + 1)C - k(\beta_k - 1)B = 0, \\ 2\beta_kA + k(\beta_k - 1)D = 0, \\ ka\tanh(ka)(-\beta_k + 1)A + (\tanh(ka)(3\beta_k + 1) + ka(\beta_k - 1))C = 0, \\ (-\tanh(ka)(3\beta_k + 1) + ka(\beta_k - 1))A + ka\tanh(ka)(-\beta_k + 1)C = 0. \end{cases} \tag{3.104}$$

System (3.104) admits a nontrivial solution (A, C, B, D) if and only if the determinant of the corresponding matrix is equal to zero. This yields

$$k^2(\beta_k-1)^2\Big((ka)^2\tanh^2(ka)(\beta_k-1)^2-\Big((ka)^2(\beta_k-1)^2-\tanh^2(ka)(3\beta_k+1)^2\Big)\Big)=0.$$

Equivalently,

$$k^2(\beta_k-1)^2((ka)^2(\beta_k-1)^2-\sinh^2(ka)(3\beta_k+1)^2)=0. \tag{3.105}$$

We have three possible values of β_k, which are solutions of (3.105) and ordered as

$$1 \text{ (multiplicity 2)} > -\frac{\sinh(ka)-ka}{3\sinh(ka)+ka} > -\frac{\sinh(ka)+ka}{3\sinh(ka)-ka}.$$

Let us take the maximal value $\beta_k=1$. Clearly, we obtain $A=C=0$ from (3.102) and

$$\phi_k(x_3)=B\sinh(kx_3)+D\cosh(kx_3).$$

Substituting the above ϕ_k into (3.94), we have

$$\int_{-a}^{0}(B\sinh(kx_3)+D\cosh(kx_3))^2dx_3+\int_{-a}^{0}(D\sinh(kx_3)+B\cosh(kx_3))^2dx_3=\frac{1}{2k^4}.$$

Equivalently,

$$(B^2+D^2)\int_{-a}^{0}\cosh(2kx_3)dx_3+2BD\int_{-a}^{0}\sinh(2kx_3)dx_3=\frac{1}{2k^4}.$$

We directly have

$$\frac{1}{2}\sinh(2ka)(B^2+D^2)-2\sinh^2(ka)BD=\frac{1}{2k^3}.$$

This yields

$$\begin{cases} D \text{ is arbitrary and} \\ B=\frac{2\sinh^2(ka)\pm\sqrt{\sinh^2(ka)(2\cosh^2(ka)+\cosh(2ka))D^2+\frac{1}{k^3}\sinh(2ka)}}{2\sinh(2ka)}. \end{cases} \tag{3.106}$$

Let us consider the minimal value $\beta_k=-\frac{\sinh(ka)+ka}{3\sinh(ka)-ka}$. It can be seen from (3.104) that

$$D=-\frac{\sinh(ka)+ka}{2k\sinh(ka)}A,\quad C=\frac{\cosh(ka)-1}{\sinh(ka)}A, \tag{3.107}$$

and

$$B=-\frac{(\sinh(ka)-ka)(\cosh(ka)-1)}{2k\sinh^2(ka)}A. \tag{3.108}$$

Hence, $\phi_k(x_3)=Az_k(x_3)$, where

$$\begin{aligned} z_k(x_3)=&\Big(x_3-\frac{(\sinh(ka)-ka)(\cosh(ka)-1)}{2k\sinh^2(ka)}\Big)\sinh(kx_3)\\ &+\Big(\frac{\cosh(ka)-1}{\sinh(ka)}x_3-\frac{\sinh(ka)+ka}{2k\sinh(ka)}\Big)\cosh(kx_3). \end{aligned}$$

To find A, we trace back to (3.94). That means

$$A^2\int_{-a}^{0}((z_k'')^2+2k^2(z_k')^2+k^4z_k^2)dx_3=1. \tag{3.109}$$

From the above cases, we conclude that

- $$\max_{\phi\in H^2((-a,0))}\frac{2k^2(\phi'(0)\phi(0)-\phi'(-a)\phi(-a))}{\int_{-a}^{0}((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_3}=1.$$

  That variational problem is attained by the function

  $$\phi(x_3)=B\sinh(kx_3)+D\cosh(kx_3),$$

  where B,D satisfy (3.106).

- $$\min_{\phi\in H^2((-a,0))}\frac{2k^2(\phi'(0)\phi(0)-\phi'(-a)\phi(-a))}{\int_{-a}^{0}((\phi'')^2+2k^2(\phi')^2+k^4\phi^2)dx_3}=-\frac{\sinh(ka)+ka}{3\sinh(ka)-ka}.$$

  That variational problem is attained by the function

  $$\phi(x_3)=(Ax_3+B)\sinh(kx_3)+(Cx_3+D)\cosh(kx_3),$$

  where A,B,C,D satisfy (3.109), (3.108) and (3.107).

3.3 *A priori* energy estimates

With a regular solution (ζ,u,q,η) of (0.99) on a finite time interval $[0,T_{\max})$, we aim at showing Proposition 0.3, i.e to prove the *a priori* energy estimate (0.128) for the nonlinear equations (0.99), which is

$$\begin{aligned}&\mathcal{E}_f^2(t)+\int_0^t\mathcal{D}_f^2(s)ds\\&\leqslant C_0\Big(\varepsilon^{-5}\mathcal{E}_f^2(0)+\varepsilon\int_0^t\mathcal{E}_f^2(s)ds+\varepsilon^{-5}\int_0^t\mathcal{E}_f(s)(\mathcal{E}_f^2(s)+\mathcal{D}_f^2(s))ds+\varepsilon^{-5}\mathcal{E}_f^3(t)\Big)\\&\qquad+C_0\varepsilon^{-59}\int_0^t(\|(\zeta,u)(s)\|_{L^2(\Omega)}^2+\|\eta(s)\|_{L^2(\Gamma)}^2)ds.\end{aligned} \tag{3.110}$$

Strategy of the proof. Respectively, we derive the *a priori* energy estimates for the space-time derivatives of η in Propositions 3.10, 3.11, for the temporal derivatives of u in Proposition 3.12, for the horizontal space-time derivatives of u in Proposition 3.13 and for the space-time derivatives of ζ in Proposition 3.14. Then, we derive some estimates thanks to the elliptic regularity theory (see Propositions 3.16, 3.17). In view of these above estimates, we obtain (3.110) and complete the proof of Proposition 0.3.

In what follows, the constants C_i ($i\geqslant 1$) are to indicate some constants, which are referred later and different to constants C_i in Chapter 2.

3.3.1 Energy estimates of the perturbation transport

We first derive the *a priori* energy estimates for η.

Proposition 3.10. *The following inequalities hold*

$$\begin{aligned}\|\eta(t)\|^2_{H^4(\Gamma)} \leqslant C_1\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\eta(s)\|^2_{H^4(\Gamma)} + \varepsilon^{-1}\|\nabla u(s)\|^2_{H^4(\Omega)})ds\Big) \\ + C_1\int_0^t \mathcal{E}_f^3(s)ds,\end{aligned} \tag{3.111}$$

$$\begin{aligned}\|\partial_t\eta(t)\|^2_{H^2(\Gamma)} \leqslant C_2\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\partial_t\eta(s)\|^2_{H^2(\Gamma)} + \varepsilon^{-1}\|\nabla \partial_t u(s)\|^2_{H^2(\Omega)})ds\Big) \\ + C_2\int_0^t \mathcal{E}_f^3(s)ds,\end{aligned} \tag{3.112}$$

and

$$\begin{aligned}\|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} \leqslant C_3\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\partial_t^2\eta(s)\|^2_{L^2(\Gamma)} + \varepsilon^{-1}\|\nabla \partial_t^2 u(s)\|^2_{L^2(\Omega)})ds\Big) \\ + C_3\int_0^t \mathcal{E}_f^3(s)ds.\end{aligned} \tag{3.113}$$

Proof. Let us prove (3.111). For any $\alpha \in \mathbb{N}^2$, $|\alpha| \leqslant 4$, we have by $(4.2)_4$,

$$\partial_t\partial^\alpha\eta = \partial^\alpha u_3 - (u_1\partial^\alpha\partial_1\eta + u_2\partial^\alpha\partial_2\eta) - \underbrace{\sum_{0\neq\beta\leqslant\alpha}(\partial^\beta u_1\partial^{\alpha-\beta}\partial_1\eta + \partial^\beta u_2\partial^{\alpha-\beta}\partial_2\eta)}_{=:R_1^\alpha}.$$

Using the integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\eta\|^2_{L^2(\Gamma)} = \frac{1}{2}\int_\Gamma(\partial_1u_1+\partial_2u_2)|\partial^\alpha\eta|^2 + \int_\Gamma(\partial^\alpha u_3 - R_1^\alpha)\partial^\alpha\eta.$$

So that, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\eta\|^2_{L^2(\Gamma)} \lesssim (\|\partial_1u_1\|_{L^\infty(\Gamma)} + \|\partial_2u_2\|_{L^\infty(\Gamma)})\|\partial^\alpha\eta\|^2_{L^2(\Gamma)} \\ + (\|\partial^\alpha u_3\|_{L^2(\Gamma)} + \|R_1^\alpha\|_{L^2(\Gamma)})\|\partial^\alpha\eta\|_{L^2(\Gamma)}.\end{aligned} \tag{3.114}$$

We make use of the trace theorem to obtain that

$$\|\partial_j u_j\|_{L^\infty(\Gamma)} \lesssim \|u\|_{H^3(\Gamma)} \lesssim \|u\|_{H^4(\Omega)}, \tag{3.115}$$

that

$$\|\partial^\alpha u_3\|_{L^2(\Gamma)} \lesssim \|\partial^\alpha u\|_{H^1(\Omega)} \tag{3.116}$$

and that

$$\|R_1^\alpha\|_{L^2(\Gamma)} \lesssim \sum_{0\neq\beta\leqslant\alpha}\|\partial^\beta u\|_{L^2(\Gamma)}\|\partial^{\alpha-\beta}\eta\|_{H^1(\Gamma)} \lesssim \|\partial^\alpha u\|_{H^1(\Omega)}\|\eta\|_{H^{|\alpha|}(\Gamma)}. \tag{3.117}$$

By summing over $\alpha \in \mathbb{N}^2$, $|\alpha| \leqslant 4$, it follows from (3.114), (3.115), (3.116) and (3.117) that

$$\begin{aligned}\frac{d}{dt}\|\eta\|^2_{H^4(\Gamma)} &\lesssim \|\nabla u\|_{H^4(\Omega)}\|\eta\|_{H^4(\Gamma)} + \|u\|_{H^4(\Omega)}\|\eta\|^2_{H^4(\Gamma)} \\ &\lesssim \|\nabla u\|_{H^4(\Omega)}\|\eta\|_{H^4(\Gamma)} + \mathcal{E}_f^3.\end{aligned}$$

Using Cauchy-Schwarz's inequality and then integrating the result inequality from 0 to t, we obtain (3.111).

We show (3.112). Let $\alpha \in \mathbb{N}^2$, $|\alpha| \leqslant 2$, we get

$$\begin{aligned}\partial_t^2\partial^\alpha\eta = \partial^\alpha\partial_t u_3 &- (u_1\partial^\alpha\partial_1\partial_t\eta + u_2\partial^\alpha\partial_2\partial_t\eta) \\ &- \underbrace{\sum_{0\neq\beta\leqslant\alpha}(\partial^\beta u_1\partial^{\alpha-\beta}\partial_1\partial_t\eta + \partial^\beta u_2\partial^{\alpha-\beta}\partial_2\partial_t\eta)}_{=:R_2^\alpha} \\ &- \underbrace{\sum_{0\leqslant\beta\leqslant\alpha}(\partial^\beta\partial_t u_1\partial^{\alpha-\beta}\partial_1\eta + \partial^\beta\partial_t u_2\partial^{\alpha-\beta}\partial_2\eta)}_{=:R_3^\alpha}.\end{aligned}$$

Via the integration by parts, this yields

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\partial_t\eta\|^2_{L^2(\Gamma)} = \frac{1}{2}\int_\Gamma(\partial_1 u_1 + \partial_2 u_2)|\partial^\alpha\partial_t\eta|^2 + \int_\Gamma(\partial^\alpha\partial_t u_3 - R_2^\alpha - R_3^\alpha)\partial^\alpha\partial_t\eta.$$

Using the trace theorem again, we have

$$\begin{aligned}\int_\Gamma(\partial_t\partial_1 u_1 + \partial_t\partial_2 u_2)\partial^\alpha\eta\partial^\alpha\partial_t\eta &\lesssim \|\partial_t u\|_{H^1(\Gamma)}\|\partial^\alpha\eta\|_{H^2(\Gamma)}\|\partial_t\partial^\alpha\eta\|_{H^2(\Gamma)} \\ &\lesssim \|\partial_t u\|_{H^2(\Omega)}\|\eta\|_{H^2(\Gamma)}\|\partial_t\eta\|_{H^2(\Gamma)},\end{aligned} \tag{3.118}$$

and

$$\|\partial^\alpha\partial_t u_3\|_{L^2(\Gamma)} \lesssim \|\partial_t u\|_{H^{|\alpha|+1}(\Omega)} \tag{3.119}$$

We follow (3.117) to get that

$$\|R_2^\alpha\|_{L^2(\Gamma)} \lesssim \sum_{0\neq\beta\leqslant\alpha}\|\partial^\beta u\|_{L^2(\Gamma)}\|\partial^{\alpha-\beta}\partial_t\eta\|_{H^1(\Gamma)} \lesssim \|u\|_{H^3(\Omega)}\|\partial_t\eta\|_{H^2(\Gamma)} \tag{3.120}$$

and that

$$\|R_3^\alpha\|_{L^2(\Gamma)} \lesssim \sum_{0\leqslant\beta\leqslant\alpha}\|\partial^\beta\partial_t u\|_{L^2(\Gamma)}\|\partial^{\alpha-\beta}\eta\|_{H^1(\Gamma)} \lesssim \|\nabla\partial_t u\|_{H^2(\Omega)}\|\eta\|_{H^3(\Gamma)}. \tag{3.121}$$

Combining (3.119), (3.120) and (3.121), we deduce that

$$\frac{d}{dt}\|\partial_t\eta\|^2_{H^2(\Gamma)} \lesssim \|\nabla\partial_t u\|_{H^2(\Omega)}\|\partial_t\eta\|_{H^2(\Gamma)} + \mathcal{E}_f^3.$$

Using Cauchy-Schwarz's inequality and then integrating from 0 to t, we obtain (3.112).

We have (3.113) by following the same strategy as for proving (3.112). The proof of Proposition 3.10 is complete. □

Proposition 3.11. *There holds*

$$\|\eta(t)\|^2_{H^{9/2}(\Gamma)} \leqslant C_4\Big(\mathcal{E}_f^2(0) + \int_0^t (\varepsilon\|\eta(s)\|^2_{H^{9/2}(\Gamma)} + \varepsilon^{-1}\|u(s)\|^2_{H^5(\Omega)})ds\Big) + C_4\int_0^t \mathcal{E}_f^3(s)ds. \tag{3.122}$$

Proof. To prove (3.122), we borrow the idea from [71, Lemma 3.9]. Let $\mathcal{J} = \sqrt{1-\partial_1^2-\partial_2^2}$. We apply $\mathcal{J}^{9/2}$ to $(4.2)_4$ and then multiply the resulting equation by $\mathcal{J}^{9/2}\eta$. Hence, we find that

$$\begin{aligned}\frac{d}{dt}\|\eta\|^2_{H^{9/2}(\Gamma)} &= -\frac{1}{2}\int_\Gamma (u_1\partial_1|\mathcal{J}^{9/2}\eta|^2 + u_2\partial_2|\mathcal{J}^{9/2}\eta|^2)\\ &\quad + \int_\Gamma (\mathcal{J}^{9/2}u_3 - [\mathcal{J}^{9/2},u_1]\partial_1\eta - [\mathcal{J}^{9/2},u_2]\partial_2\eta)\mathcal{J}^{9/2}\eta\\ &= \frac{1}{2}\int_\Gamma (\partial_1u_1+\partial_2u_2)|\mathcal{J}^{9/2}\eta|^2\\ &\quad + \int_\Gamma (\mathcal{J}^{9/2}u_3 - [\mathcal{J}^{9/2},u_1]\partial_1\eta - [\mathcal{J}^{9/2},u_2]\partial_2\eta)\mathcal{J}^{9/2}\eta.\end{aligned}$$

Thanks to (3.6), we have the following estimates,

$$\int_\Gamma \partial_ju_j|\mathcal{J}^{9/2}\eta|^2 \lesssim \|\partial_ju_j\|_{L^\infty(\Gamma)}\|\mathcal{J}^{9/2}\eta\|^2_{L^2(\Gamma)} \lesssim \|u\|_{H^3(\Gamma)}\|\eta\|^2_{H^{9/2}(\Gamma)}, \tag{3.123}$$

$$\int_\Gamma \mathcal{J}^{9/2}u_3\mathcal{J}^{9/2}\eta \lesssim \|\mathcal{J}^{9/2}u_3\|_{L^2(\Gamma)}\|\mathcal{J}^{9/2}\eta\|_{L^2(\Gamma)} \lesssim \|\mathcal{J}^4u\|_{H^1(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)}, \tag{3.124}$$

and

$$\begin{aligned}\int_\Gamma [\mathcal{J}^{9/2},u_j]\partial_j\eta\mathcal{J}^{9/2}\eta &\lesssim \|\partial_ju_j\|_{L^\infty(\Gamma)}\|\mathcal{J}^{7/2}\eta\|_{L^2(\Gamma)}\|\mathcal{J}^{9/2}\eta\|_{L^2(\Gamma)}\\ &\quad + \|\mathcal{J}^{9/2}u\|_{L^2(\Gamma)}\|\partial_j\eta\|_{L^\infty(\Gamma)}\|\mathcal{J}^{9/2}\eta\|_{L^2(\Gamma)}\\ &\lesssim \|u\|_{H^3(\Gamma)}\|\eta\|^2_{H^{9/2}(\Gamma)} + \|\mathcal{J}^4u\|_{H^1(\Omega)}\|\eta\|_{H^3(\Gamma)}\|\eta\|_{H^{9/2}(\Gamma)}.\end{aligned} \tag{3.125}$$

In view of (3.123), (3.124) and (3.125), we get

$$\begin{aligned}\frac{d}{dt}\|\eta\|^2_{H^{9/2}(\Gamma)} &\lesssim \|u\|_{H^3(\Gamma)}\|\eta\|^2_{H^{9/2}(\Gamma)} + \|\mathcal{J}^4u\|_{H^1(\Omega)}(1+\|\eta\|_{H^3(\Gamma)})\|\eta\|_{H^{9/2}(\Gamma)}\\ &\lesssim \|u\|_{H^5(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)} + \mathcal{E}_f^3.\end{aligned}$$

Using Cauchy-Schwarz's inequality and then integrating from 0 to t, we obtain (3.122). □

We provide some additional estimates on η, which will be used later.

Lemma 3.9. *We have*

$$\|\partial_t\eta\|_{H^{7/2}(\Gamma)} \lesssim \mathcal{E}_f + \mathcal{E}_f^2, \tag{3.126}$$

$$\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)} \lesssim \mathcal{E}_f + \mathcal{E}_f^2, \tag{3.127}$$

and

$$\|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2u\|_{H^1(\Omega)}(1+\mathcal{E}_f) + \mathcal{E}_f^2. \tag{3.128}$$

Proof. By $(4.2)_4$, we have that

$$\|\partial_t\eta\|_{H^{7/2}(\Gamma)} \lesssim \|u_3\|_{H^{7/2}(\Gamma)} + \|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)} \lesssim \|u_3\|_{H^4(\Omega)} + \|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}. \tag{3.129}$$

We use (3.1) and the trace theorem to estimate $\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}$ (see $\mathcal{Q}^4$ in (0.100)) as

$$\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)} \lesssim \|u\|_{H^{7/2}(\Gamma)}\|\partial_h\eta\|_{H^{7/2}(\Gamma)} \lesssim \|u\|_{H^4(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)},$$

Substituting the resulting inequality into (3.129), we have (3.126).

Using (3.1) again, we have

$$\begin{aligned}\|\partial_t\mathcal{Q}^4\|_{H^{3/2}(\Gamma)} &\lesssim \|\partial_t\partial_h\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^{7/2}(\Gamma)} + \|\partial_t u\|_{H^{3/2}(\Gamma)}\|\partial_h\eta\|_{H^{7/2}(\Gamma)}\\ &\lesssim \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|u\|_{H^4(\Omega)} + \|\eta\|_{H^{9/2}(\Gamma)}\|\partial_t u\|_{H^2(\Omega)}.\end{aligned}$$

Together with (3.126), that implies

$$\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)} \lesssim \|\partial_t u_3\|_{H^2(\Omega)} + \|\partial_t\mathcal{Q}^4\|_{H^{3/2}(\Gamma)} \lesssim \mathcal{E}_f + \mathcal{E}_f^2.$$

One thus has (3.127).

We continue using $(4.2)_4$ to have that

$$\|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2 u_3\|_{H^{1/2}(\Gamma)} + \|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2 u_3\|_{H^1(\Omega)} + \|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)}. \tag{3.130}$$

As a consequence of the product estimate (3.1) and Sobolev embedding, we obtain

$$\begin{aligned}\|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)} &\lesssim \|\partial_t^2 u\|_{H^{1/2}(\Gamma)}\|(\partial_1\eta,\partial_2\eta)\|_{H^{5/2}(\Gamma)} + \|\partial_t u\|_{H^{1/2}(\Gamma)}\|\partial_t(\partial_1\eta,\partial_2\eta)\|_{H^{5/2}(\Gamma)}\\ &\quad + \|u\|_{H^{5/2}(\Gamma)}\|\partial_t^2(\partial_1\eta,\partial_2\eta)\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|\partial_t^2 u\|_{H^1(\Omega)}\|\eta\|_{H^{7/2}(\Gamma)} + \|\partial_t u\|_{H^1(\Omega)}\|\partial_t\eta\|_{H^{7/2}(\Gamma)} + \|u\|_{H^3(\Omega)}\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)}.\end{aligned}$$

We continue using (3.126) and (3.127) to observe

$$\|\partial_t^2\mathcal{Q}^4\|_{H^{1/2}(\Gamma)} \lesssim \|\partial_t^2 u\|_{H^1(\Omega)}\mathcal{E}_f + \mathcal{E}_f^2. \tag{3.131}$$

The inequality (3.128) follows from (3.130) and (3.131). Lemma 3.9 is proven. □

3.3.2 Temporal estimates for the perturbation velocity

If we use the nonlinear equations in the perturbed form (4.2), there will be no control of the highest temporal derivative of q appearing in the nonlinear term $\mathcal{Q}^2$. Instead, we switch our original nonlinear equations (0.95) to a new formulation using a geometric transformation of the domain. The equations are

$$\begin{cases}\partial_t\zeta + \mathrm{div}_{\mathcal{A}}(\rho_0 u) = F^1 & \text{in } \Omega,\\ (\rho_0 + \rho_0'\theta + \zeta)\partial_t u + \nabla_{\mathcal{A}} q - \mu\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}} u + g\zeta e_3 = F^2 & \text{in } \Omega,\\ \mathrm{div}_{\mathcal{A}} u = 0 & \text{in } \Omega,\\ \partial_t\eta = u\cdot\mathcal{N} & \text{on } \Gamma,\\ (qId - \mu\mathbb{S}_{\mathcal{A}} u)\mathcal{N} = g\rho_+\eta\mathcal{N}, & \text{on } \Gamma.\end{cases} \tag{3.132}$$

Here,

$$\begin{aligned} F^1 &= K\partial_t\theta(\rho_0''\theta + \partial_3\zeta) - \mathrm{div}_{\mathcal{A}}((\rho_0'\theta + \zeta)u), \\ F^2 &= -(\rho_0 + \rho_0'\theta + \zeta)(-K\partial_t\theta\partial_3 u + u\cdot\nabla_{\mathcal{A}}u) - g\rho_0'(AK\theta, BK\theta, (1-K)\theta)^T. \end{aligned} \tag{3.133}$$

Applying the temporal differential operator ∂_t^l ($l \geqslant 1$) to (3.132), the resulting equations are

$$\begin{cases} \partial_t(\partial_t^l\zeta) + \mathrm{div}_{\mathcal{A}}(\rho_0\partial_t^l u) = F^{1,l} & \text{in } \Omega, \\ (\rho_0 + \rho_0'\theta + \zeta)\partial_t(\partial_t^l u) + \nabla_{\mathcal{A}}\partial_t^l q - \mu \mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^l u + g\partial_t^l\zeta e_3 = F^{2,l} & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}}\partial_t^l u = F^{3,l} & \text{in } \Omega, \\ \partial_t(\partial_t^l\eta) = \partial_t^l u\cdot\mathcal{N} + F^{4,l} & \text{on } \Gamma, \\ (\partial_t^l q Id - \mu\mathbb{S}_{\mathcal{A}}\partial_t^l u)\mathcal{N} = g\rho_+\partial_t^l\eta\mathcal{N} + F^{5,l} & \text{on } \Gamma. \end{cases} \tag{3.134}$$

The terms $F^{j,l}$ with $l \geqslant 1$ and $1 \leqslant j \leqslant 5$ are given as

$$F^{1,l} = \partial_t^l F^1 - \sum_{0<j\leqslant l} C_l^j \partial_t^j\mathcal{A}_{jk}\partial_k(\rho_0\partial_t^{l-j}u_j), \tag{3.135}$$

$$\begin{aligned} F_i^{2,l} = \partial_t^l F^2 &+ \sum_{0<j\leqslant l} C_l^j \mu(\mathcal{A}_{jk}\partial_k(\partial_t^j\mathcal{A}_{jm}\partial_t^{l-j}\partial_m u_i) + \partial_t^j\mathcal{A}_{jk}\partial_t^{l-j}\partial_k(\mathcal{A}_{jm}\partial_m u_i)) \\ &- \sum_{0<j\leqslant l} C_l^j(\rho_0\partial_t^j\mathcal{A}_{ik}\partial_k\partial_t^{l-j}\zeta + \partial_t^j(\zeta + \rho_0'\theta)\partial_t(\partial_t^{l-j}u_i)), \end{aligned} \tag{3.136}$$

$$\begin{aligned} F^{3,l} &= -\sum_{0<j\leqslant l} C_l^j\partial_t^j\mathcal{A}_{ik}\partial_k(\partial_t^{l-j}u_i), \\ F^{4,l} &= \sum_{0<j\leqslant l} C_l^j\partial_t^j\mathcal{N}\cdot\partial_t^{l-j}u, \end{aligned} \tag{3.137}$$

$$\begin{aligned} F_i^{5,l} = \mu &\sum_{0<j\leqslant l} C_l^j(\partial_t^j(\mathcal{A}_{ik}\mathcal{N}_m)\partial_k\partial_t^{l-j}u_m + \partial_t^j(\mathcal{A}_{mk}\mathcal{N}_m)\partial_k\partial_t^{l-j}u_i) \\ &+ \sum_{0<j\leqslant l} C_l^j\partial_t^j\mathcal{N}_i\partial_t^{l-j}(g\rho_+\eta - q). \end{aligned} \tag{3.138}$$

We use the convention that $F^{1,0} = F^1, F^{2,0} = F^2$ and $F^{j,0} = 0$ for $3 \leqslant j \leqslant 5$. We now derive the following proposition.

Proposition 3.12. *For $l = 0$ and 1, we have*

$$\begin{aligned} &\|\partial_t^l u(t)\|^2_{L^2(\Omega)} + \|\partial_t^l\eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla\partial_t^l u(s)\|^2_{L^2(\Omega)}ds \\ &\qquad \leqslant C_5\Big(\mathcal{E}_f^2(0) + \int_0^t \|(u,\zeta)(s)\|^2_{L^2(\Omega)}ds + \int_0^t \mathcal{E}_f^3(s)ds\Big). \end{aligned} \tag{3.139}$$

We also have

$$\begin{aligned}&\|\partial_t^2 u(t)\|^2_{L^2(\Omega)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla\partial_t^2 u(s)\|^2_{L^2(\Omega)}ds\\ &\quad\leqslant C_6\Big(\mathcal{E}_f^2(0) + \int_0^t \|(u,\zeta)(s)\|^2_{L^2(\Omega)}ds + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds + \mathcal{E}_f^3(t)\Big).\end{aligned} \tag{3.140}$$

The proof of Proposition 3.12 relies on Lemmas 3.10, 3.11 and 3.12 below.

Lemma 3.10. *Let J be defined as in (0.93). For any scalar function $\vartheta\in\mathbf{R}$ and any vector function $\varrho\in\mathbf{R}^3$, there holds*

$$\int_\Omega(\nabla_{\mathcal{A}}\vartheta)\cdot J\varrho = \int_\Gamma \vartheta(\mathcal{N}\cdot\varrho) - \int_\Omega J\vartheta\, div_{\mathcal{A}}\varrho. \tag{3.141}$$

Proof. We have from the integration by parts that

$$\int_\Omega(\nabla_{\mathcal{A}}\vartheta)\cdot J\varrho = \int_\Omega J\mathcal{A}_{ij}\partial_j\vartheta\varrho_i = \int_\Gamma \vartheta(J\mathcal{A}_{i3}\varrho_i) - \int_\Omega \vartheta\partial_j(J\mathcal{A}_{ij}\varrho_i) \tag{3.142}$$

Note that $J\mathcal{A}_{i3}\varrho_i = \mathcal{N}\cdot\varrho$, hence

$$\int_\Gamma \vartheta(J\mathcal{A}_{i3}\varrho_i) = \int_\Gamma\vartheta(\mathcal{N}\cdot\varrho) \tag{3.143}$$

Note also that

$$\partial_j(J\mathcal{A}_{ij}) = 0 \quad \text{for all } 1\leqslant i\leqslant 3,$$

this implies

$$\int_\Omega \vartheta\partial_j(J\mathcal{A}_{ij}\varrho_i) = \int_\Omega \vartheta J\mathcal{A}_{ij}\partial_j\varrho_i = \int_\Omega J\vartheta\,\mathrm{div}_{\mathcal{A}}\varrho. \tag{3.144}$$

Substituting (3.143), (3.144) into (3.142), we obtain (3.141), i.e. Lemma 3.10. ☐

Lemma 3.11. *There holds for all $l\geqslant 0$,*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2 + \int_\Gamma g\rho_+|\partial_t^l\eta|^2\Big) + \frac{1}{2}\mu\int_\Omega J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2\\ &= \frac{1}{2}\int_\Omega \partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2 + \int_\Omega J(F^{2,l}\cdot\partial_t^l u - g\partial_t^l\zeta\partial_t^l u_3 + F^{3,l}\partial_t^l q)\\ &\quad - \int_\Gamma(g\rho_+\partial_t^l\eta F^{4,l} + F^{5,l}\cdot\partial_t^l u).\end{aligned} \tag{3.145}$$

If $l\geqslant 1$, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2 + \int_\Gamma g\rho_+|\partial_t^l\eta|^2 - \int_\Omega g\rho_0'|\partial_t^{l-1}u_3|^2\Big) + \frac{1}{2}\mu\int_\Omega J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2\\ &= \frac{1}{2}\int_\Omega \partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2 + \int_\Omega J(F^{2,l}\cdot\partial_t^l u + F^{3,l}\partial_t^l q)\\ &\quad - \int_\Gamma(g\rho_+\partial_t^l\eta F^{4,l} + F^{5,l}\cdot\partial_t^l u) + \int_\Omega g\rho_0 JF^{3,l-1}\partial_t^l u_3\\ &\quad - \int_\Omega g\rho_0'(A\partial_t^{l-1}\partial_3 u_1 + B\partial_t^{l-1}\partial_3 u_2)\partial_t^l u_3 - \int_\Omega gJF^{1,l-1}\partial_t^l u_3.\end{aligned} \tag{3.146}$$

Proof. We multiply by $J\partial_t^l u$ on both sides of (3.134)$_2$ to have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2\Big)\\&=\frac{1}{2}\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2-\int_\Omega\nabla_{\mathcal{A}}\partial_t^l q\cdot J\partial_t^l u+\int_\Omega\mu(\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^l u)\cdot J\partial_t^l u\\&\quad-\int_\Omega gJ\partial_t^l\zeta\partial_t^l u_3+\int_\Omega JF^{2,l}\cdot\partial_t^l u.\end{aligned}\tag{3.147}$$

Thanks to Lemma 3.10, one deduces

$$\begin{aligned}&-\int_\Omega\nabla_{\mathcal{A}}\partial_t^l q\cdot J\partial_t^l u+\int_\Omega\mu(\mathrm{div}_{\mathcal{A}}\mathbb{S}_{\mathcal{A}}\partial_t^l u)\cdot J\partial_t^l u\\&=\int_\Gamma(\mu\mathbb{S}_{\mathcal{A}}\partial_t^l u-\partial_t^l qId)\mathcal{N}\cdot\partial_t^l u+\int_\Omega J(\mathrm{div}_{\mathcal{A}}\partial_t^l u)\partial_t^l q-\frac{1}{2}\int_\Omega\mu J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2\end{aligned}\tag{3.148}$$

Substituting (3.148) into (3.147), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t^l u|^2\Big)+\frac{1}{2}\int_\Omega\mu J|\mathbb{S}_{\mathcal{A}}\partial_t^l u|^2\\&=\frac{1}{2}\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t^l u|^2+\int_\Omega J(\mathrm{div}_{\mathcal{A}}\partial_t^l u)\partial_t^l q\\&\quad+\int_\Gamma(\mu\mathbb{S}_{\mathcal{A}}\partial_t^l u-\partial_t^l qId)\mathcal{N}\cdot\partial_t^l u-\int_\Omega gJ\partial_t^l\zeta\partial_t^l u_3+\int_\Omega JF^{2,l}\cdot\partial_t^l u.\end{aligned}\tag{3.149}$$

Using (3.134)$_{3,4,5}$, we obtain (3.145) from (3.149).

To prove (3.146), we use (3.134)$_1$ at order $l-1$ to get that

$$\begin{aligned}-\int_\Omega gJ\partial_t^l\zeta\partial_t^l u_3&=\int_\Omega gJ\mathrm{div}_{\mathcal{A}}(\rho_0\partial_t^{l-1}u)\partial_t^l u_3-\int_\Omega gJF^{1,l-1}\partial_t^l u_3\\&=\int_\Omega g\rho_0'\partial_t^{l-1}u_3\partial_t^l u_3+\int_\Omega g\rho_0 JF^{3,l-1}\partial_t^l u_3\\&\quad-\int_\Omega g\rho_0'(A\partial_t^{l-1}\partial_3 u_1+B\partial_t^{l-1}\partial_3 u_2)\partial_t^l u_3-\int_\Omega gJF^{1,l-1}\partial_t^l u_3.\end{aligned}\tag{3.150}$$

Combining (3.150) and (3.145), we obtain (3.146). Lemma 3.11 is proven. □

Lemma 3.12. *The following inequalities hold*

$$\sum_{l=0}^{1}\Big(\|(F^{1,l},F^{2,l},F^{3,l})\|_{L^2(\Omega)}+\|(F^{4,l},F^{5,l})\|_{L^2(\Gamma)}\Big)\lesssim\mathcal{E}_f^2,\tag{3.151}$$

$$\|(F^{1,2},F^{2,2})\|_{L^2(\Omega)}+\|(F^{4,2},F^{5,2})\|_{L^2(\Gamma)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}),\tag{3.152}$$

and

$$\|(F^{3,2},JF^{3,2})\|_{L^2(\Omega)}\lesssim\mathcal{E}_f^2,\quad\|\partial_t(JF^{3,2})\|_{L^2(\Omega)}\lesssim\mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}).\tag{3.153}$$

Proof. For $\Sigma = \Omega$ or Γ, all quadratic terms $\|X_1X_2\|_{L^2(\Sigma)}$ or cubic ones $\|X_1X_2X_3\|_{L^2(\Sigma)}$ appearing in $F^{j,l}$ with $1 \leqslant j \leqslant 5$ will be bounded by using Sobolev embedding, Lemma 0.2 and other inequalities in Section 3.1. Precisely, we have

$$\|X_1X_2\|_{L^2(\Sigma)} \lesssim \|X_1\|_{L^\infty(\Sigma)}\|X_2\|_{L^2(\Sigma)} \lesssim \|X_1\|_{H^2(\Sigma)}\|X_2\|_{L^2(\Sigma)}$$

and

$$\begin{aligned}\|X_1X_2X_3\|_{L^2(\Sigma)} &\lesssim \|X_1\|_{L^\infty(\Sigma)}\|X_2\|_{L^\infty(\Sigma)}\|X_3\|_{L^2(\Sigma)} \\ &\lesssim \|X_1\|_{H^2(\Sigma)}\|X_2\|_{H^2(\Sigma)}\|X_3\|_{L^2(\Sigma)}.\end{aligned}$$

We only show the estimates of the term $F^{2,l}(0 \leqslant l \leqslant 2)$ (see (3.133) and (3.136)), the estimates of others terms are proven in the same way.

For F^2 (see (3.133)), we have

$$(\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u = (\rho_0 + \rho_0'\theta + \zeta)(K-1)\partial_t\theta\partial_3 u + (\rho_0 + \rho_0'\theta + \zeta)\partial_t\theta\partial_3 u.$$

Thanks to Lemma 0.2 and (3.10), we obtain

$$\begin{aligned}&\|(\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u\|_{L^2(\Omega)} \\ &\quad\lesssim (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})(1 + \|K-1\|_{H^2(\Omega)})\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)} \\ &\quad\lesssim (1 + \|\zeta\|_{H^2(\Omega)} + \|\eta\|_{H^{3/2}(\Gamma)})(1 + \|\eta\|_{H^{5/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^1(\Omega)} \\ &\quad\lesssim \mathcal{E}_f^2.\end{aligned} \tag{3.154}$$

Note that

$$u \cdot \nabla_{\mathcal{A}} u = u \cdot \nabla_{\mathcal{A}-\mathrm{Id}} u + u \cdot \nabla u,$$

we use Lemma 0.2 and (3.12) to get that

$$\begin{aligned}&\|(\rho_0 + \rho_0'\theta + \zeta)u \cdot \nabla_{\mathcal{A}} u\|_{L^2(\Omega)} \\ &\quad\lesssim \|(\rho_0 + \rho_0'\theta + \zeta)u \cdot \nabla_{\mathcal{A}-\mathrm{Id}} u\|_{L^2(\Omega)} + \|(\rho_0 + \rho_0'\theta + \zeta)u \cdot \nabla u\|_{L^2(\Omega)} \\ &\quad\lesssim (1 + \|\zeta\|_{H^2(\Omega)} + \|\eta\|_{H^{3/2}(\Gamma)})(1 + \|\mathcal{A} - \mathrm{Id}\|_{H^2(\Omega)})\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)} \\ &\quad\lesssim (1 + \|\zeta\|_{H^2(\Omega)} + \|\eta\|_{H^{3/2}(\Gamma)})(1 + \|\eta\|_{H^{5/2}(\Gamma)})\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)} \\ &\quad\lesssim \mathcal{E}_f^2.\end{aligned} \tag{3.155}$$

Due to Lemma 0.2 again and (3.10), (3.11), we have

$$\begin{aligned}\|(AK\theta, BK\theta, (1-K)\theta)\|_{L^2(\Omega)} &\lesssim \|(AK, BK, K-1)\|_{H^2(\Omega)}\|\theta\|_{L^2(\Omega)} \\ &\lesssim \|\eta\|_{H^{5/2}(\Gamma)}\|\eta\|_{L^2(\Gamma)} \\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{3.156}$$

It follows from (3.154), (3.155) and (3.156) that $\|F^2\|_{L^2(\Omega)} \lesssim \mathcal{E}_f^2$.

For $F^{2,1}$ (see (3.136)), we obtain

$$\begin{aligned}\|F^{2,1}\|_{L^2(\Omega)} \lesssim \|\partial_t F^2\|_{L^2(\Omega)} &+ (1+\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)})\|\partial_t\mathcal{A}\|_{H^1(\Omega)}\|\nabla u\|_{H^2(\Omega)} \\ &+ (1+\|\mathcal{A}-\mathrm{Id}\|_{H^3(\Omega)})\|\nabla^2 u\|_{H^2(\Omega)}\|\partial_t\mathcal{A}\|_{H^1(\Omega)} + \|\zeta\|_{H^3(\Omega)}\|\partial_t\mathcal{A}\|_{L^2(\Omega)} \\ &+ (\|\partial_t\zeta\|_{H^2(\Omega)} + \|\partial_t\theta\|_{H^2(\Omega)})\|\partial_t u\|_{L^2(\Omega)}.\end{aligned} \tag{3.157}$$

According to Lemma 0.2 and (3.12), it follows from (3.157) that

$$\begin{aligned}\|F^{2,1}\|_{L^2(\Omega)} &\lesssim \|\partial_t F^2\|_{L^2(\Omega)} + (1+\|\eta\|_{H^{7/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^4(\Omega)} \\ &\quad + \|\zeta\|_{H^3(\Omega)}\|\partial_t\eta\|_{H^{1/2}(\Gamma)} + (\|\partial_t\zeta\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)})\|\partial_t u\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t F^2\|_{L^2(\Omega)} + \mathcal{E}_f^2.\end{aligned} \tag{3.158}$$

We calculate each term of $\partial_t F^2$,

$$\begin{aligned}\partial_t((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u) &= (\rho_0+\rho_0'\theta+\zeta)(\partial_t K\partial_t\theta\partial_3 u + K\partial_t^2\theta\partial_3 u + K\partial_t\theta\partial_t\partial_3 u) \\ &\quad + (\rho_0'\partial_t\theta + \partial_t\zeta)K\partial_t\theta\partial_3 u,\end{aligned}$$

which will be bounded as follows

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)} \\ &\lesssim (1+\|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t K\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)} \\ &\quad + (1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|K-1\|_{H^2(\Omega)}+1) \\ &\qquad\quad \times (\|\partial_3 u\|_{H^2(\Omega)}\|\partial_t^2\theta\|_{L^2(\Omega)} + \|\partial_t\theta\|_{H^2(\Omega)}\|(u,\partial_t u)\|_{H^1(\Omega)}) \\ &\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|K-1\|_{H^2(\Omega)}+1)\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}.\end{aligned}$$

Using Lemma 0.2 and (3.10), we deduce that

$$\begin{aligned}&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)} \\ &\lesssim (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})\|\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^1(\Omega)} \\ &\quad + (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(1+\|\eta\|_{H^{5/2}(\Gamma)}) \\ &\qquad\quad \times (\|u\|_{H^3(\Omega)}\|\partial_t^2\eta\|_{L^2(\Gamma)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|(u,\partial_t u)\|_{H^1(\Omega)}) \\ &\quad + (\|\partial_t\zeta\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)})(1+\|\eta\|_{H^{5/2}(\Gamma)})\|u\|_{H^1(\Omega)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)} \\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{3.159}$$

Next, we compute

$$\begin{aligned}&\partial_t((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u) \\ &= (\rho_0+\rho_0'\theta+\zeta)(\partial_t u_i\mathcal{A}_{ij}\partial_j u_k + u_i\partial_t\mathcal{A}_{ij}\partial_j u_k + u_i\mathcal{A}_{ij}\partial_t\partial_j u_k) \\ &\quad + (\rho_0'\partial_t\theta + \partial_t\zeta)u_i\mathcal{A}_{ij}\partial_j u_k.\end{aligned}$$

Hence, it follows from Lemma 0.2 and (3.12) that

$$\begin{aligned}
&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\\
&\lesssim (1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|u\|_{H^3(\Omega)}\|\partial_t u\|_{H^1(\Omega)}\\
&\quad+(1+\|(\theta,\zeta)\|_{H^2(\Omega)})\|u\|_{H^3(\Omega)}^2\|\partial_t\mathcal{A}\|_{L^2(\Omega)}\\
&\quad+\|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\
&\lesssim (1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|u\|_{H^3(\Omega)}\|\partial_t u\|_{H^1(\Omega)}\\
&\quad+(1+\|\zeta\|_{H^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)})\|u\|_{H^3(\Omega)}^2\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\\
&\quad+(\|\partial_t\zeta\|_{H^2(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|u\|_{H^2(\Omega)}\|u\|_{H^1(\Omega)}\\
&\lesssim \mathcal{E}_f^2.
\end{aligned}\tag{3.160}$$

Using again Lemma 0.2 and (3.10), (3.11), one has

$$\begin{aligned}
&\|\partial_t(AK\theta,BK\theta,(1-K)\theta)\|_{L^2(\Omega)}\\
&\lesssim \|(AK,BK,K-1)\|_{H^2(\Omega)}\|\partial_t\theta\|_{L^2(\Omega)}+\|\partial_t(AK,BK,K-1)\|_{L^2(\Omega)}\|\theta\|_{H^2(\Omega)}\\
&\lesssim \|\eta\|_{H^{5/2}(\Gamma)}\|\partial_t\eta\|_{L^2(\Gamma)}+\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\eta\|_{H^{3/2}(\Gamma)}\\
&\lesssim \mathcal{E}_f^2.
\end{aligned}\tag{3.161}$$

We deduce $\|\partial_t F^2\|_{L^2(\Omega)}\lesssim \mathcal{E}_f^2$ from (3.158), (3.159), (3.160) and (3.161). So that, $\|F^{2,1}\|_{L^2(\Omega)}\lesssim\mathcal{E}_f^2$.

For $F^{2,2}$ (see again (3.133)), we use the product estimate (3.1) and Sobolev embedding to obtain that

$$\begin{aligned}
\|F^{2,2}\|_{L^2(\Omega)}\lesssim{}& \|\partial_t^2F^2\|_{L^2(\Omega)}+\|\mathcal{A}\|_{H^2(\Omega)}(\|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\partial_t^2\mathcal{A}\|_{H^1(\Omega)}\|\nabla u\|_{H^3(\Omega)})\\
&+\|\partial_t\mathcal{A}\|_{H^2(\Omega)}(\|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla u\|_{H^2(\Omega)}+\|\mathcal{A}\|_{H^3(\Omega)}\|\nabla\partial_t u\|_{H^1(\Omega)})\\
&+\|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\mathcal{A}\|_{H^3(\Omega)}\|\nabla u\|_{H^3(\Omega)}+\|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\partial_t\zeta\|_{H^1(\Omega)}\\
&+\|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\nabla\zeta\|_{H^2(\Omega)}+\|(\partial_t\zeta,\partial_t\theta)\|_{H^2(\Omega)}\|\partial_t^2u\|_{L^2(\Omega)}\\
&+\|(\partial_t^2\zeta,\partial_t^2\theta)\|_{L^2(\Omega)}\|\partial_t u\|_{H^2(\Omega)}.
\end{aligned}$$

We make use of Lemma 0.2 and (3.12) to further get that

$$\begin{aligned}
\|F^{2,2}\|_{L^2(\Omega)}\lesssim{}& \|\partial_t^2F^2\|_{L^2(\Omega)}+\|\eta\|_{H^{3/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\nabla\partial_t u\|_{H^2(\Omega)}\\
&+\|\eta\|_{H^{3/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}\|u\|_{H^4(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}^2\|u\|_{H^3(\Omega)}\\
&+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t u\|_{H^2(\Omega)}+\|\partial_t^2\eta\|_{L^2(\Gamma)}\|\eta\|_{H^{7/2}(\Gamma)}\|u\|_{H^4(\Omega)}\\
&+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t\zeta\|_{H^1(\Omega)}+\|\partial_t^2\eta\|_{L^2(\Gamma)}\|\zeta\|_{H^3(\Omega)}\\
&+(\|\partial_t\zeta\|_{H^2(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)})\|\partial_t^2u\|_{L^2(\Omega)}\\
&+(\|\partial_t^2\zeta\|_{L^2(\Omega)}+\|\partial_t^2\eta\|_{L^2(\Gamma)})\|\partial_t u\|_{H^2(\Omega)}\\
\lesssim{}& \|\partial_t^2F^2\|_{L^2(\Omega)}+\mathcal{E}_f(\mathcal{E}_f+\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\nabla\partial_t u\|_{H^2(\Omega)}).
\end{aligned}$$

Together with (3.127), we deduce from the resulting inequality that

$$\|F^{2,2}\|_{L^2(\Omega)}\lesssim\|\partial_t^2F^2\|_{L^2(\Omega)}+\mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}).$$

Hence, in order to show that

$$\|F^{2,2}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}),$$

we will prove that

$$\|\partial_t^2 F^2\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}).$$

We now estimate each term of $\partial_t^2 F^2$. Due to Sobolev embedding, one has that

$$\begin{aligned}
&\|\partial_t^2((\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)} \\
&\lesssim (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})(\|K-1\|_{H^2(\Omega)} + 1) \\
&\qquad \times \begin{pmatrix} \|\partial_3 u\|_{H^2(\Omega)}\|\partial_t^3\theta\|_{L^2(\Omega)} + \|\partial_t\partial_3 u\|_{L^2(\Omega)}\|\partial_t^2\theta\|_{H^2(\Omega)} \\ + \|\partial_t^2\partial_3 u\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)} \end{pmatrix} \\
&\quad + (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t K\|_{H^2(\Omega)} \\
&\qquad \times \left(\|\partial_t\theta\|_{H^2(\Omega)}\|\partial_t u\|_{H^1(\Omega)} + \|\partial_t^2\theta\|_{L^2(\Omega)}\|u\|_{H^3(\Omega)}\right) \\
&\quad + (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t^2 K\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)} \\
&\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|K-1\|_{H^2(\Omega)} + 1) \\
&\qquad \times \left(\|\partial_t^2\theta\|_{L^2(\Omega)}\|u\|_{H^3(\Omega)} + \|\partial_t\theta\|_{H^2(\Omega)}\|\partial_t u\|_{H^1(\Omega)}\right) \\
&\quad + \|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}\|\partial_t K\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)} \\
&\quad + \|(\partial_t^2\theta,\partial_t^2\zeta)\|_{L^2(\Omega)}(\|K-1\|_{H^2(\Omega)} + 1)\|\partial_t\theta\|_{H^2(\Omega)}\|u\|_{H^3(\Omega)}.
\end{aligned}$$

Thanks to Lemma 0.2 and (3.10), this yields

$$\begin{aligned}
&\|\partial_t^2((\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)} \\
&\lesssim (1 + \|\eta\|_{H^{3/2}(\Gamma)} + \|\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)} + 1) \\
&\qquad \times \begin{pmatrix} \|u\|_{H^3(\Omega)}\|\partial_t^3\eta\|_{L^2(\Gamma)} + \|\partial_t u\|_{H^1(\Omega)}\|\partial_t^2\eta\|_{H^{3/2}(\Gamma)} \\ + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)} \end{pmatrix} \\
&\quad + (1 + \|\eta\|_{H^{3/2}(\Gamma)} + \|\zeta\|_{H^2(\Omega)})\|\partial_t\eta\|_{H^{5/2}(\Gamma)} \\
&\qquad \times \left(\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t u\|_{H^1(\Omega)} + \|\partial_t^2\eta\|_{L^2(\Gamma)}\|u\|_{H^3(\Omega)}\right) \\
&\quad + (1 + \|\eta\|_{H^{3/2}(\Gamma)} + \|\zeta\|_{H^2(\Omega)})(\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{1/2}(\Gamma)}^2)\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)} \\
&\quad + (\|\partial_t\eta\|_{H^{3/2}(\Gamma)} + \|\partial_t\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)} + 1) \\
&\qquad \times \left(\|\partial_t^2\eta\|_{L^2(\Gamma)}\|u\|_{H^3(\Omega)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t u\|_{H^1(\Omega)}\right) \\
&\quad + (\|\partial_t\eta\|_{H^{3/2}(\Gamma)} + \|\partial_t\zeta\|_{H^2(\Omega)})\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)} \\
&\quad + (\|\partial_t^2\eta\|_{L^2(\Gamma)} + \|\partial_t^2\zeta\|_{L^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)} + 1)\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^3(\Omega)}.
\end{aligned} \tag{3.162}$$

Using (3.127) and (3.128), we thus have from (3.162) that

$$\begin{aligned}
&\|\partial_t^2((\rho_0 + \rho_0'\theta + \zeta)K\partial_t\theta\partial_3 u)\|_{L^2(\Omega)} \\
&\quad \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t^2 u\|_{L^2(\Omega)} + \|\partial_t^3\eta\|_{L^2(\Gamma)} + \|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}) \\
&\quad \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}).
\end{aligned} \tag{3.163}$$

In a same way, we have

$$\begin{aligned}
&\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\\
&\lesssim (1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)(\|\partial_t^2 u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}+\|\partial_t u\|_{H^2(\Omega)}^2)\\
&\quad+(1+\|(\theta,\zeta)\|_{H^2(\Omega)})(\|\partial_t\mathcal{A}\|_{L^2(\Omega)}\|\partial_t u\|_{H^3(\Omega)}+\|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|u\|_{H^2(\Omega)})\|u\|_{H^3(\Omega)}\\
&\quad+\|(\partial_t\theta,\partial_t\zeta)\|_{H^2(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|\partial_t u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}\\
&\quad+\|(\partial_t^2\theta,\partial_t^2\zeta)\|_{L^2(\Omega)}(\|\mathcal{A}-\mathrm{Id}\|_{H^2(\Omega)}+1)\|u\|_{H^3(\Omega)}^2.
\end{aligned}$$

Thanks to Lemma 0.2 and (3.12), we further get that

$$\begin{aligned}
&\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\\
&\lesssim (1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)(\|\partial_t^2 u\|_{H^1(\Omega)}\|u\|_{H^3(\Omega)}+\|\partial_t u\|_{H^2(\Omega)}^2)\\
&\quad+(1+\|\eta\|_{H^{3/2}(\Gamma)}+\|\zeta\|_{H^2(\Omega)})\\
&\qquad\times(\|\partial_t\eta\|_{H^{1/2}(\Gamma)}\|\partial_t u\|_{H^3(\Omega)}+(\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\|u\|_{H^2(\Omega)})\|u\|_{H^3(\Omega)}\\
&\quad+(\|\partial_t\eta\|_{H^{3/2}(\Gamma)}+\|\partial_t\zeta\|_{H^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|\partial_t u\|_{H^3(\Omega)}\|u\|_{H^3(\Omega)}\\
&\quad+(\|\partial_t^2\eta\|_{L^2(\Gamma)}+\|\partial_t^2\zeta\|_{L^2(\Omega)})(\|\eta\|_{H^{5/2}(\Gamma)}+1)\|u\|_{H^3(\Omega)}^2\\
&\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}).
\end{aligned}$$

Due to (3.127), we obtain

$$\|\partial_t^2((\rho_0+\rho_0'\theta+\zeta)u\cdot\nabla_{\mathcal{A}}u)\|_{L^2(\Omega)}\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u\|_{L^2(\Omega)}). \tag{3.164}$$

Furthermore, thanks to (3.127) again and Lemma 0.2, (3.10), (3.11), one has

$$\begin{aligned}
&\|\partial_t^2(AK\theta,BK\theta,(1-K)\theta)\|_{L^2(\Omega)}\\
&\quad\lesssim \|\partial_t^2(AK,BK,K-1)\|_{L^2(\Omega)}\|\theta\|_{H^2(\Omega)}+\|\partial_t(AK,BK,K-1)\|_{L^2(\Omega)}\|\partial_t\theta\|_{H^2(\Omega)}\\
&\qquad+\|(AK,BK,K-1)\|_{H^2(\Omega)}\|\partial_t^2\theta\|_{L^2(\Omega)}\\
&\quad\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\|\eta\|_{H^{3/2}(\Gamma)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\\
&\qquad+\|\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{L^2(\Gamma)}\\
&\quad\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\\
&\quad\lesssim \mathcal{E}_f^2.
\end{aligned} \tag{3.165}$$

Consequently, there holds

$$\|\partial_t^2 F^2\|_{L^2(\Omega)}\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u\|_{L^2(\Omega)})$$

thanks to (3.163), (3.164) and (3.165).

We are left to prove (3.153). From the formula of $F^{3,2}$ (see (3.137)), we use Sobolev embedding and (3.12) to get

$$\begin{aligned}
\|F^{3,2}\|_{L^2(\Omega)}&\lesssim \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\nabla u\|_{H^2(\Omega)}+\|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla\partial_t u\|_{L^2(\Omega)}\\
&\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}^2)\|u\|_{H^3(\Omega)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t u\|_{H^1(\Omega)}.
\end{aligned}$$

Owing to (3.126) and (3.127), we deduce that

$$\begin{aligned}\|F^{3,2}\|_{L^2(\Omega)} &\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)})\mathcal{E}_f + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\mathcal{E}_f \\ &\lesssim \mathcal{E}_f^2.\end{aligned} \tag{3.166}$$

Together with (3.8), this yields

$$\|JF^{3,2}\|_{L^2(\Omega)} \lesssim (1 + \|J-1\|_{L^\infty(\Omega)})\|F^{3,2}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f^2. \tag{3.167}$$

We continue using (3.8), Lemma 0.2 and (3.166) to get

$$\begin{aligned}\|\partial_t(JF^{3,2})\|_{L^2(\Omega)} &\lesssim \|\partial_t J\|_{L^\infty(\Omega)}\|F^{3,2}\|_{L^2(\Omega)} + (1 + \|J-1\|_{L^\infty(\Omega)})\|\partial_t F^{3,2}\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t\partial_3\theta\|_{H^2(\Omega)}\mathcal{E}_f^2 + \|\partial_t F^{3,2}\|_{L^2(\Omega)} \\ &\lesssim \|\partial_t\eta\|_{H^{5/2}(\Omega)}\mathcal{E}_f^2 + \|\partial_t F^{3,2}\|_{L^2(\Omega)}.\end{aligned} \tag{3.168}$$

By Sobolev embedding and (3.12), let us estimate that

$$\begin{aligned}\|\partial_t F^{3,2}\|_{L^2(\Omega)} &\lesssim \|\partial_t^2\mathcal{A}\|_{L^2(\Omega)}\|\nabla\partial_t u\|_{H^2(\Omega)} + \|\partial_t\mathcal{A}\|_{H^2(\Omega)}\|\nabla\partial_t^2 u\|_{L^2(\Omega)} \\ &\quad + \|\partial_t^3\mathcal{A}\|_{L^2(\Omega)}\|\nabla u\|_{H^2(\Omega)} \\ &\lesssim (\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^2_{H^{5/2}(\Gamma)})\|\nabla\partial_t u\|_{H^2(\Omega)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\nabla\partial_t^2 u\|_{L^2(\Omega)} \\ &\quad + (\|\partial_t^3\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|_{H^{5/2}(\Gamma)}\|\partial_t^2\eta\|_{H^{1/2}(\Gamma)} + \|\partial_t\eta\|^3_{H^{5/2}(\Gamma)})\|u\|_{H^3(\Omega)}.\end{aligned} \tag{3.169}$$

Combining the resulting inequality (3.167) with Lemma 3.9, we obtain

$$\|\partial_t F^{3,2}\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}). \tag{3.170}$$

Combining (3.126), (3.168) and (3.170) gives us that

$$\|\partial_t(JF^{3,2})\|_{L^2(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)}). \tag{3.171}$$

The inequality (3.153) follows from (3.166), (3.167) and (3.171). □

We are in position to prove Proposition 3.12.

Proof of Proposition 3.12. In view of (3.145) at order $l = 0$, we have

$$\begin{aligned}&\frac{1}{2}\Big(\int_\Omega(\rho_0 + \rho_0'\theta + \zeta)J|u(t)|^2 + \int_\Gamma g\rho_+|\eta(t)|^2\Big) + \frac{1}{2}\mu\int_0^t\int_\Omega J|\mathbb{S}_{\mathcal{A}}u(s)|^2ds \\ &= \frac{1}{2}\Big(\int_\Omega(\rho_0 + \rho_0'\theta + \zeta)J|u(t)|^2 + \int_\Gamma g\rho_+|\eta(t)|^2\Big)\Big|_{t=0} \\ &+ \frac{1}{2}\int_0^t\int_\Omega\partial_t((\rho_0 + \rho_0'\theta + \zeta)J)|u|^2(s)ds + \int_0^t\int_\Omega J(F^2\cdot u - g\zeta u_3)ds\end{aligned} \tag{3.172}$$

We first estimate the l.h.s of (3.139). Notice that

$$J\|\mathbb{S}_{\mathcal{A}}u\|^2_{L^2(\Omega)} = \|\mathbb{S}u\|^2_{L^2(\Omega)} + \int_\Omega(J-1)|\mathbb{S}u|^2 + \int_\Omega J(\mathbb{S}_{\mathcal{A}}u + \mathbb{S}u) : (\mathbb{S}_{\mathcal{A}}u - \mathbb{S}u).$$

Since

$$\mathbb{S}_{\mathcal{A}}u \pm \mathbb{S}u = (\mathcal{A}_{ik} \pm \delta_{ik})\partial_k u_j + (\mathcal{A}_{jk} \pm \delta_{jk})\partial_k u_j,$$

we use (3.9) to obtain

$$\begin{aligned}\int_\Omega J(\mathbb{S}_{\mathcal{A}}u + \mathbb{S}u) : (\mathbb{S}_{\mathcal{A}}u - \mathbb{S}u) &= 4\int_\Omega (A^2(\partial_1 u_2 + \partial_2 u_1)^2 + B^2(\partial_1 u_3 + \partial_3 u_1)^2) \\ &\lesssim \|(A,B)\|^2_{H^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} \\ &\lesssim \mathcal{E}_f^4.\end{aligned}$$

Note also that $\|J-1\|_{L^\infty(\Omega)} \lesssim 1$ (see (3.8)), we use Korn's inequality (3.5) to have

$$J\|\mathbb{S}_{\mathcal{A}}u\|^2_{L^2(\Omega)} \gtrsim \|\nabla u\|^2_{L^2(\Omega)} - \mathcal{E}_f^3. \tag{3.173}$$

Due to the assumption on δ_0 (0.127) and Sobolev embedding, we then have

$$\inf_\Omega(\rho_0 + \rho_0'\theta + \zeta) \geqslant \rho_- - C_{\text{emb}}\max(1, \max_{\mathbf{R}_-}\rho_0'(x_3))\|(\theta,\zeta)\|_{H^2(\Omega)} \geqslant \frac{1}{2}\rho_-. \tag{3.174}$$

The l.h.s of (3.172) will be estimated as

$$\begin{aligned}&\int_\Omega(\rho_0 + \rho_0'\theta + \zeta)J|u(t)|^2 + \int_\Gamma g\rho_+|\eta(t)|^2 + \mu\int_0^t\int_\Omega J|\mathbb{S}_{\mathcal{A}}u(s)|^2 ds \\ &\gtrsim \|u(t)\|^2_{L^2(\Omega)} + \|\eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\Omega)}ds - \int_0^t \mathcal{E}_f^3(s)ds.\end{aligned} \tag{3.175}$$

We now estimate the r.h.s of (3.172). By Gagliardo-Nirenberg's inequality (see (3.2)) and Sobolev embedding, one has

$$\begin{aligned}&\|\partial_t((\rho_0 + \rho_0'\theta + \zeta)J)\|_{L^\infty(\Omega)} \\ &\quad\lesssim \|(\rho_0 + \rho_0'\theta + \zeta)\partial_t J\|_{H^2(\Omega)} + \|(\rho_0'\partial_t\theta + \partial_t\zeta)J\|_{L^\infty(\Omega)} \\ &\quad\lesssim (1 + \|(\theta,\zeta)\|_{H^2(\Omega)})\|\partial_t\theta\|_{H^3(\Omega)} + \|(\partial_t\theta, \partial_t\zeta)\|_{H^2(\Omega)}(1 + \|J-1\|_{L^\infty(\Omega)}).\end{aligned}$$

Together with Lemma 0.2, (3.126) and (3.8), we observe

$$\begin{aligned}&\|\partial_t((\rho_0 + \rho_0'\theta + \zeta)J)\|_{L^\infty(\Omega)} \\ &\lesssim (1 + \|\eta\|_{H^{3/2}(\Gamma)} + \|\zeta\|_{H^2(\Omega)})\|\partial_t\eta\|_{H^{5/2}(\Gamma)} + \|\partial_t\eta\|_{H^{3/2}(\Gamma)} + \|\partial_t\zeta\|_{H^2(\Omega)} \\ &\lesssim \mathcal{E}_f,\end{aligned} \tag{3.176}$$

which yields

$$\int_0^t\int_\Omega \partial_t((\rho_0 + \rho_0'\theta + \zeta)J)|u|^2(s)ds \lesssim \int_0^t \mathcal{E}_f^3(s)ds. \tag{3.177}$$

Furthermore, thanks to (3.8) and (3.151), we get

$$\begin{aligned}\int_\Omega J(F^2\cdot u - g\zeta u_3) &\lesssim (\|J-1\|_{L^\infty(\Omega)} + 1)(\|F^2\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega)}) \\ &\lesssim \mathcal{E}_f^3 + \|\zeta\|_{L^2(\Omega)}\|u_3\|_{L^2(\Omega)}.\end{aligned} \tag{3.178}$$

Substituting (3.173), (3.175), (3.177) and (3.178) into (3.172), we deduce $(3.139)_{l=0}$.

For $l = 1$, we make use of (3.146) at order $l = 1$ to have that

$$\begin{aligned}
&\frac{1}{2}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t u|^2(t)+\int_\Gamma g\rho_+|\partial_t\eta(t)|^2-\int_\Omega g\rho_0'|u_3(t)|^2\Big)\\
&\qquad+\frac{\mu}{2}\int_0^t\|\mathbb{S}_\mathcal{A}\partial_t u(s)\|^2_{L^2(\Omega)}ds\\
&=\frac{1}{2}\Big(\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t u|^2+\int_\Gamma g\rho_+|\partial_t\eta|^2-\int_\Omega g\rho_0'|u_3|^2\Big)\Big|_{t=0}\\
&+\frac{1}{2}\int_0^t\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t u|(s)^2ds+\int_0^t\int_\Omega J(F^{2,1}\cdot\partial_t u+F^{3,1}\partial_t q)(s)ds\\
&-\int_0^t\int_\Gamma(g\rho_+\partial_t\eta F^{4,1}+F^{5,1}\cdot\partial_t u)(s)ds-\int_0^t\int_\Omega g\rho_0'(A\partial_3u_1+B\partial_3u_2)\partial_t u_3(s)ds\\
&-\int_0^t\int_\Omega gJF^1\partial_t u_3(s)ds.
\end{aligned}\tag{3.179}$$

By a similar argument as the proof of (3.175), we estimate the l.h.s of (3.179) as

$$\begin{aligned}
&\int_\Omega(\rho_0+\rho_0'\theta+\zeta)J|\partial_t u|^2(t)+\int_\Gamma g\rho_+|\partial_t\eta(t)|^2-\int_\Omega g\rho_0'|u_3(t)|^2\\
&\qquad+\mu\int_0^t\|\mathbb{S}_\mathcal{A}\partial_t u(s)\|^2_{L^2(\Omega)}ds\\
&\gtrsim\|\partial_t u(t)\|^2_{L^2(\Omega)}+\|\partial_t\eta(t)\|^2_{L^2(\Gamma)}+\int_0^t\|\nabla\partial_t u(s)\|^2_{L^2(\Omega)}-\|u_3(t)\|^2_{L^2(\Omega)}-\int_0^t\mathcal{E}_f^3(s)ds.
\end{aligned}\tag{3.180}$$

For the r.h.s of (3.179), we use (3.176) to obtain that

$$\int_0^t\int_\Omega\partial_t((\rho_0+\rho_0'\theta+\zeta)J)|\partial_t u|^2(s)ds\lesssim\int_0^t\mathcal{E}_f^3(s)ds.\tag{3.181}$$

Next, thanks to (3.9), we see that

$$\int_0^t\int_\Omega g\rho_0'(A\partial_3u_1+B\partial_3u_2)\partial_t u_3(s)ds\lesssim\int_0^t\mathcal{E}_f^2\|(A,B)\|_{H^2(\Omega)}(s)ds\lesssim\int_0^t\mathcal{E}_f^3(s)ds.\tag{3.182}$$

Let us use (3.8) and (3.151) to estimate that

$$\begin{aligned}
&\int_0^t\int_\Omega J(F^{2,1}\cdot\partial_t u+F^{3,1}\partial_t q-gF^1\partial_3 u)(s)ds-\int_0^t\int_\Gamma(g\rho_+\partial_t\eta F^{4,1}+F^{5,1}\cdot\partial_t u)(s)ds\\
&\lesssim\int_0^t((\|J-1\|_{L^\infty(\Omega)}+1)\|(F^1,F^{2,1},F^{3,1})(s)\|_{L^2(\Omega)}+\|(F^{4,1},F^{5,1})(s)\|_{L^2(\Gamma)})\mathcal{E}_f(s)ds\\
&\lesssim\int_0^t\mathcal{E}_f^3(s)ds.
\end{aligned}\tag{3.183}$$

Combining (3.180), (3.181) and (3.183), we obtain

$$\begin{aligned}&\|\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_t \eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla \partial_t u(s)\|^2_{L^2(\Omega)} ds \\ &\quad\lesssim \mathcal{E}_f^2(0) + \|u_3(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{E}_f^3(s) ds.\end{aligned} \tag{3.184}$$

As a consequence of (3.184) and $(3.139)_{l=0}$, the inequality $(3.139)_{l=1}$ follows.

For $l = 2$, we use (3.146) at order $l = 2$ to have that

$$\begin{aligned}&\frac{1}{2}\Big(\int_\Omega (\rho_0 + \rho_0'\theta + \zeta) J|\partial_t^2 u|^2(t) + \int_\Gamma g\rho_+ |\partial_t^2 \eta(t)|^2 - \int_\Omega g\rho_0' |\partial_t u_3(t)|^2\Big) \\ &\qquad + \frac{1}{2}\mu \int_0^t \int_\Omega J|\mathbb{S}_{\mathcal{A}} \partial_t^2 u(s)|^2 ds \\ &= \frac{1}{2}\Big(\int_\Omega (\rho_0 + \rho_0'\theta + \zeta) J|\partial_t^2 u|^2 + \int_\Gamma g\rho_+ |\partial_t^2 \eta|^2 - \int_\Omega g\rho_0' |\partial_t u_3|^2\Big)\Big|_{t=0} \\ &\quad + \frac{1}{2}\int_0^t \int_\Omega \partial_t((\rho_0 + \rho_0'\theta + \zeta)J)|\partial_t^2 u(s)|^2 ds + \int_0^t \int_\Omega J(F^{2,2}\cdot \partial_t^2 u + F^{3,2}\partial_t^2 q)(s) ds \\ &\quad - \int_0^t \int_\Gamma (g\rho_+ \partial_t^2 \eta F^{4,2} + F^{5,2}\cdot \partial_t^2 u)(s) ds + \int_0^t \int_\Omega g\rho_0 J F^{3,1} \partial_t^2 u_3(s) ds \\ &\quad - \int_0^t \int_\Omega g\rho_0' (A\partial_t \partial_3 u_1 + B\partial_t \partial_3 u_2)\partial_t^2 u_3(s) ds - \int_0^t \int_\Omega gJF^{1,1}\partial_t^2 u_3(s) ds.\end{aligned} \tag{3.185}$$

We follow the previous arguments to observe that

$$\begin{aligned}&\|\partial_t^2 u(t)\|^2_{L^2(\Omega)} + \|\partial_t^2 \eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \|\nabla \partial_t^2 u(s)\|^2_{L^2(\Omega)} ds \\ &\lesssim \mathcal{E}_f^2(0) + \|\partial_t u_3(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{E}_f^2(s)\|(A,B)(s)\|_{H^2(\Omega)} ds \\ &\quad + \int_0^t (\|J-1\|_{L^\infty(\Omega)} + 1)\|(F^{1,1}, F^{2,2}, F^{3,1})(s)\|_{L^2(\Omega)} ds \\ &\quad + \int_0^t \|(F^{4,2}, F^{5,2})(s)\|_{L^2(\Gamma)} \mathcal{E}_f(s) ds + \int_0^t \int_\Omega (JF^{3,2}\partial_t^2 q)(s) ds.\end{aligned} \tag{3.186}$$

Since $\partial_t^2 q$ does not appear in $\mathcal{E}_f$ or $\mathcal{D}_f$, we use the integration in time to have

$$\begin{aligned}\int_0^t \int_\Omega (JF^{3,2}\partial_t^2 q)(s) ds &= \int_\Omega (\partial_t q J F^{3,2})(t) - \int_\Omega (\partial_t q JF^{3,2})(0) \\ &\quad - \int_0^t \int_\Omega \partial_t q(s) \partial_t (JF^{3,2})(s) ds.\end{aligned}$$

Thanks to (3.153), we observe

$$\int_0^t \int_\Omega (JF^{3,2}\partial_t^2 q)(s) ds \lesssim \mathcal{E}_f^2(0) + \mathcal{E}_f^3(t) + \int_0^t \mathcal{E}_f^2(\mathcal{E}_f + \|\nabla \partial_t u\|_{H^2(\Omega)} + \|\nabla \partial_t^2 u\|_{L^2(\Omega)})(s) ds. \tag{3.187}$$

It follows from (3.8), (3.151) and (3.152) that

$$\begin{aligned}&\int_0^t ((\|J-1\|_{L^\infty(\Omega)}+1)\|(F^{2,2},F^{3,1},F^{1,1})(s)\|_{L^2(\Omega)}+\|(F^{4,2},F^{5,2})(s)\|_{L^2(\Gamma)})\mathcal{E}_f(s)ds\\&\lesssim \int_0^t \mathcal{E}_f^2(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)})(s)ds.\end{aligned}\tag{3.188}$$

Using (3.186), (3.187), (3.188) and (3.9), we deduce that

$$\begin{aligned}&\|\partial_t^2 u(t)\|^2_{L^2(\Omega)}+\|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}+\int_0^t\|\nabla\partial_t^2 u(s)\|^2_{L^2(\Omega)}ds\\&\lesssim \mathcal{E}_f^2(0)+\|\partial_t u_3(t)\|^2_{L^2(\Omega)}+\mathcal{E}_f^3(t)+\int_0^t\mathcal{E}_f^2(\mathcal{E}_f+\|\nabla\partial_t u\|_{H^2(\Omega)}+\|\nabla\partial_t^2 u\|_{L^2(\Omega)})(s)ds.\end{aligned}$$

We obtain (3.140) thanks to the resulting inequality and $(3.139)_{l=1}$. □

3.3.3 Horizontal estimates of the perturbation velocity

We continue deriving the mixed horizontal space-time derivatives of u. The nonlinear terms $\mathcal{Q}^i(1\leqslant i\leqslant 5)$ in (0.100) are presented by that

$$\begin{aligned}\mathcal{Q}^1 &= -K\rho_0''\theta u_3+K\partial_t\theta(\partial_3\zeta+\rho_0''\theta)-K\rho_0'\theta(Au_1+Bu_2)\\&\quad -u_1\partial_1\zeta-u_2\partial_2\zeta-Ku_3\partial_3\zeta+K\partial_3\zeta(Au_1+Bu_2)\end{aligned}\tag{3.189}$$

that

$$\begin{aligned}\mathcal{Q}_1^2 &= -(\zeta+\rho_0'\theta)\partial_t u_1-(\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u_1+AK(\partial_3 q-g\rho_0'\theta)\\&\quad -(\zeta+\rho_0+\rho_0'\theta)\Big(u_1(\partial_1 u_1-AK\partial_3 u_1)+u_2(\partial_2 u_1-BK\partial_3 u_1)+Ku_3\partial_3 u_1\Big)\\&\quad +\mu\begin{pmatrix}(K^2+A^2+B^2-1)\partial_{33}^2 u_1-2AK\partial_{13}^2u_1-2BK\partial_{23}^2 u_1\\(K\partial_3K(A^2+B^2+1)-\partial_1(AK)-\partial_2(BK)-A\partial_1K-B\partial_2K)\partial_3 u_1\end{pmatrix},\end{aligned}\tag{3.190}$$

$$\begin{aligned}\mathcal{Q}_2^2 &= -(\zeta+\rho_0'\theta)\partial_t u_2-(\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u_1+BK(\partial_3 q-g\rho_0'\theta)\\&\quad -(\zeta+\rho_0+\rho_0'\theta)\Big(u_1(\partial_1 u_2-AK\partial_3 u_2)+u_2(\partial_2 u_2-BK\partial_3 u_2)+Ku_3\partial_3 u_2\Big)\\&\quad +\mu\begin{pmatrix}(K^2+A^2+B^2-1)\partial_{33}^2 u_2-2AK\partial_{13}^2u_2-2BK\partial_{23}^2 u_2\\(K\partial_3K(A^2+B^2+1)-\partial_1(AK)-\partial_2(BK)-A\partial_1K-B\partial_2K)\partial_3 u_2\end{pmatrix},\end{aligned}\tag{3.191}$$

$$\begin{aligned}\mathcal{Q}_3^2 = &-(\zeta+\rho_0'\theta)\partial_t u_3 - (\rho_0+\rho_0'\theta+\zeta)K\partial_t\theta\partial_3 u_3 + (1-K)(\partial_3 q - g\rho_0'\theta)\\ &- (\zeta+\rho_0+\rho_0'\theta)\Big(u_1(\partial_1 u_3 - AK\partial_3 u_3) + u_2(\partial_2 u_3 - BK\partial_3 u_3) + Ku_3\partial_3 u_3\Big)\\ &+ (K-1)(\partial_{13}^2 u_1 + \partial_{23}^2 u_2) + \partial_1 K\partial_3 u_1 + \partial_2 K\partial_3 u_2\\ &- \partial_1(AK\partial_3 u_3) - AK\partial_3(K\partial_3 u_1 + \partial_1 u_3 - AK\partial_3 u_3)\\ &- \partial_2(BK\partial_3 u_3) - BK\partial_3(K\partial_3 u_2 + \partial_2 u_3 - BK\partial_3 u_3)\\ &+ 2(K^2-1)\partial_{33}^2 u_3 + 2K\partial_3 K\partial_3 u_3,\end{aligned} \tag{3.192}$$

that

$$\begin{aligned}\mathcal{Q}^3 &= (1-K)\partial_3 u_3 + AK\partial_3 u_1 + BK\partial_3 u_2,\\ \mathcal{Q}^4 &= -u_1\partial_1\eta - u_2\partial_2\eta,\end{aligned} \tag{3.193}$$

and that

$$\begin{aligned}\mathcal{Q}^5 = \partial_1\eta &\begin{pmatrix} q - g\rho_+\eta - 2\mu(\partial_1 u_1 - AK\partial_3 u_1)\\ -\mu(\partial_1 u_2 + \partial_2 u_1 - AK\partial_3 u_2 - BK\partial_3 u_1)\\ -\mu(\partial_1 u_3 - AK\partial_3 u_3 + K\partial_3 u_1)\end{pmatrix}\\ &+ \partial_2\eta\begin{pmatrix} -\mu(\partial_1 u_2 + \partial_2 u_1 - AK\partial_3 u_2 - BK\partial_3 u_1)\\ q - g\rho_+\eta - 2\mu(\partial_2 u_2 - BK\partial_3 u_2)\\ -\mu(\partial_2 u_3 - AK\partial_3 u_3 + K\partial_3 u_2)\end{pmatrix}\\ &- \mu\begin{pmatrix}(1-K)\partial_3 u_1 + AK\partial_3 u_3\\ (1-K)\partial_3 u_2 + BK\partial_3 u_3\\ 2(1-K)\partial_3 u_3\end{pmatrix}.\end{aligned} \tag{3.194}$$

Note that ρ_0 only depends on x_3. Let $\beta = (\beta_1,\beta_2) \in \mathbb{N}^2$ and let us apply the horizontal derivative $\partial_h^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}$ to (0.99), we obtain the following equations.

$$\begin{cases}\partial_t\partial_h^\beta\zeta + \rho_0'\partial_h^\beta u_3 = \partial_h^\beta\mathcal{Q}^1 & \text{in } \Omega,\\ \rho_0\partial_t\partial_h^\beta u + \nabla\partial_h^\beta q - \mu\Delta\partial_h^\beta u + g\partial_h^\beta\zeta e_3 = \partial_h^\beta\mathcal{Q}^2 & \text{in } \Omega,\\ \mathrm{div}\partial_h^\beta u = \partial_h^\beta\mathcal{Q}^3 & \text{in } \Omega,\\ \partial_t\partial_h^\beta\eta - \partial_h^\beta u_3 = \partial_h^\beta\mathcal{Q}^4 & \text{on } \Gamma,\\ (\partial_h^\beta q\mathrm{Id} - \mu\mathbb{S}\partial_h^\beta u)e_3 = g\rho_+\partial_h^\beta\eta e_3 + \partial_h^\beta\mathcal{Q}^5 & \text{on } \Gamma.\end{cases} \tag{3.195}$$

Proposition 3.13. *The following inequalities hold*

$$\begin{aligned}\sum_{\beta\in\mathbb{N}^2, 1\leqslant|\beta|\leqslant 4}&\Big(\|\partial_h^\beta u(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla\partial_h^\beta u(s)\|_{L^2(\Omega)}^2 ds\Big)\\ &\leqslant C_7\Big(\mathcal{E}_f^2(0) + \varepsilon^3\int_0^t(\mathcal{E}_f^2(s) + \|\nabla u_3(s)\|_{H^4(\Omega)}^2)ds + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\ &\quad + C_7\varepsilon^{-27}\int_0^t(\|u_3(s)\|_{L^2(\Omega)}^2 + \|\eta(s)\|_{L^2(\Omega)}^2)ds,\end{aligned} \tag{3.196}$$

and

$$\sum_{\beta\in\mathbb{N}^2,1\leqslant|\beta|\leqslant 2}\Big(\|\partial_h^\beta\partial_t u(t)\|_{L^2(\Omega)}^2+\int_0^t\|\nabla\partial_h^\beta\partial_t u(s)\|_{L^2(\Omega)}^2 ds\Big)$$
$$\leqslant C_8\Big(\mathcal{E}_f^2(0)+\varepsilon^3\int_0^t(\mathcal{E}_f^2(s)+\|\nabla u_3(s)\|_{H^4(\Omega)}^2)ds+\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)$$
$$+C_8\varepsilon^{-27}\int_0^t(\|u_3(s)\|_{L^2(\Omega)}^2+\|\eta(s)\|_{L^2(\Omega)}^2)ds. \tag{3.197}$$

To prove Proposition 3.13, we need the following lemma.

Lemma 3.13. *The following inequalities hold*

$$\|\mathcal{Q}^1\|_{H^2(\Omega)}+\|\partial_t\mathcal{Q}^1\|_{L^2(\Omega)}+\|\partial_t\mathcal{Q}^2\|_{L^2(\Omega)}+\|\mathcal{Q}^2\|_{H^2(\Omega)}+\|\partial_t\mathcal{Q}^3\|_{H^1(\Omega)}$$
$$+\|\mathcal{Q}^3\|_{H^3(\Omega)}+\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}+\|\mathcal{Q}^5\|_{H^{5/2}(\Gamma)}+\|\partial_t\mathcal{Q}^5\|_{H^{1/2}(\Gamma)}\lesssim\mathcal{E}_f^2, \tag{3.198}$$

and

$$\|\mathcal{Q}^2\|_{H^3(\Omega)}+\|\partial_t\mathcal{Q}^2\|_{H^1(\Omega)}+\|\mathcal{Q}^3\|_{H^4(\Omega)}+\|\partial_t\mathcal{Q}^3\|_{H^2(\Omega)}+\|\mathcal{Q}^5\|_{H^{7/2}(\Gamma)}$$
$$\lesssim\mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f). \tag{3.199}$$

Proof. For (3.198), we only present estimates for some terms of the l.h.s, precisely,

$$\|\partial_t\mathcal{Q}^3\|_{H^1(\Omega)}+\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}+\|\mathcal{Q}_1^5\|_{H^{5/2}(\Gamma)}\lesssim\mathcal{E}_f^2,$$

the estimates of the other terms in the l.h.s of (3.198) follow the same way. To get $\|\partial_t\mathcal{Q}^3\|_{H^1(\Omega)}\lesssim\mathcal{E}_f^2$, we use (3.1) and (3.10), (3.11) to bound each term of $\mathcal{Q}^3$ (3.193). Indeed, we have

$$\begin{aligned}\|\partial_t((1-K)\partial_3u_3)\|_{H^1(\Omega)}&\lesssim\|\partial_tK\|_{H^1(\Omega)}\|\partial_3u_3\|_{H^3(\Omega)}+\|K-1\|_{H^3(\Omega)}\|\partial_t\partial_3u_3\|_{H^1(\Omega)}\\&\lesssim\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u_3\|_{H^4(\Omega)}+\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_tu_3\|_{H^2(\Omega)}\\&\lesssim\mathcal{E}_f^2,\end{aligned}\tag{3.200}$$

and

$$\begin{aligned}&\|\partial_t(AK\partial_3u_1+BK\partial_3u_2)\|_{H^1(\Omega)}\\&\lesssim\|\partial_t(AK,BK)\|_{H^1(\Omega)}\|\partial_3u\|_{H^3(\Omega)}+\|(AK,BK)\|_{H^3(\Omega)}\|\partial_t\partial_3u\|_{H^1(\Omega)}\\&\lesssim\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|_{H^4(\Omega)}+\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_tu\|_{H^2(\Omega)}\\&\lesssim\mathcal{E}_f^2.\end{aligned}\tag{3.201}$$

Hence, $\|\partial_t\mathcal{Q}^3\|_{H^1(\Omega)}\lesssim\mathcal{E}_f^2$ follows from (3.200) and (3.201). We apply the product estimate (3.1) and the trace theorem to have that

$$\begin{aligned}\|\mathcal{Q}^4\|_{H^{7/2}(\Gamma)}&\lesssim\|u_1\|_{H^{7/2}(\Gamma)}\|\partial_1\eta\|_{H^{7/2}(\Gamma)}+\|u_2\|_{H^{7/2}(\Gamma)}\|\partial_2\eta\|_{H^{7/2}(\Gamma)}\\&\lesssim\|u\|_{H^4(\Omega)}\|\eta\|_{H^{9/2}(\Gamma)}\\&\lesssim\mathcal{E}_f^2.\end{aligned}$$

Moreover, using (3.1), (3.10), (3.11) again and the trace theorem, we show

$$\|\mathcal{Q}_1^5\|_{H^{5/2}(\Gamma)} \lesssim \mathcal{E}_f^2.$$

From the expression of $\mathcal{Q}_1^5$ (3.194), we have that

$$\begin{aligned}
&\|\partial_1\eta(q - g\rho_+\eta - 2\mu(\partial_1 u_1 - AK\partial_3 u_1))\|_{H^{5/2}(\Gamma)} \\
&\quad \lesssim \|\partial_1\eta\|_{H^{5/2}(\Gamma)}(\|(q,\eta,\partial_1 u_1)\|_{H^{5/2}(\Gamma)} + \|AK\|_{H^{5/2}(\Gamma)}\|\partial_3 u_1\|_{H^{5/2}(\Gamma)}) \\
&\quad \lesssim \|\eta\|_{H^{7/2}(\Gamma)}(\|q\|_{H^3(\Omega)} + \|u_1\|_{H^4(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)} + \|AK\|_{H^3(\Omega)}\|u_1\|_{H^4(\Omega)}) \\
&\quad \lesssim \|\eta\|_{H^{7/2}(\Gamma)}(\|q\|_{H^3(\Omega)} + \|u_1\|_{H^4(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)} + \|\eta\|_{H^{7/2}(\Gamma)}\|u_1\|_{H^4(\Omega)}),
\end{aligned} \tag{3.202}$$

that

$$\begin{aligned}
&\|\partial_2\eta(\partial_1 u_2 + \partial_2 u_1 - AK\partial_3 u_2 - BK\partial_3 u_1)\|_{H^{5/2}(\Gamma)} \\
&\quad \lesssim \|\partial_2\eta\|_{H^{5/2}(\Gamma)}\|u\|_{H^{7/2}(\Gamma)}(1 + \|(AK,BK)\|_{H^{5/2}(\Gamma)}) \\
&\quad \lesssim \|\partial_2\eta\|_{H^{5/2}(\Gamma)}\|u\|_{H^{7/2}(\Gamma)}(1 + \|(AK,BK)\|_{H^3(\Omega)}) \\
&\quad \lesssim \|\eta\|_{H^{7/2}(\Gamma)}\|u\|_{H^4(\Omega)}(1 + \|\eta\|_{H^{7/2}(\Gamma)}),
\end{aligned} \tag{3.203}$$

and that

$$\begin{aligned}
\|(1-K)\partial_3 u_1 + AK\partial_3 u_3\|_{H^{5/2}(\Gamma)} &\lesssim \|(K-1,AK)\|_{H^{5/2}(\Gamma)}\|\partial_3 u\|_{H^{5/2}(\Gamma)} \\
&\lesssim \|(K-1,AK)\|_{H^3(\Omega)}\|u\|_{H^4(\Omega)} \\
&\lesssim \|\eta\|_{H^{7/2}(\Gamma)}\|u\|_{H^4(\Omega)}
\end{aligned} \tag{3.204}$$

Hence, the inequality $\|\mathcal{Q}_1^5\|_{H^{5/2}(\Gamma)} \lesssim \mathcal{E}_f^2$ follows from the three above estimates (3.202), (3.203) and (3.204).

Similarly, for (3.199), we show only

$$\|\partial_t\mathcal{Q}_1^2\|_{H^1(\Omega)} + \|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \mathcal{D}_f).$$

The inequality $\|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \mathcal{E}_f\mathcal{D}_f$ (see $\mathcal{Q}^3$ in (3.193)) is proven by using (3.1) and (3.10), (3.11),

$$\|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \|(AK,BK,K-1)\|_{H^4(\Omega)}\|\partial_3 u\|_{H^4(\Omega)} \lesssim \|\eta\|_{H^{9/2}(\Gamma)}\|\nabla u\|_{H^4(\Omega)}.$$

Let us prove $\|\partial_t\mathcal{Q}_1^2\|_{H^1(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f + \mathcal{D}_f)$ (see $\mathcal{Q}_1^2$ in (3.190)). In view of (3.1) and Lemma 0.2, we obtain that

$$\begin{aligned}
\|\partial_t((\zeta + \rho_0'\theta)\partial_t u_1)\|_{H^1(\Omega)} &\lesssim \|(\partial_t\zeta,\partial_t\theta)\|_{H^1(\Omega)}\|\partial_t u_1\|_{H^3(\Omega)} + \|(\zeta,\theta)\|_{H^3(\Omega)}\|\partial_t^2 u_1\|_{H^1(\Omega)} \\
&\lesssim (\|\partial_t\zeta\|_{H^1(\Omega)} + \|\partial_t\eta\|_{H^{1/2}(\Gamma)})\|\partial_t u_1\|_{H^3(\Omega)} \\
&\qquad + (\|\zeta\|_{H^3(\Omega)} + \|\eta\|_{H^{5/2}(\Gamma)})\|\partial_t^2 u_1\|_{H^1(\Omega)} \\
&\lesssim \mathcal{E}_f(\mathcal{E}_f + \|\nabla\partial_t u_1\|_{H^2(\Omega)} + \|\nabla\partial_t^2 u_1\|_{L^2(\Omega)}).
\end{aligned} \tag{3.205}$$

We further use (3.1) to have

$$
\begin{aligned}
&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)Ku_3\partial_3u_1)\|_{H^1(\Omega)}\\
&\lesssim (1+\|(\theta,\zeta)\|_{H^3(\Omega)})\|\partial_t(Ku_3\partial_3u_1)\|_{H^1(\Omega)}+\|(\partial_t\zeta,\partial_t\theta)\|_{H^1(\Omega)}\|Ku_3\partial_3u_1\|_{H^3(\Omega)}\\
&\lesssim (1+\|(\theta,\zeta)\|_{H^3(\Omega)})\|\partial_tK\|_{H^1(\Omega)}\|u_3\|_{H^3(\Omega)}\|\partial_3u_1\|_{H^3(\Omega)}\\
&\qquad+(1+\|(\theta,\zeta)\|_{H^3(\Omega)})(\|K-1\|_{H^3(\Omega)}+1)\\
&\qquad\qquad\times(\|\partial_tu_3\|_{H^1(\Omega)}\|\partial_3u_1\|_{H^3(\Omega)}+\|\partial_t\partial_3u_1\|_{H^1(\Omega)}\|u_3\|_{H^3(\Omega)})\\
&\qquad+\|(\partial_t\zeta,\partial_t\theta)\|_{H^1(\Omega)}(\|K-1\|_{H^3(\Omega)}+1)\|u_3\|_{H^3(\Omega)}\|\partial_3u_1\|_{H^3(\Omega)}.
\end{aligned}
$$

Thanks to Lemma 0.2 and (3.10), we deduce

$$
\begin{aligned}
&\|\partial_t((\rho_0+\rho_0'\theta+\zeta)Ku_3\partial_3u_1)\|_{H^1(\Omega)}\\
&\lesssim (1+\|\zeta\|_{H^3(\Omega)}+\|\eta\|_{H^{5/2}(\Gamma)})\|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|u\|^2_{H^4(\Omega)}\\
&\qquad+(1+\|\zeta\|_{H^3(\Omega)}+\|\eta\|_{H^{5/2}(\Gamma)})(\|\eta\|_{H^{7/2}(\Gamma)}+1)\|\partial_tu\|_{H^2(\Omega)}\|u\|_{H^4(\Omega)}\\
&\qquad+(\|\partial_t\zeta\|_{H^1(\Omega)}+\|\partial_t\eta\|_{H^{1/2}(\Gamma)})(\|\eta\|_{H^{7/2}(\Gamma)}+1)\|u\|^2_{H^4(\Omega)}\\
&\lesssim \mathcal{E}_f^2.
\end{aligned}
\tag{3.206}
$$

Since $K^2-1=-J^{-2}(2\partial_3\theta+(\partial_3\theta)^2)$, let us use (3.1) to obtain

$$
\begin{aligned}
&\|\partial_t((K^2+A^2+B^2-1)\partial_{33}^2u_1-2AK\partial_{13}^2u_1-2BK\partial_{23}^2u_1)\|_{H^1(\Omega)}\\
&\lesssim (\|(A^2,B^2,AK,BK)\|_{H^3(\Omega)}+\|K^2-1\|_{H^3(\Omega)})\|\partial_tu_1\|_{H^3(\Omega)}\\
&\quad+\|\partial_t(A^2,B^2,K^2-1,AK,BK)\|_{H^1(\Omega)}\|\nabla^2u_1\|_{H^3(\Omega)}\\
&\lesssim (\|(A,B)\|^2_{H^3(\Omega)}+\|(AK,BK)\|_{H^3(\Omega)}+\|\partial_3\theta\|_{H^3(\Omega)}(1+\|\partial_3\theta\|_{H^3(\Omega)}))\|\partial_tu_1\|_{H^3(\Omega)}\\
&\quad+(\|(A,B)\|_{H^3(\Omega)}\|(\partial_tA,\partial_tB)\|_{H^1(\Omega)}+\|\partial_t\partial_3\theta\|_{H^1(\Omega)}(1+\|\partial_3\theta\|_{H^3(\Omega)}))\|\nabla^2u_1\|_{H^3(\Omega)}\\
&\quad+\|\partial_t(AK,BK)\|_{H^1(\Omega)}\|\nabla^2u_1\|_{H^3(\Omega)}.
\end{aligned}
$$

Owing to (3.9) and (3.11), we deduce

$$
\begin{aligned}
&\|\partial_t((K^2+A^2+B^2-1)\partial_{33}^2u_1-2AK\partial_{13}^2u_1-2BK\partial_{23}^2u_1)\|_{H^1(\Omega)}\\
&\lesssim \|\eta\|_{H^{7/2}(\Gamma)}(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\partial_tu_1\|_{H^3(\Omega)}+\|\partial_t\eta\|_{H^{3/2}(\Gamma)}(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\nabla^2u_1\|_{H^3(\Omega)}\\
&\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_tu_1\|_{H^2(\Omega)}+\|\nabla u_1\|_{H^4(\Omega)}).
\end{aligned}
\tag{3.207}
$$

We continue using (3.1), Lemma 0.2 and (3.11) to get

$$
\begin{aligned}
&\|\partial_t(AK(\partial_3q-g\rho_0'\theta))\|_{H^1(\Omega)}\\
&\lesssim \|\partial_t(AK)\|_{H^1(\Omega)}\|(q,\theta)\|_{H^3(\Omega)}+\|AK\|_{H^3(\Omega)}(\|\partial_t\partial_3q\|_{H^1(\Omega)}+\|\partial_t\theta\|_{H^1(\Omega)})\\
&\lesssim \|\partial_t\eta\|_{H^{3/2}(\Gamma)}(\|q\|_{H^3(\Omega)}+\|\eta\|_{H^{5/2}(\Gamma)})+\|\eta\|_{H^{7/2}(\Gamma)}(\|\partial_tq\|_{H^2(\Omega)}+\|\partial_t\eta\|_{H^{1/2}(\Gamma)})\\
&\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\partial_tq\|_{H^2(\Omega)}).
\end{aligned}
\tag{3.208}
$$

From the product estimate (3.1), we obtain also

$$\begin{aligned}
&\|\partial_t((K\partial_3 K(A^2+B^2+1)-\partial_1(AK)-\partial_2(BK)-A\partial_1 K-B\partial_2 K)\partial_3 u_1)\|_{H^1(\Omega)}\\
&\lesssim (1+\|K-1\|_{H^3(\Omega)})\|\nabla K\|_{H^3(\Omega)}(\|(A,B)\|_{H^3(\Omega)}^2+1)\|\partial_t\partial_3 u_1\|_{H^1(\Omega)}\\
&\quad+(\|\nabla(AK,BK)\|_{H^3(\Omega)}+\|(A,B)\|_{H^3(\Omega)}\|\nabla K\|_{H^3(\Omega)})\|\partial_t\partial_3 u_1\|_{H^1(\Omega)}\\
&\quad+(\|\partial_t(K\partial_3 K)\|_{H^1(\Omega)}(1+\|(A,B)\|_{H^3(\Omega)}^2)\|\partial_3 u_1\|_{H^3(\Omega)}\\
&\quad+\|(\partial_t A,\partial_t B)\|_{H^1(\Omega)}\|\partial_3 K\|_{H^3(\Omega)}\|(AK,BK)\|_{H^3(\Omega)}\|\partial_3 u_1\|_{H^3(\Omega)}\\
&\quad+\|\nabla\partial_t(AK,BK)\|_{H^1(\Omega)})\|\partial_3 u_1\|_{H^3(\Omega)}\\
&\quad+(\|\nabla\partial_t K\|_{H^1(\Omega)}\|(A,B)\|_{H^3(\Omega)}+\|\partial_t(A,B)\|_{H^1(\Omega)}\|\nabla K\|_{H^3(\Omega)})\|\partial_3 u_1\|_{H^3(\Omega)}.
\end{aligned}$$

Thanks to (3.1) again and (3.10), let us estimate the term $\|\partial_t(K\partial_3 K)\|_{H^1(\Omega)}$ as follows

$$\begin{aligned}
\|\partial_t(K\partial_3 K)\|_{H^1(\Omega)} &\lesssim \|\partial_t K\|_{H^1(\Omega)}\|\partial_3 K\|_{H^3(\Omega)}+(1+\|K-1\|_{H^3(\Omega)})\|\partial_t\partial_3 K\|_{H^1(\Omega)}\\
&\lesssim \|\partial_t\eta\|_{H^{3/2}(\Gamma)}\|\eta\|_{H^{9/2}(\Gamma)}+(1+\|\eta\|_{H^{7/2}(\Gamma)})\|\partial_t\eta\|_{H^{5/2}(\Gamma)}.
\end{aligned}$$

Hence, due to (3.9), (3.10), (3.11) and note that $\|\partial_t\eta\|_{H^{5/2}(\Gamma)}\lesssim\mathcal{E}_f$ from (3.126), we have

$$\begin{aligned}
&\|\partial_t((K\partial_3 K(A^2+B^2+1)-\partial_1(AK)-\partial_2(BK)-A\partial_1 K-B\partial_2 K)\partial_3 u_1)\|_{H^1(\Omega)}\\
&\lesssim \|\eta\|_{H^{9/2}(\Gamma)}(1+\|\eta\|_{H^{9/2}(\Gamma)})\|\partial_t u_1\|_{H^2(\Omega)}+\|\partial_t\eta\|_{H^{5/2}(\Gamma)}(1+\|\eta\|_{H^{9/2}(\Gamma)})\|u_1\|_{H^4(\Omega)}\\
&\lesssim \mathcal{E}_f^2.
\end{aligned}\tag{3.209}$$

Combining (3.205), (3.206), (3.207), (3.208) and (3.209), we conclude

$$\begin{aligned}
\|\partial_t\mathcal{Q}_1^2\|_{H^1(\Omega)} &\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla\partial_t^2 u_1\|_{L^2(\Omega)}+\|\partial_t q\|_{H^2(\Omega)}+\|\nabla\partial_t u_1\|_{H^2(\Omega)})\\
&\lesssim \mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f).
\end{aligned}$$

□

We are in position to show Proposition 3.13.

Proof of Proposition 3.13. For any $\beta\in\mathbb{N}^2$ such that $1\leqslant|\beta|\leqslant 4$, multiplying by $\partial_h^\beta u$ on both sides of $(3.195)_2$ and integrating over Ω, one has the identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\partial_h^\beta u|^2+\int_\Omega(\nabla\partial_h^\beta q-\mu\Delta\partial_h^\beta u)\cdot\partial_h^\beta u+\int_\Omega g\rho_0'\partial_h^\beta\zeta\partial_h^\beta u_3=\int_\Omega\partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^2.$$

Using the integration by parts and $(3.195)_{3,5}$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\rho_0|\partial_h^\beta u|^2+\frac{1}{2}\int_\Omega\mu|\mathbb{S}\partial_h^\beta u|^2\\
&=-\int_\Omega g\rho_0'\partial_h^\beta\zeta\partial_h^\beta u_3+\int_\Omega\partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^2-\int_\Gamma g\rho_+\partial_h^\beta\eta\partial_h^\beta u_3\\
&\quad+\int_\Omega\partial_h^\beta q\partial_h^\beta\mathcal{Q}^3-\int_\Gamma\partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^5.
\end{aligned}\tag{3.210}$$

We estimate each integral in the r.h.s of (3.210). For the first integral, we use Young's inequality and (3.7) to get that

$$\begin{aligned}\int_\Omega g\rho_0'\partial_h^\beta\zeta\partial_h^\beta u_3 &\lesssim \|\zeta\|_{H^4(\Omega)}\|u_3\|_{H^4(\Omega)}\\ &\lesssim \varepsilon^3\|\zeta\|_{H^4(\Omega)}^2+\varepsilon^{-3}\|u_3\|_{H^4(\Omega)}^2\\ &\lesssim \varepsilon^3(\|\zeta\|_{H^4(\Omega)}^2+\|u_3\|_{H^5(\Omega)}^2)+\varepsilon^{-27}\|u_3\|_{L^2(\Omega)}^2.\end{aligned} \tag{3.211}$$

For the third integral, it follows from the trace theorem and Young's inequality that

$$\begin{aligned}\int_\Gamma g\rho_+\partial_h^\beta\eta\partial_h^\beta u_3 &\lesssim \|\partial_h^\beta\eta\|_{H^{-1/2}(\Gamma)}\|\partial_h^\beta u_3\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|\eta\|_{H^{|\beta|-1/2}(\Gamma)}\|u_3\|_{H^{|\beta|+1/2}(\Gamma)}\\ &\lesssim \|\eta\|_{H^{7/2}(\Gamma)}\|u_3\|_{H^5(\Omega)}\\ &\lesssim \varepsilon^3\|u_3\|_{H^5(\Omega)}^2+\varepsilon^{-3}\|\eta\|_{H^{7/2}(\Gamma)}^2.\end{aligned}$$

Thanks to (3.7) again, we have

$$\|\eta\|_{H^{7/2}(\Gamma)}^2 \lesssim \varepsilon^6\|\eta\|_{H^{9/2}(\Gamma)}^2+\varepsilon^{-21}\|\eta\|_{L^2(\Gamma)}^2.$$

Hence,

$$\int_\Gamma g\rho_+\partial_h^\beta\eta\partial_h^\beta u_3 \lesssim \varepsilon^3(\|\eta\|_{H^{9/2}(\Gamma)}^2+\|u_3\|_{H^5(\Omega)}^2)+\varepsilon^{-24}\|\eta\|_{L^2(\Omega)}^2. \tag{3.212}$$

For the fourth integral, we use Cauchy-Schwarz's inequality to have

$$\int_\Omega \partial_h^\beta q\partial_h^\beta\mathcal{Q}^3 \lesssim \|q\|_{H^4(\Omega)}\|\mathcal{Q}^3\|_{H^4(\Omega)} \lesssim \mathcal{E}_f\mathcal{D}_f(\mathcal{E}_f+\mathcal{D}_f). \tag{3.213}$$

For the second and fifth integral, we split into two cases. For $\beta\in\mathbb{N}^2$ such that $1\leqslant|\beta|\leqslant 3$, we use trace theorem to bound

$$\begin{aligned}\int_\Omega \partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^2-\int_\Gamma \partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^5 &\lesssim \|u\|_{H^3(\Omega)}\|\mathcal{Q}^2\|_{H^3(\Omega)}+\|u\|_{H^3(\Gamma)}\|\mathcal{Q}^5\|_{H^3(\Gamma)}\\ &\lesssim \|u\|_{H^3(\Omega)}\|\mathcal{Q}^2\|_{H^3(\Omega)}+\|u\|_{H^4(\Omega)}\|\mathcal{Q}^5\|_{H^3(\Gamma)}\\ &\lesssim \mathcal{E}_f^2(\mathcal{E}_f+\mathcal{D}_f).\end{aligned} \tag{3.214}$$

For $\beta=(\beta_1,\beta_2)\in\mathbb{N}^2$ such that $|\beta|=4$, we assume $\beta_1\geqslant 1$ and write

$$\beta_-=(\beta_1-1,\beta_2)\quad\text{and}\quad\beta_+=(\beta_1+1,\beta_2). \tag{3.215}$$

Hence, we estimate that

$$\begin{aligned}\Big|\int_\Omega \partial_h^\beta u\cdot\partial_h^\beta\mathcal{Q}^2\Big|=\Big|\int_\Omega \partial_h^{\beta_+}u\cdot\partial_h^{\beta_-}\mathcal{Q}^2\Big| &\lesssim \|\partial_h^{\beta_+}u\|_{L^2(\Omega)}\|\partial_h^{\beta_-}\mathcal{Q}^2\|_{L^2(\Omega)}\\ &\lesssim \|u\|_{H^5(\Omega)}\|\mathcal{Q}^2\|_{H^3(\Omega)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).\end{aligned} \tag{3.216}$$

Thanks to the trace theorem, we have

$$\begin{aligned}\Big|\int_\Gamma \partial_h^\beta u\cdot\partial_h^\beta \mathcal{Q}^5\Big| = \Big|\int_\Gamma \partial_h^{\beta_+} u\cdot\partial_h^{\beta_-} \mathcal{Q}^5\Big| &\lesssim \|\partial_h^{\beta_+}u\|_{H^{-1/2}(\Gamma)}\|\partial_h^{\beta_-}\mathcal{Q}^5\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|u\|_{H^{|\beta_+|-1/2}(\Gamma)}\|\mathcal{Q}^5\|_{H^{|\beta_-|+1/2}(\Gamma)}\\ &\lesssim \|u\|_{H^5(\Omega)}\|\mathcal{Q}^5\|_{H^{7/2}(\Gamma)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).\end{aligned} \tag{3.217}$$

Substituting (3.211), (3.212), (3.213), (3.214), (3.216), (3.217) into (3.210), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho_0|\partial_h^\beta u|^2 + \int_\Omega \mu|\mathbb{S}\partial_h^\beta u|^2 \lesssim \varepsilon^3\int_0^t(\mathcal{E}_f^2+\|\nabla u_3\|_{H^4(\Omega)}^2) + \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)\\ + \varepsilon^{-27}(\|u_3\|_{L^2(\Omega)}^2+\|\eta\|_{L^2(\Gamma)}^2).\end{aligned} \tag{3.218}$$

By Korn's inequality (3.5), one has

$$\int_\Omega \mu|\mathbb{S}\partial_h^\beta u|^2 \gtrsim \|\nabla\partial_h^\beta u\|_{L^2(\Omega)}^2.$$

Hence, we deduce from (3.218) that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho_0|\partial_h^\beta u|^2 + \|\nabla\partial_h^\beta u\|_{L^2(\Omega)}^2 \lesssim \varepsilon^3(\mathcal{E}_f^2+\|\nabla u_3\|_{H^4(\Omega)}^2) + \varepsilon^{-27}(\|u_3\|_{L^2(\Omega)}^2+\|\eta\|_{L^2(\Gamma)}^2)\\ + \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2),\end{aligned}$$

Integrating the resulting inequality in time, we obtain (3.196).

To prove (3.197), we compute from (3.195) that

$$\begin{cases}\rho_0\partial_t^2\partial_h^\beta u + \nabla\partial_t\partial_h^\beta q - \mu\Delta\partial_t\partial_h^\beta u - g\rho_0'\partial_h^\beta u_3 e_3 = -g\partial_h^\beta\mathcal{Q}^1 e_3 + \partial_t\partial_h^\beta\mathcal{Q}^2 & \text{in } \Omega,\\ \operatorname{div}\partial_t\partial_h^\beta u = \partial_t\partial_h^\beta\mathcal{Q}^3 & \text{in } \Omega,\\ (\partial_t\partial_h^\beta q\mathrm{Id} - \mu\mathbb{S}\partial_t\partial_h^\beta u)e_3 = g\rho_+\partial_h^\beta u_3 e_3 + g\rho_+\partial_h^\beta\mathcal{Q}^4 e_3 + \partial_h^\beta\mathcal{Q}^5 & \text{on } \Gamma.\end{cases} \tag{3.219}$$

For any $\beta\in\mathbb{N}^2$ with $|\beta|=1$ or 2, multiplying by $\partial_t\partial_h^\beta u$ on both sides of $(3.219)_1$ and integrating over Ω, one has the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega(\rho_0|\partial_t\partial_h^\beta u|^2 - g\rho_0'|\partial_h^\beta u_3|^2) + \int_\Omega(\nabla\partial_t\partial_h^\beta q - \mu\Delta\partial_t\partial_h^\beta u)\cdot\partial_t\partial_h^\beta u\\ &= \int_\Omega(-g\partial_h^\beta\mathcal{Q}^1\partial_t\partial_h^\beta u_3 + \partial_t\partial_h^\beta\mathcal{Q}^2\cdot\partial_t\partial_h^\beta u).\end{aligned}$$

Using the integration by parts, one has

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega\rho_0|\partial_t\partial_h^\beta u|^2 - \int_\Omega g\rho_0'|\partial_h^\beta u_3|^2\Big) + \int_\Gamma(\partial_t\partial_h^\beta q\mathrm{Id} - \mu\mathbb{S}\partial_t\partial_h^\beta u)e_3\cdot\partial_t\partial_h^\beta u\\ &= \int_\Omega\partial_t\partial_h^\beta q\operatorname{div}\partial_t\partial_h^\beta u - \frac{\mu}{2}\int_\Omega|\mathbb{S}\partial_t\partial_h^\beta u|^2 + \int_\Omega(-g\partial_h^\beta\mathcal{Q}^1\partial_t\partial_h^\beta u_3 + \partial_t\partial_h^\beta\mathcal{Q}^2\cdot\partial_t\partial_h^\beta u)\end{aligned}$$

By $(3.219)_{2,3}$, we observe

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t\partial_h^\beta u|^2+\int_\Gamma g\rho_+|\partial_h^\beta u_3|^2-\int_\Omega g\rho_0'|\partial_h^\beta u_3|^2\Big)+\frac{1}{2}\int_\Omega \mu|\mathbb{S}\partial_t\partial_h^\beta u|^2\\ &=\int_\Omega(-g\partial_h^\beta \mathcal{Q}^1\partial_t\partial_h^\beta u_3+\partial_t\partial_h^\beta\mathcal{Q}^2\cdot\partial_t\partial_h^\beta u)+\int_\Omega \partial_t\partial_h^\beta q\partial_t\partial_h^\beta\mathcal{Q}^3\\ &\quad-\int_\Gamma(\partial_t\partial_h^\beta\mathcal{Q}^5+g\rho_+\partial_h^\beta\mathcal{Q}^4e_3)\cdot\partial_t\partial_h^\beta u.\end{aligned}\tag{3.220}$$

We now estimate each integral in the r.h.s of (3.220). For the first and third integral, we use Cauchy-Schwarz's inequality to have

$$\begin{aligned}\int_\Omega \partial_h^\beta\mathcal{Q}^1\partial_t\partial_h^\beta u_3+\int_\Omega\partial_t\partial_h^\beta q\partial_t\partial_h^\beta\mathcal{Q}^3&\lesssim\|\partial_t u_3\|_{H^2(\Omega)}\|\mathcal{Q}^1\|_{H^2(\Omega)}+\|\partial_t q\|_{H^2(\Omega)}\|\partial_t\mathcal{Q}^3\|_{H^2(\Omega)}\\ &\lesssim\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).\end{aligned}\tag{3.221}$$

With the same notations $\beta_\pm$ (3.215), we bound the second integral as

$$\begin{aligned}\Big|\int_\Omega\partial_t\partial_h^\beta u\cdot\partial_t\partial_h^\beta\mathcal{Q}^2\Big|=\Big|\int_\Omega\partial_t\partial_h^{\beta_+}u\cdot\partial_t\partial_h^{\beta_-}\mathcal{Q}^2\Big|&\lesssim\|\partial_t u\|_{H^{|\beta_+|}(\Gamma)}\|\partial_t\mathcal{Q}^2\|_{H^{|\beta_-|}(\Omega)}\\ &\lesssim\|\partial_t u\|_{H^3(\Omega)}\|\partial_t\mathcal{Q}^2\|_{H^1(\Omega)}\\ &\lesssim\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).\end{aligned}\tag{3.222}$$

For the fourth integral, we have

$$\begin{aligned}\Big|\int_\Gamma\partial_t\partial_h^\beta u\cdot\partial_t\partial_h^\beta\mathcal{Q}^5\Big|=\Big|\int_\Gamma\partial_t\partial_h^{\beta_+}u\cdot\partial_t\partial_h^{\beta_-}\mathcal{Q}^5\Big|&\lesssim\|\partial_t\partial_h^{\beta_+}u\|_{H^{-1/2}(\Gamma)}\|\partial_t\partial_h^{\beta_-}\mathcal{Q}^5\|_{H^{1/2}(\Gamma)}\\ &\lesssim\|\partial_t u\|_{H^{|\beta_+|-1/2}(\Gamma)}\|\partial_t\mathcal{Q}^5\|_{H^{|\beta_-|+1/2}(\Gamma)}\\ &\lesssim\|\partial_t u\|_{H^3(\Omega)}\|\partial_t\mathcal{Q}^5\|_{H^{3/2}(\Gamma)}\\ &\lesssim\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).\end{aligned}\tag{3.223}$$

Thanks to the trace theorem and (3.198), we bound the fifth integral as

$$\begin{aligned}\int_\Gamma\partial_t\partial_h^\beta u_3\partial_h^\beta\mathcal{Q}^4&\lesssim\|\partial_t\partial^\beta u_3\|_{H^{-1/2}(\Gamma)}\|\partial^\beta\mathcal{Q}^4\|_{H^{1/2}(\Gamma)}\\ &\lesssim\|\partial_t u_3\|_{H^{|\beta|-1/2}(\Omega)}\|\mathcal{Q}^4\|_{H^{|\beta|+1/2}(\Gamma)}\\ &\lesssim\|\partial_t u_3\|_{H^2(\Omega)}\|\mathcal{Q}^4\|_{H^{5/2}(\Gamma)}\\ &\lesssim\mathcal{E}_f^3.\end{aligned}\tag{3.224}$$

In view of (3.221), (3.222), (3.223) and (3.224), we get

$$\frac{d}{dt}\Big(\int_\Omega\rho_0|\partial_t\partial_h^\beta u|^2+\int_\Gamma g\rho_+|\partial_h^\beta u_3|^2-\int_\Omega g\rho_0'|\partial_h^\beta u_3|^2\Big)+\int_\Omega|\mathbb{S}\partial_t\partial_h^\beta u|^2\lesssim\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).$$

Integrating in time and using Korn's inequality (3.5), we obtain

$$\begin{aligned}\|\partial_t\partial_h^\beta u(t)\|_{L^2(\Omega)}^2 &+ \int_0^t \|\nabla\partial_t\partial_h^\beta u(s)\|_{L^2(\Omega)}^2 ds\\ &\lesssim \mathcal{E}_f^2(0) + \|\partial_h^\beta u_3(t)\|_{L^2(\Omega)}^2 + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds.\end{aligned}$$

Combining the resulting inequality and (3.196), the inequality (3.197) follows. Proof of Proposition 3.13 is complete. □

3.3.4 Estimates of the perturbation density

We continue deriving the energy evolution of the space-time derivatives of ζ. Notice from $(4.2)_{1,3}$ that

$$\partial_t\zeta = K\partial_t\theta\partial_3\zeta - u_j\mathcal{A}_{jk}\partial_k\zeta - \rho_0' u_3 - \rho_0\mathcal{Q}^3 + \tilde{\mathcal{Q}}^1, \tag{3.225}$$

where

$$\tilde{\mathcal{Q}}^1 = \rho_0'' K\theta\partial_t\theta - q\mathcal{A}_{lk}\partial_k u_l - \mathcal{A}_{lk}\partial_k(\rho_0'\theta u_l) - (\mathcal{A}_{lk}-\delta_{lk})\partial_k(\rho_0 u_l). \tag{3.226}$$

We first present the estimate of $\tilde{\mathcal{Q}}^1$.

Lemma 3.14. *There holds*

$$\|\tilde{\mathcal{Q}}^1\|_{H^4(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f). \tag{3.227}$$

Proof. We use (3.1), (3.10) and Lemma 0.2 to have that

$$\begin{aligned}\|\rho_0'' K\theta\partial_t\theta\|_{H^4(\Omega)} &\lesssim (1+\|K-1\|_{H^4(\Omega)})\|\theta\|_{H^4(\Omega)}\|\partial_t\theta\|_{H^4(\Omega)}\\ &\lesssim (1+\|\eta\|_{H^{9/2}(\Gamma)})\|\eta\|_{H^{7/2}(\Gamma)}\|\partial_t\eta\|_{H^{7/2}(\Gamma)}.\end{aligned}$$

Combining (3.126) and the resulting inequality, we have

$$\|\rho_0'' K\theta\partial_t\theta\|_{H^4(\Omega)} \lesssim \mathcal{E}_f^2. \tag{3.228}$$

Using Lemma 0.2 again and (3.1), (3.12), one has

$$\begin{aligned}\|\mathcal{A}_{lk}\partial_k(\rho_0'\theta u_l)\|_{H^4(\Omega)} &\lesssim (1+\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)})\|\theta\|_{H^5(\Omega)}\|u\|_{H^5(\Omega)}\\ &\lesssim (1+\|\eta\|_{H^{9/2}(\Gamma)})\|\eta\|_{H^{9/2}(\Gamma)}\|u\|_{H^5(\Omega)}\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla u\|_{H^4(\Omega)})\end{aligned} \tag{3.229}$$

and

$$\|(\mathcal{A}_{lk}-\delta_{lk})\partial_k(\rho_0 u_l)\|_{H^4(\Omega)} \lesssim \|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)}\|u\|_{H^5(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f+\|\nabla u\|_{H^4(\Omega)}). \tag{3.230}$$

Thanks to Gagliardo-Nireberg's inequality also and (3.12), we obtain

$$\begin{aligned}\|q\mathcal{A}_{lk}\partial_k u_l\|_{H^4(\Omega)} &\lesssim (1+\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)})(\|q\|_{H^2(\Omega)}\|\nabla u\|_{H^4(\Omega)}+\|q\|_{H^4(\Omega)}\|\nabla u\|_{H^2(\Omega)})\\ &\lesssim \mathcal{E}_f(\mathcal{E}_f+\|q\|_{H^4(\Omega)}+\|\nabla u\|_{H^4(\Omega)}),\end{aligned} \tag{3.231}$$

Those above estimates, (3.228), (3.229), (3.230) and (3.231) imply

$$\|\tilde{\mathcal{Q}}^1\|_{H^4(\Omega)} \lesssim \mathcal{E}_f(\mathcal{E}_f+\|q\|_{H^4(\Omega)}+\|\nabla u\|_{H^4(\Omega)}) \lesssim \mathcal{E}_f(\mathcal{E}_f+\mathcal{D}_f).$$

Lemma 3.14 is proven. □

We derive the following proposition.

Proposition 3.14. *The following inequality holds*

$$\begin{aligned}\|\zeta(t)\|^2_{H^4(\Omega)} &\leqslant C_9\Big(\mathcal{E}_f^2(0)+\varepsilon^3\int_0^t(\|\zeta(s)\|^2_{H^4(\Omega)}+\|u_3(s)\|^2_{H^5(\Omega)})ds\Big)\\ &\quad + C_9\Big(\varepsilon^{-27}\int_0^t\|(u_3,\zeta)(s)\|^2_{L^2(\Omega)}ds+\int_0^t\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big).\end{aligned} \tag{3.232}$$

Proof. It can be seen from $(4.2)_1$ that

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|^2_{L^2(\Omega)} = -\int_\Omega \rho_0' u_3\zeta+\int_\Omega \mathcal{Q}^1\zeta \lesssim (\|u_3\|_{L^2(\Omega)}+\|\mathcal{Q}^1\|_{L^2(\Omega)})\|\zeta\|_{L^2(\Omega)}.$$

Due to (3.198), we thus have

$$\frac{d}{dt}\|\zeta\|^2_{L^2(\Omega)} \lesssim \|u_3\|_{L^2(\Omega)}\|\zeta\|_{L^2(\Omega)}+\mathcal{E}_f^3.$$

This yields

$$\begin{aligned}\|\zeta(t)\|^2_{L^2(\Omega)} &\lesssim \mathcal{E}_f^2(0)+\int_0^t\|(u_3,\zeta)(s)\|^2_{L^2(\Omega)}ds+\int_0^t\mathcal{E}_f^3(s)ds\\ &\lesssim \mathcal{E}_f^2(0)+\int_0^t(\varepsilon^3\|u_3(s)\|^2_{L^2(\Omega)}+\varepsilon^{-3}\|\zeta(s)\|^2_{L^2(\Omega)})ds+\int_0^t\mathcal{E}_f^3(s)ds\end{aligned} \tag{3.233}$$

For $\alpha \in \mathbb{N}^3, 1 \leqslant |\alpha| \leqslant 4$, we have from (3.225) that

$$\begin{aligned}\partial_t\partial^\alpha\zeta &= K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta+\sum_{0\neq\beta\leqslant\alpha}\partial^\beta(K\partial_t\theta)\partial^{\alpha-\beta}\partial_3\zeta\\ &\quad -\sum_{0\neq\beta\leqslant\alpha}\partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k\zeta+\partial^\alpha(-\rho_0'u_3-\rho_0\mathcal{Q}^3+\tilde{\mathcal{Q}}^1).\end{aligned} \tag{3.234}$$

We deduce from (3.234) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} &= \int_\Omega(K\partial_t\theta\partial_3\partial^\alpha\zeta-u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta)\partial^\alpha\zeta+\sum_{0\neq\beta\leqslant\alpha}\int_\Omega\partial^\beta(K\partial_t\theta)\partial^{\alpha-\beta}\partial_3\zeta\\ &\quad -\sum_{0\neq\beta\leqslant\alpha}\int_\Omega\partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k\zeta+\int_\Omega\partial^\alpha(-\rho_0'u_3-\rho_0\mathcal{Q}^3+\tilde{\mathcal{Q}}^1)\partial^\alpha\zeta.\end{aligned} \tag{3.235}$$

We bound each integral in the r.h.s of (3.235). Using the integration by parts, one has

$$2\int_\Omega (K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta)\partial^\alpha\zeta = \int_\Omega (K\partial_t\theta\partial_3|\partial^\alpha\zeta|^2 - u_j\mathcal{A}_{jk}\partial_k|\partial^\alpha\zeta|^2)$$
$$= \int_\Gamma (K\partial_t\theta - u_j\mathcal{A}_{j3})|\partial^\alpha\zeta|^2$$
$$- \int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2.$$

On Γ, we have $K\partial_t\theta - u_j\mathcal{A}_{j3} = 0$ by the definition of $\mathcal{A}$ (0.94) and by $(0.95)_4$. This yields

$$2\int_\Omega (K\partial_t\theta\partial_3\partial^\alpha\zeta - u_j\mathcal{A}_{jk}\partial_k\partial^\alpha\zeta)\partial^\alpha\zeta = -\int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2.$$

Due to Sobolev embedding and the product estimate (3.1), it can be seen that

$$\int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2$$
$$\lesssim \|\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk})\|_{H^2(\Omega)}\|\zeta\|^2_{H^4(\Omega)}$$
$$\lesssim ((\|K-1\|_{H^3(\Omega)} + 1)\|\theta\|_{H^3(\Omega)} + \|u\|_{H^3(\Omega)}(\|\mathcal{A} - \mathrm{Id}\|_{H^3(\Omega)} + 1))\|\zeta\|_{H^4(\Omega)}.$$

Owing to Lemma 0.2 and (3.10), (3.12), we have

$$\begin{aligned}\int_\Omega (\partial_3(K\theta) - \partial_k(u_j\mathcal{A}_{jk}))|\partial^\alpha\zeta|^2 &\lesssim (1 + \|\eta\|_{H^{7/2}(\Gamma)})(\|\eta\|_{H^{5/2}(\Gamma)} + \|u\|_{H^3(\Omega)})\|\zeta\|_{H^4(\Omega)} \\ &\lesssim \mathcal{E}_f^3.\end{aligned} \tag{3.236}$$

Let us bound the fourth integral. Thanks to Young's inequality, we have

$$\int_\Omega \partial^\alpha(\rho_0' u_3)\partial^\alpha\zeta \lesssim \|\partial^\alpha\zeta\|_{L^2(\Omega)}\|u_3\|_{H^{|\alpha|}(\Omega)} \lesssim \varepsilon^3\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} + \varepsilon^{-3}\|u_3\|^2_{H^4(\Omega)}$$

By Young's inequality again and (3.7), this yields

$$\begin{aligned}\int_\Omega \partial^\alpha(\rho_0' u_3)\partial^\alpha\zeta &\lesssim \varepsilon^3\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} + \varepsilon^{-3}(\varepsilon^6\|u_3\|^2_{H^5(\Omega)} + \varepsilon^{-24}\|u_3\|^2_{L^2(\Omega)}) \\ &\lesssim \varepsilon^3(\|\partial^\alpha\zeta\|^2_{L^2(\Omega)} + \|u_3\|^2_{H^5(\Omega)}) + \varepsilon^{-27}\|u_3\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.237}$$

Thanks to (3.199) and (3.227), we have

$$\begin{aligned}\int_\Omega \partial^\alpha(-\rho_0\mathcal{Q}^3 + \tilde{\mathcal{Q}}^1)\partial^\alpha\zeta &\lesssim (\|\mathcal{Q}^3\|_{H^4(\Omega)} + \|\tilde{\mathcal{Q}}^1\|_{H^4(\Omega)})\|\zeta\|_{H^4(\Omega)} \\ &\lesssim \mathcal{E}_f^2(\mathcal{E}_f + \mathcal{D}_f).\end{aligned} \tag{3.238}$$

We use (3.1), (3.10) and Lemma 0.2 also to bound the second integral in the r.h.s of (3.235) as follows

$$\sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta(K\partial_t\theta)\partial^{\alpha-\beta}\partial_3\zeta \lesssim \|\nabla K\|_{H^3(\Omega)}\|\partial_t\theta\|_{H^4(\Omega)}\|\zeta\|_{H^4(\Omega)}$$
$$\lesssim \|\eta\|_{H^{9/2}(\Gamma)}\|\partial_t\eta\|_{H^{7/2}(\Gamma)}\|\zeta\|_{H^4(\Omega)}.$$

By (3.126), we further obtain

$$\sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta(K\partial_t\theta)\partial^{\alpha-\beta}\partial_3\zeta \lesssim \mathcal{E}_f^3. \tag{3.239}$$

Next, for the third integral, one has

$$\begin{aligned}\sum_{0\neq\beta\leqslant\alpha}\int_\Omega \partial^\beta(u_j\mathcal{A}_{jk})\partial^{\alpha-\beta}\partial_k\zeta &\lesssim (\|\mathcal{A}-\mathrm{Id}\|_{H^4(\Omega)}+1)\|u\|_{H^4(\Omega)}\|\zeta\|_{H^4(\Omega)}\\ &\lesssim (1+\|\eta\|_{H^{9/2}(\Gamma)})\|u\|_{H^4(\Omega)}\|\zeta\|_{H^4(\Omega)}\\ &\lesssim \mathcal{E}_f^3,\end{aligned}\tag{3.240}$$

thanks to (3.12).

In view of (3.236), (3.237), (3.238), (3.239) and (3.240), we get

$$\frac{d}{dt}\|\partial^\alpha\zeta\|_{L^2(\Omega)}^2 \lesssim \varepsilon^3(\|\zeta\|_{H^4(\Omega)}^2+\|u_3\|_{H^5(\Omega)}^2)+\varepsilon^{-27}\|u_3\|_{L^2(\Omega)}^2+\mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2).$$

Integrating the resulting inequality from 0 to t, together with (3.233), one has (3.232). Proof of Proposition 3.14 is complete. □

In addition, we have the following estimate.

Proposition 3.15. *There holds*

$$\|\partial_t\zeta\|_{H^2(\Omega)}^2+\|\partial_t^2\zeta\|_{L^2(\Omega)}^2 \leqslant C_{10}(\|u_3\|_{H^2(\Omega)}^2+\|\partial_t u_3\|_{L^2(\Omega)}^2+\mathcal{E}_f^4). \tag{3.241}$$

Proof. It follows directly from $(4.2)_1$ and (3.198) that

$$\|\partial_t\zeta\|_{H^2(\Omega)}^2 \lesssim \|u_3\|_{H^2(\Omega)}^2+\|\mathcal{Q}^1\|_{H^2(\Omega)}^2 \lesssim \|u_3\|_{H^2(\Omega)}^2+\mathcal{E}_f^4 \tag{3.242}$$

and

$$\|\partial_t^2\zeta\|_{L^2(\Omega)}^2 \lesssim \|\partial_t u_3\|_{L^2(\Omega)}^2+\|\partial_t\mathcal{Q}^1\|_{L^2(\Omega)}^2 \lesssim \|\partial_t u_3\|_{L^2(\Omega)}^2+\mathcal{E}_f^4. \tag{3.243}$$

Hence, we obtain (3.241) by combining (3.242) and (3.243). □

3.3.5 Elliptic estimates

We use the elliptic estimate (3.3) to derive some inequalities.

Proposition 3.16. *There holds*

$$\begin{aligned}&\|u\|_{H^4(\Omega)}^2+\|q\|_{H^3(\Omega)}^2+\|\partial_t u\|_{H^2(\Omega)}^2+\|\partial_t q\|_{H^1(\Omega)}^2\\ &\quad\leqslant C_{11}\Big(\|\partial_t^2 u\|_{L^2(\Omega)}^2+\|u_3\|_{L^2(\Omega)}^2+\|\zeta\|_{H^2(\Omega)}^2+\|\eta\|_{H^{5/2}(\Gamma)}^2+\|\partial_t\eta\|_{H^{1/2}(\Gamma)}^2+\mathcal{E}_f^4\Big).\end{aligned}\tag{3.244}$$

Proof. We derive from (4.2) that

$$\begin{cases} -\mu\Delta\partial_t u + \nabla\partial_t q = -\rho_0\partial_t^2 u - g\partial_t\zeta e_3 + \partial_t\mathcal{Q}^2 & \text{in } \Omega, \\ \mathrm{div}\partial_t u = \partial_t\mathcal{Q}^3 & \text{in } \Omega, \\ (\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u)e_3 = g\rho_+\partial_t\eta e_3 + \partial_t\mathcal{Q}^5 & \text{on } \Gamma. \end{cases} \tag{3.245}$$

Applying the elliptic estimate (3.3) to (3.245), it tells us that

$$\begin{aligned} \|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)} \lesssim \|\partial_t^2 u\|^2_{L^2(\Omega)} + \|\partial_t\zeta\|^2_{L^2(\Omega)} + \|\partial_t\eta\|^2_{H^{1/2}(\Gamma)} \\ + \|\partial_t\mathcal{Q}^2\|^2_{L^2(\Omega)} + \|\partial_t\mathcal{Q}^3\|^2_{H^1(\Omega)} + \|\partial_t\mathcal{Q}^5\|^2_{H^{1/2}(\Gamma)}. \end{aligned}$$

Note that from $(4.2)_1$ that ,

$$\|\partial_t\zeta\|^2_{L^2(\Omega)} \lesssim \|u_3\|^2_{L^2(\Omega)} + \|\mathcal{Q}^1\|^2_{L^2(\Omega)}.$$

Hence, we have

$$\begin{aligned} \|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)} \lesssim \|\partial_t^2 u\|^2_{L^2(\Omega)} + \|\partial_t\eta\|^2_{H^{1/2}(\Gamma)} + \|u_3\|^2_{L^2(\Omega)} + \|\mathcal{Q}^1\|^2_{L^2(\Omega)} \\ + \|\partial_t\mathcal{Q}^2\|^2_{L^2(\Omega)} + \|\partial_t\mathcal{Q}^3\|^2_{H^1(\Omega)} + \|\partial_t\mathcal{Q}^5\|^2_{H^{1/2}(\Gamma)}. \end{aligned}$$

Due to (3.198), this yields

$$\|\partial_t u\|^2_{H^2(\Omega)} + \|\partial_t q\|^2_{H^1(\Omega)} \lesssim \|\partial_t^2 u\|^2_{L^2(\Omega)} + \|u_3\|^2_{L^2(\Omega)} + \|\partial_t\eta\|^2_{H^{1/2}(\Gamma)} + \mathcal{E}_f^4. \tag{3.246}$$

Meanwhile, we obtain from (4.2) that

$$\begin{cases} -\Delta u + \nabla q = -\rho_0\partial_t u - g\zeta e_3 + \mathcal{Q}^2 & \text{in } \Omega, \\ \mathrm{div} u = \mathcal{Q}^3 & \text{in } \Omega, \\ (q\mathrm{Id} - \mu\mathbb{S}u)e_3 = g\rho_+\eta e_3 + \mathcal{Q}^5 & \text{on } \Gamma. \end{cases} \tag{3.247}$$

Owing to (3.198) and by applying the elliptic estimate (3.3) again to (3.247), we observe that

$$\begin{aligned} \|u\|^2_{H^4(\Omega)} + \|q\|^2_{H^3(\Omega)} &\lesssim \|\partial_t u\|^2_{H^2(\Omega)} + \|\zeta\|^2_{H^2(\Omega)} + \|\mathcal{Q}^2\|^2_{H^2(\Omega)} + \|\mathcal{Q}^3\|^2_{H^3(\Omega)} \\ &\quad + \|\eta\|^2_{H^{5/2}(\Gamma)} + \|\mathcal{Q}^5\|^2_{H^{5/2}(\Gamma)} \\ &\lesssim \|\partial_t u\|^2_{H^2(\Omega)} + \|\zeta\|^2_{H^2(\Omega)} + \|\eta\|^2_{H^{5/2}(\Gamma)} + \mathcal{E}_f^4. \end{aligned} \tag{3.248}$$

Combining (3.246) and (3.248), one has (3.244). Proof of Proposition 3.16 is complete. □

Let us define the "horizontal" dissipation $\mathcal{D}_h > 0$ as follows,

$$\mathcal{D}_h^2 := \sum_{\beta\in\mathbb{N}^2, |\beta|\leqslant 4} \|\nabla\partial_h^\beta u\|^2_{L^2(\Omega)} + \sum_{\beta\in\mathbb{N}^2, |\beta|\leqslant 2} \|\nabla\partial_h^\beta\partial_t u\|^2_{L^2(\Omega)} + \|\nabla\partial_t^2 u\|^2_{L^2(\Omega)}. \tag{3.249}$$

The next proposition is to compare $\mathcal{D}_f$ (0.124) and $\mathcal{D}_h$ (3.249).

Proposition 3.17. *Assuming δ_0 sufficiently small (see (3.279)), there holds*

$$\mathcal{D}_f^2 \leqslant C_{12}\Big(\mathcal{D}_h^2 + \varepsilon^3\mathcal{E}_f^2 + \varepsilon^{-9}(\|(\zeta,u)\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Gamma)}) + \mathcal{E}_f^3\Big). \tag{3.250}$$

To prove Proposition 3.17, we use the two lemmas below.

Lemma 3.15. *For any $s \geqslant 0$, there holds*

$$\|f\|^2_{H^{s+1/2}(\Gamma)} \lesssim \|f\|^2_{H^{1/2}(\Gamma)} + \sum_{\beta\in\mathbb{N}^2,|\beta|=s} \|\partial_h^\beta f\|^2_{H^{1/2}(\Gamma)}. \tag{3.251}$$

Proof. Since $\Gamma = 2\pi L_1\mathbf{Z} \times 2\pi L_2\mathbf{Z} \times \{0\}$, we exploit the definition of the Sobolev norm on Γ to have that

$$\|f\|^2_{H^{s+1/2}(\Gamma)} = \sum_{n\in\mathbf{Z}^2} (1+|n|^2)^{s+1/2}|\hat{f}(n)|^2,$$

where $\hat{f}$ is the Fourier series of f. By Cauchy-Schwarz's inequality, one has

$$\|f\|^2_{H^{s+1/2}(\Gamma)} \lesssim \sum_{n\in\mathbf{Z}^2} (1+|n|^2)^{1/2}|\hat{f}(n)|^2 + \sum_{\beta\in\mathbb{N}^2,|\beta|=s}\ \sum_{n\in\mathbf{Z}^2} (1+|n|^2)^{1/2}|n^\beta\hat{f}(n)|^2,$$

which immediately yields (3.251). □

Lemma 3.16. *Let us write*

$$\mathcal{W} := \sqrt{\|\zeta\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} + \|\eta\|^2_{L^2(\Gamma)} + \mathcal{E}_f^3}.$$

The following inequalities holds

$$\|\partial_t u\|_{L^2(\Omega)} \lesssim \|\nabla\partial_t u\|_{L^2(\Omega)} + \mathcal{W}, \tag{3.252}$$

and

$$\|\partial_t^2 u\|_{L^2(\Omega)} \lesssim \|\nabla\partial_t u\|_{L^2(\Omega)} + \|\nabla\partial_t^2 u\|_{L^2(\Omega)} + \mathcal{W}. \tag{3.253}$$

Proof. Let us show (3.252) first. Multiplying by $\partial_t u$ on both sides of $(4.2)_2$, we obtain

$$\begin{aligned}\int_\Omega \rho_0|\partial_t u|^2 &= -\int_\Omega \nabla q\cdot\partial_t u + \mu\int_\Omega \Delta u\cdot\partial_t u - \int_\Omega g\zeta\partial_t u_3 + \int_\Omega \mathcal{Q}^2\cdot\partial_t u \\ &= -\int_\Gamma (q\mathrm{Id} - \mu\mathbb{S}u)e_3\cdot\partial_t u + \int_\Omega q\mathrm{div}\partial_t u - \frac{\mu}{2}\int_\Omega \mathbb{S}u : \mathbb{S}\partial_t u \\ &\quad - \int_\Omega g\zeta\partial_t u_3 + \int_\Omega \mathcal{Q}^2\cdot\partial_t u,\end{aligned}$$

after using the integration by parts. Using $(4.2)_{3,5}$, this yields

$$\begin{aligned}\int_\Omega \rho_0|\partial_t u|^2 = &-\int_\Gamma g\rho_+\eta\partial_t u_3 - \int_\Gamma \mathcal{Q}^5\cdot\partial_t u + \int_\Omega q\partial_t\mathcal{Q}^3 + \mu\int_\Omega \mathbb{S}u : \mathbb{S}\partial_t u \\ &- \int_\Omega g\zeta\partial_t u_3 + \int_\Omega \mathcal{Q}^2\cdot\partial_t u.\end{aligned} \tag{3.254}$$

By Young's inequality, we have

$$\frac{\mu}{2}\int_\Omega \mathbb{S}u : \mathbb{S}\partial_t u - \int_\Omega g\zeta\partial_t u_3 \lesssim \|u\|_{H^1(\Omega)}\|\partial_t u\|_{H^1(\Omega)} + \|\zeta\|_{L^2(\Omega)}\|\partial_t u_3\|_{L^2(\Omega)}. \tag{3.255}$$

Using also the trace theorem, we have

$$\int_\Gamma g\rho_+\eta\partial_t u_3 \lesssim \|\eta\|_{L^2(\Gamma)}\|\partial_t u\|_{H^1(\Omega)}. \tag{3.256}$$

Because of (3.198) and the trace theorem again, one has

$$\begin{aligned}\int_\Gamma \mathcal{Q}^5\cdot\partial_t u + \int_\Omega q\partial_t\mathcal{Q}^3 + \int_\Omega \mathcal{Q}^2\cdot\partial_t u &\lesssim (\|\mathcal{Q}^5\|_{L^2(\Gamma)} + \|\mathcal{Q}^2\|_{L^2(\Omega)})\|\partial_t u\|_{H^1(\Omega)}\\ &\quad + \|\partial_t\mathcal{Q}^3\|_{L^2(\Omega)}\|q\|_{L^2(\Omega)}\\ &\lesssim \mathcal{E}_f^3.\end{aligned} \tag{3.257}$$

Combining (3.255), (3.256) and (3.257), we obtain from (3.254) that

$$\begin{aligned}\|\partial_t u\|^2_{L^2(\Omega)} &\lesssim \|\partial_t u\|_{L^2(\Omega)}(\|\eta\|_{L^2(\Gamma)} + \|u\|_{H^1(\Omega)} + \|\zeta\|_{L^2(\Omega)})\\ &\quad + \|\nabla\partial_t u\|_{L^2(\Omega)}(\|\eta\|_{L^2(\Gamma)} + \|u\|_{H^1(\Omega)}) + \mathcal{E}_f^3.\end{aligned}$$

Using Young's inequality, we further get for any $\nu > 0$,

$$\|\partial_t u\|^2_{L^2(\Omega)} \lesssim \nu\|\partial_t u\|^2_{L^2(\Omega)} + \|\nabla\partial_t u\|^2_{L^2(\Omega)} + (1+\nu^{-1})\mathcal{W}. \tag{3.258}$$

Let $\nu > 0$ be sufficiently small, the inequality (3.252) follows from (3.258).

To prove (3.253), we differentiate $(4.2)_{2,5}$ with respect to t and then eliminate the terms $\partial_t\zeta, \partial_t\eta$ by using $(4.2)_{1,4}$ to deduce that

$$\begin{cases}\rho_0\partial_t^2 u + \nabla\partial_t q - \mu\Delta\partial_t u - g\rho_0' u_3 e_3 = \partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3 & \text{in } \Omega,\\ \mathrm{div}\partial_t u = \partial_t\mathcal{Q}^3 & \text{in } \Omega,\\ (\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u)e_3 = g\rho_+ u e_3 + g\rho_+\mathcal{Q}^4 e_3 + \partial_t\mathcal{Q}^5 & \text{on } \Gamma.\end{cases} \tag{3.259}$$

Multiplying both sides of $(3.259)_1$ by $\partial_t^2 u$, we obtain that

$$\int_\Omega \rho_0|\partial_t^2 u|^2 + \int_\Omega(\nabla\partial_t q - \mu\Delta\partial_t u)\cdot\partial_t^2 u - \int_\Omega g\rho_0' u_3\partial_t^2 u_3 = \int_\Omega(\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u. \tag{3.260}$$

Using the integration by parts, we have that

$$\begin{aligned}\int_\Omega \rho_0|\partial_t^2 u|^2 = &-\int_\Gamma(\partial_t q\mathrm{Id} - \mu\mathbb{S}\partial_t u)e_3\cdot\partial_t u + \int_\Omega \partial_t q\mathrm{div}\partial_t u - \frac{\mu}{2}\int_\Omega \mathbb{S}\partial_t u : \mathbb{S}\partial_t^2 u\\ &-\int_\Omega g\rho_0' u_3\partial_t^2 u_3 + \int_\Omega(\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u.\end{aligned}$$

Substituting (3.259)$_{2,3}$ into the resulting equality yields

$$\begin{aligned}\int_\Omega \rho_0|\partial_t^2 u|^2 = &-\int_\Gamma g\rho_+ u_3\partial_t^2 u_3 - \int_\Gamma (g\rho_+\mathcal{Q}^4 e_3 + \partial_t\mathcal{Q}^5)\cdot\partial_t^2 u + \int_\Omega \partial_t q\partial_t\mathcal{Q}^3 \\ &-\frac{\mu}{2}\int_\Omega \mathbb{S}\partial_t u : \mathbb{S}\partial_t^2 u - \int_\Omega g\rho_0' u_3\partial_t^2 u_3 + \int_\Omega (\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u.\end{aligned} \tag{3.261}$$

We estimate each integral in the r.h.s of (3.261). For the first integral, we use the trace theorem to have

$$-\int_\Gamma g\rho_+ u_3\partial_t^2 u_3 \lesssim \|u_3\|_{H^1(\Omega)}\|\partial_t^2 u_3\|_{H^1(\Omega)}. \tag{3.262}$$

For the fourth and fifth integral, we bound as

$$-\frac{\mu}{2}\int_\Omega \mathbb{S}\partial_t u : \mathbb{S}\partial_t^2 u - \int_\Omega g\rho_0' u_3\partial_t^2 u_3 \lesssim \|\partial_t u\|_{H^1(\Omega)}\|\partial_t^2 u\|_{H^1(\Omega)} + \|u_3\|_{L^2(\Omega)}\|\partial_t^2 u_3\|_{L^2(\Omega)}. \tag{3.263}$$

For the other integrals, we use (3.198) to obtain

$$\begin{aligned}&\int_\Omega (\partial_t\mathcal{Q}^2 - g\mathcal{Q}^1 e_3)\cdot\partial_t^2 u + \int_\Omega \partial_t q\partial_t\mathcal{Q}^3 \\ &\lesssim (\|\partial_t\mathcal{Q}^2\|_{L^2(\Omega)} + \|\mathcal{Q}^1\|_{L^2(\Omega)})\|\partial_t^2 u\|_{L^2(\Omega)} + \|\partial_t q\|_{L^2(\Omega)}\|\partial_t\mathcal{Q}^3\|_{L^2(\Omega)} \\ &\lesssim \mathcal{E}_f^3,\end{aligned} \tag{3.264}$$

and use the trace theorem also to obtain

$$\begin{aligned}\int_\Gamma (g\rho_+\mathcal{Q}^4 e_3 + \partial_t\mathcal{Q}^5)\cdot\partial_t^2 u &\lesssim (\|\mathcal{Q}^4\|_{L^2(\Gamma)} + \|\partial_t\mathcal{Q}^5\|_{L^2(\Gamma)})\|\partial_t^2 u\|_{H^1(\Omega)} \\ &\lesssim \mathcal{E}_f^2\|\partial_t^2 u\|_{H^1(\Omega)}.\end{aligned} \tag{3.265}$$

Substituting (3.262), (3.263), (3.264) and (3.265) into (3.261) yields

$$\begin{aligned}\|\partial_t^2 u\|_{L^2(\Omega)}^2 &\lesssim (\|u\|_{H^1(\Omega)} + \|\partial_t u\|_{H^1(\Omega)} + \mathcal{E}_f^2)\|\partial_t^2 u\|_{H^1(\Omega)} + \mathcal{E}_f^3 \\ &\lesssim (\|u\|_{H^1(\Omega)} + \|\partial_t u\|_{H^1(\Omega)} + \mathcal{E}_f^2)(\|\nabla\partial_t^2 u\|_{L^2(\Omega)} + \mathcal{E}_f).\end{aligned} \tag{3.266}$$

Combining (3.266) and (3.252) gives us that

$$\|\partial_t^2 u\|_{L^2(\Omega)}^2 \lesssim (\|\nabla\partial_t u\|_{L^2(\Omega)} + \mathcal{W})(\|\nabla\partial_t^2 u\|_{L^2(\Omega)} + \mathcal{W}). \tag{3.267}$$

Thanks to Cauchy-Schwarz's inequality, the resulting inequality (3.267) implies (3.253). Lemma 3.16 is shown. □

We are able to show Proposition 3.17.

Proof of Proposition 3.17. We apply the elliptic estimate (3.4) to

$$\begin{cases} -\mu\Delta\partial_t u + \nabla\partial_t q = -\rho_0\partial_t^2 u + g(\rho_0' u_3 - \mathcal{Q}^1) + \partial_t\mathcal{Q}^2 & \text{in } \Omega, \\ \text{div}\partial_t u = \partial_t\mathcal{Q}^3 & \text{in } \Omega, \\ \partial_t u = \partial_t u & \text{on } \Gamma, \end{cases}$$

to have that

$$\|\partial_t u\|_{H^3(\Omega)}^2 + \|\partial_t q\|_{H^2(\Omega)}^2 \lesssim \|\partial_t^2 u\|_{H^1(\Omega)}^2 + \|u_3\|_{H^1(\Omega)}^2 + \|(\mathcal{Q}^1, \partial_t \mathcal{Q}^2)\|_{H^1(\Omega)}^2 + \|\partial_t \mathcal{Q}^3\|_{H^2(\Omega)}^2 + \|\partial_t u\|_{H^{5/2}(\Gamma)}^2.$$

This yields

$$\|\partial_t u\|_{H^3(\Omega)}^2 + \|\partial_t q\|_{H^2(\Omega)}^2 \lesssim \|\partial_t^2 u\|_{H^1(\Omega)}^2 + \|u_3\|_{H^1(\Omega)}^2 + \|\partial_t u\|_{H^{5/2}(\Gamma)}^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2), \tag{3.268}$$

due to (3.199) also. It follows from (3.251) and the trace theorem that

$$\begin{aligned} \|\partial_t u\|_{H^{5/2}(\Gamma)}^2 &\lesssim \|\partial_t u\|_{H^{1/2}(\Gamma)}^2 + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\partial_h^\beta \partial_t u\|_{H^{1/2}(\Gamma)}^2 \\ &\lesssim \|\partial_t u\|_{H^1(\Omega)}^2 + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\partial_h^\beta \partial_t u\|_{H^1(\Omega)}^2. \end{aligned} \tag{3.269}$$

Combining (3.268) and (3.269) gives us that

$$\begin{aligned} \|\partial_t u\|_{H^3(\Omega)}^2 + \|\partial_t q\|_{H^2(\Omega)}^2 \lesssim \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\partial_h^\beta \partial_t u\|_{H^1(\Omega)}^2 \\ + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \end{aligned} \tag{3.270}$$

Thanks to the interpolation inequality (3.7), we get that, for $\nu > 0$,

$$\|\partial_h^\beta \partial_t u\|_{L^2(\Omega)}^2 \lesssim \|\partial_t u\|_{H^2(\Omega)}^2 \lesssim \nu \|\partial_t u\|_{H^3(\Omega)}^2 + \nu^{-2} \|\partial_t u\|_{L^2(\Omega)}^2.$$

Hence, it follows from (3.270) that

$$\begin{aligned} \|\partial_t u\|_{H^3(\Omega)}^2 + \|\partial_t q\|_{H^2(\Omega)}^2 &\lesssim \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \nu \|\partial_t u\|_{H^3(\Omega)}^2 \\ &\quad + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\nabla \partial_h^\beta \partial_t u\|_{L^2(\Omega)}^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \end{aligned}$$

Let $\nu > 0$ be sufficiently small, one has

$$\begin{aligned} \|\partial_t u\|_{H^3(\Omega)}^2 + \|\partial_t q\|_{H^2(\Omega)}^2 \lesssim \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \sum_{\beta \in \mathbb{N}^2, |\beta|=2} \|\nabla \partial_h^\beta \partial_t u\|_{L^2(\Omega)}^2 \\ + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \end{aligned} \tag{3.271}$$

Meanwhile, applying the elliptic estimate (3.4) to

$$\begin{cases} -\mu \Delta u + \nabla q = -\rho_0 \partial_t u - g\zeta e_3 + \mathcal{Q}^2 & \text{in } \Omega, \\ \mathrm{div} u = \mathcal{Q}^3 & \text{in } \Omega, \\ u = u & \text{on } \Gamma, \end{cases}$$

we have

$$\|u\|_{H^5(\Omega)}^2 + \|q\|_{H^4(\Omega)}^2 \lesssim \|\partial_t u\|_{H^3(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \|\mathcal{Q}^2\|_{H^3(\Omega)}^2 + \|\mathcal{Q}^3\|_{H^4(\Omega)}^2 + \|u\|_{H^{9/2}(\Gamma)}^2.$$

Using (3.271) and (3.199), we further obtain

$$\|u\|_{H^5(\Omega)}^2 + \|q\|_{H^4(\Omega)}^2 \lesssim \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \|u\|_{H^{9/2}(\Gamma)}^2 \\ + \sum_{\beta\in\mathbb{N}^2, |\beta|=2} \|\nabla \partial_h^\beta \partial_t u\|_{L^2(\Omega)}^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{3.272}$$

Using (3.251) again and the trace theorem, we obtain that

$$\begin{aligned} \|u\|_{H^{9/2}(\Gamma)}^2 &\lesssim \|u\|_{H^{1/2}(\Gamma)}^2 + \sum_{\beta\in\mathbb{N}^2, |\beta|=4} \|\partial_h^\beta u\|_{H^{1/2}(\Gamma)}^2 \\ &\lesssim \|u\|_{H^1(\Omega)}^2 + \sum_{\beta\in\mathbb{N}^2, |\beta|=4} \|\partial_h^\beta u\|_{H^1(\Omega)}^2. \end{aligned} \tag{3.273}$$

Notice from (3.7) again that, for $\nu > 0$,

$$\|\partial_h^\beta u\|_{L^2(\Omega)}^2 \lesssim \|u\|_{H^4(\Omega)}^2 \lesssim \nu \|u\|_{H^5(\Omega)}^2 + \nu^{-4} \|u\|_{L^2(\Omega)}^2. \tag{3.274}$$

In view of (3.273) and (3.274), we deduce from (3.272) that

$$\|u\|_{H^5(\Omega)}^2 + \|q\|_{H^4(\Omega)}^2 \lesssim \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \nu \|u\|_{H^5(\Omega)}^2 \\ + \mathcal{D}_h^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{3.275}$$

Let $\nu > 0$ be sufficiently small, the inequality (3.275) implies that

$$\|u\|_{H^5(\Omega)}^2 + \|q\|_{H^4(\Omega)}^2 \lesssim \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \mathcal{D}_h^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{3.276}$$

We obtain from (3.271) and (3.276) that

$$\|u\|_{H^5(\Omega)}^2 + \|\partial_t u\|_{H^3(\Omega)}^2 + \|q\|_{H^4(\Omega)}^2 + \|\partial_t q\|_{H^2(\Omega)}^2 \\ \lesssim \mathcal{D}_h^2 + \|(u, \partial_t u, \partial_t^2 u)\|_{H^1(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2).$$

That implies

$$\mathcal{D}_f^2 \lesssim \mathcal{D}_h^2 + \|(u, \partial_t u, \partial_t^2 u)\|_{L^2(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \mathcal{E}_f^2(\mathcal{E}_f^2 + \mathcal{D}_f^2). \tag{3.277}$$

Thanks to Lemma 3.16, we deduce from (3.277) that

$$\mathcal{D}_f^2 \lesssim \mathcal{D}_h^2 + \|u\|_{H^1(\Omega)}^2 + \|\zeta\|_{H^3(\Omega)}^2 + \|\eta\|_{L^2(\Gamma)}^2 + \mathcal{E}_f^3 + \mathcal{E}_f^2 \mathcal{D}_f^2.$$

We continue using (3.7) to further have

$$\mathcal{D}_f^2 \lesssim \mathcal{D}_h^2 + \varepsilon^3 (\|u\|_{H^2(\Omega)}^2 + \|\zeta\|_{H^4(\Omega)}^2) + \varepsilon^{-3} \|u\|_{L^2(\Omega)}^2 + \varepsilon^{-9} \|\zeta\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Gamma)}^2 \\ + \mathcal{E}_f^3 + \mathcal{E}_f^2 \mathcal{D}_f^2.$$

That means

$$\mathcal{D}_f^2 \leqslant C_{13} \Big(\mathcal{D}_h^2 + \varepsilon^3 \mathcal{E}_f^2 + \varepsilon^{-9} (\|(\zeta, u)\|_{L^2(\Omega)}^2 + \|\eta\|_{L^2(\Gamma)}^2) + \mathcal{E}_f^3 + \mathcal{E}_f^2 \mathcal{D}_f^2 \Big). \tag{3.278}$$

Restricting further

$$C_{13} \delta_0^2 \leqslant \frac{1}{2}, \tag{3.279}$$

we obtain (3.250) from (3.278). Proposition 3.17 is proven. □

3.3.6 Proof of Proposition 0.3

Let us denote

$$\begin{cases} C_{1,\varepsilon} = (C_1 + C_2 + C_3 + C_4)\varepsilon^3 + C_5 + C_6 + C_7 + C_8 + C_9, \\ C_{14} = \sum_{i=1}^{9} C_i, \\ C_{2,\varepsilon} = (C_1 + C_2 + C_3 + C_4)\varepsilon + (C_7 + C_8 + C_9)\varepsilon^3, \\ C_{3,\varepsilon} = C_5 + C_6 + (C_7 + C_8 + C_9)\varepsilon^{-27}. \end{cases}$$

We obtain from Propositions 3.10, 3.11, 3.12, 3.13, 3.14 that

$$\begin{aligned} &\varepsilon^2\Big(\|\eta(t)\|_{H^4(\Gamma)}^2 + \|\partial_t\eta(t)\|_{H^2(\Gamma)}^2 + \|\partial_t^2\eta(t)\|_{L^2(\Gamma)}^2 + \|\eta(t)\|_{H^{9/2}(\Gamma)}^2\Big) \\ &\quad + \|\zeta(t)\|_{H^4(\Omega)}^2 + \|(u, \partial_t u, \partial_t^2 u)(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla(u, \partial_t u, \partial_t^2 u)(s)\|_{L^2(\Omega)}^2 ds \\ &\quad + \sum_{\beta\in\mathbb{N}^2, 1\leqslant|\beta|\leqslant 4} \int_0^t \|\nabla\partial_h^\beta u(s)\|_{L^2(\Omega)}^2 ds + \sum_{\beta\in\mathbb{N}^2, 1\leqslant|\beta|\leqslant 2} \int_0^t \|\nabla\partial_t\partial_h^\beta u(s)\|_{L^2(\Omega)}^2 ds \\ &\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0) + C_6\mathcal{E}_f^3(t) + C_{14}\varepsilon^3 \int_0^t \mathcal{E}_f^2(s)ds + C_{2,\varepsilon}\int_0^t \mathcal{D}_f^2(s)ds \\ &\quad + C_{3,\varepsilon}\int_0^t (\|(u,\zeta)(s)\|_{L^2(\Omega)}^2 + \|\eta(s)\|_{L^2(\Gamma)}^2)ds + C_{1,\varepsilon}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds. \end{aligned} \tag{3.280}$$

Keep in mind the definition of $\mathcal{D}_h$ (3.249). It follows from (3.280) that

$$\begin{aligned} &\varepsilon^2\Big(\|\eta(t)\|_{H^4(\Gamma)}^2 + \|\partial_t\eta(t)\|_{H^2(\Gamma)}^2 + \|\partial_t^2\eta(t)\|_{L^2(\Gamma)}^2 + \|\eta(t)\|_{H^{9/2}(\Gamma)}^2\Big) \\ &\quad + \|\zeta(t)\|_{H^4(\Omega)}^2 + \|(u, \partial_t u, \partial_t^2 u)(t)\|_{L^2(\Omega)}^2 + \int_0^t \mathcal{D}_h^2(s)ds \\ &\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0) + C_6\mathcal{E}_f^3(t) + C_{14}\varepsilon^3 \int_0^t \mathcal{E}_f^2(s)ds + C_{2,\varepsilon}\int_0^t \mathcal{D}_f^2(s)ds \\ &\quad + C_{3,\varepsilon}\int_0^t (\|(u,\zeta)(s)\|_{L^2(\Omega)}^2 + \|\eta(s)\|_{L^2(\Gamma)}^2)ds + C_{1,\varepsilon}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds. \end{aligned} \tag{3.281}$$

Chaining (3.281) with (3.250) in Proposition 3.17, we get that

$$\begin{aligned} &\varepsilon^2\Big(\|\eta(t)\|_{H^4(\Gamma)}^2 + \|\partial_t\eta(t)\|_{H^2(\Gamma)}^2 + \|\partial_t^2\eta(t)\|_{L^2(\Gamma)}^2 + \|\eta(t)\|_{H^{9/2}(\Gamma)}^2 + \|\zeta(t)\|_{H^4(\Omega)}^2\Big) \\ &\quad + \|(u, \partial_t u, \partial_t^2 u)(t)\|_{L^2(\Omega)}^2 + \frac{1}{C_{12}}\int_0^t \mathcal{D}_f^2(s)ds \\ &\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0) + C_6\mathcal{E}_f^3(t) + (C_{14}+1)\varepsilon^3 \int_0^t \mathcal{E}_f^2(s)ds + C_{2,\varepsilon}\int_0^t \mathcal{D}_f^2(s)ds \\ &\quad + C_{4,\varepsilon}\int_0^t (\|(u,\zeta)(s)\|_{L^2(\Omega)}^2 + \|\eta(s)\|_{L^2(\Gamma)}^2)ds + (C_{1,\varepsilon}+1)\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds, \end{aligned} \tag{3.282}$$

where $C_{4,\varepsilon} = C_{3,\varepsilon} + \varepsilon^{-9}$. Let $0 < \varepsilon \leqslant 1$ be sufficiently small such that

$$C_{2,\varepsilon} \leqslant \frac{1}{2C_{12}}.$$

So that, the inequality (3.282) implies

$$\begin{aligned}
&\varepsilon^2\Big(\|\eta(t)\|^2_{H^4(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} + \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\zeta(t)\|^2_{H^4(\Omega)}\Big)\\
&\quad + \|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)} + \frac{1}{2C_{12}}\int_0^t \mathcal{D}_f^2(s)ds\\
&\leqslant C_{1,\varepsilon}\mathcal{E}_f^2(0) + (C_{14}+1)\varepsilon^3\int_0^t \mathcal{E}_f^2(s)ds + (C_{1,\varepsilon}+1)\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\\
&\quad + C_{4,\varepsilon}\int_0^t (\|(u,\zeta)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_6\mathcal{E}_f^3(t).
\end{aligned} \tag{3.283}$$

By dividing both sides of (3.283) by ε^2, we have

$$\begin{aligned}
&\|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\eta(t)\|^2_{H^4(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}\\
&\qquad + \|\zeta(t)\|^2_{H^4(\Omega)} + \|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds\\
&\leqslant C_{15}\Big(\varepsilon^{-2}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\
&\qquad + C_{15}\varepsilon^{-29}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{15}\varepsilon^{-2}\mathcal{E}_f^3(t).
\end{aligned} \tag{3.284}$$

Combining (3.244) and (3.284), one has

$$\begin{aligned}
&\varepsilon^{1/4}\Big(\|u(t)\|^2_{H^4(\Omega)} + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t u(t)\|^2_{H^2(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)}\Big)\\
&\quad + \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\eta(t)\|^2_{H^4(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)}\\
&\quad + \|\zeta(t)\|^2_{H^4(\Omega)} + \|(u,\partial_t u,\partial_t^2 u)(t)\|^2_{L^2(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds\\
&\leqslant C_{11}\varepsilon^{1/4}\Big(\|(\partial_t^2 u,u_3)(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{H^2(\Omega)} + \|\eta(t)\|^2_{H^{5/2}(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^{1/2}(\Gamma)} + \mathcal{E}_f^4(t)\Big)\\
&\quad + C_{15}\Big(\varepsilon^{-2}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2+\mathcal{D}_f^2)(s)ds\Big)\\
&\quad + C_{15}\varepsilon^{-29}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{15}\varepsilon^{-2}\mathcal{E}_f^3(t).
\end{aligned} \tag{3.285}$$

Let us refine ε so that

$$C_{11}\varepsilon^{1/4} \leqslant \frac{1}{2},$$

it follows from (3.285) that

$$\begin{aligned}
&\varepsilon^{1/4}\Big(\|u(t)\|^2_{H^4(\Omega)} + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t u(t)\|^2_{H^2(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)}\Big)\\
&\quad + \frac{1}{2}\Big(\|(\partial_t^2 u, u)(t)\|^2_{L^2(\Omega)} + \|\zeta(t)\|^2_{H^4(\Omega)} + \|\eta(t)\|^2_{H^{9/2}(\Gamma)} + \|\partial_t\eta(t)\|^2_{H^2(\Gamma)}\Big)\\
&\quad + \|\eta\|^2_{H^4(\Gamma)} + \|\partial_t^2\eta(t)\|^2_{L^2(\Gamma)} + \int_0^t \mathcal{D}_f^2(s)ds\\
&\leqslant C_{15}\Big(\varepsilon^{-2}\mathcal{E}_f^2(0) + \varepsilon\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds\Big)\\
&\quad + C_{15}\varepsilon^{-29}\int_0^t (\|(\zeta, u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{15}\varepsilon^{-2}\mathcal{E}_f^3(t) + C_{11}\varepsilon^{1/4}\mathcal{E}_f^4(t).
\end{aligned} \tag{3.286}$$

Dividing both sides of (3.286) by $\varepsilon^{1/4}$, one has

$$\begin{aligned}
&\sum_{j=0}^{2}(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)} + \|\partial_t^j\eta(t)\|^2_{H^{4-2j}(\Gamma)}) + \|\zeta(t)\|^2_{H^4(\Omega)} + \|\eta\|^2_{H^{9/2}(\Gamma)}\\
&\qquad + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds\\
&\leqslant C_{16}\Big(\varepsilon^{-9/4}\mathcal{E}_f^2(0) + \varepsilon^{3/4}\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-9/4}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds\Big)\\
&\qquad + C_{16}\varepsilon^{-117/4}\int_0^t (\|(\zeta, u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{16}\varepsilon^{-9/4}\mathcal{E}_f^3(t).
\end{aligned} \tag{3.287}$$

Combining (3.287) and (3.241) in Proposition 3.15, we obtain

$$\begin{aligned}
&\sum_{j=0}^{2}(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)} + \|\partial_t^j\eta(t)\|^2_{H^{4-2j}(\Gamma)}) + \|\zeta(t)\|^2_{H^4(\Omega)} + \|\eta\|^2_{H^{9/2}(\Gamma)}\\
&\qquad + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds + \varepsilon^{1/4}(\|\partial_t\zeta\|^2_{H^2(\Omega)} + \|\partial_t^2\zeta\|^2_{L^2(\Omega)})\\
&\leqslant C_{16}\Big(\varepsilon^{-9/4}\mathcal{E}_f^2(0) + \varepsilon^{3/4}\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-9/4}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds\Big)\\
&\qquad + C_{16}\varepsilon^{-117/4}\int_0^t (\|(\zeta, u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds\\
&\qquad + C_{16}\varepsilon^{-9/4}\mathcal{E}_f^3(t) + C_{10}\varepsilon^{1/4}(\|u_3\|^2_{H^2(\Omega)} + \|\partial_t u_3\|^2_{L^2(\Omega)} + \mathcal{E}_f^4(t)).
\end{aligned} \tag{3.288}$$

We continue refining ε so that

$$C_{10}\varepsilon^{1/4} \leqslant \frac{1}{2}.$$

It follows from (3.288) that

$$\begin{aligned}
&\sum_{j=0}^{2}(\|\partial_t^j u(t)\|^2_{H^{4-2j}(\Omega)} + \|\partial_t^j \eta(t)\|^2_{H^{4-2j}(\Gamma)}) + \|\zeta(t)\|^2_{H^4(\Omega)} + \|\eta\|^2_{H^{9/2}(\Gamma)} \\
&\qquad + \|q(t)\|^2_{H^3(\Omega)} + \|\partial_t q(t)\|^2_{H^1(\Omega)} + \int_0^t \mathcal{D}_f^2(s)ds + \varepsilon^{1/4}(\|\partial_t\zeta\|^2_{H^2(\Omega)} + \|\partial_t^2\zeta\|^2_{L^2(\Omega)}) \\
&\leqslant C_{17}\Big(\varepsilon^{-9/4}\mathcal{E}_f^2(0) + \varepsilon^{3/4}\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-9/4}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds\Big) \\
&\qquad + C_{17}\varepsilon^{-117/4}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{17}(\varepsilon^{-9/4}\mathcal{E}_f^3(t) + \varepsilon^{1/4}\mathcal{E}_f^4(t)).
\end{aligned} \tag{3.289}$$

Let us recall the definition of $\mathcal{E}_f$ (0.123) and divide both sides of (3.289) by $\varepsilon^{1/4}$ to deduce

$$\begin{aligned}
&\mathcal{E}_f^2(t) + \int_0^t \mathcal{D}_f^2(s)ds \\
&\leqslant C_{18}\Big(\varepsilon^{-5/2}\mathcal{E}_f^2(0) + \varepsilon^{1/2}\int_0^t \mathcal{E}_f^2(s)ds + \varepsilon^{-5/2}\int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds\Big) \\
&\qquad + C_{18}\varepsilon^{-59/2}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds + C_{18}\varepsilon^{-5/2}\mathcal{E}_f^3(t).
\end{aligned} \tag{3.290}$$

Switching $\varepsilon^{1/2}$ by ε in (3.287), one has (3.110). Proof of Proposition 0.3 is finished.

3.4 Nonlinear instability

Thanks to Proposition 3.7, we will consider a sequence of approximate solutions $e^{\lambda_n(\mathbf{k})}V_n(\mathbf{k},x)$ to the nonlinear equations (0.99), which are solutions to the linearized ones (0.101). Let us fix a $\mathbf{k} = \mathbf{k}_0 \in \tilde{B}_\Lambda$ such that (0.129) holds. We recall (0.130),

$$U^M(t,x) := \sum_{j=1}^{M} \mathsf{c}_j e^{\lambda_j(\mathbf{k}_0)t}V_j(\mathbf{k}_0,x)$$

and require that the coefficients c_j satisfying (0.131)-(0.132). From Proposition 0.4, we have that there exists a family of initial data (0.133), i.e.

$$U_0^{\delta,M}(x) = \delta U^M(0,x) + \delta^2 U_\star^{\delta,M}$$

such that

1. $\mathcal{E}_f(U_\star^{\delta,M}) \leqslant C_M^\star < \infty$ and $U_0^{\delta,M}$ satisfies the compatibility conditions (0.126),
2. the nonlinear equations (0.99) with the aobve initial data $U_0^{\delta,M}$ has a unique solution $U^{\delta,M}$ on $[0,T^{\max})$ satisfying that $\sup_{0\leqslant t<T^{\max}} \mathcal{E}_f(U^{\delta,M}(t)) < \infty$.

3.4.1 The difference functions

Set

$$U^d(t,x) = U^{\delta,M}(t,x) - \delta U^M(t,x).$$

Since $U^{\delta,M}$ solves the nonlinear equations (0.99) and U^M solves the linearized equations (0.101), we have that U^d satisfies (0.134), i.e.

$$\begin{cases} \partial_t \zeta^d + \rho_0' u_3^d = \mathcal{Q}^1(U^{\delta,M}) & \text{in } \Omega, \\ \rho_0 \partial_t u^d - \mu \Delta u^d + \nabla q^d + g\zeta^d e_3 = \mathcal{Q}^2(U^{\delta,M}) & \text{in } \Omega, \\ \text{div} u^d = \mathcal{Q}^3(U^{\delta,M}) & \text{in } \Omega, \\ \partial_t \eta^d = u_3^d + \mathcal{Q}^4(U^{\delta,M}) & \text{on } \Gamma, \\ ((q^d - g\rho_+ \eta^d)\text{Id} - \mu \mathbb{S} u^d) e_3 = \mathcal{Q}^5(U^{\delta,M}) & \text{on } \Gamma. \end{cases}$$

The initial condition for (0.134) is (0.135),

$$U^d(0) = (\zeta^d, u^d, \eta^d, q^d)(0) = \delta^2 U_\star^{\delta,M}.$$

Let $\|U\|_{\mathcal{E}_f} := \mathcal{E}_f(U)$, which is defined as in (0.123). Let $F_M(t) = \sum_{j=j_m}^M |\mathsf{c}_j| e^{\lambda_j t}$ and $0 < \epsilon_0 \ll 1$ be fixed later (3.326). There exists a unique T^δ such that $\delta F_M(T^\delta) = \epsilon_0$. Let

$$C_{19} = \|U^M(0)\|_{\mathcal{E}_f}, \quad C_{20} = \sqrt{\|(\zeta^M, u^M)(0)\|_{L^2(\Omega)}^2 + \|\eta^M(0)\|_{L^2(\Gamma)}^2}. \tag{3.291}$$

We define

$$\begin{aligned} T^\star &:= \sup \Big\{ t \in (0, T^{\max}) \big| \|U^{\delta,M}(t)\|_{\mathcal{E}_f} \leqslant 2C_{19}\delta_0 \}, \\ T^{\star\star} &:= \sup \{ t \in (0, T^{\max}) \big| \|(\zeta^{\delta,M}, u^{\delta,M})(t)\|_{L^2(\Omega)} + \|\eta^{\delta,M}(t)\|_{L^2(\Gamma)} \leqslant 2C_{20}\delta F_M(t) \Big\}. \end{aligned} \tag{3.292}$$

Note that

$$\|U^{\delta,M}(0)\|_{\mathcal{E}_f} \leqslant \delta \|U^M(0)\|_{\mathcal{E}_f} + \|U^d(0)\|_{\mathcal{E}_f} \leqslant C_{19}\delta + C_M^\star \delta^2 < 2C_{19}\delta_0,$$

we then have $T^\star > 0$. Similarly, we have $T^{\star\star} > 0$.

The aim of this part is to derive the bound in time of $\|(\zeta^d, u^d)(t)\|_{L^2(\Omega)} + \|\eta^d(t)\|_{L^2(\Gamma)}$ in the following proposition.

Proposition 3.18. *For all $t \leqslant \min(T^\delta, T^\star, T^{\star\star})$, there holds*

$$\begin{aligned} &\|(\zeta^d, u^d)(t)\|_{L^2(\Omega)}^2 + \|\eta^d(t)\|_{L^2(\Gamma)}^2 \\ &\quad \leqslant C_{21}\delta^3 \Big(\sum_{j=j_m}^N |\mathsf{c}_j| e^{\lambda_j t} + \max(0, M-N) \max_{N+1 \leqslant j \leqslant M} |\mathsf{c}_j| e^{\frac{2}{3}\Lambda t} \Big)^3. \end{aligned} \tag{3.293}$$

In order to prove Proposition 3.18, we need the following bound in time of $\|U^{\delta,M}(t)\|_{\mathcal{E}_f}$.

Proposition 3.19. *For all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$, there holds*

$$\|U^{\delta,M}(t)\|_{\mathcal{E}_f} \leqslant C_{22}\delta F_M(t) \quad \text{for all } t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}. \tag{3.294}$$

Proof. We fix a sufficiently small constant ε such that

$$C_0\varepsilon \leqslant \frac{\lambda_M}{4} \tag{3.295}$$

and Proposition 0.3 holds. Hence, it follows from (3.110) that

$$\begin{aligned} &\mathcal{E}_f^2(t) + \int_0^t \mathcal{D}_f^2(s)ds \\ &\leqslant \frac{\lambda_M}{4}\int_0^t \mathcal{E}_f^2(s)ds + C_{\lambda_M}\Big(\mathcal{E}_f^2(0) + \int_0^t \mathcal{E}_f(\mathcal{E}_f^2 + \mathcal{D}_f^2)(s)ds + \mathcal{E}_f^3(t)\Big) \\ &\qquad + C_{\lambda_M}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds. \end{aligned} \tag{3.296}$$

Refining also δ_0, we get

$$C_{\lambda_M}\delta_0 \leqslant \frac{1}{2} \quad \text{and} \quad C_{\lambda_M}\delta_0 \leqslant \frac{\lambda_M}{4}, \tag{3.297}$$

one thus has

$$\begin{aligned} \frac{1}{2}\mathcal{E}_f^2(t) + \frac{1}{2}\int_0^t \mathcal{D}_f^2(s)ds &\leqslant C_{\lambda_M}\mathcal{E}_f^2(0) + \Big(\frac{\lambda_M}{4} + \delta C_{\lambda_M}\Big)\int_0^t \mathcal{E}_f^2(s)ds \\ &\qquad + C_{\lambda_M}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds \\ &\leqslant C_{\lambda_M}\mathcal{E}_f^2(0) + \frac{\lambda_M}{2}\int_0^t \mathcal{E}_f^2(s)ds \\ &\qquad + C_{\lambda_M}\int_0^t (\|(\zeta,u)(s)\|^2_{L^2(\Omega)} + \|\eta(s)\|^2_{L^2(\Gamma)})ds. \end{aligned} \tag{3.298}$$

Consequently, for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$,

$$\begin{aligned} \|U^{\delta,M}(t)\|^2_{\mathcal{E}_f} &\leqslant 2C_{\lambda_M}\|U^{\delta,M}(0)\|^2_{\mathcal{E}_f} + \lambda_M\int_0^t \|U^{\delta,M}(s)\|^2_{\mathcal{E}_f}ds \\ &\qquad + 2C_{\lambda_M}\int_0^t (\|(\zeta^{\delta,M}, u^{\delta,M})(s)\|^2_{L^2(\Omega)} + \|\eta^{\delta,M}(s)\|^2_{L^2(\Gamma)})ds \\ &\leqslant \lambda_M\int_0^t \|U^{\delta,M}(s)\|^2_{\mathcal{E}_f}ds + C_{23}\delta^2F_M^2(t). \end{aligned}$$

Applying Gronwall's inequality, the resulting inequality tells us that

$$\|U^{\delta,M}(t)\|^2_{\mathcal{E}_f} \leqslant C_{23}\Big(\delta^2F_M^2(t) + \delta^2\int_0^t e^{\lambda_M(t-s)}F_M^2(s)ds\Big). \tag{3.299}$$

Note that $\lambda_M < \lambda_j$ for all $1 \leqslant j \leqslant M-1$, we have

$$\begin{aligned}\int_0^t e^{\lambda_M(t-s)} F_M^2(s) ds &\leqslant M^2 \sum_{j=j_m}^{M} \int_0^t e^{\lambda_M(t-s)} |\mathsf{c}_j|^2 e^{2\lambda_j s} ds \\ &\leqslant M^2 e^{\lambda_M t} \sum_{j=j_m}^{M} |\mathsf{c}_j|^2 \frac{e^{(2\lambda_j - \lambda_M)t}}{2\lambda_j - \lambda_M}.\end{aligned} \tag{3.300}$$

Substituting (3.300) into (3.299), this yields (3.294). We deduce Proposition 3.19. □

We now prove Proposition 3.18.

Proof of Proposition 3.18. Differentiating $(0.134)_{2,5}$ with respect to t and then eliminating the terms $\partial_t \zeta^d, \partial_t \eta^d$ by using $(0.134)_{1,4}$, we deduce from (0.134) that

$$\begin{cases} \rho_0 \partial_t^2 u^d + \nabla \partial_t q^d - \mu \Delta \partial_t u^d - g\rho_0' u_3^d e_3 = \partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M}) e_3 & \text{in } \Omega, \\ \operatorname{div} \partial_t u^d = \partial_t \mathcal{Q}^3(U^{\delta,M}) & \text{in } \Omega, \\ (\partial_t q^d \mathrm{Id} - \mu \mathbb{S} \partial_t u^d) e_3 = g\rho_+ u^d e_3 + g\rho_+ \mathcal{Q}^4(U^{\delta,M}) e_3 + \partial_t \mathcal{Q}^5(U^{\delta,M}) & \text{on } \Gamma. \end{cases} \tag{3.301}$$

Multiplying both sides of $(3.301)_1$ by $\partial_t u^d$, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0 |\partial_t u^d|^2 + \int_\Omega (\nabla \partial_t q^d - \mu \Delta \partial_t u^d) \cdot \partial_t u^d - \int_\Omega g\rho_0' u_3^d \partial_t u_3^d \\ &= \int_\Omega (\partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M}) e_3) \cdot \partial_t u^d.\end{aligned} \tag{3.302}$$

Using the integration by parts, we deduce from the resulting equality that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0 |\partial_t u^d|^2 - \int_\Omega g\rho_0' |u_3^d|^2\Big) + \int_\Gamma (\partial_t q \mathrm{Id} - \mu \mathbb{S} \partial_t u^d) e_3 \cdot \partial_t u^d \\ &= \int_\Omega \partial_t q^d \operatorname{div} \partial_t u^d - \frac{\mu}{2} \int_\Omega |\mathbb{S} \partial_t u^d|^2 + \int_\Omega (\partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M}) e_3) \cdot \partial_t u^d.\end{aligned}$$

Substituting $(3.301)_{2,3}$ into (3.302), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0 |\partial_t u^d|^2 - \int_\Omega g\rho_0' |u_3^d|^2\Big) + \int_\Gamma g\rho_+ \partial_t u_3^d \partial_t u_3^d \\ &\qquad + \int_\Gamma (\partial_t \mathcal{Q}^5(U^{\delta,M}) + g\rho_+ \mathcal{Q}^4(U^{\delta,M}) e_3) \cdot \partial_t u^d \\ &= \int_\Omega \partial_t q^d \partial_t \mathcal{Q}^3(U^{\delta,M}) - \frac{\mu}{2} \int_\Omega |\mathbb{S} \partial_t u^d|^2 + \int_\Omega (\partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M}) e_3) \cdot \partial_t u^d.\end{aligned}$$

This yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0 |\partial_t u^d|^2 - \int_\Omega g\rho_0' |u_3^d|^2 + \int_\Gamma g\rho_+ |u_3^d|^2\Big) + \frac{\mu}{2} \int_\Omega |\mathbb{S} \partial_t u^d|^2 \\ &= \int_\Omega (\partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M}) e_3) \cdot \partial_t u^d + \int_\Omega \partial_t q^d \partial_t \mathcal{Q}^3(U^{\delta,M}) \\ &\qquad - \int_\Gamma (\partial_t \mathcal{Q}^5(U^{\delta,M}) + g\rho_+ \mathcal{Q}^4(U^{\delta,M}) e_3) \cdot \partial_t u^d,\end{aligned} \tag{3.303}$$

Note that

$$\int_\Omega (\partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot \partial_t u^d \\ \lesssim (\|\partial_t \mathcal{Q}^2(U^{\delta,M})\|_{L^2(\Omega)} + \|\mathcal{Q}^1(U^{\delta,M})\|_{L^2(\Omega)})(\|\partial_t u^{\delta,M}\|_{L^2(\Omega)} + \delta\|\partial_t u^M\|_{L^2(\Omega)}).$$

In view of (3.198) and the definition of U^M and F_M, we have

$$\int_\Omega (\partial_t \mathcal{Q}^2(U^{\delta,M}) - g\mathcal{Q}^1(U^{\delta,M})e_3)\cdot \partial_t u^d \lesssim \|U^{\delta,M}\|^2_{\mathcal{E}_f}(\|U^{\delta,M}\|_{\mathcal{E}_f} + \delta\|\partial_t u^M\|_{L^2(\Omega)}) \\ \lesssim \delta^3 F_M^3(t). \tag{3.304}$$

Similarly, we observe

$$\int_\Omega \partial_t q^d \partial_t \mathcal{Q}^3(U^{\delta,M}) \lesssim \|\partial_t \mathcal{Q}^3(U^{\delta,M})\|_{L^2(\Omega)}(\|\partial_t q^{\delta,M}\|_{L^2(\Omega)} + \delta\|\partial_t q^M\|_{L^2(\Omega)}) \\ \lesssim \delta^3 F_M^3(t). \tag{3.305}$$

We continue applying (3.198) and use the trace theorem to get

$$\begin{aligned}\int_\Gamma &(\partial_t \mathcal{Q}^5(U^{\delta,M}) + g\rho_+\mathcal{Q}^4(U^{\delta,M})e_3)\cdot \partial_t u^d \\ &\lesssim (\|\partial_t \mathcal{Q}^5(U^{\delta,M})\|_{L^2(\Gamma)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)})\|\partial_t u^d\|_{L^2(\Gamma)} \\ &\lesssim (\|\partial_t \mathcal{Q}^5(U^{\delta,M})\|_{H^{1/2}(\Gamma)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{H^{1/2}(\Gamma)})\|\partial_t u^d\|_{H^1(\Omega)} \\ &\lesssim \delta^3 F_M^3(t).\end{aligned} \tag{3.306}$$

Substituting (3.304), (3.305) and (3.306) into (3.303), we obtain that

$$\int_\Omega \rho_0|\partial_t u^d(t)|^2 + \int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u^d(s)|^2 ds \\ \leqslant z_1 + \int_\Omega g\rho_0'|u_3^d(t)|^2 - \int_\Gamma g\rho_+|u_3(t)|^2 + C_{24}\delta^3F_M^3(t), \tag{3.307}$$

where

$$z_1 = \int_\Omega \rho_0|\partial_t u^d(0)|^2 - \int_\Omega g\rho_0'|u_3^d(0)|^2 + \int_\Gamma g\rho_+|u_3^d(0)|^2.$$

Thanks to Lemma 3.8, we deduce from (3.307) that

$$\int_\Omega \rho_0|\partial_t u^d(t)|^2 + \int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u^d(s)|^2 ds \\ \leqslant z_1 + \Lambda^2\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\Lambda\int_\Omega \mu|\mathbb{S}u^d(t)|^2 + C_{23}\delta^3F_M^3(t). \tag{3.308}$$

Using Cauchy-Schwarz's inequality, we have that

$$\int_\Omega \mu|\mathbb{S}u^d(t)|^2 = \int_\Omega \mu|\mathbb{S}u^d(0)|^2 + 2\int_0^t\int_\Omega \mu\mathbb{S}u^d(s) : \mathbb{S}\partial_t u^d(s)ds \\ \leqslant \int_\Omega \mu|\mathbb{S}u^d(0)|^2 + \frac{1}{\Lambda}\int_0^t\int_\Omega \mu|\mathbb{S}\partial_t u^d(s)|^2 ds + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds \tag{3.309}$$

and that
$$\frac{d}{dt}\int_\Omega \rho_0|u^d|^2 \leqslant \frac{1}{\Lambda}\int_\Omega \rho_0|\partial_t u^d|^2 + \Lambda\int_\Omega \rho_0|u^d|^2. \tag{3.310}$$

Three above inequalities (3.308), (3.309) and (3.310) imply that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u^d(t)|^2 &\leqslant \frac{z_1}{\Lambda} + \int_\Omega \mu|\mathbb{S}u^d(0)|^2 + 2\Lambda\int_\Omega \rho_0|u^d(t)|^2 \\ &\quad + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds + C_{24}\delta^3 F_M^3(t),\end{aligned} \tag{3.311}$$

It follows from $U^d(0) = \delta^2 U_\star^{\delta,M}$ that $z_1 \lesssim \delta^4$, this yields

$$\frac{z_1}{\Lambda} + \int_\Omega \mu|\mathbb{S}u^d(0)|^2 \lesssim \delta^4.$$

Hence, the inequality (3.311) implies

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\int_\Omega \mu|\mathbb{S}u^d(t)|^2 &\leqslant 2\Lambda\int_\Omega \rho_0|u^d(t)|^2 + \Lambda\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds \\ &\quad + C_{25}\delta^3 F_M^3(t),\end{aligned} \tag{3.312}$$

In view of Gronwall's inequality, we obtain from (3.312) that

$$\begin{aligned}\int_\Omega \rho_0|u^d(t)|^2 + \frac{1}{2}\int_0^t\int_\Omega \mu|\mathbb{S}u^d(s)|^2 ds &\leqslant C_{25}\delta^3\int_0^t e^{2\Lambda(t-s)}F_M^3(s)ds \\ &\leqslant C_{26}\delta^3\int_0^t e^{2\Lambda(t-s)}F_M(3s)ds.\end{aligned} \tag{3.313}$$

Due to (0.129), we obtain for $1 \leqslant j \leqslant N$,

$$\int_0^t e^{(3\lambda_j-2\Lambda)s}ds = \frac{1}{3\lambda_j - 2\Lambda}(e^{(3\lambda_j-2\Lambda)t} - 1) \leqslant \frac{1}{3\lambda_j - 2\Lambda}e^{(3\lambda_j-2\Lambda)t} \tag{3.314}$$

and for $j \geqslant N+1$,

$$\int_0^t e^{(3\lambda_j-2\Lambda)s}ds = \frac{1}{3\lambda_j - 2\Lambda}(e^{(3\lambda_j-2\Lambda)t} - 1) \leqslant \frac{1}{2\Lambda - 3\lambda_j}. \tag{3.315}$$

Substituting (3.314) and (3.315) into (3.313), we observe that if $M \leqslant N$,

$$\|u^d(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u^d(s)\|^2_{L^2(\Omega)}ds \leqslant C_{26}\delta^3\Big(\sum_{j=j_m}^M \frac{|\mathsf{c}_j|}{3\lambda_j - 2\Lambda}e^{3\lambda_j t}\Big) \tag{3.316}$$

and if $M \geqslant N+1$,

$$\begin{aligned}&\|u^d(t)\|^2_{L^2(\Omega)} + \int_0^t \|\nabla u^d(s)\|^2_{L^2(\Omega)}ds \\ &\qquad \leqslant C_{26}\delta^3\Big(\sum_{j=j_m}^M \frac{|\mathsf{c}_j|}{3\lambda_j - 2\Lambda}e^{3\lambda_j t} + \sum_{j=N+1}^M \frac{|\mathsf{c}_j|}{2\Lambda - 3\lambda_j}e^{2\Lambda t}\Big).\end{aligned} \tag{3.317}$$

We then estimate $\|\zeta^d(t)\|_{L^2(\Omega)}$. Due to $(0.134)_1$, we obtain

$$\|\zeta^d(t)\|^2_{L^2(\Omega)} \lesssim \|\zeta^d(0)\|^2_{L^2(\Omega)} + \int_0^t (\|u_3^d(s)\|^2_{L^2(\Omega)} + \|\mathcal{Q}^1(U^{\delta,M})(s)\|^2_{L^2(\Omega)})ds. \tag{3.318}$$

Note that $\zeta^d(0) = \delta^2\zeta_\star^{\delta,M}$ and thanks to (3.198) also, the inequality (3.318) implies

$$\begin{aligned}\|\zeta^d(t)\|^2_{L^2(\Omega)} &\lesssim \delta^4 + \int_0^t (\|u_3^d(s)\|^2_{L^2(\Omega)} + \|U^{\delta,M}(s)\|^4_{\mathcal{E}_f})ds \\ &\lesssim \delta^4 + \int_0^t (\|u_3^d(s)\|^2_{L^2(\Omega)} + \delta^4 F_M^4(s))ds.\end{aligned} \tag{3.319}$$

Note that $\delta F_M(t) \leqslant \varepsilon_0 \leqslant 1$ for any $t \leqslant T^\delta$. Hence, we have

$$\delta^4 \int_0^t F_M^4(s)ds \lesssim \delta^3 \int_0^t F_M^3(s)ds \lesssim \delta^3 \int_0^t F_M(3s)ds,$$

this yields

$$\delta^4 \int_0^t F_M^4(s)ds \lesssim \delta^3 \sum_{j=j_m}^M |\mathsf{c}_j| e^{3\lambda_j t}. \tag{3.320}$$

Combining (3.316), (3.317) and (3.320), we deduce from (3.319) that, if $M \leqslant N$,

$$\|\zeta^d(t)\|^2_{L^2(\Omega)} \leqslant C_{27}\delta^3 \sum_{j=j_m}^M |\mathsf{c}_j| e^{3\lambda_j t} \tag{3.321}$$

and if $M \geqslant N+1$,

$$\|\zeta^d(t)\|^2_{L^2(\Omega)} \leqslant C_{27}\delta^3 \Big(\sum_{j=j_m}^M |\mathsf{c}_j| e^{3\lambda_j t} + \sum_{j=N+1}^M |\mathsf{c}_j| e^{2\Lambda t}\Big). \tag{3.322}$$

To estimate $\|\eta^d(t)\|_{L^2(\Gamma)}$, we use $(0.134)_4$ and the trace theorem to obtain

$$\begin{aligned}\frac{d}{dt}\|\eta^d\|^2_{L^2(\Gamma)} &\leqslant \|\eta^d\|_{L^2(\Gamma)}(\|u_3^d\|_{L^2(\Gamma)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)}) \\ &\lesssim \|\eta^d\|_{L^2(\Gamma)}(\|u_3^d\|_{H^1(\Omega)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)}).\end{aligned}$$

This yields

$$\frac{d}{dt}\|\eta^d\|_{L^2(\Gamma)} \lesssim \|u_3^d\|_{H^1(\Omega)} + \|\mathcal{Q}^4(U^{\delta,M})\|_{L^2(\Gamma)}.$$

Thanks to (3.198), we further get

$$\begin{aligned}\|\eta^d(t)\|^2_{L^2(\Gamma)} &\lesssim \|\eta^d(0)\|^2_{L^2(\Gamma)} + \int_0^t (\|u_3^d(s)\|^2_{H^1(\Omega)} + \|U^{\delta,M}\|^2_{\mathcal{E}_f})ds \\ &\lesssim \delta^4 + \int_0^t (\|u_3^d(s)\|^2_{H^1(\Omega)} + \delta^4 F_M^4(s))ds.\end{aligned}$$

Using (3.316), (3.317) and (3.320) again, we have that $\|\eta^d(t)\|^2_{L^2(\Gamma)}$ is bounded above like (3.321) or (3.322). Together with (3.316), (3.317), (3.321) and (3.322), Proposition 3.18 is proven. □

3.4.2 Proof of Theorem 0.6

Since $j_m = \min\{j : 1 \leqslant j \leqslant N, \mathsf{c}_j \neq 0\}$, we have

$$\|u^M(t)\|^2_{L^2(\Omega)} = \sum_{i=j_m}^{M} \mathsf{c}_i^2 e^{2\lambda_i t}\|u_i\|^2_{L^2(\Omega)} + 2\sum_{j_m\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i\cdot u_j. \tag{3.323}$$

It can be seen that

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} &+ 2\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i\cdot u_j \\ &- |\mathsf{c}_{j_m}|\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)e^{(\lambda_{j_m}+\lambda_{j_m+1})t}.\end{aligned}$$

By Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}2&\sum_{j_m+1\leqslant i<j\leqslant M} \mathsf{c}_i\mathsf{c}_j e^{(\lambda_i+\lambda_j)t}\int_\Omega u_i\cdot u_j \\ &\geqslant -2\sum_{j_m\leqslant i<j\leqslant M}|\mathsf{c}_i||\mathsf{c}_j|e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_i\|_{L^2(\Omega)}\|u_j\|_{L^2(\Omega)} \\ &\geqslant -e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2.\end{aligned}$$

This implies

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} &- e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big)^2 \\ &- |\mathsf{c}_{j_m}|e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|_{L^2(\Omega)}\Big(\sum_{j=j_m+1}^{M}|\mathsf{c}_j|\|u_j\|_{L^2(\Omega)}\Big).\end{aligned}$$

Due to the assumption (0.132), we deduce that

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \sum_{j=j_m}^{M} \mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)} &- \frac14\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t}\|u_{j_m}\|^2_{L^2(\Omega)} \\ &- \frac12\mathsf{c}_{j_m}^2 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\|u_{j_m}\|^2_{L^2(\Omega)}.\end{aligned}$$

This yields

$$\begin{aligned}\|u^M(t)\|^2_{L^2(\Omega)} \geqslant \mathsf{c}_{j_m}^2\Big(e^{2\lambda_{j_m}t} - \frac14 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} - \frac12 e^{(\lambda_{j_m}+\lambda_{j_m+1})t}\Big)\|u_{j_m}\|^2_{L^2(\Omega)} \\ + \sum_{j=j_m+1}^{M}\mathsf{c}_j^2 e^{2\lambda_j t}\|u_j\|^2_{L^2(\Omega)}.\end{aligned}$$

Notice that for all $t \geqslant 0$,

$$e^{2\lambda_{j_m}t} - \frac14 e^{(\lambda_{j_m+1}+\lambda_{j_m+2})t} - \frac12 e^{(\lambda_{j_m}+\lambda_{j_m+1})t} \geqslant \frac14 e^{2\lambda_{j_m}t}.$$

Hence, we have

$$\|u^M(t)\|_{L^2(\Omega)} \geqslant C_{28}F_M(t), \tag{3.324}$$

for all $t \leqslant \min\{T^\delta, T^\star, T^{\star\star}\}$.

Let

$$\tilde{\mathsf{c}}(M) = \max_{N+1\leqslant j\leqslant M} \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|} \geqslant 0.$$

Now, we show that

$$T^\delta \leqslant \min\{T^\star, T^{\star\star}\} \tag{3.325}$$

by choosing

$$\varepsilon_0 < \min\Big(\frac{2C_{19}\delta_0}{C_{22}}, \frac{C_{20}^2}{C_{21}(1+M\tilde{\mathsf{c}}(M))^3}, \frac{C_{28}^2}{4C_{21}(1+M\tilde{\mathsf{c}}(M))^2}\Big). \tag{3.326}$$

Indeed, if $T^\star < T^\delta$, we have from (3.294) that

$$\|U^{\delta,M}(T^\star)\|_{\mathcal{E}_f} \leqslant C_{22}\delta F_M(T^\star) < C_{22}\delta F_M(T^\delta) = C_{22}\varepsilon_0 < 2C_{19}\delta_0,$$

which contradicts the definition of $T^\star$ in (3.292). If $T^{\star\star} < T^\delta$, we obtain from the definition of C_{20} (3.291) and the inequality (3.293) that

$$\begin{aligned}
&\|(\zeta^{\delta,M}, u^{\delta,M})(T^\delta)\|_{L^2(\Omega)} + \|\eta^{\delta,M}(T^\delta)\|_{L^2(\Gamma)} \\
&\leqslant \|(\zeta^d, u^d)(T^\delta)\|_{L^2(\Omega)} + \|\eta^d(T^\delta)\|_{L^2(\Omega)} + \delta(\|(\zeta^M, u^M)(T^\delta)\|_{L^2(\Omega)} + \|\eta^M(T^\delta)\|_{L^2(\Gamma)}) \\
&\leqslant \sqrt{C_{21}}\delta^{\frac{3}{2}}\Big(\sum_{j=j_m}^{N} |\mathsf{c}_j|e^{\lambda_j T^\delta} + \max(0, M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{2\Lambda T^\delta/3}\Big)^{3/2} \\
&\qquad + C_{20}\delta F_M(T^\delta).
\end{aligned} \tag{3.327}$$

Notice from (0.129) that for $N+1 \leqslant j \leqslant M$,

$$|\mathsf{c}_j|\delta e^{\frac{2}{3}\Lambda T^\delta} < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}(\delta|\mathsf{c}_{j_m}|e^{\lambda_{j_m}T^\delta}) < \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\delta F_M(T^\delta) = \frac{|\mathsf{c}_j|}{|\mathsf{c}_{j_m}|}\epsilon_0.$$

Then, it follows from (3.327) that

$$\begin{aligned}
&\|(\zeta^{\delta,M}, u^{\delta,M})(T^\delta)\|_{L^2(\Omega)} + \|\eta^{\delta,M}(T^\delta)\|_{L^2(\Gamma)} \\
&\qquad \leqslant C_{20}\delta F_M(T^\delta) + \sqrt{C_{21}}\delta^{3/2}(1+M\tilde{\mathsf{c}}(M))^{3/2}F_M^{3/2}(T^\delta) \\
&\qquad \leqslant C_{20}\epsilon_0 + \sqrt{C_{21}}(1+M\tilde{\mathsf{c}}(M))^{3/2}\epsilon_0^{3/2}.
\end{aligned}$$

Using (3.326) again, we deduce

$$\|(\zeta^{\delta,M}, u^{\delta,M})(T^\delta)\|_{L^2(\Omega)} + \|\eta^{\delta,M}(T^\delta)\|_{L^2(\Gamma)} < 2C_{20}\epsilon_0 = 2C_{20}\delta F_M(T^\delta),$$

which also contradicts the definition of $T^{\star\star}$ in (3.292). So, (3.325) holds.

Once we have (3.325), it follows from (3.293) and (3.324) that

$$\begin{aligned}
&\|u^{\delta,M}(T^\delta)\|_{L^2(\Omega)} \\
&\geqslant \delta\|u^M(T^\delta)\|_{L^2(\Omega)} - \|u^d(T^\delta)\|_{L^2(\Omega)} \\
&\geqslant C_{28}\delta F_M(T^\delta) \\
&\qquad - \sqrt{C_{21}}\delta^{3/2}\Big(\sum_{j=j_m}^{N}|\mathsf{c}_j|e^{\lambda_j T^\delta} + \max(0, M-N)\Big(\max_{N+1\leqslant j\leqslant M}|\mathsf{c}_j|\Big)e^{2\Lambda T^\delta/3}\Big)^{3/2}.
\end{aligned}$$

Thanks to (3.326) again, we conclude that

$$\|u^{\delta,M}(T^\delta)\|_{L^2(\Omega)} \geqslant C_{28}\epsilon_0 - \sqrt{C_{21}}(1+M\tilde{\mathsf{c}}(M))^{3/2}\epsilon_0^{3/2} \geqslant \frac{1}{2}C_{28}\epsilon_0 > 0.$$

Theorem 0.6 follows by taking δ_0 satisfying Propositions 0.3, 0.4 and the inequality (3.297), ε_0 satisfying (3.326) and $m_0 = \frac{1}{2}C_{28}$.

Chapter 4

Conclusions and Perspectives

4.1 Conclusions

In this thesis, we study rigorously a well-known phenomena in fluid mechanics, which is the Rayleigh-Taylor instability with an influence of constant viscosity coefficient. Of concern is the instability of the equilibrium $(\rho_0(x_d), 0, -g\int_0^{x_d}\rho_0(y)dy)$ ($d = 2$ or 3), such that the density profile ρ_0 is a smooth increasing function. There are two contributions of this thesis, that we sum up below.

The first contribution is to develop a novel method, based on the spectral theory of self-adjoint and compact operators, to prove the existence of multiple characteristic values to the linearized equations. Study on the linearized equations relies on the investigation of regular solutions to a fourth-order ordinary differential equation on a compact or non-compact interval I. In the first paper [51], we have that I is the whole line $\mathbf{R}$. The key point in our analysis is to reduce the finding of bounded solutions of the ODE on $\mathbf{R}$ to the finding of solutions of the ODE on a compact interval with boundary conditions obtained from the outer solutions. That ODE on a compact interval turns out to be equivalent to a variational problem with suitable boundary conditions, so that we can apply the spectral theory of self-adjoint and compact operators to obtain the main results. In the case ρ_0' compactly supported, there exists an *infinite* sequence of characteristic values and in the case ρ_0' positive everywhere, for any $\epsilon_\star > 0$, a *finite* number of characteristic values, greater than $\epsilon_\star$, is found.

The second paper [61] deals with the same fourth-oder ODE stated in a compact interval $I = (-1, 1)$ with the boundary conditions deduced from the Navier slip boundary conditions of the original problem. Due to the influence of the slip coefficients, we need to work on a *supercritical regime* of the viscosity μ, i.e. $\mu > \mu_c(\Xi)$ to apply our operator method. In this supercritical regime, we deduce the existence of *infinite* characteristic values to the linearized equations. In the third paper, after a change of variables, we obtain the same fourth-order ODE in the half-line $I = \mathbf{R}_-$ with the boundary conditions at 0 deduced from the boundary conditions of the nonlinear problem at the top side. For the profile ρ_0' compactly supported, we have the similar spectral analysis and we prove the existence of an *infinite* sequence of characteristic values to the linearized equations.

The second contribution of this thesis is to construct a *wide class* of the initial data to the nonlinear problem to give rise to nonlinear Rayleigh-Taylor instability in [61] and [62]. Indeed, we consider a linear combination of normal modes to set its value at initial time $t = 0$ as an initial datum of the nonlinear problem of size a small parameter $\delta > 0$. In [61], this step is straightforward. In [62], due to the complicated structure of the nonlinear equations caused by the Lagrangian transformation, a term of order δ^2 has to be constructed and added to ensure that the initial data satisfy the compatibility conditions. The nonlinear equations with the above initial data have a unique local-in-time solution with a suitable regularity. By exploiting some energy estimates thanks to Cauchy-Schwarz's inequality, Sobolev embedding and Gronwall's inequality for the nonlinear equations, we prove in [61] the nonlinear Rayleigh-Taylor instability in a high regime of viscosity $\mu > 3\mu_c(\Xi)$. Similarly, using further Gagliardo-Nirenberg's inequality for the nonlinear terms in the last case [62], these nonlinear terms being more complex than those ones in [61], we conclude on the nonlinear Rayleigh-Taylor instability for all positive viscosity.

We highlight that our nonlinear results extend the previous framework of Guo-Strauss [30] and of Grenier [35], where only the maximal normal mode was used to construct the initial data and this gives rise to the nonlinear instability.

4.2 Perspectives

In the last part of this thesis, we introduce our future works.

4.2.1 Nonlinear Rayleigh-Taylor instability for Navier-Stokes-Korteweg equations

In the book of Chandrasekhar [7], as well as viscosity, the impact of other of physical factors has been considered in the RT instability, such as, magnetic field, surface tension, etc. One of our ideas is to study the influence of capillary coefficient. The Navier–Stokes–Korteweg equations describing the motion of a nonhomogeneous incompressible viscous fluid in the presence of a uniform gravitational field in a horizontally periodic domain $\Omega = (2\pi L\mathbb{T})^2 \times \mathbf{R}$, are

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \nu \mathrm{div}(\rho \nabla u) + \sigma \nabla\rho\Delta\rho + \nabla p = -g\rho e_3, \\ \mathrm{div} u = 0, \end{cases} \tag{4.1}$$

where $t \geqslant 0, x = (x_1, x_2, x_3) \in \Omega$. The unknowns $\rho := \rho(x,t)$, $u := u(x,t)$ and $P := P(x,t)$ denote the density, the velocity and the pressure of the fluid, respectively, while $\nu > 0$ is the viscosity coefficient, $\sigma > 0$ is the capillary number, $g > 0$ being the gravity constant.

A hydrostatic state $(\rho_0(x_3), 0, P_0(x_3))$ such that

$$\nabla P_0 = -\sigma \nabla \rho_0 \Delta \rho_0 - g\rho_0 e_3,$$

is a steady state solution of (4.1). The perturbations

$$\zeta = \rho - \rho_0, \quad u = u - 0, \quad p = P - P_0$$

thus satisfy

$$\begin{cases} \partial_t \zeta + u \cdot \nabla(\rho_0 + \eta) = 0, \\ (\rho_0 + \zeta)\partial_t u + (\rho_0 + \zeta) u \cdot \nabla u + \nabla p \\ \qquad = \nu \mathrm{div}((\rho_0 + \zeta)\nabla u) - \sigma(\rho_0'' \nabla \zeta + \nabla(\rho_0 + \zeta)\Delta \zeta) - g\zeta e_3, \\ \mathrm{div} u = 0. \end{cases} \tag{4.2}$$

It implies the linearized equations

$$\begin{cases} \partial_t \zeta + \nabla \rho_0 \cdot u = 0, \\ \rho_0 \partial_t u + \nabla p = -\sigma(\nabla \rho_0 \Delta \zeta + \rho_0'' \nabla \zeta) + \nu \mathrm{div}(\rho_0 \nabla u) - g\zeta e_3, \\ \mathrm{div} u = 0. \end{cases} \tag{4.3}$$

In the same manner of Section 0.4, the search of normal modes to the linearized equations (4.3) implies to the existence of nontrivial and bounded solutions of the following ODE:

$$\begin{aligned} \lambda^2(-(\rho_0 \phi')' + k^2 \rho_0 \phi) + \lambda \nu ((\rho_0 \phi'')'' - 2k^2(\rho_0 \phi')' + k^4 \rho_0 \phi) \\ = \sigma k^2 ((|\rho_0'|^2 \phi')' - k^2 |\rho_0'|^2 \phi) + g k^2 \rho_0' \phi, \end{aligned} \tag{4.4}$$

with the limits $\lim_{|x_3| \to \infty} \phi(x) = 0$.

Let the capillary number $\sigma > 0$ be given and the effect of the viscosity be omitted ($\nu = 0$), Bresch, Desjardins, Gisclon and Sart [6] showed the asymptotic limit of the characteristic value λ under a small perturbation of wave number k in the spirit of [9].

Let us consider the density profile ρ_0, which is increasing and satisfies (0.31)-(0.32), i.e.

$$\rho_0' \text{ is a nonnegative function of class } C_0^1(\mathbf{R}), \quad \mathrm{supp}(\rho_0') = [-a, a],$$

and outside $(-a, a)$, we denote

$$\rho_0(x_3) = \begin{cases} \rho_- & \text{as } x_3 \in (-\infty, -a], \\ \rho_+ & \text{as } x_3 \in [a, +\infty), \end{cases}$$

with $0 < \rho_- < \rho_+$ are two positive constants. With the presence of the viscosity and the capillary number, the first goal is to find the effect of σ on the existence of infinitely many characteristic values in the ODE (4.4) and the second goal is to prove the nonlinear Rayleigh-Taylor instability for Eq. (4.1) in a suitable regime of capillary number.

4.2.2 Nonlinear instability of a two-fluid model

Note that, in this thesis, we focus on a fluid model, where the density is continuous. Meanwhile, the two-phase interface systems, i.e. the density has a jump, appear in lots of problems, e.g. describing propagation of waves between air and water. Hence, we propose to continue our method on a two-fluid model.

We refer to the book [74] for the description of various mathematical models for mixtures of several phases being already studied. We describe for example a model for a mixture of two phases for densities ρ_i and velocities u_i

$$\begin{cases} \partial_t \rho_i + \mathrm{div}(\rho_i u_i) = 0, \\ \partial_t(\rho_i u_i) + \mathrm{div}(\rho_i u_i \otimes u_i) = \mathrm{div}\sigma_i + \rho_i b_i + m_i. \end{cases} \tag{4.5}$$

In this equation, b_i is the specific external body force, and m_i is the momentum supply due to the interaction between the i^{th} constituent and other constituents (interactive body force). The term σ_i represents the stress term, typically of the form $\sigma_i = -p_i \mathrm{Id} + \tau_i$, where $p_i Id$ represents the pressure part and τ_i is a viscous part. These two fluids are separated by a free surface $z = \eta$ and on which, we have the kinematic relation

$$\partial_t \eta + u_{i,1}\partial_x \eta + u_{i,2}\partial_y \eta = u_{i,3},$$

and the pressure is continuous across this surface.

Instead of (4.5), a method by means of averaged equations will help us to simplify the complexity of of multiphase flows, see e.g. [44]. By using volume-averaging, one then obtains averaged equations for mass, momentum and energy expressed in terms of some volume-averaged quantities. As a part of this procedure, the volume fraction variable $0 \leqslant \alpha^+ \leqslant 1$ of the liquid (water) and $0 \leqslant \alpha^- \leqslant 1$ of the gas (air) appears and it yields a generic two-phase compressible system without free surface in $(0,T) \times \Omega$ $(\Omega \subset \mathbf{R}^3)$,

$$\begin{cases} \alpha^- + \alpha^+ = 1, \\ \partial_t(\alpha^\pm \rho^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm u^\pm) = 0, \\ \partial_t(\alpha^\pm \rho^\pm u^\pm) + \mathrm{div}(\alpha^\pm \rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm \nabla P^\pm(\rho^\pm) = \mathrm{div}(\alpha^\pm \tau^\pm), \\ P^+(\rho^+) - P^-(\rho_-) = f(\alpha^- \rho^-). \end{cases} \tag{4.6}$$

$\rho^{\pm}, u^{\pm}$ and $P^{\pm}(\rho^{\pm}) = A^{\pm}(\rho^{\pm})^{\gamma^{\pm}}$ are, respectively, the densities, the velocity of each phase, and the two pressure functions. It is assumed that are $\gamma^{\pm} \geqslant 1$ and $A^{\pm} = 1$. The capillary pressure $f \in C^3(\mathbf{R}_+)$ and $\tau^{\pm}$ are the viscous stress tensors, given by

$$\tau^{\pm} = \mu^{\pm}(\nabla u^{\pm} + \nabla^T u^{\pm}) + \lambda^{\pm}\mathrm{div}u^{\pm}\mathrm{Id}.$$

where the constants $\mu^{\pm}$ and $\lambda^{\pm}$ are shear and bulk viscosity coefficients satisfying $\mu^{\pm} > 0$ and $2\mu^{\pm} + 3\lambda^{\pm} > 0$.

In [5], Bresch, Desjardins, Ghidaglia and Grenier studied a general system of the form (4.6) by adding the influence of capillarity effects $\sigma^{\pm} > 0$, hence a third order derivative of $\alpha^{\pm}\rho^{\pm}$ which accounts for internal capillary pressure forces by the so-called Korteweg model. With additional assumptions, i.e. $P^+ = P^-$, $\mu^{\pm}$ depending on $\rho^{\pm}$ and $\lambda^{\pm} = 0$, the authors investigated the global well-posedness of that general system in $\Omega = \mathbb{T}^3$.

Let us discuss about (4.6) in $\Omega = \mathbf{R}^3$, that was treated by Evje, Wang and Wen. Being inspired by the reformulation of Bresch et. al., Evje et. al. rewrite (4.6) in a perturbed form around the constant state $(\alpha^+\rho^+, u^+, \alpha^-\rho^-, u^-) = (1, 0, 1, 0)$, that is

$$\begin{cases} \partial_t n^+ + \beta_1 \mathrm{div} u^+ = F^1, \\ \partial_t u^+ + \beta_1 \nabla n^+ + \beta_2 \nabla n^- - \nu_1^+ \Delta u^+ - \nu_2^+ \mathrm{div}\nabla u^+ = F^2, \\ \partial_t n^- + \beta_4 \mathrm{div} u^- = F^3, \\ \partial_t u^- + \beta_3 \nabla n^+ + \beta_4 \nabla n^- - \nu_1^- \Delta u^+ - \nu_2^- \mathrm{div}\nabla u^- = F^4, \end{cases} \tag{4.7}$$

with initial data

$$(n^+, u^+, n^-, u^-)|_{t=0} = (n_0^+, u_0^+, n_0^-, u_0^-)(x) \to 0 \text{ as } |x| \to \infty.$$

Here, $F^i (1 \leqslant i \leqslant 4)$ are the nonlinear terms and $\beta_i (1 \leqslant i \leqslant 4)$ are positive constants satisfying that

$$\beta_2\beta_3 - \beta_1\beta_4 \text{ has the same sign as } f'(1).$$

In [26], Evje, Wang and Wen prove the global stability of the trivial equilibrium state for (4.7) when $f'(1) < 0$. The stability is still true if $f'(1) = 0$ and has been proven recently by Wu, Yao, Zhang [72]. In contrast, if $f'(1) > 0$, the same authors show that the trivial state is nonlinearly unstable in [73].

We begin the investigation of that problem in the case, where Ω is a bounded domain in $\mathbf{R}^3$.

4.2.3 Nonlinear instability of the Zeldovitch-von Neumann-Döring detonation

Another problem we can study is the ZND detonation. The ZND system, used to study combustion, for the unknowns specific volume v, particle velocity $u = (u_x, u_y, u_z)$, entropy S, and mass fraction of reactant λ are

$$\begin{cases} \partial_t v + u \cdot \nabla v - v \nabla \cdot u = 0, \\ \partial_t u + u \cdot \nabla u + v \nabla p = 0, \\ \partial_t S + u \cdot \nabla S = -r\Delta F/T =: \Phi, \\ \partial_t \lambda + u \cdot \nabla \lambda = r. \end{cases}$$

where $p = p(v, S, \lambda)$ is pressure, T is temperature, F is the free energy increment, and $r = r(v, S, \lambda)$ is the reaction rate function.

In three space dimensions with coordinates (x, y, z) a steady planar strong detonation profile is a weak solution of this system depending only on x with a jump (the stationary von Neumann shock) at $x = 0$. Hence, we study profiles of the form $w(x) = (v, u, 0, 0, S, \lambda)(x)$. The solution u is constant and supersonic ($u > c_0$, where c_0 is the sound speed at x) in $x < 0$ (the quiescent zone) and u has an exponential decay at $+\infty$. A suitable Rankine-Hugoniot condition at $x = 0$ for u is required to make u to be a weakly well-defined solution in the vicinity of $x = 0$. Furthermore, there is a need of defining a limiting state $w_\infty = \lim_{x\to+\infty} w(x)$, which represents a state of chemical equilibrium.

Erpenbeck [17] was the first one to perform a study of linearized ZND detonations. After writing the perturbations as normal modes $e^{\tau t + ik_1 y + ik_2 z} V(x)$, he reduces the linear study into the investigation of a 5×5 system of ODEs. A different point here is that no variational formulation can be found due to the *non self-adjointness* of the ODE system of odd order. Erpenbeck defined a stability function $V(\tau, k)$ ($k = \sqrt{k_1^2 + k_2^2}$), whose zeros in the right half plane $\mathrm{Re}\tau > 0$ correspond to the characteristic values in our set-up in this thesis. For certain classes of steady ZND profiles, he provided some rigorous, and also non-rigorous arguments to prove the existence of unstable zeros of V in the high frequency regime $k \to \infty$. A method based on linear turning point theory was used by Lafitte, Williams and Zumbrun [52, 53] to complete the linear stability/instability. In particular, for the instability result, the authors obtain a sequence $\{k_n\}_{n\geqslant 1}$ such that $\mathrm{Re}\lambda(k_n) > 0$. It would be interesting to state the main results of [52, 53] as a spectral result in the same spirit of this thesis and then extend the linear study to the nonlinear one, if possible.

As a last point, we hope to revisit the second theorem of the first paper [51], that only shows the existence of possibly multiple characteristic values to the linearized equations. We expect that the method initiated by [53] would be helpful to show the existence of infinitely many characteristic values for the density profile ρ_0 such that $|\rho_0(x) - \rho_0(\infty)| \leqslant Ce^{-\beta|x|}$ for some positive constants C and β.

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