Université Sorbonne Paris Nord
École doctorale Galilée – LAGA

# Dynamiques des multisolitons pour certains champs scalaires

par Abdon DE ANDRADE MOUTINHO NETO

Thèse présentée et soutenue publiquement le 26 juillet 2023
en vue de l'obtention du grade de
Docteur de l'Université Sorbonne Paris Nord
sous la direction de Thomas Duyckaerts
et Jacek Jendrej

Membres du jury:

| | |
|---|---|
| Directeur: Thomas Duyckaerts | Université Sorbonne Paris Nord |
| Rapporteur: Philippe Gravejat | CY Cergy Paris Université |
| Rapporteur: Joachim Krieger | École Polytechnique Fédérale de Lausanne |
| Examinatrice: Anne-Sophie de Suzzoni | Université Sorbonne Paris Nord |
| President du Jury: Didier Pilod | Université de Bergen |
| Co-directeur: Jacek Jendrej | Université Sorbonne Paris Nord |
| Examinateur: Hatem Zaag | Université Sorbonne Paris Nord |
| Examinatrice: Birgit Schörkhuber | Université d'Innsbruck |

# Thanks

First of all, I would like to thank my two supervisors Thomas Duyckaerts and Jacek Jendrej for accepting my application to be a Ph.D. student at Université Sorbonne Paris Nord and for teaching me how to do mathematical research. I would like also to thank both of them for their helpful discussions and suggestions, without them I could not obtain the results and the content I have in this thesis. I would like also to thank them for inviting me to participate in conferences and helping me how to expose my work in seminars. I am also very grateful for them to teach me the importance of good writing and how to improve my writing skills, which were essential for the acceptance of my first paper to a good mathematical journal.

I would like also to thank Joachim Krieger and Philippe Gravejat, who gently desired to examine my thesis. I would like also to thank Gong Chen, Jonas Jonas Lührmann, and Didier Pilod for inviting me to give seminar talks at their universities and for the consideration they had with my work.

I would also like to thank the president of the University Sorbonne Paris Nord and the secretaries of human resources of LAGA and of Université Sorbonne Paris Nord for their patience and for their helpful support, which were essential for my adaptation in France. I would like to thank the math department of LAGA for the research seminars and for sharing their resources and contents to help me to improve my work. I would like also to thank the funding support of l'École doctorale de l'Institut Galilée. I would like also to thank all the staff of l'École doctorale de l'Institut Galilée for helping in my adaptation to the university and solving any issues that I had. I would like also to thank l'Ecole Universitaire de Recherche for the offer of graduate student that I received and for their support in helping me to find accommodation and in helping in my bureaucratic transition from Brazil to France.

The author also is grateful for the support of the French State Program "Investissement d'Avenir" managed by the "Angence Nationale de la Recherche".

I am also grateful for my friends from LAGA, especially Wassim, Tai, Mohammed Zitouni, Moussa, Mohammed Ali, Oussama, Emmanuel, Neige, Vuong, Guilhem, Mouna, Safa, Elyes, Arthur, Maissa, Hugo, and Bart, for the experience and conversations. Unfortunately, I could only meet some of them for a short time or in the final year of my Ph.D., but I still had a very good experience and funny conversations, I am very grateful.

Next, I would like to thank my mother and my father for believing in my capacity and for supporting me in achieving my dreams during my life. I would like to thank my aunt and my cousins to welcome me to Rio de Janeiro to do summer courses at IMPA and my father for helping me with accommodation and support at Rio de Janeiro while I was an undergraduate and master's student of mathematics. I would like also to thank my two brothers for their support and conversations even though we are separated by large distances for a long time period. I would like also to thank my mother and Amelia for always contacting me every week with helpful messages.

I would like also to thank my master's advisor Adán Corcho at UFRJ for his support and teaching while I was his student, which helped and motivated me more to be a mathematician. I would also like to thank all my professors at UFRJ, especially Andrew Clarke, Didier Pilod, Xavier Carvajal, Bruno Scardua, Flavio Dickstein, Graham Smith, and Seyed Hamid Hassanzadeh, all of them taught me greatly and many of them showed me what is the motivation behind doing mathematics. I would like also to thank all my friends from UFRJ for the great and funny experiences we had and for the support we gave to each other.

I would like also to thank my psychologist Bruno to help me to manage my emotions during difficult periods and still helping me to make more rational decisions. I would like also to thank the physician Dr. Wuyts for the help in treating me during all my years in Paris, without it the quality of my life would be not as great as I am now.

I would like also to thank the staff of IMPA whom I am grateful for the support they gave me when I attend graduate and summer courses there. It was a great experience to visit and work there.

I would like also to thank my friends Alex, Jamie, Tainara, and Oscar whom I had the pleasure to meet during a summer course at IMPA. I am also grateful to Felipe Linares for his patience and support, it was a great pleasure to learn the introductory theory of nonlinear dispersive equations from his book and his teaching.

Finally, I would like to thank my friends from Recife, all the conversations and experiences that we have been living are great, despite the large distance we are now.

# Résumé

Cette thèse s'inscrit dans l'étude qualitative des multi-solitons d'une équation d'onde non linéaire unidimensionnelle connue sous le nom de mòdele $\phi^6$. Ce modèle a des applications en théorie de la matière condensée, en physique des hautes énergies et en cosmologie. Les solitons associés à ce modèle sont connus sous le nom de kinks et antikinks, et tous deux sont les uniques solutions stationnaires non constantes du modèle $\phi^6$ ayant une énergie finie.

Dans la première partie de la thèse, nous décrivons toutes les solutions du modèle $\phi^6$ satisfaisant une condition aux limites avec une énergie proche du minimum. Nous allons prouver que chacune de ces solutions est une petite perturbation d'une somme de deux kinks en mouvement pendant un grand intervalle de temps. Nous analysons également le mouvement de ces solitons comme un problème à deux corps en utilisant un système différentiel ordinaire explicite. Nous prouvons que le déplacement des deux kinks est une petite perturbation de la solution de ce système différentiel ordinaire pendant un grand intervalle de temps.

Dans la deuxième partie de la thèse, nous analysons la collision entre deux kinks du modèle $\phi^6$. Nous prouvons que la collision est presque élastique, ce qui est inattendu puisque ce modèle est non intégrable. Nous estimons le défaut produit par la collision dans la vitesse de chaque soliton et dans la taille du résidu. Nous prouvons que la taille du défaut est d'ordre inférieur au polynôme pour une faible vitesse entrante.

**Mots clés :**

- Équation d'onde non linéaire unidimensionnelle
- Multi-solitons
- Kinks
- Antikinks
- Modèle $\phi^6$
- Collision
- Problème à deux corps
- Modèle non intégrable

# Abstract

This thesis is concerned with the qualitative study of multi-solitons of a one-dimensional nonlinear wave equation known as the $\phi^6$ model. This model has applications in condensed matter theory, high energy physics, and cosmology. The solitons associated with this model are known as kinks and antikinks, and both are the unique non-constant stationary solutions of the $\phi^6$ model having finite energy.

In the first part of the thesis, we describe all the solutions of the $\phi^6$ model satisfying a boundary condition with energy close to the minimum. We will prove that any of these solutions is a small perturbation of a sum of two moving kinks during a large time interval. We also analyze the movement of these solitons as a two-body problem using an explicit ordinary differential system. We prove that the displacement of the two kinks is a small perturbation of the solution of this ordinary differential system during a large time interval.

In the second part of the thesis, we analyze the collision between two kinks of the $\phi^6$ model. We prove that the collision is almost elastic, which is unexpected since this model is non-completely integrable. We estimate the defect produced by the collision in the speed of each soliton and in the size of the residue. We prove that the size of the defect is of order smaller than a polynomial for low incoming speed.

**Keywords:**

- One-dimensional nonlinear wave equation
- Multi-solitons
- Kinks
- Antikinks
- $\phi^6$ model
- Collision
- Two-body problem
- Non-integrable model

# Contents

# Chapter 1

# Introduction

We consider the following partial differential equation

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + 2\phi(t,x) - 8\phi(t,x)^2 + 6\phi(t,x)^5 = 0, \tag{$\phi^6$}$$

which is known in the physics literature also as the $\phi^6$ model. The partial differential equation ($\phi^6$) is a scalar field of dimension $1+1$ of the form

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U^{'}(\phi(t,x)) = 0,$$

for the potential function $U(\phi) = \phi^2(1-\phi^2)^2$.

First, we are interested in the study of all the solutions $\phi(t,x)$ satisfying, for any $t \in \mathbb{R}$, the following boundary condition

$$\lim_{x\to-\infty}\phi(t,x) = -1, \ \lim_{x\to+\infty}\phi(t,x) = 1, \tag{Bc}$$

and having energy slightly bigger than the minimum of the energy of all solutions of ($\phi^6$) satisfying (Bc). We are going to verify that these solutions are close to a sum of two solitons and each of them moves with a small speed. Moreover, we will see that the displacement of each soliton is very close to an explicit solution of an ordinary differential system under additional conditions.

The second topic discussed in this manuscript is the study of the elasticity of the collision between two moving solitons of the partial differential equation ($\phi^6$). More precisely, we will only consider the collision between two increasing solitons $H_1$, $H_2$ which are approaching with a sufficiently small speed $v>0$ and study their long-time behavior after they collide.

The study of nonlinear wave equation ($\phi^6$) has applications in different fields of theoretical physics. More precisely, this model has applications in condensed matter theory, see [3], which is a field of physics interested in studying the properties of a system of particles or atoms either under conditions of very low temperature or when there exist high interaction forces between the components of the system. The study of the $\phi^6$ model has also applications in cosmology, see for example [62], and high energy physics, see for example [17] and [14].

Before we state our main results, we will introduce briefly the mathematical theory of scalar fields, the concept of topological solitons with a focus on the kinks and antikinks, and the local theory of the partial differential equation ($\phi^6$).

**Notation 1.0.1.** *In this manuscript, for any* $n$, $m \in \mathbb{N}_{\geq 1}$, *we denote the space of smooth functions* $f : \mathbb{R}^n \to \mathbb{R}^m$ *with compact support by* $C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$. *In particular, when* $m = 1$, *we denote* $C_0^\infty(\mathbb{R}^n;\mathbb{R}^m)$ *by* $C_0^\infty(\mathbb{R}^n)$.

*Similarly, for any* $1 \leq p \leq +\infty$, *we denote the space* $L^p(\mathbb{R}^n;\mathbb{R}^m)$ *as the real linear space generated by all the measurable functions* $f : \mathbb{R}^n \to \mathbb{R}^m$ *satisfying*

$$\int_{\mathbb{R}^n} |f(x)|^p \, dx < +\infty.$$

*If* $p = +\infty$, *we denote* $L^\infty(\mathbb{R}^n;\mathbb{R}^m)$ *as the real linear space generated by all the measurable functions* $f : \mathbb{R}^n \to \mathbb{R}^m$ *satisfying*

$$\inf\left\{c > 0 | \lambda \left\{x \in \mathbb{R}^n \Big| \, |f(x)| > c\right\} = 0\right\} < +\infty,$$

*where* $\lambda$ *is the Lebesgue measure in the Euclidean space* $\mathbb{R}^n$. *If* $m = 1$, *we denote, for any* $0 < p < +\infty$, *each space* $L^p(\mathbb{R}^n;\mathbb{R}^m)$ *by* $L^p(\mathbb{R}^n)$.

*For any* $m$, $n \in \mathbb{N}_{\geq 1}$ *and any function* $f : \mathbb{R}^n \to \mathbb{R}^n$, *we use the following notation*

$$\Delta f(x) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f(x), \; \nabla f(x) = \left(\partial_{x_i} f(x)\right)_{i \in \{1,\dots,n\}},$$

*for every* $x \in \mathbb{R}^n$.

## 1.1 Brief introduction to Lagrangians

First, we consider the Euclidean space $\mathbb{R}^{1+n}$ with the Minkowski metric $g = -dt^2 + \sum_{j=1}^n dx_j^2$ and a complete Riemannian manifold $M$ of dimension $n$ with a Riemannian metric $\hat{g}$. We denote the set of maps $\phi : (\mathbb{R}^{1+n}, g) \to (M, \hat{g})$ by $\mathcal{O}$ and, for any function $f : \mathbb{R}^{n+1} \to M$ of class $C^1$ at least, we define, for any $\mu \in \{0, ..., n\}$ and any $(t, x) = (t, x_1, ..., x_n) \in \mathbb{R}^n$,

$$\partial^\mu f(t,x) = \begin{cases} \partial_{x_\mu} f(t,x), \text{ if } \mu \neq 0, \\ -\partial_t f(t,x), \text{ otherwise.} \end{cases}$$

Moreover, for any $x \in M$ and any $v(x)$ in the tangent space $T_x M$, we denote $|v|_{\hat{g}} = \hat{g}\left(v(x), v(x)\right)^{\frac{1}{2}}$.

Next, we consider a smooth function $U : M \subset \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ and the set $D$ as

$$D = \{\phi \in \mathcal{O} | \, \phi \in L^\infty(\mathbb{R}^{1+n}; M), \text{ and for all } t \in \mathbb{R}, \, \sum_{i=1}^n |\partial_{x_i}\phi(t,x)|_{\hat{g}}, \, |\partial_t \phi(t,x)|_{\hat{g}} \in L^2_x(\mathbb{R}^n)\},$$

Additionaly, for an interval $(t_1, t_2)$ not necessarily bounded and any functions $\phi_1$, $\phi_2 : \mathbb{R}^n \to M$ in $L^\infty$, we study the critical points of the function $L : D \cap \{\phi(t_j, x) = \phi_j(x) \text{ for } j \in \{1,2\}\} \to \mathbb{R}$ denoted by

$$L(\phi) = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \frac{1}{2} \sum_{\mu=0}^n \hat{g}\left(\partial^\mu \phi(t,x), \partial_\mu \phi(t,x)\right) + U(\phi(t,x)) \, dx dt. \tag{Ge. Lagr.}$$

It is well known that the critical points of functions (Ge. Lagr.) are solutions of nonlinear wave equations, see Chapter 2 of [36] for more information. Indeed, many dispersive models are obtained from the research of this kind of variational problem, see for example the sine-Gordon and $\phi^4$ models in Chapter 5 of [36], and the wave maps in the book [18]. The motivation of the study of these variational problems has applications in different fields of mathematical physics, for example, condensed matter theory [3] and cosmology [62], see also [36] for more information.

Actually, if we consider $M = \mathbb{R}^n$ and $\hat{g}$ the Euclidean metric of $\mathbb{R}^n$, the function $L$ can be rewritten as

$$L(\phi) = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \frac{1}{2} \left[ |\nabla \phi(t,x)|^2 - |\partial_t \phi(t,x)|^2 \right] + U(\phi(t,x))\, dxdt. \tag{Simpl. Lagr.}$$

If $\phi(t,x)$ is a critical point of $L$, then, for any function $\delta \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^n)$ such that $\operatorname{supp} \delta \subset\subset (t_1, t_2) \times K$ for some compact set $K \subset \mathbb{R}^n$, we obtain from the identity

$$\lim_{\epsilon \to 0} \frac{L(\phi + \epsilon \delta) - L(\phi)}{\epsilon} = 0,$$

and integration by parts that $\phi$ shall satisfy the following Euler-Lagrange equation

$$\partial_t^2 \phi(t,x) - \Delta \phi(t,x) + \nabla U(\phi(t,x)) = 0, \tag{1.1}$$

for any $t \in (t_1, t_2)$. The partial differential equation ($\phi^6$) studied in this thesis also satisfies equation (1.1) when $n = 1$ and $U(\phi) = \phi^2 (1 - \phi^2)^2$. See also Chapter 2 of [36] for more references about Lagrangians.

## 1.2 Scalar fields and Lagrangians

### 1.2.1 Background context

We consider, for $n \in \mathbb{N}_{\geq 1}$, a smooth potential function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying $\lim_{|y| \to \pm\infty} U(y) = +\infty$ such that the set $U^{-1}\{0\}$ is a compact manifold and every $u \in U^{-1}\{0\}$ also satisfies $U'(u) = 0$. We consider for any field $\phi : \mathbb{R}^n \to M$ such that $|\partial_{x_i} \phi(x)|$ is in $L^2(\mathbb{R}^n)$ for all $i \in \{1, ..., n\}$ the following function

$$L_U(\phi) = \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{|\partial_{x_i} \phi(x)|^2}{2} + U(\phi(x))\, dx. \tag{Stat. Lagr.}$$

We define the vacuum set by

$$\mathcal{V} = \{ y \in M |\, U(y) = 0 \}. \tag{Vacuum}$$

Clearly, if a Lipschitz field $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is in $L^\infty$ satisfying $L_U(\phi) = 0$, then it is not difficult to verify the existence of $\mu \in \mathcal{V}$ such that $\phi \equiv \mu$. Moreover, if $\phi \in C(\mathbb{R}^n; \mathbb{R}^n)$ is a Lipschitz function satisfying $L_U(\phi) < +\infty$, we would also need for any $v \in \mathbb{S}^{n-1}$ that

$$\lim_{r \to +\infty} \inf_{y \in \mathcal{V}} |\phi(vr) - y| = 0.$$

Otherwise, $\int_{\mathbb{R}^n} U(\phi(x))\,dx = +\infty$.

Furthermore, for any non-constant map $\sigma : \mathbb{S}^{n-1} \to \mathcal{V}$, we can consider the following set

$$\mathcal{V}_\sigma = \{\phi|\, \phi : \mathbb{R}^n \to \mathbb{R}^n,\, L_U(\phi) < +\infty \text{ and } \phi_\infty := \lim_{r\to+\infty} \phi(r\cdot) : \mathbb{S}^{n-1} \to \mathcal{V} \text{ is equal to } \sigma\},$$

and the following problem:

$$\text{Is there a continuous function } \phi \in \mathcal{V}_\sigma \text{ satisfying } L_U(\phi) = \inf_{\psi\in\mathcal{V}_\sigma} L_U(\psi)? \tag{P.0}$$

If there existed a minimizer $\phi$, then it should be a weak solution of the following Euler-Lagrange equation

$$\Delta\phi(x) = \nabla U(\phi). \tag{1.2}$$

When $n = 1$, we can identify the set $\mathbb{S}^{n-1}$ as the binary set $\{-1, 1\}$. In this case, we will see in the next sections that the existence of solutions of problem P.0 is possible only if there doesn't exist $v \in \mathcal{V}$ satisfying either $\sigma(-1) < v < \sigma(1)$ or $\sigma(1) < v < \sigma(-1)$.

However, when $n \geq 2$, there doesn't exist any solution of problem (P.0) for any non-constant continuous map $\sigma : \mathbb{S}^{n-1} \to \mathcal{V}$ and any continuous potential function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying the conditions $\lim_{|y|\to+\infty} U(y) = +\infty$ and $U'(u) = 0$ always when $U(u) = 0$. This result is known as Derrick's Theorem, see Section 4.2 of the book [36] for more information.

Moreover, using an argument of contradiction, the proof of Derrick's Theorem is straightforward. More precisely, If $n \geq 2$ and there exists a non-constant continuous field satisfying $\phi \in \mathcal{V}_\delta$ minimizing $L_U$, then we would have that $\phi_{(r)}(x) := \phi(rx)$ should satisfy $L_U(\phi_{(r)}) \geq L_U(\phi)$ for all $r > 0$, because the set $\{\phi_{(r)}|\, r \in \mathbb{R}_{>0}\}$ is contained in $\mathcal{V}_\delta$. But, from the change of variable $y(x) = xr$ and identity $\nabla\phi_{(r)}(x) = r\nabla\phi(rx)$, we can verify the following equations

$$\int_{\mathbb{R}^n} U(\phi_{(r)}(x))\,dx = \frac{1}{r^n}\int_{\mathbb{R}^n} U(\phi(x))\,dx,\ \int_{\mathbb{R}^n} \left|\nabla\phi_{(r)}(x)\right|^2\,dx = \frac{1}{r^{n-2}}\int_{\mathbb{R}^n} |\nabla\phi(x)|^2\,dx,$$

for every $r > 0$. Therefore, we have

$$L_U\left(\phi_{(r)}\right) = \frac{1}{2r^{n-2}}\left[\int_{\mathbb{R}^n} |\nabla\phi(x)|^2\,dx\right] + \frac{1}{r^n}\left[\int_{\mathbb{R}^n} U(\phi(x))\,dx\right].$$

If $n \geq 3$, then the function $L_U\left(\phi_{(r)}\right)$ is decreasing on $r$ unless $\phi$ is a constant function with image on $\mathcal{V}$, so $\phi$ is not a solution of problem P.0, which is a contradiction. If $n = 2$, the function $L_U\left(\phi_r\right)$ is non-decreasing only if

$$\int_{\mathbb{R}^n} U(\phi(x))\,dx = 0,$$

which would imply that the image of $\phi$ is contained in $\mathcal{V}$. But, since $\phi$ is a weak solution of equation (1.2), $\left\{U'(u)|\, u \in \mathcal{V}\right\} = \{0\}$ and $M$ is a compact set, we would have that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ should be a bounded harmonic function, therefore Liouville's Theorem would imply that $\phi$ should be a constant map, which is a contradiction of $\phi$ being in $\mathcal{V}_\sigma$.

### 1.2.2 One-dimensional scalar fields

From now on, we consider $n = 1$ and a smooth function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\lim_{|y| \to +\infty} U(y) = +\infty, \text{ and } \mathcal{V} = U^{-1}\{0\} \text{ is a compact set.}$$

In this particular case, the partial differential equation (1.2) can be rewritten as the following elliptic equation

$$\phi^{''}(x) = U^{'}(\phi(x)). \tag{1.3}$$

Since $U \in C^\infty$, we can verify using the elliptic regularity theory that if $\phi \in L^\infty(\mathbb{R})$ is a weak solution of equation (1.3), then $\phi \in C^\infty$, see Theorem 2 from Chapter 6 of [16] for more information. Clearly, if $\phi$ is a strong solution (1.3) satisfying $L_U(\phi) < +\infty$, then $\phi$ is a critical point of $L_U$.

**Definition 1.2.1.** *We say that a one-dimensional scalar field $\phi$ is a topological soliton of the differential equation* (1.3)*, if $\phi$ is a strong solution of* (1.3)*, it satisfies*

$$\int_{\mathbb{R}} \left|\phi^{'}(x)\right|^2 + U\left(\phi(x)\right)\, dx < +\infty,$$

*and $\phi_\infty \coloneqq \lim_{r \to +\infty} \phi(r\cdot) : \{-1, 1\} \to \mathcal{V}$ is a non-constant map.*

**Remark 1.2.2.** *Furthermore, when $n = 2$, we highlight that the topological solitons are critical points of Lagragians of a different form from* (1.2) *and we have verified earlier that there doesn't exist any non-constant solution of* (1.3) *satisfying*

$$\int_{\mathbb{R}^n} U(\phi(x)) + |\nabla\phi(x)|^2\, dx < +\infty,$$

*when $n \geq 2$, see also Subsection* 7.1 *of Chapter* 7 *from the book [36].*

*For example, for $n = 2$ and the potential function $U(\phi) = \left(1 - |\phi|^2\right)^2$, the topological solitons are defined as the non-constant maps*

$$(\phi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2,$$

*which are the critical points of the following Lagrangian*

$$\int_{\mathbb{R}} |\nabla_A \phi(x)|^2 + |\operatorname{curl} A(x)|^2 + \left(1 - |\phi(x)|^2\right)^2\, dx, \tag{1.4}$$

*where $\nabla \coloneqq \nabla - iA$ and*

$$\operatorname{curl}(f_1, f_2)(x) = \frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x),\ \operatorname{curl}(f) = \left(\frac{\partial f(x)}{\partial x_2}, -\frac{\partial f(x)}{\partial x_1}\right),$$

*for any functions $(f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ and all $x \in \mathbb{R}^2$. Furthermore, the Euler-Lagrange equations associated to* (1.4) *are given by*

$$\begin{aligned} \nabla_A^2 \phi(x) =& 2\left(1 - |\phi(x)|^2\right)\phi(x), \\ \operatorname{curl}^2 A =& \operatorname{Im}\left(i\overline{\phi(x)}\nabla_A \phi(x)\right), \end{aligned} \tag{1.5}$$

*where* $\nabla_A f(x) := \nabla f(x) - iA(x)f(x)$ *for any function* $f : \mathbb{R}^2 \to \mathbb{C}$*. One of the reasons to consider the Lagrangian* (1.4) *instead of* (Stat. Lagr.) *is to use, for any* $\alpha \in \mathbb{R}$*, the following transformation* $\phi_\alpha(x) = \phi(x)e^{i\alpha}$*, which is an invariance* (1.4) *and also satisfies*

$$\int_{\mathbb{R}^2} U(\phi_\alpha(x))\,dx = \int_{\mathbb{R}^2} U(\phi(x))\,dx.$$

*For more detailed information, see Subsection* 2.6 *of Chapter* 2 *and Chapter* 7 *of the book [36], see also the article [22] for more information about the partial differential equation* (1.5) *and its topological solitons.*

Since we are mainly interested in the topological solitons associated with the partial differential equation ($\phi^6$), we will describe in the next sections the properties of topological solitons associated with one-dimensional scalar field equations, which are the strong solutions of (1.3) satisfying all the conditions in Definition 1.2.1. The topological solitons associated with one-dimensional scalar fields are divided into two groups the kinks and the antikinks.

### 1.2.3 Kinks and antikinks

In this subsection, we consider $U \in C^\infty(\mathbb{R})$ satisfying $U(y) \geq 0$ for any $y \in \mathbb{R}$ and $\lim_{|y|\to+\infty} U(y) = +\infty$. In addition, we assume that $U$ satisfies the following property

$$U^{''}(x) \neq 0, \text{ for all } x \in \mathcal{V}, \tag{Non-degeneracy}$$

where $\mathcal{V}$ is defined in (Vacuum) for $n = 1$.

Next, we consider a solution $\phi \in C^\infty(\mathbb{R})$ of the ordinary differential equation

$$\begin{cases} \phi^{''}(x) = U^{'}(\phi(x)), \\ \lim_{x\to+\infty} \phi(x) \text{ and } \lim_{x\to-\infty} \phi(x) \in \mathcal{V}, \end{cases} \tag{1.6}$$

satisfying $L_U(\phi) < +\infty$, where $L_U$ is defined in (Stat. Lagr.). Now, we are going to present the properties of all the solutions $\phi$ of the ordinary differential equation (1.6) satisfying $L_U(\phi) < +\infty$.

**Lemma 1.2.3.** *If* $\lim_{x\to-\infty} \phi(x) = \lim_{x\to+\infty} \phi(x)$*, then the smooth solution* $\phi(x)$ *of the problem* (1.6) *is a constant function.*

*Proof of Lemma 1.2.3.* Since $\lim_{x\to-\infty} \phi(x)$ is equal to $\lim_{x\to+\infty} \phi(x)$, if $\phi$ is not a constant function, then there would exists $x_0 \in \mathbb{R}$ satisfying either

$$\lim_{x\to+\infty} \phi(x) < \phi(x_0) = \max_{x\in\mathbb{R}} \phi(x) \text{ or } \lim_{x\to+\infty} \phi(x) > \phi(x_0) = \min_{x\in\mathbb{R}} \phi(x) \tag{1.7}$$

and so, $\frac{d\phi(x_0)}{dx} = 0$. Furthermore, since $\phi \in C^\infty(\mathbb{R})$, we have from the ordinary differential equation (1.6) that

$$\frac{d}{dx}\left[\frac{d\phi(x)}{dx}^2 - 2U\left(\phi(x)\right)\right] = 2\left[\phi^{''}(x) - U^{'}\left(\phi(x)\right)\right]\phi^{'}(x) = 0,$$

and so, the function $\frac{d\phi(x)}{dx}^2 - 2U(\phi(x))$ is constant. Therefore, we would deduce from the Fundamental Theorem of Calculus that

$$\frac{d\phi(x)}{dx}^2 = 2\left[U(\phi(x)) - U(\phi(x_0))\right] \text{ for any } x \in \mathbb{R}.$$

Moreover, since $\lim_{x\to\pm\infty} \phi(x) \in \mathcal{V}$ and $\mathcal{V} = U^{-1}(0)$, we would obtain from the identity above that $\phi(x_0) \in \mathcal{V}$, otherwise $L_U(\phi) = +\infty$. Consequently, $\phi$ would satisfy the following ordinary differential system of equations

$$\begin{cases} \phi^{''}(x) = U^{''}(\phi(x)), \\ \phi(x_0) \in \mathcal{V},\ \frac{d\phi(x_0)}{dx} = 0. \end{cases}$$

However, from Picard-Lindelöf Existence-Uniqueness Theorem, we would obtain that $\phi(x) = \phi(x_0)$ for any $x \in \mathbb{R}$, which contradicts (1.7). In conclusion, $\phi$ shall be a constant function. □

**Lemma 1.2.4.** *The unique solutions of* (1.6) *which are topological solitons associated to U are the smooth solutions $\phi$ of only one of the following ordinary differential equations*

$$\phi^{'}(x) = \sqrt{2U(\phi(x))} \text{ or } \phi^{'}(x) = -\sqrt{2U(\phi(x))}, \tag{1.8}$$

*which satisfy $L_U(\phi) < +\infty$.*

*Proof of Lemma 1.2.4.* First, from elliptic regularity theory, Definition 1.2.1 and Lemma 1.2.3, we can verify that $\phi : \mathbb{R} \to \mathbb{R}$ is a topological soliton only if $\phi \in C^\infty(\mathbb{R})$ and $\phi$ satisfies

$$\lim_{x\to+\infty} \phi(x) \neq \lim_{x\to-\infty} \phi(x).$$

Furthermore, from the proof of Lemma 1.2.3, if $\phi$ is a smooth function satisfying $\phi^{''}(x) = U^{'}(\phi(x))$ for any $x \in \mathbb{R}$, then $\frac{d\phi(x)}{dx}^2 - 2U(\phi(x))$ is constant. Moreover, if $L_U(\phi) < +\infty$, we also would have that

$$\frac{d\phi(x)}{dx}^2 = 2U(\phi(x)), \text{ for all } x \in \mathbb{R}.$$

Consequently, (1.8) is a necessary condition for a function $\phi$ to be a topological soliton. Therefore, to conclude the proof of Lemma 1.2.4, it is enough to verify that only one of the equations in (1.8) shall be true.

We assume by contradiction that there exist $x_1$, $x_2 \in \mathbb{R}$ such that

$$\phi^{'}(x_1) = +\sqrt{2U(\phi(x_1))},\ \phi^{'}(x_2) = -\sqrt{2U(\phi(x_2))}.$$

Hence, from the Intermediate Value Theorem, there exist $x_3 \in \mathbb{R}$ satisfying $\phi^{'}(x_3) = 0$, from which we would obtain that $\phi(x_3) \in U^{-1}(0)$. However, from Picard-Lindelöf Existence-Uniqueness Theorem, we would obtain that $\phi(x) = \phi(x_3)$ for all $x \in \mathbb{R}$, which contradicts the hypothesis that $\phi$ is a topological soliton satisfying Definition 1.2.1. In conclusion, the statement of Lemma 1.2.4 is true. □

**Definition 1.2.5.** *We say that a real function $\phi : \mathbb{R} \to \mathbb{R}$ is a kink associated to the potential function $U$ if, and only if, the function $\phi$ is a non-constant solution of the following ordinary differential equation*

$$\phi^{'}(x) = \sqrt{2U\left(\phi(x)\right)}, \tag{1.9}$$

*and $L_U(\phi) < +\infty$. We say that a function $\psi$ is an antikink if, and only if, the function $\phi(x) \coloneqq \psi\left(-x\right)$ is a kink.*

**Remark 1.2.6.** *Let $\phi$ be a kink function. We consider*

$$\upsilon_{+\infty} = \lim_{x \to +\infty} \phi(x),\ \upsilon_{-\infty} = \lim_{x \to -\infty} \phi(x).$$

*From Lemma 1.2.4, $\upsilon_{+\infty} \neq \upsilon_{-\infty}$, furthermore, since*

$$\phi^{'}(x) = \sqrt{2U\left(\phi(x)\right)} \geq 0,$$

*it is not difficult to verify that $(\upsilon_{-\infty}, \upsilon_{+\infty}) \cap \mathcal{V} = \emptyset$. Otherwise, we would obtain the existence of $x_0 \in \mathbb{R}$ such that $\phi\left(x_0\right) \in \mathcal{V}$, which would imply that $\phi$ is a constant function.*

## 1.3 The $\phi^6$ model

### 1.3.1 Preliminaries

From now on, we consider the potential function $U : \mathbb{R} \to \mathbb{R}$ given by $U(\phi) = \phi^2\left(1-\phi^2\right)^2$. We consider the following nonlinear wave equation

$$\begin{cases} \partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + 2\phi(t,x) - 8\phi(t,x)^2 + 6\phi(t,x)^5 = 0, \\ \lim_{x\to+\infty}\phi(x) = 1,\ \lim_{x\to-\infty}\phi(x) = -1, \end{cases} \tag{$\phi^6$−NLW}$$

which is equivalent to the scalar field of dimension $1+1$

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U^{'}\left(\phi(t,x)\right) = 0.$$

The kinks associated with $U$ are solutions of the following ordinary differential equation

$$\phi^{'}(x) = \sqrt{2}\left|\phi(x)\left(1-\phi(x)^2\right)\right|. \tag{1.10}$$

Clearly, the vacuum set $\mathcal{V}$ associated to this potential function is $\{0, -1, +1\}$. Therefore, Lemma 1.2.4 and Remark 1.2.6 imply that the only possible kink solutions $\phi : \mathbb{R} \to \mathbb{R}$ should satisfy one of the following boundary condition

$$\lim_{x\to-\infty}\phi(x) = -1 \text{ and } \lim_{x\to+\infty}\phi(x) = 0, \text{ or } \lim_{x\to-\infty}\phi(x) = 0 \text{ and } \lim_{x\to+\infty}\phi(x) = 1.$$

By a standard application of the Fundamental Theorem of Calculus, we obtain that the following functions

$$H_{0,1}(x) = \frac{e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}},\ H_{-1,0}(x) = -\frac{e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}} \tag{1.11}$$

are solutions of ordinary differential equation (1.10). Indeed, from Picard-Lindelöf Existence-Uniqueness Theorem and since $H_{0,1}$ is a function in $C^{\infty}(\mathbb{R})$ satisfying $\lim_{x\to-\infty} H_{0,1}(x) = 0$, $\lim_{x\to+\infty} H_{0,1}(x) = 1$, we deduce that the only solutions of (1.10) satisfying the boundary conditions $\lim_{x\to-\infty}\phi(x) = 0$, $\lim_{x\to+\infty}\phi(x) = 1$ are the set of functions whose elements are the scalar fields $\phi_h : \mathbb{R} \to \mathbb{R}$ defined by $\phi_h(x) = H_{0,1}(x+h)$ for any $x,\, h \in \mathbb{R}$. Similarly, the only kinks satisfying the boundary condition $\lim_{x\to-\infty}\phi(x) = -1$ and $\lim_{x\to+\infty}\phi(x) = 0$ are the translations of the function $H_{-1,0}(x)$.

**Notation 1.3.1.** *We denote the Sobolev space $H^1_x(\mathbb{R})$ as the completion of the space $C^{\infty}_0(\mathbb{R})$ in the norm $\|\cdot\|_{H^1_x}$ satisfying*

$$\|f\|^2_{H^1_x} = \int_{\mathbb{R}} \frac{df(x)}{dx}^2 + f(x)^2\,dx,$$

*for any real function $f \in C^{\infty}_0(\mathbb{R})$. We also consider the norm $\|\cdot\|_{L^2_x}$ which satisfies*

$$\|f\|_{L^2_x} = \int_{\mathbb{R}} f(x)^2\,dx,$$

*for every $f \in L^2_x(\mathbb{R})$*

**Definition 1.3.2.** *For any $t \in \mathbb{R}$, $\cos\left(t\sqrt{-\Delta}\right)$ and $\sin\left(t\sqrt{-\Delta}\right)$ are the linear bounded maps*

$$\cos\left(t\sqrt{-\Delta}\right) : \left(L^2_x(\mathbb{R}), \|\cdot\|_{L^2_x}\right) \to \left(L^2_x(\mathbb{R}), \|\cdot\|_{L^2_x}\right),$$
$$\sin\left(t\sqrt{-\Delta}\right) : \left(L^2_x(\mathbb{R}), \|\cdot\|_{L^2_x}\right) \to \left(L^2_x(\mathbb{R}), \|\cdot\|_{L^2_x}\right),$$

*which satisfies for any $f \in C^{\infty}_0(\mathbb{R})$ the following identities*

$$\cos\left(t\sqrt{-\Delta}\right)f(x) = \int_{\mathbb{R}} \hat{f}(y)\cos(2\pi t|y|)e^{2\pi ixy}\,dy,$$
$$\sin\left(t\sqrt{-\Delta}\right)f(x) = \int_{\mathbb{R}} \hat{f}(y)\sin(2\pi t|y|)e^{2\pi ixy}\,dy,$$

*where $\hat{f}$ is the Fourier transform of $f$, which is defined by*

$$\hat{f}(x) = \int_{\mathbb{R}} f(y)e^{-2\pi ixy}\,dy, \text{ for all } x \in \mathbb{R}.$$

*We also denote $\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}$ by the bounded linear map with same domain as $\sin\left(t\sqrt{-\Delta}\right)$ which satisfies the following identity*

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}f(x) = \int_{\mathbb{R}} \frac{\hat{f}(y)}{2\pi|y|}\sin(2\pi t|y|)e^{2\pi ixy}\,dy,$$

*for any $f \in C^{\infty}_0(\mathbb{R})$.*

**Lemma 1.3.3.** *There exists $C > 0$ such that for any $f,\, g \in H^1_x(\mathbb{R})$, we have*

$$\|fg\|_{H^1_x} \leq C\,\|f\|_{H^1_x}\|g\|_{H^1_x}.$$

*Proof.* See Lemma $A.8$ and its proof in [61]. □

**Definition 1.3.4.** *We say that a real function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a solution in the energy space of the partial differential equation ($\phi^6-$NLW) if, and only if, for all $t \in \mathbb{R}$ the function $\phi(t,x)$ satisfies*

$$\|\phi(t,x) - H_{0,1}(x) - H_{-1,0}(x)\|_{H^1_x} + \|\partial_t \phi(t,x)\|_{L^2_x} < +\infty,$$

*and for any $t$, $t_0 \in \mathbb{R}$, the function $u(t,x) = \phi(t,x) - H_{0,1}(x) - H_{-1,0}(x)$ is a solution of the following integral equation*

$$\begin{aligned} u(t,x) = Fu(t,x) &:= \cos\left((t-t_0)\sqrt{-\Delta}\right)u(t_0,x) + \frac{\sin\left((t-t_0)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}}\partial_t u(t_0,x) \\ &+ \int_{t_0}^{t} \frac{\sin\left((t-s)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} \Bigg( U'\left(H_{0,1}(x)\right) + U'\left(H_{-1,0}(x)\right) \\ &\qquad -U'\left(H_{0,1}(x) + H_{-1,0}(x) + u(s,x)\right)\Bigg)\, ds \end{aligned} \tag{1.12}$$

*in the space $C(\mathbb{R}, H^1_x(\mathbb{R})) \cap C^1(\mathbb{R}, L^2_x(\mathbb{R}))$, which means that the following map*

$$f(t) := u(t, \cdot)$$

*is a continuous function from $\mathbb{R}$ to $H^1_x(\mathbb{R})$ and the derivative $\frac{df(t)}{dt}$ is a well-defined continuous map from $\mathbb{R}$ to $L^2_x(\mathbb{R})$. For a better understanding in this concept of solution, see Chapter 3 of [61].*

From now on, we are going to verify that the Definition 1.3.4 is consistent. If $\phi$ is a smooth solution of the partial differential equation ($\phi^6$), then the function $u(t,x) = \phi(t,x) - H_{0,1}(x) - H_{-1,0}(x)$ is a smooth solution of the partial differential equation

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = U'(H_{0,1}(x)) + U'(H_{-1,0}(x)) - U'\left(H_{0,1}(x) + H_{-1,0}(x) + u(t,x)\right). \tag{1.13}$$

Indeed, from the identity $U(\phi) = \phi^2\left(1-\phi^2\right)^2$ and Taylor's Theorem, we deduce for any functions $u_1$, $u_2 \in H^1_x(\mathbb{R})$ the following identity

$$\begin{aligned} &U'\left(H_{0,1}(x) + H_{-1,0}(x) + u_1(x)\right) - U'\left(H_{0,1}(x) + H_{-1,0}(x) + u_2(x)\right) = \\ &\qquad \sum_{j=2}^{6} U^{(j)}\left(H_{0,1}(x) + H_{-1,0}(x)\right) \frac{u_1(x)^{j-1} - u_2(x)^{j-1}}{(j-2)!}. \end{aligned}$$

So, from the elementary estimate,

$$|u_1(x)^{j-1} - u_2(x)^{j-1}| \leq (j-1)\left(|u_1(x)|^{j-2} + |u_2(x)|^{j-2}\right)|u_1(x) - u_2(x)|$$

obtained from the Fundamental Theorem of Calculus and the fact that $U$, $H_{0,1}$, $H_{-1,0} \in C^\infty$ and $H_{0,1}$, $H_{-1,0} \in L^\infty_x(\mathbb{R})$, we deduce using Lemma 1.3.3 for any natural number $2 \leq j \leq 6$ the existence of a constant $C_j$ satisfying

$$\begin{aligned} &\left\|U^{(j)}\left(H_{0,1}(x) + H_{-1,0}(x)\right)\left[u_1(x)^{j-1} - u_2(x)^{j-1}\right]\right\|_{H^1_x} \leq \\ &\qquad C_j\left(\|u_1\|_{H^1_x}^{j-2} + \|u_2\|_{H^1_x}^{j-2}\right)\|u_1(x) - u_2(x)\|_{H^1_x}. \end{aligned}$$

Therefore, if there exist two solutions $u$, $v$ of the integral equation (1.12) belonging to the space $C\left([-T+t_0, T+t_0], H^1_x(\mathbb{R})\right) \cap C^1\left([-T+t_0, T+t_0], L^2_x(\mathbb{R})\right)$, we deduce using Lemma 1.3.3 the existence of a constant $C>0$ independent of $u$ and $v$ satisfying for any $t \in [-T+t_0, T+t_0]$ the following inequality

$$\|u(t)-v(t)\|_{H^1_x} + \|\partial_t u(t) - \partial_t v(t)\|_{L^2_x} \leq \\ C\int_{t_0}^{t} [1+|s-t_0|]\left(1+\max\{\|u(s)\|_{H^1_x}, \|v(s)\|_{H^1_x}\}\right)^4 \|u(s)-v(s)\|_{H^1_x}\, ds.$$

Consequently, using Gronwall Lemma, we can verify that $u(s)=v(s)$ for any $s$ in the interval $[-T+t_0, T+t_0]$, from which we conclude the uniqueness of the solution of the partial differential equation (1.13) in the space $H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$.

Similarly, assuming $t_0 = 0$, using the map $F$ defined at (1.12) and considering $\delta_0 = \|(u_0, u_1)\|_{H^1_x \times L^2_x}$, we can deduce the existence of a $T_0 > 0$ depending only on $\delta_0$ such that the following restriction of $F$

$$F : \left\{u \Big| \, (u, \partial_t u) \in C\left([-T_0, T_0], H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})\right), \sup_{t\in[-T_0,T_0]} \|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x} < 2\delta_0\right\} \\ \to \left\{u \Big| \, (u, \partial_t u) \in C\left([-T_0, T_0], H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})\right), \sup_{t\in[-T_0,T_0]} \|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x} < 2\delta_0\right\}$$

is a contraction. Therefore, using Banach Fixed-Point Theorem, we can verify that (1.13) is locally well-posed in the space $H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$.

The solutions of the partial differential equation ($\phi^6$) in the energy space satisfy the following conservation laws:

$$E(\phi) = \int_{\mathbb{R}} \frac{\partial_x \phi(t,x)^2 + \partial_t \phi(t,x)^2}{2} + U(\phi(t,x))\, dx, \qquad \text{(Energy)}$$

$$P(\phi) = -\int_{\mathbb{R}} \partial_x \phi(t,x) \partial_t \phi(t,x)\, dx. \qquad \text{(Momentum)}$$

Moreover, the solutions $\phi(t,x)$ of ($\phi^6$) satisfy the following invariances:

- Time translation: For any $h \in \mathbb{R}$, $\phi(t+h, x)$ is also a solution of ($\phi^6$),
- Space translation: For any $h \in \mathbb{R}$, $\phi(t, x+h)$ is also a solution of ($\phi^6$),
- Space reflection: $\phi(t, -x)$ is also a solution of ($\phi^6$),
- Time reflection: $\phi(-t, x)$ is also a solution of ($\phi^6$).

In addition, for any $v \in (-1,1)$ and any $(t_0, x_0) \in \mathbb{R}^2$, if $\phi(t,x)$ is a solution of ($\phi^6$), then the Lorentz transformation

$$\psi(t,x) := \phi\left(\frac{t-t_0-v(x-x_0)}{\sqrt{1-v^2}}, \frac{x-x_0-v(t-t_0)}{\sqrt{1-v^2}}\right)$$

is also a solution of the partial differential equation ($\phi^6$). Consequently, if $H$ is a stationary solution of ($\phi^6$), then the following function

$$\varphi(t,x) = H\left(\frac{x-vt}{\sqrt{1-v^2}}\right)$$

is also a solution of ($\phi^6$). We observe that the kinks and anti-kinks are the unique non-constant stationary solutions of ($\phi^6$) with finite energy, see Chapter 5 of [36].

Moreover, the Space translations of the kink $H_{0,1}$ are the minimizers of the Energy function $E(\phi)$ when $\phi$ satisfies the boundary conditions $\lim_{x\to-\infty}\phi(x) = 0$ and $\lim_{x\to+\infty}\phi(x) = 1$, see Chapter 5 of [36] for the proof of this fact.

Furthermore, since the real function $U(\phi) = \phi^2(1-\phi^2)^2$ is positive and satisfies $\lim_{y\to\pm} U(y) = +\infty$, any solution $\phi$ of (2.1) having finite energy is global in time.

More precisely, if $E(\phi) < +\infty$, then there exists $C > 0$ such that $\|\phi(t,\cdot)\|_{L^\infty_x(\mathbb{R})} < C$ for any $t$ in the domain of $\phi$, from which, using the local well-posedness of partial differential equation (2.1), we obtain the global well-posedness of (2.1) in the space of solutions having finite energy. Because, if $E(\phi) < +\infty$, then, for any real $t$ in the domain of $\phi$,

$$\int_{\mathbb{R}} \partial_x\phi(t,x)^2 \leq 2E(\phi),$$

which implies with Cauchy-Schwarz inequality that

$$|\phi(t,x) - \phi(t,y)| \leq |x-y|^{\frac{1}{2}}\sqrt{2E(\phi)}.$$

Therefore, since $U$ is a non-negative function satisfying $\lim_{y\to\pm\infty} U(y) = +\infty$, if there existed a real sequence $(t_n)_{n\in\mathbb{N}}$ in the domain of $\phi$ satisfying $\lim_{n\to+\infty}\|\phi(t_n,\cdot)\|_{L^\infty_x(\mathbb{R})} = +\infty$, then there would exist a $n\in\mathbb{N}$ such that $\int_{\mathbb{R}} U(\phi(t_n,x))\,dx > E(\phi)$, which is a contradiction.

Finally, for each $t\in\mathbb{R}$, we consider the Kinetic Energy $E_k(\phi)(t)$ of a solution $\phi$ in the energy spaces by

$$E_k(\phi)(t) = \int_{\mathbb{R}} \frac{\partial_t\phi(t,x)^2}{2}\,dx,$$

and we denote the Potential Energy $E_{pot}(\phi)(t)$ by $E(\phi) - E_k(\phi)(t)$.

### 1.3.2 Previous results in the stability and dynamics of kinks

In this subsection of the thesis, we briefly describe the previous results obtained about stability and dynamics of one or two kinks for some dispersive nonlinear equations.

For the $\phi^4$ model, which is the partial differential equation

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) - \phi(t,x) + \phi(t,x)^3 = 0,\ (t,x)\in\mathbb{R}^2,$$

asymptotic stability of a single kink under odd perturbations was proved by Kowalczyk, Martel, and Muñoz in [29]. Moreover, in [13], Delort and Masmoudi obtained the decay rates for the size of the perturbations of the kink for this model.

Under assumptions on the potential function $U$, it was proved in [31], for the following partial differential equation

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U^{'}\left(\phi(t,x)\right) = 0,\ (t,x) \in \mathbb{R}^2, \tag{1.14}$$

the asymptotic stability of a kink by Kowalczyk, Martel, Muñoz, and Van Den Bosch. Indeed, the result of this article applies to the $\phi^6$ model which we studied in this thesis, therefore the kinks $H_{0,1}$ and $H_{-1,0}$ are asymptotically stable in some sense.

For the sine-Gordon model

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + \sin\left(\phi(t,x)\right) = 0,$$

Schlag and Lührmann proved asymptotic stability of a single kink under odd perturbations in [56]. Moreover, in [1], Alejo, Muñoz and Palacios, proved asymptotic stability result of a single kink in a specific manifold of perturbations.

With respect to nonlinear Schrödinger equation models, we refer the to the work [6] about orbital stability of a kink in the Gross-Pitaevskii equation. For more references in stability of solitons in nonlinear Schrödinger equations, see also the classical work [5] about orbital stability of solitary waves and [4] about asymptotic stability of solitons.

Regarding the topic of dynamics, in [26], for a certain set of potential functions $U$, Jendrej, Lawrie and Kowalczyk described the dynamics of strongly interacting kink-antikink pair solutions of (1.14). The strongly interacting kink-antikink pairs are the solutions of (1.14) which converge in infinity to a sum of kink and antikink each one moving with a speed converging asymptotically to zero. In [26], it was also obtained the existence the strongly interacting kink-antikink pairs and their uniqueness under time and space translation.

With respect to the Klein-Gordon model, Krieger, Nakanishi and Schlag proved asymptotic stability of solitary waves in the article [32]. Kowalczyk, Martel and Muñoz also proved asymptotic stability of solitons and studied their dynamics for one dimensional Klein-Gordon in [30]. See also the recent article [19] by Germain and Pusateri about asymptotic stability of solitary waves for Klein-Gordon models.

The literature about stability and dynamics of solitons for nonlinear dispersive equations is vast and not only restricted to one-dimensional nonlinear dispersive equations. For example, see the references [22], [11], [55], [27] about dynamics and stability of vortices, which are topological solitons associated with scalar fields of dimension $1+2$.

### 1.3.3 Collision of solitons for nonlinear dispersive models

The study of the collision of solitons in nonlinear dispersive equations focuses on understanding the long time behavior of a solution $\phi(t,x)$ when time variable $t$ approaches $-\infty$ knowing that this solution converges in some norm to a finite sum of solitary waves when the $t$ goes to $+\infty$. For non-integrable models, there aren't many references that study the collision between solitons for nonlinear dispersive models.

In many complete integrable models, the solutions can be described explicitly and the collision between solitons is completely elastic, see for example the results for the Korteweg-de Vries equation in [45], see also [9], [25] and the classical work of Lax in [33]. Contrary to the collision of solitons in completely integrable systems, it is expected in non-integrable models that the collision between two solitons is not elastic, which means that, after the collision instant, the solution will not converge when $t$ goes to $+\infty$ to a sum of two solitary waves with same energy and momentum as the two solitons before they collide.

In [39], Martel and Merle studied the stability of the collision between solitons for the generalized Korteweg-de Vries equation and, in [40], [41], the same authors proved inelasticity of the collision between two solitons for the quartic generalized Korteweg-de Vries equation. In [49], [50], Muñoz extended the argument used in [41] to prove the inelasticity of the collision between two solitons for other generalized Korteweg-de Vries models.

For nonlinear Schrödinger equation models, in [53], Perelman studied the collision between two solitons of different size and obtained inelasticity, indeed after the collision instant she proved that the solution doesn't preserve the two solitons' structure.

## 1.4 Main results

We recall the one-dimensional nonlinear wave equation ($\phi^6-$NLW)

$$\begin{cases}\partial_t^2\phi(t,x)-\partial_x^2\phi(t,x)+2\phi(t,x)-8\phi(t,x)^2+6\phi(t,x)^5=0,\\ \lim_{x\to+\infty}\phi(t,x)=1,\ \lim_{x\to-\infty}\phi(t,x)=-1.\end{cases}$$

In Chapter 2, we will describe all the solutions of ($\phi^6-$NLW) in the energy space with energy slightly bigger than $2E\left(H_{0,1}\right).$ Actually, from the estimate

$$\begin{aligned}\int_{\mathbb{R}}\frac{\partial_t\phi(t,x)^2}{2}+\frac{\partial_x\phi(t,x)^2}{2}+U\left(\phi(t,x)\right)\,dx\geq\int_{\mathbb{R}}\frac{\partial_x\phi(t,x)^2}{2}+U\left(\phi(t,x)\right)\,dx\\ =\int_{\mathbb{R}}\sqrt{2U\left(\phi(t,x)\right)}\left|\partial_x\phi(t,x)\right|\,dx+\frac{1}{2}\int_{\mathbb{R}}\left[\left|\frac{\partial_x\phi(t,x)}{2}\right|-\sqrt{2U\left(\phi(t,x)\right)}\right]^2\,dx\\ \geq\int_{\mathbb{R}}\sqrt{2U\left(\phi(t,x)\right)}\left|\partial_x\phi(t,x)\right|\,dx\geq\int_{-1}^{1}\sqrt{2U\left(y\right)}\,dy=2E\left(H_{0,1}\right),\end{aligned}$$

we have that $2E\left(H_{0,1}\right)$ is the minimum possible value for $E\left(\phi\right).$ This minimum value is not attained, since there isn't a non-constant solution $\phi$ with finite energy satisfying $\left|\partial_x\phi(t,x)\right|=\sqrt{2U\left(\phi(t,x)\right)}$ which is not either a kink or a antikink.

**Definition 1.4.1.** *Let $\phi$ be a solution in the energy space of the partial differential equation* ($\phi^6-$NLW). *The energy excess $\epsilon$ of $\phi$ is the following positive value:*

$$\epsilon:=E\left(\phi\right)-2E\left(H_{0,1}\right).$$

### 1.4.1 Description of the solutions with small energy excess

Our first main result is the following:

**Theorem 1.4.2.** $\exists C > 1, \delta_0 > 0$, *such that if* $\epsilon < \delta_0$ *and*

$$(\phi(0,x) - H_{0,1}(x) - H_{-1,0}(x), \partial_t\phi(0,x)) \in H^1_x(\mathbb{R}) \times L^2(\mathbb{R})$$

*with* $E_{total}(\phi(0), \partial_t\phi(0)) = 2E(H_{0,1}) + \epsilon$, *then there exist functions* $x_2, x_1 \in C^2(\mathbb{R})$ *such that the unique global time solution* $\phi(t,x)$ *of* ($\phi^6-$NLW) *is given by*

$$\phi(t,x) = H_{0,1}(x - x_2(t)) + H_{-1,0}(x - x_1(t)) + g(t,x), \tag{1.15}$$

*and for any* $t \in \mathbb{R}$,

$$\frac{\epsilon}{C} \leq e^{-\sqrt{2}(x_2(t)-x_1(t))} + \max_{j\in\{1,2\}} \dot{x}_j(t)^2 + \|(g(t), \partial_t g(t))\|^2_{H^1\times L^2} \leq C\epsilon, \tag{1.16}$$

$$\max_{j\in\{1,2\}} |\ddot{x}_j(t)| \leq C\epsilon. \tag{1.17}$$

*Furthermore, we have*

$$\|(g(t), \partial_t g(t))\|^2_{H^1\times L^2} \leq C\Big[\|(g(0), \partial_t g(0))\|^2_{H^1\times L^2} + \epsilon^2\Big] \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\left(\frac{1}{\epsilon}\right)}\right) \text{ for all } t \in \mathbb{R}. \tag{1.18}$$

The proof of Theorem 1.4.2 will be presented in the next chapter. Using an argument of contradiction, we will prove that if the energy excess $\epsilon$ of $\phi$ is small enough, then, for any $t \in \mathbb{R}$, there exist $\hat{x}_1(t)$, $\hat{x}_2(t) \in \mathbb{R}$ with $\hat{x}_2(t) \gg \hat{x}_1(t)$ such that

$$\|\phi(t,x) - H_{0,1}\left(x - \hat{x}_2(t)\right) - H_{-1,0}\left(x - \hat{x}_1(t)\right)\|_{H^1_x(\mathbb{R})} \ll 1.$$

Next, using modulation techniques similar to the one used in [54] an [26], we are going to verify that $\phi(t,x)$ has the following representation

$$\phi(t,x) = H_{0,1}\left(x - x_2(t)\right) + H_{-1,0}\left(x - x_1(t)\right) + g(t,x), \tag{1.19}$$

with $x_2(t) - x_1(t) \gg 1$, $\|g(t,x)\|_{H^1_x(\mathbb{R})} \ll 1$ for any $t \in \mathbb{R}$ and $g(t,x)$ satisfying the orthogonality conditions

$$\Big\langle g(t,x), H^{'}_{0,1}\left(x - x_2(t)\right)\Big\rangle_{L^2_x} = \Big\langle g(t,x), H^{'}_{-1,0}\left(x - x_1(t)\right)\Big\rangle_{L^2_x} = 0. \tag{1.20}$$

From the orthogonality conditions above, we will obtain the following coercive estimate in the energy

$$\begin{aligned} c_0\|g(t,x)\|^2_{H^1_x(\mathbb{R})} \leq{} & E\left(\phi\right) - E\left(H_{0,1}\left(x - x_2(t)\right) + H_{-1,0}\left(x - x_1(t)\right)\right) \\ & + O\left(\|g(t,x)\|^3_{H^1_x(\mathbb{R})} + |x_2(t) - x_1(t)|e^{-2\sqrt{2}(x_2(t)-x_1(t))}\right). \end{aligned}$$

Therefore, using a bootstrap argument and the continuity of the modulation parameters $x_1$, $x_2$, we will deduce the existence of a constant $c > 0$ such that

$$\|g(t,x)\|^2_{H^1_x(\mathbb{R})} + e^{-\sqrt{2}(x_2(t)-x_1(t))} \leq c\epsilon, \text{ for all } t \in \mathbb{R}.$$

The estimate $\|\partial_t g(t,x)\|_{L^2_x} \lesssim \epsilon^{\frac{1}{2}}$ will follow directly from the estimate of the kinetic energy of $\phi$ and the fact that $E(\phi) - 2E\left(H_{0,1}\right) = \epsilon$.

The estimate of the first and second derivatives of the modulation parameter $x_1$, $x_2$ will follow from standard analysis of the ordinary differential equations obtained from the time derivative of the orthogonality conditions (1.20) and combining this result with the estimates above we will deduce inequalities (1.16), (1.17).

The proof of inequality (1.18) will be done more carefully in Chapter 2 using refined energy estimates techniques. More precisely, it will be based on a study of a function $F(t)$ defined from the sum of the quadratic term

$$\int_{\mathbb{R}} \frac{\partial_x g(t,x)^2 + \partial_t g(t,x)^2}{2} + \frac{1}{2}U''\left(H_{0,1}\left(x - x_2(t)\right) + H_{-1,0}\left(x - x_1(t)\right)\right) g(t,x)^2\, dx$$

with correction terms. We will prove that this function has small decay in its derivative and it satisfies a coercivity inequality

$$\|(g(t), \partial_t g(t))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \lesssim F(t) + \epsilon^2.$$

Using these two observations, we will obtain the following inequality

$$\|(g(t), \partial_t g(t))\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \leq C\Big[\, \|(g(0), \partial_t g(0))\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} + \epsilon^2 \ln\left(\frac{1}{\epsilon}\right)^2\Big] \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\left(\frac{1}{\epsilon}\right)}\right), \tag{1.21}$$

for all $t \in \mathbb{R}$.

### 1.4.2 Dynamics of two kinks with small energy

Furthermore, in the second chapter, we will also prove the following theorem.

**Theorem 1.4.3.** *In notation of Theorem 1.4.2, $\exists C, \delta_0 > 0$, such that if $0 < \epsilon < \delta_0$, $\phi$ is a solution of the partial differential equation ($\phi^6-$NLW) in the energy space and $E(\phi) = 2E_{pot}(H_{0,1}) + \epsilon$, then the smooth functions $d_1$, $d_2 \in C^2(\mathbb{R})$ defined by*

$$d_1(t) = a + bt - \frac{1}{2\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{1.22}$$

$$d_2(t) = a + bt + \frac{1}{2\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{1.23}$$

*such that $d_j(0) = x_j(0)$, $\dot{d}_j(0) = \dot{x}_j(0)$ for $j \in \{1, 2\}$, satisfy*

$$\max_{j\in\{1,2\}} |d_j(t) - x_j(t)| \leq C \min(\epsilon^{\frac{1}{2}}|t|, \epsilon t^2),\ \max_{j\in\{1,2\}} |\dot{d}_j(t) - \dot{x}_j(t)| \leq C\epsilon|t|,$$

*and for $\overrightarrow{g(0)} = (g(0,x), \partial_t g(0,x))$, we have the following estimates*

$$\epsilon \max_{j\in\{1,2\}} |d_j(t) - x_j(t)| \leq C \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)^2 \ln\left(\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\left(\frac{1}{\epsilon}\right)}\right), \tag{1.24}$$

$$\epsilon^{\frac{1}{2}} \max_{j\in\{1,2\}} |\dot{d}_j(t) - \dot{x}_j(t)| \leq C \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)^2 \ln\left(\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\left(\frac{1}{\epsilon}\right)}\right), \tag{1.25}$$

$$\epsilon^{\frac{1}{2}} \max_{j\in\{1,2\}} |\ddot{d}_j(t) - \ddot{x}_j(t)| \leq C \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)^2 \ln\left(\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\left(\frac{1}{\epsilon}\right)}\right). \tag{1.26}$$

Both Theorems 1.4.2 and 1.4.3 are from the article [47].

The proof of Theorem 1.4.3 relies on the observation that the functions $x_j(t) - d_j(t)$ will be very close to a solution of a well-known linear ordinary differential system. Therefore, using the estimates (1.16), (1.17) and the inequality (1.21), we will conclude the proof of Theorem 1.4.3 using the method variation of parameters for ordinary differential equations.

Finally, the demonstration of estimate (1.18) is going to follow from the energy estimate technique using the function $F(t)$ and the estimate of the derivative $\dot{F}(t)$ using the estimates (1.25), (1.26) of Theorem 1.4.3 instead of the global estimate $\max_{j\in\{1,2\}} |\dot{x}_j(t)|^2 + |\ddot{x}_j(t)| = O(\epsilon)$.

The statement of Theorem 1.4.3 also describes with high precision the dynamics of two interacting kinks for the $\phi^6$ model, which is the behavior of the displacement solitons when initially they are very close to each other and their energy is slightly larger than the minimal value of the energy of a solution of the problem ($\phi^6-$NLW). Moreover, the conclusions of Theorem 1.4.3 allow us to understand with high precision the effect of the repulsive force of interaction between the kinks in their dynamics during a very large time interval. The methods we used to study the dynamics of two kinks for the $\phi^6$ model are not only restricted to this partial differential equation and they can be very useful to understand the dynamics and properties of multi-solitons for other non-complete integrable systems. Actually, we will also prove in the second chapter that the precision in our estimate (1.21) is optimal in an interval of size of order $O\left(\frac{\ln\left(\frac{1}{v}\right)}{v}\right)$.

### 1.4.3 Almost elasticity of the collision of two kinks

The third main result of the manuscript is the following statement:

**Theorem 1.4.4.** *For any* $0<\theta<1$ *and* $k\in\mathbb{N}_{\geq 2}$, *there exists* $0<\delta(\theta,k)<1$, *such that if* $0<v<\delta(\theta,k)$, *and* $\phi(t,x)$ *is the unique solution of* ($\phi^6-$NLW) *satisfying for all* $t\geq 4\frac{\ln\left(\frac{1}{v}\right)}{v}$

$$\left\|\phi(t,x) - H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})} + \left\|\partial_t\phi(t,x) + \frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x} \leq e^{-vt}, \tag{1.27}$$

*then there exist a real function* $v_f:(0,1)\times\mathbb{R}\to\mathbb{R}$ *and a number* $e_{v,k}$ *such that* $0<v_f<1$, $|e_{v,k}|<\ln\left(\frac{8}{v^2}\right)$ *and if* $t\leq -\frac{\ln\left(\frac{1}{v}\right)^{2-\theta}}{v}$, *then* $|v_f(t)-v|<v^k$ *and*

$$\left\|\phi(t,x) - H_{0,1}\left(\frac{x-e_{k,v}+v_f(t)t}{\sqrt{1-v_f(t)^2}}\right) - H_{-1,0}\left(\frac{x+e_{k,v}-v_f(t)t}{\sqrt{1-v_f(t)^2}}\right)\right\|_{H^1_x(\mathbb{R})} + \left\|\partial_t\phi(t,x) + \frac{v_f(t)}{\sqrt{1-v_f(t)^2}}H^{'}_{0,1}\left(\frac{x-e_{k,v}+v_f(t)t}{\sqrt{1-v_f(t)^2}}\right) - \frac{v_f(t)}{\sqrt{1-v_f(t)^2}}H^{'}_{-1,0}\left(\frac{x+e_{k,v}-v_f(t)t}{\sqrt{1-v_f(t)^2}}\right)\right\|_{L^2_x} \leq v^k.$$

*Furthermore, if* $\frac{-4\ln\left(\frac{1}{v}\right)^{2-\theta}}{v} \leq t \leq \frac{-\ln\left(\frac{1}{v}\right)^{2-\theta}}{v}$, *then*

$$\left\|\phi(t,x) - H_{0,1}\left(\frac{x-e_{k,v}+vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+e_{k,v}-vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x} + \left\|\partial_t\phi(t,x) - \frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-e_{v,k}+vt}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+e_{v,k}-vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x} \leq v^k. \tag{1.28}$$

The existence and uniqueness of two solitary kinks for the $\phi^6$ model with the energy norm of the remainder having exponential decay was proved in [8] by Chen and Jendrej. In particular, when the speed $v>0$ is small enough, we have the decay (1.27).

The statement of Theorem 1.4.4 implies that the collision between two kinks for the $\phi^6$ model is almost elastic. Indeed, for any $k\in\mathbb{N}$, if the speed $v$ of each kink is small enough, then the energy norm of the residue and the change in the speed of each kink is much smaller than $v^k$. Therefore, the collision of two kinks for the $\phi^6$ model is different, in nature than the collision of two solitons of quartic generalized Korteweg-de Vries, for which the inelasticity is polynomial with respect to the size of the speed of the solitons, compare Theorem 1.4.4 with Theorem 1 of [41]. Moreover, because of the estimate (1.28) concluded in Theorem 1.4.4, it is not possible to apply the methods of [41] to prove the inelasticity of the collision between two kinks for the $\phi^6$ model.

### 1.4.4 Sketch of the proof of Theorem 1.4.4

The demonstration of Theorem 1.4.4 is quite long and delicate, and it will be divided into Chapters 3 and 4, corresponding to the preprints [46] and [48] respectively. First, we are going to create a sequence of approximate solutions $(\phi_k)_{k\in\mathbb{N}_{\geq 2}}$ of equation ($\phi^6$) satisfying for any $v>0$ sufficiently small

$$\lim_{t\to+\infty}\left\|\phi_k(v,t,x) - H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})} + \left\|\partial_t\phi_k(v,t,x) + \frac{v}{\sqrt{1-v^2}}\left[H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right]\right\|_{L^2_x} = 0,$$

and for all $t\in\mathbb{R}$, if $0<v\ll 1$, then

$$\left\|\frac{\partial^2\phi_k(v,t,x)}{\partial t^2} - \frac{\partial^2\phi_k(v,t,x)}{\partial x^2} + U^{'}\left(\phi_k(v,t,x)\right)\right\|_{H^1_x(\mathbb{R})} \leq C(k)v^{2k}\left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{n_k} e^{-2\sqrt{2}|t|v},$$

where $C(k)>0$ and $n_k\in\mathbb{N}$ for all $k\in\mathbb{N}_{\geq 2}$.

**Definition 1.4.5.** *We define* $\Lambda : C^2(\mathbb{R}^2,\mathbb{R}) \to C(\mathbb{R}^2,\mathbb{R})$ *as the nonlinear operator satisfying*

$$\Lambda(\phi_1)(t,x) = \partial_t^2\phi_1(t,x) - \partial_x^2\phi_1(t,x) + U^{'}(\phi_1(t,x)),$$

*for any function* $\phi_1\in C^2(\mathbb{R}^2,\mathbb{R})$.

More precisely, we will prove the following theorem in Chapter 3.

**Theorem 1.4.6.** *There exist a sequence of functions $(\phi_k(v,t,x))_{k\geq 2}$, a sequence of real numbers $\delta(k)>0$ and a sequence of numbers $n_k\in\mathbb{N}$ such that for any $0<v<\delta(k)$, $\phi_k(v,t,x)$ satisfies*

$$\lim_{t\to+\infty}\left\Vert \phi_k(v,t,x)-H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\Vert_{H^1_x(\mathbb{R})}=0,$$

$$\lim_{t\to+\infty}\left\Vert \partial_t\phi_k(v,t,x)+\frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-\frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\Vert_{L^2_x}=0,$$

$$\lim_{t\to-\infty}\left\Vert \phi_k(v,t,x)-H_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\Vert_{H^1_x(\mathbb{R})}=0,$$

$$\lim_{t\to-\infty}\left\Vert \partial_t\phi_k(v,t,x)-\frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\Vert_{L^2_x(\mathbb{R})}=0,$$

*with $e_{v,k}\in\mathbb{R}$ satisfying*

$$\lim_{v\to 0}\frac{\left|e_{v,k}-\frac{\ln\left(\frac{8}{v^2}\right)}{\sqrt{2}}\right|}{v|\ln(v)|^3}=0.$$

*Moreover, if $0<v<\delta(k)$, then for any $s\geq 0$ and $l\in\mathbb{N}\cup\{0\}$, there is $C(k,s,l)>0$ such that*

$$\left\Vert\frac{\partial^l}{\partial t^l}\Lambda(\phi_k)(v,t,x)\right\Vert_{H^s_x(\mathbb{R})}\leq C(k,s,l)v^{2k+l}\left(|t|v+\ln\left(\frac{1}{v^2}\right)\right)^{n_k}e^{-2\sqrt{2}|t|v}.$$

The demonstration of Theorem 1.4.6 is very technical and requires tools from functional and complex analysis. The construction of each approximate solution follows from an argument of induction. We explain briefly the main ideas behind the proof of this theorem.

First, for any $0<v\ll 1$, we consider the function $d_v:\mathbb{R}\to\mathbb{R}$ denoted by

$$d_v(t)=\frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)\right)$$

and we consider also

$$\varphi_{1,v}(t,x)=H_{0,1}\left(\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-H_{0,1}\left(\frac{-x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right).$$

Next, we prove the existence of a Schwartz function $\mathcal{M}(x)$ orthogonal to $H^{'}_{0,1}(x)$ in $L^2_x(\mathbb{R})$ such that $\Lambda(\phi_{1,v})(t,x)$ satisfies

$$\Lambda(\varphi_{1,v})(t,x)=e^{-\sqrt{2}d_v(t)}\left[\mathcal{M}\left(\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-\mathcal{M}\left(\frac{-x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]+res(v,t,x),\qquad(1.29)$$

where, for any $v\in(0,1)$, $\mathcal{R}(v,\cdot)\in C^{\infty}(\mathbb{R}^2)$ and if $0<v\ll 1$, then

$$\left\Vert\frac{\partial^l}{\partial t^l}res(v,t,x)\right\Vert_{H^s_x(\mathbb{R})}\lesssim_{s,l} v^{4+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)e^{-2\sqrt{2}|t|v},\text{ for all }l\in\mathbb{N}\cup\{0\}.$$

Using information obtained in (1.29), we are going to consider a smooth function $\phi_{2,0,v}(t,x)$ denoted by

$$\phi_{2,0,v}(t,x)=\varphi_{1,v}(t,x)+e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-\mathcal{G}\left(\frac{-x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right],$$

where $\mathcal{G}$ is the unique Schwartz function orthogonal in $L^2_x(\mathbb{R})$ to $H^{'}_{0,1}$ satisfying the identity

$$-\frac{d^2}{dx^2}\mathcal{G}(x)+U^{''}\left(H_{0,1}(x)\right)\mathcal{G}(x)=-\mathcal{M}(x).$$

Next, for any $0<v\ll 1$, we are going to create a smooth even function $r_v:\mathbb{R}\to\mathbb{R}$ such that the function

$$\begin{aligned}\varphi_{2,v}(t,x)\coloneqq{}& H_{0,1}\left(\frac{x-\frac{d_v(t)}{2}+r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)+H_{-1,0}\left(\frac{x+\frac{d_v(t)}{2}-r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\\ &+e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x-\frac{d_v(t)}{2}+r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-\mathcal{G}\left(\frac{-x-\frac{d_v(t)}{2}+r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]\end{aligned}$$

satisfies for all $t\in\mathbb{R}$

$$\begin{aligned}&\left|\left\langle\Lambda(\varphi_{2,v}(t,x)),H^{'}_{0,1}\left(\frac{x-\frac{d_v(t)}{2}+r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right\rangle_{L^2_x}\right|\\ &+\left|\left\langle\Lambda(\varphi_{2,v}(t,x)),H^{'}_{-1,0}\left(\frac{x+\frac{d_v(t)}{2}-r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right\rangle_{L^2_x}\right|\ll v^{6-\frac{1}{2}},\end{aligned}$$

indeed we will construct $r_v$ as an solution of an explicit ordinary differential equation. Next, we will prove in the third chapter the existence of a parameter $a_{k,v}$ such that the function $\phi_2(v,t,x)\coloneqq\varphi_{2,v}(a_{k,v}+t,x)$ will satisfy Theorem 1.4.6 for $k=2$.

The remaining argument of the proof of Theorem 1.4.6 is the construction of $\phi_{k+1}$ from the function $\phi_k$ which by the principle of induction concludes the proof of Theorem 1.4.6. For all $k\in\mathbb{N}_{\geq 2}$, the argument on proof of the inductive step is similar to the method explained above to obtain $\phi_2$ from the function $\phi_{1,v}$.

More precisely, we will prove by induction on $k\in\mathbb{N}_{\geq 2}$ the existence of a sequence of approximate solutions $(\varphi_{k,v})_{k\in\mathbb{N}_{\geq 2}}$

$$\begin{aligned}\varphi_{k,v}(t,x)={}&H_{0,1}\left(\frac{x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)+H_{-1,0}\left(\frac{x-\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\\ &+e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-\mathcal{G}\left(\frac{-x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]\\ &+\sum_{i=1}^{\mathcal{M}_k}p_{i,k,v}\left(\sqrt{2}vt\right)\left[h_{i,k}\left(\frac{x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-h_{i,k}\left(\frac{-x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right],\end{aligned}$$

which satisfies for all $l \in \mathbb{N} \cup \{0\}$ and all $s \geq 0$ the inequality

$$\left\| \frac{\partial^l}{\partial t^l} \Lambda\left(\varphi_{k,v}(t,x)\right) \right\|_{H^s_x(\mathbb{R})} \leq v^{2k-\frac{1}{2}} \left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{n_k} e^{-2\sqrt{2}|t|v}, \text{ if } v \ll 1, \tag{1.30}$$

where $n_k \in \mathbb{N}$, the real function $\rho_{k,v}$ is smooth, even and, for any $1 \leq i \leq \mathcal{M}_k$, the real functions $h_{i,k} \in \mathscr{S}(\mathbb{R})$ and all the functions $p_{i,k,v}$ are smooth and even. First, assuming the existence of the approximate solution $\varphi_{k,v}$ for some $k = k_0 \in \mathbb{N}_{\geq 2}$, we are going to verify the following estimate

$$\Lambda\left(\varphi_{k,v}(t,x)\right) \sim \sum_{j \in I_k} s_{j,v}(\sqrt{2}vt) \left[ R_j\left(\frac{x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - R_j\left(\frac{-x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right],$$

where, for any $j \in I_k$, $R_j \in \mathscr{S}(\mathbb{R})$ and $s_{j,v}$ is a real even smooth function satisfying

$$\left| \frac{d^l}{dt^l} s_{j,v}(t) \right| \lesssim v^{2k-\frac{1}{2}} \left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{n_k} e^{-2\sqrt{2}|t|v}.$$

Next, for any $j \in I_k$, using Fredholm alternative in the linear self-adjoint operator $-\frac{d^2}{dx^2} + U''\left(H_{0,1}(x)\right) : H^2_x(\mathbb{R}) \subset L^2_x(\mathbb{R}) \to L^2_x(\mathbb{R})$, we will deduce the existence and uniqueness of a Schwartz function $\mathcal{Y}_j$ satisfying

$$-\frac{d^2}{dx^2}\mathcal{Y}_j(x) + U''\left(H_{0,1}(x)\right)\mathcal{Y}_j(x) = -R_j(x) + \left\langle R_j, H^{'}_{0,1}\right\rangle_{L^2_x} \frac{H^{'}_{0,1}(x)}{\left\|H^{'}_{0,1}(x)\right\|^2_{L^2_x}}.$$

The approximate solution $\varphi_{k_0+1,v}$ will be constructed using the formula of $\varphi_{k_0,v}$, more precisely:

$$\begin{aligned}\varphi_{k_0+1,v}(t,x) = {} & H_{0,1}\left(\frac{x+\rho_{k_0,v}(t)+r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) + H_{-1,0}\left(\frac{x-\rho_{k_0,v}(t)-r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\\ & + e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x+\rho_{k_0,v}(t)+r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - \mathcal{G}\left(\frac{-x+\rho_{k_0,v}(t)+r_{k_0,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]\\ & + \sum_{i=1}^{\mathcal{M}_{k_0}} p_{i,k_0,v}\left(\sqrt{2}vt\right)\left[h_{i,k_0}\left(\frac{x+\rho_{k_0,v}(t)+r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - h_{i,k_0}\left(\frac{-x+\rho_{k_0,v}(t)+r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]\\ & + \sum_{j \in I_{k_0}} s_{j,v}\left(\sqrt{2}vt\right)\left[\mathcal{Y}_j\left(\frac{x+\rho_{k_0,v}(t)+r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - \mathcal{Y}_j\left(\frac{-x+\rho_{k_0,v}(t)+r_{k_0+1,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right],\end{aligned}$$

where $r_{k_0+1,v}$ is a smooth even function satisfying an explicit linear ordinary differential equation. Finally, for each $k \in \mathbb{N}_{\geq 2}$ and $0 < v \ll 1$, we are going to prove the existence of a value $e_{k,v}$ having size of order $O\left(\frac{\ln\left(\frac{1}{v}\right)}{v}\right)$ such that $\phi_k(v,t,x) := \varphi_{k,v}(t+e_{k,v},x)$ satisfies Theorem 1.4.6.

In Chapter 4, we are going to use the results of Chapter 3 to demonstrate Theorem 1.4.4. For the proof of this theorem, we will denote the function $\phi$ by

$$\phi(t,x) = \phi_k(v,t,x) + \frac{y_1}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} H^{'}_{0,1}\left(\frac{x-\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) + \frac{y_2}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} H^{'}_{-1,0}\left(\frac{x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) + u(t,x),$$

where $\rho_{k,v}$ is an explicit function obtained in the construction of $\phi_k$ of Theorem 1.4.6 and $y_1(t)$, $y_2(t)$ are the unique real numbers satisfying

$$\left\langle u(t,x), H^{'}_{0,1}\left(\frac{x-\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right\rangle_{L^2_x} = 0, \left\langle u(t,x), H^{'}_{-1,0}\left(\frac{x+\rho_{k,v}(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right\rangle_{L^2_x} = 0. \tag{1.31}$$

Using the condition (1.27) satisfied by $\phi(t,x)$ when $t$ goes to $+\infty$, we are going to estimate the value of $\|(u(t),\partial_t u(t))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}$ using the same energy estimate methods used in the proof of the first main result Theorem 1.4.2 to estimate the energy norm of $g$ during a long time interval.

Furthermore, using the orthogonality conditions (1.31), we will deduce that the functions $y_1$, $y_2$ satisfy an ordinary differential system of equations very close to a well-known linear differential system. Therefore, using the method of variation of parameters and the estimate of the energy norm of $\|(u(t),\partial_x u(t))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}$ that we obtained, we are going to evaluate the parameters $y_1(t)$, $y_2(t)$ and their derivatives during a large time interval.

Next, using the estimates obtained for $y_1$, $y_2$, $\|(u(t),\partial_x u(t))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}$ and a bootstrap argument, we will deduce that $\|(\phi(t,x)-\phi_k(t,x),\partial_t\phi(t,x)-\partial_t\phi_k(t,x))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}$ is very small during a long time interval, which will imply estimate (1.28) of Theorem 1.4.4. The first estimate of Theorem 1.4.4 will be proved as a consequence of estimate (1.28) and a result about orbital stability of two moving kinks very similar to the Theorem 1 of the article [31] about orbital stability of a moving kink for a class of nonlinear wave equations of dimension $1+1$.

The conclusion of Theorem 1.4.4 is very unexpected since the $\phi^6$ model is non-integrable and we proved that the collision between two kinks of this model is almost elastic. Moreover, for any $k\in\mathbb{N}$, if $v>0$ is small enough, Theorems 1.4.3, 1.4.4 also allow us to describe the displacement of the two solitons during any time $t$ with precision higher than $v^k$, which is a strong result about the dynamics of multi-solitons for non-integrable systems. The result of almost inelasticity obtained in estimates (1.28) is also noteworthy and implies that the defects in the energy norm of the remainder and in the speed of the kinks after the collision can be very insignificant in comparison with the notable result of inelasticity of the collision of two solitons obtained in Theorem 1 of article [41] about generalized Korteweg-de Vries equation.

Furthermore, the results of Theorem 1.4.4 open possibilities in the investigation of the collision and the dynamics of multi-kinks for other one-dimensional wave equation models with nonlinearities of a higher order than the $\phi^6$. This topic of research has applications and interests in different fields of Physics, for example, many investigations have been made in High energy physics, see [14], [17].

## 1.5 Notation

In this section, we describe the notation that we are going to use in the following chapters.

**Notation 1.5.1.** *For any $D \subset \mathbb{R}$, any non-negative real function $f : D \subset \mathbb{R} \to \mathbb{R}$, a real function $g$ with domain $D$ is in $O(f(x))$ if and only if there is a uniform constant $C > 0$ such that $0 \leq |g(x)| \leq Cf(x)$. We denote that two real non-negative functions $f, g : D \subset \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfy*

$$f \lesssim g,$$

*if there is a constant $C > 0$ such that*

$$f(x) \leq Cg(x), \text{ for all } x \in D.$$

*If $f \lesssim g$ and $g \lesssim f$, we denote that $f \cong g$. We use the notation $(x)_+ := \max(x, 0)$. If $g(t,x) \in C^1(\mathbb{R}, L^2(\mathbb{R})) \cap C(\mathbb{R}, H^1(\mathbb{R}))$, then we define $\overrightarrow{g(t)} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ by*

$$\overrightarrow{g(t)} = (g(t), \partial_t g(t)),$$

*and we also denote the energy norm of the remainder $\overrightarrow{g(t)}$ as*

$$\left\|\overrightarrow{g(t)}\right\| = \|g(t)\|_{H^1} + \|\partial_t g(t)\|_{L^2_x}$$

*to simplify our notation in the text, where the norms $\|\cdot\|_{H^1_x}$, $\|\cdot\|_{L^2_x}$, $\|\cdot\|_{H^1_x \times L^2_x}$ are defined, respectively, by*

$$\|f_1\|_{H^1_x}^2 = \int_{\mathbb{R}} \frac{df_1(x)}{dx}^2 + f_1(x)^2 \, dx, \ \|f_2\|_{L^2_x}^2 = \int_{\mathbb{R}} f_2(x)^2 \, dx, \ \|(f_1, f_2)\|_{H^1_x \times L^2_x}^2 = \|f_1\|_{H^1_x}^2 + \|f_2\|_{L^2_x}^2,$$

*for any $f_1 \in H^1(\mathbb{R})$ and any $f_2 \in L^2(\mathbb{R})$. For any $(f_1, f_2) \in L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and any $(g_1, g_2) \in L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$, we denote*

$$\langle (f_1, f_2), (g_1, g_2) \rangle = \int_{\mathbb{R}} f_1(x) g_1(x) + f_2(x) g_2(x) \, dx.$$

*For any functions $f_1(x)$, $g_1(x) \in L^2_x(\mathbb{R})$, we denote*

$$\langle f_1, g_1 \rangle = \int_{\mathbb{R}} f_1(x) g_1(x) \, dx.$$

*We consider $\mathbb{N}$ as the set of positive integers. For any $k \in \mathbb{N}$ and any smooth function $f : \mathbb{R} \to \mathbb{R}$, we use the following notation*

$$f^{(k)}(x) = \frac{d^x}{dx^k} f(x), \text{ for all } x \in \mathbb{R}.$$

*Finally, we consider the hyperbolic functions* sech, cosh $: \mathbb{R} \to \mathbb{R}$ *and we are going to use the following notations*

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \ \operatorname{sech}(x) = (\cosh(x))^{-1}, \text{ for every } x \in \mathbb{R}.$$

# Chapter 2

# Dynamics of two interacting kinks for the $\phi^6$ model

# Abstract

We consider the nonlinear wave equation known as the $\phi^6$ model in dimension 1+1. We describe the long-time behavior of this model's solutions close to a sum of two kinks with energy slightly larger than twice the minimum energy of non-constant stationary solutions. Using the energy conservation law and spectral analysis, we prove the orbital stability of two moving kinks. We show for low energy excess $\epsilon$ that these solutions can be described for a long time of order $-\ln{(\epsilon)}\epsilon^{-\frac{1}{2}}$ as the sum of two moving kinks such that each kink's center is close to an explicit function which is a solution of an ordinary differential system of equations. These ordinary differential equations are obtained using the techniques from the previous work of M. Kowalczyk, J. Jendrej, and A. Lawrie in 2022 and a classical argument of modulation analysis. We also prove that our estimate of the energy norm of the remainder is close to the optimal during a time interval $\Delta t$ of order $-\ln{(\epsilon)}\epsilon^{-\frac{1}{2}}$.

## 2.1 Introduction

### 2.1.1 Background

We recall the partial differential equation ($\phi^6$), which, for the potential function $U(\phi) = \phi^2(1-\phi^2)^2$, is denoted by

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U^{'}(\phi(t,x)) = 0,\ (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.1}$$

The potential energy $E_{pot}$, the kinetic energy $E_{kin}$ and total energy $E_{total}$ associated to the equation (2.1) are given by

$$\begin{aligned}
E_{pot}(\phi(t)) =& \frac{1}{2}\int_{\mathbb{R}} \partial_x\phi(t,x)^2\,dx + \int_{\mathbb{R}} \phi(t,x)^2(1-\phi(t,x)^2)^2\,dx,\\
E_{kin}(\phi(t)) =& \frac{1}{2}\int_{\mathbb{R}} \partial_t\phi(t,x)^2\,dx,\\
E_{total}(\phi(t),\partial_t\phi(t)) =& \frac{1}{2}\int_{\mathbb{R}} \left[\partial_x\phi(t,x)^2 + \partial_t\phi(t,x)^2\right]\,dx\\
&+ \int_{\mathbb{R}} \phi(t,x)^2(1-\phi(t,x)^2)^2\,dx.
\end{aligned}$$

The vacuum set $\mathcal{V}$ of the potential function $U$ is the set $U^{-1}\{0\} = \{0,1,-1\}$. We say that if a solution $\phi(t,x)$ of the integral equation associated to (2.1) has $E_{total}(\phi,\partial_t\phi) < +\infty$, then it is in the energy space. The solutions of (2.1) in the energy space have constant total energy $E_{total}(\phi(t),\partial_t\phi(t))$.

From standard energy estimate techniques, the Cauchy Problem associated to (2.1) is locally well-posed in the energy space. Moreover, if $E_{total}(\phi(0),\partial_t\phi(0)) = E_0 < +\infty$, then there exists $M(E_0) > 0$ such that $\|\phi(0,x)\|_{L^\infty(\mathbb{R})} < M(E_0)$, otherwise the facts that $U \in C^\infty(\mathbb{R})$ and $\lim_{\phi\to\pm\infty} U(\phi) = +\infty$ would imply that $\int_{\mathbb{R}} U(\phi(0,x))\,dx > E_0$. Therefore, similarly to the proof of Theorem 6.1 from the book [57] of Shatah and Struwe, we can verify that the partial differential equation (2.1) is globally well-posed in the energy space since $U$ is a Lipschitz function when restricted to the space of real functions $\phi$ satisfying $\|\phi\|_{L^\infty(\mathbb{R})} < K_0$ for some positive number $K_0$.

We recall that the stationary solutions of (2.1) are the critical points of the potential energy. From Chapter 1, the only non-constant stationary solutions of (2.1) with finite total energy are the topological solitons called kinks and anti-kinks. Moreover, Remark 1.2.6 implies that each topological soliton $H$ connects different numbers $v_1,\ v_2 \in \mathcal{V}$, more precisely,

$$\lim_{x\to-\infty} H(x) = v_1,\ \lim_{x\to+\infty} H(x) = v_2,\ \mathcal{V}\cap\{H(x)|\,x\in\mathbb{R}\} = \emptyset.$$

We recall from (1.11) that all kinks of (2.1) are given by

$$H_{0,1}(x-a) = \frac{e^{\sqrt{2}(x-a)}}{\sqrt{1+e^{2\sqrt{2}(x-a)}}},\ H_{-1,0}(x-a) = -H_{0,1}(-x+a),$$

for any real $a$. The anti-kinks of (2.1) are given by $-H_{0,1}(x-a)$, $H_{0,1}(-x+a)$ for any $a\in\mathbb{R}$.

In the article [35], for the $\phi^6$ model, Manton did approximate computations to verify that the force between two static kinks is repulsive and the force between a kink and anti-kink is attractive. Furthermore, it was also obtained by approximate computations in [35] that the force of interaction between two topological solitons of the $\phi^6$ model has an exponential decay with the distance between the solitons.

The study of kink and multi-kink solutions of nonlinear wave equations has applications in many domains of mathematical physics. More precisely, the model (2.1) that we study has applications in condensed matter physics [3] and cosmology [62], [23], [20].

It is well known that the set of solutions in energy space of (2.1) for any potential $U$ is invariant under space translation, time translation, and space reflection. Moreover, if $H$ is a stationary solution of (2.1) and $-1 < v < 1$, then the function

$$\phi(t,x) = H\left(\frac{x - vt}{(1-v^2)^{\frac{1}{2}}}\right),$$

which is denominated the Lorentz transformation of $H$, is also a solution of the partial differential equation (2.1).

The problem of stability of multi-kinks is of great interest in mathematical physics, see for example [17], [14]. For the integrable model $mKdV$, Muñoz proved in [51] the $H^1$ stability and asymptotic stability of multi-kinks. However, for many non-integrable models such as the $\phi^6$ nonlinear wave equation, the asymptotic and long-time dynamics of multi-kinks after the instant where the collision or interaction happens are still unknown, even though there are numerical studies of kink-kink collision for the $\phi^6$ model, see [17], which motivate our research on the topic of the description of long time behavior of a kink-kink pair.

For one-dimensional nonlinear wave equation models, results of stability of a single kink were obtained, for example, asymptotic stability under odd perturbations of a single kink of $\phi^4$ model was proved in [29] and the study of the decay rate of this odd perturbation during a long time was studied in [13]. Also, in [31], Martel, Muñoz, Kowalczyk, and Van Den Bosch proved asymptotic stability of a single kink for a general class of nonlinear wave equations, including the model which we study here.

The main purpose of this chapter is to prove Theorem 1.4.2 and Theorem 1.4.3. Moreover, we will describe the long time behavior of solutions $\phi(t,x)$ of (2.1) in the energy space such that

$$\lim_{x\to+\infty} \phi(t,x) = 1,$$
$$\lim_{x\to-\infty} \phi(t,x) = -1,$$

with total energy equal to $2E_{pot}(H_{01})+\epsilon$, for $0<\epsilon\ll 1$. More precisely, in Theorem 1.4.2, we proved orbital stability for a sum of two moving kinks with total energy $2E_{pot}(H_{0,1})+\epsilon$ and we verified that the remainder has a better estimate during a long time interval which goes to $\mathbb{R}$ as $\epsilon \to 0$, indeed we proved that the estimate of the remainder during this long time interval is optimal. In Theorem 1.4.3, we proved that the dynamics of the kinks' movement is very close to two explicit functions $d_j : \mathbb{R} \to \mathbb{R}$ during a long time interval.

Theses results are very important to understand the behavior of two kinks after the instant of collision, which happens when the kinetic energy is minimal. Numerically, the study of interaction and collision between kinks for the $\phi^6$ model was done in [17], in which it was verified that the collision of kinks is close to an elastic collision when the speed of each kink is low and smaller than a critical speed $v_c$.

For nonlinear wave equation models in dimension $2+1$, there are similar results obtained in the dynamics of topological multi-solitons. For the Higgs Model, there are results in the description of the dynamics of multi-vortices in [58] obtained by Stuart and in [22] obtained by Gustafson and Sigal. Indeed, we took inspiration from the proof and statement of Theorem 2 of [22] to construct our main results. Also, in [59], Stuart described the dynamics of monopole solutions for the Yang-Mills-Higgs equation. For more references, see also [60], [15], [37] and [21].

In [2], Bethuel, Orlandi, and Smets described the asymptotic behavior of solutions of a parabolic Ginzburg-Landau equation closed to multi-vortices in the initial instant. For more references, see also [27] and [55].

There are also results in the dynamics of multi-vortices for nonlinear Schrödinger equation, for example, the description of the dynamics of multi-vortices for the Gross-Pitaevski equation was obtained in [52] by Ovchinnikov and Sigal and results in the dynamics of vortices for the Ginzburg-Landau-Schrödinger equations were proved in [11] by Colliander and Jerrard, see also [28] for more information about Gross-Pitaevski equation.

### 2.1.2 Main results

We recall that the objective of this chapter is to show orbital stability for the solutions of the equation (2.1) which are close to a sum of two interacting kinks in an initial instant and estimate the size of the time interval where better stability properties hold. The main techniques of the proof are modulation techniques adapted from [26], [43], and [54] and a refined energy estimate method to control the size of the remainder term.

**Definition 2.1.1.** *We define $S$ as the set $g \in L^{\infty}(\mathbb{R})$ such that*

$$\|g(x) - H_{0,1}(x) - H_{-1,0}(x)\|_{H^1_x} < +\infty.$$

From the observations made about the global well-posedness of partial differential equation (2.1) in the energy space and, since $1, -1$ are in $\mathcal{V}$, we have that (2.1) is also globally well-posed in the affine space $S \times L^2_x(\mathbb{R})$. Motivated by the proof and computations that we are going to present, we consider

**Definition 2.1.2.** *We define for $x_1$, $x_2 \in \mathbb{R}$*

$$H^{x_2}_{0,1}(x) \coloneqq H_{0,1}(x - x_2) \text{ and } H^{x_1}_{-1,0}(x) \coloneqq H_{-1,0}(x - x_1),$$

*and we say that $x_2$ is the kink center of $H_{0,1}^{x_2}(x)$ and $x_1$ is the kink center of $H_{-1,0}^{x_1}(x)$.*

In Chapter 1, we verified for any $a \in \mathbb{R}$ that the kinks $H_{0,1}(x-a)$ are the unique functions minimizing the potential energy in the set of functions satisfying

$$\lim_{x\to+\infty} \phi(t,x) = 1, \lim_{x\to-\infty} \phi(t,x) = 0, \tag{2.2}$$

since they also satisfy the partial differential equation (1.3) which is the Euler-Lagrange equation associated to the potential energy. Moreover, using the Bogomolny equation (1.9) satisfied by the kinks, we can verify that all functions $\phi(x) \in S$ have $E_{pot}(\phi) > 2E_{pot}(H_{0,1})$, see also the Subsection 2.2 of [26].

**Definition 2.1.3.** *We define the energy excess $\epsilon$ of a solution $(\phi(t), \partial_t\phi(t)) \in S \times L^2_x(\mathbb{R})$ as the following value*

$$\epsilon = E_{total}(\phi(t), \partial_t\phi(t)) - 2E_{pot}(H_{0,1}).$$

We recall the notation $(x)_+ \coloneqq \max(x, 0)$. It's not difficult to verify the following inequalities

(D1) $|H_{0,1}(x)| \leq e^{-\sqrt{2}(-x)_+}$,

(D2) $|H_{-1,0}(x)| \leq e^{-\sqrt{2}(x)_+}$,

(D3) $\left|H'_{0,1}(x)\right| \leq \sqrt{2}e^{-\sqrt{2}(-x)_+}$,

(D4) $\left|H'_{-1,0}(x)\right| \leq \sqrt{2}e^{-\sqrt{2}(x)_+}$.

Moreover, since

$$H''_{0,1}(x) = U'(H_{0,1}(x)), \tag{2.3}$$

we can verify by induction the following estimate

$$\left|\frac{d^k H_{0,1}(x)}{dx^k}\right| \lesssim_k \min\left(e^{-2\sqrt{2}x},\, e^{\sqrt{2}x}\right) \tag{2.4}$$

for all $k \in \mathbb{N}\setminus\{0\}$. The following result is crucial in the framework of Chapter 2 :

**Lemma 2.1.4** (Modulation Lemma)**.** *There exist $C_0, \delta_0 > 0$, such that if $0 < \delta \leq \delta_0$, $x_1$, $x_2$ are real numbers with $x_2 - x_1 \geq \frac{1}{\delta}$ and $g \in H^1(\mathbb{R})$ satisfies $\|g\|_{H^1_x} \leq \delta$, then for $\phi(x) = H_{-1,0}(x-x_1) + H_{0,1}(x-x_2) + g(x)$, there exist unique $y_1$, $y_2$ such that for*

$$g_1(x) = \phi(x) - H_{-1,0}(x-y_1) - H_{0,1}(x-y_2),$$

*the four following statements are true*

*1* $\langle g_1,\, \partial_x H_{-1,0}(x-y_1)\rangle = 0$,

*2* $\langle g_1,\, \partial_x H_{0,1}(x-y_2)\rangle = 0$, *3* $\|g_1\|_{H^1_x} \leq C_0\delta$,

*4* $|y_2 - x_2| + |y_1 - x_1| \leq C_0\delta$.

*We will refer to the first and second statements as the orthogonality conditions of the Modulation Lemma.*

*Proof.* The proof follows from the implicit function theorem for Banach spaces. □

Now, we recall our main results:

**Theorem 2.1.5.** *There exist* $C, \delta_0 > 0$*, such that if* $\epsilon < \delta_0$ *and*

$$(\phi(0), \partial_t\phi(0)) \in S \times L^2_x(\mathbb{R})$$

*with* $E_{total}(\phi(0), \partial_t\phi(0)) = 2E_{pot}(H_{0,1}) + \epsilon$*, then there exist functions* $x_1, x_2 \in C^2(\mathbb{R})$ *such that, for all* $t \in \mathbb{R}$*, the unique global time solution* $\phi(t,x)$ *of* (2.1) *is given by*

$$\phi(t) = H_{0,1}(x - x_2(t)) + H_{-1,0}(x - x_1(t)) + g(t), \tag{2.5}$$

*with* $g(t)$ *satisfying, for any* $t \in \mathbb{R}$*, the orthogonality conditions of the Modulation Lemma and*

$$e^{-\sqrt{2}(x_2(t)-x_1(t))} + \max_{j\in\{1,2\}} |\ddot{x}_j(t)| + \max_{j\in\{1,2\}} \dot{x}_j(t)^2 + \|(g(t), \partial_t g(t))\|^2_{H^1_x \times L^2_x} \leq C\epsilon. \tag{2.6}$$

*Furthermore, we have that*

$$\|(g(t), \partial_t g(t))\|^2_{H^1_x \times L^2_x} \leq C \min\left(\epsilon, \left[\|(g(0), \partial_t g(0)))\|^2 + \epsilon^2\right] \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right) \text{ for all } t \in \mathbb{R}. \tag{2.7}$$

**Remark 2.1.6.** *In notation of the statement of Theorem 2.1.5, for any* $\delta > 0$*, there exists* $K(\delta) \in (0,1)$ *such that if* $0 < \epsilon < K(\delta)$*,* $E_{total}(\phi(0), \partial_t\phi(0)) = 2E_{pot}(H_{0,1}) + \epsilon$*, then we have that* $\|(g(0), \partial_t g(0))\|_{H^1_x \times L^2_x} < \delta$ *and* $x_2(0) - x_1(0) > \frac{1}{\delta}$*, for the proof see Lemma A.1.3 and Corollary A.1.4 in the Appendix Section A.1.*

**Theorem 2.1.7.** *In notation of Theorem 2.1.5, there exist constants* $\delta, \kappa > 0$ *such that if* $0 < \epsilon < \delta$*, then* $\frac{\epsilon}{\kappa+1} \leq \|(g(T), \partial_t g(T))\|_{H^1_x \times L^2_x}$ *for some* $T \in \mathbb{R}$ *satisfying* $0 \leq T \leq (\kappa+1)\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

*Proof.* See the Appendix Section A.2. □

**Remark 2.1.8.** *Theorem 2.1.7 implies that estimate* (2.7) *is relevant in a time interval* $(-T, T)$ *for a* $T > 0$ *of order* $-\epsilon^{-\frac{1}{2}}\ln(\epsilon)$*. More precisely, for any function* $r : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{h\to 0} r(h) = 0$*, there is a positive value* $\delta(r)$ *such that if* $0 < \epsilon < \delta(r)$ *and* $\|(g(0), \partial_t g(0))\|_{H^1_x \times L^2_x} \leq r(\epsilon)\epsilon$*, then* $\epsilon \lesssim \|(g(t), \partial_t g(t))\|_{H^1_x \times L^2_x}$ *for some* $0 < t = O\left(\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}\right)$.

**Remark 2.1.9.** *Theorem 2.1.7 also implies the existence of a $\delta_0 > 0$ such that if $0 < \epsilon < \delta_0$, then, for any $(\phi(0,x), \partial_t\phi(0,x)) \in S \times L^2_x(\mathbb{R})$ with $E_{total}(\phi(0), \partial_t\phi(0))$ equals to $2E_{pot}(H_{0,1}) + \epsilon$, $g(t,x)$ defined in identity (2.5) satisfies $\epsilon \lesssim \limsup_{t\to+\infty} \|(g(t), \partial_t g(t)))\|_{H^1_x\times L^2_x}$, similarly we have that $\epsilon \lesssim \limsup_{t\to-\infty} \|(g(t), \partial_t g(t))\|_{H^1_x\times L^2_x}$.*

**Theorem 2.1.10.** *Let $\phi$ satisfy the assumptions in Theorem 2.1.5 and $x_1$, $x_2$, and $g$ be as in the conclusion of this theorem. Let the functions $d_1$, $d_2$ be defined for any $t \in \mathbb{R}$ by*

$$d_1(t) = a + bt - \frac{1}{2\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{2.8}$$

$$d_2(t) = a + bt + \frac{1}{2\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{2.9}$$

*where $a$, $b$, $c \in \mathbb{R}$ and $v \in (0,1)$ are the unique real values satisfying $d_j(0) = x_j(0)$, $\dot{d}_j(0) = \dot{x}_j(0)$ for $j \in \{1, 2\}$. Let $d(t) = d_2(t) - d_1(t)$, $z(t) = x_2(t) - x_1(t)$. Then, for all $t \in \mathbb{R}$, we have*

$$|z(t) - d(t)| \leq C\min(\epsilon^{\frac{1}{2}}|t|, \epsilon t^2),\ |\dot{z}(t) - \dot{d}(t)| \leq C\epsilon|t|.$$

*Furthermore, for any $t \in \mathbb{R}$,*

$$\epsilon \max_{j\in\{1,2\}} |d_j(t) - x_j(t)| = O\left(\max\left(\|(g(0), \partial_t g(0))\|_{H^1_x\times L^2_x}, \epsilon\right)^2 \left(\ln\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right), \tag{2.10}$$

$$\epsilon^{\frac{1}{2}} \max_{j\in\{1,2\}} \left|\dot{d}_j(t) - \dot{x}_j(t)\right| = O\left(\max\left(\|(g(0), \partial_t g(0))\|_{H^1_x\times L^2_x}, \epsilon\right)^2 \left(\ln\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right). \tag{2.11}$$

**Remark 2.1.11.** *If $\|(g(0), \partial_t g(0))\|_{H^1_x\times L^2_x} = O(\epsilon)$, then the estimates* (2.10) *and* (2.11) *imply that the functions $x_j(t)$, $\dot{x}_j(t)$ are very close to $d_j(t)$, $\dot{d}_j(t)$ during a time interval of order $-\ln(\epsilon)\epsilon^{-\frac{1}{2}}$.*

**Remark 2.1.12.** *The proof of Theorem 2.1.5 and Theorem 2.1.10 for $t \leq 0$ is analogous to the proof for $t \geq 0$, so we will only prove them for $t \geq 0$.*

Theorem 2.1.10 describes the repulsive behavior of the kinks. More precisely, if the kinetic energy of the kinks and the energy norm of the remainder $g$ are small enough in the initial instant $t = 0$, then the kinks will move away with displacement $z(t) \cong \epsilon^{\frac{1}{2}}t + \ln\frac{1}{\epsilon}$ when $t > 0$ is big enough belonging to a large time interval.

Furthermore, using Theorem 2.1.10, we can also deduce the following corollary.

**Corollary 2.1.13.** *With the same hypotheses as in Theorem 2.1.10, we have that*

$$\begin{aligned}\max_{j\in\{1,2\}} \left|\ddot{d}_j(t) - \ddot{x}_j(t)\right| =& O\left(\max\left(\|(g(0), \partial_t g(0))\|_{H^1_x\times L^2_x}, \epsilon\right)\epsilon^{\frac{1}{2}} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right) \\ &+ O\left(\max\left(\|(g(0), \partial_t g(0))\|_{H^1_x\times L^2_x}, \epsilon\right)^2 \left(\ln\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right).\end{aligned}$$

*Proof of Corollary 2.1.13.* It follows directly from Theorem 2.1.10 and from Lemma A.1.1 presented in the Appendix Section A.1. □

### 2.1.3 Resume of the proof

In this subsection, we present how Chapter 2 is organized and explain briefly the content of each section.

**Section 2.** In this section, we prove the orbital stability of a perturbation of a sum of two kinks. Moreover, we prove that if the initial data $(\phi(0,x),\partial_t\phi(0,x))$ satisfies the hypotheses of Theorem 2.1.5, then there are real functions $x_1$, $x_2$ of class $C^2$ such that for all $t\geq 0$

$$\left\|\phi(t,x)-H_{0,1}^{x_2(t)}-H_{-1,0}^{x_1(t)}\right\|_{H^1_x}\lesssim \epsilon^{\frac{1}{2}},$$
$$\left\|\partial_t\left(\phi(t,x)-H_{0,1}^{x_2(t)}-H_{-1,0}^{x_1(t)}\right)\right\|_{L^2_x}\lesssim \epsilon^{\frac{1}{2}}.$$

First, for every $z>0$, we are going to demonstrate the following estimate

$$E_{pot}\left(H_{0,1}(x-z)+H_{-1,0}(x)\right)=2E_{pot}\left(H_{0,1}\right)+2\sqrt{2}e^{-\sqrt{2}z}+O\left((z+1)e^{-2\sqrt{2}z}\right). \tag{2.12}$$

The proof of this inequality is similar to the demonstration of Lemma 2.7 of [26] and it follows using the Fundamental Theorem of Calculus.

The proof of the orbital stability will follow from studying the expression

$$E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g)-E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}),$$

using the fact that the kinks are critical points of $E_{pot}$ and the spectral properties of the operator $D^2E_{pot}\left(H_{0,1}\right)$, which is also non-negative. Moreover, from the modulation lemma, we will introduce the functions $x_2$, $x_1$ that will guarantee the following coercivity property

$$\|(g(t),\partial_t g(t))\|_{H^1_x\times L^2_x}^2\lesssim E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g)-E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}).$$

Therefore, the estimate above and (2.12) will imply that

$$e^{-\sqrt{2}(x_2(t)-x_1(t))}+\|(g(t),\partial_t g(t))\|_{H^1_x\times L^2_x}^2\lesssim\epsilon. \tag{2.13}$$

From the orthogonality conditions of the Modulation Lemma and standard ordinary differential equation techniques, we also obtain uniform bounds for $\|\dot{x}_j(t)\|_{L^\infty(\mathbb{R})}$, $\|\ddot{x}_j(t)\|_{L^\infty(\mathbb{R})}$ for $j\in\{1,\,2\}$. More precisely, the modulation parameters $x_1$ and $x_2$ are going to satisfying the following estimate

$$\max_{j\in\{1,2\}}\|\dot{x}_j(t)\|_{L^\infty(\mathbb{R})}^2+\|\ddot{x}_j(t)\|_{L^\infty(\mathbb{R})}\lesssim\epsilon. \tag{2.14}$$

The main techniques of this section are an adaption of sections 2 and 3 of [26].

**Section 3.** In this section, we study the long-time behavior of $\dot{x}_j(t)$, $x_j(t)$ for $j\in\{1,\,2\}$. More precisely, we prove that the parameters $x_1$ and $x_2$ satisfy the following system of differential inequalities

$$\dot{x}_j(t)=p_j(t)+O\left(\zeta(t)\right), \tag{2.15}$$
$$\dot{p}_j(t)=(-1)^{j+1}\frac{1}{\left\|H_{0,1}'\right\|_{L^2_x}^2}\frac{d}{dz}\Big|_{z=x_2(t)-x_1(t)}E_{pot}\left(H_{0,1}^z+H_{-1,0}\right)+O\left(\alpha(t)\right), \tag{2.16}$$

for $j\in\{1,2\}$, where $\alpha(t),\zeta(t)$ are non-negative functions depending only on the functions $(x_j(t))_{j\in\{1,2\}}$, $(\dot{x}_j(t))_{j\in\{1,2\}}$, $\|(g(t),\partial_t g(t))\|_{H^1_x\times L^2_x}$ and satisfying

$$\alpha(t)\lesssim\frac{\epsilon}{\ln\ln\frac{1}{\epsilon}},\ \zeta(t)\lesssim\epsilon\ln\frac{1}{\epsilon},\ \text{for all } t\in\mathbb{R}, \tag{2.17}$$

because of the estimates (2.13) and (2.14). However, the estimates (2.17) can be improved during a large time interval if we could use the estimate (2.7) in the place of $\left\|\overrightarrow{g(t)}\right\|=O(\epsilon^{\frac{1}{2}})$.

Our proof of estimates (2.15), (2.16) is based on the proof of Lemma 3.5 from [26]. First, for each $j\in\{1,2\}$, the estimate (2.15) is obtained from the time derivative of the equations

$$\langle\phi(t,x)-H_{-1,0}(x-x_1(t))-H_{0,1}\left(x-x_2(t)\right),\partial_x H_{0,1}(x-x_2(t))\rangle=0,$$
$$\langle\phi(t,x)-H_{-1,0}(x-x_1(t))-H_{0,1}\left(x-x_2(t)\right),\partial_x H_{-1,0}(x-x_1(t))\rangle=0,$$

which are the orthogonality conditions of the Modulation Lemma. Indeed, we are going to obtain that

$$\dot{x}_1(t)=-\frac{\left\langle\partial_t\phi(t,x),\partial_x H^{x_1(t)}_{-1,0}(x)\right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}}+O\left(\zeta(t)\right),$$
$$\dot{x}_2(t)=-\frac{\left\langle\partial_t\phi(t,x),\partial_x H^{x_2(t)}_{0,1}(x)\right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}}+O\left(\zeta(t)\right).$$

Next, we are going to construct a smooth cut-off function $0\leq\chi\leq 1$ satisfying

$$\chi(x)=\begin{cases}1, \text{ if } x\leq\theta(1-\gamma),\\ 0, \text{ if } x\geq\theta,\end{cases}$$

where $0<\gamma,\ \theta<1$ are parameters that will be chosen later with the objective of minimizing the modulus of the time derivative of

$$p_1(t)=-\frac{\left\langle\partial_t\phi(t),\ \partial_x H^{x_1(t)}_{-1,0}(x)+\partial_x\left(\chi\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right)g(t)\right)\right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}},$$
$$p_2(t)=-\frac{\left\langle\partial_t\phi(t),\ \partial_x H^{x_2(t)}_{0,1}(x)+\partial_x\left(\left[1-\chi\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right)\right]g(t)\right)\right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}},$$

from which with the second time derivative of the orthogonality conditions of Modulation Lemma and the partial differential equation (2.1), we will deduce the estimate (2.16) for $j\in\{1,\ 2\}$.

**Section 4.** In Section 4, we introduce a function $F(t)$ with the objective of controlling $\left\|\|(g(t),\partial_t g(t))\|_{H^1_x\times L^2_x}\right\|$ for a long time interval. More precisely, we show that the function $F(t)$ satisfies for a constant $K>0$ the global estimate $\left\|\|(g(t),\partial_t g(t))\|_{H^1_x\times L^2_x}\right\|^2\lesssim F(t)+K\epsilon^2$ and we show that $|\dot{F}(t)|$ is small enough for a long time interval. We start the function from the quadratic part of the total energy of $\phi(t)$, more precisely with

$$D(t)=\|\partial_t g(t,x)\|^2_{L^2_x}+\|\partial_x g(t,x)\|^2_{L^2_x}+\int_{\mathbb{R}}U^{(2)}(H^{x_2(t)}_{0,1}(x)+H^{x_1(t)}_{-1,0}(x))g(t,x)^2\,dx.$$

However, we obtain that the terms of worst decay that appear in the computation of $\dot{D}(t)$ are of the form

$$\int_{\mathbb{R}} \left[\partial_t \left(g(t,x)^k\right)\right] J(x_1, x_2, \dot{x}_1, \dot{x}_2, x)\, dx, \tag{2.18}$$

where $k \in \{1,2,3\}$ and the function $J$ satisfies for some $l \in \mathbb{Q}_{\geq 0}$ the following estimates

$$\sup_{t\in\mathbb{R}} \max_{j\in\{1,2\}} \left\Vert \frac{\partial}{\partial x_j} J(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x)\right\Vert_{L^2_x} \lesssim \epsilon^l,$$

$$\sup_{t\in\mathbb{R}} \max_{j\in\{1,2\}} \left\Vert \frac{\partial}{\partial \dot{x}_j} J(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x)\right\Vert_{L^2_x} \lesssim \epsilon^{l-\frac{1}{2}},$$

and

$$\sup_{t\in\mathbb{R}} \Vert J\left(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x\right)\Vert_{L^2_x} \lesssim \epsilon^l \text{ if } k=1, \text{ otherwise}$$
$$\sup_{t\in\mathbb{R}} \Vert J\left(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x\right)\Vert_{L^\infty_x(\mathbb{R})} \lesssim \epsilon^l \text{ when } k \in \{2,3\}.$$

But, we can cancel these bad terms after we add to the function $D(t)$ correction terms of the form

$$-\int_{\mathbb{R}} \left(g(t,x)^k\right) J(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x)\, dx, \tag{2.19}$$

and now, in the time derivative of the sum of $D(t)$ with these correction terms, we obtain an expression with a size of order $\epsilon^{l+\frac{1}{2}} \Vert (g(t), \partial_t g(t))\Vert^k_{H^1_x \times L^2_x}$ which is much smaller than $\epsilon^l \Vert (g(t), \partial_t g(t))\Vert^k_{H^1_x\times L^2_x}$ because of inequality (2.14) obtained in Section 2 of this chapter. Next, we consider a smooth cut-off function $0 \leq \omega \leq 1$ satisfying

$$\omega(x) = \begin{cases} 1, \text{ if } x \leq \frac{1}{2}, \\ 0, \text{ if } x \geq \frac{3}{4}, \end{cases}$$

and $\omega_1(t,x) = \omega\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right)$. Based on the argument in the proof of Lemma 4.2 of [26], we aggregate the last correction term

$$2\int_{\mathbb{R}} \partial_t g(t,x) \partial_x g(t,x) \left[\dot{x}_1(t)\omega_1(t,x) + \dot{x}_2(t)\left(1 - \omega_1(t,x)\right)\right] dx,$$

whose time derivative will cancel with the term

$$-\int_{\mathbb{R}} U^{(3)}(H^{x_2(t)}_{0,1}(x) + H^{x_1(t)}_{-1,0}(x))(\dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1} + \dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}) g(t,x)^2\, dx,$$

which comes from $\dot{D}(t)$, since we cannot remove this expression using the correction terms similar to (2.19). Finally, we evaluate the time derivative of the function $F(t)$ obtained from the sum $D(t)$ with all the correction terms described above.

**Remaining Sections.** In the remaining part of this chapter, we prove our main results, the estimate (2.7) of Theorem 2.1.5 is a consequence of the energy estimate obtained in Section 4 and the estimates with high precision of the modulation parameters $x_1(t)$, $x_2(t)$ which are obtained in Section 5. In Section 5, we prove the result of Theorem 2.1.10, where we study the evolution of the precision of the modulation parameters estimates by comparing it with a solution of a system of ordinary differential equations. Complementary information for Chapter 2 is given in Appendix Section A.1 and the proof of Theorem 2.1.7 is in the Appendix Section A.2.

## 2.2 Global Stability of two moving kinks

Before the presentation of the proofs of the main theorems, we define a function to study the potential energy of a sum of two kinks.

**Definition 2.2.1.** *The function $A : \mathbb{R}_+ \to \mathbb{R}$ is defined by*

$$A(z) \coloneqq E_{pot}(H^z_{0,1}(x) + H_{-1,0}(x)). \tag{2.20}$$

The study of the function $A$ is essential to obtain global control of the norm of the remainder $g$ and the lower bound of $x_2(t) - x_1(t)$ in Theorem 2.1.5.

**Remark 2.2.2.** *It is easy to verify that $E_{pot}(H_{0,1}(x - x_2) + H_{-1,0}(x - x_1)) = E_{pot}(H_{0,1}(x - (x_2 - x_1)) + H_{-1,0}(x))$.*

We will use several times the following elementary estimate from the Lemma 2.5 of [26] given by:

**Lemma 2.2.3.** *For any real numbers $x_2, x_1$, such that $x_2 - x_1 > 0$ and $\alpha$, $\beta > 0$ with $\alpha \neq \beta$ the following bound holds:*

$$\int_{\mathbb{R}} e^{-\alpha(x-x_1)_+} e^{-\beta(x_2-x)_+} \, dx \lesssim_{\alpha,\beta} e^{-\min(\alpha,\beta)(x_2-x_1)},$$

*For any $\alpha > 0$, the following bound holds*

$$\int_{\mathbb{R}} e^{-\alpha(x-x_1)_+} e^{-\alpha(x_2-x)_+} \, dx \lesssim_{\alpha} (1 + (x_2 - x_1)) e^{-\alpha(x_2-x_1)}.$$

The main result of this section is the following

**Lemma 2.2.4.** *The function $A$ is of class $C^2$ and there is a constant $C > 0$, such that*

1. $\left| A''(z) - 4\sqrt{2} e^{-\sqrt{2}z} \right| \leq C(z+1) e^{-2\sqrt{2}z}$,
2. $\left| A'(z) + 4 e^{-\sqrt{2}z} \right| \leq C(z+1) e^{-2\sqrt{2}z}$,
3. $\left| A(z) - 2E_{pot}(H_{0,1}) - 2\sqrt{2} e^{-\sqrt{2}z} \right| \leq C(z+1) e^{-2\sqrt{2}z}$.

*Proof.* By the definition of $A$, it's clear that

$$\begin{aligned} A(z) &= \frac{1}{2} \int_{\mathbb{R}} \left( \partial_x \left[ H^z_{0,1}(x) + H_{-1,0}(x) \right] \right)^2 dx + \int_{\mathbb{R}} U(H^z_{0,1}(x) + H_{-1,0}(x)) \, dx \\ &= \|\partial_x H_{0,1}\|^2_{L^2_x} + \int_{\mathbb{R}} \partial_x H^z_{0,1}(x) \partial_x H_{-1,0}(x) \, dx + \int_{\mathbb{R}} U(H^z_{0,1}(x) + H_{-1,0}(x)) \, dx. \end{aligned}$$

Since the functions $U$ and $H_{0,1}$ are smooth and $\partial_x H_{0,1}(x)$ has exponential decay when $|x| \to +\infty$, it is possible to differentiate $A(z)$ in $z$. More precisely, we obtain

$$\begin{aligned} A'(z) =& - \int_{\mathbb{R}} \partial^2_x H^z_{0,1}(x) \partial_x H_{-1,0}(x) \, dx - \int_{\mathbb{R}} U'(H^z_{0,1}(x) + H_{-1,0}(x)) \partial_x H^z_{0,1}(x) \, dx \\ =& \int_{\mathbb{R}} \partial_x H^z_{0,1}(x) \left[ U'(H_{-1,0})(x) - U'(H_{-1,0}(x) + H^z_{0,1}(x)) \right] dx. \end{aligned}$$

For similar reasons, it is always possible to differentiate $A(z)$ twice, precisely, we obtain

$$A^{''}(z) = \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)^2 U^{''}(H_{-1,0}(x) + H^z_{0,1}(x)) \\ -\partial^2_x H^z_{0,1}(x)\Big[U^{'}(H_{-1,0}(x)) - U^{'}\Big(H_{-1,0}(x) + H^z_{0,1}(x)\Big)\Big]\,dx. \quad (2.21)$$

Then, using integrating by parts, we obtain

$$A^{''}(z) = \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\partial_x H_{-1,0}(x)\Big[U^{''}(H_{-1,0}(x)) - U^{''}(H_{-1,0}(x) + H^z_{0,1}(x))\Big]\,dx. \quad (2.22)$$

Now, we consider the function

$$B(z) = \int_{\mathbb{R}} \partial_x H_{0,1}(x)\partial_x H_{-1,0}(x+z)\Big[U^{''}(0) - U^{''}(H_{0,1}(x))\Big]\,dx. \quad (2.23)$$

Then, we have

$$A^{''}(z) - B(z) = \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\partial_x H_{-1,0}(x)\Big[U^{''}(H_{-1,0}(x)) - U^{''}(H_{-1,0}(x) + H^z_{0,1}(x))\Big]\,dx \\ - \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\partial_x H_{-1,0}(x)\Big[U^{''}(0) - U^{''}(H^z_{0,1}(x))\Big]dx. \quad (2.24)$$

Also, it is not difficult to verify the following identity

$$\Big[U^{''}(H_{-1,0}(x)) - U^{''}(H_{-1,0}(x) + H^z_{0,1}(x))\Big] - \Big[U^{''}(0) - U^{''}(H^z_{0,1}(x))\Big] = \\ -\int_0^{H_{-1,0}(x)}\int_0^{H^z_{0,1}(x)} U^{(4)}(\omega_1+\omega_2)\,d\omega_1\,d\omega_2. \quad (2.25)$$

So, the identities (2.25) and (2.24) imply the following inequality

$$\Big|A^{''}(z) - B(z)\Big| \leq \int_{\mathbb{R}} \Big|\partial_x H^z_{0,1}(x)\partial_x H_{-1,0}(x)\Big| \left|\int_0^{H_{-1,0}(x)}\int_0^{H^z_{0,1}(x)} U^{(4)}(\omega_1+\omega_2)\,d\omega_1\,d\omega_2\right|\,dx.$$

Since $U$ is smooth and $\|H_{0,1}\|_{L^\infty} = 1$, we have that there is a constant $C > 0$ such that

$$\Big|A^{''}(z) - B(z)\Big| \leq C\int_{\mathbb{R}} \Big|\partial_x H^z_{0,1}(x)\partial_x H_{-1,0}(x)H_{-1,0}(x)H^z_{0,1}(x)\Big|\,dx. \quad (2.26)$$

Now, using the inequalities from (D1) to (D4) and Lemma 2.2.3 to inequality (2.26), we obtain that there exists a constant $C_1$ independent of $z$ such that

$$\Big|A^{''}(z) - B(z)\Big| \leq C_1(z+1)e^{-2\sqrt{2}z}. \quad (2.27)$$

Also, it is not difficult to verify that the estimate

$$\Big|\partial_x H_{-1,0}(x) - \sqrt{2}e^{-\sqrt{2}x}\Big| \leq C\min(e^{-3\sqrt{2}x}, e^{-\sqrt{2}x}), \quad (2.28)$$

and the identity (2.23) imply the inequality

$$\left|B(z) - \sqrt{2}e^{-\sqrt{2}z}\int_{\mathbb{R}} e^{-\sqrt{2}x}\partial_x H_{0,1}(x)(U^{''}(0) - U^{''}(H_{0,1}(x)))\,dx\right| \\ \lesssim \int_{\mathbb{R}} H_{0,1}(x)\partial_x H_{0,1}(x)\min\Big(e^{-3\sqrt{2}(x+z)}, e^{-\sqrt{2}(x+z)}\Big)\,dx \\ \lesssim \int_{\mathbb{R}} e^{-2\sqrt{2}(-x)_+}\min\Big(e^{-3\sqrt{2}(x+z)}, e^{-\sqrt{2}(x+z)}\Big)\,dx \\ \lesssim \int_{-\infty}^0 e^{-2\sqrt{2}(z-x)_+}e^{-\sqrt{2}x}\,dx + \int_0^{+\infty} e^{-2\sqrt{2}(z-x)_+}e^{-3\sqrt{2}(x)_+}\,dx.$$

Since we have the following identity and estimate from Lemma 2.2.3

$$\int_{-\infty}^{0} e^{-2\sqrt{2}(z-x)}e^{-\sqrt{2}x}\,dx = \frac{e^{-2\sqrt{2}z}}{\sqrt{2}}, \tag{2.29}$$

$$\int_{0}^{+\infty} e^{-2\sqrt{2}(z-x)_+}e^{-3\sqrt{2}(x)_+} \lesssim e^{-2\sqrt{2}z}, \tag{2.30}$$

we obtain then:

$$\left|B(z) - \sqrt{2}e^{-\sqrt{2}z}\int_{\mathbb{R}} e^{-\sqrt{2}x}\partial_x H_{0,1}(x)\Big[U''(0) - U''(H_{0,1}(x))\Big]\,dx\right| \lesssim e^{-2\sqrt{2}z}, \tag{2.31}$$

which clearly implies with (2.27) the inequality

$$\left|A''(z) - \sqrt{2}e^{-\sqrt{2}z}\int_{\mathbb{R}} e^{-\sqrt{2}x}\partial_x H_{0,1}(x)\Big[U''(0) - U''(H_{0,1}(x))\Big]\,dx\right| \lesssim (z+1)e^{-2\sqrt{2}z}. \tag{2.32}$$

Also, we have the identity

$$\int_{\mathbb{R}} \Big(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\Big)e^{-\sqrt{2}x}\,dx = 2\sqrt{2}, \tag{2.33}$$

for the proof see the end of Appendix A.1. Since we have the identity $U^{(2)}(0) - U^{(2)}(\phi) = 24\phi^2 - 30\phi^4$, by integration by parts, we obtain

$$\int_{\mathbb{R}} \frac{e^{-\sqrt{2}x}}{\sqrt{2}}\partial_x H_{0,1}(x)\Big[U''(0) - U''(H_{0,1}(x))\Big]\,dx = \int_{\mathbb{R}} \Big(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\Big)e^{-\sqrt{2}x}\,dx.$$

In conclusion, inequality (2.32) is equivalent to $\left|A''(z) - 4\sqrt{2}e^{-\sqrt{2}z}\right| \lesssim (z+1)e^{-2\sqrt{2}z}$.

The identities

$$U'(\phi) + U'(\theta) - U'(\phi+\theta) = 24\phi\theta(\phi+\theta) - 6\Big(\sum_{j=1}^{4}\binom{5}{j}\phi^j\theta^{5-j}\Big),$$

$$A'(z) = -\int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\Big[U'(H^z_{0,1}(x) + H_{-1,0}(x)) + U'(H_{-1,0}(x)) - U'(H^z_{0,1}(x))\Big]\,dx$$

and Lemma 2.2.3 imply the following estimate for $z>0$ $\left|A'(z)\right| \lesssim e^{-\sqrt{2}z}$, so $\lim_{|z|\to+\infty}\left|A'(z)\right| = 0$. In conclusion, integrating inequality $\left|A''(z) - 4\sqrt{2}e^{-\sqrt{2}z}\right| \lesssim (z+1)e^{-2\sqrt{2}z}$ from $z$ to $+\infty$ we obtain the second result of the lemma

$$\left|A'(z) + 4e^{-\sqrt{2}z}\right| \lesssim (z+1)e^{-2\sqrt{2}z}. \tag{2.34}$$

Finally, from the fact that $\lim_{z\to+\infty} E_{pot}(H_{-1,0} + H^z_{0,1}(x)) = 2E_{pot}(H_{0,1})$, we obtain the last estimate integrating inequality (2.34) from $z$ to $+\infty$, which is

$$\left|2E_{pot}(H_{0,1}) + 2\sqrt{2}e^{-\sqrt{2}z} - A(z)\right| \lesssim (z+1)e^{-2\sqrt{2}z}.$$

□

It is not difficult to verify that the Fréchet derivative of $E_{pot}$ as a linear functional from $H^1(\mathbb{R})$ to $\mathbb{R}$ is given by

$$(DE_{pot}(\phi))(v) := \int_{\mathbb{R}} \partial_x \phi(x)\partial_x v(x) + U^{'}(\phi(x))v(x)\,dx. \tag{2.35}$$

Also, for any $v$, $w \in H^1(\mathbb{R})$, it is not difficult to verify that

$$\left\langle D^2 E_{pot}(\phi)v,\, w\right\rangle = \int_{\mathbb{R}} \partial_x v(x)\partial_x w(x)\,dx + \int_{\mathbb{R}} U^{''}(\phi(x))v(x)w(x)\,dx. \tag{2.36}$$

Moreover, the operator $D^2 E_{pot}\left(H_{0,1}\right) : H^2_x(\mathbb{R}) \subset L^2_x(\mathbb{R}) \to L^2_x(\mathbb{R})$ satisfies the following property.

**Lemma 2.2.5.** *The operator $D^2 E_{pot}\left(H_{0,1}\right)$ satisfies:*

$$\ker\left(D^2 E_{pot}\left(H_{0,1}\right)\right) = \{c\partial_x H_{0,1}(x) |\, c \in \mathbb{R}\},$$
$$\left\langle D^2 E_{pot}\left(H_{0,1}\right) g, g\right\rangle \geq c\left[\|g\|^2_{L^2_x} - \langle g, \partial_x H_{0,1}\rangle^2 \frac{1}{\|\partial_x H_{0,1}\|^2_{L^2_x}}\right],$$

*for a constant $c > 0$ and any $g \in H^1(\mathbb{R})$.*

*Proof.* See Proposition 2.2 from [26], see also [34]. □

**Lemma 2.2.6.** *[Coercivity Lemma] There exist $C, c, \delta > 0$, such that if $x_2 - x_1 \geq \frac{1}{\delta}$, then for any $g \in H^1(\mathbb{R})$ we have*

$$\left\langle D^2 E_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0})g,\, g\right\rangle \geq c\,\|g\|^2_{H^1_x} - C\left[\langle g,\, \partial_x H^{x_1}_{-1,0}\rangle^2 + \langle g,\, \partial_x H^{x_2}_{0,1}\rangle^2\right]. \tag{2.37}$$

*Proof of Coercivity Lemma.* The proof of this Lemma is analogous to the proof of Lemma 2.4 in [26]. □

**Lemma 2.2.7.** *There is a constant $C_2$, such that if $x_2 - x_1 > 0$, then*

$$\left\|DE_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0})\right\|_{L^2_x} \leq C_2 e^{-\sqrt{2}(x_2 - x_1)}. \tag{2.38}$$

*Proof.* By the definition of the potential energy, the equation (2.3), and the exponential decay of the two kinks functions, we have that

$$DE_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) = U^{'}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) - U^{'}(H^{x_2}_{0,1}) - U^{'}(H^{x_1}_{-1,0})$$

as a bounded linear operator from $L^2_x(\mathbb{R})$ to $\mathbb{C}$. So, we have that

$$DE_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) = -24 H^{x_2}_{0,1} H^{x_1}_{-1,0}\left[H^{x_2}_{0,1} + H^{x_1}_{-1,0}\right] + 6\left[\sum_{j=1}^{4}\binom{5}{j}(H^{x_1}_{-1,0})^j (H^{x_2}_{0,1})^{5-j}\right],$$

and, then, the conclusion follows directly from Lemma 2.2.3, (D1) and (D2). □

**Theorem 2.2.8** (Orbital Stability of a sum of two moving kinks)**.** *There exists $\delta_0 > 0$ such that if the solution $\phi$ of* (2.1) *satisfies $(\phi(0), \partial_t\phi(0)) \in S \times L^2_x(\mathbb{R})$ and the energy excess $\epsilon = E_{total}(\phi) - 2E_{pot}(H_{0,1})$ is smaller than $\delta_0$, then there exist $x_1$, $x_2 : \mathbb{R} \to \mathbb{R}$ functions of class $C^2$, such that for all $t \in \mathbb{R}$ denoting $g(t) = \phi(t) - H_{0,1}(x - x_2(t)) - H_{-1,0}(x - x_1(t))$ and $z(t) = x_2(t) - x_1(t)$, we have:*

1. $\|g(t)\|_{H^1_x} = O(\epsilon^{\frac{1}{2}})$,

2. $z(t) \geq \frac{1}{\sqrt{2}}\left[\ln\frac{1}{\epsilon} + \ln 2\right]$,

3. $\|\partial_t\phi(t)\|^2_{L^2_x} \leq 2\epsilon$,

4. $\max_{j\in\{1,2\}} |\dot{x}_j(t)|^2 + \max_{j\in\{1,2\}} |\ddot{x}_j(t)| = O(\epsilon)$.

*Proof.* First, from the fact that $E_{total}(\phi(x)) > 2E_{pot}(H_{0,1})$, we deduce, from the conservation of total energy, the estimate

$$\|\partial_t\phi(t)\|^2_{L^2_x} \leq 2\epsilon. \tag{2.39}$$

From Remark 2.1.6, we can assume if $\epsilon \ll 1$ that there exist $w_1$, $w_2 \in \mathbb{R}$ such that

$$\phi(0, x) = H_{0,1}(x - w_2) + H_{-1,0}(x - w_1) + g_1(x),$$

and

$$\|g_1\|_{H^1_x} < \delta,\; w_2 - w_1 > \frac{1}{\delta},$$

for a small constant $\delta > 0$. Since the equation (2.1) is locally well-posed in the space $S \times L^2_x(\mathbb{R})$, we conclude that there is a $\delta_1 > 0$ depending only on $\delta$ and $\epsilon$ such that if $-\delta_1 \leq t \leq \delta_1$, then

$$\|\phi(t, x) - H_{0,1}(x - w_2) - H_{-1,0}(x - w_1)\|_{H^1_x} \leq 2\delta. \tag{2.40}$$

If $\delta, \epsilon > 0$ are small enough, then, from the inequality (2.40) and the Modulation Lemma, we obtain in the time interval $[-\delta_1, \delta_1]$ the existence of modulation parameters $x_1(t)$, $x_2(t)$ such that for

$$g(t) = \phi(t) - H_{0,1}(x - x_2(t)) - H_{-1,0}(x - x_1(t)),$$

we have

$$\langle g(t),\, \partial_x H_{0,1}(x - x_2(t))\rangle = \langle g(t),\, \partial_x H_{-1,0}(x - x_1(t))\rangle = 0, \tag{2.41}$$

$$\frac{1}{|x_2(t) - x_1(t)|} + \|g(t)\|_{H^1_x} \lesssim \delta. \tag{2.42}$$

From now on, we denote $z(t) = x_2(t) - x_1(t)$. From the conservation of the total energy, we have for $-\delta_1 \leq t \leq \delta_1$ that

$$\begin{aligned} E_{total}(\phi(t)) =& \frac{\|\partial_t \phi(t)\|_{L^2_x}^2}{2} + E_{pot}\Big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\Big) \\ &+\Big\langle DE_{pot}\Big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\Big),\, g(t)\Big\rangle \\ &+\frac{\Big\langle D^2E_{pot}\Big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\Big)g(t),\, g(t)\Big\rangle}{2} + O(\|g(t)\|_{H^1_x}^3). \end{aligned}$$

From Lemma 2.2.4 and (2.42), the above identity implies that

$$\begin{aligned} \epsilon =& \frac{\|\partial_t \phi(t)\|_{L^2_x}^2}{2} + 2\sqrt{2}e^{-\sqrt{2}z(t)} + \Big\langle DE_{pot}\Big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\Big),\, g(t)\Big\rangle \\ &+\frac{\Big\langle D^2E_{pot}\Big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\Big)g(t),\, g(t)\Big\rangle}{2} + O\Big(\,\|g(t)\|_{H^1_x}^3 + z(t)e^{-2\sqrt{2}z(t)}\Big) \end{aligned} \tag{2.43}$$

for any $t \in [-\delta_1, \delta_1]$. From (2.38), we can verify that $\Big|\langle DE_{pot}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}),\, g(t)\rangle\Big| \leq C_2 e^{-\sqrt{2}z(t)}\, \|g(t)\|_{H^1(\mathbb{R})}$. So, the equation (2.43) and the Coercivity Lemma imply, while $-\delta_1 \leq t \leq \delta_1$, the following inequality

$$\begin{aligned} \epsilon + C_2 e^{-\sqrt{2}z(t)}\, \|g(t)\|_{H^1_x} \geq & \frac{\|\partial_t \phi(t)\|_{L^2_x}^2}{2} + 2\sqrt{2}e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1_x}^2}{2} \\ &+ O\left(\|g(t)\|_{H^1_x}^3 + z(t)e^{-2\sqrt{2}z(t)}\right). \end{aligned} \tag{2.44}$$

Finally, applying the Young inequality in the term $C_2 e^{-\sqrt{2}z(t)}\, \|g(t)\|_{H^1(\mathbb{R})}$, we obtain that the inequality (2.44) can be rewritten in the form

$$\epsilon \geq \frac{\|\partial_t \phi(t)\|_{L^2_x}^2}{2} + 2\sqrt{2}e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1_x}^2}{4} + O\left(\|g(t)\|_{H^1_x}^3 + (z(t)+1)e^{-2\sqrt{2}z(t)}\right). \tag{2.45}$$

Then, the estimates (2.45), (2.42) imply for $\delta > 0$ small enough the following inequality

$$\epsilon \geq \frac{\|\partial_t \phi(t)\|_{L^2_x}^2}{2} + 2e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1_x}^2}{8}. \tag{2.46}$$

So, the inequality (2.46) implies the estimates

$$e^{-\sqrt{2}z(t)} < \frac{\epsilon}{2},\ \|g(t)\|_{H^1_x}^2 \lesssim \epsilon, \tag{2.47}$$

for $t \in [-\delta_1, \delta_1]$. In conclusion, if $\frac{1}{\delta} \lesssim \ln\left(\frac{1}{\epsilon}\right)^{\frac{1}{2}}$, we can conclude by a bootstrap argument that the inequalities (2.39), (2.47) are true for all $t \in \mathbb{R}$. More precisely, we study the set

$$C = \left\{ b \in \mathbb{R}_{>0} \middle|\, \epsilon \geq \frac{\|\partial_t \phi(t)\|_{L^2_x}^2}{2} + 2e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1_x}^2}{8}, \text{ if } |t| \leq b. \right\}$$

and prove that $M = \sup_{b \in C} b = +\infty$. We already have checked that $C$ is not empty, also $C$ is closed by its definition. Now from the previous argument, we can verify that $C$ is open. So, by connectivity, we obtain that $C = \mathbb{R}_{>0}$.

In conclusion, it remains to prove that the modulation parameters $x_1(t)$, $x_2(t)$ are of class $C^2$ and that the fourth item of the statement of Theorem 2.2.8 is true.

**(Proof of the $C^2$ regularity of $x_1$, $x_2$, and of the fourth item.)**

For $\delta_0 > 0$ small enough, we denote $(y_1(t),\, y_2(t))$ to be the solution of the following system of ordinary differential equations, with the function $g_1(t) = \phi(t,x) - H_{0,1}^{y_2(t)}(x) - H_{-1,0}^{y_1(t)}(x)$,

$$\Big( \|\partial_x H_{0,1}\|_{L^2_x}^2 - \Big\langle g_1(t),\, \partial_x^2 H_{-1,0}^{y_1(t)} \Big\rangle \Big)\dot{y}_1(t) + \Big( \Big\langle \partial_x H_{0,1}^{y_2(t)},\, \partial_x H_{-1,0}^{y_1(t)} \Big\rangle \Big)\dot{y}_2(t) = \\ -\Big\langle \partial_t \phi(t),\, \partial_x H_{-1,0}^{y_1(t)}(x) \Big\rangle, \quad (2.48)$$

$$\Big( \Big\langle \partial_x H_{0,1}^{y_2(t)},\, \partial_x H_{-1,0}^{y_1(t)} \Big\rangle \Big)\dot{y}_1(t) + \Big( \|\partial_x H_{0,1}(t)\|_{L^2_x}^2 - \Big\langle g_1(t),\, \partial_x^2 H_{0,1}^{y_2} \Big\rangle \Big)\dot{y}_2(t) = \\ -\Big\langle \partial_t \phi(t),\, \partial_x H_{0,1}^{y_2(t)}(x) \Big\rangle, \quad (2.49)$$

with initial condition $(y_2(0), y_1(0)) = (x_2(0), x_1(0))$. This system of ordinary differential equations is motivated by the time derivative of the orthogonality conditions of the Modulation Lemma.

Since we have the estimate $\ln\left(\frac{1}{\epsilon}\right) \lesssim x_2(0) - x_1(0)$ and $g_1(0) = g(0)$, Lemma 2.2.3 and the inequalities in (2.47) imply that the matrix

$$\begin{bmatrix} \|\partial_x H_{0,1}\|_{L^2_x}^2 - \Big\langle g_1(0),\, \partial_x^2 H_{-1,0}^{y_1(0)} \Big\rangle & \Big\langle \partial_x H_{0,1}^{y_2(0)},\, \partial_x H_{-1,0}^{y_1(0)} \Big\rangle \\ \Big\langle \partial_x H_{0,1}^{y_2(0)},\, \partial_x H_{-1,0}^{y_1(0)} \Big\rangle & \|\partial_x H_{0,1}\|_{L^2_x}^2 - \Big\langle g_1(0),\, \partial_x^2 H_{0,1}^{y_2} \Big\rangle \end{bmatrix} \quad (2.50)$$

is positive, so we have from Picard-Lindelöf Theorem that $y_2(t)$, $y_1(t)$ are of class $C^1$ for some interval $[-\delta, \delta]$, with $\delta > 0$ depending on $|x_2(0) - x_1(0)|$ and $\epsilon$. From the fact that $(y_2(0), y_1(0)) = (x_2(0), x_1(0))$, we obtain, from the equations (2.48) and (2.49), that $(y_2(t), y_1(t))$ also satisfies the orthogonality conditions of Modulation Lemma for $t \in [-\delta, \delta]$. In conclusion, the uniqueness of Modulation Lemma implies that $(y_2(t), y_1(t)) = (x_2(t), x_1(t))$ for $t \in [-\delta, \delta]$. From this argument, we also have for $t \in [-\delta, \delta]$ that $e^{-\sqrt{2}(y_2(t) - y_1(t))} \leq \frac{\epsilon}{2\sqrt{2}}$. By bootstrap, we can show, repeating the argument above, that

$$\sup\{C > 0 |\, (y_2(t), y_1(t)) = (x_2(t), x_1(t)), \text{for } t \in [-C, C]\} = +\infty. \quad (2.51)$$

Also, the argument above implies that if $(y_1(t), y_2(t)) = (x_1(t), x_2(t))$ in an instant $t$, then $y_1$, $y_2$ are of class $C^1$ in a neighborhood of $t$. In conclusion, $x_1$, $x_2$ are functions in $C^1(\mathbb{R})$. Finally, since $\|g(t)\|_{H^1_x} = O(\epsilon^{\frac{1}{2}})$ and $e^{-\sqrt{2}z(t)} = O(\epsilon)$, the following matrix

$$M(t) := \begin{bmatrix} \|\partial_x H_{0,1}\|_{L^2_x}^2 - \Big\langle g(t),\, \partial_x^2 H_{-1,0}^{x_1(t)} \Big\rangle & \Big\langle \partial_x H_{0,1}^{x_2(t)},\, \partial_x H_{-1,0}^{x_1(t)} \Big\rangle \\ \Big\langle \partial_x H_{0,1}^{x_2(t)},\, \partial_x H_{-1,0}^{x_1(t)} \Big\rangle & \|\partial_x H_{0,1}\|_{L^2_x}^2 - \Big\langle g(t),\, \partial_x^2 H_{0,1}^{x_2(t)} \Big\rangle \end{bmatrix} \quad (2.52)$$

is uniformly positive for all $t \in \mathbb{R}$. So, from the estimate $\|\partial_t \phi(t)\|_{L^2_x} = O(\epsilon^{\frac{1}{2}})$, the identities $x_j(t) = y_j(t)$ for $j = 1, 2$ and the equations (2.48) and (2.49), we obtain

$$\max_{j \in \{1,2\}} |\dot{x}_j(t)| = O(\epsilon^{\frac{1}{2}}). \quad (2.53)$$

Since the matrix $M(t)$ is invertible for any $t \in \mathbb{R}$, we can obtain from the equations (2.48), (2.49) that the functions $\dot{x}_1(t), \dot{x}_2(t)$ are given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = M(t)^{-1} \begin{bmatrix} -\left\langle \partial_t \phi(t),\, \partial_x H^{x_1(t)}_{-1,0}(x) \right\rangle \\ -\left\langle \partial_t \phi(t),\, \partial_x H^{x_2(t)}_{0,1}(x) \right\rangle \end{bmatrix}. \tag{2.54}$$

Now, since we have that $(\phi(t), \partial_t \phi(t)) \in C(\mathbb{R}, S \times L^2_x(\mathbb{R}))$ and $x_1(t)$, $x_2(t)$ are of class $C^1$, we can deduce that $(g(t), \partial_t g(t)) \in C(\mathbb{R}, H^1(\mathbb{R}) \times L^2_x(\mathbb{R}))$. So, by definition, we can verify that $M(t) \in C^1(\mathbb{R}, \mathbb{R}^4)$.

Also, since $\phi(t,x)$ is the solution in distributional sense of (2.1), we have that for any $y_1$, $y_2 \in \mathbb{R}$ the following identities hold

$$\begin{aligned} \left\langle \partial_x H^{y_2}_{0,1},\, \partial_t^2 \phi(t) \right\rangle &= -\left\langle \partial_x^2 H^{y_2}_{0,1},\, \partial_x \phi(t) \right\rangle - \left\langle \partial_x H^{y_2(t)}_{0,1},\, U'(\phi(t)) \right\rangle, \\ \left\langle \partial_x H^{y_1}_{-1,0},\, \partial_t^2 \phi(t) \right\rangle &= -\left\langle \partial_x^2 H^{y_1}_{-1,0},\, \partial_x \phi(t) \right\rangle - \left\langle \partial_x H^{y_1}_{-1,0},\, U'(\phi(t)) \right\rangle. \end{aligned}$$

Since (2.1) is locally well-posed in $S \times L^2_x(\mathbb{R})$, we obtain from the identities above that the following functions $h(t,y) := \left\langle \partial_x H^{y}_{0,1},\, \partial_t^2 \phi(t) \right\rangle$ and $l(t,y) := \left\langle \partial_x H^{y}_{-1,0},\, \partial_t^2 \phi(t) \right\rangle$ are continuous in the domain $\mathbb{R} \times \mathbb{R}$.

So, from the continuity of the functions $h(t,y)$, $l(t,y)$ and from the fact that $x_1$, $x_2 \in C^1(\mathbb{R})$, we obtain that the functions

$$h_1(t) := -\left\langle \partial_t \phi(t),\, \partial_x H^{x_1(t)}_{-1,0}(x) \right\rangle,\; h_2(t) := -\left\langle \partial_t \phi(t),\, \partial_x H^{x_2(t)}_{0,1}(x) \right\rangle$$

are of class $C^1$. In conclusion, from the equation (2.54), by chain rule and product rule, we verify that $x_1$, $x_2$ are in $C^2(\mathbb{R})$.

Now, since $x_1$, $x_2 \in C^2(\mathbb{R})$ and $\dot{x}_1$, $\dot{x}_2$ satisfy (2.54), we deduce after differentiate in time the function

$$M(t) \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$$

the following equations

$$\begin{aligned} \ddot{x}_1(t)\Big( \|\partial_x H_{0,1}\|^2_{L^2_x} + \left\langle \partial_x g(t),\, \partial_x H^{x_1(t)}_{-1,0} \right\rangle \Big) + \ddot{x}_2(t) \left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_x H^{x_2(t)}_{0,1} \right\rangle \\ = \dot{x}_1(t)^2 \left\langle \partial_x^2 H^{x_1(t)}_{-1,0},\, \partial_x g(t) \right\rangle + \dot{x}_1(t) \left\langle \partial_x^2 H^{x_1(t)}_{-1,0},\, \partial_t g(t) \right\rangle \\ + \dot{x}_2(t)^2 \left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_x^2 H^{x_2(t)}_{0,1} \right\rangle + \dot{x}_1(t)\dot{x}_2(t) \left\langle \partial_x^2 H^{x_1(t)}_{-1,0},\, \partial_x H^{x_2(t)}_{0,1} \right\rangle \\ + \dot{x}_1(t) \left\langle \partial_x^2 H^{x_1(t)}_{-1,0},\, \partial_t \phi(t) \right\rangle - \left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_t^2 \phi(t) \right\rangle, \end{aligned} \tag{2.55}$$

$$\begin{aligned} \ddot{x}_2(t)\Big( \|\partial_x H_{0,1}\|^2_{L^2_x} + \left\langle \partial_x g(t),\, \partial_x H^{x_2(t)}_{0,1} \right\rangle \Big) + \ddot{x}_1(t) \left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_x H^{x_2(t)}_{0,1} \right\rangle \\ = \dot{x}_2(t)^2 \left\langle \partial_x^2 H^{x_2(t)}_{0,1},\, \partial_x g(t) \right\rangle + \dot{x}_2(t) \left\langle \partial_x^2 H^{x_2(t)}_{0,1},\, \partial_t g(t) \right\rangle \\ + \dot{x}_1(t)\dot{x}_2(t) \left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_x^2 H^{x_2(t)}_{0,1} \right\rangle + \dot{x}_1(t)^2 \left\langle \partial_x H^{x_2(t)}_{0,1},\, \partial_x^2 H^{x_1(t)}_{-1,0} \right\rangle \\ + \dot{x}_2(t) \left\langle \partial_x^2 H^{x_2(t)}_{0,1},\, \partial_t \phi(t) \right\rangle - \left\langle \partial_x H^{x_2(t)}_{0,1},\, \partial_t^2 \phi(t) \right\rangle. \end{aligned} \tag{2.56}$$

Also, from the identity $g(t) = \phi(t) - H^{x_1(t)}_{-1,0} - H^{x_2(t)}_{0,1}$, we obtain that $\partial_t g(t) = \partial_t \phi(t,x) + \dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} + \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}$, so, from the estimates (2.39) and (2.53), we obtain that

$$\|\partial_t g(t)\|_{L^2_x} = O(\epsilon^{\frac{1}{2}}). \tag{2.57}$$

Now, since $\phi(t)$ is a distributional solution of (2.1), we also have, from the global equality $\phi(t) = H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)$, the following identity

$$\begin{aligned}\left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_t^2 \phi(t)\right\rangle =& \left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_x^2 g(t) - U''\left(H^{x_1(t)}_{-1,0}\right) g(t)\right\rangle \\ &- \left\langle \partial_x H^{x_1(t)}_{-1,0}, \left[U''\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) - U''\left(H^{x_1(t)}_{-1,0}\right)\right] g(t)\right\rangle \\ &+ \left\langle \partial_x H^{x_1(t)}_{-1,0},\, U'\left(H^{x_1(t)}_{-1,0}\right) + U'\left(H^{x_2(t)}_{0,1}\right) - U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right\rangle \\ &- \left\langle \partial_x H^{x_1(t)}_{-1,0}, U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right) - U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right\rangle \\ &+ \left\langle \partial_x H^{x_1(t)}_{-1,0}, U''\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) g(t)\right\rangle.\end{aligned}$$

Since $\partial_x H^{x_1(t)}_{-1,0} \in \ker\left(D^2 E_{pot}\left(H^{x_1(t)}_{-1,0}\right)\right)$, we have by integration by parts that

$$\left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_x^2 g(t) - U''\left(H^{x_1(t)}_{-1,0}\right) g(t)\right\rangle = 0.$$

Since we have

$$\begin{aligned}&U'\left(H^{x_1(t)}_{-1,0}\right) + U'\left(H^{x_2(t)}_{0,1}\right) - U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) \\ &\quad = 24 H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) - 6\sum_{j=1}^{4}\binom{5}{j}\left(H^{x_1(t)}_{-1,0}\right)^j\left(H^{x_2(t)}_{0,1}\right)^{5-j},\end{aligned} \tag{2.58}$$

Lemma 2.2.3 implies that

$$\left\langle \partial_x H^{x_1(t)}_{-1,0},\, U'\left(H^{x_1(t)}_{-1,0}\right) + U'\left(H^{x_2(t)}_{0,1}\right) - U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right\rangle = O\Big(e^{-\sqrt{2}(z(t))}\Big).$$

Also, from Taylor's Expansion Theorem, we have the estimate

$$\begin{aligned}&\left\langle \partial_x H^{x_1(t)}_{-1,0}, U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right) - U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right\rangle \\ &\qquad - \left\langle \partial_x H^{x_1(t)}_{-1,0}, U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) g(t)\right\rangle = O(\|g(t)\|^2_{H^1_x}).\end{aligned}$$

From Lemma 2.2.3, the fact that $U$ is a smooth function and $H_{0,1} \in L^\infty(\mathbb{R})$, we can obtain

$$\begin{aligned}\left\langle \partial_x H^{x_1(t)}_{-1,0}, \left[U''\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) - U''\left(H^{x_1(t)}_{-1,0}\right)\right] g(t)\right\rangle =& O\Big(\int_{\mathbb{R}} \partial_x H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\,|g(t)|\,dx\Big) \\ =& O\Big(e^{-\sqrt{2}z(t)}\,\|g(t)\|_{H^1_x}\,z(t)^{\frac{1}{2}}\Big).\end{aligned}$$

In conclusion, we have

$$\left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_t^2 \phi(t)\right\rangle = O\Big(\,\|g(t)\|^2_{H^1_x} + e^{-\sqrt{2}z(t)}\Big), \tag{2.59}$$

and by similar arguments, we have

$$\left\langle \partial_x H^{x_2(t)}_{0,1},\, \partial_t^2 \phi(t)\right\rangle = O\Big(\,\|g(t)\|^2_{H^1_x} + e^{-\sqrt{2}z(t)}\Big). \tag{2.60}$$

Also, the equations (2.55) and (2.56) form a linear system with $\ddot{x}_1(t)$, $\ddot{x}_2(t)$. Recalling that the Matrix $M(t)$ is uniformly positive, we obtain from the estimates (2.47), (2.53), (2.57), (2.59) and (2.60) that

$$\max_{j\in\{1,2\}} |\ddot{x}_j(t)| = O(\epsilon). \tag{2.61}$$

□

The Theorem 2.2.8 can also be improved when the kinetic energy of the solution is included in the computation and additional conditions are added, more precisely:

**Theorem 2.2.9.** *There exist $C$, $c$, $\delta_0 > 0$, such that if $0 < \epsilon \leq \delta_0$, $(\phi(0,x), \partial_t\phi(0,x)) \in S \times L^2_x(\mathbb{R})$ and $E_{total}((\phi(0,x), \partial_t\phi(0,x))) = 2E_{pot}(H_{0,1}) + \epsilon$, then there are $x_2$, $x_1 \in C^2(\mathbb{R})$ such that $g(t,x) = \phi(t,x) - H^{x_2(t)}_{0,1}(x) - H^{x_1(t)}_{-1,0}(x)$ satisfies*

$$\left\langle g(t,x), \partial_x H^{x_2(t)}_{0,1}(x)\right\rangle = 0,\ \left\langle g(t,x), \partial_x H^{x_1(t)}_{-1,0}(x)\right\rangle = 0,$$

*and, for all $t \in \mathbb{R}$,*

$$c\epsilon \leq e^{-\sqrt{2}(x_2(t)-x_1(t))} + \|(g(t), \partial_t g(t))\|^2_{H^1_x\times L^2_x} + |\dot{x}_1(t)|^2 + |\dot{x}_2(t)|^2 \leq C\epsilon. \tag{2.62}$$

*Proof.* From Modulation Lemma and Theorem 2.2.8, we can rewrite the solution $\phi(t)$ in the form

$$\phi(t,x) = H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x) + g(t,x)$$

with $x_1(t)$, $x_2(t)$, $g(t)$ satisfying the conclusion of Theorem 2.2.8. First, we denote

$$\phi_\sigma(t) = \left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x), -\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} - \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}\right) \in S \times L^2_x(\mathbb{R}), \tag{2.63}$$

then we apply Taylor's Expansion Theorem in $E(\phi(t))$ around $\phi_\sigma(t)$. More precisely, for $R_\sigma(t)$ the residue of quadratic order of Taylor's Expansion of $E(\phi(t), \partial_t\phi(t))$ around $\phi_\sigma(t)$, we have:

$$\begin{aligned} 2E_{pot}(H_{0,1}) + \epsilon =& E_{total}(\phi_\sigma(t)) + \langle DE_{total}(\phi_\sigma(t)),\ (g(t), \partial_t g(t))\rangle \\ &+ \frac{\left\langle D^2E_{total}(\phi_\sigma(t))\big(g(t), \partial_t g(t)\big),\ \big(g(t), \partial_t g(t)\big)\right\rangle}{2} + R_\sigma(t), \end{aligned} \tag{2.64}$$

such that for $(\nu_1, \nu_2) \in S \times L^2_x(\mathbb{R})$ and $(v_1, v_2) \in H^1(\mathbb{R}) \times L^2_x(\mathbb{R})$, we have the identities

$$\begin{aligned} E_{total}(\nu_1, \nu_2) =& \frac{\|\partial_x\nu_1\|^2_{L^2_x} + \|\nu_2\|^2_{L^2_x}}{2} + \int_{\mathbb{R}} U(\nu_1(x))\,dx, \\ \langle DE_{total}(\nu_1, \nu_2), (v_1, v_2)\rangle =& \int_{\mathbb{R}} \partial_x\nu_1(x)\partial_x v_1(x) + U^{'}(\nu_1)v_1 + \nu_2(x)v_2(x)\,dx, \qquad (2.65) \\ D^2E_{total}(\nu_1, \nu_2) =& \begin{bmatrix} -\partial^2_x + U^{''}(\nu_1) & 0 \\ 0 & \mathbb{I} \end{bmatrix} \qquad (2.66) \end{aligned}$$

with $D^2E_{total}(\nu_1, \nu_2)$ defined as a linear operator from $H^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ to $L^2_x(\mathbb{R}) \times L^2_x(\mathbb{R})$.

So, from identities (2.65) and (2.66), it is not difficult to verify that

$$
\begin{aligned}
R_\sigma(t) =& \int_{\mathbb{R}} U\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x) + g(t,x)\right) - U\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right)\,dx \\
&- \int_{\mathbb{R}} U'\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right) g(t,x)\,dx \\
&- \int_{\mathbb{R}} \frac{U''\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right) g(t,x)^2}{2}\,dx,
\end{aligned}
$$

and, so,

$$
|R_\sigma(t)| = O\left(\|g(t)\|^3_{H^1_x}\right). \tag{2.67}
$$

Also, we have

$$
\begin{aligned}
\langle DE_{total}(\phi_\sigma(t)),\,(g(t),\partial_t g(t))\rangle =& \left\langle DE_{pot}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right),\,g(t)\right\rangle \\
&- \left\langle \dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} + \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1},\,\partial_t g(t)\right\rangle.
\end{aligned} \tag{2.68}
$$

The orthogonality conditions satisfied by $g(t)$ also imply for all $t \in \mathbb{R}$ that

$$
\left\langle \partial_t g(t),\,\partial_x H^{x_1(t)}_{-1,0}\right\rangle = \dot{x}_1(t)\left\langle g(t),\,\partial^2_x H^{x_1(t)}_{-1,0}\right\rangle, \tag{2.69}
$$

$$
\left\langle \partial_t g(t),\,\partial_x H^{x_2(t)}_{0,1}\right\rangle = \dot{x}_2(t)\left\langle g(t),\,\partial^2_x H^{x_2(t)}_{0,1}\right\rangle. \tag{2.70}
$$

So, the inequality (2.38) and the identities (2.68), (2.69), (2.70) imply that

$$
|\langle DE_{total}(\phi_\sigma(t)),\,(g(t),\partial_t g(t))\rangle| = O\Big(\|g(t)\|_{H^1_x}\sup_{j\in\{1,2\}}|\dot{x}_j(t)|^2 + \|g(t)\|_{H^1_x} e^{-\sqrt{2}z(t)}\Big). \tag{2.71}
$$

From the Coercivity Lemma and the definition of $D^2E_{total}(\phi_\sigma(t))$, we have that

$$
\left\langle D^2E_{total}(\phi_\sigma(t))(g(t),\partial_t g(t)),\,(g(t),\partial_t g(t))\right\rangle \cong \|(g(t),\partial_t g(t))\|^2_{H^1_x\times L^2_x}. \tag{2.72}
$$

Finally, there is the identity

$$
\begin{aligned}
&\left\|\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}(x) + \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}(x)\right\|^2_{L^2_x} \\
&\quad = 2\dot{x}_1(t)\dot{x}_2(t)\left\langle \partial_x H^{z(t)}_{0,1},\,\partial_x H_{-1,0}\right\rangle + \dot{x}_1(t)^2\|\partial_x H_{0,1}\|^2_{L^2_x} \\
&\quad\quad + \dot{x}_2(t)^2\|\partial_x H_{0,1}\|^2_{L^2_x}.
\end{aligned} \tag{2.73}
$$

From Lemma 2.2.3, we have that $\left|\langle \partial_x H^z_{0,1},\,\partial_x H_{-1,0}\rangle\right| = O\Big(ze^{-\sqrt{2}z}\Big)$ for $z$ big enough. Then, it is not difficult to verify that Lemma 2.2.4, (2.67), (2.71), (2.72) and (2.73) imply directly the statement of the Theorem 2.2.9 which finishes the proof. □

**Remark 2.2.10.** *Theorem 2.2.9 implies that it is possible to have a solution $\phi$ of the equation* (2.1) *with energy excess $\epsilon > 0$ small enough to satisfy all the hypotheses of Theorem 2.1.5. More precisely, in notation of Theorem 2.1.5, if $\|(g(0,x),\partial_t g(0,x))\|_{H^1_x\times L^2_x} \ll \epsilon^{\frac{1}{2}}$ and*

$$
e^{-\sqrt{2}z(0)} + \dot{x}_1(0)^2 + \dot{x}_2(0)^2 \cong \epsilon,
$$

*then we would have that $E_{total}(\phi(0),\partial_t\phi(0)) - 2E_{pot}(H_{0,1}) \cong \epsilon$.*

## 2.3 Long Time Behavior of Modulation Parameters

Even though Theorem 2.2.8 implies the orbital stability of a sum of two kinks with low energy excess, this theorem does not explain the movement of the kinks' centers $x_2(t)$, $x_1(t)$ and their speed for a long time. More precisely, we still don't know if there is an explicit smooth real function $d(t)$, such that $(z(t), \dot{z}(t))$ is close to $(d(t), \dot{d}(t))$ in a large time interval.

But, the global estimates on the modulus of the first and second derivatives of $x_1(t)$, $x_2(t)$ obtained in Theorem 2.2.8 will be very useful to estimate with high precision the functions $x_1(t)$, $x_2(t)$ during a very large time interval. Moreover, we first have the following auxiliary lemma.

**Lemma 2.3.1.** *Let* $0 < \theta,\, \gamma < 1$. *We recall the function*

$$A(z) = E_{pot}(H_{0,1}^{z} + H_{-1,0})$$

*for any* $z > 0$. *We assume all the hypotheses of Theorem 2.2.8 and let* $\chi(x)$ *be a smooth function satisfying*

$$\chi(x) = \begin{cases} 1, & \text{if } x \leq \theta(1-\gamma), \\ 0, & \text{if } x \geq \theta, \end{cases} \tag{2.74}$$

*and* $0 \leq \chi(x) \leq 1$ *for all* $x \in \mathbb{R}$. *In notation of Theorem 2.2.8, we denote*

$$\chi_0(t,x) = \chi\Big(\frac{x - x_1(t)}{z(t)}\Big),\ \overrightarrow{g(t)} = (g(t), \partial_t g(t)) \in H^1(\mathbb{R}) \times L^2_x(\mathbb{R})$$

*and* $\left\|\overrightarrow{g(t)}\right\| = \|(g(t), \partial_t g(t))\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}$,

$$\begin{aligned}\alpha(t) = {} & \left\|\overrightarrow{g(t)}\right\| \max_{j\in\{1,2\}} |\dot{x}_j(t)| \left[1 + \frac{1}{z(t)\gamma} + \frac{1}{z(t)^2\gamma^2} \max_{j\in\{1,2\}} |\dot{x}_j(t)|\right]\Big(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big) \\ & + \max_{j\in\{1,2\}} \dot{x}_j(t)^2 z(t) e^{-\sqrt{2}z(t)} + \frac{\max_{j\in\{1,2\}} \dot{x}_j(t)^2}{z(t)\gamma}\Big(e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big) \\ & + \left\|\overrightarrow{g(t)}\right\|^2 \left[\frac{1}{\gamma^2 z(t)^2} + \frac{1}{\gamma z(t)} + \Big(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big)\right].\end{aligned} \tag{2.75}$$

*Then, for* $\theta = \frac{1-\gamma}{2-\gamma}$ *and the correction terms*

$$\begin{aligned} p_1(t) = {} & -\frac{\Big\langle \partial_t\phi(t),\, \partial_x H_{-1,0}^{x_1(t)}(x) + \partial_x(\chi_0(t,x) g(t))\Big\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}}, \\ p_2(t) = {} & -\frac{\Big\langle \partial_t\phi(t),\, \partial_x H_{0,1}^{x_2(t)}(x) + \partial_x([1-\chi_0(t,x)] g(t))\Big\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}}, \end{aligned}$$

*we have the following estimates, for* $j \in \{1,2\}$,

$$\begin{aligned} |\dot{x}_j(t) - p_j(t)| \lesssim {} & \left[1 + \frac{\left\|\chi'\right\|_{L^\infty}}{z(t)}\right]\Big(\max_{j\in\{1,2\}} |\dot{x}_j(t)| \left\|\overrightarrow{g(t)}\right\| + \left\|\overrightarrow{g(t)}\right\|^2\Big) \\ & + \max_{j\in\{1,2\}} |\dot{x}_j(t)|\, z(t) e^{-\sqrt{2}z(t)}, \end{aligned} \tag{2.76}$$

$$\left|\dot{p}_j(t) + (-1)^j \frac{A'(z(t))}{\|\partial_x H_{0,1}\|^2_{L^2_x}}\right| \lesssim \alpha(t). \tag{2.77}$$

**Remark 2.3.2.** *We will take* $\gamma = \frac{\ln\ln\left(\frac{1}{\epsilon}\right)}{\ln\left(\frac{1}{\epsilon}\right)}$. *With this value of* $\gamma$ *and the estimates of Theorem 2.2.8, we will see in Lemma 2.5.1 that* $\exists C > 0$ *such that*

$$\alpha(t) \lesssim \frac{\left(\left\Vert\overrightarrow{g(0)}\right\Vert_{H^1_x \times L^2_x} + \epsilon \ln\frac{1}{\epsilon}\right)^2}{\ln\ln\left(\frac{1}{\epsilon}\right)} \exp\left(\frac{2C\,|t|\,\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right).$$

*Proof.* For $\gamma \ll 1$ enough and from the definition of $\chi(x)$, it is not difficult to verify that

$$\left\Vert\chi^{'}\right\Vert_{L^\infty(\mathbb{R})} \lesssim \frac{1}{\gamma},\ \left\Vert\chi^{''}\right\Vert_{L^\infty(\mathbb{R})} \lesssim \frac{1}{\gamma^2}. \tag{2.78}$$

We will only do the proof of the estimates (2.76) and (2.77) for $j = 1$, the proof for the case $j = 2$ is completely analogous. From the proof of Theorem 2.2.8, we know that $\dot{x}_1(t)$, $\dot{x}_2(t)$ solve the linear system

$$M(t)\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-\langle\partial_t\phi(t),\, \partial_x H^{x_1(t)}_{-1,0}\rangle\\ -\langle\partial_t\phi(t),\, \partial_x H^{x_2(t)}_{0,1}\rangle\end{bmatrix},$$

where $M(t)$ is the matrix defined by (2.52). Then, from Cramer's rule, we obtain that

$$\dot{x}_1(t) = \frac{-\left\langle\partial_t\phi(t),\, \partial_x H^{x_1(t)}_{-1,0}\right\rangle\left(\left\langle\partial_x H^{x_2(t)}_{0,1},\, \partial_x g(t)\right\rangle + \Vert\partial_x H_{0,1}\Vert^2_{L^2_x}\right)}{\det(M(t))} \\ + \frac{\left\langle\partial_t\phi(t),\, \partial_x H^{x_2(t)}_{0,1}\right\rangle\left\langle\partial_x H^{x_2(t)}_{0,1},\, \partial_x H^{x_1(t)}_{-1,0}\right\rangle}{\det(M(t))}. \tag{2.79}$$

Using the definition (2.52) of the matrix $M(t)$, $\left\Vert\overrightarrow{g(t)}\right\Vert = O(\epsilon^{\frac{1}{2}})$ and Lemma 2.2.3 which implies the following estimate

$$\left\langle\partial_x H^{x_2(t)}_{0,1},\, \partial_x H^{x_1(t)}_{-1,0}\right\rangle = O\left(z(t)e^{-\sqrt{2}z(t)}\right), \tag{2.80}$$

we obtain that

$$\left|\det(M(t)) - \Vert\partial_x H_{0,1}\Vert^4_{L^2_x}\right| = O\Big(\left\Vert\overrightarrow{g(t)}\right\Vert + z(t)^2 e^{-2\sqrt{2}z(t)}\Big) = O(\epsilon^{\frac{1}{2}}). \tag{2.81}$$

So, from the estimate (2.81) and the identity (2.79), we obtain that

$$\left|\dot{x}_1(t) + \frac{\left\langle\partial_t\phi(t),\, \partial_x H^{x_1(t)}_{-1,0}\right\rangle}{\Vert\partial_x H_{0,1}\Vert^2_{L^2_x}}\right| \\ = O\Big(\left|\left\langle\partial_x H^{x_1(t)}_{-1,0},\, \partial_x H^{x_2(t)}_{0,1}\right\rangle\left\langle\partial_t\phi(t),\, \partial_x H^{x_2(t)}_{0,1}\right\rangle\right|\Big) \\ + O\left(\left|\left\langle\partial_t\phi(t),\, \partial_x H^{x_1(t)}_{-1,0}(x)\right\rangle\right|\left[\left\Vert\overrightarrow{g(t)}\right\Vert + z(t)^2 e^{-2\sqrt{2}z(t)}\right]\right). \tag{2.82}$$

Finally, from the definition of $g(t,x)$ in Theorem 2.2.8 we know that

$$\partial_t\phi(t,x) = -\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}(x) - \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}(x) + \partial_t g(t,x),$$

from the Modulation Lemma, we also have verified that

$$\left\langle \partial_t g(t),\, \partial_x H^{x_1(t)}_{-1,0} \right\rangle = O\Big( \left\| \overrightarrow{g(t)} \right\| |\dot{x}_1(t)| \Big),$$
$$\left\langle \partial_t g(t),\, \partial_x H^{x_2(t)}_{0,1} \right\rangle = O\Big( \left\| \overrightarrow{g(t)} \right\| |\dot{x}_2(t)| \Big),$$

and from Theorem 2.2.8 we have that $\left\| \overrightarrow{g(t)} \right\| + \max_{j\in\{1,2\}} |\dot{x}_j(t)| \ll 1$. In conclusion, we can rewrite the estimate (2.82) as

$$\begin{aligned}\left| \dot{x}_1(t) + \frac{\left\langle \partial_t \phi(t),\, \partial_x H^{x_1(t)}_{-1,0} \right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} \right| =& O\Big( \max_{j\in\{1,2\}} |\dot{x}_j(t)| \left\| \overrightarrow{g(t)} \right\| + \left\| \overrightarrow{g(t)} \right\|^2 \Big) \\ &+ O\Big( z(t) e^{-\sqrt{2} z(t)} \max_{j\in\{1,2\}} |\dot{x}_j(t)| \Big). \end{aligned} \tag{2.83}$$

By similar reasoning, we can also deduce that

$$\begin{aligned}\left| \dot{x}_2(t) + \frac{\left\langle \partial_t \phi(t),\, \partial_x H^{x_2(t)}_{0,1} \right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} \right| =& O\Big( \max_{j\in\{1,2\}} |\dot{x}_j(t)| \left\| \overrightarrow{g(t)} \right\| + \left\| \overrightarrow{g(t)} \right\|^2 \Big) \\ &+ O\Big( z(t) e^{-\sqrt{2} z(t)} \max_{j\in\{1,2\}} |\dot{x}_j(t)| \Big). \end{aligned} \tag{2.84}$$

Following the reasoning of Lemma 3.5 of [26], we will use the terms $p_1(t)$, $p_2(t)$ with the objective of obtaining the estimates (2.77), which have high precision and will be useful later to approximate $x_j(t)$, $\dot{x}_j(t)$ by explicit smooth functions during a long time interval.

First, it is not difficult to verify that

$$\langle \partial_t \phi(t),\, \partial_x(\chi_0(t) g(t)) \rangle = O\Big( \Big[ 1 + \frac{\left\| \chi' \right\|_{L^\infty}}{z(t)} \Big] \left\| \overrightarrow{g(t)} \right\|^2 + \max_{j\in\{1,2\}} |\dot{x}_j(t)| \left\| \overrightarrow{g(t)} \right\| \Big),$$

which clearly implies with estimate (2.83) the inequality (2.76) for $j = 1$. The proof of inequality (2.76) for $j = 2$ is completely analogous.

Now, the demonstration of the inequality (2.77) is similar to the proof of the second inequality of Lemma 3.5 of [26]. First, we have

$$\begin{aligned} \dot{p}_1(t) =& -\frac{\left\langle \partial_t \phi(t),\, \partial_t \Big( \partial_x H^{x_1(t)}_{-1,0}(x) \Big) \right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} - \frac{\left\langle \partial_t \phi(t),\, \partial_x \Big( \partial_t \chi_0(t) g(t) \Big) \right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} \\ &- \frac{\left\langle \partial_x \Big( \chi_0(t) \partial_t g(t) \Big),\, \partial_t \phi(t) \right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} - \frac{\left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_t^2 \phi(t) \right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} \\ &- \frac{\langle \partial_x \chi_0(t) g(t),\, \partial_t^2 \phi(t) \rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} - \frac{\langle \chi_0(t) \partial_x g(t),\, \partial_t^2 \phi(t) \rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} \end{aligned} \tag{2.85}$$
$$= I + II + III + IV + V + VI, \tag{2.86}$$

and we will estimate each term one by one. More precisely, from now on, we will work with a general cut-off function $\chi(x)$, that is a smooth function $0 \leq \chi \leq 1$ satisfying

$$\chi(x) = \begin{cases} 1, & \text{if } x \leq \theta(1-\gamma), \\ 0, & \text{if } x \geq \theta. \end{cases} \tag{2.87}$$

with $0<\theta$, $\gamma<1$ and

$$\chi_0(t,x)=\chi\Big(\frac{x-x_1(t)}{z(t)}\Big). \tag{2.88}$$

The reason for this notation is to improve the precision of the estimate of $\dot{p}_1(t)$ by the searching of the $\gamma$, $\theta$ which minimize $\alpha(t)$.

**Step 1.**(Estimate of $I$) We will only use the identity $I=\dot{x}_1(t)\frac{\left\langle \partial_t\phi(t),\partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}$.

**Step 2.**(Estimate of $II$.) We have, by chain rule and definition of $\chi_0$, that

$$\begin{aligned}
II=&-\frac{\left\langle \partial_t\phi(t),\,\partial_x\Big(\partial_t\chi_0(t)g(t)\Big)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}\\
=&-\frac{\left\langle \partial_t\phi(t),\,\partial_x\Big(\chi^{'}\Big(\frac{x-x_1(t)}{z(t)}\Big)\frac{d}{dt}\Big[\frac{x-x_1(t)}{z(t)}\Big]g(t)\Big)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}\\
=&\frac{\left\langle \partial_t\phi(t),\,\partial_x\Big(\chi^{'}\Big(\frac{x-x_1(t)}{z(t)}\Big)\Big[\frac{\dot{x}_1(t)z(t)+(x-x_1(t))\dot{z}(t)}{z(t)^2}\Big]g(t)\Big)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
II=&\frac{\left\langle \partial_t\phi(t),\,\chi^{''}\Big(\frac{x-x_1(t)}{z(t)}\Big)\Big[\frac{\dot{x}_1(t)}{z(t)}+\frac{(x-x_1(t))\dot{z}(t)}{z(t)^2}\Big]g(t)\right\rangle}{z(t)\,\|\partial_x H_{0,1}\|_{L^2_x}^2}\\
&+\frac{\left\langle \partial_t\phi(t),\,\chi^{'}\Big(\frac{x-x_1(t)}{z(t)}\Big)\frac{\dot{z}(t)}{z(t)^2}g(t)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}\\
&+\frac{\left\langle \partial_t\phi(t),\,\chi^{'}\Big(\frac{x-x_1(t)}{z(t)}\Big)\Big[\frac{\dot{x}_1(t)}{z(t)}+\frac{(x-x_1(t))\dot{z}(t)}{z(t)^2}\Big]\partial_x g(t)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}.
\end{aligned} \tag{2.89}$$

First, since the support of $\chi^{'}$ is contained in $[\theta(1-\gamma),\,\theta]$, from the estimates (D3) and (D4) we obtain that

$$\Big\|\partial_x H_{-1,0}^{x_1(t)}\Big\|_{L^2_x\Big(\operatorname{supp}\partial_x\chi_0(t,x)\Big)}^2=O\Big(e^{-2\sqrt{2}\theta(1-\gamma)z(t)}\Big), \tag{2.90}$$

$$\Big\|\partial_x H_{0,1}^{x_2(t)}\Big\|_{L^2_x\Big(\operatorname{supp}\partial_x\chi_0(t,x)\Big)}^2=O\Big(e^{-2\sqrt{2}(1-\theta)z(t)}\Big), \tag{2.91}$$

Now, we recall the identity $\partial_t\phi(t,x)=-\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}-\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}+\partial_t g(t)$, by using the estimates (2.90), (2.91) in the identity (2.89), we deduce that

$$\begin{aligned}
II =& O\Bigg(\left\|\chi^{'}\right\|_{L^{\infty}(\mathbb{R})}\frac{\max_{j\in\{1,2\}}|\dot{x}_j(t)|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\\
&+\left\|\chi^{''}\right\|_{L^{\infty}(\mathbb{R})}\left\|\overrightarrow{g(t)}\right\|^2\frac{\max_{j\in\{1,2\}}|\dot{x}_j(t)|}{z(t)^2}\\
&+e^{-\sqrt{2}z(t)\min((1-\theta),\theta(1-\gamma))}\left\|\chi^{''}\right\|_{L^{\infty}(\mathbb{R})}\frac{\max_{j\in\{1,2\}}\dot{x}_j(t)^2}{z(t)^2}\left\|\overrightarrow{g(t)}\right\|\\
&+\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)\min((1-\theta),\theta(1-\gamma))}\left[\frac{\left\|\chi^{''}\right\|_{L^{\infty}(\mathbb{R})}}{z(t)^2}+\frac{\left\|\chi^{'}\right\|_{L^{\infty}(\mathbb{R})}}{z(t)}\right]\max_{j\in\{1,2\}}\dot{x}_j(t)^2\Bigg).
\end{aligned}\tag{2.92}$$

Since $\frac{1-\gamma}{2-\gamma}\leq\max((1-\theta),\theta(1-\gamma))$ for $0<\gamma,\theta<1$, we have that the estimate (2.92) is minimal when $\theta=\frac{1-\gamma}{2-\gamma}$. So, from now on, we consider

$$\theta=\frac{1-\gamma}{2-\gamma},\tag{2.93}$$

which implies with (2.78) and (2.92) that $II=O(\alpha(t))$.

**Step 3.**(Estimate of $III$.) We deduce from the identity

$$III=-\frac{\langle\partial_x(\chi_0(t)\partial_t g(t)),\,\partial_t\phi(t)\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}$$

that

$$\begin{aligned}
III =& -\frac{\left\langle\chi^{'}\left(\frac{x-x_1(t)}{z(t)}\right)\partial_t g(t),\,-\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}-\dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}+\partial_t g(t)\right\rangle}{z(t)\left\|\partial_x H_{0,1}\right\|_{L^2_x}^2}\\
&-\frac{\left\langle\chi_0(t,x)\partial^2_{t,x}g(t),\,-\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}-\dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}+\partial_t g(t,x)\right\rangle}{\left\|\partial_x H_{0,1}\right\|_{L^2_x}^2}\\
=& III.1+III.2.
\end{aligned}\tag{2.94}$$

The identity (2.93) and the estimates (2.78), (2.90) and (2.91) imply by Cauchy-Schwarz inequality that

$$III.1=O\left(\frac{\max_{j\in\{1,2\}}|\dot{x}_j(t)|\,e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|+\frac{1}{z(t)\gamma}\left\|\overrightarrow{g(t)}\right\|^2\right).\tag{2.95}$$

In conclusion, we have estimated that $III.1=O(\alpha(t))$.

Also, from condition (2.87) and the estimate (2.4), we can deduce that

$$\left\|(1-\chi_0(t))\partial_x^2 H^{x_1(t)}_{-1,0}\right\|_{L^2_x}+\left\|\chi_0(t)\partial_x^2 H^{x_2(t)}_{0,1}\right\|_{L^2_x}=O\left(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right).\tag{2.96}$$

Additionally, we have that

$$III.2=-\frac{\left\langle\chi_0(t,x)\left[\partial^2_{t,x}\phi(t)+\dot{x}_1(t)\partial_x^2 H^{x_1(t)}_{-1,0}+\dot{x}_2(t)\partial_x^2 H^{x_2(t)}_{0,1}\right],\,\partial_t\phi(t)\right\rangle}{\left\|\partial_x H_{0,1}\right\|_{L^2_x}^2}.\tag{2.97}$$

By integration by parts, we have that

$$\left|\left\langle \chi\Big(\frac{x-x_1(t)}{z(t)}\Big)\partial^2_{t,x}\phi(t,x),\,\partial_t\phi(t,x)\right\rangle\right| = O\Big(\frac{1}{\gamma z(t)}\,\|\partial_t\phi(t)\|^2_{L^2_x(\operatorname{supp}\partial_x\chi_0(t))}\Big).$$

In conclusion, from the estimates (2.78), (2.90), (2.91) and identity (2.93), we obtain that

$$\begin{aligned}\left|\left\langle \chi\Big(\frac{x-x_1(t)}{z(t)}\Big)\partial^2_{t,x}\phi(t,x),\,\partial_t\phi(t,x)\right\rangle\right|&\\ = O\left(\frac{1}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|^2 + \max_{j\in\{1,2\}}\frac{\dot{x}_j(t)^2}{\gamma z(t)}\Big[e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big]\right).&\quad (2.98)\end{aligned}$$

Also, from Lemma (2.2.3), the estimate (2.4) and the fact of $0\leq\chi_0\leq 1$, we deduce that

$$\left|\left\langle \chi_0(t,x)\partial^2_x H^{x_2(t)}_{0,1},\,\partial_x H^{x_1(t)}_{-1,0}\right\rangle\right| = O\Big(z(t)e^{-\sqrt{2}z(t)}\Big), \quad (2.99)$$

$$\left|\left\langle (1-\chi_0(t,x))\partial^2_x H^{x_1(t)}_{-1,0},\,\partial_x H^{x_2(t)}_{0,1}\right\rangle\right| = O\Big(z(t)e^{-\sqrt{2}z(t)}\Big). \quad (2.100)$$

From the estimates (2.90), (2.91) and identity (2.93), we can verify by integration by parts the following estimates

$$\left\langle (1-\chi_0(t))\dot{x}_1(t)\partial^2_x H^{x_1(t)}_{-1,0},\,\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}\right\rangle = O\Big(\frac{\dot{x}_1(t)^2}{\gamma z(t)}e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big), \quad (2.101)$$

$$\left\langle \chi_0(t)\dot{x}_2(t)\partial^2_x H^{x_2(t)}_{0,1},\,\dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}\right\rangle = O\Big(\frac{\dot{x}_2(t)^2}{\gamma z(t)}e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big). \quad (2.102)$$

Finally, from Cauchy-Schwarz inequality and the estimate (2.96) we obtain that

$$\left\langle (1-\chi_0(t))\dot{x}_1(t)\partial^2_x H^{x_1(t)}_{-1,0},\,\partial_t g(t)\right\rangle = O\Big(\,|\dot{x}_1(t)|\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big), \quad (2.103)$$

$$\left\langle \chi_0(t)\dot{x}_1(t)\partial^2_x H^{x_2(t)}_{0,1},\,\partial_t g(t)\right\rangle = O\Big(\,|\dot{x}_2(t)|\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big). \quad (2.104)$$

In conclusion, we obtain from the estimates (2.99), (2.100), (2.101), (2.102) (2.103) and (2.104) that

$$III.2 = -\dot{x}_1(t)\frac{\left\langle \partial^2_x H^{x_1(t)}_{-1,0},\,\partial_t\phi(t)\right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} + O(\alpha(t)). \quad (2.105)$$

This estimate of $III.2$ and the estimate (2.95) of $III.1$ imply

$$III = -\dot{x}_1(t)\frac{\left\langle \partial^2_x H^{x_1(t)}_{-1,0},\,\partial_t\phi(t)\right\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}} + O(\alpha(t)). \quad (2.106)$$

In conclusion, from the estimates $II = O(\alpha(t))$, (2.106) and the definition of $I$, we have that $I+II+III = O(\alpha(t))$.

**Step 4.**(Estimate of $V$.) We recall that $V = -\frac{\langle \partial_x\chi_0(t)g(t),\,\partial^2_t\phi(t)\rangle}{\|\partial_x H_{0,1}\|^2_{L^2_x}}$, and that

$$\begin{aligned}\partial^2_t\phi(t) = \partial^2_x g(t) + \Big[U'\left(H^{x_1(t)}_{-1,0}\right) + U'\left(H^{x_2(t)}_{0,1}\right) - U''\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\Big]&\\ + \Big[U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) - U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right)\Big].&\quad (2.107)\end{aligned}$$

First, by integration by parts, using estimate (2.78), we have the following estimate

$$-\frac{1}{\|\partial_x H_{0,1}\|_{L^2_x}^2}\langle \partial_x\chi_0(t)\partial_x^2 g(t),\, g(t)\rangle = O\Big(\Big[\frac{1}{\gamma z(t)}+\frac{1}{\gamma^2 z(t)^2}\Big]\left\|\overrightarrow{g(t)}\right\|^2\Big) = O(\alpha(t)). \quad (2.108)$$

Second, since $U$ is smooth and $\|g(t)\|_{L^\infty} = O\big(\epsilon^{\frac{1}{2}}\big)$ for all $t\in\mathbb{R}$, we deduce that

$$\Big|\Big\langle U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right) - U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}+g(t)\right), \partial_x\chi_0(t)g(t)\Big\rangle\Big| \lesssim \frac{\left\|\overrightarrow{g(t)}\right\|^2}{z(t)\gamma} = O(\alpha(t)). \quad (2.109)$$

Next, from equation (2.58) and Lemma 2.2.3, we have that

$$\left\|U'\left(H^{x_1(t)}_{-1,0}\right)+U'\left(H^{x_2(t)}_{0,1}\right)-U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\right\|_{L^2_x} = O(e^{-\sqrt{2}z(t)}), \quad (2.110)$$

then, by Hölder inequality we have that

$$\Big\langle U'\left(H^{x_1(t)}_{-1,0}\right)+U'\left(H^{x_2(t)}_{0,1}\right)-U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right), \partial_x\chi_0(t)\partial_x g(t)\Big\rangle \lesssim \frac{\left\|\overrightarrow{g(t)}\right\|}{\gamma z(t)}e^{-\sqrt{2}z(t)} = O(\alpha(t)). \quad (2.111)$$

Clearly, the estimates (2.108), (2.109) and (2.111) imply that $V = O(\alpha(t))$.
**Step 5.**(Estimate of $VI$.) We know that

$$VI = -\frac{\left\langle \partial_x g(t)\chi_0(t),\, \partial_t^2\phi(t)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}.$$

We recall the equation (2.107) which implies that

$$\begin{aligned}\|\partial_x H_{0,1}\|_{L^2_x}^2 VI &= \Big\langle \partial_x g(t)\chi_0(t),\, U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}+g(t)\right)-U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\Big\rangle \\ &\quad + \Big\langle \partial_x g(t)\chi_0(t),\, U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)-U'\left(H^{x_1(t)}_{-1,0}\right)-U'\left(H^{x_2(t)}_{0,1}\right)\Big\rangle \\ &\quad - \Big\langle \partial_x g(t)\chi_0(t),\, \partial_x^2 g(t)\Big\rangle.\end{aligned}$$

By integration by parts, we have from estimate (2.78) that

$$\langle \partial_x g(t,x)\chi_0(t,x),\, \partial_x^2 g(t,x)\rangle = O\Big(\frac{1}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|^2\Big). \quad (2.112)$$

From the estimate (2.110) and Cauchy-Schwarz inequality, we can obtain the following estimate

$$\Big\langle \partial_x g(t)\chi_0(t),\, U'\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)-U'\left(H^{x_1(t)}_{-1,0}\right)-U'\left(H^{x_2(t)}_{0,1}\right)\Big\rangle = O\Big(e^{-\sqrt{2}z(t)}\left\|\overrightarrow{g(t)}\right\|\Big). \quad (2.113)$$

Then, to conclude the estimate of $VI$ we just need to study the following term $C(t) := \left\langle \partial_x g(t)\chi_0(t),\, U'(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)) - U'(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)})\right\rangle$. Since we have from Taylor's theorem that

$$U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) = \sum_{k=2}^{6} U^{(k)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\frac{g(t)^{k-1}}{(k-1)!},$$

from estimate (2.78), we can deduce using integration by parts that

$$C(t) + \left\langle \chi_0(t)\partial_x\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right),\, \sum_{k=3}^{6} U^{(k)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\frac{g(t)^{k-1}}{(k-1)!}\right\rangle = O(\alpha(t)).$$

Since

$$\left\|\chi_0(t)\partial_x H_{0,1}^{x_2(t)}\right\|_{L^\infty} + \left\|(1-\chi_0(t))\partial_x H_{-1,0}^{x_1(t)}\right\|_{L^\infty} = O\Big(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big),$$

we obtain that

$$C(t) = -\left\langle \partial_x H_{-1,0}^{x_1(t)},\, \sum_{k=3}^{6} U^{(k)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\frac{g(t)^{k-1}}{(k-1)!}\right\rangle \\ + O\left(\frac{1}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|^2 + e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\left\|\overrightarrow{g(t)}\right\|^2\right).$$

Also, from Lemma 2.2.3 and the fact that $\|g(t)\|_{L^\infty} \lesssim \left\|\overrightarrow{g(t)}\right\|$, we deduce that

$$\left\langle \partial_x H_{-1,0}^{x_1},\left[U''\left(H_{-1,0}^{x_1(t)}\right) - U''\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right]g(t)\right\rangle = O\Big(e^{-\sqrt{2}z(t)}\left\|\overrightarrow{g(t)}\right\|\Big). \tag{2.114}$$

In conclusion, we obtain that

$$C(t) = -\int_{\mathbb{R}} \partial_x H_{-1,0}^{x_1(t)}\Big(U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\Big)\,dx \\ + \int_{\mathbb{R}} \partial_x H_{-1,0}^{x_1(t)} U''\left(H_{-1,0}^{x_1(t)}\right) g(t,x)\,dx + O(\alpha(t)). \tag{2.115}$$

So

$$VI = \frac{-\int_{\mathbb{R}} \partial_x H_{-1,0}^{x_1(t)}\Big(U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\Big)\,dx}{\|\partial_x H_{0,1}\|_{L^2_x}^2} \\ + \frac{\int_{\mathbb{R}} \partial_x H_{-1,0}^{x_1(t)} U''\left(H_{-1,0}^{x_1(t)}\right) g(t,x)\,dx}{\|\partial_x H_{0,1}\|_{L^2_x}^2} + O(\alpha(t)). \tag{2.116}$$

**Step 6.**(Sum of $IV$, $VI$.) From the identities (2.107) and

$$IV = -\frac{\left\langle \partial_x H_{-1,0}^{x_1(t)},\, \partial_t^2\phi(t)\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2},$$

we obtain that

$$IV = -\frac{\left\langle U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right),\, \partial_x H_{-1,0}^{x_1(t)}\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2} \\ - \frac{\left\langle \partial_x^2 g(t) - \left(U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right),\, \partial_x H_{-1,0}^{x_1(t)}\right\rangle}{\|\partial_x H_{0,1}\|_{L^2_x}^2}. \tag{2.117}$$

In conclusion, from the identity

$$\left[\partial_x^2 - U''\left(H_{-1,0}^{x_1(t)}\right)\right]\partial_x H_{-1,0}^{x_1(t)} = 0$$

and by integration by parts, we have that

$$IV + VI = -\frac{\left\langle U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right), \partial_x H_{-1,0}^{x_1(t)}\right\rangle}{\|\partial_x H_{0,1}\|_{L_x^2}^2} + O(\alpha(t)).$$

From our previous results, we conclude that

$$\begin{aligned} I + II + III + IV + V + VI = \\ -\frac{\left\langle U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right), \partial_x H_{-1,0}^{x_1(t)}\right\rangle}{\|\partial_x H_{0,1}\|_{L_x^2}^2} \\ + O(\alpha(t)). \quad (2.118) \end{aligned}$$

The conclusion of the lemma follows from estimate (2.118) with identity

$$\dot{A}(z(t)) = -\left\langle U'\left(H_{-1,0}\right) + U'\left(H_{0,1}^{z(t)}\right) - U'\left(H_{-1,0} + H_{0,1}^{z(t)}\right), \partial_x H_{-1,0}\right\rangle,$$

which can be obtained from (2.21) by integration by parts with the fact that

$$\left\langle U'\left(H_{-1,0} + H_{0,1}^{z(t)}\right), \partial_x H_{-1,0} + \partial_x H_{0,1}^{z(t)}\right\rangle = 0.$$

□

**Remark 2.3.3.** *Since, we know from Lemma 2.2.3 that*

$$\left|\dot{A}(z(t)) + 4e^{-\sqrt{2}z(t)}\right| \lesssim z(t)e^{-2\sqrt{2}z(t)},$$

*and, by elementary calculus with change of variables, that* $\|\partial_x H_{0,1}\|_{L_x^2}^2 = \frac{1}{2\sqrt{2}}$, *then the estimates* (2.76) *and* (2.77) *obtained in Lemma 2.3.1 motivate us to study the following ordinary differential equation*

$$\ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)}. \quad (2.119)$$

*Clearly, the solution of* (2.119) *satisfies the equation*

$$\frac{d}{dt}\left[\frac{\dot{d}(t)^2}{4} + 8e^{-\sqrt{2}d(t)}\right] = 0. \quad (2.120)$$

*As a consequence, it can be verified that if* $d(t_0) > 0$ *for some* $t_0 \in \mathbb{R}$, *then there are real constants* $v > 0$, $c$ *such that*

$$d(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right) \text{ for all } t \in \mathbb{R}. \quad (2.121)$$

*In conclusion, the solution of the equations*

$$\begin{aligned}\ddot{d}_1(t) =& -8\sqrt{2}e^{-\sqrt{2}d(t)},\\ \ddot{d}_2(t) =& 8\sqrt{2}e^{-\sqrt{2}d(t)},\\ d_2(t)-d_1(t) =& d(t)>0,\end{aligned}$$

*are given by*

$$d_2(t) = a + bt + \frac{1}{2\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt+c\right)^2\right), \tag{2.122}$$
$$d_1(t) = a + bt - \frac{1}{2\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt+c\right)^2\right), \tag{2.123}$$

*for* $a$, $b$ *real constants. So, we now are motivated to study how close the modulation parameters* $x_1$, $x_2$ *of Theorem 2.2.8 can be to functions* $d_1$, $d_2$ *satisfying, respectively the identities* (2.123) *and* (2.122) *for constants* $v \neq 0$, $a$, $b$, $c$.

At first view, the statement of the Lemma 2.3.1 seems too complex and unnecessary for use and that a simplified version should be more useful for our objectives. However, we will show later that for a suitable choice of $\gamma$ depending on the energy excess of the solution $\phi(t)$, we can get a high precision in the approximation of the modulation parameters $x_1$, $x_2$ by smooth functions $d_1$, $d_2$ satisfying (2.123) and (2.122) for a large time interval.

## 2.4 Energy Estimate Method

Before applying Lemma 2.3.1, we need to construct a function $F(t)$ to get better estimate on the value of $\|(g(t), \partial_t g(t))\|_{H^1_x \times L^2_x}$ than that obtained in Theorem 2.2.8.

From now on, we consider $\phi(t) = H_{0,1}(x-x_2(t)) + H_{-1,0}(x-x_1(t)) + g(t,x)$, with $x_1(t)$, $x_2(t)$ satisfying the orthogonality conditions of the Modulation Lemma and $x_1$, $x_2$, $(g(t), \partial_t g(t))$ and $\epsilon > 0$ satisfying all the properties of Theorem 2.2.8. Before we enunciate the main theorem of this section, we consider the following notation

$$\begin{aligned}&\left\langle D^2 E_{total}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\overrightarrow{g(t)}, \overrightarrow{g(t)}\right\rangle\\ &\qquad = \int_{\mathbb{R}} \partial_x g(t,x)^2 + \partial_t g(t,x)^2 + U''\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) g(t,x)^2\, dx.\end{aligned}$$

We also denote $\omega_1(t,x) = \omega\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right)$ for $\omega$ a smooth cut-off function with the image contained in the interval $[0,1]$ and satisfying the following condition

$$\omega(x) = \begin{cases} 1, & \text{if } x \leq \frac{3}{4},\\ 0, & \text{if } x \geq \frac{4}{5}.\end{cases}$$

We consider now the following function

$$
\begin{aligned}
F(t) =&\left\langle D^2E_{total}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\overrightarrow{g(t)},\,\overrightarrow{g(t)}\right\rangle_{L^2\times L^2}\\
&+2\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\Big[\dot{x}_1(t)\omega_1(t,x)+\dot{x}_2(t)(1-\omega_1(t,x))\Big]\,dx\\
&-2\int_{\mathbb{R}}g(t)\Big(U'(H_{-1,0}^{x_1(t)})+U'(H_{0,1}^{x_2(t)})-U'(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)})\Big)\,dx\\
&+2\int_{\mathbb{R}}g(t)\Big[\dot{x}_1(t)^2\partial_x^2H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2H_{0,1}^{x_2(t)}\Big]\,dx\\
&+\frac{1}{3}\int_{\mathbb{R}}U^{(3)}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)})g(t)^3\,dx.
\end{aligned}
\tag{2.124}
$$

Since $x_1$, $x_2$ are functions of class $C^2$, it is not difficult to verify that $(g(t),\partial_t g(t))$ solves the integral equation associated to the following partial differential equation

$$
\begin{aligned}
\partial_t^2 g(t,x)-\partial_x^2 g(t,x)&+U^{(2)}(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x))g(t,x)\\
=&-\Big[U'(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x)+g(t,x))-U'(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x))\\
&-U''(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x))g(t,x)\Big]\\
&+U'(H_{-1,0}^{x_1(t)}(x))+U'(H_{0,1}^{x_2(t)}(x))-U'(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x))\\
&-\dot{x}_1(t)^2\partial_x^2H_{-1,0}^{x_1(t)}(x)-\dot{x}_2(t)^2\partial_x^2H_{0,1}^{x_2(t)}(x)\\
&+\ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x)+\ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x)
\end{aligned}
\tag{II}
$$

in the space $H^1(\mathbb{R})\times L^2_x(\mathbb{R})$.

**Theorem 2.4.1.** *Assuming the hypotheses of Theorem 2.2.8 and recalling its notation, let* $\left\|\overrightarrow{g(t)}\right\| = \|(g(t),\partial_t g(t))\|_{H^1_x\times L^2_x}$ *and let* $\delta(t)$ *be the following quantity*

$$
\begin{aligned}
\delta(t) =&\left\|\overrightarrow{g(t)}\right\|\left(e^{-\sqrt{2}z(t)}\max_{j\in\{1,2\}}|\dot{x}_j(t)|+\max_{j\in\{1,2\}}|\dot{x}_j(t)|^3e^{-\frac{\sqrt{2}z(t)}{5}}\right)\\
&+\left\|\overrightarrow{g(t)}\right\|^2\left(\frac{\max_{j\in\{1,2\}}|\dot{x}_j(t)|}{z(t)}+\max_{j\in\{1,2\}}\dot{x}_j(t)^2+\max_{j\in\{1,2\}}|\ddot{x}_j(t)|\right)\\
&+\left\|\overrightarrow{g(t)}\right\|^4+\left\|\overrightarrow{g(t)}\right\|\max_{j\in\{1,2\}}|\dot{x}_j(t)\ddot{x}_j(t)|.
\end{aligned}
$$

*Then, there exist positive constants $A_1$, $A_2$, $A_3$ such that the function $F(t)$ satisfies the inequalities*

$$F(t)+A_1\epsilon^2\geq A_2\left\|\overrightarrow{g(t)}\right\|^2,\ \left|\dot{F}(t)\right|\leq A_3\delta(t).$$

**Remark 2.4.2.** *Theorem 2.2.8 and Theorem 2.4.1 imply*

$$\left|\dot{F}(t)\right|\lesssim\frac{\epsilon^{\frac{1}{2}}}{\ln\left(\frac{1}{\epsilon}\right)}\left\|\overrightarrow{g(t)}\right\|^2+\left\|\overrightarrow{g(t)}\right\|\epsilon^{\frac{3}{2}}.$$

*Proof.* Since the formula defining function $F(t)$ is very large, we decompose the function in a sum of five terms $F_1$, $F_2$, $F_3$, $F_4$ and $F_5$. More specifically:

$$\begin{aligned}
F_1(t) =& \int_{\mathbb{R}} \partial_t g(t)^2 + \partial_x g(t)^2 + U''\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) g(t,x)^2\,dx,\\
F_2(t) =& -2\int_{\mathbb{R}} g(t)\left[U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\right]\,dx,\\
F_3(t) =& 2\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\,dx,\\
F_4(t) =& 2\int_{\mathbb{R}} \partial_t g(t)\partial_x g(t)(\dot{x}_1(t)\omega_1(t) + \dot{x}_2(t)(1-\omega_1(t)))\,dx,\\
F_5(t) =& \frac{1}{3}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^3\,dx.
\end{aligned}$$

First, we prove that $\left|\dot{F}(t)\right| \lesssim \delta(t)$. The main idea of the proof of this item is to estimate each derivative $\frac{dF_j(t)}{dt}$, for $1 \leq j \leq 5$, with an error of size $O(\delta(t))$, then we will check that the sum of these estimates are going to be a value of order $O(\delta(t))$, which means that the main terms of the estimates of these derivatives cancel.

**Step 1.**(The derivative of $F_1(t)$.) By definition of $F_1(t)$, we have that

$$\begin{aligned}
\frac{dF_1(t)}{dt} =& 2\int_{\mathbb{R}} \left(\partial_t^2 g(t,x) - \partial_x^2 g(t,x) + U''(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x))g(t,x)\right)\partial_t g(t,x)\,dx\\
&- \int_{\mathbb{R}} \dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x) U^{(3)}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) g(t,x)^2\,dx\\
&- \int_{\mathbb{R}} \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x) U^{(3)}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) g(t,x)^2\,dx.
\end{aligned}$$

Moreover, from the identity (II) satisfied by $g(t,x)$, we can rewrite the value of $\frac{dF_1(t)}{dt}$ as

$$\begin{aligned}
\frac{dF_1(t)}{dt} =& 2\int_{\mathbb{R}} \left[U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}} \left[U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)} + g(t)\right) - U'\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}} U''\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}} \left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}} \left[\ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx\\
&- \int_{\mathbb{R}} \left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^2\,dx,
\end{aligned}$$

and, from the orthogonality conditions of the Modulation Lemma, we obtain

$$\begin{aligned}
&\frac{dF_1(t)}{dt}\\
&\quad=2\int_{\mathbb{R}} U''\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right) g(t)\partial_t g(t)\,dx\\
&\quad-2\int_{\mathbb{R}}\left[U'\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g(t)\right)-U'\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&\quad+2\int_{\mathbb{R}}\left[U'\left(H_{-1,0}^{x_1(t)}\right)+U'\left(H_{0,1}^{x_2(t)}\right)-U'\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&\quad-2\int_{\mathbb{R}}\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx\\
&\quad+2\int_{\mathbb{R}}\left[\ddot{x}_1(t)\dot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)}+\ddot{x}_2(t)\dot{x}_2(t)\partial_x^2 H_{0,1}^{x_2(t)}\right]g(t)\,dx\\
&\quad-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{dF_1(t)}{dt}=&2\int_{\mathbb{R}} U''\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right) g(t)\partial_t g(t,x)\,dx\\
&-2\int_{\mathbb{R}}\left[U'\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g(t)\right)-U'\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}}\left[U'\left(H_{-1,0}^{x_1(t)}\right)+U'\left(H_{0,1}^{x_2(t)}\right)-U'\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}}\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx\\
&-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\,dx\\
&+O\left(\delta(t)\right).
\end{aligned}\tag{2.125}$$

**Step 2.**(The derivative of $F_2(t)$.) It is not difficult to verify that

$$\begin{aligned}
\frac{dF_2(t)}{dt}=&2\int_{\mathbb{R}} g(t)U''\left(H_{-1,0}^{x_1(t)}\right)\partial_x H_{-1,0}^{x_1(t)}\dot{x}_1(t)\,dx\\
&+2\int_{\mathbb{R}} g(t)U''\left(H_{0,1}^{x_2(t)}\right)\partial_x H_{0,1}^{x_2(t)}\dot{x}_2(t)\,dx\\
&-2\int_{\mathbb{R}}\partial_t g(t)\left[U'\left(H_{-1,0}^{x_1(t)}\right)+U'\left(H_{0,1}^{x_2(t)}\right)-U'\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]\,dx\\
&-2\int_{\mathbb{R}} U''\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\left[\partial_x H_{-1,0}^{x_1(t)}\dot{x}_1(t)+\partial_x H_{0,1}^{x_2(t)}\dot{x}_2(t)\right]g(t)\,dx.
\end{aligned}$$

From the definition of the function $U$, we can deduce that

$$U''\left(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x)\right)-U''\left(H_{-1,0}^{x_1(t)}(x)\right)=O\Big(\left|H_{-1,0}^{x_1(t)}(x)H_{0,1}^{x_2(t)}(x)\right|+\left|H_{0,1}^{x_2(t)}(x)\right|^2\Big),$$

$$U''\left(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x)\right)-U''\left(H_{0,1}^{x_2(t)}(x)\right)=O\Big(\left|H_{-1,0}^{x_1(t)}(x)H_{0,1}^{x_2(t)}(x)\right|+\left|H_{-1,0}^{x_1(t)}(x)\right|^2\Big),$$

therefore, we obtain from Lemma 2.2.3 and Cauchy-Schwarz inequality that

$$\left|\int_{\mathbb{R}}\left[U''\left(H_{0,1}^{x_2(t)}\right)-U''\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]\partial_x H_{0,1}^{x_2(t)}g(t)\,dx\right|\lesssim\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)},$$
$$\left|\int_{\mathbb{R}}\left[U''\left(H_{-1,0}^{x_1(t)}\right)-U''\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]\partial_x H_{-1,0}^{x_1(t)}g(t)\,dx\right|\lesssim\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)}.$$

In conclusion, we obtain from the identity satisfied by $\frac{dF_2(t)}{dt}$ that

$$\frac{dF_2(t)}{dt} = -2\int_{\mathbb{R}} \partial_t g(t)\left[U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right)\right]dx \\ +2\int_{\mathbb{R}} \partial_t g(t,x) U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)dx + O(\delta(t)). \quad (2.126)$$

**Step 3.**(The derivative of $F_3(t)$.) From the definition of $F_3(t)$, we obtain that

$$\begin{aligned}\frac{dF_3(t)}{dt} =& 2\int_{\mathbb{R}} \partial_t g(t)\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]dx\\ &-2\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\right]dx\\ &+4\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)\ddot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\ddot{x}_2(t)\partial_x^2 H_{0,1}^{x_2(t)}\right]dx,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\frac{dF_3(t)}{dt} =& 2\int_{\mathbb{R}} \partial_t g(t)\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]dx\\ &-2\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\right]dx + O(\delta(t)).\end{aligned} \quad (2.127)$$

**Step 4.**(Sum of $\frac{dF_1}{dt}, \frac{dF_2}{dt}, \frac{dF_3}{dt}$.) If we sum the estimates (2.125), (2.126) and (2.127), we obtain that

$$\begin{aligned}\sum_{i=1}^{3}\frac{dF_i(t)}{dt} =& 2\int_{\mathbb{R}} U''\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)\partial_t g(t)\,dx\\ &-2\int_{\mathbb{R}}\left[U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)} + g(t)\right) - U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\right]\partial_t g(t)\,dx\\ &-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)g(t)^2 dx\\ &-2\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\right]dx + O(\delta(t)).\end{aligned}$$

More precisely, from Taylor's Expansion Theorem and since $\left\|\overrightarrow{g(t)}\right\|^4 \leq \delta(t)$,

$$\begin{aligned}\sum_{i=1}^{3}\frac{dF_i(t)}{dt} =& -\int_{\mathbb{R}}\left[U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)g(t)^2\right]\partial_t g(t)\,dx\\ &-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)g(t)^2 dx\\ &-2\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\right]dx + O(\delta(t)).\end{aligned} \quad (2.128)$$

**Step 5.**(The derivative of $F_4(t)$.) The computation of the derivative of $F_4(t)$ will be more careful since the motivation for the addition of this term is to cancel with the expression

$$-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)g(t)^2 dx$$

of (2.128). The construction of functional $F_4(t)$ is based on the *momentum correction term* of Lemma 4.2 of [26]. To estimate $\frac{dF_4(t)}{dt}$ with precision of $O(\delta(t))$, it is just necessary to study the time derivative of

$$2\int_{\mathbb{R}} \partial_t g(t)\partial_x g(t)\dot{x}_1(t)\omega_1(t)\,dx, \quad (2.129)$$

since the estimate of the other term in $F_4(t)$ is completely analogous. First, we have the identity

$$\frac{d}{dt}\left[2\int_{\mathbb{R}} \partial_t g(t)\partial_x g(t)\dot{x}_1(t)\omega_1(t)\,dx\right] = 2\ddot{x}_1(t)\int_{\mathbb{R}} \omega_1(t,x)\partial_t g(t)\partial_x g(t)\,dx \\ +2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t,x)\partial_t^2 g(t)\partial_x g(t)\,dx \\ +2\dot{x}_1(t)\int_{\mathbb{R}} \partial_t\omega_1(t)\partial_t g(t)\partial_x g(t)\,dx \\ +2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t,x)\partial_{t,x}^2 g(t,x)\partial_t g(t)\,dx.$$

From the definition of $\omega_1(t,x) = \omega\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right)$, we have

$$\partial_t\omega_1(t,x) = \omega^{'}\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right)\left(\frac{-\dot{x}_1(t)z(t)-\dot{z}(t)(x-x_1(t))}{z(t)^2}\right). \tag{2.130}$$

Since in the support of $\omega^{'}(x)$ is contained in the set $\frac{3}{4}\leq x\leq\frac{4}{5}$, we obtain the following estimate:

$$2\dot{x}_1(t)\int_{\mathbb{R}} \partial_t\omega_1(t)\partial_t g(t)\partial_x g(t)\,dx = O\left(\max_{j\in\{1,2\}}\frac{|\dot{x}_j(t)|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\right) = O(\delta(t)). \tag{2.131}$$

Clearly, from integration by parts, we deduce that

$$2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)\partial_{t,x}^2 g(t)\partial_t g(t)\,dx = O\left(\max_{j\in\{1,2\}}\frac{|\dot{x}_j(t)|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\right) = O(\delta(t)). \tag{2.132}$$

Also, we have

$$2\ddot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)\partial_t g(t)\partial_x g(t)\,dx = O\Big(\max_{j\in\{1,2\}}|\ddot{x}_j(t)|\left\|\overrightarrow{g(t)}\right\|^2\Big) = O(\delta(t)). \tag{2.133}$$

So, to estimate the time derivative of (2.129) with precision $O(\delta(t))$, it is enough to estimate

$$2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t,x)\partial_t^2 g(t,x)\partial_x g(t,x)\,dx.$$

We have that

$$2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)\partial_t^2 g(t)\partial_x g(t)\,dx = 2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)\partial_x^2 g(t)\partial_x g(t)\,dx \\ -2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)U''\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)g(t)\partial_x g(t)\,dx \\ +2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)\left[\partial_t^2 g(t)-\partial_x^2 g(t)\right]\partial_x g(t)\,dx \\ +2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)U''\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)g(t)\partial_x g(t)\,dx. \tag{2.134}$$

From integration by parts, the first term of the right-hand side of equation (2.134) satisfies

$$2\dot{x}_1(t)\int_{\mathbb{R}} \omega_1(t)\partial_x^2 g(t)\partial_x g(t)\,dx = O\left(\max_{j\in\{1,2\}}\frac{|\dot{x}_j(t)|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\right) = O(\delta(t)). \tag{2.135}$$

From Taylor's Expansion Theorem, we have that

$$\left\| U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)} + g(t)\right) - \sum_{j=1}^{3} U^{(j)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\frac{g(t)^{j-1}}{(j-1)!}\right\|_{L^2_x} = O\left(\left\|\overrightarrow{g(t)}\right\|^3\right). \quad (2.136)$$

Also, we have verified the identity

$$U'(\phi) + U'(\theta) - U'(\phi+\theta) = 24\phi\theta(\phi+\theta) - 6\Big(\sum_{j=1}^{4}\binom{5}{j}\phi^j\theta^{5-j}\Big),$$

which clearly implies with the inequalities $(D1), (D2)$ and Lemma 2.2.3 the estimate

$$\left\|U'\left(H_{0,1}^{x_2(t)}\right) + U'\left(H_{-1,0}^{x_1(t)}\right) - U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\right\|_{L^2_x} = O\left(e^{-\sqrt{2}z(t)}\right). \quad (2.137)$$

Finally, it is not difficult to verify that

$$\begin{aligned}\Big\|-\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} - \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)} + \ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\Big\|_{L^2_x}& \\ = O\left(\max_{j\in\{1,2\}}|\dot{x}_j(t)|^2 + |\ddot{x}_j(t)|\right).& \quad (2.138)\end{aligned}$$

Then, from estimates (2.136), (2.137) and (2.138) and the partial differential equation (II) satisfied by $g(t,x)$, we can obtain the estimate

$$\begin{aligned}2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\left[\partial_t^2 g(t) - \partial_x^2 g(t) + U''\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)g(t)\right]\partial_x g(t)\,dx&\\ =-\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)g(t)^2\partial_x g(t)\,dx&\\ -2\dot{x}_1(t)^3\int_{\mathbb{R}}\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx - 2\dot{x}_1(t)\dot{x}_2(t)^2\int_{\mathbb{R}}\omega_1(t)\partial_x^2 H_{0,1}^{x_2(t)}\partial_x g(t)\,dx&\\ -2\dot{x}_1(t)^3\int_{\mathbb{R}}(\omega_1(t)-1)\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx + O\left(\left\|\overrightarrow{g(t)}\right\|^4\max_{j\in\{1,2\}}|\dot{x}_j(t)|\right)&\\ +O\left(\max_{j\in\{1,2\}}|\ddot{x}_j(t)\dot{x}_j(t)|\left\|\overrightarrow{g(t)}\right\| + e^{-\sqrt{2}z(t)}\max_{j\in\{1,2\}}|\dot{x}_j(t)|\left\|\overrightarrow{g(t)}\right\|\right),&\end{aligned}$$

which, by integration by parts and by Cauchy-Schwarz inequality using the estimate (2.96) for $\omega_1$, we obtain that

$$\begin{aligned}2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\left[\partial_t^2 g(t) - \partial_x^2 g(t) + U''\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)g(t)\right]\partial_x g(t)\,dx&\\ = \frac{\dot{x}_1(t)}{3}\int_{\mathbb{R}}\omega_1(t)U^{(4)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\left[\partial_x H_{-1,0}^{x_1(t)} + \partial_x H_{0,1}^{x_2(t)}\right]g(t)^3\,dx&\\ -2\dot{x}_1(t)^3\int_{\mathbb{R}}\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx + O\left(\max_{j\in\{1,2\}}\frac{|\dot{x}_j(t)|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^3\right)&\\ + O\left(\max_{j\in\{1,2\}}|\dot{x}_j(t)|^3 e^{-\frac{\sqrt{2}z(t)}{5}}\left\|\overrightarrow{g(t)}\right\|\right) + O(\delta(t)).& \quad (2.139)\end{aligned}$$

Now, to finish the estimate of $2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\partial_t^2 g(t,x)\partial_x g(t,x)\,dx$, it remains to study the integral given by

$$-2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U''\left(H_{-1,0}^{x_1(t)}(x) + H_{0,1}^{x_2(t)}(x)\right)g(t)\partial_x g(t)\,dx, \quad (2.140)$$

which by integration by parts is equal to

$$\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\partial_x H^{x_1(t)}_{-1,0}g(t)^2\,dx$$
$$+\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\partial_x H^{x_2(t)}_{0,1}g(t)^2\,dx+O(\delta(t)). \quad (2.141)$$

Since the support of $\omega_1(t,x)$ is included in $\{x|\,(x-x_2(t))\leq -\frac{z(t)}{5}\}$ and the support of $1-\omega_1(t,x)$ is included in $\{x|\,(x-x_1(t))\geq \frac{3z(t)}{4}\}$, from the exponential decay properties of the kink solutions in (D1), (D2), (D3), (D4) we obtain the estimates

$$\left|\dot{x}_1(t)\int_{\mathbb{R}}(\omega_1(t)-1)U^{(3)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\partial_x H^{x_1(t)}_{-1,0}g(t)^2\,dx\right|=O(\delta(t)), \quad (2.142)$$
$$\left|\dot{x}_2(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\partial_x H^{x_2(t)}_{0,1}g(t)^2\,dx\right|=O(\delta(t)), \quad (2.143)$$
$$\left|\frac{1}{3}\dot{x}_1(t)\int_{\mathbb{R}}(1-\omega_1(t))U^{(4)}(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1})\partial_x H^{x_1(t)}_{-1,0}g(t)^3\,dt\right|=O(\delta(t)), \quad (2.144)$$
$$\left|\frac{1}{3}\dot{x}_2(t)\int_{\mathbb{R}}(\omega_1(t))U^{(4)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\partial_x H^{x_2(t)}_{0,1}g(t)^3\,dt\right|=O(\delta(t)). \quad (2.145)$$

In conclusion, we obtain that the estimates (2.142), (2.143) imply the following estimate

$$-2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)U''\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)g(t)\partial_x g(t)\,dx$$
$$=\int_{\mathbb{R}}\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}U^{(3)}\left(H^{x_2(t)}_{0,1}+H^{x_1(t)}_{-1,0}\right)g(t)^2\,dx+O(\delta(t)). \quad (2.146)$$

Then, the estimates (2.134), (2.139), (2.144), (2.145) and (2.146) imply that

$$2\frac{d}{dt}\left(\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\dot{x}_1(t)\omega_1(t)\,dx\right)$$
$$=-2\dot{x}_1(t)^3\int_{\mathbb{R}}\partial_x^2 H^{x_1(t)}_{-1,0}\partial_x g(t)\,dx$$
$$+\frac{1}{3}\int_{\mathbb{R}}U^{(4)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\left(\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}\right)g(t)^3\,dx$$
$$+\int_{\mathbb{R}}\left(\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}\right)U^{(3)}(H^{x_2(t)}_{0,1}+H^{x_1(t)}_{-1,0})g(t)^2\,dx+O(\delta(t)).$$

By an analogous argument, we deduce that

$$2\frac{d}{dt}\left(\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\dot{x}_2(t)(1-\omega_1(t))\,dx\right)$$
$$=-2\dot{x}_2(t)^3\int_{\mathbb{R}}\partial_x^2 H^{x_2(t)}_{0,1}\partial_x g(t)\,dx$$
$$+\frac{\dot{x}_2(t)}{3}\int_{\mathbb{R}}U^{(4)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\partial_x H^{x_2(t)}_{0,1}g(t)^3\,dx$$
$$+\int_{\mathbb{R}}\dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}U^{(3)}\left(H^{x_2(t)}_{0,1}+H^{x_1(t)}_{-1,0}\right)g(t)^2\,dx$$
$$+O(\delta(t)).$$

In conclusion, we have that

$$\begin{aligned}\frac{dF_4(t)}{dt} &= \int_{\mathbb{R}} \left[\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} + \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}\right] U^{(3)}\left(H^{x_2(t)}_{0,1} + H^{x_1(t)}_{-1,0}\right) g(t)^2\,dx \\ &-2\dot{x}_2(t)^3 \int_{\mathbb{R}} \partial_x^2 H^{x_2(t)}_{0,1}\partial_x g(t)\,dx - 2\dot{x}_1(t)^3 \int_{\mathbb{R}} \partial_x^2 H^{x_1(t)}_{-1,0}\partial_x g(t)\,dx \\ &+\int_{\mathbb{R}} \frac{1}{3}U^{(4)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\left[\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} + \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}\right] g(t)^3\,dx \\ &+O(\delta(t)). \end{aligned} \tag{2.147}$$

**Step 6.**(The derivative of $F_5(t)$.) We have that

$$\begin{aligned}\frac{dF_5(t)}{dt} &= \int_{\mathbb{R}} U^{(3)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) g(t)^2\partial_t g(t)\,dx \\ &-\frac{1}{3}\int_{\mathbb{R}} U^{(4)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\left[\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} + \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}\right] g(t)^3\,dx. \end{aligned} \tag{2.148}$$

**Step 7.**(Conclusion of estimate of $|\dot{F}(t)|$) From the identities (2.147) and (2.148), we obtain that

$$\begin{aligned}\frac{dF_4(t)}{dt} + \frac{dF_5(t)}{dt} &= \int_{\mathbb{R}} \dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} U^{(3)}\left(H^{x_2(t)}_{0,1} + H^{x_1(t)}_{-1,0}\right) g(t)^2\,dx \\ &+\int_{\mathbb{R}} \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1} U^{(3)}\left(H^{x_2(t)}_{0,1} + H^{x_1(t)}_{-1,0}\right) g(t)^2\,dx \\ &-2\dot{x}_1(t)^3 \int_{\mathbb{R}} \partial_x^2 H^{x_1(t)}_{-1,0}\partial_x g(t)\,dx - 2\dot{x}_2(t)^3 \int_{\mathbb{R}} \partial_x^2 H^{x_2(t)}_{0,1}\partial_x g(t)\,dx \\ &+\int_{\mathbb{R}} U^{(3)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) g(t)^2\partial_t g(t)\,dx + O(\delta(t)). \end{aligned} \tag{2.149}$$

Then, the sum of identities (2.128) and (2.149) implies $\sum_{i=1}^5 \frac{dF_i(t)}{dt} = O(\delta(t))$, this finishes the proof of inequality $\left|\dot{F}(t)\right| = O(\delta(t))$.

**Proof of** $F(t) + A_1\epsilon^2 \geq A_2\epsilon^2$. The Coercivity Lemma implies that $\exists\, c > 0$, such that $F_1(t) \geq c\left\|\overrightarrow{g(t)}\right\|^2$. Also, from Theorem 2.2.8, we have the global estimate

$$\max_{j\in\{1,2\}} |\dot{x}_j(t)|^2 + |\ddot{x}_j(t)| + e^{-\sqrt{2}z(t)} + \left\|\overrightarrow{g(t)}\right\|^2 = O(\epsilon), \tag{2.150}$$

which implies that $|F_3(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\|\epsilon\Big)$, $|F_4(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\|^2\epsilon^{\frac{1}{2}}\Big)$, $|F_5(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\|^2\epsilon^{\frac{1}{2}}\Big)$. Also, since

$$\begin{aligned}&\left|U'\left(H^{x_1(t)}_{-1,0}(x)\right) + U'\left(H^{x_2(t)}_{0,1}(x)\right) - U'\left(H^{x_2(t)}_{0,1}(x) + H^{x_1(t)}_{-1,0}(x)\right)\right| = \\ &\qquad O\left(\left|H^{x_1(t)}_{-1,0}(x)H^{x_2(t)}_{0,1}(x)\left[H^{x_2(t)}_{0,1}(x) + H^{x_1(t)}_{-1,0}(x)\right]\right|\right),\end{aligned}$$

Lemma 2.2.3 and Cauchy-Schwarz inequality imply that

$$|F_2(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\| e^{-\sqrt{2}z(t)}\Big).$$

Then, the conclusion of $F(t) + A_1\epsilon^2 \geq A_2\left\|\overrightarrow{g(t)}\right\|^2$ follows from Young inequality for $\epsilon$ small enough. □

**Remark 2.4.3.** *In the proof of Theorem 2.4.1, from Theorem 2.2.8 we have* $|F_2(t)|+|F_3(t)| = O\left(\left\Vert\overrightarrow{g(t)}\right\Vert \epsilon\right)$. *Since* $|F_4(t)| + |F_5(t)| = O\left(\left\Vert\overrightarrow{g(t)}\right\Vert^2 \epsilon^{\frac{1}{2}}\right)$ *and* $|F_1(t)| \lesssim \left\Vert\overrightarrow{g(t)}\right\Vert^2$, *then Young inequality implies that*

$$|F(t)| \lesssim \left\Vert\overrightarrow{g(t)}\right\Vert^2 + \epsilon^2.$$

**Remark 2.4.4** (General Energy Estimate)**.** *For any* $0 < \theta, \gamma < 1$, *we can create a smooth cut-off function* $0 \leq \chi(x) \leq 1$ *such that*

$$\chi(x) = \begin{cases} 0, & \text{if } x \leq \theta(1-\gamma), \\ 1, & \text{if } x \geq \theta. \end{cases}$$

*We define*

$$\chi_0(t,x) = \chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right).$$

*If we consider the following function*

$$\begin{aligned} L(t) = & \left\langle D^2 E_{total}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)})\overrightarrow{g(t)}, \overrightarrow{g(t)}\right\rangle_{L^2 \times L^2} \\ & +2\int_{\mathbb{R}} \partial_t g(t) \partial_x g(t) \Big[\dot{x}_1(t)\chi_0(t) + \dot{x}_2(t)\left(1 - \chi_0(t)\right)\Big]\, dx \\ & -2\int_{\mathbb{R}} g(t)\left(U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\right) dx \\ & +2\int_{\mathbb{R}} g(t)\Big[\dot{x}_1(t)^2 \partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2 \partial_x^2 H_{0,1}^{x_2(t)}\Big]\, dx \\ & +\frac{1}{3}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^3\, dx, \end{aligned}$$

*then, by a similar proof to the Theorem 2.4.1, we obtain that if* $0 < \epsilon \ll 1$ *and*

$$\begin{aligned} \delta_1(t) = \delta(t) + \max_{j\in\{1,2\}} |\dot{x}_j(t)|^3 \max(e^{-\sqrt{2}z(t)(1-\theta)}, e^{-\sqrt{2}z(t)\theta(1-\gamma)}) \left\Vert\overrightarrow{g(t)}\right\Vert \\ - \max_{j\in\{1,2\}} |\dot{x}_j(t)|^3 e^{-\frac{\sqrt{2}}{5}z(t)} \left\Vert\overrightarrow{g(t)}\right\Vert, \end{aligned} \tag{2.151}$$

*then there are positive constants* $A_1$, $A_2 > 0$ *such that*

$$\left|\dot{L}(t)\right| = O(\delta_1(t)),\ L(t) + A_1\epsilon^2 \geq A_2\epsilon^2.$$

Our first application of Theorem 2.4.1 is to estimate the size of the remainder $\left\Vert\overrightarrow{g(t)}\right\Vert$ during a long time interval. More precisely, this corresponds to the following theorem, which is a weaker version of Theorem 2.1.5.

**Theorem 2.4.5.** *There is* $\delta > 0$, *such that if* $0 < \epsilon < \delta$, $(\phi(0), \partial_t\phi(0)) \in S \times L_x^2(\mathbb{R})$ *and* $E_{total}(\phi(0), \partial_t\phi(0)) = 2E_{pot}(H_{0,1}) + \epsilon$, *then there exist* $x_1, x_2 \in C^2(\mathbb{R})$ *such that the unique solution of* (2.1) *is given, for any* $t \in \mathbb{R}$, *by*

$$\phi(t) = H_{0,1}(x - x_2(t)) + H_{-1,0}(x - x_1(t)) + g(t), \tag{2.152}$$

*with $g(t)$ satisfying orthogonality conditions of the Modulation Lemma and*

$$\|(g(t),\partial_t g(t))\|^2_{H^1_x\times L^2_x}\leq C\left[\|(g(0),\partial_t g(0))\|^2_{H^1_x\times L^2_x}+\left(\epsilon\ln\frac{1}{\epsilon}\right)^2\right]\exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right), \tag{2.153}$$

*for all $t\in\mathbb{R}$.*

*Proof of Theorem 2.4.5.* In notation of Theorem 2.4.1, from Theorem 2.4.1 and Remark 2.4.3, there are uniform positive constants $A_2$, $A_1$ such that for all $t\geq 0$

$$A_2\left\|\overrightarrow{g(t)}\right\|^2\leq F(t)+A_1\epsilon^2\leq C\Big(\left\|\overrightarrow{g(t)}\right\|^2+\epsilon^2\Big). \tag{2.154}$$

From now on, we denote $G(t):=F(t)+A_1\left(\epsilon\ln\frac{1}{\epsilon}\right)^2$. From the inequality (2.154) and Remark 2.4.2, there is a constant $C>0$ such that, for all $t\geq 0$, $G(t)$ satisfies

$$G(t)\leq G(0)+C\left(\int_0^t G(s)\frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}ds\right).$$

In conclusion, from Gronwall Lemma, we obtain that $G(t)\leq G(0)\exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)$ for all $t\geq 0$. Then, from the definition of $G$ and inequality (2.154), we verify the inequality (2.153) for any $t\geq 0$. The proof of inequality (2.153) for the case $t<0$ is completely analogous. □

## 2.5 Global Dynamics of Modulation Parameters

**Lemma 2.5.1.** *In notation of Theorem 2.1.5, $\exists C>0$, such that if the hypotheses of Theorem 2.1.5 are true, then for $\overrightarrow{g(0)}=(g(0,x),\partial_t g(0,x))$ we have that there are functions $p_1(t)$, $p_2(t)\in C^1(\mathbb{R}_{\geq 0})$, such that for $j\in\{1,\,2\}$ and any $t\geq 0$, we have:*

$$|\dot{x}_j(t)-p_j(t)|\lesssim\Big(\left\|\overrightarrow{g(0)}\right\|_{H^1_x\times L^2_x}+\epsilon\ln\frac{1}{\epsilon}\Big)\epsilon^{\frac{1}{2}}\exp\Big(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\Big), \tag{2.155}$$

$$\left|\dot{p}_j(t)-(-1)^j8\sqrt{2}e^{-\sqrt{2}z(t)}\right|\lesssim\frac{\Big(\left\|\overrightarrow{g(0)}\right\|_{H^1_x\times L^2_x}+\epsilon\ln\frac{1}{\epsilon}\Big)^2}{\ln\ln\frac{1}{\epsilon}}\exp\Big(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\Big). \tag{2.156}$$

*Proof.* In the notation of Lemma 2.3.1, we consider the functions $p_j(t)$ for $j\in\{1,\,2\}$ and we consider $\theta=\frac{1-\gamma}{2-\gamma}$, the value of $\gamma$ will be chosen later. From Lemma 2.3.1, we have that

$$|\dot{x}_j(t)-p_j(t)|\lesssim\left[1+\frac{1}{\gamma z(t)}\right]\Big(\max_{j\in\{1,2\}}|\dot{x}_j(t)|\left\|\overrightarrow{g(t)}\right\|+\left\|\overrightarrow{g(t)}\right\|^2\Big)+\max_{j\in\{1,2\}}|\dot{x}_j(t)|\,z(t)e^{-\sqrt{2}z(t)}.$$

We recall from Theorem 2.2.8 the estimates $\max_{j\in\{1,2\}}|\dot{x}_j(t)|=O(\epsilon^{\frac{1}{2}})$, $e^{-\sqrt{2}z(t)}=O(\epsilon)$. From Theorem 2.4.5, we have that

$$\left\|\overrightarrow{g(t)}\right\|\lesssim\Big(\left\|\overrightarrow{g(0)}\right\|+\epsilon\ln\frac{1}{\epsilon}\Big)\exp\Big(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\Big).$$

To simplify our computations, we denote $c_0 = \frac{\left\|\overrightarrow{g(0)}\right\| + \epsilon \ln \frac{1}{\epsilon}}{\epsilon \ln \frac{1}{\epsilon}}$. Then, we obtain for any $j \in \{1, 2\}$ and all $t \geq 0$ that

$$|\dot{x}_j(t) - p_j(t)| \lesssim \left[1 + \frac{1}{\gamma \ln \frac{1}{\epsilon}}\right] c_0 \epsilon^{\frac{3}{2}} \ln \frac{1}{\epsilon} \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) + \left[1 + \frac{1}{\gamma \ln \frac{1}{\epsilon}}\right] \left(c_0 \epsilon \ln \frac{1}{\epsilon}\right)^2 \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right). \quad (2.157)$$

Since $e^{-\sqrt{2} z(t)} \lesssim \epsilon$, we deduce for $\epsilon \ll 1$ that $z(t) e^{-\sqrt{2} z(t)} \lesssim \epsilon \ln \frac{1}{\epsilon} < \epsilon^{1 - \frac{\gamma}{(2-\gamma)2}} \ln \frac{1}{\epsilon}$. Then, for any $t \geq 0$, we obtain from the same estimates and the definition (2.75) of $\alpha(t)$ that

$$\begin{aligned} \alpha(t) \lesssim & c_0^2 \left(\epsilon \ln \frac{1}{\epsilon}\right)^2 \left[\max_{k \in \{1,2\}} \left(\frac{1}{\gamma z(t)}\right)^k + \epsilon^{\frac{1-\gamma}{2-\gamma}}\right] \exp\left(2 \frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) \\ & + c_0 \left[\epsilon^{2 - \frac{\gamma}{(2-\gamma)2}} \ln \frac{1}{\epsilon}\right] \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) \left[1 + \frac{1}{\gamma z(t)} + \frac{\epsilon^{\frac{1}{2}}}{(\gamma z(t))^2}\right] + \frac{\epsilon^{1 + \frac{2(1-\gamma)}{2-\gamma}}}{z(t)\gamma}. \end{aligned} \quad (2.158)$$

However, if $\gamma \ln \frac{1}{\epsilon} \leq 1$ and $z(0) \cong \ln \frac{1}{\epsilon}$, which is possible, then the right-hand side of inequality (2.158) is greater than or equivalent to $\left(\epsilon \ln \frac{1}{\epsilon}\right)^2$ while $0 \leq t \lesssim \frac{\ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$. But, it is not difficult to verify for $\gamma = \frac{\ln \ln \frac{1}{\epsilon}}{\ln \frac{1}{\epsilon}}$ that the right-hand side of inequality (2.158) is smaller than $\left(\epsilon \ln \frac{1}{\epsilon}\right)^2$.

Therefore, from now on, we are going to study the right-hand side of (2.158) for $\frac{1}{\ln(\frac{1}{\epsilon})} < \gamma < 1$. Since we know that $\ln\left(\frac{1}{\epsilon}\right) \lesssim z(t)$ from Theorem 2.2.8, the inequality (2.158) implies for $\frac{1}{\ln(\frac{1}{\epsilon})} < \gamma < 1$ and $t \geq 0$ that

$$\begin{aligned} \alpha(t) \lesssim \beta(t) := & \left(c_0 \epsilon \ln \frac{1}{\epsilon}\right)^2 \left[\frac{1}{\gamma \ln \frac{1}{\epsilon}} + \epsilon^{\frac{1-\gamma}{2-\gamma}}\right] \exp\left(2 \frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) \\ & + c_0 \epsilon^{2 - \frac{\gamma}{2(2-\gamma)}} \ln \frac{1}{\epsilon} \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) + \frac{\epsilon^{1 + \frac{2(1-\gamma)}{2-\gamma}}}{\gamma \ln \frac{1}{\epsilon}} \\ & = \beta_1(t) + \beta_2(t) + \beta_3(t), \text{ respectively.} \end{aligned} \quad (2.159)$$

For $\epsilon > 0$ small enough, it is not difficult to verify that if $\beta_3(t) \geq \beta_1(t)$, then $\gamma \geq \frac{\ln \ln \frac{1}{\epsilon}}{\ln \frac{1}{\epsilon}}$. Moreover, if we have that $1 > \gamma > 8 \frac{\ln \ln \frac{1}{\epsilon}}{\ln \frac{1}{\epsilon}}$, we obtain from the following estimate

$$\beta_3(t) = \frac{\epsilon^2 \epsilon^{\frac{-\gamma}{2-\gamma}}}{\gamma \ln \frac{1}{\epsilon}} > \frac{\epsilon^2}{\ln \frac{1}{\epsilon}} \exp\left(\frac{8 \ln \ln \frac{1}{\epsilon}}{2-\gamma}\right) = \frac{\epsilon^2}{\ln \frac{1}{\epsilon}} \left(\ln \frac{1}{\epsilon}\right)^{\frac{8}{2-\gamma}},$$

that $\beta_3(t) > \frac{\left(\epsilon \ln\left(\frac{1}{\epsilon}\right)\right)^2}{\ln \ln \frac{1}{\epsilon}}$. If $\gamma \leq \frac{\ln \ln\left(\frac{1}{\epsilon}\right)}{\ln \frac{1}{\epsilon}}$, then $\frac{\left(\epsilon \ln \frac{1}{\epsilon}\right)^2}{\ln \ln \frac{1}{\epsilon}} \lesssim \beta_1(t)$ for any $t \geq 0$.

In conclusion, for any case we have that $\frac{\left(\epsilon^2 \ln \frac{1}{\epsilon}\right)^2}{\ln \ln \frac{1}{\epsilon}} \lesssim \beta(t)$ when $t \geq 0$, so we choose $\gamma = \frac{\ln \ln \frac{1}{\epsilon}}{\ln \frac{1}{\epsilon}}$. As a consequence, there exists a constant $C_1 > 0$ such that, for any $t \in \mathbb{R}_{\geq 0}$,

$$\alpha(t) \leq C_1 c_0^2 \frac{\left(\epsilon \ln \frac{1}{\epsilon}\right)^2}{\ln \ln \frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right). \quad (2.160)$$

So, the estimates (2.157), (2.160), Remark 2.3.3 and our choice of $\gamma$ imply the inequalities (2.155) and (2.156). $\square$

**Remark 2.5.2.** *If* $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^m} \lesssim \left\Vert\overrightarrow{g(0)}\right\Vert$ *for a constant* $m > 0$*, then, for* $\gamma = \frac{1}{8}$*, we have from Lemma 2.3.1 that there is* $p(t) \in C^2(\mathbb{R})$ *satisfying for all* $t \geq 0$

$$|\dot{z}(t) - p(t)| \lesssim \epsilon^{\frac{1}{2}} \left\Vert\overrightarrow{g(0)}\right\Vert, \tag{2.161}$$

$$\left|\dot{p}(t) - 16\sqrt{2}e^{-\sqrt{2}z(t)}\right| \lesssim \frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{z(t)}. \tag{2.162}$$

*Then, for the smooth real function* $d(t)$ *satisfying*

$$\ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)},\ (d(0), \dot{d}(0)) = (z(0), \dot{z}(0)),$$

*and since* $e^{-\sqrt{2}z(t)} \lesssim \epsilon$, $\ln\frac{1}{\epsilon} \lesssim z(t)$*, we can deduce for any* $t \geq 0$ *that* $Y(t) = (z(t) - d(t))$ *satisfies the following integral inequality for a constant* $K > 0$

$$|Y(t)| \leq K\left(\epsilon^{\frac{1}{2}}\left\Vert\overrightarrow{g(0)}\right\Vert t + \frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\ln\frac{1}{\epsilon}}t^2 + \int_0^t\int_0^s \epsilon\,|Y(s_1)|\,ds_1\,ds\right) = \Lambda\left(|Y|\right)(t),$$
$$Y(0) = 0,\ \dot{Y}(0) = 0.$$

*Indeed, for any* $k \in \mathbb{N}$ *and all* $t \geq 0$, $|Y(t)| \leq \Lambda^{(k)}\left(|Y|\right)(t)$*. We also can verify for any* $T > 0$ *that* $\Lambda^{(k)}\left(|Y|\right)(t)$ *is a Cauchy sequence in the Banach space* $L^\infty[0,T]$*. In conclusion, we can deduce for any* $t \geq 0$ *that* $|Y(t)| \lesssim Q(tK^{\frac{1}{2}})$*, where* $Q(t)$ *is the solution of the following integral equation*

$$Q(t) = \epsilon^{\frac{1}{2}}\left\Vert\overrightarrow{g(0)}\right\Vert t + \frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\ln\frac{1}{\epsilon}}t^2 + \int_0^t\int_0^s \epsilon Q(s_1)\,ds_1\,ds.$$

*By standard ordinary differential equation techniques, we deduce for any* $t \geq 0$ *that*

$$\begin{aligned}|z(t) - d(t)| \lesssim Q(tK^{\frac{1}{2}}) = &\left(\frac{\left\Vert\overrightarrow{g(0)}\right\Vert}{2} + \frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\epsilon\ln\frac{1}{\epsilon}}\right)e^{\epsilon^{\frac{1}{2}}tK^{\frac{1}{2}}} \\ &+\left(\frac{-\left\Vert\overrightarrow{g(0)}\right\Vert}{2} + \frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\epsilon\ln\frac{1}{\epsilon}}\right)e^{-\epsilon^{\frac{1}{2}}tK^{\frac{1}{2}}} - 2\frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\epsilon\ln\frac{1}{\epsilon}},\end{aligned} \tag{2.163}$$

*and from* $\dot{z}(0) = \dot{d}(0)$ *and the estimates* (2.161) *and* (2.162)*, we obtain that*

$$\left|\dot{z}(t) - \dot{d}(t)\right| \lesssim |p(0) - \dot{z}(0)| + \int_0^t \epsilon\,|z(s) - d(s)|\,ds, \tag{2.164}$$

*from which with* (2.163) *we obtain for all* $t \geq 0$ *that*

$$\left|\dot{z}(t) - \dot{d}(t)\right| \lesssim e^{\epsilon^{\frac{1}{2}}tK^{\frac{1}{2}}}\epsilon^{\frac{1}{2}}\left(\left\Vert\overrightarrow{g(0)}\right\Vert + \frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\epsilon\ln\frac{1}{\epsilon}}\right). \tag{2.165}$$

*However, the precision of the estimates* (2.163) *and* (2.165) *is very bad when* $\epsilon^{-\frac{1}{2}} \ll t$, *which motivate us to apply Lemma 2.3.1 to estimate the modulation parameters* $x_1(t)$, $x_2(t)$ *for* $|t| \lesssim \frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

**Remark 2.5.3.** *We recall from Theorem 2.1.10 the definitions of the functions* $d_1(t)$, $d_2(t)$. *If* $\left\|\overrightarrow{g(0)}\right\| \geq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5}$, *then, using estimates*

$$\max_{j\in\{1,2\}} |d_j(t) - x_j(t)| = O(\min(\epsilon|t|, \epsilon^{\frac{1}{2}}|t|)), \ \max_{j\in\{1,2\}} \left|\dot{d}_j(t) - \dot{x}_j(t)\right| = O(\epsilon|t|),$$

*we deduce for a positive constant* $C$ *large enough the inequalities* (2.10) *and* (2.11) *of Theorem 2.1.10.*

**Remark 2.5.4.** *If*

$$\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5},$$

*the estimates of* $\max_{j\in\{1,2\}} |x_j(t) - d_j(t)|$, $\max_{j\in\{1,2\}} \left|\dot{x}_j(t) - \dot{d}_j(t)\right|$ *can be done by studying separated cases depending on the initial data* $z(0)$, $\dot{z}(0)$.

**Lemma 2.5.5.** *In notation of Theorem 2.4.1, there exists* $K > 0$ *such that if* $\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5}$, *all the hypotheses of Theorem 2.1.10 are true and* $\frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8} \lesssim e^{-\sqrt{2}z(0)} \lesssim \epsilon$, *then we have for* $t \geq 0$ *that*

$$\max_{j\in\{1,2\}} |x_j(t) - d_j(t)| = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right), \tag{2.166}$$

$$\max_{j\in\{1,2\}} \left|\dot{x}_j(t) - \dot{d}_j(t)\right| = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2 \frac{\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \tag{2.167}$$

*Proof of Lemma 2.5.5.* First, in notation of Lemma 2.5.1, we consider

$$p(t) := p_2(t) - p_1(t), \ z(t) := x_2(t) - x_1(t), \ \dot{z}(t) := \dot{x}_2(t) - \dot{x}_1(t).$$

Also, motivated by Remark 2.3.3, we consider the smooth function $d(t)$ solution of the following ordinary differential equation

$$\begin{cases} \ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)}, \\ (d(0), \dot{d}(0)) = (z(0), \dot{z}(0)). \end{cases}$$

**Step 1.**(Estimate of $z(t)$, $\dot{z}(t)$) From now on, we denote the functions $W(t) = z(t) - d(t)$, $V(t) = p(t) - \dot{d}(t)$. Then, Lemma 2.5.1 implies that $W, V$ satisfy for any $t \in \mathbb{R}_{\geq 0}$ the following differential estimates

$$\begin{aligned} \left|\dot{W}(t) - V(t)\right| =& O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right), \\ \left|\dot{V}(t) + 16\sqrt{2}e^{-\sqrt{2}d(t)} - 16\sqrt{2}e^{-\sqrt{2}z(t)}\right| =& O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\left(\frac{1}{\epsilon}\right)} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \end{aligned}$$

From the above estimates and Taylor's Expansion Theorem, we deduce for $t \geq 0$ the following system of differential equations, while $|W(t)| < 1$ :

$$\begin{cases} \dot{W}(t) = V(t) + O\Big( \max\left( \left\| \overrightarrow{g(0)} \right\|, \epsilon \ln \frac{1}{\epsilon} \right) \epsilon^{\frac{1}{2}} \exp\left( \frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}} \right) \Big), \\ \dot{V}(t) = -32 e^{-\sqrt{2} d(t)} W(t) + O\left( e^{-\sqrt{2} d(t)} W(t)^2 \right) \\ \qquad + O\left( \frac{\max\left( \left\| \overrightarrow{g(0)} \right\|, \epsilon \ln \frac{1}{\epsilon} \right)^2}{\ln \ln \frac{1}{\epsilon}} \exp\left( \frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}} \right) \right). \end{cases} \tag{2.168}$$

Recalling Remark 2.3.3, we have that

$$d(t) = \frac{1}{\sqrt{2}} \ln\left( \frac{8}{v^2} \cosh\left( \sqrt{2} v t + c \right)^2 \right), \tag{2.169}$$

where $v > 0$ and $c \in \mathbb{R}$ are chosen such that $(d(0), \dot{d}(0)) = (z(0), \dot{z}(0))$. Moreover, it is not difficult to verify that

$$v = \left( \frac{\dot{z}(0)^2}{4} + 8 e^{-\sqrt{2} z(0)} \right)^{\frac{1}{2}},\ c = \operatorname{arctanh}\left( \frac{\dot{z}(0)}{\left[ 32 e^{-\sqrt{2} z(0)} + \dot{z}(0)^2 \right]^{\frac{1}{2}}} \right).$$

Moreover, since $8e^{-\sqrt{2}z(0)} = v^2 \operatorname{sech}(c)^2 \leq 4v^2 e^{-2|c|}$, we obtain from the hypothesis for $e^{-\sqrt{2}z(0)}$ that $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^4} \lesssim v \lesssim \epsilon^{\frac{1}{2}}$ and as a consequence the estimate $|c| \lesssim \ln\left(\ln\left(\frac{1}{\epsilon}\right)\right)$.

Also, it is not difficult to verify that the functions

$$n(t) = (\sqrt{2} v t + c) \tanh(\sqrt{2} v t + c) - 1,\ m(t) = \tanh(\sqrt{2} v t + c)$$

generate all solutions of the following ordinary differential equation

$$\ddot{y}(t) = -32 e^{-\sqrt{2} d(t)} y(t), \tag{2.170}$$

which is obtained from the linear part of the system (2.168).

To simplify our computations, we use the following notation

$$\begin{aligned} error_1(t) &= \max\left( \left\| \overrightarrow{g(0)} \right\|, \epsilon \ln \frac{1}{\epsilon} \right) \epsilon^{\frac{1}{2}} \exp\left( \frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}} \right), \\ error_2(t) &= e^{-\sqrt{2} d(t)} (z(t) - d(t))^2 + \frac{\max\left( \left\| \overrightarrow{g(0)} \right\|, \epsilon \ln \frac{1}{\epsilon} \right)^2}{\ln \ln \frac{1}{\epsilon}} \exp\left( \frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}} \right). \end{aligned}$$

From the variation of parameters technique for ordinary differential equations, we can write that

$$\begin{bmatrix} W(t) \\ V(t) \end{bmatrix} = c_1(t) \begin{bmatrix} m(t) \\ \dot{m}(t) \end{bmatrix} + c_2(t) \begin{bmatrix} n(t) \\ \dot{n}(t) \end{bmatrix}, \tag{2.171}$$

such that for any $t \geq 0$

$$\begin{cases} \begin{bmatrix} m(t) & n(t) \\ \dot{m}(t) & \dot{n}(t) \end{bmatrix} \begin{bmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{bmatrix} = \begin{bmatrix} O(error_1(t)) \\ O(error_2(t)) \end{bmatrix}, \\ \\ \begin{bmatrix} m(0) & n(0) \\ \dot{m}(0) & \dot{n}(0) \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ O\left(\left[\left\|\overrightarrow{g(0)}\right\| + \epsilon \ln \frac{1}{\epsilon}\right] \epsilon^{\frac{1}{2}}\right) \end{bmatrix}. \end{cases}$$

The presence of an error in the condition of the initial data $c_1(0)$, $c_2(0)$ comes from estimate (2.155) of Lemma 2.5.1. Since for all $t \in \mathbb{R}$ $m(t)\dot{n}(t) - \dot{m}(t)n(t) = \sqrt{2}v$, we can verify by Cramer's rule and from the fact that $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^4} \lesssim v$ that

$$c_1(0) = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right) |c \tanh(c) - 1| \left(\ln \frac{1}{\epsilon}\right)^4\right), \tag{2.172}$$

$$c_2(0) = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\left(\frac{1}{\epsilon}\right)\right) |\tanh(c)| \left(\ln \frac{1}{\epsilon}\right)^4\right), \tag{2.173}$$

and, for all $t \geq 0$, the estimates

$$\begin{aligned} |\dot{c}_1(t)| = & O\left(|\dot{n}(t)| \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right) \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right)\right) \\ & + O\left(|n(t)| v \operatorname{sech}(\sqrt{2}vt + c)^2 |W(t)|^2\right) \\ & + O\left(|n(t)| \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2}{v \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right)\right), \end{aligned} \tag{2.174}$$

$$\begin{aligned} |\dot{c}_2(t)| = & O\left(|m(t)| v \operatorname{sech}(\sqrt{2}vt + c)^2 |W(t)|^2\right) \\ & + O\left(|m(t)| \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2}{v \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right)\right) \\ & + O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right) \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) \epsilon^{\frac{1}{2}} \operatorname{sech}(\sqrt{2}vt + c)^2\right). \end{aligned} \tag{2.175}$$

Since we have for all $x \geq 0$ that

$$\begin{aligned} \frac{d}{dx}\left(-\frac{\operatorname{sech}(x)^2 x}{2} + \frac{3\tanh(x)}{2}\right) = & \frac{\operatorname{sech}(x)^2}{2} + x \tanh(x) \operatorname{sech}(x)^2 \\ & \geq \frac{|x \tanh(x) - 1| \operatorname{sech}(x)^2}{2} = \frac{|n(x)| \operatorname{sech}(x)^2}{2}, \end{aligned}$$

we deduce from the Fundamental Theorem of Calculus, the fact that $n(t) = (\sqrt{2}vt +$

$c)\tanh(\sqrt{2}vt+c)-1$, estimate $\frac{\epsilon^{\frac{1}{2}}}{\ln\left(\frac{1}{\epsilon}\right)^4}\lesssim v\lesssim\epsilon^{\frac{1}{2}}$ and the estimates (2.174), (2.175) that

$$\begin{aligned}|c_1(t)-c_1(0)| =&O\left(\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{2Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right)\\ &+O\left(\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\Vert n(s)\Vert_{L^{\infty}_{s}[0,t]}\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)^2\frac{\left(\ln\frac{1}{\epsilon}\right)^5}{\epsilon\ln\ln\frac{1}{\epsilon}}\right)\\ &+O\left(\left\vert-\frac{\operatorname{sech}(x)^2x}{2}+\frac{3\tanh(x)}{2}\Big\vert_{c}^{\sqrt{2}vt+c}\right\vert\Vert W(s)\Vert^2_{L^{\infty}_{s}[0,t]}\right),\end{aligned}\tag{2.176}$$

for any $t\geq 0$. From a similar argument, we deduce that

$$\begin{aligned}|c_2(t)-c_2(0)| =&O\left(\Vert W(s)\Vert^2_{L^{\infty}_{s}[0,t]}\left[\tanh(\sqrt{2}vt+c)-\tanh(c)\right]\right)\\ &+O\left(\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)^2\left[\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)-1\right]\frac{\left(\ln\frac{1}{\epsilon}\right)^5}{\epsilon\ln\ln\frac{1}{\epsilon}}\right)\\ &+O\left(\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)\left(\ln\frac{1}{\epsilon}\right)\exp\left(2Ct\frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right),\end{aligned}\tag{2.177}$$

for any $t\geq 0$.

From the estimates $v\lesssim\epsilon^{\frac{1}{2}}$, $|c|\lesssim\ln\ln\frac{1}{\epsilon}$, we obtain for $\epsilon\ll 1$ while $t\geq 0$ and

$$\Vert W(s)\Vert_{L^{\infty}_{s}[0,t]}\left[\epsilon^{\frac{1}{2}}t+\ln\ln\frac{1}{\epsilon}\right]\ln\ln\frac{1}{\epsilon}\leq 1,\tag{2.178}$$

that

$$\Vert W(s)\Vert^2_{L^{\infty}_{s}[0,t]}(1+|n(t)|)\lesssim\Vert W(s)\Vert_{L^{\infty}_{s}[0,t]}\frac{1}{\ln\ln\frac{1}{\epsilon}}.\tag{2.179}$$

Also, from $|n(t)|\leq(\sqrt{2}v|t|+|c|)$, we deduce for any $t\geq 0$ that

$$|n(t)|\lesssim\epsilon^{\frac{1}{2}}t+\ln\ln\frac{1}{\epsilon}\lesssim\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\tag{2.180}$$

In conclusion, the estimates (2.176), (2.177), (2.179), (2.180) and the definition of $W(t)=z(t)-d(t)$ imply that while $t\geq 0$ and the condition (2.178) is true, then

$$|W(t)|\lesssim f(t)=\frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)^2\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(2C+1)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right).\tag{2.181}$$

Then, from the expression for $V(t)$ in the equation (2.171) and the estimates (2.176), (2.177), (2.180), we obtain that if inequality (2.181) is true and $t\geq 0$, then

$$\begin{aligned}|V(t)|\lesssim&\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2\frac{\ln\left(\frac{1}{\epsilon}\right)^6}{\epsilon^{\frac{1}{2}}\ln\ln\left(\frac{1}{\epsilon}\right)}\exp\left(\frac{(4C+3)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\\ &+\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)^4\frac{\left(\ln\frac{1}{\epsilon}\right)^{12}}{\epsilon^{\frac{3}{2}}\left[\ln\ln\frac{1}{\epsilon}\right]^2}\exp\left(\frac{(4C+3)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right),\end{aligned}\tag{2.182}$$

which implies the following estimate

$$\left|\dot{W}(t)\right| \lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2 \frac{\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{(4C+3)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \tag{2.183}$$

Indeed, from the bound $\left\|\overrightarrow{g(0)}\right\| \lesssim \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4}$, we deduce that (2.178) is true if $0 \leq t \leq \frac{\left[\ln\ln\frac{1}{\epsilon}\right]\ln\frac{1}{\epsilon}}{(4C+2)\epsilon^{\frac{1}{2}}}$. As a consequence, the estimates (2.181) and (2.183) are true if $0 \leq t \leq \frac{\left[\ln\ln\frac{1}{\epsilon}\right]\ln\frac{1}{\epsilon}}{(4C+2)\epsilon^{\frac{1}{2}}}$.

But, for $t \geq 0$, we have that

$$|W(t)| \lesssim \epsilon^{\frac{1}{2}}t \lesssim 3\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right), \left|\dot{W}(t)\right| \lesssim \epsilon t \lesssim 3\epsilon^{\frac{1}{2}}\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right). \tag{2.184}$$

Since $f(t)$ defined in inequality (2.181) is strictly increasing and $f(0) \lesssim \frac{1}{\left(\ln\frac{1}{\epsilon}\right)^2\ln\ln\frac{1}{\epsilon}}$, there is an instant $T_M > 0$ such that

$$\exp\left(\frac{\epsilon^{\frac{1}{2}}T_M}{\ln\frac{1}{\epsilon}}\right)f(T_M) = \frac{1}{\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2}, \tag{2.185}$$

from which with estimate (2.181) and condition (2.178) we deduce that (2.181) is true for $0 \leq t \leq T_M$. Also, from the identity (2.185) and the fact that $\left\|\overrightarrow{g(0)}\right\| \lesssim \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4}$ we deduce

$$\frac{1}{\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2} \lesssim \frac{1}{\left(\ln\frac{1}{\epsilon}\right)^2\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(2C+2)\epsilon^{\frac{1}{2}}T_M}{\ln\frac{1}{\epsilon}}\right),$$

from which we obtain that $T_M \geq \frac{3}{8(C+1)}\frac{\ln\ln\frac{1}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)}{\epsilon^{\frac{1}{2}}}$ for $\epsilon \ll 1$. In conclusion, since $f(t)$ is an increasing function, we have for $t \geq T_M$ and $\epsilon \ll 1$ that

$$\begin{aligned} f(t)\exp\left(\frac{[17(C+1)+4]\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right) &\geq \frac{1}{\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2}\exp\left(\frac{[17(C+1)+1]\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right) \\ &\geq \frac{\left(\ln\frac{1}{\epsilon}\right)^{1+\frac{1}{8}}}{\left(\ln\ln\frac{1}{\epsilon}\right)^2}\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right), \end{aligned}$$

from which with the estimates (2.184) and (2.181) we deduce for all $t \geq 0$ that

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(8C+9)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \tag{2.186}$$

As consequence, we obtain from the estimates (2.172), (2.173), (2.176), (2.177) and (2.186) that

$$\left|\dot{W}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(16C+18)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{2.187}$$

for all $t\geq 0$.

**Step 2.**(Estimate of $|x_1(t)+x_2(t)|\,,\,|\dot{x}_1(t)+\dot{x}_2(t)|\,.$) First, we define

$$M(t)\coloneqq(x_1(t)+x_2(t))-(d_1(t)+d_2(t)),\; N(t)\coloneqq(p_1(t)+p_2(t))-(\dot{d}_1(t)+\dot{d}_2(t)). \quad (2.188)$$

From the inequalities (2.155), (2.156) of Lemma 2.5.1, we obtain for all $t\geq 0$, respectively:

$$\left|\dot{M}(t)-N(t)\right|\lesssim \max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}\exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right),$$

$$\left|\dot{N}(t)\right|\lesssim\frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right).$$

Also, from inequality (2.155) and the fact that for $j\in\{1,2\}$ $d_j(0)=x_j(0),\,\dot{d}_j(0)=\dot{x}_j(0)$, we deduce that $M(0)=0$ and $|N(0)|\lesssim\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert,\epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}$. Then, from the Fundamental Theorem of Calculus, we obtain for all $t\geq 0$ that

$$N(t)=O\left(\frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)^2\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{4C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right), \quad (2.189)$$

$$M(t)=O\left(\frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)^2\left(\ln\frac{1}{\epsilon}\right)^2}{\epsilon\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{4C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \quad (2.190)$$

In conclusion, for $K=16C+18$, we verify from triangle inequality that the estimates (2.186) and (2.190) imply (2.166) and the estimates (2.187) and (2.189) imply (2.167). $\square$

**Remark 2.5.6.** *The estimates* (2.190) *and* (2.189) *are true for any initial data* $\overrightarrow{g(0)}\in H^1(\mathbb{R})\times L^2_x(\mathbb{R})$ *such that the hypotheses of Theorem 2.1.10 are true.*

**Remark 2.5.7** (Similar Case)**.** *If we add the following conditions*

$$e^{-\sqrt{2}z(0)}\ll\frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8},\,\frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4}\lesssim v\lesssim\epsilon^{\frac{1}{2}},\,-\left(\ln\frac{1}{\epsilon}\right)^2<c<0,$$

*to the hypotheses of Theorem 2.1.10, then, by repeating the above proof of Lemma 2.5.5, we would still obtain for any* $t\geq 0$ *the estimates* (2.174), (2.175), (2.176) *and* (2.177).

*However, since now* $|c|\leq\left(\ln\frac{1}{\epsilon}\right)^2$, *if* $\epsilon\ll 1$ *enough, we can verify while* $t\geq 0$ *and*

$$\Vert W(s)\Vert_{L^\infty_s[0,t]}\left(\epsilon^{\frac{1}{2}}t+\left(\ln\frac{1}{\epsilon}\right)^2\right)\ln\ln\frac{1}{\epsilon}\leq 1, \quad (2.191)$$

*that*

$$\Vert W(s)\Vert^2_{L^\infty_s[0,t]}\left(1+|n(t)|\right)\lesssim\Vert W(s)\Vert_{L^\infty_s[0,t]}\frac{1}{\ln\ln\frac{1}{\epsilon}},$$

*which implies by a similar reasoning to the proof of Lemma 2.5.5 for a uniform constant* $C > 1$ *and any* $t \in \mathbb{R}_{\geq 0}$ *the following estimates*

$$|W(t)| \lesssim \frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon \ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^7}{\epsilon \ln\ln\frac{1}{\epsilon}} \exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right) = f_1(t, C), \tag{2.192}$$

$$\left|\dot{W}(t)\right| \lesssim \max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon \ln\frac{1}{\epsilon}\right)^2 \frac{\left(\ln\frac{1}{\epsilon}\right)^7}{\epsilon^{\frac{1}{2}} \ln\ln\frac{1}{\epsilon}} \exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right) = f_2(t, C). \tag{2.193}$$

*From the estimates* (2.192), (2.193) *and* $\left\Vert\overrightarrow{g(0)}\right\Vert \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5}$, *we deduce that the condition* (2.191) *holds while* $0 \leq t \leq \frac{\ln\ln\frac{1}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)}{4(C+1)\epsilon^{\frac{1}{2}}}$. *Indeed, since* $\left\Vert\overrightarrow{g(0)}\right\Vert^2 \leq \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^{10}}$, *we can verify that there is an instant* $\frac{\ln\ln\frac{1}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)}{4(C+1)\epsilon^{\frac{1}{2}}} \leq T_M$ *such that* (2.191) *and* (2.192) *are true for* $0 \leq t \leq T_M$ *and*

$$f_1(T_M, C) \exp\left(\frac{\epsilon^{\frac{1}{2}} T_M}{\ln\frac{1}{\epsilon}}\right) = \frac{1}{\left(\ln\frac{1}{\epsilon}\right)^{2+\frac{1}{2}} \ln\ln\frac{1}{\epsilon}}.$$

*In conclusion, we can repeat the argument in the proof of step* 1 *of Lemma 2.5.5 and deduce that there is* $1 < K \lesssim C + 1$ *such that for all* $t \geq 0$

$$|W(t)| \lesssim f_1(t, K), \ \left|\dot{W}(t)\right| \lesssim f_2(t, K). \tag{2.194}$$

**Lemma 2.5.8.** *In notation of Theorem 2.1.10,* $\exists K > 1$, $\delta > 0$ *such that if* $0 < \epsilon < \delta$, $0 < v \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4}$, $\overrightarrow{g(0)} = (g(0, x), \partial_t g(0, x))$ *and* $\left\Vert\overrightarrow{g(0)}\right\Vert \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5}$, *then we have for all* $t \geq 0$ *that*

$$\max_{j\in\{1,2\}} |d_j(t) - x_j(t)| = O\left(\frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\epsilon\ln\ln\frac{1}{\epsilon}} \left(\ln\frac{1}{\epsilon}\right)^2 \exp\left(\frac{Kt\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right), \tag{2.195}$$

$$\max_{j\in\{1,2\}} \left|\dot{d}_j(t) - \dot{x}_j(t)\right| = O\left(\frac{\max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}} \left(\ln\frac{1}{\epsilon}\right) \exp\left(\frac{Kt\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right). \tag{2.196}$$

*Proof of Lemma 2.5.8.* First, we recall that

$$d(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)\right),$$

which implies that

$$e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt + c\right)^2. \tag{2.197}$$

We recall the notation $W(t) = z(t) - d(t)$, $V(t) = p(t) - \dot{d}(t)$. From the first inequality of Lemma 2.5.1, we have that

$$|V(0)| \lesssim \max\left(\left\Vert\overrightarrow{g(0)}\right\Vert, \epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}. \tag{2.198}$$

We already verified that $W$, $V$ satisfy the following ordinary differential system

$$\begin{cases} \dot{W}(t) = V(t) + O\Big( \max\left( \left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right) \epsilon^{\frac{1}{2}} \exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\Big), \\ \dot{V}(t) = -32 e^{-\sqrt{2}d(t)} W(t) + O\left(e^{-\sqrt{2}z(t)} W(t)^2\right) \\ \qquad + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \end{cases} \tag{2.199}$$

However, since $v^2 \leq \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$, we deduce from (2.197) that $e^{-\sqrt{2}d(t)} \lesssim \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$ for all $t \geq 0$. So, while $\|W(s)\|_{L^\infty[0,t]} < 1$, we have from the system of ordinary differential equations above for some constant $C > 0$ independent of $\epsilon$ that

$$\left|\dot{V}(t)\right| \lesssim \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8} \|W(s)\|_{L^\infty[0,t]} + \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \text{for all } t \geq 0,$$

from which we deduce the following estimate for any $t \geq 0$

$$|V(t) - V(0)| = O\left(\frac{\epsilon t}{\left(\ln\frac{1}{\epsilon}\right)^8} \|W(s)\|_{L^\infty[0,t]}\right) + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2 \ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right).$$

In conclusion, while $\|W(s)\|_{L^\infty[0,t]} < 1$, we have that

$$\left|\dot{W}(t)\right| \leq |V(0)| + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2 \ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right) + O\left(\frac{\epsilon t}{\left(\ln\frac{1}{\epsilon}\right)^8} \|W(s)\|_{L^\infty[0,t]}\right). \tag{2.200}$$

Finally, since $W(0) = 0$, the Fundamental Theorem of Calculus and (2.200) imply the following estimate for all $t \geq 0$

$$\|W(s)\|_{L^\infty[0,t]} \leq |V(0)|\, t + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2 \ln\left(\frac{1}{\epsilon}\right)^2}{\epsilon\ln\ln\left(\frac{1}{\epsilon}\right)} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\left(\frac{1}{\epsilon}\right)}\right)\right) + O\left(\frac{\epsilon t^2}{\ln\left(\frac{1}{\epsilon}\right)^8} \|W(s)\|_{L^\infty[0,t]}\right). \tag{2.201}$$

Then, the estimates (2.198) and (2.201) imply if $\epsilon \ll 1$ that

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{(2C+1)\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right), \tag{2.202}$$

for $0 \leq t \leq \frac{\left(\ln \frac{1}{\epsilon}\right) \ln \ln \frac{1}{\epsilon}}{(8C+4)\epsilon^{\frac{1}{2}}}$. From (2.202) and (2.200), we deduce for $0 \leq t \leq \frac{\left(\ln \frac{1}{\epsilon}\right) \ln \ln \frac{1}{\epsilon}}{(8C+4)\epsilon^{\frac{1}{2}}}$ that

$$\left|\dot{W}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}} \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{(2C+1)\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right). \tag{2.203}$$

Since $|W(t)| \lesssim \epsilon^{\frac{1}{2}} t$, $\left|\dot{W}(t)\right| \lesssim \epsilon t$ for all $t \geq 0$, we can verify by a similar argument to the proof of Step 1 of Lemma 2.5.5 that for all $t \geq 0$ there is a constant $1 < K \lesssim (C+1)$ such that

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right), \tag{2.204}$$

$$\left|\dot{W}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}} \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right). \tag{2.205}$$

In conclusion, estimates (2.195) and (2.196) follow from Remark 2.5.6, inequalities (2.204), (2.205) and triangle inequality. □

**Remark 2.5.9.** *We recall the definition* (2.169) *of* $d(t)$. *It is not difficult to verify that if* $\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^5}$, $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^4} \lesssim v$ *and one of the following statements*

1. $e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^8}$ *and* $c > 0$,
2. $e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^8}$ *and* $c \leq -\left(\ln \frac{1}{\epsilon}\right)^2$

*were true, then we would have that* $e^{-\sqrt{2}d(t)} \ll \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^8}$ *for* $0 \leq t \lesssim \frac{\left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}}}$. *Moreover, assuming* $e^{-\sqrt{2}z(0)}\left(\ln \frac{1}{\epsilon}\right)^8 \ll \epsilon$, *if* $c > 0$, *then we have for all* $t \geq 0$ *that*

$$e^{-\sqrt{2}d(t)} = \frac{v^2}{8} \operatorname{sech}\left(\sqrt{2}vt + c\right)^2 \leq \frac{v^2}{8} \operatorname{sech}(c)^2 = e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^8},$$

*otherwise if* $c \leq -\left(\ln \frac{1}{\epsilon}\right)^2$, *since* $0 < v \lesssim \epsilon^{\frac{1}{2}}$, *then there is* $1 \lesssim K$ *such that for* $0 \leq t \leq \frac{K\left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}}}$, *then* $2\left|\sqrt{2}vt + c\right| > |c|$, *and so*

$$e^{-\sqrt{2}d(t)} \leq v^2 \operatorname{sech}\left(-\frac{c}{2}\right)^2 \ll \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^8}.$$

*In conclusion, the result of Lemma 2.5.8 would be true for these two cases.*

From the following inequality

$$\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right) \leq \left(\ln\frac{1}{\epsilon}\right)\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right),$$

we deduce from Lemmas 2.5.5, 2.5.8 and Remarks 2.5.6, 2.5.7 and 2.5.9 the statement of Theorem 2.1.10.

## 2.6 Proof of Theorem 2.1.5

If $\left\|\overrightarrow{g(0)}\right\| \geq \epsilon\ln\frac{1}{\epsilon}$, the result of Theorem 2.1.5 is a direct consequence of Theorem 2.4.5. So, from now on, we assume that $\left\|\overrightarrow{g(0)}\right\| < \epsilon\ln\frac{1}{\epsilon}$.

We recall from Theorem 2.1.10 the notations $v$, $c$, $d_1(t)$, $d_2(t)$ and we denote $d(t) = d_2(t) - d_1(t)$ that satisfies

$$d(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right),\ e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt + c\right)^2.$$

From the definition of $d_1(t)$, $d_2(t)$, $d(t)$, we know that

$$\max_{j\in\{1,\,2\}}\left|\ddot{d}_j(t)\right| + e^{-\sqrt{2}d(t)} = O\Big(v^2\operatorname{sech}\left(\sqrt{2}vt + c\right)^2\Big),$$

and since $z(0) = d(0)$, $\dot{z}(0) = \dot{d}(0)$, we have that $v$, $c$ satisfy the following identities

$$v = \left(e^{-\sqrt{2}z(0)} + \left(\frac{\dot{x}_2(0) - \dot{x}_1(0)}{2}\right)^2\right)^{\frac{1}{2}},\ c = \operatorname{arctanh}\left(\frac{\dot{x}_2(0) - \dot{x}_1(0)}{2v}\right),$$

so Theorem 2.2.8 implies that $v \lesssim \epsilon^{\frac{1}{2}}$.

From the Corollary 2.1.13 and the Theorem 2.1.10, we deduce that $\exists C > 0$ such that if $\epsilon \ll 1$ and $0 \leq t \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, then we have that

$$\max_{j\in\{1,\,2\}}|\ddot{x}_j(t)| = O\Big(\max_{j\in\{1,2\}}\left|\ddot{d}_j(t)\right|\Big) + O\left(\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^9\exp\left(\frac{Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right), \tag{2.206}$$

$$\begin{aligned} e^{-\sqrt{2}z(t)} &= e^{-\sqrt{2}d(t)} + O\left(\max\left(e^{-\sqrt{2}d(t)}, e^{-\sqrt{2}z(t)}\right)|z(t) - d(t)|\right) \\ &= e^{-\sqrt{2}d(t)} + O\left(\epsilon^2\left(\ln\frac{1}{\epsilon}\right)^9\exp\left(\frac{Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right). \end{aligned} \tag{2.207}$$

Next, we consider a smooth function $0 \leq \chi_2(x) \leq 1$ that satisfies

$$\chi_2(x) = \begin{cases} 1, & \text{if } x \leq \frac{9}{20}, \\ 0, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

We denote

$$\chi_2(t, x) = \chi_2\Big(\frac{x - x_1(t)}{x - x_2(t)}\Big).$$

From Theorem 2.4.1 and Remark 2.4.4, the estimates (2.206) and (2.207) of the modulation parameters imply that for the following function

$$
\begin{aligned}
L_1(t) =& \left\langle D^2 E_{total}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\overrightarrow{g(t)},\, \overrightarrow{g(t)}\right\rangle_{L^2\times L^2}\\
&+2\int_{\mathbb{R}} \partial_t g(t)\partial_x g(t)\Big[\dot{x}_1(t)\chi_2(t,x) + \dot{x}_2(t)\left(1-\chi_2(t)\right)\Big]\,dx\\
&-2\int_{\mathbb{R}} g(t,x)\Big(U'\left(H_{-1,0}^{x_1(t)}\right) + U'\left(H_{0,1}^{x_2(t)}\right) - U'\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\Big)\,dx\\
&+2\int_{\mathbb{R}} g(t,x)\Big[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}(x)\Big]\,dx\\
&+\frac{1}{3}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)g(t)^3\,dx,
\end{aligned}
$$

and the following quantity $\delta_1(t)$ denoted by

$$
\begin{aligned}
\delta_1(t) =& \left\|\overrightarrow{g(t)}\right\|\left(e^{-\sqrt{2}z(t)}\max_{j\in\{1,2\}}|\dot{x}_j(t)| + \max_{j\in\{1,2\}}|\dot{x}_j(t)|^3 e^{-\frac{9\sqrt{2}z(t)}{20}}\right)\\
&+\left\|\overrightarrow{g(t)}\right\|\max_{j\in\{1,2\}}|\dot{x}_j(t)|\,|\ddot{x}_j(t)| + \left\|\overrightarrow{g(t)}\right\|^2\frac{\max_{j\in\{1,2\}}|\dot{x}_j(t)|}{z(t)}\\
&+\left\|\overrightarrow{g(t)}\right\|^2\left(\max_{j\in\{1,2\}}\dot{x}_j(t)^2 + \max_{j\in\{1,2\}}|\ddot{x}_j(t)|\right) + \left\|\overrightarrow{g(t)}\right\|^4,
\end{aligned}
$$

we have $\left|\dot{L}_1(t)\right| = O(\delta_1(t))$ for $t\geq 0$. Moreover, estimates (2.206), (2.207) and the bound $\dot{L}_1(t) = O(\delta_1(t))$ imply that for

$$
\begin{aligned}
\delta_2(t) =& \left\|\overrightarrow{g(t)}\right\| v^2\epsilon^{\frac{1}{2}}\operatorname{sech}\left(\sqrt{2}vt+c\right)^2 + \left\|\overrightarrow{g(t)}\right\|\epsilon^2\left(\ln\frac{1}{\epsilon}\right)^9\exp\left(\frac{Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\\
&+\epsilon^{\frac{3}{2}}e^{-\frac{9\sqrt{2}z(t)}{20}}\left\|\overrightarrow{g(t)}\right\| + \max_{j\in\{1,2\}}\frac{|\dot{x}_j(t)|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2 + \left\|\overrightarrow{g(t)}\right\|^4,
\end{aligned}
$$

$\left|\dot{L}_1(t)\right| = O(\delta_2(t))$ if $0\leq t\leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

Now, similarly to the proof of Theorem 2.4.5, we denote $G(s) = \max\left(\left\|\overrightarrow{g(s)}\right\|, \epsilon\right)$. From Theorem 2.4.1 and Remark 2.4.4, we have that there are positive constants $K, k>0$ independent of $\epsilon$ such that

$$k\left\|\overrightarrow{g(t)}\right\|^2 \leq L_1(t) + K\epsilon^2.$$

We recall that Theorem 2.2.8 implies that

$$\ln\left(\frac{1}{\epsilon}\right)\lesssim z(t),\, e^{-\sqrt{2}z(t)} + \max_{j\in\{1,2\}}|\dot{x}_j(t)|^2 + \max_{j\in\{1,2\}}|\ddot{x}_j(t)| = O(\epsilon),$$

from which with the definition of $G(s)$ and estimates (2.206) and (2.207) we deduce that

$$\delta_2(t)\lesssim G(t)v^2\operatorname{sech}\left(\sqrt{2}vt+c\right)^2\epsilon^{\frac{1}{2}} + G(t)\epsilon^{\frac{39}{20}} + G(t)^2\frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}},$$

while $0\leq t\leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

In conclusion, the Fundamental Theorem of Calculus implies that $\exists K > 0$ independent of $\epsilon$ such that

$$G(t)^2 \leq K\left(G(0)^2 + \int_0^t G(s)v^2 \operatorname{sech}(\sqrt{2}vs + c)^2 \epsilon^{\frac{1}{2}} + G(s)\epsilon^{\frac{39}{20}} + G(s)^2 \frac{\epsilon^{\frac{1}{2}}}{\ln \frac{1}{\epsilon}}\, ds\right), \tag{2.208}$$

while $0 \leq t \leq \frac{\left(\ln \ln \frac{1}{\epsilon}\right) \ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

Since $\frac{d}{dt}[\tanh(\sqrt{2}vt + c)] = \sqrt{2}v \operatorname{sech}(\sqrt{2}vt + c)^2$, we verify that while the term

$$G(s)v^2 \operatorname{sech}(\sqrt{2}vt + c)^2 \epsilon^{\frac{1}{2}}$$

is dominant in the integral of the estimate (2.208), then $G(t) \lesssim G(0)$. The remaining case corresponds when $G(s)^2 \frac{\epsilon^{\frac{1}{2}}}{\ln\left(\frac{1}{\epsilon}\right)}$ is the dominant term in the integral of (2.208) from an instant $0 \leq t_0 \leq \frac{\left(\ln \ln \frac{1}{\epsilon}\right) \ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$. Similarly to the proof of 2.4.5, we have for $t_0 \leq t \leq \frac{\left(\ln \ln \frac{1}{\epsilon}\right) \ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that $G(t) \lesssim G(t_0) \exp\left(C \frac{(t - t_0)\epsilon^{\frac{1}{2}}}{\ln \frac{1}{\epsilon}}\right)$.

In conclusion, in any case, we have for $0 \leq t \leq \frac{\left(\ln \ln \frac{1}{\epsilon}\right) \ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that

$$G(t) \lesssim G(0) \exp\left(C \frac{t\epsilon^{\frac{1}{2}}}{\ln \frac{1}{\epsilon}}\right). \tag{2.209}$$

But, for $T \geq \frac{\left(\ln \ln \frac{1}{\epsilon}\right) \ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ and $K > 2$ we have that

$$\epsilon\left(\ln \frac{1}{\epsilon}\right) \exp\left(K \frac{\epsilon^{\frac{1}{2}} T}{\ln \frac{1}{\epsilon}}\right) \leq \epsilon \exp\left(\frac{2K\epsilon^{\frac{1}{2}} T}{\ln \frac{1}{\epsilon}}\right).$$

In conclusion, from the result of Theorem 2.4.5, we can exchange the constant $C > 0$ by a larger constant such that estimate (2.209) is true for all $t \geq 0$.

# Chapter 3

# Approximate kink-kink solutions for the $\phi^6$ model in the low-speed limit

# Abstract

This chapter is the first part of a series of two chapters that study the problem of elasticity and stability of the collision of two kinks with low speed $v$ for the nonlinear wave equation known as the $\phi^6$ model in dimension $1+1$. In this paper, we construct a sequence of approximate solutions $(\phi_k(v,t,x))_{k\in\mathbb{N}_{\geq 2}}$ for this nonlinear wave equation such that each function $\phi_k(v,t,x)$ converges in the energy norm to the traveling kink-kink with speed $v$ when $t$ goes to $+\infty$. The methods used in this chapter are not restricted only to the $\phi^6$ model.

## 3.1 Introduction

We recall for the potential function $U(\phi)=\phi^2(1-\phi^2)^2$ the partial differential equation

$$\partial_t^2\phi(t,x)-\partial_x^2\phi(t,x)+U^{'}(\phi(t,x))=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{3.1}$$

From Chapter 1, we also recall the energy and the momentum quantities given by (Energy) and (Momentum) respectively. We recall the potential energy formula, which is

$$E_{pot}(\phi)(t)=\int_{\mathbb{R}}\frac{\partial_x\phi(t,x)^2}{2}+U(\phi(t,x))\,dx.$$

If the solution of the partial differential equation (3.1) has finite energy, the quantities (Energy) and (Momentum) are preserved for all $t\in\mathbb{R}$.

Moreover, if $H$ is a stationary solution of (3.1), then, for any $-1<v<1$, the Lorentz transformation of $H$ given by

$$\phi(t,x)=H\left(\frac{x-vt}{\sqrt{1-v^2}}\right) \tag{3.2}$$

is also a solution of (3.1).

The only non-constant stationary solutions of (3.1) with finite energy are the topological solitons denominated kinks and anti-kinks. The kinks of (3.1) are the space translation of the functions denoted in (1.11) and the anti-kinks are the space reflection around 0 of the kinks. Moreover, from Chapter 2, we recall the estimate (2.4) which implies the existence of a constant $C(k)>0$ for any $k\in\mathbb{N}$ such that

$$\left|\frac{d^k}{dx^k}H_{0,1}(x)\right|\leq C(k)\min\left(e^{\sqrt{2}x},e^{-2\sqrt{2}x}\right)\text{ for all }x\in\mathbb{R}. \tag{3.3}$$

Finally, since $H_{0,1}^{'}(x)=\sqrt{2}\frac{e^{\sqrt{2}x}}{\left(1+e^{2\sqrt{2}x}\right)^{\frac{3}{2}}}$, we have that $\left\|H_{0,1}^{'}(x)\right\|_{L^2_x}^2=\frac{1}{2\sqrt{2}}$.

In [8], it was obtained for any $-1<v<1$ the existence of a solution $\phi(t,x)$ of (3.1) satisfying

$$\lim_{t\to+\infty}\left\|\phi(t,x)-H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})}=0, \tag{3.4}$$

$$\lim_{t\to+\infty}\left\|\partial_t\phi(t,x)+\frac{v}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-\frac{v}{\sqrt{1-v^2}}H_{-1,0}^{'}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x}=0. \tag{3.5}$$

However, the uniqueness of a solution $\phi(t,x)$ satisfying (3.4) and (3.5) is still an open problem. In Chapter 2, we studied the dynamics of two kinks of (3.1) with energy slightly bigger than two times the energy of a kink. The asymptotic stability of a kink for the $\phi^6$ model was obtained in [31]. See also the references [19], [29], [32] and [56] for more information on the stability and asymptotic stability of a kink for other one-dimension nonlinear wave equation models. For more information about kinks and other topological solitons, see the book [36].

The objective of this chapter is to construct a sequence of approximate solutions $\phi_k(v,t,x)$ satisfying for any $0<v\ll1$ and $s>0$

$$\left\|\partial_t^2\phi_k(v,t,x)-\partial_x^2\phi_k(v,t,x)+U^{'}(\phi_k(v,t,x))\right\|_{L^\infty_tH^s_x}\ll v^{2k-\frac{1}{2}},$$

and

$$\lim_{t\to+\infty}\left\|\overrightarrow{\phi_k}(v,t,x)-\overrightarrow{H_{0,1}}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-\overrightarrow{H_{-1,0}}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^s_x}=0,$$

with $\overrightarrow{f}(t,x)=(f(t,x),\partial_t f(t,x))$ for any function $f\in C^1(\mathbb{R}^2)$. This result is the first part of our work about the study of the collision of two kinks with low speed $v$.

The study of dynamics of multi-kink solutions for the $\phi^6$ is motivated from condensed matter, see [3], and cosmology [62]. Also, there is a large literature about the numerical study of collision of multi-kinks for the $\phi^6$, for example in high energy physics see [14] and [17]. More precisely, in the article [17] it was numerically proved that there is a critical velocity $v_c$, so that if two kinks collide with a velocity smaller than $v_c$, the collision is very close to an elastic collision.

Motivated by [17], we theoretically study the high elasticity of the collision of two kinks with low speed for the $\phi^6$ model. The sequence of approximate solutions $\phi_k(v,t,x)$ will be useful later in the next chapter to study the elasticity of collision of two kinks with low speed. Since the $\phi^6$ model is a non-integrable system, there are many issues and difficulties in the studying of the collision problem for two kinks of this model.

There exist few mathematical results about the inelasticity of the collision of two solitons for other dispersive models. In [41], Martel and Merle proved the inelasticity of the collision of two solitons with low speed for the quartic $gKdV$. There are results on the elasticity and inelasticity of the collision of solitons for $gKdV$ for a certain class of nonlinearities, see [49] and [50] by Muñoz, see also [39] by Martel and Merle. For nonlinear Schrödinger equation, in [53], Perelman studied the collision of two solitons of different sizes and obtained that after the collision the solution doesn't preserve the two solitons' structure.

### 3.1.1 Main Results

**Definition 3.1.1.** *We define* $\Lambda: C^2(\mathbb{R}^2,\mathbb{R})\to C(\mathbb{R}^2,\mathbb{R})$ *as the nonlinear operator satisfying*

$$\Lambda(\phi_1)(t,x)=\partial_t^2\phi_1(t,x)-\partial_x^2\phi_1(t,x)+U^{'}(\phi_1(t,x)),$$

*for any function* $\phi_1\in C^2(\mathbb{R}^2,\mathbb{R})$. *And, for any smooth functions* $w:(0,1)\times\mathbb{R}\to\mathbb{R}$, $\phi:\mathbb{R}^2\to\mathbb{R}$, *let* $\phi_2(t,x):=\phi\left(t,w(t,x)\right)$, *then we define*

$$\Lambda\left(\phi\left(t,w(t,x)\right)\right)=\Lambda\left(\phi_2\right)(t,x),\ \textit{for all}\ (t,x)\in\mathbb{R}^2.$$

From Chapter 1, we recall Theorem 1.4.6 which is the main result of Chapter 3 :

**Theorem 3.1.2.** *There exist a sequence of functions* $\left(\phi_k(v,t,x)\right)_{k\geq 2}$, *a sequence of real values*

*$\delta(k) > 0$ and a sequence of numbers $n_k \in \mathbb{N}$ such that for any $0 < v < \delta(k)$, $\phi_k(v,t,x)$ satisfies*

$$\lim_{t\to+\infty}\left\Vert \phi_k(v,t,x) - H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\Vert_{H^1_x} = 0,$$

$$\lim_{t\to+\infty}\left\Vert \partial_t\phi_k(v,t,x) + \frac{v}{\sqrt{1-v^2}}H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - \frac{v}{\sqrt{1-v^2}}H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\Vert_{L^2_x} = 0,$$

$$\lim_{t\to-\infty}\left\Vert \phi_k(v,t,x) - H_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\Vert_{H^1_x} = 0,$$

$$\lim_{t\to-\infty}\left\Vert \partial_t\phi_k(v,t,x) - \frac{v}{\sqrt{1-v^2}}H_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\Vert_{L^2_x} = 0,$$

*with $e_{v,k} \in \mathbb{R}$ satisfying*

$$\lim_{v\to 0}\frac{\left\vert e_{v,k} - \frac{\ln\left(\frac{8}{v^2}\right)}{\sqrt{2}}\right\vert}{v\vert\ln(v)\vert^3} = 0.$$

*Moreover, if $0 < v < \delta(k)$, then for any $s \geq 0$ and $l \in \mathbb{N}\cup\{0\}$, there exists $C(k,s,l) > 0$ such that*

$$\left\Vert \frac{\partial^l}{\partial t^l}\Lambda(\phi_k)(v,t,x)\right\Vert_{H^s_x} \leq C(k,s,l)v^{2k+l}\left(\vert t\vert v + \ln\left(\frac{1}{v^2}\right)\right)^{n_k} e^{-2\sqrt{2}\vert t\vert v}.$$

### 3.1.2 Organization of Chapter 3

In this chapter, we denote by $\mathcal{G} \in \mathscr{S}(\mathbb{R})$ the following function

$$\mathcal{G}(x) = e^{-\sqrt{2}x} - \frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}} + 2\sqrt{2}\frac{xe^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}} + k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}, \tag{3.6}$$

where $k_1 \in \mathbb{R}$ is the unique real number such that $\left\langle \mathcal{G}(x),\, H^{'}_{0,1}(x)\right\rangle = 0$. The function $\mathcal{G}$ satisfies

$$-\frac{\partial^2}{\partial x^2}\mathcal{G}(x) + U^{(2)}\left(H_{0,1}(x)\right)\mathcal{G}(x) = \left(U^{(2)}(H_{0,1}(x)) - 2\right)e^{-\sqrt{2}x} + 8\sqrt{2}H^{'}_{0,1}(x), \tag{3.7}$$

see Remark A.3.2 in the Appendix for the proof. Next, from Chapter 2, we recall, for $0 < v < 1$, the following function

$$d_v(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right), \tag{3.8}$$

which is a solution to the ordinary differential equation

$$\ddot{d}_v(t) = 16\sqrt{2}e^{-\sqrt{2}d_v(t)}. \tag{3.9}$$

In Section 3.2, we are going to develop the main techniques necessary to construct each approximate solution $\phi_k$ of Theorem 3.1.2. More precisely, we are going to construct function spaces in Subsection 3.2.1 and study the applications of Fredholm alternative of the linear operator $-\partial_x^2 + U^{(2)}(H_{0,1}(x))$ restricted to these function spaces in Subsection 3.2.2.

In Section 3.3, we will prove auxiliary estimates with the objective of simplifying, in the next sections, the computation and evaluation of $\Lambda(\phi_k)(v,t,x)$ for each $k \in \mathbb{N}_{\geq 2}$ and $0 < v \ll 1$.

In Section 3.4, we are going to prove Theorem 3.1.2 for the case $k = 2$. More precisely, for $v > 0$ small enough, we will first choose the function

$$\begin{aligned}\varphi_{2,0}\left(t,x\right) =& H_{0,1}\left(\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - H_{0,1}\left(\frac{-x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\\ &+e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(\frac{-x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\end{aligned}$$

as a candidate for the case $k = 2$. The next argument is to use the main results of Subsection 3.2.1 and Section 3.3 to estimate $\Lambda(\varphi_{2,0})(t,x)$, see also Lemma 3.5.6 and Corollary 3.5.7 for a better understanding of the main ideas behind this argument. More precisely, we are going to verify the existence of two finite sets of Schwartz functions with exponential decay in both directions $(h_i(x))_{i\in I}$ and $(p_i(t))_{i\in I}$ such that

$$\Lambda(\varphi_{2,0})(t,x) = \sum_{i\in I} p_i(\sqrt{2}vt)\left[h_i\left(\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - h_i\left(\frac{-x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right] + u_v(t,x),$$

where the function $u_v : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth and satisfies, for a real constant $q > 0$, any $l \in \mathbb{N}\cup\{0\}$ and any $s > 0$, the estimate

$$\left\Vert\frac{\partial^l}{\partial t^l}u_v(t,x)\right\Vert_{H^s_x} \leq C(s,l)v^{6+l}\left[\ln\left(\frac{1}{v}\right)+|t|v\right]^q e^{-2\sqrt{2}|t|v}, \text{ for all } t\in\mathbb{R}, \text{ if } 0<v\ll 1,$$

where $C(s,l)$ is a positive number depending only on $l$ and $s$. Next, using the estimate above of $\Lambda(\varphi_{2,0})(t,x)$, we are going to construct a linear ordinary differential equation with a solution being a smooth function $r_v(t)$ with $L^\infty(\mathbb{R})$ norm of order $v^2\ln\left(\frac{1}{v}\right)$. Using the function $r_v(t)$, we are going to verify, for

$$\begin{aligned}\varphi_{2,1}(t,x) =& H_{0,1}\left(\frac{x+r_v(t)-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - H_{0,1}\left(\frac{-x+r_v(t)-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\\ &+e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(\frac{x+r_v(t)-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(\frac{-x+r_v(t)-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right),\end{aligned}$$

and an explicit real value $e_{2,v}$, that the function $\phi_2(v,t,x) := \varphi_{2,1}(t+e_{2,v},x)$ satisfies Theorem 3.1.2 for the case $k = 2$, if $v > 0$ is small enough.

In Section 3.5, we are going to prove Theorem 3.1.2 by an argument of induction on $k \in \mathbb{N}_{\geq 2}$. The proof of complementary information is done in the Subsection A.3 of the Appendix.

### 3.1.3 Notation

In this subsection, we will present the notations that are going to be used in the next sections of this chapter.

**Notation 3.1.3.** *For any pair of functions* $w : \mathbb{R}^2 \to \mathbb{R}$, $h \in L^\infty_x(\mathbb{R})$ *we denote* $h^w(t,x)$ *by the following function*

$$h^w(t,x) = h\left(w(t,x)\right) - h\left(w(t,-x)\right) \text{ for any } (t,x) \in \mathbb{R}^2.$$

*Next, for any* $s \geq 0$*, we consider the norm* $\|\cdot\|_{H^s_x}$ *given by*

$$\|f\|_{H^s_x} = \|f\|_{H^s_x} = \left(\int_{\mathbb{R}} (1+|x|)^{2s} |\hat{f}(x)|^2 \, dx\right)^{\frac{1}{2}}, \text{ for any } f \in H^s_x(\mathbb{R}),$$

*where* $\hat{f}$ *is the Fourier transform of the function* $f$*. Finally, we denote* $\mathbb{D}$ *as the set given by* $\{z \in \mathbb{C} \big| \, |z| < 1\}$.

## 3.2 Functional analysis methods

### 3.2.1 Asymptotic analysis methods

We will use the following Lemma in several occasions.

**Lemma 3.2.1.** *For any real numbers* $x_2, x_1$*, such that* $\zeta = x_2 - x_1 > 0$ *and* $\alpha$, $\beta$, $m > 0$ *with* $\alpha \neq \beta$ *the following bound holds:*

$$\int_{\mathbb{R}} |x - x_1|^m e^{-\alpha(x-x_1)_+} e^{-\beta(x_2-x)_+} \lesssim_{\alpha,\beta,m} \max\left(\left(1+\zeta^m\right) e^{-\alpha\zeta}, e^{-\beta\zeta}\right),$$

*For any* $\alpha > 0$*, the following bound holds*

$$\int_{\mathbb{R}} |x - x_1|^m e^{-\alpha(x-x_1)_+} e^{-\alpha(x_2-x)_+} \lesssim_{\alpha} \left[1+\zeta^{m+1}\right] e^{-\alpha\zeta}.$$

*Proof.* Elementary computations. □

Next, we define the function spaces $S^+$ and $S^-$. They will be used to construct the approximate solutions $\phi_k(v,t,x)$ of Theorem 3.1.2 for each $k \in \mathbb{N}_{\geq 2}$.

**Definition 3.2.2.** $S^+$ *is the linear subspace of* $L^\infty(\mathbb{R})$ *such that* $f \in S^+$*, if and only if all the following conditions are true*

- $f' \in \mathscr{S}(\mathbb{R})$ *and there is a holomorphic function* $F : \{z \in \mathbb{C} | -1 < \operatorname{Im}(z) < 1\} \to \mathbb{C}$ *such that* $F(e^{\sqrt{2}x}) = f(x)$ *for all* $x \in \mathbb{R}$.
- $F$ *satisfies* $F(z) = \sum_{k=0}^{+\infty} a_k z^{2k+1}$*, for some sequence of real numbers* $(a_k)$ *and all* $z \in \mathbb{D}$.

**Definition 3.2.3.** $S^-$ *is the linear subspace of* $L^\infty(\mathbb{R})$ *such that* $g \in S^-$*, if and only if all the following conditions are true*

- $g^{'} \in \mathscr{S}(\mathbb{R})$ *and there is a holomorphic function* $G : \{z \in \mathbb{C}| -1 < \mathrm{Im}\,(z) < 1\} \to \mathbb{C}$ *such that* $G(e^{-\sqrt{2}x}) = g(x)$ *for all* $x \in \mathbb{R}$.

- $G$ *satisfies* $G(z) = \sum_{k=1}^{+\infty} b_k z^{2k}$, *for some sequence of real numbers* $(b_k)$ *and all* $z \in \mathbb{D}$.

**Remark 3.2.4.** *In Definitions 3.2.2 and 3.2.3, from standard complex analysis theory, the holomorphic functions* $F$ *and* $G$ *are unique.*

**Remark 3.2.5.** *From Definition 3.2.3, if* $f_1,\, f_2 \in S^-$, *then* $f_1 f_2 \in S^-$. *Therefore,* $S^-$ *is an algebra.*

**Remark 3.2.6.** *From Definitions* (3.2.2) *and 3.2.3, if* $f \in S^+$ *and* $g \in S^-$, *then, for any* $l \in \mathbb{N}$, $f^{(l)} \in S^+$ *and* $g^{(l)} \in S^-$.

The following Lemma is a direct consequence of Definitions 3.2.2 and 3.2.3.

**Lemma 3.2.7** (Multiplicative Lemma)**.** *If* $f_1,\, f_2,\, f_3 \in S^+$, *then the function* $g_1(x) \coloneqq f_1(-x)f_2(-x)$ *is in* $S^-$ *and the function* $g_2(x) \coloneqq f_1(x)f_2(x)f_3(x)$ *is in* $S^+$.

**Definition 3.2.8.** *We define, for any* $n \in \mathbb{N}\cup\{0\}$, *the linear spaces* $S^{+,n} = \{x^n f(x)|\, f(x) \in S^+ \cap \mathscr{S}(\mathbb{R})\}$ *and* $S^{-,n} = \{x^n f(x)|\, f(x) \in S^- \cap \mathscr{S}(\mathbb{R})\}$, *and for any* $m \in \mathbb{N}\cup\{0\}$, *we define*

$$S^+_m = \bigoplus_{n=0}^{m} S^{+,n},\ S^-_m = \bigoplus_{n=0}^{m} S^{-,n},\ S^+_\infty = \bigoplus_{n=0}^{+\infty} S^{+,n},\ S^-_\infty = \bigoplus_{n=0}^{+\infty} S^{-,n}.$$

**Remark 3.2.9.** *From Definition 3.2.8, for any* $m \in \mathbb{N}\cup\{0\}$, *it is not difficult to verify that*

$$\frac{d}{dx}\left\{S^+_m\right\} = \left\{\frac{df}{dx}|\, f \in S^+_m\right\} \subset S^+_m,$$
$$\frac{d}{dx}\left\{S^-_m\right\} = \left\{\frac{df}{dx}|\, f \in S^-_m\right\} \subset S^-_m.$$

**Remark 3.2.10.** *From Remark 3.2.5,* $S^-_\infty$ *is an algebra. Furthermore, the result of Lemma 3.2.7 is also true if we replace the function spaces* $S^+$ *and* $S^-$, *respectively, with* $S^+_\infty$ *and* $S^-_\infty$ *in the statement of this lemma.*

**Remark 3.2.11.** *We will prove later in Lemma 3.2.22 that the linear space generated by the union of all subspaces* $S^+_n \subset \mathscr{S}(\mathbb{R})$ *is a direct sum. By analogy, the same result is true for the union of all subspaces* $S^-_n$.

**Remark 3.2.12.** *In the definition 3.2.2, we can verify that if* $F(z)$ *is a polynomial function, then* $F \equiv 0$. *Otherwise, the identity* $f(x) = F(e^{\sqrt{2}x})$ *would imply that* $\lim_{x\to+\infty} |f(x)| = \lim_{x\to+\infty} |F(e^{\sqrt{2}x})| = +\infty$, *if* $F(z)$ *is a non-trivial polynomial, which contradicts first condition in definition 3.2.2. Similarly, we can verify that* $G(z)$ *in definition 3.2.3 cannot be a non-zero polynomial.*

**Remark 3.2.13.** *For any number* $l \in \mathbb{N}\cup\{0\}$, *any odd number* $m$ *and any even number* $n$, *we have that* $\frac{d^l}{dx^l}\left[H_{0,1}(x)^m\right] \in S^+$ *and* $\frac{d^l}{dx^l}\left[H_{-1,0}(x)^n\right] \in S^-$.

**Definition 3.2.14.** *In notation of definition 3.2.2, If* $f \in S^+$, *we define*

$$\mathrm{val}_+(f) = \min\{2k+1 |\, k \in \mathbb{N} \cup \{0\},\, a_k \neq 0\}.$$

*And in notation of definition 3.2.3, if* $g \in S^-$, *we define*

$$\mathrm{val}_-(g) = \min\{2k |\, k \in \mathbb{N},\, b_k \neq 0\}.$$

**Remark 3.2.15.** *The exponential decay of the functions in* $S^+ \cap \mathscr{S}(\mathbb{R})$, $S^- \cap \mathscr{S}(\mathbb{R})$ *and* $S^+_m$ *are going to be very important to obtain high precision in the approximate solutions of the main theorem.*

Now, we can prove the main proposition of this subsection.

**Lemma 3.2.16** (Separation Lemma)**.** *If* $f \in S^+$, $g \in S^-$, *then there exist a sequence of pairs* $(h_n, d_n)_{n\geq 1}$ *and a set* $\Delta \subset \mathbb{N}$ *such that* $h_n(x) \in S^+ \cap \mathscr{S}(\mathbb{R})$ *for all* $n \in \Delta$, $h_n(-x)$ *is in* $S^+ \cap \mathscr{S}(\mathbb{R})$ *for all* $n \in \Omega = \mathbb{N} \setminus \Delta$ *and* $(d_n)_{n\geq 1} \subset \mathbb{N}$ *is a strictly increasing sequence satisfying, for any* $\mathcal{M} \in \mathbb{N}$ *and any* $\zeta \geq 1$, *the following equation*

$$f(x-\zeta)g(x) = \sum_{\substack{1\leq n\leq \mathcal{M},\\ n\in\Delta}} h_n(x-\zeta)e^{-\sqrt{2}d_n\zeta} + \sum_{\substack{1\leq n\leq \mathcal{M},\\ n\in\Omega}} h_n(x)e^{-\sqrt{2}d_n\zeta} + e^{-\sqrt{2}d_{\mathcal{M}}\zeta} f_{\mathcal{M}}(x-\zeta)g_{\mathcal{M}}(x), \tag{3.10}$$

*where* $f_{\mathcal{M}} \in S^+$, $g_{\mathcal{M}} \in S^-$ *with* $f_{\mathcal{M}}$ *or* $g_{\mathcal{M}}$ *in* $\mathscr{S}(\mathbb{R})$. *Also,* $\|f_{\mathcal{M}}(x-\zeta)g_{\mathcal{M}}(x)\|_{H^k_x(\mathbb{R})} \lesssim_{k,\mathcal{M}} 1$ *for any* $\zeta \geq 1$.

**Lemma 3.2.17.** *Let* $f \in S^+$, $g \in S^-$, *then:*

- *If* $\mathrm{val}_+(f) > \mathrm{val}_-(g)$, *then there exist* $h_1 \in S^+ \cap \mathscr{S}(\mathbb{R})$ *and functions* $f_1 \in S^+$, $g_1 \in S^-$ *satisfying, for any* $\zeta \geq 1$, *the following identity*

$$f(x-\zeta)g(x) = h_1(x-\zeta)e^{-\mathrm{val}_-(g)\zeta} + e^{-\sqrt{2}\,\mathrm{val}_-(g)\zeta} f_1(x-\zeta)g_1(x),$$

  *and at least one of the functions* $f_1$, $g_1$ *is in* $\mathscr{S}(\mathbb{R})$.

- *If* $\mathrm{val}_-(g) > \mathrm{val}_+(f)$, *then there exist* $\hat{h}_1 \in \mathscr{S}(\mathbb{R}) \cap S^+$ *and functions* $f_1 \in S^+$, $g_1 \in S^-$ *satisfying, for any* $\zeta \geq 1$, *the following identity*

$$f(x-\zeta)g(x) = \hat{h}_1(-x)e^{-\mathrm{val}_+(f)\zeta} + e^{-\sqrt{2}\,\mathrm{val}_+(f)\zeta} f_1(-x+\zeta)g_1(-x),$$

  *and at least one of the functions* $f_1$, $g_1$ *is in* $\mathscr{S}(\mathbb{R})$.

*Proof of Lemma 3.2.17.* We consider the notation of Definition 3.2.2 and Definition 3.2.3. For $2w_1 + 1 = \mathrm{val}_+(f)$ and $2w_2 = \mathrm{val}_-(g)$ there are only two cases to consider, which are $2w_1 + 1 > 2w_2$ and $2w_1 + 1 < 2w_2$.

First, we consider the case where $2w_1 + 1 > 2w_2$.

$$
\begin{aligned}
f(x-\zeta)g(x) =& f(x-\zeta)b_{w_2}e^{-2w_2\sqrt{2}x} + f(x-\zeta)\left[g(x) - b_{w_2}e^{-2w_2\sqrt{2}x}\right] \\
=& f(x-\zeta)b_{w_2}e^{-2\sqrt{2}w_2(x-\zeta)}e^{-2\sqrt{2}w_2\zeta} \\
&+ e^{-2\sqrt{2}w_2\zeta}\left[f(x-\zeta)e^{-2\sqrt{2}w_2(x-\zeta)}\left(g(x)e^{+2\sqrt{2}w_2x} - b_{w_2}\right)\right].
\end{aligned}
$$

Because $2w_1 + 1 > 2w_2$ and $f \in S^+$, we have that $f(x)e^{-2w_2\sqrt{2}x} \in S^+ \cap \mathscr{S}(\mathbb{R})$. Clearly, if $g(x)e^{+2\sqrt{2}w_2x} - b_{w_2} \in S^-$, then, from the identity above, Lemma 3.2.17 would be true for the case where $\mathrm{val}_+(f) > \mathrm{val}_-(g)$. Moreover, for any $x > 0$, we have that

$$
g(x)e^{+2\sqrt{2}w_2x} - b_{w_2} = \sum_{n=w_2+1}^{+\infty} b_n e^{-2(n-w_2)\sqrt{2}x}. \tag{3.11}
$$

Since $G(z)$ is analytic in the region $\mathbb{D}$, we clearly have that the following function

$$
Q(z) = \frac{G(z)}{z^{2w_2}} - b_{w_2} = \sum_{n=w_2+1}^{+\infty} b_n z^{2(n-w_2)} \tag{3.12}
$$

is analytic in $\mathbb{D}$, from which, using the product rule of the derivative, for any $x > 1$ and $l,\, m \in \mathbb{N}$, we deduce that

$$
\left|(1+|x|^m)\frac{d^l}{dx^l}\left[g(x)e^{+2\sqrt{2}w_2x} - b_{w_2}\right]\right| \lesssim_{l,m} 1. \tag{3.13}
$$

From equation (3.12) and from the fact that $G(z)$ has a holomorphic extension in the region $\mathcal{B} = \{z \mid -1 < \operatorname{Im} z < 1\}$ since $g \in S^-$, we conclude that $Q(z)$ has a holomorphic extension in the region $\mathcal{B}$. Moreover, since $g \in S^-$, then $g \in L^\infty_x(\mathbb{R}) \cap C^\infty(\mathbb{R})$ and $g^{'} \in \mathscr{S}(\mathbb{R})$, from which we deduce the following estimate

$$
\left|(1+|x|^m)\frac{d^l}{dx^l}\left[g(x)e^{+2\sqrt{2}w_2x}\right]\right| \lesssim_{l,m} 1 \text{ for any } x < -1 \text{ and } l \in \mathbb{N}_{\geq 1},
$$

and so, we conclude that $\frac{d}{dx}\left[g(x)e^{+2\sqrt{2}w_2x} - b_{w_2}\right] \in \mathscr{S}(\mathbb{R})$.

Analogously, if $2w_2 = val_-(g) > val_+(f) = 2w_1 + 1$, then we can deduce from the Definition 3.2.2 and Definition 3.2.3 that

$$
h_1(x) = g(-x)e^{-(2w_1+1)\sqrt{2}x} \in S^+ \cap \mathscr{S}(\mathbb{R}),\; g_1(x) = f(-x)e^{(2w_1+1)\sqrt{2}x} - a_{w_1} \in S^-,
$$

and

$$
\begin{aligned}
f(x-\zeta)g(x) = & g(x)a_{w_1}e^{\sqrt{2}(2w_1+1)x}e^{-\sqrt{2}(2w_1+1)\zeta} \\
&+ e^{-\sqrt{2}(2w_1+1)\zeta}\left[g(x)e^{\sqrt{2}(2w_1+1)x}\left(f(x-\zeta)e^{-\sqrt{2}(2w_1+1)(x-\zeta)} - a_{w_1}\right)\right].
\end{aligned} \tag{3.14}
$$

□

*Proof of Proposition 3.2.16.* If $f \equiv 0$ or $g \equiv 0$, we can take $h_n = 0$ and $d_n = n$ for all $n \in \mathbb{N}$. From now on, we consider the case where both $f$ and $g$ are not identically zero. Clearly, Lemma 3.2.17 implies Proposition 3.2.16 for the case where $\mathcal{M} = 1$.

Moreover, if Proposition 3.2.16 is true when $\mathcal{M} = M_0 \in \mathbb{N}$, we can repeat the argument above of the proof of Lemma 3.2.17 using $f_{M_0}$, $g_{M_0}$ in the place of $f$, $g$ and conclude that Proposition 3.2.16 is also true when $\mathcal{M} = M_0 + 1$, so by induction on $\mathcal{M}$, Proposition 3.2.16 is true for all $\mathcal{M} \in \mathbb{N}$. □

**Corollary 3.2.18.** *If $f \in S^+, g \in S^-$ and $f \not\equiv 0$, $g \not\equiv 0$, then the sequence $(h_n, d_n)_{n\in\mathbb{N}}$ satisfying Proposition 3.2.16 is unique. Furthermore,* $d_1 = \min(\mathrm{val}_+(f), \mathrm{val}_-(g))$.

*Proof.* From an argument of analogy, it is enough to consider the case where $2w_1 + 1 = val_+(f) > val_-(g) = 2w_2$. In this case, from the proof of Proposition 3.2.16, we have that the real function $\hat{h}_1(x) = b_{w_2} f(x) e^{-2\sqrt{2}w_2 x}$ satisfies $\hat{h}_1 \in S^+ \cap \mathscr{S}(\mathbb{R})$, $\hat{h}_1 \not\equiv 0$ and the following identity

$$f(x-\zeta)g(x) = \hat{h}_1(x-\zeta)e^{-2\sqrt{2}w_2\zeta} + e^{-2\sqrt{2}w_2\zeta} f_1(x-\zeta)g_1(x), \tag{3.15}$$

where $f_1 \in S^+$, $g_1 \in S^-$ and either $f_1$ or $g_1$ is in $\mathscr{S}(\mathbb{R})$. In conclusion, Lemma 3.2.1 and equation (3.15) imply for any $s \geq 0$ that

$$\lim_{\zeta \to +\infty} \left\| f(x-\zeta)g(x)e^{2\sqrt{2}w_2\zeta} - \hat{h}_1(x-\zeta) \right\|_{H^s_x} = 0, \tag{3.16}$$

and so,

$$0 < \lim_{\zeta \to +\infty} \left\| f(x-\zeta)g(x)e^{2\sqrt{2}w_2\zeta} \right\|_{H^s_x} < \infty. \tag{3.17}$$

Since $\hat{h}_1 \in \mathscr{S}(\mathbb{R})$ and $\hat{h}_1 \not\equiv 0$, $\left\|\hat{h}_1\right\|_{H^s_x} \neq 0$ for all $s \geq 0$. Therefore, using equations (3.16) and (3.17), we can verify that the unique possible choice for $d_1$ is $2w_2$. And so, the function $h_1$ satisfying Proposition 3.2.16 for $f$ and $g$ is unique and equal to $\hat{h}_1$, otherwise (3.16) would be false. Similarly, we can repeat the argument above for the case $\mathrm{val}_+(f) < \mathrm{val}_-(g)$ and obtain in this situation that $d_1 = \mathrm{val}_+(f)$ and $h_1(x) = a_{w_1} g(x) e^{(2w_1+1)\sqrt{2}x}$.

Next, assuming that $(h_n, d_n)$ is unique for all $1 \leq n \leq \mathcal{M}_0 \in \mathbb{N}$, we can repeat the argument above in $f_{\mathcal{M}_0}(x-\zeta)g_{\mathcal{M}_0}(x)$ and conclude that $(h_{\mathcal{M}_0+1}, d_{\mathcal{M}_0+1})$ is unique too. In conclusion, from the principle of finite induction applied on $n \in \mathbb{N}$, we obtain the uniqueness of $(h_n, d_n)_{n\in\mathbb{N}}$ satisfying Proposition 3.2.16 when both functions $f$ and $g$ are not identically zero. □

**Remark 3.2.19.** *When $f \not\equiv 0$ and $g \not\equiv 0$, we can find explicitly the sequence $(h_n, d_n)$ satisfying* (3.10) *from the proof of Lemma 3.2.17.*

**Remark 3.2.20.** *If $f(x) = x^m f_0(x)$, $g(x) = x^l g_0(x)$ such that $m$, $l \in \mathbb{N}$, $f_0 \in S^+ \cap \mathscr{S}(\mathbb{R})$ and $g_0 \in S^- \cap \mathscr{S}(\mathbb{R})$, then there exist a sequence of pairs $(h_n, d_n)_{n\geq 1}$ and a set $\Delta \subset \mathbb{N}$ satisfying $h_n(x) \in S^+ \cap \mathscr{S}(\mathbb{R})$ for all $n \in \Delta$, $h_n(-x)$ is in $S^+ \cap \mathscr{S}(\mathbb{R})$ for all $n \in \Omega = \mathbb{N}_{\geq 1} \setminus \Delta$, $d_n \in \mathbb{N}$*

*is strictly increasing such that for any $\zeta \geq 1$, $x \neq 0$, $x \neq \zeta$ and $\mathcal{M} \in \mathbb{N}$ we have the following equation*

$$\frac{f(x-\zeta)g(x)}{(x-\zeta)^m x^l} = \sum_{\substack{1\leq n\leq \mathcal{M},\\ n\in\Delta}} h_n(x-\zeta)e^{-\sqrt{2}d_n\zeta} + \sum_{\substack{1\leq n\leq \mathcal{M},\\ n\in\Omega}} h_n(x)e^{-\sqrt{2}d_n\zeta} + e^{-\sqrt{2}d_{\mathcal{M}}\zeta} f_{\mathcal{M}}(x-\zeta)g_{\mathcal{M}}(x),$$

*where $f_{\mathcal{M}} \in S^+$, $g_{\mathcal{M}} \in S^-$ and $f_{\mathcal{M}}$ or $g_{\mathcal{M}}$ is in $\mathscr{S}(\mathbb{R})$. Furthermore, the sequence $(h_n, d_n)_{n\in\mathbb{N}}$ is unique.*

**Remark 3.2.21.** *From Proposition 3.2.16, we can deduce if $f(-x) \in S^+$, $g(-x) \in S^-$, $f \not\equiv 0$ and $g \not\equiv 0$, then there exists a sequence of pairs $(h_n, d_n)_{n\geq 1}$ and a set $\Delta \subset \mathbb{N}$ such that $h_n(x)$ is in $S^+ \cap \mathscr{S}(\mathbb{R})$ for all $n \in \Delta$, $h_n(-x)$ is in $S^+ \cap \mathscr{S}(\mathbb{R})$ for all $n \in \Omega = \mathbb{N} \setminus \Delta$ and $(d_n)_{n\geq 1} \subset \mathbb{N}$ is a strictly increasing sequence satisfying, for any $\mathcal{M} \in \mathbb{N}$ and any $\zeta \geq 1$, the following equation*

$$f(-x+\zeta)g(-x) = \sum_{\substack{1\leq n\leq \mathcal{M},\\ n\in\Delta}} h_n(x-\zeta)e^{-\sqrt{2}d_n\zeta} + \sum_{\substack{1\leq n\leq \mathcal{M},\\ n\in\Omega}} h_n(x)e^{-\sqrt{2}d_n\zeta} + e^{-\sqrt{2}d_{\mathcal{M}}\zeta} f_{\mathcal{M}}(x-\zeta)g_{\mathcal{M}}(x), \tag{3.18}$$

*where $f_{\mathcal{M}} \in S^+$, $g_{\mathcal{M}} \in S^-$ and $f_{\mathcal{M}}$ or $g_{\mathcal{M}}$ is in $\mathscr{S}(\mathbb{R})$. Furthermore, the sequence $(h_n, d_n)_{n\in\mathbb{N}}$ is unique.*

We also demonstrate the following lemma, which will be essential to obtain the results in the next subsection.

**Lemma 3.2.22.** *Let $m \in \mathbb{N}$ and $f_j \in S^+ \cap \mathscr{S}(\mathbb{R})$ for $0 \leq j \leq m$, $\sum_{j=0}^m x^j f_j(x) = 0$, if and only if $f_j \equiv 0$ for all $0 \leq j \leq m$.*

*Proof.* For each $0 \leq j \leq m$, since $f_j \in S^+$, we have that either $f_j \equiv 0$ or there exists a natural $d_j \in \mathbb{N} \cup \{0\}$ and $a_j \in \mathbb{R}$ with $a_j \neq 0$ such that $f_j(x) = a_j e^{(2d_j+1)\sqrt{2}x} + O\left(e^{(2d_j+3)\sqrt{2}x}\right)$ for all $x \leq -1$. So, there are only two possible cases to consider.
**Case 1.**($\exists f_j$ such that $f_j(x) \neq 0$ for some $x \leq -1$.) In this situation, we have that there is a natural $d_{min} \geq 0$ and a non-trivial real polynomial $p(x)$ of degree at most $m$ such that

$$0 = \sum_{j=0}^m x^j f_j(x) = e^{(2d_{\min}+1)\sqrt{2}x} p(x) + O\left(e^{(2d_{\min}+3)\sqrt{2}x}|x|^{m+1}\right) \text{ for all } x \leq -1, \tag{3.19}$$

which is not possible since if $p(x)$ is a non-identically zero polynomial, then $p(x) = c$ for $c \neq 0$ or $\lim_{|x|\to+\infty} |p(x)| = +\infty$, but both cases contradict identity (3.19).
**Case 2.**($f_j \equiv 0$ for all $0 \leq j \leq m$) Clearly, the second case is the only possible. $\square$

### 3.2.2 Applications of Fredholm alternative

We consider the self-adjoint unbounded linear operator $L : H_x^2(\mathbb{R}) \subset L_x^2(\mathbb{R}) \to L_x^2(\mathbb{R})$ defined by

$$L(f)(x) = -\frac{d^2 f(x)}{dx^2} + U^{(2)}(H_{0,1}(x))f(x) \text{ for all } x \in \mathbb{R}. \tag{3.20}$$

From Lemma 2.6 of [47] we know for a constant $\lambda > 0$ that $\sigma(L) \subset \{0\} \cup [\lambda, +\infty)$, $\ker(L) = \{cH_{0,1}^{'}(x) | \, c \in \mathbb{C}\}$. From this, in the proof of Lemma 2.5 of [47], we have deduced the existence of a constant $k > 0$ such that if $g \in H_x^1(\mathbb{R})$ satisfies $\langle g, \, H_{0,1}^{'} \rangle = 0$, we have that

$$\langle L(g), \, g \rangle \geq k \, \|g\|_{H_x^1}^2 \, . \tag{3.21}$$

Next, we consider the linear space

$$Ort(H_{0,1}^{'}) = \left\{ g \in L_x^2(\mathbb{R}) | \, \langle g, H_{0,1}^{'} \rangle = 0 \right\}.$$

Since $0 < H_{0,1} < 1$ and $U$ is a smooth function, Cauchy-Schwarz inequality implies for any $u, \, \mu \in Ort(H_{0,1}^{'}) \cap H_x^1(\mathbb{R})$ that

$$|\langle L(u), \mu \rangle| \leq \left\| \frac{du}{dx} \right\|_{L_x^2} \left\| \frac{d\mu}{dx} \right\|_{L_x^2} + \left\| U^{(2)} \right\|_{L_x^{\infty}[-1,1]} \|u\|_{L_x^2} \|\mu\|_{L_x^2} \, . \tag{3.22}$$

In conclusion, from Lax-Milgram Theorem and inequalities (3.21), (3.22), we obtain for any bounded linear map $\mathcal{A} : \left( Ort(H_{0,1}^{'}) \cap H_x^1(\mathbb{R}) \, , \|\cdot\|_{H_x^1(\mathbb{R})} \right) \to \mathbb{R}$ the existence of a unique $h_{\mathcal{A}} \in Ort(H_{0,1}^{'}) \cap H_x^1(\mathbb{R})$ such that, for any $u \in Ort(H_{0,1}^{'}) \cap H_x^1(\mathbb{R})$, we have

$$\langle L(h_{\mathcal{A}}), \, u \rangle = \mathcal{A}(u). \tag{3.23}$$

As a consequence, we can obtain, for any $\mu \in L_x^2$, the existence of a unique $h(\mu) \in Ort\left(H_{0,1}^{'}\right) \cap H_x^1(\mathbb{R})$ satisfying for any $u \in H_x^1(\mathbb{R})$ the following identity

$$\langle L(h(\mu)), \, u \rangle = \langle \mu, u \rangle \, .$$

Then, inequalities (3.21), (3.22) imply the existence of $\beta > 0$ such that for any $\mu \in Ort\left(H_{0,1}^{'}\right) \cap H_x^1(\mathbb{R})$, $\|h(\mu)\|_{H_x^1(\mathbb{R})} \leq \beta \|\mu\|_{L_x^2}$. In conclusion, from the density of $H_x^1(\mathbb{R})$ in $L_x^2$ and the fact that $h(\mu) \in Ort\left(H_{0,1}^{'}\right) \cap H_x^1(\mathbb{R})$, we deduce the following lemma:

**Lemma 3.2.23.** *There is a unique injective and bounded linear map*

$$L_1 : \left( Ort\left(H_{0,1}^{'}\right), \|\cdot\|_{L_x^2} \right) \to \left( Ort\left(H_{0,1}^{'}\right) \cap H_x^1(\mathbb{R}) \, , \|\cdot\|_{H_x^1} \right),$$

*such that for any* $\mu \in Ort\left(H_{0,1}^{'}\right)$, $L(L_1(\mu)) = \mu$.

Now, for all $m \in \mathbb{N} \cup \{0\}$, we are going to consider the linear spaces $S_m^+ \cap Ort\left(H_{0,1}^{'}\right)$ and study the applications of the operator $L_1$ in these subspaces. More precisely, we are going to prove the following lemma:

**Lemma 3.2.24.** *The map* $L_1$ *defined in Lemma 3.2.23 satisfies* $L_1\left(S_m^+ \cap Ort\left(H_{0,1}^{'}\right)\right) \subset S_{m+1}^+ \cap Ort\left(H_{0,1}^{'}\right)$ *for all* $m \in \mathbb{N} \cup \{0\}$.

*Proof.* From Lemma A.3.3 in Appendix section, we have that if $f \in \mathscr{S}(\mathbb{R}) \cap Ort\left(H_{0,1}^{'}\right)$, then $L_1(f) \in \mathscr{S}(\mathbb{R})$. Since $L_1$ is a linear map, it is enough to prove for any $g(x) \in S^{+} \cap \mathscr{S}(\mathbb{R})$ and any $m \in \mathbb{N} \cup \{0\}$ that

$$L_1\left(x^m g(x) - \kappa H_{0,1}^{'}(x)\right) \in S_{m+1}^{+}, \tag{3.24}$$

with $\kappa$ satisfying $\left\langle x^m g(x), H_{0,1}^{'}(x)\right\rangle = \kappa \left\Vert H_{0,1}^{'}\right\Vert_{L_x^2}^2$. To simplify our notation, we denote $h(x) = x^m g(x) - kH_{0,1}^{'}(x)$. From Lemma 3.2.23, $L_1\left(x^m g(x) - kH_{0,1}^{'}(x)\right)$ is well defined, so it is only necessary to prove (3.24) by induction on $m \in \mathbb{N} \cup \{0\}$. We also observe that we can apply a change of variable $z(x) = e^{\sqrt{2}x}$ to rewrite the ordinary differential equation

$$-f^{(2)}(x) + U^{(2)}(H_{0,1}(x))f(x) = h(x) \tag{3.25}$$

as

$$-2z^2\frac{d^2F_0(z)}{dz^2} - 2z\frac{dF_0(z)}{dz} + \left(2 + E(z)\right)F_0(z) = H(z), \tag{3.26}$$

where $F_0(e^{\sqrt{2}x}) = f(x)$, $H(e^{\sqrt{2}x}) = h(x)$ and

$$E : \{z \in \mathbb{C} | -1 < \operatorname{Im}(z) < 1\} \to \mathbb{C}$$

is the analytic function

$$E(z) = -24\frac{z^2}{1+z^2} + 30\frac{z^4}{(1+z^2)^2},$$

because of the following identity $U^{(2)}(H_{0,1}(x)) = 2 - 24\frac{e^{2\sqrt{2}x}}{(1+e^{2\sqrt{2}x})} + 30\frac{e^{4\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^2}$.

We also recall that the operator $L$ defined in (3.20) satisfies $L\left(H_{0,1}^{'}\right) = 0$ and $H_{0,1}^{'}(x) = \frac{\sqrt{2}e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}$. Also, using the method of variation of parameters, we have that the real function

$$c(x) = \frac{1-e^{-2\sqrt{2}x}}{4\sqrt{2}} + \frac{3x}{2} + \frac{3(e^{2\sqrt{2}x}-1)}{4\sqrt{2}} + \frac{e^{4\sqrt{2}x}-1}{8\sqrt{2}}, \tag{3.27}$$

satisfies $L\left(c(x)H_{0,1}^{'}(x)\right) = 0$. In conclusion, from the Picard–Lindelöf Theorem, we deduce that

$$L^{-1}\{0\} = \left\{\left[c_1\left(\frac{-e^{-2\sqrt{2}x}}{4\sqrt{2}} + \frac{3x}{2} + \frac{3e^{2\sqrt{2}x}}{4\sqrt{2}} + \frac{e^{4\sqrt{2}x}}{8\sqrt{2}}\right) + c_2\right]\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}\middle| c_1,\, c_2 \in \mathbb{R}\right\}. \tag{3.28}$$

Moreover, we can verify that $c(x)H_{0,1}^{'}(x)$ satisfies

$$\int_0^{+\infty} c(x)^2 H_{0,1}^{'}(x)^2\,dx = +\infty,\ \int_{-\infty}^{0} c(x)^2 H_{0,1}^{'}(x)^2\,dx = +\infty,$$

from which we deduce with identity (3.28) that

$$L^{-1}\{0\} \cap L_x^2(\mathbb{R}_{\leq -1}) = L^{-1}\{0\} \cap L_x^2 = \left\{c_1 H_{0,1}^{'}(x) | \, c_1 \in \mathbb{R}\right\}. \tag{3.29}$$

In conclusion, from Theorem 3.2.23 and identity (3.29), we deduce that if $h \in Ort\left(H_{0,1}^{'}\right)$, $f \in L_x^2(\mathbb{R}_{\leq -1})$ and $-f^{(2)}(x) + U^{(2)}(H_{0,1}(x))f(x) = h(x)$ for all $x \in \mathbb{R}$, then there exists a constant $\kappa_1 \in \mathbb{R}$ such that $L_1(h)(x) - f(x) = \kappa_1 H_{0,1}^{'}(x)$ for all $x \in \mathbb{R}$. So, to prove Lemma 3.2.24 it is enough to find one $f \in S_{m+1}^+$ such that $L(f)(x) = h(x)$.

**Case** $(m = 0.)$ If $h \in S_0^+$, there exist an analytic function

$$H : \{z \in \mathbb{C} | -1 < \operatorname{Im}(z) < 1\} \to \mathbb{C},$$

and a sequence $(h_k)_{k\in\mathbb{N}}$ such that $H(z) = \sum_{k=0}^{+\infty} h_k z^{2k+1}$ for any $z \in \mathbb{D}$ and $h(x) = H\left(e^{\sqrt{2}x}\right)$ for all $x \in \mathbb{R}$. We are going to construct a sequence $(c_k)_{k\in\mathbb{N}\cup\{0\}}$ such that there exists a solution $f \in S_1^+ \cap \mathscr{S}(\mathbb{R})$ of $L(f)(x) = h(x)$ satisfying for all $x < 0$

$$f(x) = c_0 x H_{0,1}^{'}(x) + \sum_{k=0}^{+\infty} c_k e^{(2k+1)\sqrt{2}x}. \tag{3.30}$$

First, since $L(H_{0,1}^{'})(x) = 0$, we have for any smooth function $g(x)$ that

$$L(g)(x) = -2c_0 H_{0,1}^{''}(x) - \frac{d^2}{dx^2}\left[g(x) - c_0 x H_{0,1}^{'}(x)\right] + U^{(2)}(H_{0,1}(x))\left[g(x) - c_0 x H_{0,1}^{''}(x)\right]. \tag{3.31}$$

Next, if $(c_k)_{k\in\mathbb{N}}$ is a real sequence such that the function $F_1(z) = \sum_{k=0}^{+\infty} c_k z^{2k+1}$ is analytic in the open unitary disk $\mathbb{D}$, then the chain rule of derivative implies for any $x < 0$ that

$$\frac{dF_1(e^{\sqrt{2}x})}{dx} = \sqrt{2}\sum_{k=0}^{+\infty} c_k(2k+1)e^{(2k+1)\sqrt{2}x}, \quad \frac{d^2F_1(e^{\sqrt{2}x})}{dx^2} = 2\sum_{k=0}^{+\infty} c_k(2k+1)^2 e^{(2k+1)\sqrt{2}x}. \tag{3.32}$$

We also denote the analytic expansion of $E(z)$ in the open complex unitary disk as

$$E(z) = \sum_{k=1}^{+\infty} p_k z^{2k}, \tag{3.33}$$

and since $H_{0,1}^{''} = 2H_{0,1} - 8H_{0,1}^3 + 6H_{0,1}^5 \in S^+$, we have for $x < 0$ that $H_{0,1}^{''}(x) = 2e^{\sqrt{2}x} + \sum_{k=1}^{+\infty} u_k e^{(2k+1)\sqrt{2}x}$, with $\mathcal{U}(z) = \sum_{k=1}^{+\infty} u_k z^k$ analytic in $\mathbb{D}$.

Moreover, using identity (3.31), we would obtain that if $L(g) = h$,

$$g(x) = c_0 x H_{0,1}^{'}(x) + \sum_{k=0}^{+\infty} c_k e^{(2k+1)\sqrt{2}x}, \text{ for any } x < 0,$$

and $\limsup_{k\to+\infty} |c_k|^{\frac{1}{k}} \leq 1$, then $(c_k)_{k\in\mathbb{N}\cup\{0\}}$ should satisfy the following equations:

$$\begin{cases} -4c_0 = h_0, \\ \left(2 - 2(2k+1)^2\right) c_k = \left[h_k + 2c_0 u_k - \sum_{j+m=k,\, j\geq 1} c_m p_j\right], \text{ for any } k \geq 1. \end{cases} \tag{3.34}$$

From now on, we consider the sequence $(c_k)_{k\in\mathbb{N}\cup\{0\}}$ to be the unique solution of the linear recurrence (3.34). Clearly, for any $0 < \epsilon < 1$, we have that $\lim_{k\to+\infty} |c_k|\epsilon^k = 0$, which implies

$$\limsup_{k\to+\infty} |c_k|^{\frac{1}{k}} \leq 1.$$

Otherwise, $(c_k\epsilon^{\frac{k}{2}})_{k\in\mathbb{N}}$ would be an unbounded sequence and there would be a subsequence $(c_{k_j})_{j\in\mathbb{N}}$, so that $|c_l|\epsilon^{\frac{l}{2}} < |c_{k_j}|\epsilon^{\frac{k_j}{2}}$ for all $0 \leq l < k_j$, from which we would obtain with the identities $\lim_{n\to+\infty} p_n\epsilon^{\frac{n}{2}} = \lim_{n\to+\infty} h_n\epsilon^{\frac{n}{2}} = \lim_{n\to+\infty} u_n\epsilon^{\frac{n}{2}} = 0$ that

$$\epsilon^{\frac{k_j}{2}}|c_{k_j}|(2(2k_j+1)^2 - 2) \gg 2|c_0 u_{k_j}|\epsilon^{\frac{k_j}{2}} + |h_{k_j}|\epsilon^{\frac{k_j}{2}} + 2(k_j+1)|c_{k_j}|\epsilon^{\frac{k_j}{2}}\left\|(\epsilon^{\frac{j}{2}}p_j)\right\|_{L^\infty(\mathbb{N})},$$

but this estimate would contradict (3.34). So, we deduced that

$$F_1(z) = \sum_{k=0}^{+\infty} c_k z^{2k+1}$$

is analytic in $\mathbb{D}$. In conclusion, the recurrence (3.34) implies that the function $f(x)$ denoted in (3.30) satisfies $L(f)(x) = h(x)$ for all $x < 0$.

Moreover, because $E(z)$, $\frac{1}{z}$ are analytic in the simply connected regions

$$\mathcal{B}_{\delta,+} = \left\{z \in \mathbb{C} \mid -1 < \mathrm{Im}(z) < 1,\, |z| > \delta,\, \mathrm{Re}(z) > -\frac{4}{5}\delta\right\},$$
$$\mathcal{B}_{\delta,-} = \left\{z \in \mathbb{C} \mid -1 < \mathrm{Im}(z) < 1,\, |z| > \delta,\, \mathrm{Re}(z) < \frac{4}{5}\delta\right\}$$

for any $0 < \delta < 1$, we obtain, from $h \in S^+$ and the ordinary differential equation (3.26), the existence of a unique holomorphic function $F_+$ in the region $\mathcal{B}_{\delta,+}$ which is a solution of (3.26) and satisfies $F_1(e^{\sqrt{2}x}) + c_0 x H^{'}_{0,1}(x) = F_+(e^{\sqrt{2}x})$ for all $e^{\sqrt{2}x} \in \mathcal{B}_{\delta,+} \cap \mathbb{D}$, see Chapter 3.7 of [10]. By analogy, there exists a unique holomorphic function $F_-$ with domain $\mathcal{B}_{\delta,-}$ which is a solution of (3.26) and satisfies $F_1(e^{\sqrt{2}x}) + c_0 x H^{'}_{0,1}(x) = F_-(e^{\sqrt{2}x})$ for all $e^{\sqrt{2}x} \in \mathcal{B}_{\delta,-} \cap \mathbb{D}$. In conclusion, there exists a unique analytic function $F_2$ in the region $\mathcal{B} = \{z \in \mathbb{C} \mid -1 < \mathrm{Im}(z) < 1\}$ such that $F_2(z) = F_1(z)$ for all $z \in \mathbb{D}$ and the real function

$$c_0 x H^{'}_{0,1}(x) + F_2\left(e^{\sqrt{2}x}\right) \in L^{-1}\{h\}.$$

Indeed, from the recurrence relation (3.34) and identities (3.31), (3.32), (3.33), we conclude that if

$$f(x) = c_0 x H^{'}_{0,1}(x) + F_2(e^{\sqrt{2}x}), \tag{3.35}$$

then $f(x) \in L^2_x(\mathbb{R}_{\leq -1})$, and $L(f)(x) = h(x)$ for all $x \in \mathbb{R}$. In conclusion, there exists $\tau \in \mathbb{R}$ such that $L_1(h)(x) = f(x) - \tau H^{'}_{0,1}(x)$, and since $L_1(h)(x) \in \mathscr{S}(\mathbb{R})$, identity (3.35) implies that $L_1(h)(x)$ is in $S^+_1$.

**General case**($m \geq 1$.) Based on the observation made in (3.24), it suffices to check for any $g \in S^+ \cap \mathscr{S}(\mathbb{R})$ that

$$T_m(g) := L_1\left(x^m g - \left\langle x^m g,\, H^{'}_{0,1}\right\rangle \frac{H^{'}_{0,1}}{\left\|H^{'}_{0,1}\right\|^2_{L^2_x}}\right) \in S^+_{m+1}, \tag{3.36}$$

for all $m \in \mathbb{N} \cup \{0\}$. Clearly, we checked (3.36) when $m = 0$ in the first case. Now, we assume that (3.36) is true for all $m \in \mathbb{N} \cup \{0\}$ satisfying $0 \leq m \leq M$, for some number $M \in \mathbb{N} \cup \{0\}$. From the inductive hypothesis, if $g \in S^+ \cap \mathscr{S}(\mathbb{R})$, then $T_M(g) \in S^+_{M+1}$, which implies the existence of a finite set of functions $(f_m)_{0 \leq m \leq M+1} \subset S^+ \cap \mathscr{S}(\mathbb{R})$ such that

$$T_M(g) = \sum_{m=0}^{M+1} x^m f_m. \tag{3.37}$$

Moreover, since $L(T_M(g)) \in S^+_M$, we derive from Lemma 3.2.22 and identity (3.37) the identity $x^{M+1}L(f_{M+1})(x) = 0$, which is possible only if $f_{M+1} = \sigma H'_{0,1}$ for a real number $\sigma$. Therefore, we have

$$T_M(g)(x) = \sigma x^{M+1} H'_{0,1}(x) + \sum_{m=0}^{M} x^m f_m(x) \text{ for any } x \in \mathbb{R}. \tag{3.38}$$

Consequently,

$$\frac{d}{dx} T_M(g)(x) - \sigma x^{M+1} H''_{0,1}(x) \text{ is in } S^+_M,$$

from which, using (3.36) and identity $L(H'_{0,1})(x) = 0$, we obtain that

$$-\frac{d^2}{dx^2}\Big[xT_M(g)(x)\Big] + U^{(2)}(H_{0,1}(x))xT_M(g)(x) - \Big[x^{M+1}g(x) - \tau_M x H'_{0,1}(x) - 2\sigma x^{M+1} H''_{0,1}(x)\Big] \text{ is in } S^+_M,$$

where

$$\tau_M = \frac{\left\langle x^M g,\, H'_{0,1}\right\rangle}{\left\|H'_{0,1}\right\|^2_{L^2_x}}.$$

Using identity $L(H'_{0,1})(x) = 0$, we also obtain that

$$-\frac{d^2}{dx^2}\left[-\frac{\sigma x^{M+2} H'_{0,1}(x)}{M+2}\right] + U^{(2)}(H_{0,1}(x))\left[-\frac{\sigma x^{M+2} H'_{0,1}(x)}{M+2}\right] = 2\sigma x^{M+1} H''_{0,1}(x) + \sigma(M+1)x^M H'_{0,1}(x).$$

Therefore, using that $xH'_{0,1}$ and $x^M H'_{0,1}$ are in $S^+_M$, we deduce

$$L\left(xT_M(g)(x) - \frac{\sigma x^{M+2} H'_{0,1}(x)}{M+2}\right) - x^{M+1} g(x) \text{ is in } S^+_M,$$

from which, for $\tau_{M+1}\left\|H'_{0,1}\right\|^2_{L^2} = \langle H'_{0,1}, x^{M+1}g(x)\rangle$, we obtain that

$$L_1\left(x^{M+1}g - \tau_{M+1}H'_{0,1}\right) - \left[xT_M(g) - \frac{\sigma x^{M+2} H'_{0,1}}{M+2}\right] \text{ is in } S^+_{M+1}. \tag{3.39}$$

In conclusion, we obtain that (3.36) is true for $m = M+1$, so by induction, it is true for all $m \in \mathbb{N} \cup \{0\}$, so Lemma 3.2.24 is true for all $m \in \mathbb{N} \cup \{0\}$. □

## 3.3 Auxiliary estimates

In this section, we will prove useful lemmas, which will be used later to estimate $\frac{\partial^l}{\partial t^l}\Lambda(\phi_k)(v,t,x)$ for all $k \in \mathbb{N}_{\geq 2}$ and $l \in \mathbb{N} \cup \{0\}$.

First, we can verify by induction that $|d^{(l)}(t)| \lesssim_l v^l$, for any $l \in \mathbb{N}$, more precisely:

**Lemma 3.3.1.** *For any $v \in (0,1)$, the function $d_v(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right)$ satisfies $\left\|\dot{d}_v(t)\right\|_{L^\infty(\mathbb{R})} = 2v$ and*

$$\left|d_v(t) - 2v|t| - \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\right) + \sqrt{2}\ln 2\right| \lesssim e^{-2\sqrt{2}|t|v},$$

$$|d_v^{(l)}(t)| \lesssim_l v^l e^{-2\sqrt{2}|t|v} \text{ for all natural number } l \geq 2.$$

*Proof.* The proof of the first inequality follows directly from the definition of $d_v$ and the following estimate

$$|\ln(1+x)| \lesssim |x|, \text{ for all } x \in (0,1).$$

From $\dot{d}_v(t) = 2v\tanh\left(\sqrt{2}vt\right)$, we obtain that $\left\|\dot{d}_v(t)\right\|_{L^\infty(\mathbb{R})} = 2v$. Moreover, because

$$\ddot{d}_v(t) = 16\sqrt{2}e^{-\sqrt{2}d_v(t)} = 2\sqrt{2}v^2\operatorname{sech}(\sqrt{2}vt)^2,$$

Lemma 3.3.1 is also true for $l=2$.

Next, since the following function $q : \mathbb{C}\setminus\{i,-i\} \to \mathbb{C}$

$$q(z) := \frac{2z}{1+z^2}$$

satisfies $q(z) = q\left(z^{-1}\right)$ and it is analytic when restricted to the set $\mathcal{B} = \{z \in \mathbb{C} | -1 < \operatorname{Im} z < 1\}$, we have

$$\left\|q^{(l)}(x)\right\|_{L^\infty_x(\mathbb{R})} = \left\|q^{(l)}(x)\right\|_{L^\infty_x(\{x\in\mathbb{R}|\,|x|\leq 1\})} < +\infty \text{ for all } l \in \mathbb{N}\cup\{0\}.$$

In conclusion, since

$$\operatorname{sech}(x) = \frac{2e^{-x}}{1+e^{-2x}},$$

then, for all $l \in \mathbb{N}\cup\{0\}$,

$$\left|\frac{d^l}{dx^l}\operatorname{sech}(x)\right| \lesssim_l e^{-|x|}. \tag{3.40}$$

Furthermore, since $\ddot{d}_v(t) = 2\sqrt{2}v^2\operatorname{sech}\left(\sqrt{2}vt\right)^2$, we have for any $l \geq 2$ that

$$d_v^{(l)}(t) = 2\sqrt{2}v^2\frac{d^{l-2}}{dt^{l-2}}\left[\operatorname{sech}\left(\sqrt{2}vt\right)^2\right] = 2\sqrt{2}(\sqrt{2})^{l-2}v^l\frac{d^{l-2}}{dx^{l-2}}\Big|_{x=\sqrt{2}vt}\left[\operatorname{sech}(x)^2\right]. \tag{3.41}$$

In conclusion, we obtain that Lemma 3.3.1 is also true for any $l \in \mathbb{N}_{\geq 2}$, and so, it is true for all $l \in \mathbb{N}\cup\{0\}$. □

From now on, we denote the function $w_0 : \mathbb{R}^2 \to \mathbb{R}$ by

$$w_0(t,x) = \frac{x - \frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} \tag{3.42}$$

We will use several times the function $w_0(t,x)$ in the next sections too. Clearly, from (3.9), for any $h \in C^\infty(\mathbb{R})$, we have the following identity

$$\frac{\partial}{\partial t}\left[h\left(w_0(t,x)\right)\right] = -\frac{\dot{d}_v(t)}{\sqrt{4-\dot{d}_v(t)^2}}h'\left(w_0(t,x)\right) + \frac{16\sqrt{2}\dot{d}_v(t)}{4-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}w_0(t,x)h'\left(w_0(t,x)\right). \tag{3.43}$$

Moreover, we have:

**Lemma 3.3.2.** *If $f \in S^+_m$ for some $m \in \mathbb{N} \cup \{0\}$, then for any numbers $l$, $k_1 \in \mathbb{N} \cup \{0\}$ the function $f(w_0(t,x))$ satisfies the following estimate*

$$\left\| \frac{\partial^l f(w_0(t,x))}{\partial t^l} \right\|_{H^{k_1}_x} \lesssim_{l,k_1} v^l \left\| (1+|x|)^l \max_{0\leq j\leq k_1+l} \left| f^{(j)}(x) \right| \right\|_{L^2_x} \lesssim_{f,l,k_1} v^l. \tag{3.44}$$

*More precisely, there exist a natural number $N_l$ and a finite set $\{(h_{i,l}, p_{i,l,v}) \in S^+_{m+l} \times C^\infty | 1 \leq i \leq N_l\}$ such that*

$$\frac{\partial^l f(w_0(t,x))}{\partial t^l} = \sum_{i=1}^{N_l} h_{i,l}(w_0(t,x)) p_{i,l,v}(t), \tag{3.45}$$

*and, for all $1 \leq i \leq N_l$ and all $k_1 \in \mathbb{N} \cup \{0\}$*

$$\left| \frac{\partial^{k_1} h_{i,l}(x)}{\partial x^{k_1}} \right| \lesssim_{k_1,l} (1+|x|)^l \max_{0\leq j\leq k_1+l} \left| f^{(j)}(x) \right|, \quad \left\| \frac{\partial^{k_1} p_{i,l,v}(t)}{\partial t^{k_1}} \right\|_{L^\infty(\mathbb{R})} \lesssim_{l,k_1} v^{k_1+l}, \text{ if } 0 < v \ll 1. \tag{3.46}$$

*Furthermore, if $l$ is odd, then $p_{i,l,v}(t)$ is an odd function for all $1 \leq i \leq N_l$, otherwise they are all even functions.*

*Proof.* We will prove by induction for all $l \in \mathbb{N} \cup \{0\}$ the existence of $N_l \in \mathbb{N}$ such that (3.45) holds, and for all $1 \leq i \leq N_l$ $h_{i,l} \in S^+_{m+l}$, $p_{i,l,v}(t) = (-1)^l p_{i,l,v}(-t)$ and they also satisfy (3.46) for all $1 \leq i \leq N_l$ and all $k_1 \in \mathbb{N} \cup \{0\}$.

The case $l = 0$ is trivial, we can just take the unitary set $\{(f, 1)\} \subset S^+_m \times C^\infty$. So, there exists $l_0 \in \mathbb{N} \cup \{0\}$ such that Lemma 3.3.2 is true for all $l \in \mathbb{N} \cup \{0\}$ satisfying $0 \leq l \leq l_0$. In conclusion, using the identity (3.45) for $l = l_0$ and identity (3.43), we obtain that

$$\begin{aligned}
&\frac{\partial^{l_0+1} f(w_0(t,x))}{\partial t^{l_0+1}} \\
&\quad = \sum_{i=1}^{N_{l_0}} \frac{\partial h_{i,l_0}(w_0(t,x))}{\partial t} p_{i,l_0}(t) + h_{i,l_0}(w_0(t,x)) \dot{p}_{i,l_0,v}(t) \qquad (3.47) \\
&\quad = \sum_{i=1}^{N_{l_0}} -h'_{i,l_0}(w_0(t,x)) \frac{\dot{d}_v(t) p_{i,l_0,v}(t)}{\sqrt{4 - \dot{d}_v(t)^2}} \\
&\qquad + \sum_{i=1}^{N_{l_0}} h_{i,l_0}(w_0(t,x)) \dot{p}_{i,l_0,v}(t) + w_0(t,x) h'_{i,l_0}(w_0(t,x)) \frac{16\sqrt{2}\dot{d}_v(t) p_{i,l_0,v}(t)}{4 - \dot{d}_v(t)^2} e^{-\sqrt{2} d_v(t)}. \qquad (3.48)
\end{aligned}$$

Since $h_{i,l_0} \in S^+_{m+l_0}$, we deduce that $h'_{i,l_0} \in S^+_{m+l_0} \subset S^+_{m+l_0+1}$ and $x h'_{i,l_0} \in S^+_{m+l_0+1}$. Also, we recall that the function $d_v(t) = \frac{1}{\sqrt{2}} \ln \left( \frac{8}{v^2} \cosh \left( \sqrt{2} v t \right)^2 \right)$ satisfies for all $l \in \mathbb{N}$

$$\left\| d^{(l)}_v(t) \right\|_{L^\infty(\mathbb{R})} \lesssim_l v^l, \text{ if } 0 < v \ll 1. \tag{3.49}$$

Moreover, for any $m \in \mathbb{N} \cup \{0\}$ and any $0 < \delta < 1$,

$$\left\| \frac{d^m}{d\theta^m} \left[ \frac{1}{\sqrt{1-\theta^2}} \right] \right\|_{L^\infty_\theta(|\theta|<\delta)} < +\infty, \tag{3.50}$$

because the function $q(\theta)=(1-\theta^2)^{-\frac{1}{2}}$ is smooth in the set $\{\theta|\,|\theta|\leq\delta\}$. Therefore, since the functions $h_{i,l_0}$ and $p_{i,l_0,v}$ satisfy (3.46), using the chain rule of derivative, estimate (3.49) and (3.47), (3.48), we deduce the existence of a natural number $N_{l_0+1}$ such that

$$F_{l_0+1,t}(x)=\sum_{i=1}^{N_{l_0+1}} h_{i,l_0+1}(x)p_{i,l_0+1,v}(t),$$

and, for all $1\leq i\leq N_{l_0+1}$, the functions $h_{i,l_0+1}$, $p_{i,l_0+1,v}$ satisfy (3.46), $h_{i,l_0+1}\in S^+_{m+l_0+1}$. More precisely, from (3.47) and (3.48), we choose $N_{l_0+1}=3N_{l_0}$ and

$$\begin{cases}\Big(h_{i,l_0+1}(x),p_{i,l_{0+1},v}(t)\Big)=\Big(-h^{'}_{i,l_0}(x),\frac{\dot{d}_v(t)p_{i,l_0,v}(t)}{\sqrt{4-\dot{d}(t)^2}}\Big), \text{ if }\leq i\leq N_{l_0},\\ \Big(h_{i,l_0+1}(x),p_{i,l_{0+1},v}(t)\Big)=\Big(xh^{'}_{i-N_{l_0},l_0}(x),\frac{16\sqrt{2}\dot{d}_v(t)p_{i,l_0,v}(t)}{1-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}\Big), \text{ if } N_{l_0}+1\leq i\leq 2N_{l_0},\\ \Big(h_{i,l_0+1}(x),p_{i,l_{0+1},v}(t)\Big)=\Big(h_{i-2N_{l_0}}(x),\dot{p}_{i,l_0,v}(t)\Big), \text{ if } 2N_{l_0}+1\leq i\leq 3N_{l_0},\end{cases}$$

for all $(t,x)\in\mathbb{R}^2$. In conclusion, (3.45), (3.46) are true for $l=l_0+1$ and $h_{i,l_0+1}\in S^+_{m+l_0+1}$ for all $1\leq i\leq N_{l_0+1}$. Finally, since $d_v(t)$ is an even smooth function and, for any $1\leq i\leq N_{l_0}$, $p_{i,l_0,v}(t)=(-1)^{l_0}p_{i,l_0,v}(-t)$, then, from (3.47) and (3.48), we deduce that $p_{i,l_0+1,v}(t)=(-1)^{l_0+1}p_{i,l_0+1,v}(-t)$ for all $1\leq i\leq N_{l_0+1}$. In conclusion, the statement of Lemma 3.3.2 is true for $l=l_0+1$, and so, by induction, it is true for all $l\in\mathbb{N}\cup\{0\}$. □

**Remark 3.3.3.** *If $\gamma:(0,1)\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $\gamma(v,\cdot):\mathbb{R}\to\mathbb{R}$ is smooth for all $0<v<1$ and*

$$\left|\frac{\partial^l\gamma(v,t)}{\partial t^l}\right|\lesssim_l v^l \text{ for any } l\in\mathbb{N}\cup\{0\} \text{ and all } t\in\mathbb{R}, \text{ if } 0<v\ll 1,$$

*then for any Schwartz function $f$ and*

$$\omega(t,x)=\frac{x-\frac{d_v(t)}{2}+\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}},$$

*we obtain similarly to the proof of Lemma 3.3.2 that if $v\ll 1$, then, for all $l\in\mathbb{N}\cup\{0\}$ and $k_1\in\mathbb{N}$,*

$$\left\|\frac{\partial^l f(\omega(t,x))}{\partial t^l}\right\|_{H^{k_1}_x}\lesssim_{l,k_1} v^l\left\|(1+|x|)^l\max_{0\leq j\leq k_1+l}\left|f^{(j)}(x)\right|\right\|_{L^2_x}\lesssim_{f,l,k_1} v^l. \tag{3.51}$$

*Furthermore, if $f\in C^\infty(\mathbb{R})$ and $f^{'}\in\mathscr{S}(\mathbb{R})$, for example $f=H_{0,1}$, then from identity*

$$\frac{\partial}{\partial t}f\left(\omega(t,x)\right)=\left[\partial_t\gamma(v,t)-\frac{\dot{d}_v(t)}{2}\right]\frac{1}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}f^{'}\left(\omega(t,x)\right)$$
$$+\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}\left(\frac{d}{dt}\left[\frac{1}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right]\right)\omega(t,x)f^{'}(\omega(t,x)),$$

*we obtain from the same argument above any $l$, $k_1\in\mathbb{N}$ that estimate* (3.51) *holds. We are going to use this remark later in Section 3.5.*

**Lemma 3.3.4.** *For any $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N} \cup \{0\}$, let $r : (0,1) \times \mathbb{R} \to \mathbb{R}$ be a function such that $r_v := r(v, \cdot) : \mathbb{R} \to \mathbb{R}$ is smooth for all $0 < v < 1$ and satisfies for $n_1 \in \mathbb{N}$, $n_2 \in \mathbb{N} \cup \{0\}$*

$$\left|\frac{d^l r_v(t)}{dt^l}\right| \lesssim_l v^{n_1+l} \ln\left(\frac{1}{v}\right)^{n_2},$$

*for all $l \in \mathbb{N} \cup \{0\}$, if $0 < v \ll 1$. Then, for any $s \geq 1$ and any smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $H^{'} \in \mathscr{S}(\mathbb{R})$, we have*

$$\left\|\frac{\partial^l}{\partial t^l}\left[h\left(w_0(t, x + r_v(t))\right) - h\left(w_0(t,x)\right)\right]\right\|_{H^s_x} \lesssim_{h,s,l} v^{n_1+l} \ln\left(\frac{1}{v}\right)^{n_2},$$

$$\left\|\frac{\partial^l}{\partial t^l}\left[h\left(w_0(t, x + r_v(t))\right) - h\left(w_0(t,x)\right) - \frac{r_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} h^{'}\left(w_0(t,x)\right)\right]\right\|_{H^s_x} \lesssim_{h,s,l} v^{2n_1+l} \ln\left(\frac{1}{v}\right)^{2n_2},$$

*if $0 < v \ll 1$.*

*Proof of Lemma 3.3.4.* From the Fundamental Theorem of Calculus and the definition of $w_0(t,x)$, we have

$$h\left(w_0(t, x + r_v(t))\right) - h\left(w_0(t,x)\right) = \frac{r_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} \int_0^1 h^{'}\left(\frac{x - \frac{d_v(t)}{2} + \theta r_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) d\theta, \tag{3.52}$$

and

$$\begin{aligned} h\left(w_0(t, x + r_v(t))\right) - h\left(w_0(t,x)\right) &- \frac{r_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} h^{'}\left(w_0(t,x)\right) \\ &= \frac{r_v(t)^2}{1-\frac{\dot{d}(t)^2}{4}} \int_0^1 h^{''}\left(\frac{x - \frac{d_v(t)}{2} + \theta r_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)(1-\theta)\, d\theta. \end{aligned} \tag{3.53}$$

From Remark 3.3.3, we obtain for all $0 \leq \theta \leq 1$ and $0 < v \ll 1$ that

$$\left\|\frac{\partial^l}{\partial t^l}\left[h^{''}\left(w_0(t, x + \theta r_v(t))\right)\right]\right\|_{H^s_x} + \left\|\frac{\partial^l}{\partial t^l}\left[h^{'}\left(w_0(t, x + \theta r_v(t))\right)\right]\right\|_{H^s_x} \lesssim_l v^l \text{ for all } l \in \mathbb{N} \cup \{0\}.$$

In conclusion, from identities (3.52) and (3.53), we conclude Lemma 3.3.4 using the product rule of derivative and Lemma 3.3.1. □

**Lemma 3.3.5.** *For any $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N} \cup \{0\}$ and for $0 < v < 1$, let $r_v : \mathbb{R} \to \mathbb{R}$ being a smooth function satisfying*

$$\left|\frac{d^l r_v(t)}{dt^l}\right| \lesssim_l v^{n_1+l} \ln\left(\frac{1}{v}\right)^{n_2}, \text{ if } 0 < v \ll 1$$

*for all $l \in \mathbb{N} \cup \{0\}$. For any $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N} \cup \{0\}$ and $m_3 \in \mathbb{Z}$, let $p : (0,1) \times \mathbb{R} \to \mathbb{R}$ be the function*

$$p(v,t) = \left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_3}{2}} \exp\left(\frac{-m_1\sqrt{2}(d_v(t) + r_v(t))}{\left(1-\frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_2}{2}}}\right) - e^{-m_1\sqrt{2}d_v(t)}.$$

*If $m_2 = m_3 = 0$ and $0 < v \ll 1$, then for all $l \in \mathbb{N} \cup \{0\}$*

$$\left|\frac{\partial^l}{\partial t^l} p(v,t)\right| \lesssim_{m_1,l} v^{2m_1+n_1+l} \left(\ln\left(\frac{1}{v}\right) + |t|v\right)^{n_2} e^{-2\sqrt{2}|t|v}. \tag{3.54}$$

*If $m_3 \neq 0$, $m_2 = 0$ and $0 < v \ll 1$, then for all $l \in \mathbb{N} \cup \{0\}$*

$$\left|\frac{\partial^l}{\partial t^l} p(v,t)\right| \lesssim_{l,m_1} \max\left(v^{2m_1+2+l}, v^{2m_1+n_1+l}\left(\ln\left(\frac{1}{v}\right) + |t|v\right)^{n_2}\right) e^{-2\sqrt{2}|t|v}. \tag{3.55}$$

*If $m_2 \neq 0$ and $0 < v \ll 1$, then for all $l \in \mathbb{N} \cup \{0\}$*

$$\left|\frac{\partial^l}{\partial t^l} p(v,t)\right| \lesssim_{l,m_1} \max\left(v^{2m_1+2+l}\left(|t|v + \ln\left(\frac{1}{v}\right)\right), v^{2m_1+n_1+l}\left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{n_2}\right) e^{-2\sqrt{2}|t|v}. \tag{3.56}$$

*Proof.* If $m_2 = m_3 = 0$, then, from the Fundamental Theorem of Calculus, we have

$$p(v,t) = -\sqrt{2}m_1 \int_0^1 e^{-\sqrt{2}m_1(d_v(t)+\theta r_v(t))} r_v(t)\, d\theta.$$

So, for all $l \in \mathbb{N} \cup \{0\}$, we deduce that

$$\begin{aligned}\frac{\partial^l}{\partial t^l} p(v,t) =& -\sqrt{2} \int_0^1 \frac{d^l}{dt^l}\left[e^{-\sqrt{2}m_1(d_v(t)+\theta r_v(t))} r_v(t)\right] d\theta = \\ & -\sqrt{2}\int_0^1 \sum_{j=0}^{l} \binom{l}{j} \frac{d^j}{dt^j}\left[e^{-\sqrt{2}m_1(d_v(t)+\theta r_v(t))}\right] \frac{d^{l-j}}{dt^{l-j}} r_v(t)\, d\theta.\end{aligned}$$

From the hypothesis of $r_v(t)$, $\left|e^{-\theta\sqrt{2}r_v(t)}\right| \lesssim 1$ for any $0 \leq \theta \leq 1$ if $0 < v \ll 1$, so, using the chain and product rules, we obtain that

$$\left|\frac{d^l}{dt^l} e^{-\sqrt{2}\theta r_v(t)}\right| \lesssim_l v^l, \text{ for any } l \in \mathbb{N} \text{ and any } 0 \leq \theta \leq 1. \tag{3.57}$$

Moreover, since $8^{m_1} e^{-\sqrt{2}m_1 d_v(t)} = v^{2m_1} \operatorname{sech}\left(\sqrt{2}vt\right)^{2m_1} = \ddot{d}_v(t)^{m_1} 2^{-\frac{3m_1}{2}}$, we have from Lemma 3.3.1 and the product rule of derivative that

$$\left|\frac{d^l}{dt^l} e^{-\sqrt{2}m_1 d_v(t)}\right| \lesssim_{l,m_1} v^{2m_1+l} e^{-2\sqrt{2}m_1|t|v} \lesssim v^{2m_1+l} e^{-2\sqrt{2}|t|v}, \text{ for all } l \in \mathbb{N} \cup \{0\}, \text{ if } 0 < v \ll 1. \tag{3.58}$$

In conclusion, using the hypotheses satisfied by the function $r_v$ and the estimates above, we obtain inequality (3.54).

If $m_3 \neq 0$ and $m_2 = 0$, we have

$$\begin{aligned}p(v,t) =& \left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_3}{2}} e^{-m_1\sqrt{2}(d_v(t)+r_v(t))} - e^{-m_1\sqrt{2}d_v(t)} \\ =& e^{-m_1\sqrt{2}(d_v(t)+r_v(t))} - e^{-m_1\sqrt{2}d_v(t)} + e^{-m_1\sqrt{2}(d_v(t)+r_v(t))}\left[\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_3}{2}} - 1\right].\end{aligned}$$

From the argument above, we have for any $l \in \mathbb{N} \cup \{0\}$ that

$$\left|\frac{d^l}{dt^l}\left[e^{-m_1\sqrt{2}(d_v(t)+r_v(t))} - e^{-\sqrt{2}m_1 d_v(t)}\right]\right| \lesssim_{l,m_1} v^{2m_1+n_1+l}\left(\ln\left(\frac{1}{v}\right) + |t|v\right)^{n_2} e^{-2\sqrt{2}|t|v},$$

if $0 < v \ll 1$. Moreover, since the function $q : (-1, 1) \to \mathbb{R}$ denoted by

$$q(x) = (1 - x^2)^{\frac{m_3}{2}} - 1$$

is smooth when restricted to the compact set $[-1 + \delta, 1 - \delta]$ for any $0 < \delta < 1$, we conclude from Lemma 3.3.1, the chain rule and product rule of derivative that if $0 < v \ll 1$, then

$$\left|\frac{d^l}{dt^l}\left[\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_3}{2}} - 1\right]\right| \lesssim_{l,m_3} v^{2+l}, \text{ for all } l \in \mathbb{N} \cup \{0\}. \tag{3.59}$$

In conclusion, using the product rule of derivative, we obtain (3.55) from (3.57), (3.58) and (3.59)

Finally, we will prove now (3.56). Clearly, using estimates (3.55) and (3.59), if the function

$$p_1(v,t) = \exp\left(\frac{-m_1\sqrt{2}(d_v(t) + r_v(t))}{\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_2}{2}}}\right) - e^{-m_1\sqrt{2}(d_v(t)+r_v(t))}$$

satisfies, for any $m_1$, $m_2 \in \mathbb{N}$ and $0 < v \ll 1$, the following inequality

$$\left|\frac{\partial^l}{\partial t^l}p_1(v,t)\right| \lesssim_{l,m_1,m_2} v^{2m_1+2+l}\left(|t|v + \ln\left(\frac{1}{v}\right)\right) e^{-2\sqrt{2}|t|v}, \text{ for all } l \in \mathbb{N} \cup \{0\}, \tag{3.60}$$

then (3.56) is true. From the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned} p_1(v,t) =& -m_1\sqrt{2}(r_v(t) + d_v(t)) \int_0^1 \exp\left(-m_1\sqrt{2}(d_v(t) + r_v(t))\left[1 - \theta + \frac{\theta}{\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_2}{2}}}\right]\right) d\theta \\ &+ \frac{m_1\sqrt{2}(r_v(t) + d_v(t))}{\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_2}{2}}} \int_0^1 \exp\left(-m_1\sqrt{2}(d_v(t) + r_v(t))\left[1 - \theta + \frac{\theta}{\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{\frac{m_2}{2}}}\right]\right) d\theta. \end{aligned}$$

Similarly to the proof of (3.59), we deduce if $0 < v \ll 1$, then

$$\left|\frac{d^l}{dt^l}\left(1 - \frac{\dot{d}_v(t)^2}{4}\right)^{-\frac{m_2}{2}}\right| \lesssim_{l,m_2} v^{2+l}e^{-2\sqrt{2}|t|v} \text{ for all } l,\, m_2 \in \mathbb{N}. \tag{3.61}$$

Moreover, from the hypotheses satisfied by $r_v$, we obtain using Lemma 3.3.1, estimate (3.61) and the product rule of derivative that if $0 < v \ll 1$, then

$$\left|\frac{d^l}{dt^l}\exp\left(-m_1\sqrt{2}r_v(t)\left[1 - \theta + \frac{\theta}{\left(1 - \frac{\dot{d}(t)^2}{4}\right)^{m_2}}\right]\right)\right| \lesssim_{l,m_2,m_1} v^l, \text{ for all } 0 \leq \theta \leq 1 \text{ and } l \in \mathbb{N} \cup \{0\}.$$

Similarly, since $e^{-\sqrt{2}d_v(t)} \lesssim v^2 \ll 1$ and $d_v(t) \lesssim v|t| + \ln\left(\frac{1}{v}\right)$ we obtain from Lemma 3.3.1, estimate (3.61) and the product rule of derivative that

$$\left|\frac{d^l}{dt^l}\exp\left(-m_1\sqrt{2}d_v(t)\theta\left[\frac{1}{\left(1-\frac{\dot{d}(t)^2}{4}\right)^{m_2}}-1\right]\right)\right| \lesssim_{l,m_2,m_1} v^l, \text{ for all } 0\leq\theta\leq 1 \text{ and } l\in\mathbb{N}\cup\{0\}.$$

In conclusion, using (3.58), Lemma 3.3.1, and the product rule of derivative, we obtain (3.60), and so (3.56) is true. □

**Lemma 3.3.6.** *Let* $m,\, n \in \mathbb{N}\cup\{0\}$, $f\in S^+$, $g\in S^-$. *Let* $\gamma : (0,1)\times\mathbb{R}\to\mathbb{R}$ *be a continuous function satisfying for any* $l\in\mathbb{N}\cup\{0\}$

$$\left|\frac{d^l}{dt^l}\gamma(v,t)\right| \lesssim_l v^l \text{ if } 0<v\ll 1. \tag{3.62}$$

*Then, for*

$$\omega(t,x) = w_0(t,x+\gamma(v,t)) = \frac{x-\frac{d_v(t)}{2}+\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}, \tag{3.63}$$

*if* $0<v\ll 1$, *then, for any* $s\geq 0$ *and all* $l\in\mathbb{N}\cup\{0\}$, *we have*

$$\left\|\frac{\partial^l}{\partial t^l}\left[\omega(t,x)^m f\left(\omega(t,x)\right)\omega(t,-x)^n g\left(-\omega(t,-x)\right)\right]\right\|_{H^s_x} \lesssim_{s,l,m,n} v^{2\min(\mathrm{val}_+(f),\mathrm{val}_-(g))+l}\left(\ln\left(\frac{1}{v}\right)+|t|v\right)^{m+n}e^{-2\sqrt{2}|t|v}. \tag{3.64}$$

*Furthermore, if* $0<v\ll 1$, $\mathrm{val}_+(f)+1\neq \mathit{val}_-(g)$ *and* $\mathrm{val}_-(g)+1\neq\mathrm{val}_+(f)$, *then for all* $l\in\mathbb{N}\cup\{0\}$

$$\left|\frac{d^l}{dt^l}\left\langle\omega(t,x)^m f\left(\omega(t,x)\right)\omega(t,-x)^n g\left(-\omega(t,-x)\right), H^{'}_{0,1}\left(\omega(t,x)\right)\right\rangle\right| \lesssim_{l,m,n} v^{l+2\min(\mathrm{val}_+(f)+1,\mathrm{val}_-(g))}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{m+n}e^{-2\sqrt{2}|t|v}, \tag{3.65}$$

*and*

$$\left|\frac{d^l}{dt^l}\left\langle\omega(t,x)^m f\left(\omega(t,x)\right)\omega(t,-x)^n g\left(-\omega(t,-x)\right), H^{'}_{0,1}\left(\omega(t,-x)\right)\right\rangle\right| \lesssim_{l,m,n} v^{l+2\min(\mathrm{val}_+(f),\mathrm{val}_-(g)+1)}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{m+n}e^{-2\sqrt{2}|t|v}. \tag{3.66}$$

*Otherwise, if* $0<v\ll 1$ *and* $\mathrm{val}_+(f)+1=\mathrm{val}_-(g)$, *then for any* $l\in\mathbb{N}\cup\{0\}$

$$\left|\frac{d^l}{dt^l}\left\langle\omega(t,x)^m f\left(\omega(t,x)\right)\omega(t,-x)^n g\left(-\omega(t,-x)\right), H^{'}_{0,1}\left(\omega(t,x)\right)\right\rangle\right| \lesssim_{l,m,n} v^{l+2\,\mathrm{val}_-(g)}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{m+n+1}e^{-2\sqrt{2}|t|v}. \tag{3.67}$$

*If* $0<v\ll 1$ *and* $\mathrm{val}_+(f)=\mathrm{val}_-(g)+1$, *then*

$$\left|\frac{d^l}{dt^l}\left\langle\omega(t,x)^m f\left(\omega(t,x)\right)\omega(t,-x)^n g\left(-\omega(t,-x)\right), H^{'}_{0,1}\left(\omega(t,-x)\right)\right\rangle\right| \lesssim_{l,m,n} v^{l+2\,\mathrm{val}_+(f)}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{m+n+1}e^{-2\sqrt{2}|t|v}. \tag{3.68}$$

*Proof of the Lemma 3.3.6.* First, by an argument of analogy, it is enough to prove that estimate (3.64) is true for the case $\text{val}_{+}(f) = 2w_1 + 1 > 2w_2 = \text{val}_{-}(g)$, such that $w_1, w_2 \in \mathbb{N}$. From the Separation Lemma and Corollary 3.2.18, we have that there exists functions $h_1 \in S^+ \cap \mathscr{S}(\mathbb{R})$, $f_1 \in S^+$, $g_1 \in S^-$ with either $f_1$ or $g_1 \in \mathscr{S}(\mathbb{R})$ such that

$$f(x-\zeta)g(x) = h_1(x-\zeta)e^{-2\sqrt{2}w_2\zeta} + e^{-2\sqrt{2}w_2\zeta}f_1(x-\zeta)g_1(x),$$

for all $x \in \mathbb{R}$ and $\zeta \geq 1$. Moreover, after a change of variables, we obtain that

$$\begin{aligned}
&\omega(t,x)^m\omega(t,-x)^n f\left(\omega(t,x)\right) g\left(-\omega(t,-x)\right) \\
&\quad =\omega(t,x)^m\omega(t,-x)^n h_1\left(\omega(t,x)\right)\exp\left(\frac{-2w_2\sqrt{2}(d_v(t)-2\gamma(v,t))}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) \\
&\qquad +\omega(t,x)^m\omega(t,-x)^n\exp\left(\frac{-2\sqrt{2}w_2(d_v(t)-2\gamma(v,t))}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) f_1\left(\omega(t,x)\right) g_1\left(-\omega(t,-x)\right).
\end{aligned}$$

Since $f_1$ or $g_1 \in \mathscr{S}(\mathbb{R})$ and $f_1 \in S^+$, $g_1 \in S^-$, then either $x^{k_1}f_1(x) \in S^+_\infty \subset \mathscr{S}(\mathbb{R})$ for all $k_1 \in \mathbb{N}\cup\{0\}$ or $x^{k_1}g_1(x) \in S^-_\infty \subset \mathscr{S}(\mathbb{R})$ for all $k_1 \in \mathbb{N}\cup\{0\}$. Consequently, from Remark 3.3.3, if $0 < v \ll 1$, then for all $l$, $k_1 \in \mathbb{N}\cup\{0\}$ and $s \geq 1$ either

$$\left\Vert \frac{\partial^l}{\partial t^l}\left[\omega(t,x)^{k_1} f_1\left(\omega(t,x)\right)\right]\right\Vert_{H^s_x} \lesssim_{s,l,k_1} v^l,$$

or

$$\left\Vert \frac{\partial^l}{\partial t^l}\left[\omega(t,-x)^{k_1} g_1\left(-\omega(t,-x)\right)\right]\right\Vert_{H^s_x} \lesssim_{s,l,k_1} v^l.$$

From Lemma 3.3.1, if $0 < v \ll 1$, then we also have the following estimate for all $l \in \mathbb{N}$

$$\left|\frac{d^l}{dt^l}\left[\frac{1}{\sqrt{4-\dot{d}_v(t)^2}}\right]\right| \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v}, \tag{3.69}$$

which with the hypotheses satisfied by $\gamma(v,t)$ and the product rule of derivative implies that if $0 < v \ll 1$, then

$$\left|\frac{\partial^l}{\partial t^l}\left[\frac{d_v(t)-2\gamma(v,t)}{\sqrt{4-\dot{d}_v(t)^2}}\right]\right| \lesssim_l v^l, \text{ for all } l \in \mathbb{N}. \tag{3.70}$$

Therefore, since

$$\omega(t,x)+\omega(t,-x) = \frac{-2d_v(t)+4\gamma(v,t)}{\sqrt{4-\dot{d}_v(t)^2}}, \tag{3.71}$$

we deduce, from the product rule of derivative, the hypotheses (3.62) and Cauchy-Schwarz inequality, that if $0 < v \ll 1$, then for all $k_1$, $l \in \mathbb{N}\cup\{0\}$

$$\begin{aligned}
&\left\Vert \frac{\partial^l}{\partial t^l}\left[\omega(t,x)^m\omega(t,-x)^n f_1\left(\omega(t,x)\right) g_1\left(-\omega(t,-x)\right)\right]\right\Vert_{H^{k_1}_x} \\
&\qquad\qquad \lesssim_{m,n,l,f_1,g_1} d_v(t)^{\max(m,n)}v^l.
\end{aligned} \tag{3.72}$$

Moreover, since $d_v(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right)$ and $\sup_{t\in\mathbb{R}} |\gamma(v,t)| \lesssim 1$ when $0 < v \ll 1$, then

$$\mu(t) = \exp\left(\frac{-2\sqrt{2}w_2(d_v(t) - 2\gamma(v,t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \lesssim v^{4w_2}\operatorname{sech}\left(\sqrt{2}vt\right)^2, \text{ if } 0 < v \ll 1, \tag{3.73}$$

from which with estimate (3.70) implies for all $l \in \mathbb{N}\cup\{0\}$ that if $0 < v \ll 1$, then $\left|\frac{d^l\mu(t)}{dt^l}\right| \lesssim_{l,w_2} v^{l+4w_2}e^{-2\sqrt{2}v|t|}$. In conclusion, estimate (3.72) implies, if $0 < v \ll 1$, that for all $m, n, l \in \mathbb{N}\cup\{0\}$ we have

$$\begin{aligned}\left\|\frac{\partial^l}{\partial t^l}\Big[\mu(t)\omega(t,x)^m\omega(t,-x)^n f_1(\omega(t,x))\, g_1(-\omega(t,-x))\Big]\right\|_{H^{k_1}_x} \\ \lesssim_{w_1,w_2,m,n,l} \left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{\max(m,n)} v^{4w_2+l}e^{-2\sqrt{2}|t|v}.\end{aligned} \tag{3.74}$$

Finally, since $h_1 \in S^+\cap\mathscr{S}(\mathbb{R})$, we have $\left\|x^{k_1}h_1(x)\right\|_{H^s_x} \lesssim_{s,k_1} 1$ for all $s,\ k_1 \in \mathbb{N}\cup\{0\}$. Therefore, Remark 3.3.3 implies for $0 < v \ll 1$ that

$$\left\|\frac{\partial^l}{\partial t^l}\left[\omega(t,x)^{k_1}h_1(\omega(t,x))\right]\right\|_{H^s_x} \lesssim_{s,k_1,l} v^l \text{ for all } k_1 \in \mathbb{N}\cup\{0\}.$$

In conclusion, if $0 < v \ll 1$, then, using (3.71), (3.73) and Lemma 3.3.1, we obtain from the product rule of derivative for any $k_1,\ l \in \mathbb{N}\cup\{0\}$ that

$$\left\|\frac{\partial^l}{\partial t^l}\left[\omega(t,x)^m\omega(t,-x)^n h_1(\omega(t,x))\,\mu(t)\right]\right\|_{H^{k_1}_x} \lesssim_{l,k_1} v^{l+4w_2}\left(|t|v + \ln\left(\frac{1}{v}\right)\right)^n e^{-2\sqrt{2}|t|v},$$

from which with inequality (3.74) and triangle inequality, we deduce (3.64).

From now on, we will prove estimates (3.65), (3.66), (3.67) and (3.68). Indeed, it is sufficient to demonstrate estimates (3.65) and (3.68), because the proof of the other inequalities follows from a similar argument.

Since $\omega$ satisfies (3.63), we obtain after a change of variables that

$$\begin{aligned}&\left\langle\omega(t,x)^m f(\omega(t,x))\,\omega(t,-x)^n g(-\omega(t,-x)),\, H^{'}_{0,1}(\omega(t,x))\right\rangle \\ &= \sqrt{1-\frac{\dot{d}_v(t)^2}{4}}\left\langle x^m f(x)\left(-x-\frac{d_v(t)-2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)^n g\left(x+\frac{d_v(t)-2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right),\, H^{'}_{0,1}(x)\right\rangle.\end{aligned} \tag{3.75}$$

Moreover, since $f \in S^+$ and $g \in S^-$, we deduce from Lemma 3.2.1 for any $\zeta \geq 1$ and all $l \in \mathbb{N}\cup\{0\}$ that if $\mathrm{val}_+(f)+1 \neq \mathrm{val}_-(g)$, then

$$\left|\frac{d^l}{d\zeta^l}\left\langle x^m f(x)(x+\zeta)^n g(x+\zeta), H^{'}_{0,1}(x)\right\rangle\right| \lesssim_l \zeta^{m+n}\max\left(e^{-\sqrt{2}(1+\mathrm{val}_+(f))\zeta}, e^{-\sqrt{2}\,\mathrm{val}_-(g)\zeta}\right),$$

otherwise

$$\left|\frac{d^l}{d\zeta^l}\left\langle x^m f(x)(x+\zeta)^n g(x+\zeta), H^{'}_{0,1}(x)\right\rangle\right| \lesssim_l \zeta^{m+n+1}e^{-\sqrt{2}\,\mathrm{val}_-(g)\zeta}.$$

Finally, from Lemma 3.3.1 and the hypotheses satisfied by $\gamma(v,t)$, we obtain if $0<v\ll 1$, then for all $l\in\mathbb{N}$

$$\left|\frac{\partial^l}{\partial t^l}\left[\frac{2d_v(t)-4\gamma(v,t)}{\sqrt{4-\dot{d}_v(t)^2}}\right]\right|+\left|\frac{d^l}{dt^l}\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}\right|\lesssim_l v^l.$$

In conclusion, the product rule of derivative and identity (3.75) imply (3.65), if $\operatorname{val}_+(f)+1\neq \operatorname{val}_-(g)$, otherwise they imply (3.68). The inequalities (3.66) and (3.67) can be demonstrated using an analogous argument. □

**Remark 3.3.7.** *For any $m, n\in\mathbb{N}\cup\{0\}$ and any $f\in S^+_m$, $g\in S^+_n$, we have the following identity*

$$H(v,t)=\langle f\left(w(t,x)\right),g((w(t,-x)))\rangle=\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}\left\langle f\left(x-\frac{d_v(t)-2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right),g(-x)\right\rangle.$$

*So, we can use Lemmas 3.2.1, 3.3.1 and Remark 3.3.3 to conclude that if $0<v\ll 1$, then, for all $l\in\mathbb{N}\cup\{0\}$,*

$$\left|\frac{\partial^l}{\partial t^l}H(v,t)\right|\lesssim_l v^{2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{m+n+1}e^{-2\sqrt{2}v|t|}.$$

## 3.4 Approximate solution for $k=2$

First, we recall the function $w_0:\mathbb{R}^2\to\mathbb{R}$ denoted by

$$w_0(t,x)=\frac{x-\frac{d_v(t)}{2}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}},$$

and the function $\varphi_{2,0}$ denoted by

$$\varphi_{2,0}(t,x)=H_{0,1}(w_0(t,x))-H_{0,1}(w_0(t,-x))+e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}(w_0(t,x))-\mathcal{G}(w_0(t,-x))\right]. \quad (3.76)$$

Using the results of the last section, we will estimate with high precision precision the function $\Lambda(\phi_{2,0})(t,x)$. We recall the identity (3.7) satisfied by the function $\mathcal{G}$

$$-\frac{d^2}{dx^2}\mathcal{G}(x)+U^{(2)}(H_{0,1}(x))\mathcal{G}(x)=\left[-24H_{0,1}(x)^2+30H_{0,1}(x)^4\right]e^{-\sqrt{2}x}+8\sqrt{2}H_{0,1}^{'}(x). \quad (3.77)$$

Since $H_{0,1}^{'}$ is in the kernel of the linear self-adjoint operator $-\frac{d^2}{dx^2}+U^{(2)}(H_{0,1})$, we can deduce using (3.77) that

$$\int_{\mathbb{R}}\left[24H_{0,1}(x)^2-30H_{0,1}(x)^4\right]e^{-\sqrt{2}x}H_{0,1}^{'}(x)\,dx=8\sqrt{2}\left\|H_{0,1}^{'}\right\|_{L^2_x}^2=4. \quad (3.78)$$

The main objective of this section is to demonstrate the following theorem.

**Theorem 3.4.1.** *Let $d_v(t)$ be the function defined in* (3.8)*. If $0 < v \ll 1$, then there is a smooth even function $r_v(t)$ and a value $e(v)$ such that for the following approximate solution*

$$\begin{aligned}\varphi_2(t,x) = {} & H_{0,1}\left(w_0(t,x+r_v(t))\right) - H_{0,1}\left(w_0(t,-x+r_v(t))\right) \\ & + e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(w_0(t,x+r_v(t))\right) - \mathcal{G}\left(w_0(t,-x+r_v(t))\right)\right], \quad (3.79)\end{aligned}$$

*$\phi_2(v,t,x) = \varphi_2(t+e(v),x)$ satisfies the conclusion of Theorem 3.1.2 for $k=2$ and there exists $n_2 \in \mathbb{N}$ such that if $0 < v \ll 1$, then*

$$\left|\frac{d^l}{dt^l}\left\langle \Lambda(\varphi_2)(t,x), H_{0,1}^{'}(w_0(t,\pm x + r_v(t)))\right\rangle\right| \lesssim_l v^{6+l}\left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{n_2+1} e^{-2\sqrt{2}|t|v}, \qquad (3.80)$$

*for all $l \in \mathbb{N}\cup\{0\}$. Furthermore, if $v \ll 1$, the function $r_v$ satisfies*

$$\|r_v\|_{L^\infty(\mathbb{R})} \lesssim v^2 \ln\left(\frac{1}{v^2}\right), \ \left|\frac{d^l}{dt^l} r_v(t)\ \right| \lesssim_l v^{2+l}\left[\ln\left(\frac{1}{v}\right) + |t|v\right] e^{-2\sqrt{2}|t|v},$$

*for all $l \in \mathbb{N}$.*

From now on, we say that any two smooth functions $f$, $g : \mathbb{R}^2 \to \mathbb{R}$ satisfy the relation of equivalence $f \cong_6 g$ if, and only if, for any $s \geq 0$ and $l \in \mathbb{N}\cup\{0\}$ there exists a positive number $C(s,l)$ such that

$$\left\|\frac{\partial^l}{\partial t^l}\left[f(t,x) - g(t,x)\right]\right\|_{H^s_x} \leq C(s,l) v^{6+l}\left[|t|v + \ln\left(\frac{1}{v^2}\right)\right]^2 e^{-2\sqrt{2}|t|v},$$

for all $t \in \mathbb{R}$. With the objective of simplifying our reasoning, we also say in this section that two functions $f$, $g$ are equivalent if, and only if, $f \cong_6 g$ and that a function $f$ is negligible if $f \cong_6 0$.

### 3.4.1 Estimate of non interacting terms of $\Lambda(\phi_{2,0})(t,x)$.

In this subsection, we only focus on estimating the main terms of order $O(v^2)$ of

$$\Lambda\Big(H_{0,1}\left(w_0\left(t,x\right)\right) + e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(w_0(t,x)\right)\Big).$$

**Lemma 3.4.2.** *For any $(t,x) \in \mathbb{R}^2$, we have*

$$\Lambda\left(H_{0,1}(w_0(t,x))\right) = -\frac{8\sqrt{2}e^{-\sqrt{2}d_v(t)}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} H_{0,1}^{'}(w_0(t,x)) + R_{1,v}\left(t, w_0(t,x)\right), \qquad (3.81)$$

*where the function $R_{1,v}(t,x)$ in* (3.81) *is a finite sum of functions $h_i(x)p_{i,v}(t)$, with $h_i(x) \in S^+_2$ and $p_{i,v}(t) \in C^\infty(\mathbb{R})$ being an even function satisfying $|\frac{d^l p_{i,v}(t)}{dt^l}| \lesssim_l v^{4+l}e^{-2\sqrt{2}v|t|}$ for all $l \in \mathbb{N}\cup\{0\}$. Furthermore, for any $s \geq 1$ and any $l \in \mathbb{N}\cup\{0\}$,*

$$\left\|\frac{\partial^l}{\partial t^l} R_{1,v}(t, w_0(t,x))\right\|_{H^s_x} \lesssim_{s,l} v^{4+l}e^{-2\sqrt{2}v|t|}. \qquad (3.82)$$

*Proof.* First, from identities $\frac{\partial^2}{\partial x^2}H_{0,1}(w_0(t,x)) = \frac{1}{\left(1-\frac{\dot{d}(t)^2}{4}\right)}H^{''}_{0,1}(w_0(t,x)),\ H^{''}_{0,1}(x) = U^{'}(H_{0,1}(x)),$ we have the following equation

$$\frac{\dot{d}_v(t)^2}{4-\dot{d}_v(t)^2}H^{''}_{0,1}(w_0(t,x)) - \frac{\partial^2}{\partial x^2}H_{0,1}(w_0(t,x)) + U^{'}(H_{0,1}(w_0(t,x))) = 0. \tag{3.83}$$

Next, from (3.43), we have

$$\frac{\partial}{\partial t}H_{0,1}(w_0(t,x)) = -\frac{\dot{d}_v(t)}{\sqrt{4-\dot{d}_v(t)^2}}H^{'}_{0,1}(w_0(t,x)) + \frac{16\sqrt{2}\dot{d}_v(t)}{4-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}w_0(t,x)H^{'}_{0,1}(w_0(t,x)). \tag{3.84}$$

Now, since $f(x) = xH^{'}_{0,1}(x) \in S^{+}_1$ and $d_v(t)$ is an even smooth function, we obtain from Lemma 3.3.2 the existence of $N_1 \in \mathbb{N}$ satisfying

$$\frac{\partial}{\partial t}\left[w_0(t,x)H^{'}_{0,1}(w_0(t,x))\right] = \sum_{i=1}^{N_1} h_{i,1}\left(w_0(t,x)\right)p_{i,1,v}(t), \tag{3.85}$$

such that for all $1 \leq i \leq N_1$ $h_{i,1} \in S^{+}_2$, $p_{i,1,v} \in C^{\infty}(\mathbb{R})$ and $p_{i,1,v}$ is an odd function. They also satisfy for any $1 \leq i \leq N_1$

$$\left|\frac{d^l h_{i,1}(x)}{dx^l}\right| \lesssim_l (1+|x|)^2 \max_{0\leq j\leq 2+l}\left|f^{(j)}(x)\right|, \quad \left\|\frac{d^l p_{i,1,v}(t)}{dt^l}\right\|_{L^{\infty}(\mathbb{R})} \lesssim_l v^{1+l} \text{ for all } l \in \mathbb{N}\cup\{0\}. \tag{3.86}$$

In conclusion, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left[\frac{16\sqrt{2}\dot{d}_v(t)e^{-\sqrt{2}d_v(t)}}{4-\dot{d}_v(t)^2}w_0(t,x)H^{'}_{0,1}(w_0(t,x))\right] &= w_0(t,x)H^{'}_{0,1}(w_0(t,x))\frac{d}{dt}\left[\frac{16\sqrt{2}\dot{d}_v(t)e^{-\sqrt{2}d_v(t)}}{4-\dot{d}_v(t)^2}\right]\\ &+\frac{16\sqrt{2}\dot{d}_v(t)e^{-\sqrt{2}d_v(t)}}{4-\dot{d}_v(t)^2}\sum_{i=1}^{N_1}h_{i,1}(w_0(t,x))p_{i,1,v}(t).\end{aligned} \tag{3.87}$$

Moreover, from estimate (3.50) and Lemma 3.3.1, we deduce using the chain and product rule of derivative that

$$\left|\frac{d^l}{dt^l}\left[\frac{\dot{d}_v(t)e^{-\sqrt{2}d_v(t)}}{4-\dot{d}_v(t)^2}\right]\right| \lesssim_l v^{3+l}e^{-2\sqrt{2}|t|v} \text{ for all } l \in \mathbb{N}\cup\{0\}.$$

Next, since $w_0(t,x) = \left(x - \frac{d_v(t)}{2}\right)\left(1-\frac{\dot{d}_v(t)^2}{4}\right)^{-\frac{1}{2}}$ and $\ddot{d}_v(t) = 16\sqrt{2}e^{-\sqrt{2}d_v(t)}$, we can verify that

$$\begin{aligned}\frac{\partial}{\partial t}\left[-\frac{\dot{d}_v(t)}{\sqrt{4-\dot{d}_v(t)^2}}H^{'}_{0,1}(w_0(t,x))\right] =& -\frac{8\sqrt{2}e^{-\sqrt{2}d_v(t)}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}H^{'}_{0,1}(w_0(t,x)) + \frac{\dot{d}_v(t)^2}{4-\dot{d}_v(t)^2}H^{''}_{0,1}(w_0(t,x)) \\ &-\dot{d}_v(t)\left[\frac{d}{dt}\left(4-\dot{d}_v(t)^2\right)^{-\frac{1}{2}}\right]H^{'}_{0,1}(w_0(t,x))\\ &-\dot{d}_v(t)\left[\frac{d}{dt}\left(4-\dot{d}_v(t)^2\right)^{-\frac{1}{2}}\right]w_0(t,x)H^{''}_{0,1}(w_0(t,x)).\end{aligned} \tag{3.88}$$

Using the Remarks 3.2.9 and 3.2.13, we can verify that $H^{'}_{0,1} \in S^{+} \cap \mathscr{S}(\mathbb{R})$, and $xH^{''}_{0,1} \in S^{+}_{1}$. We also recall the estimate (3.69) which is given by

$$\left|\frac{d^l}{dt^l}\left(1-\frac{\dot{d}_v(t)^2}{4}\right)^{-\frac{1}{2}}\right| \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v} \text{ for all } l \in \mathbb{N}.$$

In conclusion, from Lemmas 3.3.1, 3.3.2, identities (3.83), (3.84), equations (3.87) and (3.88), we obtain that $R_{1,v}(t,x)$ is a finite sum of functions $p_{i,v}(t)h_i(x)$ with $h_i \in S^{+}_{2}$ and $p_{i,v}$ satisfying

$$\left|\frac{d^l}{dt^l}p_{i,v}(t)\right| \lesssim_l v^{4+l}e^{-2\sqrt{2}|t|v} \text{ for all } l \in \mathbb{N}\cup\{0\}.$$

Since $d_v(t)$ is an even function, equations (3.87) and (3.88) imply that all the functions $p_{i,v}(t)$ are also even. Estimate (3.82) is obtained from Lemma 3.3.1 and the product rule of derivative on time applied to each function $p_{i,v}(t)h_i\left(w_0(t,x)\right)$. ☐

**Lemma 3.4.3.** *The function $\mathcal{G}$ defined in* (3.6) *satisfies the following identity*

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}(w_0(t,x))\right)\right]\left(e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(w_0(t,x)\right)\right)\\
&\quad=8\sqrt{2}H^{'}_{0,1}(w_0(t,x))e^{-\sqrt{2}d_v(t)}\\
&\qquad-\left[24H_{0,1}(w_0(t,x))^2-30H_{0,1}(w_0(t,x))^4\right]e^{-\sqrt{2}w_0(t,x)}e^{-\sqrt{2}d_v(t)}+R_{2,v}\left(t,w_0(t,x)\right),
\end{aligned} \tag{3.89}$$

*where $R_{2,v}(t,x)$ is a finite sum of functions $h_i(x)p_{i,v}(t)$ with $h_i(x) \in S^{+}_{3}$ and $p_{i,v}(t) \in C^{\infty}(\mathbb{R})$ being an even function such that, for any $l \in \mathbb{N}\cup\{0\}$, $\left|\frac{d^l p_{i,v}(t)}{dt^l}\right| \lesssim_l v^{4+l}e^{-2\sqrt{2}v|t|}$. Furthermore, if $0 < v \ll 1$, then for any $s \geq 1$, $l \in \mathbb{N}\cup\{0\}$, we have*

$$\left\Vert\frac{\partial^l}{\partial t^l}R_{2,v}\left(t,w_0(t,x)\right)\right\Vert_{H^s_x} \lesssim_{s,l} v^{4+l}e^{-2\sqrt{2}|t|v}. \tag{3.90}$$

*Proof.* First, using equation (3.77), we deduce that

$$\begin{aligned}
&\frac{\dot{d}_v(t)^2}{4-\dot{d}_v(t)^2}\mathcal{G}^{(2)}(w_0(t,x))-\frac{\partial^2}{\partial x^2}\mathcal{G}(w_0(t,x))+U^{(2)}\left(H_{0,1}(w_0(t,x))\right)\mathcal{G}(w_0(t,x))\\
&\quad=-\left[24H_{0,1}(w_0(t,x))^2-30H_{0,1}(w_0(t,x))^4\right]e^{-\sqrt{2}w_0(t,x)}+8\sqrt{2}H^{'}_{0,1}(w_0(t,x)).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
R_{2,v}\left(t,w_0(t,x)\right)=&\left[\frac{d^2}{dt^2}e^{-\sqrt{2}d_v(t)}\right]\mathcal{G}\left(w_0(t,x)\right)+2\left[\frac{d}{dt}e^{-\sqrt{2}d_v(t)}\right]\frac{\partial}{\partial t}\mathcal{G}\left(w_0(t,x)\right)\\
&+e^{-\sqrt{2}d_v(t)}\frac{\partial^2}{\partial t^2}\mathcal{G}\left(w_0(t,x)\right)-\frac{\dot{d}_v(t)^2}{4-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}\mathcal{G}^{(2)}\left(w_0(t,x)\right).
\end{aligned} \tag{3.91}$$

Clearly identities (3.43) and $\ddot{d}_v(t) = 16\sqrt{2}e^{-\sqrt{2}d_v(t)}$ imply the following equations

$$\frac{\partial}{\partial t}\mathcal{G}(w_0(t,x)) = \left[-\frac{\dot{d}_v(t)}{\sqrt{4-\dot{d}_v(t)^2}} + \frac{16\sqrt{2}\dot{d}_v(t)}{4-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}w_0(t,x)\right]\mathcal{G}^{(1)}(w_0(t,x)), \tag{3.92}$$

$$\frac{\partial}{\partial t}\mathcal{G}^{(1)}(w_0(t,x)) = \left[-\frac{\dot{d}_v(t)}{\sqrt{4-\dot{d}_v(t)^2}} + \frac{16\sqrt{2}\dot{d}_v(t)}{4-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}w_0(t,x)\right]\mathcal{G}^{(2)}(w_0(t,x)), \tag{3.93}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left[w_0(t,x)\mathcal{G}^{(1)}(w_0(t,x))\right] =& -\frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}}\left[w_0(t,x)\mathcal{G}^{(2)}(w_0(t,x)) + \mathcal{G}^{(1)}(w_0(t,x))\right] \\ &+\frac{16\sqrt{2}\dot{d}_v(t)}{4-\dot{d}_v(t)^2}e^{-\sqrt{2}d_v(t)}w_0(t,x)\left[w_0(t,x)\mathcal{G}^{(2)}(w_0(t,x)) + \mathcal{G}^{(1)}(w_0(t,x))\right].\end{aligned} \tag{3.94}$$

Moreover, since $\mathcal{G} \in S_1^+$, then $\mathcal{G}^{(2)}(x)$, and $x^2\mathcal{G}^{(2)}(x)$ are in $S_3^+$. Therefore, using estimates (3.69), Lemma 3.3.1 and identities (3.91), (3.92), (3.93), (3.94), we deduce from the time derivative of (3.92) and the product rule that $R_{2,v}(t,x)$ is a finite sum of functions $h_i(x)p_{i,v}(t)$ satisfying, for any index $i$, the conditions $h_i \in S_+^3$ and

$$\left|\frac{d^l}{dt^l}p_{i,v}(t)\right| \lesssim_l v^{4+l}e^{-2\sqrt{2}|t|v} \text{ for all } l \in \mathbb{N}\cup\{0\}.$$

Therefore, estimate (3.90) follows from Lemmas 3.3.1, 3.3.2 and the product rule of derivative. Finally, Since $d_v(t)$ is an even function, we can deduce from Lemma 3.3.2 applied on $\mathcal{G}$ and identity (3.92) that all the functions $p_{i,v}$ are even. □

### 3.4.2 Applications of Proposition 3.2.16 .

This subsection contains lemmas that are consequences of Proposition 3.2.16 and Remarks 3.2.20, 3.2.21. These lemmas are going to be used later to estimate the remaining terms of $\Lambda(\phi_{2,0})(t,x)$. From now on, we denote

$$M(x) = \frac{H_{0,1}(x)}{\sqrt{1+e^{2\sqrt{2}x}}},\ N(x) = \frac{H_{0,1}(x)^3}{\sqrt{1+e^{2\sqrt{2}x}}},\ V(x) = \frac{H_{0,1}(x)}{1+\sqrt{1+e^{2\sqrt{2}x}}}. \tag{3.95}$$

**Remark 3.4.4.** *From* (3.77), $-\frac{d^2}{dx^2}\mathcal{G}(x)+U^{(2)}(H_{0,1})\mathcal{G}(x) = -24M(x)+30N(x)+8\sqrt{2}H_{0,1}'(x)$.

**Lemma 3.4.5.** *For any* $\zeta > 1$, *we have that*

$$\begin{aligned}&U'\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right) - U'\left(H_{0,1}(x-\zeta)\right) - U'\left(H_{-1,0}(x)\right)\\ &\quad= 24e^{-\sqrt{2}\zeta}\left[M(x-\zeta)-M(-x)\right] - 30e^{-\sqrt{2}\zeta}\left[N(x-\zeta)-N(-x)\right]\\ &\qquad+24e^{-2\sqrt{2}\zeta}\left[V(x-\zeta)-V(-x)\right] + \frac{60e^{-2\sqrt{2}\zeta}}{\sqrt{2}}\left[H_{0,1}'(x-\zeta)-H_{-1,0}'(x)\right]\\ &\qquad+R(x,\zeta),\end{aligned} \tag{3.96}$$

*where* $R(x,\zeta)$ *is a finite sum of terms* $m_i(x-\zeta)n_i(x)e^{-(2+d_i)\sqrt{2}\zeta}$, *with* $m_i \in S^+$, $n_i \in S^-$ *and* $d_i \in \mathbb{N}\cup\{0\}$.

**Remark 3.4.6.** *In notation of Lemma 3.4.5, if we replace $x$, $\zeta$, respectively, with $-w_0(t,-x)$ and $\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}$, we obtain the following estimate*

$$\begin{aligned}
&U'\left(H^{w_0}_{0,1}(t,x)\right) - U'\left(H_{0,1}\left(w_0(t,x)\right)\right) - U'\left(-H_{0,1}\left(w_0(t,-x)\right)\right)\\
&\qquad\cong_6 24\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)M^{w_0}(t,x) - 30\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)N^{w_0}(t,x)\\
&\qquad\quad+24\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)V^{w_0}(t,x) + \frac{60}{\sqrt{2}}\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\left(H_{0,1}'\right)^{w_0}(t,x).
\end{aligned}$$

*Moreover, using Lemma 3.3.1 and the chain rule of derivative, we deduce that*

$$\left|\frac{d^l}{dt^l}\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right| \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v},$$

*for any $l\in\mathbb{N}\cup\{0\}$ if $0<v\ll 1$. Therefore, using Lemma 3.3.6 and the product rule, we deduce from Lemma 3.4.5 that*

$$\left\|\frac{\partial^l}{\partial t^l}R\left(-w_0(t,-x),\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right\|_{H^s_x} \lesssim_{l,s} v^{6+l}e^{-2\sqrt{2}v|t|},$$

*for all $s\geq 0$ and $l\in\mathbb{N}\cup\{0\}$.*

*Proof of Lemma 3.4.5.* From the definition of the potential function $U$ we have for any $\zeta>1$ that

$$\begin{aligned}
&U'\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right) - U'\left(H_{0,1}(x-\zeta)\right) - U'\left(H_{-1,0}(x)\right)\\
&=-24H_{0,1}(x-\zeta)^2H_{-1,0}(x) - 24H_{0,1}(x-\zeta)H_{-1,0}(x)^2 + 30H_{0,1}(x-\zeta)^4H_{-1,0}(x)\\
&\quad+30H_{0,1}(x-\zeta)H_{-1,0}(x)^4 + 60H_{0,1}(x-\zeta)^3H_{-1,0}(x)^2 + 60H_{0,1}(x-\zeta)^2H_{-1,0}(x)^3,
\end{aligned}\tag{3.97}$$

and so Lemma 3.96 follows directly from Proposition 3.2.16 applied to each term of the right-hand side of (3.97). Indeed, from Remark 3.2.21, we need only to apply Proposition 3.2.16 in the expressions

$$-24H_{0,1}(x-\zeta)H_{-1,0}(x)^2,\quad 30H_{0,1}(x-\zeta)H_{-1,0}(x)^4,\quad 60H_{0,1}(x-\zeta)^3H_{-1,0}(x)^2.$$

First, since $val_+\left(H_{0,1}(x)\right) < val_-\left(H_{-1,0}(x)^2\right)$, we obtain applying Lemma 3.2.17 two times that

$$\begin{aligned}
-24H_{0,1}(x-\zeta)H_{-1,0}(x)^2 =& -24H_{-1,0}(x)^2e^{\sqrt{2}x}e^{-\sqrt{2}\zeta} - 24H_{-1,0}(x)^2\left[H_{0,1}(x-\zeta)-e^{\sqrt{2}(x-\zeta)}\right]\\
=& -24H_{-1,0}(x)^2e^{\sqrt{2}x}e^{-\sqrt{2}\zeta} - 24e^{-2\sqrt{2}x}\left[H_{0,1}(x-\zeta)-e^{\sqrt{2}(x-\zeta)}\right]\\
&-24\left[H_{-1,0}(x)^2-e^{-2\sqrt{2}x}\right]\left[H_{0,1}(x-\zeta)-e^{\sqrt{2}(x-\zeta)}\right]\\
=& -24M(-x)e^{-\sqrt{2}\zeta} + 24e^{-2\sqrt{2}\zeta}V(x-\zeta)\\
&-24e^{-2\sqrt{2}\zeta}\left[H_{1,0}(x)^2e^{2\sqrt{2}x}-1\right]\left[H_{0,1}(x-\zeta)e^{-2\sqrt{2}(x-\zeta)}-e^{-\sqrt{2}(x-\zeta)}\right],
\end{aligned}$$

and since

$$H_{-1,0}(x)^2e^{2\sqrt{2}x}-1=-H_{-1,0}(x)^2,\quad H_{0,1}(x)e^{-2\sqrt{2}x}-e^{-\sqrt{2}x}=-\frac{e^{\sqrt{2}x}}{1+e^{2\sqrt{2}x}+\sqrt{1+e^{2\sqrt{2}x}}},$$

we have that $H_{0,1}(x)e^{-2\sqrt{2}x}-e^{-\sqrt{2}x}\in S^{+}\cap\mathscr{S}(\mathbb{R})$ and $H_{-1,0}(x)^2e^{2\sqrt{2}x}-1\in S^{-}$.

Furthermore, since $val_{-}\left(H_{-1,0}(x)^4\right)>val_{+}(H_{0,1}(x))$, we obtain from Lemma 3.2.17 that

$$\begin{aligned}
&30H_{0,1}(x-\zeta)H_{-1,0}(x)^4\\
&\qquad=30e^{-\sqrt{2}\zeta}N(-x)+30e^{-\sqrt{2}\zeta}H_{-1,0}(x)^4e^{\sqrt{2}x}\left[H_{0,1}(x-\zeta)e^{-\sqrt{2}(x-\zeta)}-1\right]\\
&\qquad=30e^{-\sqrt{2}\zeta}N(-x)+30e^{-2\sqrt{2}\zeta}H_{-1,0}(x)^4e^{2\sqrt{2}x}\left[H_{0,1}(x-\zeta)e^{-2\sqrt{2}(x-\zeta)}-e^{-\sqrt{2}(x-\zeta)}\right],\\
&\qquad\quad\text{and } H_{-1,0}(x)^4e^{2\sqrt{2}x}\in S^{-}\cap\mathscr{S}(\mathbb{R}),\quad H_{0,1}(x)e^{-2\sqrt{2}x}-e^{-\sqrt{2}x}\in S^{+}.
\end{aligned}$$

Similarly, since $val_{+}\left(H_{0,1}(x)^3\right)>val_{-}\left(H_{-1,0}(x)^2\right)$, we obtain from Lemma 3.2.17 that

$$\begin{aligned}
60H_{0,1}(x-\zeta)^3H_{-1,0}(x)^2=&60e^{-2\sqrt{2}\zeta}H_{0,1}(x-\zeta)^3e^{-2\sqrt{2}(x-\zeta)}\\
&+60e^{-2\sqrt{2}\zeta}H_{0,1}(x-\zeta)^3e^{-2\sqrt{2}(x-\zeta)}\left[H_{-1,0}(x)^2e^{2\sqrt{2}x}-1\right]\\
=&\frac{60e^{-2\sqrt{2}\zeta}}{\sqrt{2}}H_{0,1}^{'}(x-\zeta)+\frac{60e^{-2\sqrt{2}\zeta}}{\sqrt{2}}H_{0,1}^{'}(x-\zeta)\left[H_{-1,0}(x)^2e^{2\sqrt{2}x}-1\right],\\
&\text{and } H_{0,1}^{'}(x)\in S^{+}\cap\mathscr{S}(\mathbb{R}),\quad H_{-1,0}(x)^2e^{2\sqrt{2}x}-1\in S^{-}.
\end{aligned}$$

In conclusion, using all the estimates above and Remark 3.2.21, we obtain the conclusion of Lemma 3.4.5. □

**Lemma 3.4.7.** *There exist $A$, $B$, $C$, $D\in S^{+}\cap\mathscr{S}(\mathbb{R})$ and there exists a finite set of quadruples $(h_{i,+},\,h_{i,-},\,d_i,\,l_i)\in S^{+}\times S^{-}\times\mathbb{N}^2$, with $h_{i,+}$ or $h_{i,-}$ in $\mathscr{S}(\mathbb{R})$, $l_i\in\{0,1\}$ and $d_i\geq 0$, satisfying the following identity*

$$\begin{aligned}
&\left[U^{(2)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\right]e^{-\sqrt{2}\zeta}\mathcal{G}(x-\zeta)\\
&\quad=(x-\zeta)A(x-\zeta)e^{-2\sqrt{2}\zeta}+(x-\zeta)B(-x)e^{-2\sqrt{2}\zeta}+C(x-\zeta)e^{-2\sqrt{2}\zeta}+D(-x)e^{-2\sqrt{2}\zeta}\\
&\qquad+\sum_i(x-\zeta)^{l_i}h_{i,+}(x-\zeta)h_{i,-}(x)e^{-(2+d_i)\sqrt{2}\zeta},
\end{aligned}\tag{3.98}$$

*for all $x\in\mathbb{R}$ and any $\zeta>1$.*

**Remark 3.4.8.** *In notation of Lemma 3.4.7, for all $(x,\zeta)\in\mathbb{R}^2$, we denote the real function $\mathcal{Q}:\mathbb{R}^2\to\mathbb{R}$ by*

$$\mathcal{Q}(x,\zeta)=(x-\zeta)A(x-\zeta)e^{-\sqrt{2}\zeta}+(x-\zeta)B(-x)e^{-\sqrt{2}\zeta}+C(x-\zeta)e^{-\sqrt{2}\zeta}+D(-x)e^{-\sqrt{2}\zeta},\tag{3.99}$$

*and the function $R_q:\mathbb{R}^2\to\mathbb{R}$ by*

$$R_q(x,\zeta)=\sum_i(x-\zeta)^{l_i}h_{i,+}(x-\zeta)h_{i,-}(x)e^{-(1+d_i)\sqrt{2}\zeta},\tag{3.100}$$

*for any* $(x,\zeta)\in\mathbb{R}^2$*. If we change the variables* $x$*,* $\zeta$*, respectively, with* $-w_0(t,-x)$ *and* $\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}$*, we obtain using Lemmas 3.3.5 and 3.4.7 that*

$$\begin{aligned}&\left[U^{(2)}\left(H^{w_0}_{0,1}(t,x)\right)-U^{(2)}\left(H_{0,1}\left(w_0(t,x)\right)\right)\right]e^{-\sqrt{2}d_v(t)}\mathcal{G}\left(w_0(t,x)\right)\\&\quad=\mathcal{Q}\left(-w_0(t,-x),\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)e^{-\sqrt{2}d_v(t)}+R_q\left(-w_0(t,-x),\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)e^{-\sqrt{2}d_v(t)}\\&\quad\cong_6\mathcal{Q}\left(-w_0(t,-x),d_v(t)\right)e^{-\sqrt{2}d_v(t)}.\end{aligned}$$

*Indeed, from Lemma 3.3.6, we also have for all index* $i$*,* $s\geq 1$ *and any* $m\in\mathbb{N}\cup\{0\}$ *that*

$$\left\Vert\frac{\partial^m}{\partial t^m}\left[w_0(t,x)^{l_i}h_{i,+}\left(w_0(t,x)\right)h_{i,-}\left(-w_0\left(t,-x\right)\right)\right]\right\Vert_{H^s_x}\lesssim_{s,m}v^{2+m}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)e^{-2\sqrt{2}|t|v},$$

*if* $0<v\ll 1$*, since* $l_i\in\{0,1\}$ *for all* $i$*, which implies with Lemma 3.3.1 that*

$$R_q\left(-w_0(t,-x),\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)e^{-\sqrt{2}d_v(t)}\cong_6 0.$$

*Proof of Lemma 3.4.7.* The identity (3.6) implies that $\mathcal{G}_1(x)=\mathcal{G}(x)-2xH^{'}_{0,1}(x)\in S^+\cap\mathscr{S}(\mathbb{R})$. So, the proof follows from Remark 3.2.13, applications of Proposition 3.2.16, and Remark 3.2.21 in the following expressions

$$\begin{aligned}&\left(U^{(2)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\right)\mathcal{G}_1(x-\zeta)e^{-\sqrt{2}\zeta},\\2&\left(U^{(2)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\right)(x-\zeta)H^{'}_{0,1}(x-\zeta)e^{-\sqrt{2}\zeta}.\end{aligned}$$

More precisely, since

$$\begin{aligned}U^{(2)}&\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\\&=-24H_{-1,0}(x)^2+30H_{-1,0}(x)^4\\&\quad-48H_{0,1}(x-\zeta)H_{-1,0}(x)+30\sum_{i=1}^{3}\binom{4}{i}H_{-1,0}(x)^iH_{0,1}(x-\zeta)^{4-i},\end{aligned}$$

we obtain that $\left(U^{(2)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\right)\mathcal{G}_1(x-\zeta)$ is a linear combination of functions

$$H_{0,1}(x-\zeta)^{m_i}H_{-1,0}(x)^{l_i}h_i(x-\zeta),$$

such that $h_i\in S^+\cap\mathscr{S}(\mathbb{R})$, $m_i\in\mathbb{N}\cup\{0\}$, $l_i\in\mathbb{N}$ and $0<m_i+n_i$ is an even number. By similar reasoning, we can verify that

$$2\left[U^{(2)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\right](x-\zeta)H^{'}_{0,1}(x-\zeta)$$

is also a linear combination of functions $(x-\zeta)H_{0,1}(x-\zeta)^{m_i}H_{-1,0}(x)^{l_i}H^{'}_{0,1}(x-\zeta)$, such that $m_i\in\mathbb{N}\cup\{0\}$, $l_i\in\mathbb{N}$ and $0<m_i+l_i$ is an even number. Therefore, using Lemma 3.2.7, we can verify that

$$\left[U^{(2)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)-U^{(2)}\left(H_{0,1}(x-\zeta)\right)\right]\mathcal{G}(x-\zeta)$$

is a linear combination of functions

$$(x-\zeta)^{\alpha_i} h_{i,1}(x-\zeta)\, h_{i,2}(x)$$

such that $\alpha_i \in \{0,1\}$, $h_{i,1}$ or $h_{i,2} \in \mathscr{S}(\mathbb{R})$ and either $h_{i,1}(x) \in S^{+}$ and $h_{i,2}(x) \in S^{-}$ or $h_{i,1}(-x) \in S^{-}$ and $h_{i,2}(-x) \in S^{+}$. In conclusion, the statement of Lemma 3.64 is a consequence of Proposition 3.2.16 and Remarks 3.2.20, 3.2.21. □

**Lemma 3.4.9.** *For all* $\zeta \geq 1$,

$$\mathcal{D}_1(x,\zeta) = \sum_{j=4}^{6} \frac{1}{(j-1)!} U^{(j)}\left(H_{0,1}(x-\zeta) + H_{-1,0}(x)\right)\left(\mathcal{G}(x-\zeta) - \mathcal{G}(-x)\right)^{j-1} e^{-(j-1)\sqrt{2}\zeta}$$

*satisfies for any* $l_1,\, l_2 \in \mathbb{N}\cup\{0\}$ *the following estimate*

$$\left\Vert \frac{\partial^{l_1+l_2}}{\partial x^{l_1}\partial\zeta^{l_2}} \mathcal{D}_1(x,\zeta)\right\Vert_{L^2_x} \lesssim_{l_1+l_2} e^{-3\sqrt{2}\zeta}.$$

**Remark 3.4.10.** *Indeed, using Lemmas 3.3.1, 3.3.2 and the product rule of derivative, we have that*

$$\left\Vert \frac{\partial^{l_1+l_2}}{\partial x^{l_1}\partial t^{l^2}} \left[\mathcal{D}_1\left(-w_0(t,-x), \frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]\right\Vert_{L^2_x} \lesssim_{l_1,l_2} v^{l_2+6} e^{-2\sqrt{2}|t|v}.$$

*In conclusion, the following function*

$$\mathcal{D}_{1,1}(t,x) = \sum_{j=4}^{6} \frac{1}{(j-1)!} U^{(j)}\left(H^{w_0}_{0,1}(t,x)\right) \mathcal{G}^{w_0}(t,x)^{j-1} e^{-(j-1)\sqrt{2}d_v(t)}$$

*satisfies* $D_{1,1} \cong_6 0$.

*Proof of Lemma 3.4.9.* First, since $U \in C^\infty(\mathbb{R})$, $0 < H_{0,1} < 1$ and $H^{'}_{0,1} \in \mathscr{S}(\mathbb{R})$, we obtain for all $\zeta \in \mathbb{R}$ and any $l_1,\, l_2,\, l_3 \in \mathbb{N}\cup\{0\}$ that

$$\left\Vert \frac{\partial^{l_1+l_2}}{\partial x^{l_1}\partial\zeta^{l_2}} U^{(l_3)}\left(H_{0,1}(x-\zeta) + H_{-1,0}(x)\right)\right\Vert_{L^\infty_x(\mathbb{R})} \lesssim_{l_1,l_2,l_3} 1.$$

In conclusion, since $\mathcal{G} \in \mathscr{S}(\mathbb{R})$ and

$$\Vert fg\Vert_{H^s_x} \lesssim_s \Vert f\Vert_{H^s_x} \Vert g\Vert_{L^\infty_x(\mathbb{R})} + \Vert f\Vert_{H^s_x} \Vert g\Vert_{L^\infty_x(\mathbb{R})},$$

$\Vert fg\Vert_{H^s_x} \lesssim_s \Vert f\Vert_{H^s_x}\Vert g\Vert_{H^s_x}$ for all $f,\, g \in H^s_x$ when $s \geq 1$, we deduce for any $l_1,\, l_2 \in \mathbb{N}\cup\{0\}$ and all $\zeta \geq 1$ that

$$\left\Vert \frac{\partial^{l_1+l_2}}{\partial x^{l_1}\partial\zeta^{l_2}} \mathcal{D}_1(x,\zeta)\right\Vert_{L^2_x} \lesssim_{l_1+l_2} \left[\Vert\mathcal{G}\Vert^3_{H^{l_1+l_2+1}_x} + \Vert\mathcal{G}\Vert^5_{H^{l_1+l_2+1}_x}\right] e^{-3\sqrt{2}\zeta} \lesssim e^{-3\sqrt{2}\zeta}.$$

□

Next, we consider the following lemma.

**Lemma 3.4.11.** *There exists a finite set of elements* $(W_i, \mathcal{W}_i, d_i, j_i, l_i) \in S^+ \times S^- \times (\mathbb{N} \cup \{0\})^3$ *such that* $W_i$ *or* $\mathcal{W}_i$ *is in* $\mathscr{S}(\mathbb{R})$, $j_i$, $l_i$ *satisfy* $0 \leq j_i + l_i \leq 2$ *for all* $i$ *and we have the following identity*

$$\begin{aligned}\mathcal{D}_2(x,\zeta) =& \frac{1}{2}U^{(3)}\left(H_{0,1}(x-\zeta) + H_{-1,0}(x)\right)\left(\mathcal{G}(x-\zeta) - \mathcal{G}(-x)\right)^2 e^{-2\sqrt{2}\zeta} \\ =& \frac{1}{2}\left[U^{(3)}\left(H_{0,1}(x-\zeta)\right)\mathcal{G}(x-\zeta)^2 e^{-2\sqrt{2}\zeta} + U^{(3)}\left(H_{-1,0}(x)\right)\mathcal{G}(-x)^2 e^{-2\sqrt{2}\zeta}\right] \\ &+ \sum_i (x-\zeta)^{j_i}(-x)^{l_i} W_i(x-\zeta)\mathcal{W}_i(x) e^{-(2+d_i)\sqrt{2}\zeta} \\ &- \sum_i (-x)^{j_i}(x-\zeta)^{l_i} W_i(-x)\mathcal{W}_i(-x+\zeta) e^{-(2+d_i)\sqrt{2}\zeta},\end{aligned}$$

*for all* $\zeta \geq 1$.

**Remark 3.4.12.** *In notation of Lemma 3.4.11, for any* $t \in \mathbb{R}$, *if we change the variables* $x$ *and* $\zeta$, *respectively, with* $-w_0(t,-x)$ *and* $\frac{d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}$, *we can deduce that*

$$\begin{aligned}&\frac{1}{2}U^{(3)}\left(H_{0,1}^{w_0}(t,x)\right)\mathcal{G}^{w_0}(t,x)^2 e^{-2\sqrt{2}d_v(t)} \\ &= \frac{1}{2}\left[U^3\left(H_{0,1}\right)\mathcal{G}^2\right]^{w_0}(t,x) e^{-2\sqrt{2}d_v(t)} \\ &+ \sum_i w_0(t,x)^{j_i} w_0(t,-x)^{l_i} W_i\left(w_0(t,x)\right)\mathcal{W}_i\left(-w_0(t,-x)\right) e^{-2\sqrt{2}d_v(t)} \exp\left(\frac{-d_i\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) \\ &- \sum_i w_0(t,x)^{l_i} w_0(t,-x)^{j_i} \mathcal{W}_i\left(w_0(t,x)\right) W_i\left(-w_0(t,-x)\right) e^{-2\sqrt{2}d_v(t)} \exp\left(\frac{-d_i\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right).\end{aligned}$$

*Furthermore, in notation of Lemma 3.4.11, Lemma 3.3.6 implies for any* $i$ *that*

$$\left\|\frac{\partial^l}{\partial t^l}\left[w_0(t,x)^{j_i} w_0(t,-x)^{l_i} W_i\left(w_0(t,x)\right)\mathcal{W}_i\left(-w_0(t,-x)\right)\right]\right\|_{H^s_x} \lesssim_{s,l} v^{2+l}\left(|t|v + \ln\left(\frac{1}{v}\right)\right)^2 e^{-2\sqrt{2}|t|v},$$

*for all* $l \in \mathbb{N} \cup \{0\}$, *if* $0 < v \ll 1$. *In conclusion, we have that*

$$\frac{1}{2}U^{(3)}\left(H_{0,1}^{w_0}(t,x)\right)\mathcal{G}^{w_0}(t,x)^2 e^{-2\sqrt{2}d_v(t)} \cong_6 \frac{1}{2}\left[U^3\left(H_{0,1}\right)\mathcal{G}^2\right]^{w_0}(t,x) e^{-2\sqrt{2}d_v(t)}.$$

*Proof of Lemma 3.4.11.* The proof follows from Proposition 3.2.16 and Remarks 3.2.20, 3.2.21. More precisely, since

$$\begin{aligned}U^{(3)}\left(H_{0,1}(x-\zeta) + H_{-1,0}(x)\right) =& U^{(3)}\left(H_{0,1}(x-\zeta)\right) + U^{(3)}\left(H_{-1,0}(x)\right) \\ &+ 360\left[H_{0,1}(x-\zeta)^2 H_{-1,0}(x) + H_{0,1}(x-\zeta)H_{-1,0}(x)^2\right],\end{aligned}$$

we deduce that

$$\begin{aligned}
&\frac{1}{2}U^{(3)}\left(H_{0,1}(x-\zeta)+H_{-1,0}(x)\right)\left(\mathcal{G}(x-\zeta)-\mathcal{G}(-x)\right)^2 e^{-2\sqrt{2}\zeta}\\
&\qquad -\frac{1}{2}U^{(3)}\left(H_{0,1}(x-\zeta)\right)\mathcal{G}(x-\zeta)^2 e^{-2\sqrt{2}\zeta}-\frac{1}{2}U^{(3)}\left(H_{-1,0}(x)\right)\mathcal{G}(-x)^2 e^{-2\sqrt{2}\zeta}\\
&=\frac{1}{2}U^{(3)}\left(H_{0,1}(x-\zeta)\right)\left(\mathcal{G}(-x)^2-2\mathcal{G}(x-\zeta)\mathcal{G}(-x)\right)e^{-2\sqrt{2}\zeta}\\
&\quad+\frac{1}{2}U^{(3)}\left(H_{-1,0}(x)\right)\left(\mathcal{G}(x-\zeta)^2-2\mathcal{G}(x-\zeta)\mathcal{G}(-x)\right)e^{-2\sqrt{2}\zeta}\\
&\quad+\left[360H_{0,1}(x-\zeta)^2H_{-1,0}(x)+360H_{0,1}(x-\zeta)H_{-1,0}(x)^2\right]\left(\mathcal{G}(x-\zeta)^2+\mathcal{G}(-x)^2\right)e^{-2\sqrt{2}\zeta}\\
&\quad-2\left[360H_{0,1}(x-\zeta)^2H_{-1,0}(x)+360H_{0,1}(x-\zeta)H_{-1,0}(x)^2\right]\mathcal{G}(x-\zeta)\mathcal{G}(-x)e^{-2\sqrt{2}\zeta}.
\end{aligned} \tag{3.101}$$

Moreover, since $U^{(3)}(\phi)=-48\phi+120\phi^3$ is an odd polynomial and $H_{-1,0}(x)=-H_{0,1}(-x)$, the right-hand side of (3.101) is a finite sum of functions

$$\mathcal{G}(x-\zeta)^{l_1}\mathcal{G}(-x)^{l_2}H_{0,1}^{\zeta}(x)^{l_3}\,H_{0,1}(-x)^{l_4}-\mathcal{G}(x-\zeta)^{l_2}\mathcal{G}(-x)^{l_1}H_{0,1}^{\zeta}(x)^{l_4}\,H_{0,1}(-x)^{l_3},$$

such that $l_1,\,l_2,\,l_3,\,l_4\in\mathbb{N}\cup\{0\}$, $l_1+l_2=2$, $\sum_{i=1}^4 l_i$ is odd and $\min\left(l_1+l_3,l_2+l_4\right)>0$. Therefore, using Lemma 3.2.7 and Remark 3.2.10, we deduce that (3.101) is a finite sum of functions

$$\mathcal{J}_i\left(x-\zeta\right)\mathcal{N}_i\left(x\right)-\mathcal{J}_i(-x)\mathcal{N}_i\left(-x+\zeta\right),$$

where $\mathcal{J}_i\in S^+\cup S^+_\infty$ and $\mathcal{N}_i\in S^-\cup S^-_\infty$. In conclusion, we obtain the statement of Lemma 3.4.11 from the Proposition 3.2.16 and Remarks 3.2.20, 3.2.21 applied in the right-hand side of (3.101). □

Now, we can start the estimate of $\Lambda(\varphi_{2,0})(t,x)$. First, from the definition of $\varphi_{2,0}(t,x)$ in (3.76), we have that

$$\begin{aligned}
\Lambda\left(\varphi_{2,0}\right)(t,x)&=\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\left(H^{w_0}_{0,1}(t,x)+e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right)+U'\left(H^{w_0}_{0,1}(t,x)+e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right)\\
&=\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\left[e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right]+\Lambda\left(H_{0,1}\left(w_0(t,x)\right)\right)-\Lambda\left(H_{0,1}\left(w_0(t,-x)\right)\right)\\
&\quad+U'\left(H^{w_0}_{0,1}(t,x)+e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right)-U'\left(H_{0,1}\left(w_0(t,x)\right)\right)-U'\left(-H_{0,1}\left(w_0(t,-x)\right)\right).
\end{aligned}$$

Therefore, using Taylor's Expansion Theorem, we deduce that

$$\begin{aligned}
&\Lambda(\varphi_{2,0})(t,x)-\Lambda(H_{0,1}\left(w_0(t,x)\right))+\Lambda\left(H_{0,1}\left(w_0(t,-x)\right)\right)\\
&\qquad=\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\left[e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right]\\
&\qquad\quad+U'\left(H^{w_0}_{0,1}(t,x)\right)-U'\left(H_{0,1}\left(w_0(t,x)\right)\right)-U'\left(-H_{0,1}\left(w_0(t,-x)\right)\right)\\
&\qquad\quad+\sum_{j=2}^{6}\frac{U^{(j)}\left(H^{w_0}_{0,1}(t,x)\right)}{(j-1)!}\left[e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}\left(t,x\right)\right]^{j-1}.
\end{aligned} \tag{3.102}$$

Consequently, we deduce using Lemma 3.4.2 that

$$\Lambda(\varphi_{2,0})(t,x) = -\frac{8\sqrt{2}e^{-\sqrt{2}d_v(t)}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\left(H_{0,1}^{'}\right)^{w_0}(t,x) + R_1\left(t,w_0(t,x)\right) - R_1\left(t,w_0(t,-x)\right)$$

$$+e^{-\sqrt{2}d_v(t)}\left[U^{(2)}\left(H_{0,1}^{w_0}(t,x)\right)\mathcal{G}^{w_0}(t,x) - \left(U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right)^{w_0}(t,x)\right] \tag{3.103}$$

$$+\sum_{j=4}^{6}\frac{U^{(j)}\left(H_{0,1}^{w_0}(t,x)\right)}{(j-1)!}\left[e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right]^{j-1} \tag{3.104}$$

$$+U^{'}\left(H_{0,1}^{w_0}(t,x)\right) - U^{'}\left(H_{0,1}\left(w_0(t,x)\right)\right) - U^{'}\left(-H_{0,1}\left(w_0(t,-x)\right)\right) \tag{3.105}$$

$$+\frac{U^{(3)}\left(H_{0,1}^{w_0}(t,x)\right)}{2}\left[e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right]^2 \tag{3.106}$$

$$+\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\left[e^{-\sqrt{2}d_v(t)}\mathcal{G}^{w_0}(t,x)\right] + e^{-\sqrt{2}d_v(t)}\left[U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right]^{w_0}(t,x). \tag{3.107}$$

Next, from Remark 3.4.8, we have that the expression (3.103) is equivalent to

$$e^{-\sqrt{2}d_v(t)}\left[\mathcal{Q}\left(-w_0(t,-x),d_v(t)\right) - \mathcal{Q}\left(-w_0(t,x),d_v(t)\right)\right].$$

Moreover, Remark 3.4.10 implies that the term (3.104) is negligible.

Additionally, using Remark 3.4.6, we obtain that the expression (3.105) is equivalent to

$$24\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)M^{w_0}(t,x) - 30\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)N^{w_0}(t,x)$$
$$+24\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)V^{w_0}(t,x) + \frac{60}{\sqrt{2}}\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\left(H_{0,1}^{'}\right)^{w_0}(t,x).$$

Finally, Remark 3.4.12 implies that the term (3.106) is equivalent to

$$\frac{e^{-2\sqrt{2}d_v(t)}}{2}\left[U^{(3)}\left(H_{0,1}\right)\mathcal{G}^2\right]^{w_0}(t,x),$$

and Lemma 3.4.3 implies the equivalence between the expression (3.107) with

$$-e^{-\sqrt{2}d_v(t)}\left[24M^{w_0}(t,x) - 30N^{w_0}(t,x) - 8\sqrt{2}\left(H_{0,1}^{'}\right)^{w_0}(t,x)\right] + R_{2,v}(t,w_0(t,x)) - R_{2,v}(t,w_0(t,-x)).$$

Consequently, we have the following estimate

$$\Lambda\left(\varphi_{2,0}\right)(t,x) \cong_6 -\frac{8\sqrt{2}e^{-\sqrt{2}d_v(t)}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\left(H_{0,1}^{'}\right)^{w_0}(t,x) + R_{1,v}\left(t,w_0(t,x)\right) - R_1\left(t,w_0(t,-x)\right)$$
$$+e^{-\sqrt{2}d_v(t)}\left[\mathcal{Q}\left(-w_0(t,-x),d_v(t)\right) - \mathcal{Q}\left(-w_0(t,x),d_v(t)\right)\right]$$
$$+24\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)M^{w_0}(t,x) - 30\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)N^{w_0}(t,x)$$
$$+24\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)V^{w_0}(t,x) + \frac{60}{\sqrt{2}}\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\left(H_{0,1}^{'}\right)^{w_0}(t,x)$$
$$-e^{-\sqrt{2}d_v(t)}\left[24M^{w_0}(t,x) - 30N^{w_0}(t,x) - 8\sqrt{2}\left(H_{0,1}^{'}\right)^{w_0}(t,x)\right]$$
$$+R_{2,v}(t,w_0(t,x)) - R_{2,v}(t,w_0(t,-x))$$
$$+\frac{e^{-2\sqrt{2}d_v(t)}}{2}\left[U^{(3)}\left(H_{0,1}\right)\mathcal{G}^2\right]^{w_0}(t,x).$$

Furthermore, using Lemma 3.3.5 the following result, we deduce the following estimate

$$\left|\frac{d^l}{dt^l}\left[\exp\left(\frac{-2\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-e^{-2\sqrt{2}d_v(t)}\right]\right|\lesssim_l v^{6+l}\left[|t|v+\ln\left(\frac{1}{v}\right)\right]e^{-2\sqrt{2}|t|v},$$

for any $l\in\mathbb{N}\cup\{0\}$ and $t\in\mathbb{R}$, if $0<v\ll 1$. In conclusion, from Lemma 3.3.2, Remark 3.4.8 and the estimate above of $\Lambda(\varphi_{2,0})$, we deduce the following result:

**Lemma 3.4.13.** *The function $\varphi_{2,0}(t,x)$ satisfies if $1<v\ll 1$, for all $l\in\mathbb{N}\cup\{0\}$ and $s\geq 0$,*

$$\left\|\frac{\partial^l}{\partial t^l}\Lambda(\varphi_{2,0})(t,x)\right\|_{H^s_x}\lesssim_{l,s} v^{4+l}\left[|t|v+\ln\left(\frac{1}{v^2}\right)\right]e^{-2\sqrt{2}|t|v}.$$

*Furthermore, we have that*

$$\Lambda\left(\varphi_{2,0}\right)(t,x)\cong_6 Sym\left(t,w_0(t,x)\right)-Sym\left(t,w_0(t,-x)\right),$$

*where, for $0<v\ll 1$, the function $Sym:\mathbb{R}^2\to\mathbb{R}$ satisfies, for all $(t,x)\in\mathbb{R}$, the following identity*

$$\begin{aligned}Sym(t,x)=&8\sqrt{2}H^{'}_{0,1}(x)\left[e^{-\sqrt{2}d_v(t)}-\frac{e^{-\sqrt{2}d_v(t)}}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right]+\frac{1}{2}U^{(3)}\left(H_{0,1}(x)\right)\mathcal{G}(x)^2e^{-2\sqrt{2}d_v(t)}\\&+\left[-24H_{0,1}(x)^2+30H_{0,1}(x)^4\right]e^{-\sqrt{2}x}\left[e^{-\sqrt{2}d_v(t)}-\exp\left(\frac{-\sqrt{2}d_v(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right]\\&+R_{1,v}(t,x)+R_{2,v}(t,x)\\&+e^{-2\sqrt{2}d_v(t)}\left[xA(x)+xB(x)-d(t)B(x)+C(x)-D(x)+24V(x)+\frac{60}{\sqrt{2}}H^{'}_{0,1}(x)\right].\end{aligned}$$

Now, we can start the demonstration of Theorem 3.4.1.

### 3.4.3 Proof of Theorem 3.4.1.

*Proof of Theorem 3.4.1.* **Step 1.**(Construction of $r_v(t)$ for $k=2$.)

First, we recall $R_{1,v}(t,x)$, $R_{2,v}(t,x)$ defined, respectively, in equation (3.81) of Lemma 3.4.2 and in equation (3.89) of Lemma 3.4.3. To lighten more our notation, we denote $R_{1,v}$, $R_{2,v}$, $d_v(t)$ by $R_1$, $R_2$, $d(t)$ from now on. Also, we recall the functions $M(x)$, $N(x)$ and $V(x)$ from (3.95) and the functions $A$, $B$, $C$, $D$ from Lemma 3.4.7. Next, based on Lemma 3.4.13, we consider the following ordinary differential equation

$$\begin{cases}\left\|H^{'}_{0,1}\right\|^2_{L^2_x}\ddot{r}(t)=-32e^{-\sqrt{2}d(t)}\left\|H^{'}_{0,1}\right\|^2_{L^2_x}r(t)-\left\langle H^{'}_{0,1}(x),\,Sym(t,x)\right\rangle,\\ r(t)=r(-t).\end{cases}\tag{3.108}$$

Indeed, from the definition of $Sym$ in the statement of Lemma 3.4.13, identities (3.78) and $\left\|H^{'}_{0,1}\right\|^2_{L^2_x}=\frac{1}{2\sqrt{2}}$, the ordinary differential equation (3.108) can be rewritten for fixed constants

$c_1,\, c_2 \in \mathbb{R}$ as

$$\begin{cases} \left\Vert H_{0,1}^{'}\right\Vert_{L^2_x}^2 \ddot{r}(t) = -32e^{-\sqrt{2}d(t)}\left\Vert H_{0,1}^{'}\right\Vert_{L^2_x}^2 r(t) - \left\langle H_{0,1}^{'}(x),\, R_1(t,x)+R_2(t,x)\right\rangle + c_1 d(t)e^{-2\sqrt{2}d(t)} \\ \qquad\qquad + c_2 e^{-2\sqrt{2}d(t)} + 4\left[\frac{e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} - \exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right], \\ r(t) = r(-t). \end{cases} \tag{3.109}$$

Since $d(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right)$, we have that all the solutions of the linear ordinary differential equation $\ddot{r}_0(t) = -32e^{-\sqrt{2}d(t)}r_0(t)$ are a linear combination of

$$sol_1(t) = \tanh\left(\sqrt{2}vt\right) \text{ and } sol_2(t) = \sqrt{2}vt\tanh\left(\sqrt{2}vt\right) - 1.$$

From Lemma 3.3.1, we obtain if $0 < v \ll 1$, and $l \in \mathbb{N}\cup\{0\}$,

$$\left|\frac{d^l}{dt^l}d(t)e^{-2\sqrt{2}d(t)}\right| \lesssim_l v^{4+l}\left(v|t| + \ln\left(\frac{8}{v^2}\right)\right)e^{-4\sqrt{2}|t|v}. \tag{3.110}$$

Next, to simplify more our notation, we denote

$$NL(t) = -\left\langle H_{0,1}^{'}(x),\, R_1(t,x)+R_2(t,x)\right\rangle + c_1 d(t)e^{-2\sqrt{2}d(t)} + c_2 e^{-2\sqrt{2}d(t)} \\ -4\left[\exp\left(\frac{-\sqrt{2}d(t)}{\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}}\right) - \frac{e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right]. \tag{3.111}$$

Using the variation of parameters technique, we can write any $C^2$ solution $r(t)$ of (3.109) as $r(t) = \theta_1(t)sol_1(t) + \theta_2(t)sol_2(t)$ such that $\theta_1(t)$ and $\theta_2(t)$ satisfy for any $t \in \mathbb{R}$

$$\begin{bmatrix} sol_1(t) & sol_2(t) \\ \dot{sol}_1(t) & \dot{sol}_2(t) \end{bmatrix}\begin{bmatrix}\dot{\theta}_1(t) \\ \dot{\theta}_2(t)\end{bmatrix} = \frac{1}{\left\Vert H_{0,1}^{'}\right\Vert_{L^2_x}^2}\begin{bmatrix} 0 \\ NL(t)\end{bmatrix} = \begin{bmatrix} 0 \\ 2\sqrt{2}NL(t)\end{bmatrix}.$$

In conclusion, since for all $t \in \mathbb{R}$

$$\det\begin{bmatrix} sol_1(t) & sol_2(t) \\ \dot{sol}_1(t) & \dot{sol}_2(t) \end{bmatrix} = \sqrt{2}v,$$

we have

$$\dot{\theta}_2(t) = \frac{2}{v}NL(t)\tanh\left(\sqrt{2}vt\right),\, \dot{\theta}_1(t) = \frac{-2}{v}NL(t)\left[\sqrt{2}vt\tanh\left(\sqrt{2}vt\right) - 1\right]. \tag{3.112}$$

From Lemmas 3.4.2 and 3.4.3, we have that $R_1(t,x)$ and $R_2(t,x)$ are even in $t$, so $NL(t)$ is also even. Since we are interested in an even solution $r(t)$ of (3.109), we need $\theta_1$ odd and $\theta_2$ even, so we must choose

$$\theta_2(t) = \frac{1}{\sqrt{2}v}\int_{-\infty}^t NL(s)\tanh\left(\sqrt{2}vs\right)ds,\, \theta_1(t) = \frac{-1}{\sqrt{2}v}\int_0^t NL(s)\left[\sqrt{2}vs\tanh\left(\sqrt{2}vs\right) - 1\right]ds. \tag{3.113}$$

From Lemmas 3.4.2 and 3.4.3, we deduce for any $j \in \{1,2\}$ that if $0 < v \ll 1$, then

$$\left|\frac{d^l}{dt^l}\left\langle R_j(t,x),\, H^{'}_{0,1}(x)\right\rangle\right| \lesssim_l v^{4+l}\operatorname{sech}\left(\sqrt{2}vt\right)^2 \text{ for all } l \in \mathbb{N}\cup\{0\}, \tag{3.114}$$

and so, from the equations (3.110),(3.111) and Lemma 3.3.5, we deduce for all $0 < v \ll 1$ and any $l \in \mathbb{N}\cup\{0\}$ that

$$\left|\frac{d^l}{dt^l}NL(t)\right| \lesssim_l v^{4+l}\left(v|t| + \ln\left(\frac{1}{v}\right)\right)e^{-2\sqrt{2}|t|v}. \tag{3.115}$$

Therefore, from the definition of $d(t)$, the identities (3.111), (3.112) and the estimates (3.114), (3.115), using the Fundamental Theorem of Calculus, we deduce the existence of a constant $C > 0$ such that if $0 < v \ll 1$, then

$$\|\theta_1\|_{L^\infty(\mathbb{R})} < Cv^2\ln\left(\frac{1}{v^2}\right). \tag{3.116}$$

Furthermore, since $NL(t)$ is an even function and $\tanh\left(\sqrt{2}vs\right)$ is an odd function, we have that

$$\int_{-\infty}^{t} NL(s)\tanh\left(\sqrt{2}vs\right)ds = -\int_{t}^{+\infty} NL(s)\tanh\left(\sqrt{2}vs\right)ds,$$

from which with identity (3.113), we deduce the following estimate

$$|\theta_2(t)| \leq \frac{1}{\sqrt{2}v}\int_{|t|}^{+\infty}|NL(s)|\tanh\left(\sqrt{2}vs\right)ds, \text{ for all } t \in \mathbb{R}. \tag{3.117}$$

Therefore, the estimate (3.115) implies that

$$|\theta_2(t)| \lesssim v^2\left[\ln\left(\frac{8}{v^2}\right) + v|t|\right]e^{-2\sqrt{2}|t|v}, \text{ for any } t \in \mathbb{R}.$$

Finally, since $r(t) = \theta_1(t)sol_1(t) + \theta_2(t)sol_2(t)$ and $\dot{r}(t) = \theta_1(t)\dot{sol}_1(t) + \theta_2(t)\dot{sol}_2(t)$, we deduce for all $t \in \mathbb{R}$ that

$$|r(t)| \lesssim v^2\ln\left(\frac{1}{v^2}\right),\ |\dot{r}(t)| \lesssim v^3\left[\ln\left(\frac{1}{v^2}\right) + |t|v\right]\operatorname{sech}\left(\sqrt{2}vt\right)^2. \tag{3.118}$$

Moreover, (3.115) and the definitions of $sol_1$ and $sol_2$, we can verify by induction on $l \in \mathbb{N}$ for any $0 < v \ll 1$ that

$$\left|\frac{d^l r}{dt^l}(t)\right| \lesssim_l v^{l+2}\left[\ln\left(\frac{1}{v^2}\right) + |t|v\right]\operatorname{sech}\left(\sqrt{2}vt\right)^2 \text{ for all integers } l \geq 1 \text{ and } t \in \mathbb{R}. \tag{3.119}$$

**Step 2.**(Estimate of $\Lambda(\varphi_2)(t,x)$.) From now on, we define the function $w_1 : \mathbb{R}^2 \to \mathbb{R}$ as the unique function satisfying

$$w_1(t,x) = w_0(t, x + r_v(t)) = \frac{x - \frac{d_v(t)}{2} + r_v(t)}{\sqrt{1 - \frac{\dot{d}_v(t)^2}{4}}},$$

for every $(t,x)\in\mathbb{R}^2$. Furthermore, similarly to the identity (3.102), we have the following equation

$$\Lambda(\varphi_2)(t,x) = \Lambda(H_{0,1}(w_1(t,x))) - \Lambda(H_{0,1}(w_1(t,-x))) \tag{3.120}$$
$$+\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]\left[e^{-\sqrt{2}d(t)}\mathcal{G}^{w_1}(t,x)\right]+\left[U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right]^{w_1}(t,x)e^{-\sqrt{2}d(t)} \tag{3.121}$$
$$+U^{(2)}\left(H_{0,1}^{w_1}(t,x)\right)\mathcal{G}^{w_1}(t,x)e^{-\sqrt{2}d(t)}-\left[U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right]^{w_1}(t,x)e^{-\sqrt{2}d(t)} \tag{3.122}$$
$$+U^{'}\left(H_{0,1}^{w_1}(t,x)\right)-U^{'}\left(H_{0,1}(w_1(t,x))\right)-U^{'}\left(-H_{0,1}(w_1(t,-x))\right) \tag{3.123}$$
$$+\frac{U^{(3)}\left(H_{0,1}^{w_1}(t,x)\right)}{2}\left[e^{-\sqrt{2}d(t)}\mathcal{G}^{w_1}(t,x)\right]^2 \tag{3.124}$$
$$+\sum_{j=4}^{6}\frac{U^{(j)}\left(H_{0,1}^{w_1}(t,x)\right)}{(j-1)!}\left[e^{-\sqrt{2}d(t)}\mathcal{G}^{w_1}(t,x)\right]^{(j-1)}. \tag{3.125}$$

From identity $\left\Vert H_{0,1}^{'}\right\Vert_{L^2_x}^2=\frac{1}{2\sqrt{2}}$, the definitions of $M(x)$, $N(x)$ in (3.95) and identity (3.78), we have

$$\left\langle\left[24M(w_0(t,x))-30N(w_0(t,x))\right],H_{0,1}^{'}(w_0(t,x))\right\rangle=4\sqrt{1-\frac{\dot{d}(t)^2}{4}}.$$

Therefore, we deduce the following identity

$$\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}})}\right)\left\langle 24M(w_0(t,x))-30N(w_0(t,x)),H_{0,1}^{'}(w_0(t,x))\right\rangle-4e^{-\sqrt{2}d(t)}$$
$$=4\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\sqrt{1-\frac{\dot{d}(t)^2}{4}}-4e^{-\sqrt{2}d(t)}$$
$$=4\left[\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)-e^{-\sqrt{2}(d(t)-2r(t))}\right]\sqrt{1-\frac{\dot{d}(t)^2}{4}} \tag{3.126}$$
$$+4e^{-\sqrt{2}(d(t)-2r(t))}\left[\sqrt{1-\frac{\dot{d}(t)^2}{4}}-1\right] \tag{3.127}$$
$$+4\left[e^{-\sqrt{2}(d(t)-2r(t))}-e^{-\sqrt{2}d(t)}-2\sqrt{2}e^{-\sqrt{2}d(t)}r(t)\right] \tag{3.128}$$
$$+8\sqrt{2}e^{-\sqrt{2}d(t)}r(t).$$

Since $e^{-\sqrt{2}d(t)}=\frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt\right)^2$, using estimates (3.118) and (3.119) of the function $r$, we deduce from an application of Lemma 3.3.5 in the expressions (3.126), (3.127) and from an application of Taylor's Expansion Theorem in the term (3.128) that the following function

$$Rem(t)=\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\left\langle 24M(w_0(t,x))-30N(w_0(t,x)),\,H_{0,1}^{'}(w_0(t,x))\right\rangle$$
$$-4e^{-\sqrt{2}d(t)}-8\sqrt{2}e^{-\sqrt{2}d(t)}r(t)$$

satisfies $\left|\frac{d^l Rem(t)}{dt^l}\right|\lesssim_l v^{l+4}\left[|t|\,v+\ln\left(\frac{8}{v^2}\right)\right]e^{-2\sqrt{2}v|t|}$ for all $t\in\mathbb{R}$ and any $l\in\mathbb{N}\cup\{0\}$.

**Substep 2.1.**(Estimate of $\Lambda\left(H_{0,1}(w_1(t,x))\right)$.) From now on, we use the following notation

$$\varphi_2(t,x)=H_{0,1}^{w_1}(t,x)+e^{-\sqrt{2}d(t)}\mathcal{G}^{w_1}(t,x),\text{ for all }(t,x)\in\mathbb{R}^2.$$

First, for all $(t,x)\in\mathbb{R}^2$, the following identity

$$\begin{aligned}\frac{\partial^2}{\partial t^2}H_{0,1}(w_1(t,x)) =&\frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}H_{0,1}(w_0(t_1,x+r(t)))+\frac{\ddot r(t)}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{'}(w_1(t,x))\\ &-\frac{\dot d(t)\dot r(t)}{1-\frac{\dot d(t)^2}{4}}H_{0,1}^{''}(w_1(t,x))+\frac{8\sqrt{2}\dot d(t)\dot r(t)e^{-\sqrt{2}d(t)}}{\left(1-\frac{\dot d(t)^2}{4}\right)^{\frac{3}{2}}}w_1(t,x)H_{0,1}^{'}(w_1(t,x))\\ &+\frac{8\sqrt{2}\dot r(t)\dot d(t)e^{-\sqrt{2}d(t)}}{\left(1-\frac{\dot d(t)^2}{4}\right)^{\frac{3}{2}}}w_1(t,x)H_{0,1}^{''}(w_1(t,x))\end{aligned}$$

implies with the product rule, estimates (3.69), (3.118), (3.119), Lemmas 3.3.1 and Remark 3.3.3 that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}H_{0,1}(w_1(t,x))\cong_6\frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}H_{0,1}(w_0(t_1,x+r(t)))+\frac{\ddot r(t)}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{'}(w_1(t,x))\\ -\frac{\dot d(t)\dot r(t)}{1-\frac{\dot d(t)^2}{4}}H_{0,1}^{''}(w_1(t,x)).\end{aligned}$$

Therefore, from Lemma 3.3.4, we deduce from the estimate above and the decay estimates (3.118), (3.119) of $r$ that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}H_{0,1}(w_1(t,x))\cong_6\frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}H_{0,1}(w_0(t_1,x+r(t)))+\frac{\ddot r(t)}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{'}(w_0(t,x))\\ -\frac{\dot d(t)\dot r(t)}{1-\frac{\dot d(t)^2}{4}}H_{0,1}^{''}(w_0(t,x)).\end{aligned}\tag{3.129}$$

Moreover, Lemma 3.4.2 implies that

$$\begin{aligned}\frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}H_{0,1}(w_0(t_1,x+r(t)))-\frac{\partial^2}{\partial x^2}\left[H_{0,1}\left(w_1(t,x)\right)\right]+U^{'}\left(H_{0,1}\left(w_1(t,x)\right)\right)\\ =-\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{(1-\frac{\dot d(t)^2}{4})^{\frac{1}{2}}}H_{0,1}^{'}\left(w_1(t,x)\right)+R_1\left(t,w_1(t,x)\right),\end{aligned}$$

from which with Lemma 3.3.4 and estimates (3.118), (3.119), we obtain the following estimate

$$\begin{aligned}&\frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}H_{0,1}(w_0(t_1,x+r(t)))-\frac{\partial^2}{\partial x^2}\left[H_{0,1}\left(w_1(t,x)\right)\right]+U^{'}\left(H_{0,1}\left(w_1(t,x)\right)\right)\\ &\cong_6-\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{'}\left(w_0(t,x)\right)-\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{''}\left(w_0(t,x)\right)+R_1\left(t,w_0(t,x)\right).\end{aligned}\tag{3.130}$$

Therefore, we obtain using estimates (3.129) and (3.130) that

$$\begin{aligned}\Lambda\left(H_{0,1}\left(w_1(t,x)\right)\right)\cong_6&-\frac{8\sqrt{2}e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{'}\left(w_0(t,x)\right)-\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{''}\left(w_0(t,x)\right)\\ &+\frac{\ddot r(t)}{\sqrt{1-\frac{\dot d(t)^2}{4}}}H_{0,1}^{'}\left(w_0(t,x)\right)-\frac{\dot d(t)\dot r(t)}{1-\frac{\dot d(t)^2}{4}}H_{0,1}^{''}\left(w_0(t,x)\right)+R_1(t,w_0(t,x)).\end{aligned}$$

Consequently, using Lemma 3.4.2, we deduce the following estimate

$$\begin{aligned}\Lambda\left(H_{0,1}\left(w_1(t,x)\right)\right)-\Lambda\left(H_{0,1}\left(w_1(t,-x)\right)\right)\cong_6 &\Lambda\left(H_{0,1}\left(w_0(t,x)\right)\right)-\Lambda\left(H_{0,1}\left(w_0(t,-x)\right)\right)\\ &-\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H_{0,1}^{''}\right)^{w_0}(t,x)\\ &+\frac{\ddot{r}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H_{0,1}^{'}\right)^{w_0}(t,x)\\ &-\frac{\dot{d}(t)\dot{r}(t)}{1-\frac{\dot{d}(t)^2}{4}}\left(H_{0,1}^{''}\right)^{w_0}(t,x).\end{aligned}\tag{3.131}$$

**Substep 2.2.**(Estimate of (3.121).) Next, from Lemmas 3.3.1, 3.3.4, we deduce with estimates (3.118), (3.119) and the product rule that

$$\frac{\partial^2}{\partial t^2}\left[e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_1(t,x)\right)\right]\cong_6\frac{\partial^2}{\partial t^2}\left[e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_0(t,x)\right)\right]\cong_6\left.\frac{\partial^2}{\partial t_1^2}\right|_{t_1=t}\left[e^{-\sqrt{2}d(t_1)}\mathcal{G}(w_0(t_1,x+r(t)))\right].$$

Therefore, we deduce from Lemma 3.4.3 the following estimate

$$\begin{aligned}&\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}(w_1(t,x))\right)\right]\left(e^{-\sqrt{2}d(t)}\mathcal{G}(w_1(t,x))\right)\\ &\quad\cong_6-\Big[24M\left(w_1(t,x)\right)-30N\left(w_1(t,x)\right)\Big]e^{-\sqrt{2}d(t)}+8\sqrt{2}H_{0,1}^{'}\left(w_1(t,x)\right)e^{-\sqrt{2}d(t)}\\ &\qquad+R_2(t,w_1(t,x)),\end{aligned}$$

from which with Lemma 3.3.4 and the decay estimates (3.118), (3.119) of $r$, we deduce that

$$\begin{aligned}&\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}(w_1(t,x))\right)\right]\left(e^{-\sqrt{2}d(t)}\mathcal{G}(w_1(t,x))\right)\\ &\quad\cong_6-\Big[24M\left(w_0(t,x)\right)-30N\left(w_0(t,x)\right)\Big]e^{-\sqrt{2}d(t)}+8\sqrt{2}H_{0,1}^{'}\left(w_0(t,x)\right)e^{-\sqrt{2}d(t)}\\ &\qquad-\Big[24M^{'}\left(w_0(t,x)\right)-30N^{'}\left(w_0(t,x)\right)\Big]\frac{r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}+\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H_{0,1}^{''}\left(w_0(t,x)\right)\\ &\qquad+R_2(t,w_0(t,x)).\end{aligned}$$

Hence, using Lemma 3.4.3, we obtain the following estimate

$$\begin{aligned}&\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}(w_1(t,x))\right)\right]\left(e^{-\sqrt{2}d(t)}\mathcal{G}(w_1(t,x))\right)\\ &\qquad\cong_6\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}(w_0(t,x))\right)\right]\left(e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_0(t,x)\right)\right)\\ &\qquad\quad-\Big[24M^{'}\left(w_0(t,x)\right)-30N^{'}\left(w_0(t,x)\right)\Big]\frac{r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\\ &\qquad\quad+\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H_{0,1}^{''}\left(w_0(t,x)\right).\end{aligned}\tag{3.132}$$

**Substep 2.3.**(Estimate of (3.123).) In notation of Lemma 3.4.5, we have the following identity

$$
\begin{aligned}
U'\left(H_{0,1}^{w_1}(t,x)\right) - U'\left(H_{0,1}(w_1(t,x))\right) - U'\left(-H_{0,1}(w_1(t,-x))\right)\\
= \exp\left(-\frac{\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\left[24M^{w_1}(t,x) - 30N^{w_1}(t,x)\right]\\
+\exp\left(-\frac{2\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4})}}\right)\left[24V^{w_1}(t.x) + \frac{60}{\sqrt{2}}\left(H_{0,1}'\right)^{w_1}(t,x)\right] \quad (3.133)\\
+R\left(-w_1(t,x), \frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right).
\end{aligned}
$$

Moreover, similarly to the proof of Remark 3.4.6, Lemmas 3.3.4 and 3.4.5 imply that

$$R\left(-w_1(t,x), \frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 0.$$

Therefore, identity (3.133) and Lemmas 3.3.4, 3.3.5 imply the following estimate

$$
\begin{aligned}
U'\left(H_{0,1}^{w_1}(t,x)\right) - U'\left(H_{0,1}(w_1(t,x))\right) - U'\left(-H_{0,1}(w_1(t,-x))\right)\\
\cong_6 \exp\left(-\frac{\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\left[24M^{w_1}(t,x) - 30N^{w_1}(t,x)\right]\\
+e^{-2\sqrt{2}d(t)}\left[24V^{w_0}(t,x) + \frac{60}{\sqrt{2}}\left(H_{0,1}'\right)^{w_0}(t,x)\right]. \quad (3.134)
\end{aligned}
$$

Next, using the decay estimates (3.118), (3.119) of $r$, we deduce from Lemma 3.3.4 that

$$\left[M^{w_1}(t,x) - M^{w_0}(t,x) - \frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(M'\right)^{w_0}(t,x)\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 0, \quad (3.135)$$

$$\left[N^{w_1}(t,x) - N^{w_0}(t,x) - \frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(N'\right)^{w_0}(t,x)\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 0. \quad (3.136)$$

We also deduce from Taylor's Expansion Theorem and the decay estimates (3.118), (3.119) of the function $r$ that

$$
\begin{aligned}
M^{w_1}(t,x)\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 M^{w_1}(t,x)\left[1+\frac{2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
N^{w_1}(t,x)\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 N^{w_1}(t,x)\left[1+\frac{2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right),
\end{aligned}
$$

therefore, using now Lemma 3.3.4, we conclude the following estimates

$$\begin{aligned}
M^{w_1}(t,x)\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 & M^{w_1}(t,x)\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
&+M^{w_0}(t,x)\frac{2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right),\\
N^{w_1}(t,x)\exp\left(\frac{-\sqrt{2}(d(t)-2r(t))}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) \cong_6 & N^{w_1}(t,x)\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
&+N^{w_0}(t,x)\frac{2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right).
\end{aligned}$$

As a consequence, we obtain from estimate (3.133) and Lemma 3.3.5 that

$$\begin{aligned}
&U'\left(H^{w_1}_{0,1}(t,x)\right)-U'\left(H_{0,1}\left(w_1(t,x)\right)\right)-U'\left(-H_{0,1}\left(w_1(t,-x)\right)\right)\\
&\quad\cong_6\left[24M^{w_0}(t,x)-30N^{w_0}(t,x)\right]\frac{2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
&\qquad+\left[24M^{w_0}(t,x)-30N^{w_0}(t,x)\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
&\qquad+\frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[24\left(M'\right)^{w_0}(t,x)-30\left(N'\right)^{w_0}(t,x)\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
&\qquad+\left[24V^{w_0}(t,x)+\frac{60}{\sqrt{2}}\left(H_{0,1}'\right)^{w_0}(t,x)\right]e^{-2\sqrt{2}d(t)}\\
&\quad\cong_6 U'\left(H^{w_0}_{0,1}(t,x)\right)-U'\left(H_{0,1}\left(w_0(t,x)\right)\right)-U'\left(-H_{0,1}\left(w_0(t,-x)\right)\right)\\
&\qquad+\frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[24\left(M'\right)^{w_0}(t,x)-30\left(N'\right)^{w_0}(t,x)\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\
&\qquad+\frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[48M^{w_0}(t,x)-60N^{w_0}(t,x)\right]\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right).
\end{aligned}$$

Therefore, using Remark 3.4.6, we conclude that

$$\begin{aligned}
&U'\left(H^{w_1}_{0,1}(t,x)\right)-U'\left(H_{0,1}\left(w_1(t,x)\right)\right)-U'\left(-H_{0,1}\left(w_1(t,-x)\right)\right)\\
&\quad\cong_6 U'\left(H^{w_0}_{0,1}(t,x)\right)-U'\left(H_{0,1}\left(w_0(t,x)\right)\right)-U'\left(-H_{0,1}\left(w_0(t,-x)\right)\right)\\
&\qquad+\frac{r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[24\left(M'\right)^{w_0}(t,x)-30\left(N'\right)^{w_0}(t,x)\right]\\
&\qquad+\frac{r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[48M^{w_0}(t,x)-60N^{w_0}(t,x)\right].
\end{aligned}\tag{3.137}$$

**Substep 2.4.**(Estimate of (3.122).)

Now, using identities (3.99) and (3.100), Lemma 3.4.7 also implies the following equation

$$\begin{aligned}&\left[U^{(2)}\left(H^{w_1}_{0,1}(t,x)\right)-U^{(2)}\left(H_{0,1}(w_1(t,x))\right)\right]e^{-\sqrt{2}d(t)}\mathcal{G}(w_1(x,t))\\&\quad=\mathcal{Q}\left(-w_1(t,-x),\frac{d_v(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)e^{-\sqrt{2}d(t)}+R_q\left(-w_1(t,-x),\frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)e^{-\sqrt{2}d(t)}.\end{aligned}\tag{3.138}$$

Furthermore, from Lemma 3.3.4 and the definition of $\mathcal{Q}$ in (3.99), we deduce that

$$\mathcal{Q}\left(-w_1(t,-x),\frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)e^{-\sqrt{2}d(t)}\cong_6\mathcal{Q}\left(-w_0(t,-x),\frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)e^{-\sqrt{2}d(t)},$$

from which with Lemmas 3.3.1, 3.3.5 and identity (3.99), we obtain that

$$\mathcal{Q}\left(-w_1(t,-x),\frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)e^{-\sqrt{2}d(t)}\cong_6\mathcal{Q}\left(-w_0(t,-x),d(t)\right)e^{-\sqrt{2}d(t)}.\tag{3.139}$$

Using identity (3.100) and Remark 3.4.8, we can deduce similarly to the proof of estimate (3.139) that

$$R_q\left(-w_1(t,-x),\frac{d(t)-2r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)e^{-\sqrt{2}d(t)}\cong_6 R_q\left(-w_0(t,-x),d(t)\right)e^{-\sqrt{2}d(t)}\cong_6 0.$$

Consequently, in notation of Lemma 3.4.7, we have from identity (3.99) that

$$\begin{aligned}&\left[U^{(2)}\left(H_{0,1}(w_1(t,x))-H_{0,1}(w_1(t,-x))\right)-U^{(2)}\left(H_{0,1}(w_1(t,x))\right)\right]e^{-\sqrt{2}d(t)}\mathcal{G}(w_1(t,x))\\&\quad\cong_6 w_0(t,x)A(w_0(t,x))e^{-2\sqrt{2}d(t)}+w_0(t,x)B(w_0(t,-x))e^{-2\sqrt{2}d(t)}+C(w_0(t,x))e^{-2\sqrt{2}d(t)}\\&\qquad+D(w_0(t,-x))e^{-2\sqrt{2}d(t)},\end{aligned}$$

from which, using Remark 3.4.8, we deduce

$$\begin{aligned}&\left[U^{(2)}\left(H_{0,1}(w_1(t,x))-H_{0,1}(w_1(t,-x))\right)-U^{(2)}\left(H_{0,1}(w_1(t,x))\right)\right]e^{-\sqrt{2}d(t)}\mathcal{G}(w_1(t,x))\\&\quad\cong_6\left[U^{(2)}\left(H^{w_0}_{0,1}(t,x)\right)-U^{(2)}\left(H_{0,1}(w_0(t,x))\right)\right]e^{-\sqrt{2}d(t)}\mathcal{G}(w_0(t,x)).\end{aligned}$$

In conclusion, since $U^{(2)}$ is an even function, we have

$$\begin{aligned}&e^{-\sqrt{2}d_v(t)}U^{(2)}\left(H^{w_1}_{0,1}(t,x)\right)\mathcal{G}^{w_1}(t,x)-e^{-\sqrt{2}d_v(t)}\left[U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right]^{w_1}(t,x)\\&\quad\cong_6 e^{-\sqrt{2}d_v(t)}U^{(2)}\left(H^{w_0}_{0,1}(t,x)\right)\mathcal{G}^{w_0}(t,x)-e^{-\sqrt{2}d_v(t)}\left[U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right]^{w_0}(t,x).\end{aligned}\tag{3.140}$$

**Substep 2.5.**(Estimate of (3.124).) Next, using Lemma 3.3.4, we can verify that

$$\begin{aligned}&\frac{1}{2}U^{(3)}\left(H_{0,1}(w_1(t,x))-H_{0,1}(w_1(t,-x))\right)\left[\mathcal{G}(w_1(t,x))-\mathcal{G}(w_1(t,-x))\right]^2e^{-2\sqrt{2}d(t)}\\&\quad\cong_6\frac{1}{2}U^{(3)}\left(H_{0,1}(w_0(t,x))-H_{0,1}(w_0(t,-x))\right)\left[\mathcal{G}(w_0(t,x))-\mathcal{G}(w_0(t,-x))\right]^2e^{-2\sqrt{2}d(t)}.\end{aligned}$$

Therefore, from Remark 3.4.11, we obtain

$$\frac{1}{2}U^{(3)}\left(H_{0,1}^{w_1}(t,x)\right)\left[\mathcal{G}^{w_1}(t,x)\right]^2 e^{-2\sqrt{2}d(t)} \cong_6 \frac{1}{2}\left[U^{(3)}\left(H_{0,1}\right)\mathcal{G}^2\right]^{w_0}(t,x)e^{-2\sqrt{2}d(t)}. \tag{3.141}$$

**Substep 2.6.**(Estimate of (3.125).) Furthermore, similarly to the proof of estimate (3.141), we can verify that

$$\sum_{j=4}^{6}\frac{U^{(j)}\left(H_{0,1}^{w_1}(t,x)\right)}{(j-1)!}\left[\mathcal{G}^{w_1}(t,x)\right]^{j-1}e^{-\sqrt{2}d(t)(j-1)} \cong_6 \sum_{j=4}^{6}\frac{U^{(j)}\left(H_{0,1}^{w_0}(t,x)\right)}{(j-1)!}\left[\mathcal{G}^{w_0}(t,x)\right]^{j-1}e^{-\sqrt{2}d(t)(j-1)}.$$

Hence, we obtain using Remark 3.4.10 that

$$\sum_{j=4}^{6}\frac{U^{(j)}\left(H_{0,1}^{w_1}(t,x)\right)}{(j-1)!}\left[\mathcal{G}^{w_1}(t,x)\right]^{j-1}e^{-\sqrt{2}d(t)(j-1)} \cong_6 0. \tag{3.142}$$

**Substep 2.7.**(Conclusion of estimate of $\Lambda\left(\varphi_2\right)(t,x)$.) From identity (3.102), after we use the estimates (3.131), (3.132), (3.140), (3.137), (3.141), (3.142) respectively, in the terms (3.120), (3.121), (3.122), (3.123), (3.124), (3.125), we obtain

$$\begin{aligned}\Lambda(\varphi_2)(t,x)-\Lambda(\phi_{2,0})(t,x) \cong_6 &\frac{\ddot{r}(t)}{(1-\frac{\dot{d}(t)^2}{4})^{\frac{1}{2}}}\left(H_{0,1}^{'}\right)^{w_0}(t,x)-\frac{\dot{d}(t)\dot{r}(t)}{1-\frac{\dot{d}(t)^2}{4}}\left(H_{0,1}^{''}\right)^{w_0}(t,x)\\&+\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H_{0,1}^{''}\right)^{w_0}(t,x)\\&+\left[48M^{w_0}(t,x)-60N^{w_0}(t,x)\right]\frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\exp\left(\frac{-\sqrt{2}d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right).\end{aligned}$$

In conclusion, we deduce from Lemma 3.3.5 and the estimate above that

$$\begin{aligned}\Lambda(\varphi_2)(t,x)-\Lambda(\phi_{2,0})(t,x) \cong_6 &\frac{\ddot{r}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H_{0,1}^{'}\right)^{w_0}(t,x)-\frac{\dot{d}(t)\dot{r}(t)}{1-\frac{\dot{d}(t)^2}{4}}\left(H_{0,1}^{''}\right)^{w_0}(t,x)\\&+\frac{8\sqrt{2}r(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H_{0,1}^{''}\right)^{w_0}(t,x)\\&+\left[48M^{w_0}(t,x)-60N^{w_0}(t,x)\right]\frac{r(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}e^{-\sqrt{2}d(t)}.\end{aligned} \tag{3.143}$$

**Step 3.**(Conclusion of the proof of Theorem 3.4.1.)

Using Lemmas 3.3.1, 3.3.2 and estimates (3.118), (3.119), we conclude from the product rule of derivative and estimate (3.143) that if $0<v\ll 1$, then

$$\left\Vert\frac{\partial^l}{\partial t^l}\left[\Lambda(\varphi_2)(t,x)-\Lambda(\phi_{2,0})(t,x)\right]\right\Vert_{H^s_x}\lesssim_{l,s} v^{4+l}\left(|t|v+\ln\left(\frac{1}{v^2}\right)\right)e^{-2\sqrt{2}|t|v}, \tag{3.144}$$

for all $t\in\mathbb{R}$, $s\geq 0$ and $l\in\mathbb{N}\cup\{0\}$.

Moreover, Remark 3.3.7 implies for all $m,l\in\mathbb{N}\cup\{0\}$ and $t\in\mathbb{R}$ that if $h\in S^+_m$, then

$$\left|\frac{d^l}{dt^l}\left\langle h\left(w_0(t,x)\right),H_{0,1}^{'}(w_0(t,-x))\right\rangle\right|\lesssim_{h,l} v^{2+l}\left[|t|v+\ln\left(\frac{1}{v}\right)\right]^{m+1}e^{-2\sqrt{2}|t|v}.$$

Consequently, using Lemma 3.4.13, the ordinary differential equation (3.108), identity (3.78) and estimate (3.143), if $0 < v \ll 1$, there exists $n_2 \in \mathbb{N}$ satisfying for all $l \in \mathbb{N} \cup \{0\}$ the following estimate

$$\left|\frac{d^l}{dt^l}\left\langle \Lambda(\varphi_2)(t,x),\, H_{0,1}^{'}\left(\frac{x-\frac{d(t)}{2}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}})}\right)\right\rangle\right| \lesssim_l v^{l+6}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{n_2+1} e^{-2\sqrt{2}|t|v}.$$

Therefore, if $0 < v \ll 1$, Lemmas 3.3.4, 3.4.13, inequality (3.144) and estimates (3.118), (3.119) of $r(t)$ imply for any $l \in \mathbb{N} \cup \{0\}$ and all $t \in \mathbb{R}$ that

$$\left|\frac{d^l}{dt^l}\left\langle \Lambda(\varphi_2)(t,x),\, H_{0,1}^{'}\left(\frac{x+r(t)-\frac{d(t)}{2}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right\rangle\right| \lesssim_l v^{l+6}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{n_2+1} e^{-2\sqrt{2}|t|v}.$$

Since $\varphi_2$ is an odd function on $x$, the estimate above implies (3.80).

Finally, since $d(t)$ and $r(t)$ are even functions and $\lim_{t\to+\infty} r(t)$ exists, there exists a number $e(v)$ such that $\phi_2(v,t,x) = \varphi_2(t+e(v),x)$ satisfies Theorem 3.1.2 for $k=2$. More precisely, because $d(t) = 2vt + \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\right) + O(e^{-2\sqrt{2}vt})$ when $t \gg 1$ and $\lim_{s\to\pm\infty} r(s) = e_r = O(v^2\ln\left(\frac{1}{v}\right)^2)$, we consider $e(v) = \frac{-1}{2v}\left[\frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\right)+e_r\right]$. □

## 3.5 Approximate solutions for $k > 2$

We will prove the following theorem, which implies Theorem 3.1.2:

**Theorem 3.5.1.** *There exist a sequence of approximate solutions $(\varphi_{k,v}(t,x))_{k\geq 2}$, functions $r_k(v,t)$ that are smooth and even in $t$, and numbers $n_k \in \mathbb{N}$ such that if $0 < v \ll 1$, then for any $k \in \mathbb{N}_{\geq 2}$, $m \in \mathbb{N}$*

$$|r_k(v,t)| \lesssim_k v^{2(k-1)}\left[\ln\left(\frac{1}{v}\right)\right]^{n_k},\ \left|\frac{\partial^m}{\partial t^m} r_k(v,t)\right| \lesssim_{k,m} v^{2(k-1)+m}\left[\ln\left(\frac{1}{v}\right)+|t|v\right]^{n_k} e^{-2\sqrt{2}|t|v}, \tag{3.145}$$

*$\varphi_{k,v}(t,x)$ satisfies for $\rho_k(v,t) = -\frac{d_v(t)}{2} + \sum_{j=2}^{k} r_j(v,t)$ the identity*

$$\begin{aligned}\varphi_{k,v}(t,x) = {} & H_{0,1}\left(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) + H_{-1,0}\left(\frac{x-\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) \\ & + e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - \mathcal{G}\left(\frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right] \\ & + \mathcal{T}_{k,v}\left(vt, \frac{x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - \mathcal{T}_{k,v}\left(vt, \frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right),\end{aligned} \tag{3.146}$$

*the following estimates for any $l \in \mathbb{N} \cup \{0\}$ and $s \geq 1$*

$$\left\|\frac{\partial^l}{\partial t^l}\Lambda(\varphi_{k,v}(t,x)\right\|_{H^s_x} \lesssim_{k,l,s} v^{2k+l}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{n_k} e^{-2\sqrt{2}|t|v}, \tag{3.147}$$

*and*

$$\left|\frac{d^l}{dt^l}\left[\left\langle \Lambda(\varphi_{k,v})(t,x),\, H^{'}_{0,1}\left(\frac{\pm x+\rho_k(v,t)}{(1-\frac{\dot{d}_v(t)^2}{4})^{\frac{1}{2}}}\right)\right\rangle\right]\right| \lesssim_{k,l} v^{2k+l+2}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{n_k+1} e^{-2\sqrt{2}|t|v}, \tag{3.148}$$

*where $\mathcal{T}_k(t,x)$ is a finite sum of functions $p_{k,i,v}(t)h_{k,i}(x)$ with $h_{k,i,v}\in\mathscr{S}(\mathbb{R})\cap S^{+}_{\infty}$ and each $p_{k,i,v}(t)$ being an even function satisfying*

$$\left|\frac{d^m p_{k,i,v}(t)}{dt^m}\right| \lesssim_{k,m} v^4\left(\ln\left(\frac{1}{v^2}\right)+|t|\right)^{n_{k,i}} e^{-2\sqrt{2}|t|}$$

*for a positive number $n_{k,i}\in\mathbb{N}$ and all $m\in\mathbb{N}\cup\{0\}$.*

**Remark 3.5.2.** *From the result of the subsection before, we have that $\varphi_2(t,x)$ and $r(t)$ satisfy all the properties (3.146), (3.145) and (3.148) for $k=2$ if $v\ll 1$, so Theorem 3.5.1 is true for $k=2$. We are going to prove that if, for any $2\leq k\leq\mathcal{M}$, there exists a smooth function $\varphi_{k,v}(t,x)$ denoted by (3.146) that satisfies the conclusion of Theorem 3.5.1 if $0<v\ll 1$, then there exists also $\varphi_{\mathcal{M}+1,v}(t,x)$ satisfying (3.146), (3.148) and Theorem 3.5.1 if $v\ll 1$. Next, after a time translation of order $O\left(\frac{\ln\left(\frac{1}{v}\right)}{v}\right)$, this function will satisfy Theorem 3.1.2.*

**Remark 3.5.3.** *Furthermore, from Theorem 3.4.1, we also have that $r_2$ satisfies, if $v>0$ is small enough, the following estimates*

$$\|r_2(v,\cdot)\|_{L^\infty(\mathbb{R})}\lesssim v^2\ln\left(\frac{1}{v^2}\right),\ \left|\frac{\partial^l}{\partial t^l}r_2(v,t)\ \right|\lesssim_l v^{2+l}\left[\ln\left(\frac{1}{v}\right)+|t|v\right]e^{-2\sqrt{2}|t|v},$$

*for all $l\in\mathbb{N}$.*

### 3.5.1 Auxiliary lemmas.

From now on, we assume that Theorem 3.5.1 is true for $2\leq k\leq\mathcal{M}$. We also consider the following defintion.

**Definition 3.5.4.** *We say that function $\mathcal{F}:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ is negligible of order $(n,m)\in\mathbb{N}^2$ if there exist a constant $M(n)$ satisfying such that $\mathcal{F}$ satisfies for any $v\in(0,1)$ small enough the following estimate*

$$\left\|\frac{\partial^l}{\partial t^l}\mathcal{F}(v,t,x)\right\|_{H^s_x}\lesssim_{l,s} v^{n+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^m e^{-2\sqrt{2}|t|v},$$

*for all $t\in\mathbb{R}$, any $l\in\mathbb{N}$ and all $s\geq 0$. Moreover, we also say for any $n\in\mathbb{N}_{>6}$ that any two real functions $f,g:(0,1)\times\mathbb{R}^2\to\mathbb{R}^2$ satisfy $f\cong_n g$ if $f-g$ is a negligible function of order $(n,m)$ for some $m\in\mathbb{N}$.*

The demonstration of Theorem 3.5.1 will be done by induction on $k$. However, before the beginning of this proof, we need to prove three lemmas necessary to demonstrate Theorem 3.5.1. The first lemma is the following:

**Lemma 3.5.5.** *In notation of Theorem 3.5.1, there exist natural numbers $N_1, N_2$ satisfying, for $0 < v \ll 1$, the following estimate*

$$\Lambda(\varphi_{\mathcal{M},v})(t,x) \cong_{2\mathcal{M}+4} \sum_{i=1}^{N_1} s_{i,v}(\sqrt{2}vt)\left[\mathcal{R}_i\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right) - \mathcal{R}_i\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]$$

*such that for all $1 \leq i,j \leq N_1$ we have $\langle \mathcal{R}_i, \mathcal{R}_j\rangle = \delta_{i,j}$, $\mathcal{R}_i \in S^+_\infty \cap \mathscr{S}(\mathbb{R})$, $s_{i,v} \in C^\infty(\mathbb{R})$ satisfies, for all $l \in \mathbb{N}\cup\{0\}$, $\left|\frac{d^l}{dt^l}s_{i,v}(t)\right| \lesssim_l v^{2\mathcal{M}}\left[|t| + \ln\left(\frac{1}{v^2}\right)\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|}$.*

Our demonstration of Lemma 3.5.5 will need the following result.

**Lemma 3.5.6.** *For any $\zeta > 1$, let $\phi : \mathbb{R}_{\geq 1}\times\mathbb{R}^2 \to \mathbb{R}$ be a function of the form*

$$\phi(\zeta,t,x) = H_{0,1}(x-\zeta) - H_{0,1}(-x) + \sum_{i=1}^{\mathcal{N}} p_i(t)\left[I_i(x-\zeta) - I_i(-x)\right],$$

*where $\mathcal{N} < +\infty$, all the functions $p_i(t)$ are smooth with all their non-zero derivatives being in $\mathscr{S}(\mathbb{R})$, and for all $1 \leq i \leq \mathcal{N}$, $I_i \in \mathscr{S}(\mathbb{R}) \cap S^{+,m_i}$ for some $m_i \in \mathbb{N}\cup\{0\}$. Let $Z_\zeta : \mathbb{R}^2 \to \mathbb{R}$ be the following function*

$$\mathcal{Z}_\zeta(t,x) = U'(\phi(\zeta,t,x)) - U'(H_{0,1}(x-\zeta)) - U'(H_{-1,0}(x)),$$

*for any $(t,x)\in\mathbb{R}^2$, and $\zeta > 1$. For any $k \in \mathbb{N}$, there exist $\mathcal{N}_1(k) \in \mathbb{N}$, functions $h_i \in S^+_\infty$, and numbers $n_i$, $l_i \in \mathbb{N}\cup\{0\}$, $\alpha_{i,j} \in \mathbb{N}\cup\{0\}$ for all $1 \leq i \leq \mathcal{N}_1(k)$ and $1 \leq j \leq \mathcal{N}$ such that the following function*

$$\mathcal{Z}_{k,\zeta}(t,x) = \sum_{i=1}^{\mathcal{N}_1(k)}\left[\zeta^{l_i}e^{-\sqrt{2}n_i\zeta}\left(h_i(x-\zeta) - h_i(-x)\right)\prod_{j=1}^{\mathcal{N}} p_j(t)^{\alpha_{i,j}}\right], \text{ for all } (\zeta,x)\in\mathbb{R}_{\geq 1}\times\mathbb{R},$$

*satisfies for any $s \geq 0$ and every $(\zeta,t)\in\mathbb{R}_{\geq 1}\times\mathbb{R}$ the estimate*

$$\left\|\mathcal{Z}_\zeta(t,x) - \mathcal{Z}_{k,\zeta}(t,x)\right\|_{H^s_x} \leq C(\phi,s,k)e^{-\sqrt{2}k\zeta},$$

*where $C(\phi,s,k)$ is a positive value depending only on $k$ and $s$ and the function $\phi$.*

*Proof.* Proposition 3.2.16 and Remarks 3.2.20, 3.2.21 can be applied to estimate with higher precision the function

$$\mathcal{Z}_\zeta(t,x) = U'(\phi(\zeta,t,x)) - U'(H_{0,1}(t,x-\zeta)) - U'(-H_{0,1}(-x)), \tag{3.149}$$

since $U'(\phi) = 2\phi - 8\phi^3 + 6\phi^5$. More precisely, since $U'$ is an odd polynomial, it is not difficult to verify from the definition of $\phi(\zeta,t,x)$ and the multinomial formula that $\mathcal{Z}_\zeta(t,x)$ is a finite sum of functions of the following kind

$$\begin{aligned}\mathcal{X}_\zeta(t,x) = &\left[H_{0,1}(x-\zeta)^{\alpha_0}\Big(-H_{0,1}(-x)\Big)^{\beta_0}\prod_{i,j=1}^{\mathcal{N}} p_j(t)^{\alpha_j}I_j(x-\zeta)^{\alpha_j}p_i(t)^{\beta_i}\Big(-I_i(-x)\Big)^{\beta_i}\right]\\ &+\left[H_{0,1}(x-\zeta)^{\beta_0}\Big(-H_{0,1}(-x)\Big)^{\alpha_0}\prod_{i,j=1}^{\mathcal{N}} p_i(t)^{\beta_i}I_i(x-\zeta)^{\beta_i}p_j(t)^{\alpha_j}\Big(-I_j(-x)\Big)^{\alpha_j}\right],\end{aligned}$$

such that

- $\alpha_i,\ \beta_i \in \mathbb{N}\cup\{0\}$ for all $0 \leq i \leq \mathcal{N}$,

- $\sum_{i=0}^{\mathcal{N}} \alpha_i + \beta_i$ is odd,

- either $\sum_{i=1}^{\mathcal{N}} \alpha_i + \beta_i \neq 0$ or $\min(\alpha_0, \beta_0) > 0$.

Since every $I_j \in S^{+}_{\infty}$, we can apply Lemma 3.2.7 and deduce for any natural number $1 \leq j \leq \mathcal{N}$ and any $k \in \mathbb{N}$ that $I_j(-x)^{2k} \in S^{-}_{\infty}$ and $I_j(x)^{2k-1} \in S^{+}_{\infty}$. Moreover, Lemma 3.2.7 also implies for all $k \in \mathbb{N}$ that if $(f_i)_{1\leq i\leq 2k-1} \subset S^{+}_{\infty}$, then $\prod_{i=1}^{2k-1} f_i \in S^{+}_{\infty}$, and if $(f_i)_{1\leq i\leq 2k} \subset S^{-}_{\infty}$, then $\prod_{i=1}^{2k} f_i \in S^{-}_{\infty}$. Therefore, we deduce that either

$$H_{0,1}\left(x\right)^{\alpha_0}\prod_{j=1}^{\mathcal{N}} I_j\left(x\right)^{\alpha_j} \in S^{+}_{\infty},\ H_{0,1}\left(-x\right)^{\beta_0}\prod_{i=1}^{\mathcal{N}} I_i\left(-x\right)^{\beta_i} \in S^{-}_{\infty}\cup\{1\} \text{ or}$$
$$H_{0,1}\left(-x\right)^{\alpha_0}\prod_{j=1}^{\mathcal{N}} I_j\left(-x\right)^{\alpha_j} \in S^{-}_{\infty}\cup\{1\},\ H_{0,1}\left(x\right)^{\beta_0}\prod_{i=1}^{\mathcal{N}} I_i\left(x\right)^{\beta_i} \in S^{+}_{\infty}.$$

Consequently, we can apply the Separation Lemma and Remark 3.2.21 in the expression

$$H_{0,1}\left(x-\zeta\right)^{\alpha_0}\prod_{j=1}^{\mathcal{N}} I_j\left(x-\zeta\right)^{\alpha_j}\left[\left(-H_{0,1}\left(-x\right)\right)^{\beta_0}\prod_{i=1}^{\mathcal{N}}\left(-I_i\left(-x\right)\right)^{\beta_i}\right]$$
$$+\left[\left(-H_{0,1}\left(-x\right)\right)^{\alpha_0}\prod_{j=1}^{\mathcal{N}}\left(-I_j\left(-x\right)\right)^{\alpha_j}\right]H_{0,1}\left(x-\zeta\right)^{\beta_0}\prod_{i=1}^{\mathcal{N}} I_i\left(x-\zeta\right)^{\beta_i},$$

and deduce for any $k \in \mathbb{N}$ the existence of $\mathcal{N}_2(k) \in \mathbb{N}$, a set of numbers $l_{i,1},\ n_{i,1} \in \mathbb{N}\cup\{0\}$ and a set of functions $h_{i,1} \in S^{+}_{\infty}\cap\mathscr{S}(\mathbb{R})$, such that the function

$$\mathcal{X}_{k,\zeta}(t,x) = \left[\sum_{i=1}^{\mathcal{N}_2(k)} \zeta^{l_{i,1}} e^{-\sqrt{2}n_{i,1}\zeta}\Big(h_{i,1}\left(x-\zeta\right) - h_{i,1}\left(-x\right)\Big)\right]\prod_{j=1}^{\mathcal{N}} p_j(t)^{\alpha_j+\beta_j}$$

satisfies, if $\zeta$ is large enough, the estimate

$$\left\Vert \mathcal{X}_{\zeta}(t,x) - \mathcal{X}_{k,\zeta}(t,x)\right\Vert_{H^s_x} \lesssim_{s,k} e^{-\sqrt{2}(k+1)\zeta}\prod_{j=1}^{\mathcal{N}} \left|p_j(t)\right|^{\alpha_j+\beta_j}.$$

In conclusion, using triangle inequality, we obtain the result of Lemma 3.5.6. □

**Corollary 3.5.7.** *Let the functions $I_i \in \mathscr{S}(\mathbb{R})$, $p_i \in C^{\infty}(\mathbb{R})$ be as defined in the statement of Lemma 3.5.6. Let $\gamma : (0,1)\times\mathbb{R} \to \mathbb{R}$ be a function satisfying*

$$\left\Vert \frac{\partial^l}{\partial t^l}\gamma(v,t)\right\Vert_{L^{\infty}_t(\mathbb{R})} \lesssim_l v^l, \textit{ for any } l \in \mathbb{N}\cup\{0\},\ \textit{if } 0 < v \ll 1,$$

*and $w : (0,1)\times\mathbb{R}^2 \to \mathbb{R}$ be the following smooth function*

$$\omega(v,t,x) = \frac{x - \frac{d_v(t)}{2} + \gamma(v,t)}{\sqrt{1 - \frac{\dot{d}_v(t)^2}{4}}}.$$

*In addition, let $\phi_{app} : \mathbb{R}^2 \to \mathbb{R}$ be the following function*

$$\phi_{app}(t,x) = H_{0,1}\left(w(v,t,x)\right) - H_{0,1}\left(w(v,t,-x)\right) + \sum_{i=1}^{\mathcal{N}} p_i(t)\left[I_i\left(w(v,t,x)\right) - I_i\left(w(v,t,-x)\right)\right],$$

*for all $(t,x) \in \mathbb{R}^2$ and $\mathcal{Z}(t,x)$ be denoted by*

$$\mathcal{Z}(t,x) = U^{'}(\phi_{app}(t,x)) - U^{'}\left(H_{0,1}\left(w(v,t,x)\right)\right) - U^{'}\left(-H_{0,1}\left(w(v,t,-x)\right)\right),$$

*for any $(t,x) \in \mathbb{R}^2$. If $v \ll 1$ and the functions $p_i$ also satisfy the following decay estimate*

$$\max_{1\leq i\leq \mathcal{N}} \left\Vert p_i^{(l)}(t)\right\Vert \lesssim_l v^l, \text{ for every } l \in \mathbb{N},$$

*then, for any $k \in \mathbb{N}_{\geq 2}$, there exist $\mathcal{N}_1(k) \in \mathbb{N}$, functions $h_i \in S^{+}_{\infty}$, and numbers $n_i, l_i \in \mathbb{N}\cup\{0\}$, $\alpha_{i,j} \in \mathbb{N}\cup\{0\}$ for all $1 \leq i \leq \mathcal{N}_1(k)$ and $1 \leq j \leq \mathcal{N}$ such that the following function*

$$\mathcal{Z}_k(t,x) = \left[\sum_{i=1}^{\mathcal{N}_1(k)} \left[\frac{d_v(t) - 2\gamma(v,t)}{\sqrt{1-\dot{d}_v(t)^2}}\right]^{l_i} \exp\left(\frac{-2\sqrt{2}n_i\left[d_v(t) - 2\gamma(v,t)\right]}{\sqrt{1-v(t)^2}}\right)\prod_{j=1}^{\mathcal{N}} p_j(t)^{\alpha_{j,i}}\Big(h_i\left(w(v,t,x)\right) - h_i\left(w(v,t,-x)\right)\Big)\right], \text{ for any } (t,x) \in \mathbb{R}^2,$$

*satisfies*

$$\left\Vert \frac{\partial^l}{\partial t^l}\left[\mathcal{Z}_k(t,x) - \mathcal{Z}(t,x)\right]\right\Vert_{H^s_x} \leq \hat{C}v^l e^{-2\sqrt{2}kd_v(t)}d_v(t)^{M_2(k)},$$

*for every $l \in \mathbb{N}\cup\{0\}$ and $s \geq 0$, where $\hat{C} > 0$ is a constant depending only on the functions $(p_i)_{1\leq i\leq\mathcal{N}}$ and the numbers $l$, $s$ and $k$.*

*Proof of Corollary 3.5.7.* First, from Lemma 3.5.6, if we replace the variables $x$ and $\zeta$, respectively, with $-w(t,-x)$ and

$$\frac{d_v(t) - 2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}},$$

we deduce for any $k \in \mathbb{N}_{\geq 2}$ the existence of a set of functions $(h_i)_{i\in\mathbb{N}} \subset S^{+}_{\infty}$, a set of numbers $(\alpha_{j,i})_{(j,i)\in\mathbb{N}^2} \subset \mathbb{N}\cup\{0\}$ and two sequences of numbers $(l_i)_{\in\mathbb{N}} \subset \mathbb{N}\cup\{0\}$, $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ such that if $0 < v \ll 1$, the following function

$$\mathcal{Z}_k(t,x) = \left[\sum_{i=1}^{\mathcal{N}_1(k)} \left[\frac{d_v(t) - 2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right]^{l_i} \exp\left(\frac{-2\sqrt{2}n_i\left(d(t) - 2\gamma(v,t)\right)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\prod_{j=1}^{\mathcal{N}} p_j(t)^{\alpha_{j,i}}\Big(h_i\left(w(t,x)\right) - h_i\left(w(t,-x)\right)\Big)\right]$$

satisfies, for a constant $M_2(k) \in \mathbb{N}$ any $m \in \mathbb{N}$, the following estimate

$$\left\Vert \mathcal{Z}_k(t,x) - \mathcal{Z}(t,x)\right\Vert_{H^m_x} \lesssim_{m,k} e^{-2\sqrt{2}ky(t)}\left(1+y(t)\right)^{M_2(k)}.$$

Furthermore, Separation Lemma also implies the existence of $M_1(k) \in \mathbb{N}$, for any $k \in \mathbb{N}$, such that

$$\mathcal{Z}(t,x) - \mathcal{Z}_k(t,x) = \sum_{i=1}^{M_1(k)} \exp\left(\frac{-\sqrt{2}N_i\left(d_v(t)-2\gamma(v,t)\right)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\left(\frac{d_v(t)-2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)^{n_i}\prod_{j=1}^{\mathcal{N}} p_j(t)^{\beta_{j,i}} h_{i,1}\left(w(t,x)\right) h_{i,2}\left(w(t,-x)\right), \tag{3.150}$$

where for any $1 \leq i \leq M_1(k)$, $n_i \in \mathbb{N}\cup\{0\}$ and $N_i$ in $\mathbb{N}_{\geq k}$, the functions $h_{i,1}$, $h_{i,2} \in L^\infty_x(\mathbb{R})$ are smooth and all $\beta_{j,i} \in \mathbb{N}\cup\{0\}$.

In fact, from Proposition 3.2.16, we could also say for all $1 \leq i \leq M_1(k)$ that $2k \leq N_i$, $n_i \in \mathbb{N}\cup\{0\}$, either $h_{i,1}$ or $h_{i,2}$ is in $\mathscr{S}(\mathbb{R})$ and either $h_{i,1}(x) \in S^+ \cup S^+_\infty$, $h_{j,2}(x) \in S^- \cup S^-_\infty$ or $h_{i,1}(-x) \in S^+ \cup S^+_\infty$, $h_{i,2}(-x) \in S^- \cup S^-_\infty$. Moreover, since $\gamma$ satisfies the condition of Remark 3.3.3, and

$$\max_{1\leq j\leq \mathcal{N}}\left|\frac{d^l}{dt^l}p_j(t)\right| \lesssim_l v^l, \text{ for all } l \in \mathbb{N} \text{ and } t \in \mathbb{R},$$

we deduce from Remark 3.3.3 and the product rule of derivative that if $v > 0$ is small enough, then

$$\left\Vert \frac{\partial^l}{\partial t^l}\left[\mathcal{Z}_k(t,x) - \mathcal{Z}(t,x)\right]\right\Vert_{H^s_x} \lesssim_{s,k,l} v^{k+l} e^{-2\sqrt{2}|t|v}, \text{ for any } l \in \mathbb{N}\cup\{0\} \text{ and } s \geq 0. \tag{3.151}$$

Actually, using the product rule of derivative, for every $1 \leq i \leq M_1(k)$, we have if $v > 0$ is small enough that

$$\left|\frac{d^l}{dt^l}\left[\prod_{j=1}^{\mathcal{N}} p_j(t)^{\beta_{j,i}} \exp\left(\frac{-2\sqrt{2}N_i(d_v(t)-2\gamma(v,t))}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\left(\frac{d_v(t)-2\gamma(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)^{n_i}\right]\right| \lesssim_{l,k} v^{k+l} e^{-2\sqrt{2}|t|v},$$

for all $l \in \mathbb{N}\cup\{0\}$ and every $t \in \mathbb{R}$. Therefore, since Remark 3.3.3 implies

$$\left\Vert\frac{\partial^l}{\partial t^l} h_{i,1}(w(t,x))\right\Vert_{H^s_x} + \left\Vert\frac{\partial^l}{\partial t^l} h_{i,2}(w(t,x))\right\Vert_{H^s_x} \lesssim_{l,s} v^l,$$

for every $1 \leq i \leq M_1(k)$, we conclude estimate (3.151) from the product rule, triangle inequality and identity (3.150). $\square$

*Proof of Lemma 3.5.5.* First, we consider $0 < v \ll 1$ and recall that $\Lambda(\cdot) = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U'(\cdot)$. From Lemma 3.3.2 and Remark 3.3.3, if $h \in S^+_\infty$ and $p_v(t)$ satisfies for constants $q_1$, $q_2 \in \mathbb{N}$ the following estimate

$$\left|\frac{d^l}{dt^l}p_v(t)\right| \lesssim_l v^{2q_1}\left[\ln\left(\frac{1}{v}\right) + |t|\right]^{q_2} e^{-2\sqrt{2}|t|}, \text{ for all } l \in \mathbb{N}\cup\{0\},$$

then

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\left(p_v(\sqrt{2}vt) h\left(\frac{x+\rho_M(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right)$$

is a finite sum of functions $p_{i,v}(\sqrt{2}vt)h_i\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)$ with $h_i \in S^{+}_{\infty}$ and $p_{i,v}$ satisfying for some natural numbers $m_i > 0$, $w_i$ the following decay

$$\left|\frac{d^l}{dt^l}\left[p_i(\sqrt{2}vt)\right]\right| \lesssim_l v^{2m_i+l}\left[\ln\left(\frac{1}{v}\right)+|t|v\right]^{w_i} e^{-2\sqrt{2}|t|v}, \text{ for all } l \in \mathbb{N}\cup\{0\}. \tag{3.152}$$

Next, using Lemma 3.3.1, Remark 3.3.3 and identity $H^{''}_{0,1}(x) = U^{'}(H_{0,1}(x))$, we can verify similarly to the proof of Lemma 3.4.2 the following estimate

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right]H_{0,1}\left(\frac{x-\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) = -U^{'}\left(H_{0,1}\left(\frac{x-\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right)+residue_0(t,x), \tag{3.153}$$

where $residue_0(t,x)$ is a finite sum of functions

$$q_{i,v}(\sqrt{2}vt)h_i\left(\frac{x-\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right),$$

with $h_i \in S^2_+$ and

$$\left|\frac{d^l q_{i,v}(t)}{dt^l}\right| \lesssim_l v^2\left(|t|+\ln\left(\frac{1}{v^2}\right)\right)e^{-2|t|}, \text{ for all } l \in \mathbb{N}\cup\{0\}.$$

Therefore, to finish the proof of Lemma 3.5.5 we need only to study the expression

$$DU(t,x) = U^{'}(\varphi_{\mathcal{M},v}(t,x)) - U^{'}\left(H_{0,1}\left(\frac{x-\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right) - U^{'}\left(H_{-1,0}\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right). \tag{3.154}$$

Furthermore, from Corollary 3.5.7, we can obtain for any natural $N \gg 1$ the existence of natural numbers $N_1$, $N_2$, a set of functions $h_{\mathcal{M},j} \in S^{+}_{\infty}$ and a set of functions $p_{\mathcal{M},j,v}(t)$ satisfying property (3.152) such that $DU(t,x)$ satisfies

$$DU(t,x) \cong_{2N} \sum_{j=1}^{N_1} p_{\mathcal{M},j,v}(\sqrt{2}vt)\left[h_{\mathcal{M},j}\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) - h_{\mathcal{M},j}\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right]. \tag{3.155}$$

Moreover, if two functions $p_1(t)$, $p_2(t)$ satisfy property (3.152), then, from the product rule of derivative, $p_1(t)p_2(t)$ have much smaller decay than the right-hand side of (3.152) as $|t| \to +\infty$, because of the $e^{-4\sqrt{2}|t|}$ contribution obtained in the product of these functions.

In conclusion, we proved that there exist a finite subset $I_0$ of $\mathbb{N}$, functions $p_{j,v}$ satisfying property (3.152) and $h_j \in \mathscr{S}(\mathbb{R}) \cap S^{+}_{\infty}$ such that

$$\Lambda(\varphi_{\mathcal{M},v})(t,x) \cong_{2N} \sum_{j\in I_0} p_{j,v}(\sqrt{2}vt)\left[h_j\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) - h_j\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right]. \tag{3.156}$$

Moreover, after a finite number of applications of Proposition 3.2.16, it is possible to obtain an estimate of the form (3.156) for any $N \gg 1$ if we assume $v \ll 1$.

From Gram-Schmidt, we can exchange the functions $h_j$ in (3.156) by functions $\mathcal{R}_j \in S^{+}_{\infty} \cap \mathscr{S}(\mathbb{R})$ such that $\langle \mathcal{R}_j,\, \mathcal{R}_i\rangle = \delta_{i,j}$ and

$$\Lambda(\varphi_{\mathcal{M},v})(t,x) \cong_{2N} \sum_{j\in I} s_{j,v}(\sqrt{2}vt)\left[\mathcal{R}_j\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) - \mathcal{R}_j\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right], \tag{3.157}$$

for a finite set $I$ with the functions $s_{j,v}(t)$ also satisfying property (3.152). In conclusion, from the assumption that the conclusion of Theorem 3.5.1 is true when $k = \mathcal{M}$, we deduce from Lemma 3.2.1 and condition $\langle \mathcal{R}_j,\, \mathcal{R}_i\rangle = \delta_{i,j}$ that, for any $j \in I$, we have

$$\begin{aligned}\left\langle \Lambda(\varphi_{\mathcal{M},v})(t,x), \mathcal{R}_j\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right\rangle = &\left[\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}} + O(v)\right] s_{j,v}(\sqrt{2}vt)\\ &+ \sum_{i\neq j,\, i\in I} s_{i,v}(\sqrt{2}vt)O(v)\\ &+O\left(v^{2N}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{N_2} e^{-2\sqrt{2}|t|v}\right). \end{aligned} \tag{3.158}$$

Since $N > \mathcal{M}+1$, using the identities(3.158) for all $j \in I$ and estimate (3.147), we deduce that

$$|s_{j,v}(t)| \lesssim v^{2\mathcal{M}}\left[|t| + \ln\left(\frac{1}{v^2}\right)\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|}, \tag{3.159}$$

for all $j \in I$, and $t \in \mathbb{R}$.

Furthermore, we can assume the existence of $m_0 \in \mathbb{N}\cup\{0\}$ such that

$$\left|\frac{d^l s_{j,v}(t)}{dt^l}\right| \lesssim_l v^{2\mathcal{M}}\left[|t| + \ln\left(\frac{1}{v^2}\right)\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|}, \text{ for all } j\in I,\ l \in \mathbb{N}\cup\{0\} \text{ satisfying } 0\leq l\leq m_0. \tag{3.160}$$

But, from estimate (3.157), assumption (3.160), Lemma 3.3.1 and Remark 3.3.3, we deduce using the product rule of derivative that

$$\begin{aligned}\sum_{j\in I} 2^{\frac{m_0+1}{2}} v^{m_0+1} s^{(m_0+1)}_{j,v}\left(\sqrt{2}vt\right)\left[\mathcal{R}_j\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right) - \mathcal{R}_j\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right]\\ \cong_{2\mathcal{M}+m_0+1} \frac{\partial^{m_0+1}\Lambda\left(\phi_{\mathcal{M},v}\right)(t,x)}{\partial t^{m_0+1}}.\end{aligned}$$

Therefore, similarly to the proof of (3.159) for all $j \in I$ and using Remark 3.3.7 in the expressions

$$\left\langle \mathcal{R}_j\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right), \mathcal{R}_i\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right\rangle \text{ for all } i,\, j\in I,$$

we obtain the following estimate

$$\left|\frac{d^{m_0+1} s_{j,v}(t)}{dt^l}\right| \lesssim_{m_0+1} v^{2\mathcal{M}}\left[|t| + \ln\left(\frac{1}{v^2}\right)\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|}.$$

In conclusion, from induction on $l$, the estimate of the decay of the derivatives of $s_{j,v}$ in Lemma 3.5.5 is true for all $l \in \mathbb{N}\cup\{0\}$. □

The third lemma necessary to the proof of the existence of $\phi_{\mathcal{M}+1,v}(t,x)$ is the following:

**Lemma 3.5.8.** *In notation of Lemma 3.5.5, there is a positive number $n_{\mathcal{M}}$ such that the following function*

$$Proj(t) = \left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}\left[\sum_{i=1}^{N_1} s_{i,v}(\sqrt{2}vt)\left\langle \mathcal{R}_i(x), H_{0,1}^{'}(x)\right\rangle\right]$$

*satisfies*

$$\left|\frac{d^l}{dt^l}Proj(t)\right| \lesssim_l v^{2M+l+2}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v} \text{ for all } l\in\mathbb{N}\cup\{0\}.$$

*Proof.* From Lemma 3.5.5, there exists a function $res:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ such that $res\cong_{2\mathcal{M}+4}0$ and

$$\Lambda(\varphi_{\mathcal{M},v})(t,x)=\sum_{j\in I}s_{j,v}(\sqrt{2}vt)\left[\mathcal{R}_j\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)-\mathcal{R}_j\left(\frac{-x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right]+res(v,t,x).$$

Therefore, we have the following identity

$$\begin{aligned}\Bigg\langle \Lambda(\varphi_{\mathcal{M},v})(t,x),\, &H_{0,1}^{'}\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\Bigg\rangle\\ =&Proj(t)+\left\langle H_{0,1}^{'}\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right),res(v,t,x)\right\rangle\\ &-\sum_{j}s_{j,v}(\sqrt{2}tv)\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}\left\langle H_{0,1}^{'}(x),\,\mathcal{R}_j\left(-x+2\frac{\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right\rangle.\end{aligned} \tag{3.161}$$

First, we recall the function $d(t)=\frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right)$, which satisfies

$$\left\|\dot{d}(t)\right\|_{L^\infty(\mathbb{R})}\lesssim v,\ \left\|d^{(k)}(t)\right\|_{L^\infty(\mathbb{R})}\lesssim v^k e^{-2\sqrt{2}|t|v} \text{ if } k\geq 2.$$

We also recall $\rho_{\mathcal{M}}(v,t)=\sum_{j=2}^{\mathcal{M}}r_{j,v}(t)-\frac{d(t)}{2}$. Since we are assuming the veracity of estimates (3.145) for any natural number $k$ satisfying $2\leq k\leq\mathcal{M}$, we deduce, from Remark 3.3.3, Lemma 3.5.5, the product rule of derivative and Cauchy-Schwarz inequality, the existence of $N_2\geq 0$ satisfying for any $l\in\mathbb{N}\cup\{0\}$ the following inequalities

$$\begin{aligned}\left|\frac{d^l}{dt^l}\left[\left\langle H_{0,1}^{'}\left(\frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right),res(v,t,x)\right\rangle\right]\right|\lesssim_l&\sum_{j=0}^{l}\left\|\frac{\partial^j}{\partial t^j}res(v,t,x)\right\|_{L^2_x}v^{l-j}\\ \lesssim_l&v^{2\mathcal{M}+4+l}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{N_2}e^{-2\sqrt{2}|t|v}.\end{aligned}$$

Furthermore, Lemma 3.3.1 and Remark 3.3.7 imply for any $n\in\mathbb{N}\cup\{0\}$ that if $\mathcal{R}_j\in S_n^+$ and $0<v\ll 1$, then

$$\begin{aligned}&\left|\frac{d^l}{dt^l}\left[\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}\left\langle H_{0,1}^{'}(x),\,\mathcal{R}_j\left(-x+2\frac{\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right\rangle\right]\right|\\ &\lesssim_l v^{2+l}\left[|t|v+\ln\left(\frac{1}{v^2}\right)\right]^{n+1}e^{-2\sqrt{2}v|t|},\text{ for all } l\in\mathbb{N}\cup\{0\}.\end{aligned}$$

Consequently, from Lemma 3.5.5 and the product rule of derivative, we deduce the existence of a sufficiently large number $n_{\mathcal{M}}$ satisfying for $v \ll 1$ the following inequality

$$\left|\frac{d^l}{dt^l}\left[\sum_j s_{j,v}(\sqrt{2}tv)\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}\left\langle H_{0,1}^{'}(x),\,\mathcal{R}_j\left(-x+2\frac{\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\right\rangle\right]\right| \\ \lesssim_{l,\mathcal{M}} v^{2\mathcal{M}+2+l}\left[|t|v+\ln\left(\frac{1}{v^2}\right)\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v}, \quad (3.162)$$

for all $l \in \mathbb{N}\cup\{0\}$.

In conclusion, we obtain Lemma 3.5.8 from the estimates above, Lemma 3.5.5 and triangle inequality. □

From now on, for $\rho_{\mathcal{M}}(v,t) = -\frac{d(t)}{2} + \sum_{j=2}^{\mathcal{M}} r_j(v,t)$, we consider

$$w_{\mathcal{M}}(t,x) = \frac{x+\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}. \quad (3.163)$$

To simplify our notation, we denote the function $r_{l,v}$ as $r_l$ for every $l \in \mathbb{N}_{\geq 2}$. Using the notation of Lemma 3.5.5 and Lemma 3.5.8, we define the function

$$\Gamma(t,x) = \sum_{i=1}^{N_1} s_{i,v}(\sqrt{2}vt)\mathcal{R}_i(x) - H_{0,1}^{'}(x)\frac{Proj(t)}{\left\|H_{0,1}^{'}\right\|_{L^2}^2\sqrt{1-\frac{\dot{d}(t)^2}{4}}}. \quad (3.164)$$

Lemmas 3.5.5 and 3.5.8 imply $\left\langle\Gamma\left(t,x\right),\,H_{0,1}^{'}\left(w_{\mathcal{M}}\left(x\right)\right)\right\rangle = 0$ for all $t \in \mathbb{R}$, and for any $(t,x) \in \mathbb{R}^2$

$$\Lambda(\varphi_{M,v})(t,x) \cong_{2\mathcal{M}+4} \left[H_{0,1}^{'}\left(w_{\mathcal{M}}(t,x)\right)\frac{Proj(t)}{\left\|H_{0,1}^{'}\right\|_{L^2}^2\sqrt{1-\frac{\dot{d}(t)^2}{4}}} \right. \\ \left. -H_{0,1}^{'}\left(w_{\mathcal{M}}(t,-x)\right)\frac{Proj(t)}{\left\|H_{0,1}^{'}\right\|_{L^2}^2\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right] \\ +\Gamma\left(t,w_{\mathcal{M}}(t,x)\right) - \Gamma\left(t,w_{\mathcal{M}}(t,-x)\right). \quad (3.165)$$

Moreover, from Lemma 3.2.23 and Lemma 3.2.24, we can define the function $L_1(\Gamma(t,\cdot))(x) \in \mathscr{S}(\mathbb{R})\cap S^{+}_{\infty}$, more precisely, from the linearity of $L_1$, we have for any $(t,x)\in\mathbb{R}^2$ the following identity

$$L_1(\Gamma(t,\cdot))(x) = \sum_{i=1}^{N_1} s_{i,v}(\sqrt{2}vt)L_1\left(\mathcal{R}_i - \frac{H_{0,1}^{'}}{\left\|H_{0,1}^{'}\right\|_{L^2}^2}\langle H_{0,1}^{'},\,\mathcal{R}_i\rangle\right)(x), \quad (3.166)$$

and so, from Lemma 3.5.5, we have for any $t\in\mathbb{R}$, $s>0$ and $l\in\mathbb{N}\cup\{0\}$ that

$$\left\|\frac{\partial^l}{\partial t^l}L_1(\Gamma(t,\cdot))(x)\right\|_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+l}\left(|t|v+\ln\left(\frac{8}{v^2}\right)\right)^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v}. \quad (3.167)$$

Next, we recall from the inductive hypothesis of Theorem 3.5.1 that $\varphi_{\mathcal{M},v}(t,x)$ also has the representation (3.146) given by

$$\varphi_{\mathcal{M},v}(t,x) = H_{0,1}\left(w_{\mathcal{M}}(t,x)\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right) + e^{-\sqrt{2}d(t)}\left[\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right) - \mathcal{G}\left(w_{\mathcal{M}}(t,-x)\right)\right] \\ +\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,x)\right) - \mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,-x)\right), \quad (3.168)$$

where $\mathcal{T}_{\mathcal{M}}(t,x)$ is a function even on $t$ satisfying for a sufficiently large number $n_{\mathcal{M},1} \in \mathbb{N}$ and any $s>0$ the following inequality

$$\left\Vert \frac{\partial^l}{\partial t^l}\mathcal{T}_{\mathcal{M}}(t,x)\right\Vert_{H^s_x} \lesssim_{l,s} v^4\left(|t|+\ln\left(\frac{1}{v^2}\right)\right)^{n_{\mathcal{M},1}} e^{-2\sqrt{2}|t|} \text{ for all } l\in\mathbb{N}\cup\{0\}, \text{ if } 0<v\ll 1. \tag{3.169}$$

### 3.5.2 Construction of $r_{\mathcal{M}+1}(v,t)$.

From now on, for $j\in\{1,2,3,4\}$, we consider the smooth functions $I_j:\mathbb{R}\to\mathbb{R}$ defined by

$$I_1(t) = e^{-\sqrt{2}d(t)}\left\langle U^{(3)}(H_{0,1}(x))e^{-\sqrt{2}x}L_1\left(\Gamma(t,\cdot)\right)(x), H_{0,1}^{'}(x)\right\rangle, \tag{3.170}$$

$$I_2(t) = \left\langle L_1\left(\Gamma(t,\cdot)\right)\left(-x+\frac{d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right), \left[2-U^{(2)}(H_{0,1}\left(x\right))\right]H_{0,1}^{'}\left(x\right)\right\rangle \tag{3.171}$$

$$I_3(t) = -e^{-\sqrt{2}d(t)}\left\langle U^{(3)}\left(H_{0,1}(x)\right)\mathcal{G}(x)L_1\left(\Gamma(t,\cdot)\right)(x), H_{0,1}^{'}(x)\right\rangle, \tag{3.172}$$

$$I_4(t) = -\left\langle\left[\frac{\partial^2}{\partial t^2}-\frac{\dot{d}(t)^2}{4-\dot{d}(t)^2}\frac{\partial^2}{\partial x^2}\right]L_1\left(\Gamma(t,\cdot)\right)(x), H_{0,1}^{'}(x)\right\rangle. \tag{3.173}$$

Denoting the function $NL_{\mathcal{M}}:\mathbb{R}\to\mathbb{R}$ by

$$NL_{\mathcal{M}}(t) = \sum_{i=1}^{4} I_i(t), \text{ for any } t\in\mathbb{R},,$$

and recalling the function $Proj:\mathbb{R}\to\mathbb{R}$ defined in Lemma 3.5.8, we consider

$$Res_{\mathcal{M}}(t) = NL_{\mathcal{M}}(t) - Proj(t),$$

for any $t\in\mathbb{R}$, and the following ordinary differential equation

$$\begin{cases}\left\Vert H_{0,1}^{'}\right\Vert_{L^2_x}^2 \ddot{r}_{\mathcal{M}+1}(t) = -32\left\Vert H_{0,1}^{'}\right\Vert_{L^2_x}^2 e^{-\sqrt{2}d(t)}r_{\mathcal{M}+1}(t)+Res_{\mathcal{M}}(t),\\ r_{\mathcal{M}+1}(t) = r_{\mathcal{M}+1}(-t).\end{cases} \tag{3.174}$$

From Lemma 3.5.5, we recall the existence of $n_{\mathcal{M}}>0$ such that, for any $l\in\mathbb{N}\cup\{0\}$ and $1\leq i\leq N_1$, $\left|\frac{d^l}{dt^l}s_{i,v}(t)\right| \lesssim_l v^{2\mathcal{M}}\left[|t|+\ln\left(\frac{1}{v^2}\right)\right]^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|}$, if $0<v\ll 1$. Therefore, for $0<v\ll 1$ and using Remark 3.3.7 and identities (3.166), (3.171), we deduce the existence of $n_{\mathcal{M},2}\in\mathbb{N}\cup\{0\}$ satisfying

$$\left|I_2^{(l)}(t)\right| \lesssim_l v^{2\mathcal{M}+2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M},2}} e^{-2\sqrt{2}|t|v}, \text{ for every } t\in\mathbb{R} \text{ and any } l\in\mathbb{N}\cup\{0\}.$$

Next, from estimate (3.167), Lemma 3.3.1, identity (3.170) and Cauchy-Schwarz inequality, we obtain using the product rule of derivative that

$$\left|I_1^{(l)}(t)\right| \lesssim_l v^{2\mathcal{M}+2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v}, \text{ for every } t\in\mathbb{R} \text{ and any } l\in\mathbb{N}\cup\{0\}. \tag{3.175}$$

Similarly to the proof of estimate (3.175), we deduce that

$$\left|I_3^{(l)}(t)\right| \lesssim_l v^{2\mathcal{M}+2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v}, \text{ for every } t\in\mathbb{R} \text{ and any } l\in\mathbb{N}\cup\{0\}.$$

Furthermore, using Lemma 3.3.1, estimate (3.167) and the product rule of derivative, we obtain the following decay estimate

$$\left|I_4^{(l)}(t)\right| \lesssim_l v^{2\mathcal{M}+2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v}, \text{ for every } t\in\mathbb{R} \text{ and any } l\in\mathbb{N}\cup\{0\}.$$

In conclusion, using Lemma 3.5.8, we obtain that the function $Res_{\mathcal{M}}(t)$ defined in the ordinary differential equation (3.174) satisfies for some number $n_{\mathcal{M}+1}\geq 0$ the following decay estimate

$$\left|\frac{d^l}{dt^l}Res_{\mathcal{M}}(t)\right| \lesssim_l v^{l+2\mathcal{M}+2}\left(|t|v+\ln\left(\frac{1}{v^2}\right)\right)^{n_{\mathcal{M}+1}} e^{-2\sqrt{2}v|t|}, \text{ for every } t\in\mathbb{R} \text{ and any } l\in\mathbb{N}\cup\{0\}. \tag{3.176}$$

Repeating the argument in the first step of the proof of Theorem 3.4.1, we have for the following functions

$$\theta_{\mathcal{M}+1,2}(t) = \frac{1}{\sqrt{2}v}\int_{-\infty}^{t} Res_{\mathcal{M}}(s)\tanh(\sqrt{2}vs)\,ds, \tag{3.177}$$

$$\theta_{\mathcal{M}+1,1}(t) = \frac{-1}{\sqrt{2}v}\int_{0}^{t} Res_{\mathcal{M}}(s)\left[\sqrt{2}vs\tanh(\sqrt{2}vs)-1\right]\,ds, \tag{3.178}$$

that $r_{\mathcal{M}+1}(t)=\theta_{\mathcal{M}+1,1}(t)\tanh(\sqrt{2}vt)+\theta_{\mathcal{M}+1,2}(t)\left[\sqrt{2}vt\tanh(\sqrt{2}vt)-1\right]$ is even and satisfies the ordinary differential equation (3.174). Moreover, from the decay estimates of $Res_{\mathcal{M}}(t)$ in (3.176), we can deduce by induction on $l\in\mathbb{N}$ the existence of a number $n_{\mathcal{M}+1}\geq 0$ satisfying

$$\left|\frac{d^l}{dt^l}r_{\mathcal{M}+1}(t)\right| \lesssim_l v^{2\mathcal{M}+l}\left(|t|v+\ln\left(\frac{1}{v^2}\right)\right)^{n_{\mathcal{M}+1}} e^{-2\sqrt{2}|t|v}, \text{ for every } t\in\mathbb{R} \text{ and any } l\in\mathbb{N}, \tag{3.179}$$

so $\lim_{t\to+\infty} r_{\mathcal{M}+1}(t)$ exists and $\|r_{\mathcal{M}+1}(t)\|_{L^\infty_t(\mathbb{R})}\lesssim v^{2\mathcal{M}}\ln\left(\frac{1}{v}\right)^{n_{\mathcal{M}+1}}$.

Next, we are going to denote, for all $(t,x)\in\mathbb{R}^2$ and $0<v\ll 1$, $\phi_{\mathcal{M}+1,v,0}:\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}\phi_{\mathcal{M}+1,v,0}(t,x) =& H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\\ &+e^{-\sqrt{2}d(t)}\left[\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)-\mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right]\\ &+\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)-\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right),\end{aligned} \tag{3.180}$$

and use this function to construct $\varphi_{\mathcal{M}+1,v}:\mathbb{R}^2\to\mathbb{R}$ satisfying Theorem 3.5.1 for $k=\mathcal{M}+1$, which will imply the statement of this theorem for all $k\in\mathbb{N}_{\geq 2}$ by induction.

Since we assume Theorem 3.5.1 is true for $k=\mathcal{M}$, we deduce from Lemma 3.3.4 and estimates (3.179) of $r_{\mathcal{M}+1}$ that the following function

$$\begin{aligned}\phi_{\mathcal{M}+1,v,1}(t,x) =& H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\\ &+e^{-\sqrt{2}d(t)}\left[\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)-\mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right]\\ &+\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,x)\right)-\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,-x)\right), \text{ for every } (t,x)\in\mathbb{R}^2,\end{aligned} \tag{3.181}$$

satisfies

$$\Lambda(\phi_{\mathcal{M}+1,v,1})(t,x)\cong_{2\mathcal{M}+4}\Lambda\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right). \tag{3.182}$$

**Lemma 3.5.9.** *For any function $h \in L^{\infty}(\mathbb{R})$ such that $h^{'} \in \mathscr{S}(\mathbb{R})$, we have*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left[h\left(w_{\mathcal{M}}\left(t, x+r_{\mathcal{M}+1}(t)\right)\right)\right] \cong_{2\mathcal{M}+4} & \frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}\left[h\left(w_{\mathcal{M}}\left(t_1, x+r_{\mathcal{M}+1}(t)\right)\right)\right] \\ & +\frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} h^{'}\left(w_{\mathcal{M}}\left(t, x+r_{\mathcal{M}+1}(t)\right)\right) \\ & -\frac{\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}} h^{''}\left(w_{\mathcal{M}}\left(t, x+r_{\mathcal{M}+1}(t)\right)\right).\end{aligned} \tag{3.183}$$

*Proof of Lemma 3.5.9.* First, using (3.163) and the product rule of derivative, we can verify the following identity

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left[h\left(w_{\mathcal{M}}\left(t, x+r_{\mathcal{M}+1}(t)\right)\right)\right] = & \frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}\left[h\left(w_{\mathcal{M}}\left(t_1, x+r_{\mathcal{M}+1}(t)\right)\right)\right] \\ & +2\frac{\dot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\frac{\partial}{\partial t_1}\Big|_{t_1=t}\left[h^{'}\left(w_{\mathcal{M}}(t_1, x+r_{\mathcal{M}+1}(t))\right)\right] \\ & +2\dot{r}_{\mathcal{M}+1}(t)\left[\frac{d}{dt}\left(1-\frac{\dot{d}(t)^2}{4}\right)^{-\frac{1}{2}}\right] h^{'}\left(w_{\mathcal{M}}(t, x+r_{\mathcal{M}+1}(t))\right) \\ & +\frac{\dot{r}_{\mathcal{M}+1}(t)^2}{1-\frac{\dot{d}(t)^2}{4}} h^{''}\left(w_{\mathcal{M}}(t, x+r_{\mathcal{M}+1})\right)+\frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} h^{'}\left(w_{\mathcal{M}}(t, x+r_{\mathcal{M}+1})\right).\end{aligned}$$

We recall that the function $w_{\mathcal{M}}$ satisfies, for all $(t,x) \in \mathbb{R}^2$, the equation

$$w_{\mathcal{M}}(t,x) = w_0\left(t, x-\frac{d(t)}{2}+\sum_{j=2}^{\mathcal{M}} r_j(t)\right),$$

and the estimates in (3.145) are true for any $2 \leq k \leq \mathcal{M}$ from the inductive hypotheses of Theorem 3.1.2.

Using estimate (3.179) and the product rule of derivative, we deduce that

$$\left|\frac{d^l}{dt^l}\left[\dot{r}_{\mathcal{M}+1}(t)^2\right]\right| \lesssim_l v^{4\mathcal{M}+2+l}\left(|t|+\ln\left(\frac{1}{v}\right)\right)^{2n_{\mathcal{M}+1}} e^{-4\sqrt{2}|t|v}, \text{ for every } t \in \mathbb{R} \text{ and any } l \in \mathbb{N}\cup\{0\}.$$

Therefore, the estimate above, Lemma 3.3.1, Remark 3.3.3 and the product rule of derivative imply that

$$\frac{\dot{r}_{\mathcal{M}+1}(t)^2}{2-\frac{\dot{d}(t)^2}{4}} h^{''}\left(w_{\mathcal{M}}\left(t, x+r_{\mathcal{M}+1}\right)\right) \cong_{2\mathcal{M}+4} 0.$$

Moreover, from estimates (3.179), we deduce using Lemma 3.3.1, the chain and product rule of derivative that if $0 < v \ll 1$, then

$$\left|\frac{d^l}{dt^l}\left[\dot{r}_{\mathcal{M}+1}(t)\left[\frac{d}{dt}\left(1-\frac{\dot{d}(t)^2}{4}\right)^{-\frac{1}{2}}\right]\right]\right| \lesssim_l v^{2\mathcal{M}+4+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}+1}} e^{-2\sqrt{2}|t|v},$$

for every $t \in \mathbb{R}$ and any $l \in \mathbb{N}\cup\{0\}$. So, using Remark 3.3.3 and the product rule of derivative, we obtain that

$$\dot{r}_{\mathcal{M}+1}(t)\left[\frac{d}{dt}\left(1-\frac{\dot{d}(t)^2}{4}\right)^{-\frac{1}{2}}\right] h^{'}\left(w_{\mathcal{M}}(t, x+r_{\mathcal{M}+1}(t))\right) \cong_{2\mathcal{M}+4} 0.$$

Next, from estimates (3.179) and $\|r_{\mathcal{M}+1}(t)\|_{L^\infty} \lesssim v^{2\mathcal{M}} \ln\left(\frac{1}{v}\right)^{n_{\mathcal{M}+1}}$, we deduce using Lemma 3.3.4 for all $s \geq 1$ and $l \in \mathbb{N} \cup \{0\}$ that

$$\left\|\frac{\partial^l}{\partial t^l}\left[w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))h''\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right) - w_{\mathcal{M}}(t, x)h''\left(w_{\mathcal{M}}(t, x)\right)\right]\right\|_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+l}\left[\ln\frac{1}{v}\right]^{n_{\mathcal{M}+1}},$$

$$\left\|\frac{\partial^l}{\partial t^l}\left[h''\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right) - h''\left(w_{\mathcal{M}}(t, x)\right)\right]\right\|_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+l}\left[\ln\frac{1}{v}\right]^{n_{\mathcal{M}+1}}.$$

Therefore, since we are assuming the veracity of estimates (3.145) for any $2 \leq j \leq \mathcal{M}$, using the identity

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\Big|_{t_1=t} h'\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right) \\
&\quad = \left[\frac{d}{dt}\left(1 - \frac{\dot{d}(t)^2}{4}\right)^{\frac{-1}{2}}\right]\left(1 - \frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}} w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))h''\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right) \\
&\qquad + \frac{\dot{r}_{\mathcal{M}+1}(t) + \sum_{j=2}^{\mathcal{M}} \dot{r}_j(t) - \frac{\dot{d}(t)}{2}}{\sqrt{1 - \frac{\dot{d}(t)^2}{4}}} h''\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right),
\end{aligned}$$

estimate (3.179), Lemma 3.3.1 and the product rule of derivative, we deduce that

$$\frac{2\dot{r}_{\mathcal{M}+1}(t)}{\sqrt{1 - \frac{\dot{d}(t)^2}{4}}} \frac{\partial}{\partial t_1}\Big|_{t_1=t}\left[h'\left(w_{\mathcal{M}}(t_1, x + r_{\mathcal{M}+1}(t))\right)\right] \cong_{2\mathcal{M}+4} \frac{-\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t)}{1 - \frac{\dot{d}(t)^2}{4}} h''\left(w_{\mathcal{M}}(t_1, x)\right).$$

In conclusion, estimate (3.183) is true. □

### 3.5.3 Proof of Theorem 3.5.1.

*Proof of Theorem 3.5.1.* From the observations made at the beginning of this section, we need only to construct $\varphi_{\mathcal{M}+1,v}$ satisfying Theorem 3.5.1 from the function $\varphi_{\mathcal{M},v}$ denoted in (3.168). Let $\varphi_{\mathcal{M}+1,v} : \mathbb{R}^2 \to \mathbb{R}$ be the function satisfying the following identity

$$\varphi_{\mathcal{M}+1,v}(t, x) = \phi_{\mathcal{M}+1,v,0}(t, x) - L_1(\Gamma(t, \cdot))\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right) + L_1(\Gamma(t, \cdot))\left(w_{\mathcal{M}}(t, -x + r_{\mathcal{M}+1}(t)\right),$$

for every $(t, x) \in \mathbb{R}^2$, where $\phi_{\mathcal{M}+1,v,0}(t, x)$ is defined in (3.180).

From the definition of $\Lambda$, we have that

$$\begin{aligned}
\Lambda\left(\varphi_{\mathcal{M}+1,v}\right)(t, x) = &\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\phi_{\mathcal{M}+1,v,0}(t, x) + U'\left(\varphi_{\mathcal{M}+1,v}(t, x)\right) \\
&+ \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\left[-L_1(\Gamma(t, \cdot))\left(w_{\mathcal{M}}(t, x + r_{\mathcal{M}+1}(t))\right)\right] \quad (3.184) \\
&+ \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right] L_1(\Gamma(t, \cdot))\left(w_{\mathcal{M}}(t, -x + r_{\mathcal{M}+1}(t)\right)
\end{aligned}$$

Moreover, since $\left[-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}(x)\right)\right]L_1\left(\Gamma(t,\cdot)\right)(x)=\Gamma(t,x)$, and $w_{\mathcal{M}}(t,x)=\frac{x-\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}$, we have the following identity

$$\left[-\frac{4-\dot{d}(t)^2}{4}\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\right]L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right) \\ =\Gamma(t,w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))). \quad (3.185)$$

Moreover, from identity (3.184),we deduce that $\varphi_{\mathcal{M}+1,v}(t,x)$ satisfies

$$\begin{aligned}&\Lambda(\varphi_{\mathcal{M}+1,v})(t,x)-\Lambda(\phi_{\mathcal{M}+1,v,0})(t,x)\\ &\quad=\mathcal{L}_{\mathcal{M}+1,0}(t,x)+\mathcal{L}_{\mathcal{M}+1,1}(t,x)-\mathcal{L}_{\mathcal{M}+1,1}(t,-x)+\mathcal{L}_{\mathcal{M}+1,2}(t,x)-\mathcal{L}_{\mathcal{M}+1,2}(t,-x),\end{aligned} \quad (3.186)$$

for all $(t,x)\in\mathbb{R}^2$, where, for $0\leq j\leq 2$, the functions $\mathcal{L}_{\mathcal{M}+1,j}:\mathbb{R}^2\to\mathbb{R}$ satisfy for any $(t,x)\in\mathbb{R}^2$ the following identities:

$$\begin{aligned}\mathcal{L}_{M+1,0}(t,x)=&U^{'}\left(\varphi_{\mathcal{M}+1,v}(t,x)\right)-U^{'}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right)\\ &-U^{(2)}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right)\Big[L_1(\Gamma(t,\cdot))\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\\ &-L_1(\Gamma(t,\cdot))\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t)\right)\Big],\end{aligned} \quad (3.187)$$

$$\mathcal{L}_{\mathcal{M}+1,1}(t,x)=-\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\right]L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right), \quad (3.188)$$

$$\begin{aligned}\mathcal{L}_{M+1,2}(t,x)=&-\Big[U^{(2)}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right)\\ &-U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\Big]L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right).\end{aligned} \quad (3.189)$$

Next, for $3 \leq j \leq 6$, we denote the functions $\mathcal{L}_{\mathcal{M}+1,j} : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,3}(t,x) =& U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right) \\ & -U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right) - U^{'}\left(-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right),\end{aligned} \tag{3.190}$$

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,4}(t,x) =& \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\right]\left[e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right] \\ & -\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\left[e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right] \\ & -U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,\,-x+r_{\mathcal{M}+1}(t))\right)\right)e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right),\end{aligned} \tag{3.191}$$

$$\begin{aligned}\mathcal{L}_{M+1,5}(t,x) =& \Big[U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t)) - H_{0,1}\left(t,-x+r_{\mathcal{M}+1}(t)\right)\right)\right) \\ & -U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\Big]e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right) \\ & -\Big[U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t)) - H_{0,1}\left(t,-x+r_{\mathcal{M}+1}(t)\right)\right)\right) \\ & -U^{(2)}\left(-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right)\Big]e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right),\end{aligned} \tag{3.192}$$

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,6}(t,x) =& U^{'}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right) - U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right) \\ & -U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right)\times \\ & \Big[\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right) - \mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\Big]e^{-\sqrt{2}d(t)},\end{aligned} \tag{3.193}$$

and they satisfy the following equation

$$\begin{aligned}&\sum_{j=3}^{6}\mathcal{L}_{\mathcal{M}+1,j}(t,x) \\ &\quad = U^{'}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right) - U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right) - U^{'}\left(-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right) \\ &\qquad + \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\left[e^{-\sqrt{2}d(t)}\left(\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right) - \mathcal{G}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\right)\right].\end{aligned}$$

We recall the function $\phi_{\mathcal{M}+1,v,1}$ defined in (3.181) and obtain from (3.180), the identity above and estimate (3.182) that

$$\begin{aligned}\Lambda(\phi_{\mathcal{M}+1,v,1})(t,x) \cong_{2\mathcal{M}+4}& \Lambda(\phi_{\mathcal{M}+1,v,0})(t,x) \\ \cong_{2\mathcal{M}+4}& \Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right) + \Lambda\left(-H_{0,1}\left(w_{\mathcal{M}+1}(t,-x+r_{\mathcal{M}+1})\right)\right) \\ & + \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right]\left(\mathcal{T}_{\mathcal{M}}(vt,x+r_{\mathcal{M}+1}) - \mathcal{T}_{\mathcal{M}}(vt,-x+r_{\mathcal{M}+1})\right) \\ & + \sum_{j=3}^{6}\mathcal{L}_{\mathcal{M}+1,j}(t,x).\end{aligned} \tag{3.194}$$

Next, using Lemma 3.5.9, we obtain the following estimate

$$\begin{aligned}\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\cong_{2\mathcal{M}+4}&\left[\frac{\partial^2}{\partial t_1^2}\Big|_{t_1=t}-\frac{\partial^2}{\partial x^2}\right]H_{0,1}\left(w_{\mathcal{M}}(t_1,x+r_{\mathcal{M}+1}(t))\right)\\&+U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\\&+\frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H_{0,1}^{'}\left(w_{\mathcal{M}}\left(t,x+r_{\mathcal{M}+1}(t)\right)\right)\\&-\frac{\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}H_{0,1}^{''}\left(w_{\mathcal{M}}\left(t,x+r_{\mathcal{M}+1}(t)\right)\right).\end{aligned}\tag{3.195}$$

Consequently, using estimates (3.179) and Lemma 3.3.4 in the right-hand side of (3.195), we obtain the following estimate

$$\begin{aligned}\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\cong_{2\mathcal{M}+4}&\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\\&+\frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H_{0,1}^{'}\left(w_{\mathcal{M}}\left(t,x\right)\right)-\frac{\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}H_{0,1}^{''}\left(w_{\mathcal{M}}\left(t,x\right)\right)\\&+r_{\mathcal{M}+1}(t)\frac{\partial}{\partial x}\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right).\end{aligned}\tag{3.196}$$

Actually, since $H_{0,1}^{''}(x)=U^{'}(H_{0,1})$, we have

$$-\frac{\partial^2}{\partial x^2}H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)+U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)=\frac{-\dot{d}(t)^2}{4-\dot{d}(t)^2}H_{0,1}^{''}\left(w_{\mathcal{M}}(t,x)\right),$$

which implies the following equation

$$\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)=\frac{-\dot{d}(t)^2}{4-\dot{d}(t)^2}H_{0,1}^{''}\left(w_{\mathcal{M}}(t,x)\right)+\frac{\partial^2}{\partial t^2}H_{0,1}\left(w_{\mathcal{M}}(t,x)\right).$$

Consequently, since we are assuming that the estimates in (3.145) are true every $k\in\mathbb{N}$ satisfying $2\leq k\leq\mathcal{M}$, we deduce from Lemma 3.3.4 and estimate (3.179) that

$$\begin{aligned}r_{\mathcal{M}+1}(t)\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)=&\frac{-r_{\mathcal{M}+1}(t)\dot{d}(t)^2}{4-\dot{d}(t)^2}\ddot{H}_{0,1}\left(w_{\mathcal{M}}(t,x)\right)+r_{\mathcal{M}+1}(t)\frac{\partial^2}{\partial t^2}H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\\\cong_{2\mathcal{M}+4}&\frac{-r_{\mathcal{M}+1}(t)\dot{d}(t)^2}{4-\dot{d}(t)^2}\ddot{H}_{0,1}\left(w_0(t,x)\right)+r_{\mathcal{M}+1}(t)\frac{\partial^2}{\partial t^2}H_{0,1}\left(w_0(t,x)\right)\\\cong_{2\mathcal{M}+4}&\,r_{\mathcal{M}+1}(t)\Lambda\left(H_{0,1}\left(w_0(t,x)\right)\right).\end{aligned}$$

Therefore, from Lemma 3.4.2 and the above estimate above, we deduce that

$$\begin{aligned}r_{\mathcal{M}+1}(t)\frac{\partial}{\partial x}\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\cong_{2\mathcal{M}+4}&-\frac{r_{\mathcal{M}+1}(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}}H_{0,1}^{''}\left(w_0(t,x)\right)\\\cong_{2\mathcal{M}+4}&-\frac{r_{\mathcal{M}+1}(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}}H_{0,1}^{''}\left(w_{\mathcal{M}}(t,x)\right)\end{aligned}$$

due to Lemma 3.3.4 and the assumption that estimates (3.145) are true for $2 \leq k \leq \mathcal{M}$. In conclusion, we have the following estimate

$$\Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right) \cong_{2\mathcal{M}+4} \Lambda\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right) + \frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H_{0,1}^{'}\left(w_{\mathcal{M}}(t,x)\right)$$
$$- \frac{\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t) + r_{\mathcal{M}+1}(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}} H_{0,1}^{''}\left(w_{\mathcal{M}}(t,x)\right). \quad (3.197)$$

From now, we are going to divide the remaining part of the proof on different steps.

**Step.1**(Estimate of $\mathcal{L}_{\mathcal{M}+1,0}(t,x)$.) First, we recall the inequality $\|fg\|_{H^s_s} \lesssim_s \|f\|_{H^s_s}\|g\|_{H^s_x}$ for all $s \geq 1$. So, using Remark 3.3.3, Lemma 3.3.6, estimate (3.167) and the facts that $U \in C^{\infty}(\mathbb{R})$ and $\phi_{\mathcal{M}+1,v,0} \in L^{\infty}(\mathbb{R}^2) \cap C^{\infty}(\mathbb{R}^2)$, we obtain for any natural number $j \geq 3$ that the function

$$\mathcal{E}_{j,\mathcal{M}}(t,x) = U^{(j)}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right)\Big[L_1(\Gamma(t,\cdot))\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)$$
$$- L_1(\Gamma(t,\cdot))\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)\Big]^{j-1}$$

satisfies, for all $s \geq 1$, $\|\mathcal{E}_{j,\mathcal{M}}(t,x)\|_{H^s_x} \lesssim_s \|L_1(\Gamma(t,\cdot))(x)\|_{H^x_s}^{(j-1)} \lesssim v^{2\mathcal{M}+4}\left(\ln\left(\frac{1}{v}\right)\right)^{2n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v(j-1)}$ if $0 < v \ll 1$. Indeed, using Remark 3.3.3, estimate (3.167) and the product rule of derivative, we obtain similarly for all natural number $j \geq 3$ that

$$\left\|\frac{\partial^l}{\partial t^l}\mathcal{E}_{j,\mathcal{M}}(t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+4+l}\left(\ln\left(\frac{1}{v}\right)\right)^{2n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v} \text{ for all } l \in \mathbb{N}\cup\{0\}, \text{ if } 0 < v \ll 1.$$

Therefore, since $U(\phi) = \phi^2(1-\phi^2)^2$, the following function

$$\mathcal{L}_{M+1,0}(t,x) = -U^{(2)}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right)\Big[L_1(\Gamma(t,\cdot))\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right)$$
$$-L_1(\Gamma(t,\cdot))\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\Big]$$
$$+U^{'}\left(\varphi_{\mathcal{M}+1,v}(t,x)\right) - U^{'}\left(\phi_{\mathcal{M}+1,v,0}(t,x)\right)$$

satisfies $\mathcal{L}_{\mathcal{M}+1,0} \cong_{2\mathcal{M}+4} 0$.

**Step 2.**(Estimate of $\mathcal{L}_{\mathcal{M}+1,3}$.) In notation of Lemma 3.4.5, from the definition of $w_{\mathcal{M}}$ in

(3.163) and Remark 3.4.6, we have

$$
\begin{aligned}
U'&\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right)\\
&\qquad -U'\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)-U'\left(-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right)\\
=&24\exp\left(\frac{2\sqrt{2}\left(\rho_{\mathcal{M}}+r_{\mathcal{M}+1}\right)}{\sqrt{1-\frac{\dot{d}(t)^{2}}{4}}}\right)\left[M\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)-M\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right]\\
&-30\exp\left(\frac{2\sqrt{2}\left(\rho_{\mathcal{M}}+r_{\mathcal{M}+1}\right)}{\sqrt{1-\frac{\dot{d}(t)^{2}}{4}}}\right)\left[N\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)-N\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right]\\
&+24\exp\left(\frac{4\sqrt{2}\left(\rho_{\mathcal{M}}+r_{\mathcal{M}+1}\right)}{\sqrt{1-\frac{\dot{d}(t)^{2}}{4}}}\right)\left[V\left(w_{0}(t,x+r_{\mathcal{M}+1})\right)-V\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right]\\
&+\frac{60}{\sqrt{2}}\exp\left(\frac{4\sqrt{2}\left(\rho_{\mathcal{M}}+r_{\mathcal{M}+1}\right)}{\sqrt{1-\frac{\dot{d}(t)^{2}}{4}}}\right)\left[H_{0,1}'\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)-H_{0,1}'\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right]\\
&+R\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}),\frac{-4\rho_{\mathcal{M}}-4r_{\mathcal{M}+1}}{\sqrt{4-\dot{d}(t)^{2}}}\right).
\end{aligned}
$$

Moreover, Lemma 3.4.5 implies that $R\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}),\frac{-4\rho_{\mathcal{M}}(v,t)-4r_{\mathcal{M}+1}(t)}{\sqrt{4-\dot{d}(t)^{2}}}\right)$ is a finite sum of functions

$$
\exp\left(\frac{-4\left(2+d_{i}\right)\sqrt{2}\left(\rho_{\mathcal{M}}(v,t)-r_{\mathcal{M}+1}(t)\right)}{\sqrt{1-\frac{\dot{d}(t)^{2}}{4}}}\right)m_{i}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)n_{i}\left(-w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}(t))\right),
$$

where any $d_i\in\mathbb{N}$, every $m_i\in S^{+}$ and every $n_i\in S^{-}$. Consequently, using the decay estimates 3.179 of $r_{\mathcal{M}+1}$ and estimate (3.145) for any $2\leq k\leq\mathcal{M}$, Lemmas 3.3.4 and 3.3.5 imply that

$$
R\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}),\frac{-4\rho_{\mathcal{M}}(v,t)-4r_{\mathcal{M}+1}(t)}{\sqrt{4-\dot{d}(t)^{2}}}\right)\cong_{2\mathcal{M}+4}R\left(w_{\mathcal{M}}(t,x),\frac{-4\rho_{\mathcal{M}}(t)}{\sqrt{4-\dot{d}(t)^{2}}}\right).
$$

Furthermore, since we are assuming the veracity of Theorem 3.5.1 for any $k\leq\mathcal{M}$ belonging to $\mathbb{N}_{\geq 2}$, we deduce from the Fundamental Theorem of Calculus, Lemma 3.3.1, estimates (3.145) for $2\leq k\leq\mathcal{M}$ and estimate (3.179) that if $v\ll 1$, then

$$
\left|\frac{d^{l}}{dt^{l}}\left[e^{-\sqrt{2}(2\rho_{\mathcal{M}}(v,t)-2r_{\mathcal{M}+1}(t))}-e^{-2\sqrt{2}\rho_{\mathcal{M}}(v,t)}-2\sqrt{2}r_{\mathcal{M}+1}(t)e^{-2\sqrt{2}\rho_{\mathcal{M}}(v,t)}\right]\right|\lesssim_{l}v^{4\mathcal{M}+2+l}e^{-2\sqrt{2}v|t|},\text{ and}
$$

$$
\left|\frac{d^{l}}{dt^{l}}\left[e^{-2\sqrt{2}\rho_{\mathcal{M}}(v,t)}-e^{-\sqrt{2}d(t)}\right]\right|\lesssim_{l}v^{4}\left(\ln\frac{1}{v}\right)^{n_{2}}e^{-2\sqrt{2}v|t|},
$$

for any $l\in\mathbb{N}\cup\{0\}$. Therefore, using estimates $\|r_{\mathcal{M}}(t)\|_{L^{\infty}}\lesssim v^{2\mathcal{M}}\ln\left(\frac{1}{v}\right)^{n_{\mathcal{M}}}$, (3.179) and

(3.145) for $2 \leq k \leq \mathcal{M}$, we deduce from Lemmas 3.3.5, 3.3.4 the following estimate

$$\begin{aligned}
\mathcal{L}_{\mathcal{M}+1,3}(t,x) \cong_{2\mathcal{M}+4} & U^{'}\left(H_{0,1}^{w_{\mathcal{M}}}(t,x)\right) - U^{'}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right) - U^{'}\left(-H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right)\right) \\
& +2\sqrt{2}r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\left[24M\left(w_{\mathcal{M}}(t,x)\right) - 30N\left(w_{\mathcal{M}}(t,x)\right)\right] \\
& -2\sqrt{2}r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\left[24M\left(w_{\mathcal{M}}(t,-x)\right) - 30N\left(w_{\mathcal{M}}(t,-x)\right)\right] \\
& +\frac{r_{\mathcal{M}+1}e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[24M^{'}\left(w_{\mathcal{M}}(t,x)\right) - 30N^{'}\left(w_{\mathcal{M}}(t,x)\right)\right] \\
& -\frac{r_{\mathcal{M}+1}e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[24M^{'}\left(w_{\mathcal{M}}(t,-x)\right) - 30N^{'}\left(w_{\mathcal{M}}(t,-x)\right)\right]. \quad (3.198)
\end{aligned}$$

**Step 3.**(Estimate of $\mathcal{L}_{\mathcal{M}+1,4}$.) From Lemma 3.3.4, if $0 < v \ll 1$, we deduce for every $s \geq 1$ and every $l \in \mathbb{N} \cup \{0\}$ that

$$\left\Vert \frac{\partial^l}{\partial t^l}\left[\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right) - \mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)\right]\right\Vert_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+l}\left(\ln\left(\frac{1}{v}\right)+|t|v\right)^{n_{\mathcal{M}+1}},$$

which implies with Lemma 3.3.1 the following estimate

$$\frac{\partial^2}{\partial t^2}\left[e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right] \cong \frac{\partial^2}{\partial t^2}\left[e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)\right].$$

Moreover, using Lemma 3.3.1 and estimate 3.179, Lemma 3.3.4 also implies

$$\begin{aligned}
& e^{-\sqrt{2}d(t)}\left(\left[-\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\right]\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right) \\
& \cong_{2\mathcal{M}+4} e^{-\sqrt{2}d(t)}\left(\left[-\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\right]\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)\right) \\
& \quad + r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\frac{\partial}{\partial x}\left(\left[-\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\right]\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right)\right]\left(e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right) \\
& \cong_{2\mathcal{M}+4}\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\right]\left(e^{-\sqrt{2}d(t)}\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)\right) \\
& \quad +r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\frac{\partial}{\partial x}\left(\left[-\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\right]\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)\right).
\end{aligned}$$

In conclusion, recalling the notation $h^{w_{\mathcal{M}}}(t,x) = f\left(w_{\mathcal{M}}(t,x)\right) - h\left(w_{\mathcal{M}}(t,-x)\right)$ for any function $h : \mathbb{R} \to \mathbb{R}$, using Lemma 3.3.1, identity (3.191) and estimate (3.179), we obtain the following estimate

$$\begin{aligned}
\mathcal{L}_{\mathcal{M}+1,4}(t,x) \cong_{2\mathcal{M}+4} & r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\frac{\partial}{\partial x}\left(\left[-\mathcal{G}^{(2)} + U^{(2)}(H_{0,1})\mathcal{G}\right]^{w_{\mathcal{M}}}(t,x)\right) \\
& +e^{-\sqrt{2}d(t)}\left[-\mathcal{G}^{(2)} + U^{(2)}(H_{0,1})\mathcal{G}\right]^{w_{\mathcal{M}}}(t,x) + \frac{\partial^2}{\partial t^2}\left(e^{-\sqrt{2}d(t)}\mathcal{G}^{w_{\mathcal{M}}}(t,x)\right). \quad (3.199)
\end{aligned}$$

**Step 4.**(Estimate of $\mathcal{L}_{\mathcal{M}+1,1}$.) Since Lemma 3.5.5 implies for all $s\geq 1,\, l\in\mathbb{N}\cup\{0\}$ that $\left\Vert\frac{\partial^l}{\partial t^l}L_1\left(\Gamma(t,\cdot)\right)(x)\right\Vert_{H^s}\lesssim_{s,l} v^{2\mathcal{M}+l}\left(v|t|+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}}e^{-2\sqrt{2}v|t|}$ if $0<v\ll 1$, we can repeat the argument in the second step and obtain, from Lemma 3.3.4 and estimates 3.179, that

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,1}(t,x)=&-\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)\right]L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\\ \cong_{2\mathcal{M}+4}&-\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\right]L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)\\ \cong_{2\mathcal{M}+4}&-\Gamma\left(t,w_{\mathcal{M}}(t,x)\right)+\frac{\dot{d}(t)^2}{4-\dot{d}(t)^2}\frac{\partial^2}{\partial y^2}\Big|_{y=w_{\mathcal{M}}(t,x)}L_1\left(\Gamma(t,\cdot)\right)(y)\\ &-\frac{\partial^2}{\partial t^2}L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right).\end{aligned}$$

**Step 5.**(Estimate of $\mathcal{L}_{\mathcal{M}+1,5}$.) Lemma 3.3.4 and estimate (3.179) imply for all $m\in\mathbb{N},\, l\in\mathbb{N}\cup\{0\}$ the following estimates

$$\left\Vert\frac{\partial^l}{\partial t^l}\left[H_{0,1}\left(w_{\mathcal{M}}(t,\pm x+r_{\mathcal{M}+1})\right)^m-H_{0,1}\left(w_{\mathcal{M}}(t,\pm x)\right)^m\right]\right\Vert_{H^s_x}\lesssim_{m,s,l} v^{2\mathcal{M}+l}\left[\ln\frac{1}{v}\right]^{n_{\mathcal{M}+1}},\quad(3.200)$$

$$\left\Vert\frac{\partial^l}{\partial t^l}\left[\mathcal{G}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)^m-\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)^m\right]\right\Vert_{H^s_x}\lesssim_{m,s,l} v^{2\mathcal{M}+l}\left[\ln\frac{1}{v}\right]^{n_{\mathcal{M}+1}},\quad(3.201)$$

if $0<v\ll 1$. Therefore, since

$$U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})-H_{0,1}\left(t,-x+r_{\mathcal{M}+1}\right)\right)\right)-U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)$$

is a real linear combination of functions $H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}\right)^m H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}\right)^n$ such that $m\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$, we deduce using the identity (3.192) and Lemma 3.3.1 the following estimate

$$\mathcal{L}_{M+1,5}(t,x)\cong_{2\mathcal{M}+4} e^{-\sqrt{2}d(t)}U^{(2)}\left(H_{0,1}^{w_{\mathcal{M}}}(t,x)\right)\mathcal{G}^{w_{\mathcal{M}}}(t,x)-e^{-\sqrt{2}d(t)}\left[U^{(2)}(H_{0,1})\mathcal{G}\right]^{w_{\mathcal{M}}}(t,x),$$

where $f^{w_{\mathcal{M}}}(t,x)=f\left(w_{\mathcal{M}}(t,x)\right)-f\left(w_{\mathcal{M}}(t,-x)\right)$ for any function $f:\mathbb{R}\to\mathbb{R}$ and $(t,x)\in\mathbb{R}^2$.

**Step 6.**(Estimate of $\mathcal{L}_{\mathcal{M}+1,6}$.) From the definition of the functions $\varphi_{\mathcal{M},v}$, $\phi_{\mathcal{M}+1,0,v}$, $\mathcal{L}_{\mathcal{M}+1,6}$ respectively in (3.168), (3.180), (3.193) and using the notation

$$\mathcal{S}(v,t,x)=\phi_{\mathcal{M}+1,v,0}(t,x)-H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)+H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right),$$

we have the following identity

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,6}(t,x)=&\sum_{j=3}^{6}\frac{U^{(j)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right)}{(j-1)!}\Big[\mathcal{S}(v,t,x)\Big]^{j-1}\\ &+U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)-H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right)\\ &\times\left[\mathcal{T}_{\mathcal{M}}(vt,w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}))-\mathcal{T}_{\mathcal{M}}(vt,w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1}))\right].\end{aligned}$$

Furthermore, from the assumption that Theorem 3.5.1 is true for any $k\in\mathbb{N}$ satisfying $2\leq k\leq\mathcal{M}$, we have the following estimate

$$\left\Vert\frac{\partial^l}{\partial t^l}\left[\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}}(t,x)\right)\right]\right\Vert_{H^s_x}\lesssim_{s,l} v^{4+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{c_k}e^{-2\sqrt{2}|t|v},$$

for some positive constant $c_k$, all $s \geq 0$, and any $l \in \mathbb{N} \cup \{0\}$ if $0 < v \ll 1$. Therefore, using Lemma 3.3.4 and estimate (3.179), we obtain that if $0 < v \ll 1$, then the following inequality

$$\left\Vert \frac{\partial^l}{\partial t^l} \left[\mathcal{T}_{\mathcal{M}}\left(vt, w_{\mathcal{M}}(t, \pm x + r_{\mathcal{M}+1}(t))\right)\right]\right\Vert_{H^s_x} \lesssim_{s,l} v^{4+l} \left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{c_k} e^{-2\sqrt{2}|t|v}$$

is true for every $s \geq 0$ and any $l \in \mathbb{N} \cup \{0\}$. Thus, using estimates (3.200), (3.201) and the following algebraic property of $H^s_x$ for any $s > \frac{1}{2}$

$$\Vert fg \Vert_{H^s_x} \lesssim_s \Vert f \Vert_{H^s_x} \Vert g \Vert_{H^s_x}, \text{ for all } f,\, g \in H^s_x,$$

we deduce that

$$\begin{aligned}\mathcal{L}_{M+1,6}(t,x) \cong_{2\mathcal{M}+4}\, & U^{(2)}\left(H^{w_{\mathcal{M}}}_{0,1}(t,x)\right)\left[\mathcal{T}_{\mathcal{M}}(vt, w_{\mathcal{M}}(t,x)) - \mathcal{T}_{\mathcal{M}}(vt, w_{\mathcal{M}}(t,-x))\right] \\ & + \sum_{j=3}^{6} \frac{U^{(j)}\left(H^{w_{\mathcal{M}}}_{0,1}(t,x)\right)\left[\phi_{\mathcal{M},v}(t,x) - H^{w_{\mathcal{M}}}_{0,1}(t,x)\right]^{(j-1)}}{(j-1)!}. \quad (3.202)\end{aligned}$$

**Step 7.**(Estimate of $\mathcal{L}_{\mathcal{M}+1,2}$.) Finally, we will estimate the last term, which is

$$\begin{aligned}\mathcal{L}_{M+1,2}(t,x) = & \Big[-U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right) \\ & + U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)\Big] L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) \\ & - \Big[U^{(2)}\Big(\phi_{\mathcal{M},v,0}(t,x)\Big) \\ & - U^{(2)}\Big(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\Big)\Big] \\ & \times L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right).\end{aligned}$$

To simplify the estimate of this function, we are going to estimate separately the functions

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,2,1}(t,x) = & -\Big[U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\right) \\ & - U^{(2)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)\Big] L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,2,2}(t,x) = & \Big[U^{(2)}\Big(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right) - H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\Big) \\ & - U^{(2)}\Big(\phi_{\mathcal{M},v,0}(t,x)\Big)\Big] L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right),\end{aligned}$$

we also recall that $U^{(2)}(\phi) = 2 - 24\phi^2 + 30\phi^4$.

First, from Taylor's Theorem, we have

$$\begin{aligned}& \mathcal{L}_{\mathcal{M}+1,2,1}(t,x) \\ & = L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\Big[U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right) H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right) \\ & \quad + \sum_{j=4}^{6} \frac{(-1)^{(j-1)}}{(j-2)!} U^{(j)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right) H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)^{j-2}\Big]\end{aligned}$$

Next, from estimate (3.179) and Lemma 3.3.6, we have for any $f \in S^{+}_{\infty}$, $l$ and $m$ in $\mathbb{N}\cup\{0\}$, and $j \in \mathbb{N}$ that there exists $n_0 \in \mathbb{N}$ satisfying

$$\left\Vert \frac{\partial^l}{\partial t^l}\left[f\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)^{2m} H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)^{2j}\right]\right\Vert_{H^s_x} \lesssim_{s,l,m,j,f} v^{2+l}\left(\ln\left(\frac{1}{v}\right)+|t|v\right)^{n_0} e^{-2\sqrt{2}|t|v},$$

for any $s \geq 0$ if $0 < v \ll 1$. Therefore, using Lemmas 3.3.4, 3.5.5, identity (3.166), estimate (3.179) and the product rule of derivative, we deduce

$$\begin{aligned}&\mathcal{L}_{\mathcal{M}+1,2,1}(t,x)\\ \cong_{2\mathcal{M}+4}\,& L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\right)H_{0,1}\left(w_{\mathcal{M}}(t,-x+r_{\mathcal{M}+1})\right)\\ &\cong_{2\mathcal{M}+4} L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right).\end{aligned}$$

Moreover, since $\left|\frac{d^k}{dx^k}\left[H_{0,1}(x)-e^{\sqrt{2}x}\right]\right| \lesssim_k \min\left(e^{2\sqrt{2}x}, e^{\sqrt{2}x}\right)$, we can verify using Lemmas 3.2.1, 3.3.4, 3.3.6 and estimate (3.200) that

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,2,1}(t,x) &\cong_{2\mathcal{M}+4} L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)e^{\sqrt{2}w_{\mathcal{M}}(t,-x)}\\ &\cong_{2\mathcal{M}+4} L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)e^{-\sqrt{2}w_{\mathcal{M}}(t,x)}\exp\left(\frac{2\sqrt{2}\rho_{\mathcal{M}}(v,t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)\\ &\cong_{2\mathcal{M}+4} L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)e^{-\sqrt{2}w_{\mathcal{M}}(t,x)}e^{-\sqrt{2}d(t)},\end{aligned}$$

so $\left\Vert\frac{\partial^l}{\partial t^l}\mathcal{L}_{\mathcal{M}+1,2,1}(t,x)\right\Vert_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+2}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v}$.

Next, let $w_{\mathcal{M}+1}:\mathbb{R}^2\to\mathbb{R}$ be the unique function satisfying

$$w_{\mathcal{M}+1}(t,x) = w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t)), \text{ for all } (t,x)\in\mathbb{R}^2. \tag{3.203}$$

Since we are assuming that Theorem 3.5.1 is true for $k=\mathcal{M}$, Lemmas 3.3.6, 3.5.5 and the following identity

$$\begin{aligned}U^{(2)}\left(\phi_{\mathcal{M},v,0}(t,x)\right) =& U^{(2)}\left(H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right)+e^{-\sqrt{2}d(t)}U^{(3)}\left(H^{w_{\mathcal{M}+1}}(t,x)\right)\mathcal{G}^{w_{\mathcal{M}+1}}(t,x)\\ &+U^{(3)}\left(H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right)\left[\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}+1}(t,x)\right)-\mathcal{T}_{\mathcal{M}}\left(vt,w_{\mathcal{M}+1}(t,-x)\right)\right]\\ &+\sum_{j=4}^{6}\frac{1}{(j-2)!}U^{(j)}\left(H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right)\left[\phi_{\mathcal{M},v,0}(t,x)-H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right]^{j-2}\end{aligned}$$

imply

$$\begin{aligned}&L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)U^{(2)}\left(\phi_{\mathcal{M},v,0}(t,x)\right)\\ &\cong_{2\mathcal{M}+4} U^{(2)}\left(H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right)\\ &\quad+e^{-\sqrt{2}d(t)}U^{(3)}\left(H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right)\mathcal{G}^{w_{\mathcal{M}+1}}(t,x)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right).\end{aligned}$$

Thus, we obtain that

$$\mathcal{L}_{\mathcal{M}+1,2,2}(t,x) \cong_{2\mathcal{M}+4} -e^{-\sqrt{2}d(t)}U^{(3)}\left(H^{w_{\mathcal{M}+1}}_{0,1}(t,x)\right)\mathcal{G}^{w_{\mathcal{M}+1}}(t,x)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1})\right).$$

Indeed, using Lemma 3.3.4 and estimates (3.179), we deduce from the estimate above that

$$\mathcal{L}_{\mathcal{M}+1,2,2}(t,x) \cong_{2\mathcal{M}+4} -e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}^{w_{\mathcal{M}}}(t,x)\right)\mathcal{G}^{w_{\mathcal{M}}}(t,x)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right).$$

Furthermore, Lemmas 3.5.5 and 3.3.1 implies, for any $1 \leq i \leq N_1$,

$$\left|\frac{d^l}{dt^l}\left[s_{i,v}\left(\sqrt{2}vt\right)e^{-\sqrt{2}d(t)}\right]\right| \lesssim_l v^{2\mathcal{M}+2+l}\left(|t|v + \ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}} e^{-2\sqrt{2}|t|v},$$

for all $l \in \mathbb{N}\cup\{0\}$, if $0 < v \ll 1$. Also, Lemma 3.3.6 implies if $f \in S^{+}_{\infty}$, then there exists of $n_0 \in \mathbb{N}$ satisfying

$$\begin{aligned}\left\|\frac{\partial^l}{\partial t^l}\left[f\left(w_{\mathcal{M}}\left(t,x\right)\right)\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)^{\alpha}H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right)^{\beta}\right]\right\|_{H^s_x}\\ \lesssim_{s,\alpha,\beta,l} v^{2+l}\left(\ln\left(\frac{1}{v}\right)+|t|v\right)^{n_0}e^{-2\sqrt{2}|t|v},\end{aligned}$$

for all $s \geq 0$, every $l \in \mathbb{N}\cup\{0\}$, and any $\alpha \in \mathbb{N}\cup\{0\}$, $\beta \in \mathbb{N}$ with $\alpha+\beta$ odd, if $0 < v \ll 1$. By similar reasoning, if $f \in S^{+}_{\infty}$, there exists $n_0 \in \mathbb{N}$ satisfying

$$\begin{aligned}\left\|\frac{\partial^l}{\partial t^l}\left[f\left(w_{\mathcal{M}}\left(t,x\right)\right)\mathcal{G}\left(w_{\mathcal{M}}(t,-x)\right)H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)^{\alpha}H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right)^{\beta}\right]\right\|_{H^s_x}\\ \lesssim_{s,\alpha,\beta,l} v^{2+l}\left(\ln\left(\frac{1}{v}\right)+|t|v\right)^{n_0}e^{-2\sqrt{2}|t|v},\end{aligned}$$

for all $s \geq 0$, $l \in \mathbb{N}\cup\{0\}$ and any $\alpha$, $\beta \in \mathbb{N}\cup\{0\}$ with $\alpha+\beta$ odd, if $0<v \ll 1$. Therefore, using the estimate above, inequality (3.167) and the inequality

$$\|fg\|_{H^s_x} \lesssim_s \|f\|_{H^{s+1}_x}\|g\|_{H^{s+1}_x},$$

for any $f$, $g \in \mathscr{S}(\mathbb{R})$ and all $s \geq 0$, we deduce that

$$\mathcal{L}_{\mathcal{M}+1,2,2}(t,x) \cong_{2\mathcal{M}+4} -e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right).$$

As a consequence, we obtain that

$$\left\|\frac{\partial^l}{\partial t^l}\mathcal{L}_{\mathcal{M}+1,2}(t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{2\mathcal{M}+2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}}e^{-2\sqrt{2}|t|v}, \tag{3.204}$$

and

$$\begin{aligned}\mathcal{L}_{\mathcal{M}+1,2}(t,x) \cong_{2\mathcal{M}+4} & -e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)\\ & +e^{-\sqrt{2}d(t)}L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)e^{-\sqrt{2}w_{\mathcal{M}}(t,x)}.\end{aligned}$$

**Step 8.**(Estimate of $\Lambda(\phi_{\mathcal{M}+1,v})$.) From the equation (3.186) and the conclusions obtained in all the steps before, we deduce

$$\begin{aligned}
&\Lambda(\varphi_{\mathcal{M}+1,v})(t,x)-\Lambda(\phi_{\mathcal{M}+1,v,0})(t,x)\\
&\quad\cong_{2\mathcal{M}+4}-\Gamma\left(t,w_{\mathcal{M}}(t,x)\right)+\Gamma\left(t,w_{\mathcal{M}}(t,-x)\right)\\
&\qquad+\frac{\dot{d}(t)^2}{4-\dot{d}(t)^2}\frac{\partial^2}{\partial y^2}\Big|_{y=w_{\mathcal{M}}(t,x)}L_1\left(\Gamma(t,\cdot)\right)(y)-\frac{\dot{d}(t)^2}{4-\dot{d}(t)^2}\frac{\partial^2}{\partial y^2}\Big|_{y=w_{\mathcal{M}}(t,-x)}L_1\left(\Gamma(t,\cdot)\right)(y)\\
&\qquad-\frac{\partial^2}{\partial t^2}L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)+\frac{\partial^2}{\partial t^2}L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,-x)\right)\\
&\qquad+e^{-\sqrt{2}d(t)}L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)e^{-\sqrt{2}w_{\mathcal{M}}(t,x)}\\
&\qquad-e^{-\sqrt{2}d(t)}L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,-x)\right)U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right)\right)e^{-\sqrt{2}w_{\mathcal{M}}(t,x)}\\
&\qquad-e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,x)\right)\right)\mathcal{G}\left(w_{\mathcal{M}}(t,x)\right)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,x)\right)\\
&\qquad+e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}\left(w_{\mathcal{M}}(t,-x)\right)\right)\mathcal{G}\left(w_{\mathcal{M}}(t,-x)\right)L_1\left(\Gamma(t,\cdot)\right)\left(w_{\mathcal{M}}(t,-x)\right).
\end{aligned}$$

Furthermore, from (3.194) and the estimates of $\mathcal{L}_{\mathcal{M}+1,j}$ for $3\leq j\leq 6$, we deduce

$$\begin{aligned}
&\Lambda(\phi_{\mathcal{M}+1,v,0})(t,x)\\
&\quad\cong_{2\mathcal{M}+4}\Lambda(\varphi_{\mathcal{M},v})(t,x)+2\sqrt{2}r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\left[24M^{w_{\mathcal{M}}}(t,x)-30N^{w_{\mathcal{M}}}(t,x)\right]\\
&\qquad+\frac{r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[24\left(M^{'}\right)^{w_{\mathcal{M}}}(t,x)-30\left(N^{'}\right)^{w_{\mathcal{M}}}(t,x)\right]\\
&\qquad+r_{\mathcal{M}+1}(t)\frac{\partial}{\partial x}\left(\left[-\mathcal{G}^{(2)}+U^{(2)}\left(H_{0,1}\right)\mathcal{G}\right]^{w_{\mathcal{M}}}(t,x)\right)e^{-\sqrt{2}d(t)}\\
&\qquad+\frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left[H^{'}_{0,1}\left(w_{\mathcal{M}}\left(t,x\right)\right)-H^{'}_{0,1}\left(w_{\mathcal{M}}\left(t,-x\right)\right)\right]\\
&\qquad-\frac{\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}\left[H^{''}_{0,1}\left(w_{\mathcal{M}}\left(t,x\right)\right)-H^{''}_{0,1}\left(w_{\mathcal{M}}\left(t,-x\right)\right)\right],
\end{aligned}$$

from which with Remark (3.4.4) we deduce that

$$\begin{aligned}
&\Lambda(\phi_{\mathcal{M}+1,v,0})(t,x)\\
&\quad\cong_{2\mathcal{M}+4}\Lambda(\varphi_{\mathcal{M},v})(t,x)+2\sqrt{2}r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}\left[24M^{w_{\mathcal{M}}}(t,x)-30N^{w_{\mathcal{M}}}(t,x)\right]\\
&\qquad+\frac{8\sqrt{2}r_{\mathcal{M}+1}(t)e^{-\sqrt{2}d(t)}-\dot{r}_{\mathcal{M}+1}(t)\dot{d}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H^{''}_{0,1}\right)^{w_{\mathcal{M}}}(t,x)\\
&\qquad+\frac{\ddot{r}_{\mathcal{M}+1}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left(H^{'}_{0,1}\right)^{w_{\mathcal{M}}}(t,x),
\end{aligned}$$

We also have, from Lemmas 3.5.5, 3.5.8, for all $l\in\mathbb{N}\cup\{0\}$ and any $s\geq 0$ that if $0<v\ll 1$, then

$$\begin{aligned}
&\left\Vert\frac{\partial^l}{\partial t^l}\left[\Lambda\left(\varphi_{\mathcal{M},v}\right)(t,x)-\Gamma\left(t,w_{\mathcal{M}}(t,x)\right)+\Gamma\left(t,w_{\mathcal{M}}(t,-x)\right)\right]\right\Vert_{H^s_x}\\
&\qquad\qquad\lesssim_{s,l}v^{2\mathcal{M}+2+l}\left(|t|v+\ln\left(\frac{1}{v}\right)\right)^{n_{\mathcal{M}}}e^{-2\sqrt{2}|t|v}.
\end{aligned}$$

Therefore, from the estimates above, inequalities (3.204), (3.179), Lemmas 3.3.1, 3.5.5 and Remark 3.3.3, we obtain that the estimate (3.147) of Theorem 3.5.1 is true for $k = \mathcal{M}+1$.

Furthermore, Lemma 3.3.4 and (3.179) imply that if $h \in S^{+}_{\infty}$, then we have for all $l \in \mathbb{N}\cup\{0\}$ the following inequality

$$\left|\frac{d^l}{dt^l}\left\langle h\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)-h\left(w_{\mathcal{M}}(t,x)\right), H^{'}_{0,1}\left(w_{\mathcal{M}}(t,x+r_{\mathcal{M}+1}(t))\right)\right\rangle\right| \lesssim_l v^{2\mathcal{M}+l}\left[\ln\frac{1}{v}\right]^{n_{\mathcal{M}+1}}.$$

Therefore, the estimates above, Remark 3.3.7, the ordinary differential equation (3.174) satisfied by $r_{\mathcal{M}+1}$ and estimate (3.179) of the derivatives of $r_{\mathcal{M}+1}$ imply (3.148) for $k = \mathcal{M}+1$. In conclusion, by induction on $k$, we deduce that Theorem 3.5.1 is true for all $k \in \mathbb{N}_{\geq 2}$. □

**Remark 3.5.10.** *From Theorem 3.5.1, we have that if $v \ll 1$, then*

$$\lim_{t\to+\infty}\sum_{k=1}^{\mathcal{M}} r_k(v,t) \text{ exists.}$$

### 3.5.4 Proof of Theorem 3.1.2

*Proof of Theorem 3.1.2.* The Theorem 3.5.1 implies the existence, for any $k \in \mathbb{N}_{\geq 2}$, of a smooth function $\varphi_{k,v}(t,x)$ and a even function $r(t) \in L^{\infty}(\mathbb{R})$ such that if $v \ll 1$, then

$$\lim_{t\to\pm\infty}\left\|\varphi_{k,v}(t,x)-H_{0,1}\left(\frac{x\mp vt+r(t)}{\sqrt{1+v^2}}\right)-H_{-1,0}\left(\frac{x\pm vt-r(t)}{\sqrt{1+v^2}}\right)\right\|_{H^1_x}=0,$$

$$\lim_{t\to\pm\infty}\left\|\partial_t\varphi_{k,v}(t,x)\pm\frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x\mp vt+r(t)}{\sqrt{1+v^2}}\right)\mp\frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x\pm vt-r(t)}{\sqrt{1+v^2}}\right)\right\|_{L^2_x}=0,$$

and $\lim_{t\to+\infty}|r(t)| \lesssim v^2\ln\left(\frac{1}{v}\right)$. In conclusion, from Lemma 3.3.1, Remark 3.5.10 and Theorem 3.5.1, the function

$$\phi_k(v,t,x)=\varphi_{k,v}\left(t+\frac{\ln(v^2)-\ln(8)}{2\sqrt{2}v}+\lim_{s\to+\infty}\frac{r(s)}{v},x\right)$$

satisfies Theorem 3.1.2. □

# Chapter 4

# On the kink-kink collision problem for the $\phi^6$ model with low speed

# Abstract

We study the elasticity of the collision of two kinks with an incoming low speed $v \in (0,1)$ for the nonlinear wave equation in dimension $1+1$ known as the $\phi^6$ model. We prove for any $k \in \mathbb{N}$ that if the incoming speed $v$ is small enough, then, after the collision, the two kinks will move away with a velocity $v_f$ such that $|v_f - v| \leq v^k$ and the energy of the remainder will also be smaller than $v^k$. This chapter is the continuation of the work done in Chapter 3 where we constructed a sequence $\phi_k$ of approximate solutions for the $\phi^6$ model. The proof of our main result relies on the use of the set of approximate solutions from Chapter 3, modulation analysis, and a refined energy estimate method to evaluate the precision of our approximate solutions during a large time interval.

## 4.1 Introduction

### 4.1.1 Background

First, we recall the potential function $U(\phi) = \phi^2(1-\phi^2)^2$ and the partial differential equation ($\phi^6$)

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U^{'}(\phi(t,x)) = 0.$$

From Chapter 1, we have verified that all solutions $\phi(t,x)$ of ($\phi^6$) in the energy space preserve the following quantities

$$E(\phi)(t) = \int_{\mathbb{R}} \frac{[\partial_t\phi(t,x)]^2 + [\partial_x\phi(t,x)]^2}{2} + U(\phi(t,x))\, dx, \qquad \text{(Energy)}$$

$$P(\phi) =- \int_{\mathbb{R}} \partial_t\phi(t,x)\partial_x\phi(t,x)\, dx. \qquad \text{(Momentum)}$$

We also recall the kinetic energy and potential energy, which are given respectively by

$$E_{kin}(\phi)(t) = \int_{\mathbb{R}} \frac{[\partial_t\phi(t,x)]^2}{2}\, dx, \quad E_{pot}(\phi)(t) = \int_{\mathbb{R}} \frac{[\partial_x\phi(t,x)]^2}{2} + U(\phi(t,x))\, dx.$$

We recall that all the kinks associated with the partial differential equation ($\phi^6$) are given by the space translation of the following functions

$$H_{0,1}(x) = \frac{e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}}, \quad H_{-1,0}(x) = -H_{0,1}(-x) = \frac{-e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}},$$

and the anti-kinks are the space translation of the following functions

$$H_{1,0}(x) = H_{0,1}(-x) = \frac{e^{-\sqrt{2}x}}{\sqrt{1+e^{-2\sqrt{2}x}}}, \quad H_{0,-1}(x) = -H_{0,1}(x) = \frac{-e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}}.$$

From the previous chapters, we recall the following identity

$$\left\|\frac{d}{dx}H_{0,1}(x)\right\|_{L^2_x}^2 = \frac{1}{2\sqrt{2}}, \tag{4.1}$$

and the following estimates for any $k \geq 1$

$$\left|\frac{d^k}{dx^k}H_{0,1}(x)\right| \lesssim_k \min\left(e^{\sqrt{2}x}, e^{-2\sqrt{2}x}\right), \tag{4.2}$$

and

$$|H_{0,1}(x)| \leq e^{\sqrt{2}\min(x,0)}. \tag{4.3}$$

In this chapter, we study the traveling kink-kink solutions of ($\phi^6$) with speed $0 < v < 1$ small enough. More precisely, we consider the following definition.

**Definition 4.1.1.** *The traveling kink-kink with speed $v \in (0,1)$ is the unique solution $\phi(t,x)$ that satisfying for some positive constants $K$, $c$ and any $t \geq K$ the following decay estimate*

$$\left\|(\phi(t,x), \partial_t\phi(t,x)) - \overrightarrow{H_{0,1}}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - \overrightarrow{H_{-1,0}}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \leq e^{-ct}, \tag{4.4}$$

*where, for any* $-1 < v < 1$ *and any* $y \in \mathbb{R}$,

$$\overrightarrow{H_{0,1}}\left(\frac{x - vt + y}{\sqrt{1 - v^2}}\right) = \begin{bmatrix} H_{0,1}\left(\frac{x-vt+y}{\sqrt{1-v^2}}\right) \\ \frac{-v}{\sqrt{1-v^2}} H_{0,1}^{'}\left(\frac{x-vt+y}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{4.5}$$

$$\overrightarrow{H_{-1,0}}\left(\frac{x + vt - y}{\sqrt{1 - v^2}}\right) = \begin{bmatrix} H_{-1,0}\left(\frac{x+vt-y}{\sqrt{1-v^2}}\right) \\ \frac{v}{\sqrt{1-v^2}} H_{-1,0}^{'}\left(\frac{x+vt-y}{\sqrt{1-v^2}}\right) \end{bmatrix}. \tag{4.6}$$

The existence and uniqueness for any $0 < v < 1$ of solutions $\phi(t, x)$ satisfying (4.4) was obtained in [8], but the uniqueness of the solution of $(\phi^6)$ satisfying for $0 < v < 1$

$$\lim_{t \to +\infty} \left\| \overrightarrow{\phi}(t, x) - \overrightarrow{H_{0,1}}\left(\frac{x - vt}{\sqrt{1 - v^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x + vt}{\sqrt{1 - v^2}}\right) \right\|_{H^1_x \times L^2_x} = 0$$

is still an open problem. For references on the existence and uniqueness of multi-soliton solutions of other nonlinear dispersive partial differential equations, see for example [38] and [12].

For non-integrable dispersive models, there exist previous results about the inelasticity of the collision of two solitons. For example, in the article [41], Martel and Merle verified that the collision between two solitons with nearly equal speed is not elastic. More precisely, they obtained that the incoming speed of the two solitons is different of their outgoing speed after their collision.

Since the $\phi^6$ model is a non-integrable system, the collision of two kinks with low speed $0 < v < 1$ is expected to be inelastic. More precisely, we were expecting the existence of a value $k > 1$ such that if $0 < v \ll 1$ and $\phi(t, x)$ is a solution $(\phi^6)$ satisfying the condition (4.4), then $\phi(t, x)$ should have inelasticity of order $v^k$, which means the existence of $t < 0$ with $|t| \gg 1$ such that

$$(\phi(t, x), \partial_t \phi(t, x)) = \overrightarrow{H_{0,1}}\left(\frac{x + v_f t + y_1(t)}{\sqrt{1 - v_f^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x - v_f t + y_2(t)}{\sqrt{1 - v_f^2}}\right) + ro(t, x), \tag{4.7}$$

with $v^k \ll \|ro(t)\|_{H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})} \ll v$ and $v_f(t)$, $y_1$, $y_2$ satisfying

$$v^k \ll |v_f(t) - v| + \max_{j \in \{1,2\}} |\dot{y}_j(t)| \ll v, \tag{4.8}$$

for all $t < 0$ satisfying $|t| \gg 1$. Actually, in the quartic $gKdV$, the collision of the two solitons satisfies a similar property than our previous expectations in (4.7) and (4.8), see Theorem 1 in the article [41] of Martel and Merle for more details.

However, in this chapter, we prove for the $\phi^6$ model and any $k > 1$ that if $0 < v \ll 1$ and $t$ is close to $-\infty$, both estimates (4.7) and (4.8) are not possible. Indeed, we demonstrate that if $v \ll 1$ and $\phi(t, x)$ satisfies (4.4), then there exists a number $e_{k,2v} \in \mathbb{R}$ satisfying, for all $t$ close to $-\infty$,

$$(\phi(t, x), \partial_t \phi(t, x)) = \overrightarrow{H_{0,1}}\left(\frac{x + v_f t - e_{k,2v}}{\sqrt{1 - v_f^2}}\right) + \overrightarrow{H_{-1,0}}\left(\frac{x - v_f t + e_{k,2v}}{\sqrt{1 - v_f^2}}\right) + r_{c,v}(t, x),$$

$\limsup_{t\to-\infty}\|r_{c,v}(t)\|_{H^1_x\times L^2_x}\leq v^{2k}$ and

$$\limsup_{t\to-\infty}|v_f(v,t)-v|\leq v^{2k}. \tag{4.9}$$

In conclusion, the inelasticity of the collision of two kinks cannot be of any order $v^k$ for any $1\ll k\in\mathbb{N}$, if the incoming speed $v$ of the kinks is small enough. The problem to verify the inelasticity of the collision of kinks for the $\phi^6$ model is still open. But, because of the conclusion obtained in this paper, the change $|v-v_f|$ in the speeds of each soliton is much smaller than any monomial function $v^k$, more precisely for all $k>0$

$$\lim_{v\to0^+}\limsup_{t\to-\infty}\frac{|v_f(v,t)-v|}{v^k}=0, \tag{4.10}$$

which is a new result.

The study of collision of kinks for the $\phi^6$ model is important for high energy physics, see for example [17] and [14]. Actually, in the article [17], it was obtained numerically the existence of a critical speed $v_c$ such that if each of the two kinks moves with speed $v$ with absolute value less than $v_c$ and they approach each other, then they will collide and the collision will be very elastic, which is exactly the result we obtained rigorously in this chapter. The study of the dynamics of multi-soliton solutions of the $\phi^6$ model has also applications in condensed matter physics, see [3], and cosmology, see [62].

For other nonlinear dispersive equations, there exist rigorous results of inelasticity and stability of collision of solitons. For $gKdV$ models, the inelasticity of collision of solitons was proved for the quartic $gKdV$ in [41], and, for a certain class of generalized $gKdV$, inelasticity of collision between solitons was also proved in [49] and [50] by Muñoz, see also the article [39] of Martel and Merle. For nonlinear Schrödinger equation, in [53], Perelman studied the collision of two solitons of different sizes and obtained that after that the solution does not preserve the two solitons' structure after the collision. See also the work [42] by Martel and Merle about the inelasticity of the collision of two solitons for the fifth-dimensional energy critical wave equation.

### 4.1.2 Main Results

The main theorem obtained in Chapter 4 is the following result:

**Theorem 4.1.2.** *There exists a continuous function* $v_f:(0,1)\times\mathbb{R}\to(0,1)$ *and, for any* $0<\theta<1$ *and* $k\in\mathbb{N}_{\geq2}$, *there exists* $0<\delta(\theta,k)<1$, *such that if* $0<v<\delta(\theta,k)$, *and* $\phi(t,x)$ *is a traveling kink-kink solution of* $(\phi^6)$ *with speed* $v$, *then there exists a number* $e_{v,k}$ *such that* $|e_{v,k}|<\ln\left(\frac{8}{v^2}\right)$ *and if* $t\leq-\frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v}$, *then* $|v_f(v,t)-v|<v^k$ *and*

$$\left\|\phi(t,x)-H_{0,1}\left(\frac{x-e_{k,v}+v_ft}{\sqrt{1-v_f^2}}\right)-H_{-1,0}\left(\frac{x+e_{k,v}-v_ft}{\sqrt{1-v_f^2}}\right)\right\|_{H^1_x(\mathbb{R})}$$
$$+\left\|\partial_t\phi(t,x)-\frac{v_f}{\sqrt{1-v_f^2}}H^{'}_{0,1}\left(\frac{x-e_{v,k}+v_ft}{\sqrt{1-v_f^2}}\right)+\frac{v_f}{\sqrt{1-v_f^2}}H^{'}_{-1,0}\left(\frac{x+e_{v,k}-v_ft}{\sqrt{1-v_f^2}}\right)\right\|_{L^2_x(\mathbb{R})}\leq v^k.$$

*If* $\frac{-4\left(\ln\frac{1}{v}\right)^{2-\theta}}{v} \leq t \leq \frac{-\left(\ln\frac{1}{v}\right)^{2-\theta}}{v}$, *then*

$$\left\Vert \phi(t,x) - H_{0,1}\left(\frac{x-e_{k,v}+vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+e_{k,v}-vt}{\sqrt{1-v^2}}\right)\right\Vert_{H^1_x(\mathbb{R})}$$
$$+\left\Vert \partial_t\phi(t,x) - \frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-e_{v,k}+vt}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+e_{v,k}-vt}{\sqrt{1-v^2}}\right)\right\Vert_{L^2_x(\mathbb{R})} \leq v^k.$$

Clearly, Theorem 4.1.2 implies (4.10). Actually, the first inequality of Theorem 4.1.2 is a consequence of the second inequality of this theorem and the following result about the orbital stability of two moving kinks.

**Theorem 4.1.3.** *There exists a constant* $c > 0$ *and, for any* $\theta \in (0,1)$, *there exists* $\delta(\theta) \in (0,1)$ *such that if* $0 < v < \delta(\theta)$, *and* $(u_1(x), u_2(x)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ *is an odd function satisfying*

$$\Vert (u_1,u_2)\Vert_{H^1_x\times L^2_x} < v^{2+\theta}, \tag{4.11}$$

*and* $y_0 \geq -4\ln v$, *then the solution* $(\phi(t,x), \partial_t\phi(t,x))$ *of the Cauchy problem*

$$\begin{cases}\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U^{'}(\phi(t,x)) = 0,\\ \begin{bmatrix}\phi(0,x)\\ \partial_t\phi(0,x)\end{bmatrix} = \begin{bmatrix} H_{0,1}\left(\frac{x-y_0}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+y_0}{\sqrt{1-v^2}}\right) + u_1(x)\\ \frac{-v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-y_0}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+y_0}{\sqrt{1-v^2}}\right) + u_2(x)\end{bmatrix}\end{cases} \tag{4.12}$$

*is given for all* $t \geq 0$ *by*

$$\begin{bmatrix}\phi(t,x)\\ \partial_t\phi(t,x)\end{bmatrix} = \begin{bmatrix} H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right) + \psi_1(t,x)\\ \frac{-v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right) + \psi_2(t,x)\end{bmatrix}, \tag{4.13}$$

*such that*

$$\begin{aligned}|y(0)-y_0| + \Vert(\psi_1(t,x),\psi_2(t,x))\Vert_{H^1_x\times L^2_x} \leq & c\,\Vert(u_1(x),u_2(x))\Vert^{\frac{1}{2}}_{H^1_x\times L^2_x} + c(1+y_0)^{\frac{1}{2}}e^{-\sqrt{2}y_0},\\ |\dot{y}(t)-v| \leq & c\,\Vert(u_1(x),u_2(x))\Vert^{\frac{1}{2}}_{H^1_x\times L^2_x} + ce^{-\sqrt{2}y_0}y_0^{\frac{1}{2}},\end{aligned} \tag{4.14}$$

*for all* $t \in \mathbb{R}_{\geq 0}$.

### 4.1.3 Notation

In this subsection, we explain the notation that we are going to use in the next sections of Chapter 4.

**Notation 4.1.4.** *First, for any real function* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfying the conditions* $f(t,\cdot) \in L^\infty_x(\mathbb{R})$, *and* $\partial_t f(t,\cdot) \in L^2_x(\mathbb{R})$, *we denote the function* $\overrightarrow{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$\overrightarrow{f}(t,x) = (f(t,x), \partial_t f(t,x))\,, \text{ for every } (t,x) \in \mathbb{R}^2.$$

*Next, for any subset $\mathcal{D}\subset\mathbb{R}$, any $v\in(0,1)$ and any function $y:\mathcal{D}\to\mathbb{R}$, we define the functions $\overrightarrow{H_{0,1,v,y}}:\mathcal{D}\times\mathbb{R}\to\mathbb{R}^2$, $\overrightarrow{H_{-1,0,v,y}}:\mathcal{D}\times\mathbb{R}\to\mathbb{R}^2$ by*

$$\overrightarrow{H_{0,1,v,y}}(t,x)=\begin{bmatrix}H_{0,1}\left(\frac{x-vt+y(t)}{\sqrt{1-v^2}}\right)\\ \frac{-v}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x-vt+y(t)}{\sqrt{1-v^2}}\right)\end{bmatrix},$$

$$\overrightarrow{H_{-1,0,v,y}}(t,x)=\begin{bmatrix}H_{-1,0}\left(\frac{x+vt-y(t)}{\sqrt{1-v^2}}\right)\\ \frac{v}{\sqrt{1-v^2}}H_{-1,0}^{'}\left(\frac{x+vt-y(t)}{\sqrt{1-v^2}}\right)\end{bmatrix}.$$

*We say that two non-negative functions $f_1(\alpha_1,...,\alpha_n,x)$ and $f_2(\alpha_1,...,\alpha_n,x)$ both with domain $\mathcal{D}\times\mathbb{R}\subset\mathbb{R}^{n+1}$ satisfy $f_1\lesssim_{\alpha_1,...,\alpha_n}f_2$ if there is a positive function $L:\mathcal{D}\to\mathbb{R}_{\geq 1}$ such that*

$$f_1(\alpha_1,...,\alpha_n,x)\leq L(\alpha_1,\,...,\alpha_n)f_2(\alpha_1,...,\alpha_n,x) \text{ for all } (\alpha_1,...,\alpha_m,x)\in\mathcal{D}\times\mathbb{R}.$$

*Moreover, for any $s\geq 0$, we consider the norm $\lVert\cdot\rVert_{H^s_x}$ given by*

$$\lVert f\rVert_{H^s_x}=\lVert f\rVert_{H^s_x}=\left(\int_{\mathbb{R}}(1+|x|)^{2s}|\hat{f}(x)|^2\,dx\right)^{\frac{1}{2}},\text{ for any } f\in H^s_x(\mathbb{R}),$$

*where $\hat{f}$ is the Fourier transform of the function $f$.*

*Finally, for any $n\in\mathbb{N}$ and any $a$, $b\in\mathbb{R}^n$, we denote the scalar product in the Euclidean space $\mathbb{R}^n$ by*

$$\langle a:b\rangle=\sum_{j=1}^{n}a_jb_j,$$

*where $a=(a_1,...,a_n)$ and $b=(b_1,...,b_n)$.*

### 4.1.4 Approximate solutions

We recall the following definition and theorem from Chapter 3.

**Definition 4.1.5.** *We define $\Lambda$ as the nonlinear operator with domain $C^2(\mathbb{R}^2,\mathbb{R})$ that satisfies:*

$$\Lambda(\phi_1)(t,x)=\partial_t^2\phi_1(t,x)-\partial_x^2\phi_1(t,x)+U^{'}(\phi_1(t,x)),$$

*for any $\phi_1(t,x)\in C^2(\mathbb{R}^2,\mathbb{R})$.*

**Theorem 4.1.6.** *There exist a sequence of functions $(\phi_k(v,t,x))_{k\geq 2}$, a sequence of real values $\delta(k)>0$ and a sequence of numbers $n_k\in\mathbb{N}$ such that for any $0<v<\delta(k)$, $\phi_k(v,t,x)$ satisfies*

$$\lim_{t\to+\infty}\left\lVert\phi_k(v,t,x)-H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\rVert_{H^1_x}=0,$$

$$\lim_{t\to+\infty}\left\lVert\partial_t\phi_k(v,t,x)+\frac{v}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)-\frac{v}{\sqrt{1-v^2}}H_{-1,0}^{'}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right\rVert_{L^2_x}=0,$$

$$\lim_{t\to-\infty}\left\lVert\phi_k(v,t,x)-H_{0,1}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right)-H_{-1,0}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\rVert_{H^1_x}=0,$$

$$\lim_{t\to-\infty}\left\lVert\partial_t\phi_k(v,t,x)-\frac{v}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x+vt-e_{v,k}}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H_{-1,0}^{'}\left(\frac{x-vt+e_{v,k}}{\sqrt{1-v^2}}\right)\right\rVert_{L^2_x}=0,$$

*with* $e_{v,k} \in \mathbb{R}$ *satisfying*

$$\lim_{v\to 0}\frac{\left|e_{v,k}-\frac{\ln\left(\frac{8}{v^2}\right)}{\sqrt{2}}\right|}{v|\ln(v)|^3}=0.$$

*Moreover, if* $0<v<\delta(k)$, *then for any* $s\geq 0$ *and* $l\in\mathbb{N}\cup\{0\}$, *there is* $C(k,s,l)>0$ *such that*

$$\left\|\frac{\partial^l}{\partial t^l}\Lambda(\phi_k(v,t,x))\right\|_{H^s_x(\mathbb{R})}\leq C(k,s,l)v^{2k+l}\left(|t|v+\ln\left(\frac{1}{v^2}\right)\right)^{n_k}e^{-2\sqrt{2}|t|v}.$$

From Chapter 3, we recall the Schwartz function $\mathcal{G}$ defined by

$$\mathcal{G}(x)=e^{-\sqrt{2}x}-\frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}+2\sqrt{2}\frac{xe^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}+k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}},\tag{4.15}$$

for all $x\in\mathbb{R}$, where $k_1$ is the unique real number such that $\mathcal{G}$ satisfies $\left\langle\mathcal{G}(x),\,H^{'}_{0,1}(x)\right\rangle=0.$ Moreover, we recall identity 3.7

$$-\frac{d^2}{dx^2}\mathcal{G}(x)+U^{(2)}(H_{0,1}(x))\mathcal{G}(x)=\left[-24H_{0,1}(x)^2+30H_{0,1}(x)^4\right]e^{-\sqrt{2}x}+8\sqrt{2}H^{'}_{0,1}(x).$$

Next, for any $v\in(0,1)$, we recall the function (2.121) defined in Chapter 2 and consider

$$d_v(t)=\frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right),\text{ for any }t\in\mathbb{R}.$$

From the statement of Theorem 2.1.10 of Chapter 2, we have that the function $d_v$ describes the movement between two kinks for the $\phi^6$ model during a large time interval when their total energy is small and their initial speeds are both zero.

Furthermore, from the proof of Theorem 3.5.1 in the previous chapter, we can construct inductively an explicit sequence of smooth functions $(\varphi_{k,v})_{k\in\mathbb{N}_{\geq 2}}$, and, for each $k\in\mathbb{N}_{\geq 2}$, there exists a real number $\tau_{k,v}$ satisfying $|\tau_{k,v}|<\frac{\sqrt{2}}{v}\ln\left(\frac{8}{v^2}\right)$ such that $\phi_k(v,t,x):=\varphi_{k,v}(t+\tau_{k,v},x)$ satisfies Theorem 4.1.6 for all $k\in\mathbb{N}_{\geq 2}$. More precisely, the statement of Theorem 3.5.1 is the following:

**Theorem 4.1.7.** *There exist a function* $\mathcal{C}:\mathbb{R}^4\to\mathbb{R}_{>0}$, *a sequence of approximate solutions* $\varphi_{k,v}(t,x)$, *functions* $r_k(v,t)$ *that are smooth and even on* $t$, *and numbers* $n_k\in\mathbb{N}$ *such that if* $0<v\ll 1$, *then for any* $m\in\mathbb{N}_{\geq 1}$

$$\frac{|r_k(v,t)|}{\mathcal{C}(k,0,0,0)}\leq v^{2(k-1)}\left(\ln\frac{1}{v}\right)^{n_k},\ \frac{\left|\frac{\partial^m}{\partial t^m}r_k(v,t)\right|}{\mathcal{C}(k,0,l,0)}\leq v^{2(k-1)+m}\left[\ln\frac{1}{v}+|t|v\right]^{n_k}e^{-2\sqrt{2}|t|v},\tag{4.16}$$

$\varphi_{k,v}(t,x)$ *satisfies for* $\rho_k(v,t)=-\frac{d_v(t)}{2}+\sum_{j=2}^k r_j(v,t)=-\frac{d_v(t)}{2}+c_k(v,t)$ *the identity*

$$\begin{aligned}\varphi_{k,v}(t,x)=&H_{0,1}\left(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)+H_{-1,0}\left(\frac{x-\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\\&+e^{-\sqrt{2}d_v(t)}\left[\mathcal{G}\left(\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-\mathcal{G}\left(\frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\right]\\&+\mathcal{R}_{k,v}\left(vt,\frac{x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)-\mathcal{R}_{k,v}\left(vt,\frac{-x+\rho_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}}\right)\end{aligned}\tag{4.17}$$

*the following estimates for any $l \in \mathbb{N} \cup \{0\}$ and $s \geq 1$*

$$\left\| \frac{\partial^l}{\partial t^l} \Lambda(\varphi_{k,v}(t,x)) \right\|_{H^s} \leq \mathcal{C}(k,s,l,1) v^{2k+l} \left[ \ln\left(\frac{1}{v^2}\right) + |t|v \right]^{n_k} e^{-2\sqrt{2}|t|v}, \tag{4.18}$$

*and*

$$\left| \frac{d^l}{dt^l} \left[ \left\langle \Lambda(\varphi_{k,v})(t,x),\, H^{'}_{0,1}\left( \frac{x+\rho_k(v,t)}{(1-\frac{\dot{d}_v(t)^2}{4})^{\frac{1}{2}}} \right) \right\rangle \right] \right| \leq \mathcal{C}(k,2,l,2) v^{2k+l+2} \left[ \ln\left(\frac{1}{v^2}\right) + |t|v \right]^{n_k+1} e^{-2\sqrt{2}|t|v}, \tag{4.19}$$

*where $\mathcal{R}_k(t,x)$ is a finite sum of functions $p_{k,i,v}(t) h_{k,i}(x)$ with $h_{k,i} \in \mathscr{S}(\mathbb{R})$ and each $p_{k,i,v}(t)$ being an even function satisfying, for all $m \in \mathbb{N}$,*

$$\left| \frac{d^m p_{k,i,v}(t)}{dt^m} \right| \leq \mathcal{C}(k,0,m,3) v^4 \left( \ln\left(\frac{1}{v^2}\right) + |t| \right)^{n_{k,i}} e^{-2\sqrt{2}|t|},$$

*where $n_{k,i} \in \mathbb{N}$ depends only on $k$ and $i$.*

**Remark 4.1.8.** *At first look, the statement of Theorem 4.1.7 seems to contain excessive information about the approximate solutions $\varphi_{k,v}(t,x)$. However, we are going to need every information of Theorem 4.1.7 to study the elasticity and stability of the collision of two kinks with low speed $0 < v < 1$.*

### 4.1.5 Organization of Chapter 4

First, from the global well-posedness of the partial differential equation ($\phi^6$), we recall that if $\phi$ is a strong solution of ($\phi^6$) with finite energy satisfying $\lim_{x \to \pm\infty} \phi(t_0, x) = \pm 1$ for some $t_0 \in \mathbb{R}$, then the function $\phi$ satisfies

$$\| \phi(t,x) - H_{0,1}(x) - H_{-1,0}(x) \|_{H^1_x} < +\infty,$$

for all $t \in \mathbb{R}$.

In Section 4.2, using the notation of Theorem 4.1.7, we are going to verify that any solution of ($\phi^6$) with finite energy close to a sum of two kinks can be written as

$$\begin{aligned} \phi(t,x) = \varphi_{k,v}(t,x) &+ \frac{y_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H^{'}_{0,1}\left( \frac{x - \frac{d_v(v,t)}{2} + c_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} \right) \\ &+ \frac{y_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H^{'}_{0,1}\left( \frac{-x - \frac{d_v(t)}{2} + c_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} \right) + u(t,x), \end{aligned} \tag{4.20}$$

such that, for any $t \in \mathbb{R}$, $u(t) \in H^1_x(\mathbb{R})$ satisfies the following orthogonality conditions

$$\left\langle u(t,x),\, H^{'}_{0,1}\left( \frac{x - \frac{d_v(t)}{2} + c_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} \right) \right\rangle = 0,$$

$$\left\langle u(t,x),\, H^{'}_{0,1}\left( \frac{-x - \frac{d_v(t)}{2} + c_k(v,t)}{\sqrt{1-\frac{\dot{d}_v(t)^2}{4}}} \right) \right\rangle = 0.$$

Moreover, using $\Lambda(\phi) \equiv 0$, we can verify that $y_1,\, y_2 \in C^2(\mathbb{R})$. Furthermore, using the formula (4.20), we will estimate $\Lambda(\phi)(t,x)$. More precisely, we estimate the expression $\Lambda\left(\phi\right)(t,x) - \Lambda\left(\varphi_{k,v}\right)(t,x)$, in function of $y_1(t),\ y_2(t),\ d_v(t),\ u(t,x)$ and the estimate of the term $\Lambda\left(\varphi_{k,v}\right)(t,x)$ will follow from the main results of Subsection 4.1.4 about the decay of approximate solutions. The function $c_k(v,t)$ will not appear in the evaluation of $\Lambda(\phi)(t,x)$, since we are going to use only its decay.

Next, in Section 4.3, we are going to construct a function $L(t)$ to estimate $\|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x}$ during a large time interval. The main argument in this section is analogous to the ideas of Section 2.4 of Chapter 1. More precisely, for

$$w_{k,v}(t,x) = \frac{x - \frac{d_v(t)}{2} + c_k(v,t)}{\sqrt{1 - \frac{\dot{d}_v(t)^2}{4}}},$$

we consider first

$$L_1(t) = \int_{\mathbb{R}} \partial_t u(t,x)^2 + \partial_x u(t,x)^2 + U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right) u(t,x)^2 \, dx.$$

From the orthogonality conditions satisfied by $u(t,x)$, if $v \ll 1$, we deduce the following coercivity inequality

$$\|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x} \lesssim L_1(t).$$

The function $L(t)$ will be constructed after correction terms $L_2(t)$ and $L_3(t)$ are added to $L_1(t)$. The motivation for the usage of the correction term $L_3(t)$ is to reduce the growth of the modulus of the following expression

$$2\int_{\mathbb{R}} \left[\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,x)\right)\right) u(t,x)\right] \partial_t u(t,x)\, dx$$

in $\dot{L}_1(t)$. The time derivative of $L_2(t)$ will cancel with the expression

$$\int_{\mathbb{R}} \left[\frac{\partial}{\partial t} U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,x)\right)\right)\right] u(t,x)^2 \, dx,$$

from $\dot{L}_1(t)$. Finally, under additional conditions in the growth of the functions $y_1(t),\, y_2(t)$, if $0 < v \ll 1$, the function $L(t) = \sum_{j=1}^{3} L_j(t)$ will satisfy for a constant $C(k)$ depending only on $k$ the following estimates

$$\left|\dot{L}(t)\right| \lesssim \frac{v}{\ln \frac{1}{v}} \|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x},$$

$$\|(u(t), \partial_t u(t))\|^2_{H^1_x \times L^2_x} \lesssim L(t) + C(k) v^{4k} \left(\ln \frac{1}{v}\right)^{2n_k},$$

for all $t$ in a large time interval, $n_k$ is the number denoted in Theorem 4.1.7. Therefore, using Gronwall Lemma and the two estimates above, we are going to obtain an upper bound for $\|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x}$ when $t$ belongs to a large time interval.

In Section 4.4, we are going to estimate $\left\|\overrightarrow{\phi}(t) - \overrightarrow{\varphi_{k,v}}(t)\right\|_{H^1_x \times L^2_x}$ during a large time interval. This estimate follows from the study of a linear ordinary differential system whose solutions $\hat{y}_1$, $\hat{y}_2$ are close to $y_1$, $y_2$ during a time interval of size much larger than $\frac{-\ln(v)}{v}$ and from the conclusions of the last section. Indeed, the closeness of the functions $y_1$, $y_2$ with $\hat{y}_1$, $\hat{y}_2$ during this large time interval is guaranteed because of the upper bound obtained for $\|(u(t), \partial_t u(t))\|_{H^1_x \times L^2_x}$ from the control of $L(t)$, which implies that $y_1$, $y_2$ will satisfy an ordinary differential system very close to the linear ordinary differential system satisfied by $\hat{y}_1$ and $\hat{y}_2$.

In Section 4.5, we are going to prove Theorem 4.1.3, the proof of this result is inspired by the demonstration of Theorem 1 of [31] and Theorem 1 of [44]. This result will imply in the next section the second inequality of Theorem 4.1.2. In addition, the main techniques used in this section are modulation techniques based on Section 2 of [31] and based on [44], the use of conservation of energy of $\phi(t,x)$ and the monotonicity of the localized momentum given by

$$P_+\left(\phi(t), \partial_t\phi(t)\right) = -\int_0^{+\infty} \partial_t\phi(t,x)\partial_x\phi(t,x)\,dx.$$

Finally, in Section 4.6, we will show that the demonstration of Theorem 4.1.2 is a direct consequence of the main results of Sections 4.4 and 4.5. For complementary information, see Section A.4 and Section A.5 of the Appendix.

## 4.2 Auxiliary estimates

First, we recall the following lemma from Chapter 3.

**Lemma 4.2.1.** *In notation of Theorem 4.1.7, for* $0 < v \ll 1$, *let* $w_{k,v} : \mathbb{R}^2 \to \mathbb{R}$ *be the following function*

$$w_{k,v}(t,x) = \frac{x + \rho_k(v,t)}{\sqrt{1 - \frac{d_v(t)^2}{4}}},$$

*and let* $f \in L^\infty_x(\mathbb{R})$ *be a function satisfying* $f^{'} \in \mathscr{S}(\mathbb{R})$. *Then, if* $0<v \ll 1$, *we have for any* $l \in \mathbb{N}$ *that*

$$\frac{\partial^l}{\partial t^l} f\left(w_{k,v}(t,x)\right)$$

*is a finite sum of functions* $q_{k,l,i,v}(t)h_i\left(w_{k,v}(t,x)\right)$ *with each* $h_i \in \mathscr{S}(\mathbb{R})$ *and any* $q_{k,l,i,v}(t)$ *is a smooth real function satisfying*

$$\|q_{k,l,i,v}\|_{L^\infty(\mathbb{R})} \lesssim v^l.$$

*Furthermore, if* $0 < v \ll 1$, *we have for all* $l \in \mathbb{N}$ *and any* $s \geq 0$ *that*

$$\left\|\frac{\partial^l}{\partial t^l} f\left(w_{k,v}(t,x)\right)\right\|_{H^s_x} \lesssim_{k,s,l} v^l.$$

Moreover, we are going to use the following result several times in the computation of the estimates of this chapter.

**Lemma 4.2.2.** *For any $s \geq 1$, we have for any functions $f$, $g \in \mathscr{S}(\mathbb{R})$ that*

$$\|fg\|_{H^s_x(\mathbb{R})} \lesssim_s \|f\|_{H^s_x} \|g\|_{L^\infty_x} + \|g\|_{H^s_x} \|f\|_{L^\infty_x} \lesssim_s \|f\|_{H^s_x} \|g\|_{H^s_x}.$$

*As a consequence,*

$$\|fg\|_{H^s_x} \lesssim_{s_0} \|f\|_{H^{s+1}_x} \|g\|_{H^{s+1}_x},$$

*for all $s \geq 0$.*

*Proof.* See the proof of Lemma $A.8$ in the book [61]. □

In Chapter 4, to simplify our notation, we denote $d_v(t)$ by $d(t)$, which means that

$$d(t) = \frac{1}{\sqrt{2}} \ln\Big(\frac{8}{v^2}\cosh\big(\sqrt{2}vt\big)^2\Big). \tag{4.21}$$

In Lemma 3.1 of [46], we have verified by induction the following estimates

$$|\dot{d}(t)| \lesssim v, \text{ and for any } l \in \mathbb{N}_{\geq 2} \left|d^{(l)}(t)\right| \lesssim_l v^l e^{-2\sqrt{2}|t|v}. \tag{4.22}$$

From now on, we consider for each $k \in \mathbb{N}_{\geq 2}$ the function $\phi_{k,v}(t,x)$ satisfying Theorem 4.1.7. Next, for $T_{0,k} > 0$ to be chosen later, we consider the following kind of Cauchy problem

$$\begin{cases} \partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \\ \|(\phi(T_{0,k},x), \partial_t\phi(T_{0,k},x)) - (\phi_{k,v}(T_{0,k},x), \partial_t\phi_{k,v}(T_{0,k},x))\|_{H^1_x \times L^2_x} < v^{8k}. \end{cases} \tag{4.23}$$

Our first objective is to prove the following theorem.

**Theorem 4.2.3.** *There is a constant $C > 0$ and for any for any $0 < \theta < \frac{1}{4}$, $k \in \mathbb{N}_{\geq 3}$ there exist $C_1(k) > 0$, $\delta_{k,\theta} > 0$ and $\eta_k \in \mathbb{N}$ such that if $0 < v < \delta_{k,\theta}$ and $T_{0,k} = \frac{32k}{2\sqrt{2}}\frac{\ln\left(\frac{1}{v^2}\right)}{v}$, then any solution $\phi(t,x)$ of (4.23) satisfies:*

$$\|(\phi(t,x), \partial_t\phi(t,x)) - (\varphi_{k,v}(t,x), \partial_t\varphi_{k,v}(t,x))\|_{H^1_x \times L^2_x} < C_1(k) v^{2k} \left(\ln\frac{1}{v}\right)^{\eta_k} \exp\left(C\frac{v|t - T_{0,k}|}{\ln\frac{1}{v}}\right), \tag{4.24}$$

*if*

$$|t - T_{0,k}| < \frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v}.$$

Clearly, we can obtain from Theorem 4.2.3 and Theorem 4.1.7 the following result:

**Corollary 4.2.4.** *There is a constant $C > 0$ and for any $0 < \theta < \frac{1}{4}$, $k \in \mathbb{N}_{\geq 3}$ there exist $C_1(k) > 0$, $\delta_{k,\theta} > 0$ and $\eta_k \in \mathbb{N}$ such that if $0 < v < \delta_{k,\theta}$ and $T_{0,k} = \frac{32k}{2\sqrt{2}}\frac{\ln\left(\frac{1}{v^2}\right)}{v}$, then any solution $\phi(t,x)$ of*

$$\begin{cases} \partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + U'(\phi(t,x)) = 0, \\ \|(\phi(T_{0,k},x), \partial_t\phi(T_{0,k},x)) - (\phi_k(v,T_{0,k},x), \partial_t\phi_k(v,T_{0,k},x))\|_{H^1_x \times L^2_x} < v^{8k} \end{cases}$$

*satisfies*

$$\|(\phi(t,x),\partial_t\phi(t,x))-(\phi_k(v,t,x),\partial_t\phi_k(v,t,x))\|_{H^1_x\times L^2_x}<C_1(k)v^{2k}\left(\ln\frac{1}{v}\right)^{\eta_k}\exp\left(C\frac{v|t-T_{0,k}|}{\ln(v)}\right), \tag{4.25}$$

*if*

$$|t-T_{0,k}|<\frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v}.$$

*Proof of Corollary 4.2.4.* It follows from Theorem 4.2.3 and Theorems 4.1.6, 4.1.7. □

With the objective of simplifying the demonstration of Theorem 4.2.3, we are going to elaborate on necessary lemmas before the proof of Theorem 4.2.3. From now on, to simplify our notation, we will use $d(t)$, $c_k(t)$ in the place of $d_v(t)$, $c_k(v,t)$ respectively for any $k\in\mathbb{N}_{\geq 2}$, every $t\in\mathbb{R}$ and $v\in(0,1)$ small enough. For any $k\in\mathbb{N}_{\geq 2}$, We also consider the following function

$$w_{k,v}(t,x)=\frac{x-\frac{d(t)}{2}+c_k(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}},\text{ for all }(t,x)\in\mathbb{R}^2. \tag{4.26}$$

Moreover, we denote any solution $\phi(t,x)$ of the partial differential equation (4.23) as

$$\phi(t,x)=\varphi_{k,v}(t,x)+\frac{y_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,x)\right)+\frac{y_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,-x)\right)+u(t,x), \tag{4.27}$$

such that

$$\left\langle u(t,x),H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle=\left\langle u(t,x),H^{'}_{0,1}\left(w_{k,v}(t,-x)\right)\right\rangle=0. \tag{4.28}$$

Furthermore, since Theorem 4.1.7 implies that $\zeta_k(t)=d(t)-2c_k(t)\gg 1$ when $v$ is small enough, we deduce from the orthogonal conditions (4.28) satisfied by $u(t,x)$ the following identity

$$\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}=M_1(t)^{-1}\begin{bmatrix}\left\langle\phi(t,x)-\varphi_{k,v}(t,x),H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle\\\left\langle\phi(t,x)-\varphi_{k,v}(t,x),H^{'}_{-1,0}\left(w_{k,v}(t,-x)\right)\right\rangle\end{bmatrix}. \tag{4.29}$$

where, for any $t\in\mathbb{R}$, $M(t)$ is the matrix denoted by

$$M_1(t)=\begin{bmatrix}\left\|H^{'}_{0,1}\right\|^2_{L^2_x} & \left\langle H^{'}_{0,1}(x-\zeta_k(t)),H^{'}_{-1,0}(x)\right\rangle\\\left\langle H^{'}_{0,1}(x-\zeta_k(t)),H^{'}_{-1,0}(x)\right\rangle & \left\|H^{'}_{0,1}\right\|^2_{L^2_x}\end{bmatrix},$$

which is uniformly positive since $\zeta_k(t)\gg 1$.

Moreover, since $\ln\frac{1}{v}\lesssim\zeta_k$ when $v>0$ is small enough, we obtain from Lemma 3.2.1 that $\left\langle H^{'}_{0,1}(x-\zeta_k(t)),H^{'}_{-1,0}(x)\right\rangle\ll 1$. Therefore, since the matrix $M(t)$ is a smooth function with domain $\mathbb{R}$, then $M(t)^{-1}$ is also smooth on $\mathbb{R}$.

Next, for $\psi(t,x)=\phi(t,x)-\varphi_{k,v}(t,x)$, we obtain from the partial differential equation (4.23) that $\psi(t,x)$ satisfies the following partial differential equation

$$\frac{\partial^2}{\partial t^2}\psi(t,x)-\frac{\partial^2}{\partial x^2}\psi(t,x)+\Lambda(\varphi_{k,v})(t,x)+\sum_{j=2}^{6}\frac{U^{(j)}\left(\varphi_{k,v}(t,x)\right)}{(j-1)!}\psi(t,x)^{j-1}=0. \tag{4.30}$$

Since $\varphi_{k,v}$ satisfies Theorem 4.1.7 and the partial differential equation ($\phi^6$) is globally well-posed in the energy space, we can verify for any initial data $(\psi_0(x), \psi_1(x)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ that there exists a unique solution $\psi(t,x)$ of (4.30) satisfying $(\psi(0,x), \partial_t\psi(0,x)) = (\psi_0(x), \psi_1(x))$ and

$$(\psi(t,x), \partial_t\psi(t,x)) \in C\left(\mathbb{R}; H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})\right). \tag{4.31}$$

Therefore, for any function $h \in \mathscr{S}(\mathbb{R})$, we deduce from (4.30) that

$$\frac{d}{dt}\langle \psi(t,x), h(x)\rangle = \langle \partial_t\psi(t,x), h(x)\rangle,$$
$$\frac{d^2}{dt^2}\langle \psi(t,x), h(x)\rangle = \left\langle \frac{\partial^2}{\partial x^2}\psi(t,x) - \Lambda(\varphi_{k,v})(t,x) - U^{'}\left(\varphi_{k,v}(t,x) + \psi(t,x)\right) + U^{'}\left(\varphi_{k,v}(t,x)\right), h(x)\right\rangle,$$

which implies that the real function $\mathcal{P}_1(t) = \left\langle \psi(t,x), H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle$ and the real function $\mathcal{P}_2(t) = \left\langle \psi(t,x), H^{'}_{-1,0}\left(w_{k,v}(t,-x)\right)\right\rangle$ are in $C^2(\mathbb{R})$. In conclusion, using equation (4.29) and the product rule of derivative, we deduce that $y_1$, $y_2 \in C^2(\mathbb{R})$.

In conclusion, we obtain the following lemma:

**Lemma 4.2.5.** *Assuming the same hypotheses of Theorem 4.2.3, there exist functions $y_1$, $y_2 : \mathbb{R} \to \mathbb{R}$ of class $C^2$ such that any solution $\phi(t,x)$ of (4.23) satisfies for any $t \in \mathbb{R}$ the following identity*

$$\phi(t,x) = \varphi_{k,v}(t,x) + \frac{y_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,x)\right) + \frac{y_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,-x)\right) + u(t,x),$$

*where $(u(t), \partial_t u(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ and the function $u$ satisfies the following orthogonality conditions:*

$$\left\langle u(t,x), H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle = 0,$$
$$\left\langle u(t,x), H^{'}_{0,1}\left(w_{k,v}(t,-x)\right)\right\rangle = 0.$$

**Remark 4.2.6.** *Using Lemmas 4.2.1, 4.2.2, 4.2.5 , $\Lambda(\phi) = 0$, Theorem 4.1.7, Remark 3.5.3 and identities $H^{(3)}_{0,1}(x) = U^{(2)}\left(H_{0,1}(x)\right)H^{'}_{0,1}(x)$, $\ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)}$, we can deduce that $u$ satisfies the following partial differential equation*

$$\begin{aligned}
&\Lambda\left(\varphi_{k,v}\right)(t,x) + \partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U^{(2)}\left(\varphi_{k,v}(t,x)\right)\left(\phi(t,x) - \varphi_{k,v}(t,x)\right)\\
&+\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,x)\right) + \frac{\ddot{y}_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,-x)\right) - \frac{y_1(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,x)\right)\\
&-\frac{y_2(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,-x)\right) - \frac{\dot{y}_1(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(x,t)\right) - \frac{\dot{y}_2(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,-x)\right)\\
&- y_1(t)\frac{U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)\right)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,x)\right) - y_2(t)\frac{U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,-x)\right)\right)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,-x)\right)\\
&= \mathcal{Q}(t,x),
\end{aligned} \tag{4.32}$$

*where $\mathcal{Q}(t,\cdot)$ is a function in $H^1_x(\mathbb{R})$ satisfying for all $t\in\mathbb{R}$*

$$\|\mathcal{Q}(t,x)\|_{H^1_x(\mathbb{R})} \lesssim \|u(t)\|^2_{H^1_x} + \|u(t)\|^6_{H^1_x} + \max_{j\in\{1,2\}} |y_j(t)|^2 + \max_{j\in\{1,2\}} |y_j(t)|^6$$
$$+ \left[\max_{j\in\{1,2\}} |\dot{y}_j(t)| + v \max_{j\in\{1,2\}} |y_j(t)|\right] v^3 \left(\ln\left(\frac{1}{v^2}\right) + |t|v\right) e^{-2\sqrt{2}|t|v},$$

*if $v>0$ is small enough.*

Next, from equation (4.32) of Remark 4.2.6, we consider the terms

$$Y_1(t,x) = \left[U^{(2)}\left(\varphi_{k,v}(t,x)\right) - U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)\right)\right] \frac{y_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H^{'}_{0,1}\left(w_{k,v}(t,x)\right), \tag{4.33}$$

$$Y_2(t,x) = \left[U^{(2)}\left(\varphi_{k,v}(t,x)\right) - U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\right] \frac{y_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H^{'}_{0,1}\left(w_{k,v}(t,-x)\right). \tag{4.34}$$

Now, we will estimate the expressions

$$\left\langle Y_1(t), H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle, \left\langle Y_2(t), H^{'}_{0,1}\left(w_{k,v}(t,-x)\right)\right\rangle.$$

**Lemma 4.2.7.** *In notation of Theorem 4.1.7 and Lemma 4.2.5, the functions $Y_1(t)$ and $Y_2(t)$ satisfy*

$$\left\langle Y_1(t),\, H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle = 4\sqrt{2}e^{-\sqrt{2}d(t)}y_1(t) + y_1(t)Res_1(v,t),$$
$$\left\langle Y_2(t),\, H^{'}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle = -4\sqrt{2}e^{-\sqrt{2}d(t)}y_2(t) + y_2(t)Res_2(v,t),$$

*where, for any $j\in\{1,2\}$ and all $v\in(0,1)$, the function $Res_j(v,t)$ is a Schwartz function on $t$ satisfying for any $l\in\mathbb{N}\cup\{0\}$, if $0<v\ll 1$, the following estimate*

$$\left|\frac{\partial^l}{\partial t^l} Res_j(v,t)\right| \lesssim_l v^{l+4}\left[\ln\left(\frac{1}{v^2}\right) + |t|v\right]^{\eta_k} e^{-2\sqrt{2}|t|v}, \tag{4.35}$$

*for a number $\eta_k\geq 0$ depending only on $k\in\mathbb{N}_{\geq 2}$.*

*Proof of Lemma 4.2.7.* First, we observe that

$$\left|\frac{d^l}{dt^l} e^{-\sqrt{2}d(t)}\right| = \left|\frac{d^l}{dt^l}\frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt\right)^2\right| \lesssim_l v^{2+l}e^{-2\sqrt{2}|t|v}.$$

Using Taylor's Expansion Theorem, Theorem 4.1.7 and Lemma 4.2.2, we deduce that

$$\begin{aligned} &U^{(2)}\left(\varphi_{k,v}(t,x)\right)\\ &\quad= U^{(2)}\left(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\\ &\qquad + e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}(w_{k,v}(t,x)) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\left[\mathcal{G}(w_{k,v}(t,x)) - \mathcal{G}(w_{k,v}(t,-x))\right]\\ &\qquad + res_1(v,t,x), \end{aligned}$$

where, if $0<v\ll 1$, $res_1(v,t,x)$ is a smooth function on the variables $(t,x)$ which satisfies for some $\eta_k\in\mathbb{N}$ and any $s\geq 0$, $l\in\mathbb{N}\cup\{0\}$ the following inequality

$$\left\|\frac{\partial^l}{\partial t^l} res_1(v,t,x)\right\|_{H^s_x} \lesssim_{s,l} v^{4+l}\left[\ln\left(\frac{1}{v^2}\right) + |t|v\right]^{\eta_k} e^{-2\sqrt{2}|t|v}. \tag{4.36}$$

Therefore, using identity

$$\begin{aligned}
&U^{(2)}\left(\varphi_{k,v}(t,x)\right)-U^{(2)}\left(H_{0,1}(w_{k,v}(t,x)\right)\\
&\qquad=U^{(2)}\left(\varphi_{k,v}(t,x)\right)-U^{(2)}\left(H_{0,1}(w_{k,v}(t,x)-H_{0,1}(w_{k,v}(t,-x))\right)\\
&\qquad\quad+U^{(2)}\left(H_{0,1}(w_{k,v}(t,x)-H_{0,1}(w_{k,v}(t,-x))\right)-U^{(2)}\left(H_{0,1}(w_{k,v}(t,x)\right),
\end{aligned}$$

we obtain that

$$\begin{aligned}
&Y_1(t,x)\sqrt{1-\frac{\dot{d}(t)^2}{4}}\\
&=\left[U^{(2)}\left(H_{0,1}(w_{k,v}(t,x))-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(H_{0,1}(w_{k,v}(t,x))\right)\right]y_1(t)H_{0,1}^{'}\left(w_{k,v}(x,t)\right)\\
&\quad+y_1(t)e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}(w_{k,v}(t,x))-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\mathcal{G}(w_{k,v}(t,x))H_{0,1}^{'}\left(w_{k,v}(t,x)\right)\\
&\quad-y_1(t)e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}(w_{k,v}(t,x))-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\mathcal{G}(w_{k,v}(t,-x))H_{0,1}^{'}\left(w_{k,v}(t,x)\right)\\
&\quad+y_1(t)res_1(v,t,x).
\end{aligned}\tag{4.37}$$

By a similar reasoning, we obtain that

$$\begin{aligned}
&Y_2(t,x)\sqrt{1-\frac{\dot{d}(t)^2}{4}}\\
&=\left[U^{(2)}\left(H_{0,1}(w_{k,v}(t,x))-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(H_{0,1}(w_{k,v}(t,-x))\right)\right]y_2(t)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)\\
&\quad+y_2(t)e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}(w_{k,v}(t,x))-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\mathcal{G}(w_{k,v}(t,x))H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)\\
&\quad-y_2(t)e^{-\sqrt{2}d(t)}U^{(3)}\left(H_{0,1}(w_{k,v}(t,x))-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\mathcal{G}(w_{k,v}(t,-x))H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)\\
&\quad+y_2(t)res_2(v,t,x),
\end{aligned}\tag{4.38}$$

where if $0<v\ll 1$, $res_2(v,t,x)$ is a smooth function on $t$, $x$ satisfying, for some constant $\eta_k\geq 0$, any $l\in\mathbb{N}\cup\{0\}$ and $s\geq 0$, the following estimate

$$\left\Vert\frac{\partial^l}{\partial t^l}res_2(v,t,x)\right\Vert_{H^s_x}\lesssim_{s,l} v^{4+l}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{\eta_k}e^{-2\sqrt{2}|t|v}.\tag{4.39}$$

Next, from the Fundamental Theorem of Calculus, we have for any $\zeta>1$ that

$$\begin{aligned}
&\left[U^{(2)}\left(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)\right)-U^{(2)}\left(H^{\zeta}_{0,1}(x)\right)\right]\partial_x H^{\zeta}_{0,1}(x)\\
&=U^{(3)}\left(H^{\zeta}_{0,1}(x)\right)H_{-1,0}(x)\partial_x H^{\zeta}_{0,1}(x)+\int_0^1 U^{(4)}\left(H^{\zeta}_{0,1}+\theta H_{-1,0}\right)(1-\theta)H_{-1,0}(x)^2\partial_x H^{\zeta}_{0,1}(x)\,d\theta,
\end{aligned}$$

from which with Lemma 3.2.1, estimates (4.2), (4.3) and

$$\left|\frac{d^l}{dx^l}\left[H_{-1,0}(x)+e^{-\sqrt{2}x}\right]\right|\lesssim_l \min\left(e^{-\sqrt{2}x},e^{-3\sqrt{2}x}\right),$$

we obtain that

$$\begin{aligned}
&\left\langle\left[U^{(2)}\left(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)\right)-U^{(2)}\left(H^{\zeta}_{0,1}(x)\right)\right]\partial_x H^{\zeta}_{0,1}(x),\,\partial_x H^{\zeta}_{0,1}(x)\right\rangle\\
&\qquad=-e^{-\sqrt{2}\zeta}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}(x)\right)H_{0,1}^{'}(x)^2e^{-\sqrt{2}x}\,dx+res_3(\zeta),
\end{aligned}\tag{4.40}$$

with $res_3 \in C^{\infty}(\mathbb{R}_{\geq 1})$ satisfying for all $l \in \mathbb{N} \cup \{0\}$ and $\zeta \geq 1$

$$\left|res_3^{(l)}(\zeta)\right| \lesssim_l \zeta e^{-2\sqrt{2}\zeta}.$$

Next, since $U \in C^{\infty}(\mathbb{R})$ and we have estimates (4.2), (4.3), we deduce for al $\zeta \geq 1$ and any $l \in \mathbb{N} \cup \{0\}$ that

$$\left|\frac{\partial^l}{\partial \zeta^l}\left[U^{(3)}\left(H_{0,1}^{\zeta}(x) + H_{-1,0}(x)\right) - U^{(3)}\left(H_{0,1}^{\zeta}(x)\right)\right]\right| \lesssim_l |H_{-1,0}(x)|.$$

Therefore, since $\mathcal{G}$ defined in (4.15) is a Schwartz function, Lemma 3.2.1 implies that

$$int(\zeta) = \left\langle \left[U^{(3)}\left(H_{0,1}^{\zeta}(x) + H_{-1,0}(x)\right) - U^{(3)}\left(H_{0,1}^{\zeta}(x)\right)\right] \mathcal{G}(x-\zeta)\partial_x H_{0,1}^{\zeta}(x),\, \partial_x H_{0,1}^{\zeta}(x)\right\rangle$$

satisfies for all $\zeta \geq 1$ and any $l \in \mathbb{N} \cup \{0\}$ the following inequality $\left|int^{(l)}(\zeta)\right| \lesssim_l e^{-\sqrt{2}\zeta}$. Moreover, using the following identity

$$U^{(3)}(\phi) = -48\phi + 120\phi^3, \tag{4.41}$$

we can deduce similarly that

$$int_2(\zeta) = \left\langle U^{(3)}\left(H_{0,1}^{\zeta}(x) + H_{-1,0}(x)\right) \mathcal{G}(-x) H_{-1,0}^{'}(x),\, \partial_x H_{0,1}^{\zeta}(x)\right\rangle$$

satisfies $\left|int_2^{(l)}(\zeta)\right| \lesssim_l e^{-\sqrt{2}\zeta}$ for any $l \in \mathbb{N} \cup \{0\}$ and $\zeta \geq 1$. As a consequence, we deduce that there exists a real function $int_3 : \mathbb{R}_{\geq 1} \to \mathbb{R}$ satisfying for any $l \in \mathbb{N} \cup \{0\}$

$$\left|int_3^{(l)}(\zeta)\right| \lesssim_l e^{-\sqrt{2}\zeta},$$

where the function $int_3$ satisfies the following identity

$$\begin{aligned}\Big\langle U^{(3)}\left(H_{0,1}^{\zeta}(x) + H_{-1,0}(x)\right) &\mathcal{G}(x-\zeta)\partial_x H_{0,1}^{\zeta}(x), \partial_x H_{0,1}^{\zeta}(x)\Big\rangle \\ &- \left\langle U^{(3)}\left(H_{0,1}^{\zeta}(x) + H_{-1,0}(x)\right) \mathcal{G}(-x) H_{0,1}^{'}(-x), \partial_x H_{0,1}^{\zeta}(x)\right\rangle \\ &= \int_{\mathbb{R}} U^{(3)}\left(H_{0,1}(x)\right) H_{0,1}^{'}(x)^2 \mathcal{G}(x)\, dx + int_3(\zeta).\end{aligned} \tag{4.42}$$

From Theorem 4.1.7, estimates (4.22) and identity $e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}|t|v\right)^2$, it is not difficult to verify for any $l \in \mathbb{N} \cup \{0\}$ that if $0 < v \ll 1$, then

$$\frac{d^l}{dt^l}\exp\left(\frac{2\rho_{k,v}(t)}{\sqrt{1 - \frac{\dot{d}(t)^2}{4}}}\right) \lesssim_l v^{2+l} e^{-2\sqrt{2}|t|v}. \tag{4.43}$$

In conclusion, using estimates (4.37), (4.40), (4.42) and Lemma A.4.3 of Appendix Section A.4, identity

$$w_{k,v}(t,x) = \frac{x - \frac{d(t)}{2} + c_k(t)}{\sqrt{1 - \frac{\dot{d}(t)^2}{4}}},$$

and Theorem 4.1.7, we obtain that $Y_1(t)$ satisfies Lemma 4.2.7.

The proof that $Y_2(t)$ satisfies Lemma 4.2.7 is similar. First, from the Fundamental Theorem of Calculus, we have for any real number $\zeta \geq 1$ the following identity

$$\begin{aligned}
&\left[U^{(2)}\left(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)\right)-U^{(2)}(H_{-1,0}(x))\right]H^{'}_{-1,0}(x)\\
&\quad=\left[U^{(2)}\left(H^{\zeta}_{0,1}(x)\right)-2\right]H^{'}_{-1,0}(x)+U^{(3)}\left(H^{\zeta}_{0,1}(x)\right)H_{-1,0}(x)H^{'}_{-1,0}(x)\\
&\qquad+\int_0^1\left[U^{(4)}\left(H^{\zeta}_{0,1}(x)+\theta H_{-1,0}(x)\right)-U^{(4)}\left(\theta H_{-1,0}(x)\right)\right]H_{-1,0}(x)^2H^{'}_{-1,0}(x)(1-\theta)\,d\theta.
\end{aligned}$$

Therefore, estimates (4.2), (4.3), identity (4.41) and Lemma 3.2.1 imply for any $\zeta \geq 1$ the following estimate

$$\left|\frac{d^l}{d\zeta^l}\left\langle U^{(2)}\left(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)\right)-U^{(2)}(H_{-1,0}(x))-U^{(2)}\left(H^{\zeta}_{0,1}(x)\right)+2,\,H^{'}_{-1,0}(x)\partial_x H^{\zeta}_{0,1}(x)\right\rangle\right| \lesssim_l \zeta e^{-2\sqrt{2}\zeta}. \quad (4.44)$$

Similarly, Lemma 3.2.1 and identity (4.41) imply that the functions

$$\begin{aligned}
int_4(\zeta) &= \left\langle U^{(3)}\left(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)\right)\mathcal{G}(x-\zeta)H^{'}_{-1,0}(x),\,\partial_x H^{\zeta}_{0,1}(x)\right\rangle,\\
int_5(\zeta) &= \left\langle U^{(3)}\left(H^{\zeta}_{0,1}(x)+H_{-1,0}(x)\right)\mathcal{G}(-x)H^{'}_{-1,0}(x),\,\partial_x H^{\zeta}_{0,1}(x)\right\rangle
\end{aligned}$$

satisfy the estimates

$$\left|int_4^{(l)}(\zeta)\right|+\left|int_5^{(l)}(\zeta)\right| \lesssim_l e^{-\sqrt{2}\zeta}, \quad (4.45)$$

for all $\zeta \geq 1$ and any $l \in \mathbb{N}\cup\{0\}$. Therefore, from estimates (4.43), (4.38), (4.44), (4.45), Lemma 3.2.1 and Theorem 4.1.7 imply that

$$\left\langle Y_2(t,x),\,\dot{H}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle = y_2(t)\int_{\mathbb{R}}\left[U^{(2)}(H_{0,1}(x))-2\right]H^{'}_{0,1}(x)H^{'}_{-1,0}\left(x+\frac{d(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\right)dx + y_2(t)res_6(v,t), \quad (4.46)$$

where $res_6(v,t)$ is a real function, which satisfies for some constant $\eta_k \geq 0$, if $0<v\ll 1$,

$$\left|\frac{\partial^l}{\partial t^l}res_6(v,t)\right| \lesssim_l v^{4+l}\left[\ln\left(\frac{1}{v^2}\right)+|t|v\right]^{\eta_k}e^{-2\sqrt{2}|t|v},$$

for all $l \in \mathbb{N}\cup\{0\}$. So, from identity (A.56) of Appendix Section, estimates (4.22),

$$\left|\frac{d^l}{dx^l}\left[H_{-1,0}(x)+e^{-\sqrt{2}x}\right]\right| \lesssim_l \min\left(e^{-\sqrt{2}x},e^{-3\sqrt{2}x}\right),$$

and Lemma 3.2.1, we conclude the proof of Lemma 4.2.7 for $Y_2(t)$. □

**Remark 4.2.8.** *If $v \ll 1$, using the formula $U^{(2)}(\phi)=2-24\phi^2+30\phi^4$, Lemmas 3.2.1, 4.2.1, the estimates (4.36), (4.37), (4.38) and (4.39) of the proof of Lemma 4.2.7 imply for any*

*$s \geq 0$ that*

$$\max_{j\in\{1,2\}} \|Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^2 e^{-2\sqrt{2}|t|v},$$
$$\max_{j\in\{1,2\}} \|\partial_t Y_j(t)\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^2 e^{-2\sqrt{2}|t|v},$$
$$\max_{j\in\{1,2\}} \left\|\partial_t^2 Y_j(t)\right\|_{H^s_x} \lesssim_s \max_{j\in\{1,2\}} |y_j(t)| v^4 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}} |\ddot{y}_j(t)| v^2 e^{-2\sqrt{2}|t|v}.$$

*These estimates above don't depend on $k$, because from Theorem 4.1.7 we can verify for any $l \in \mathbb{N}\cup\{0\}$ the existence of $0 < \delta_{k,l} < 1$ such that if $0 < v < \delta_{k,l}$, then*

$$\left\|\frac{\partial^l}{\partial t^l} c_k(v,t)\right\|_{L^\infty_t(\mathbb{R})} \lesssim_l v^{2+l} \ln\frac{1}{v},$$

*which implies for any $l \in \mathbb{N}$ and any $v \ll 1$*

$$\left\|\frac{\partial^l}{\partial t^l}\left[-\frac{d(t)}{2} + c_k(v,t)\right]\right\|_{L^\infty_t(\mathbb{R})} \lesssim_l v^l, \quad \frac{d(t)}{2} - v < \left|-\frac{d(t)}{2} + c_k(v,t)\right|.$$

## 4.3 Energy Estimate Method

In this section, we are going to repeat the main argument of Section 4 of Chapter 2 to construct a function $L : \mathbb{R} \to \mathbb{R}$, which is going to be used to estimate the energy norm of $(u(t), \partial_t u(t))$ during a large time interval.

First, we consider a smooth cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 1, \text{if } x \leq \frac{49}{100}, \\ 0, \text{if } x \geq \frac{1}{2}. \end{cases} \tag{4.47}$$

Next, using the notation of Theorem 4.1.7, we denote

$$x_1(t) = -\frac{d(t)}{2} + \sum_{j=2}^{k} r_j(v,t),\; x_2(t) = \frac{d(t)}{2} - \sum_{j=2}^{k} r_j(v,t). \tag{4.48}$$

Actually, Theorem 4.1.7 and estimates 4.22 imply that

$$\max_{j\in\{1,2\}} |\dot{x}_j(t)| \lesssim v,\; \ln\frac{1}{v} \lesssim x_2(t) - x_1(t),\; \max_{j\in\{1,2\}} |\ddot{x}_j(t)| \lesssim v^2 e^{-2\sqrt{2}|t|v}. \tag{4.49}$$

From now on, we define the function $\chi_1 : \mathbb{R}^2 \to \mathbb{R}$ by

$$\chi_1(t,x) = \chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right). \tag{4.50}$$

Clearly, using the identities

$$\frac{\partial}{\partial t}\chi_1(t,x) = \frac{-\dot{x}_1(t)}{x_2(t) - x_1(t)}\chi'\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right) - \frac{(\dot{x}_2(t) - \dot{x}_1(t))(x - x_1(t))}{(x_2(t) - x_1(t))^2}\chi'\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right),$$
$$\frac{\partial}{\partial x}\chi_1(t,x) = \frac{1}{x_2(t) - x_1(t)}\chi'\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right),$$

we obtain the following estimates

$$\left\Vert\frac{\partial}{\partial t}\chi_1(t,x)\right\Vert_{L^\infty_x(\mathbb{R})}\lesssim\frac{v}{\ln\frac{1}{v}},\ \left\Vert\frac{\partial}{\partial x}\chi_1(t,x)\right\Vert_{L^\infty_x(\mathbb{R})}\lesssim\frac{1}{\ln\frac{1}{v}}. \tag{4.51}$$

Finally, using the notation (4.27) and the functions $Y_1(t)$, $Y_2(t)$ denoted respectively by (4.33) and (4.34), we define the function $A:\mathbb{R}^2\to\mathbb{R}$ by

$$\begin{aligned}A(t,x)=-\Lambda(\varphi_{k,v})(t,x)-&\frac{y_1(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,x)\right)-\frac{y_2(t)8\sqrt{2}e^{-\sqrt{2}d(t)}}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,-x)\right)\\ -Y_1(t,x)-Y_2(t,x)+&\frac{\dot{y}_1(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,x)\right)+\frac{\dot{y}_2(t)\dot{d}(t)}{1-\frac{\dot{d}(t)^2}{4}}H^{(2)}_{0,1}\left(w_{k,v}(t,-x)\right),\end{aligned} \tag{4.52}$$

for any $(t,x)\in\mathbb{R}^2$. Clearly, in notation of Remark 4.2.6, we have the following identity

$$\begin{aligned}&\partial_t^2u(t,x)-\partial_x^2u(t,x)+U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t,x)\\ &\quad=-\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,x)\right)-\frac{\ddot{y}_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}H^{'}_{0,1}\left(w_{k,v}(t,-x)\right)+A(t,x)+\mathcal{Q}(t,x)\\ &\qquad+\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}(\varphi_{k,v}(t,x))\right]u(t,x).\end{aligned} \tag{4.53}$$

Next, we consider

$$\begin{aligned}L(t)=&\int_{\mathbb{R}}\partial_t u(t,x)^2+\partial_x u(t,x)^2+U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\right)u(t,x)^2\,dx\\ &+2\int_{\mathbb{R}}\partial_t u(t,x)\partial_x u(t,x)\left[\dot{x}_1(t)\chi_1(t,x)+\dot{x}_2(t)\left(1-\chi_1(t,x)\right)\right]\,dx\\ &-2\int_{\mathbb{R}}u(t,x)A(t,x)\,dx.\end{aligned} \tag{4.54}$$

From now on, we use the notation $\overrightarrow{u}(t)=(u(t),\partial_t u(t))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$. The main objective of the Section 3 is to demonstrate the following theorem.

**Theorem 4.3.1.** *There exist constants $K$, $c>0$ and, for any $k\in\mathbb{N}_{\geq3}$, there exists $0<\delta(k)<1$ such that if $0<v\leq\delta(k)$, then the function $L(t)$ denoted in (4.54) satisfies, while the following condition*

$$\max_{j\in\{1,2\}}v^2|y_j(t)|+v|\dot{y}_j(t)|<v^{2k}\left(\ln\frac{1}{v}\right)^{n_k} \tag{4.55}$$

*is true, the estimates*

$$c\left\Vert\overrightarrow{u}(t)\right\Vert^2_{H^1_x\times L^2_x}\leq L(t)+C(k)v^{4k}\left(\ln\frac{1}{v}\right)^{2n_k},$$

*and*

$$\begin{aligned}\left|\dot{L}(t)\right|\leq K&\left[\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert^2_{H^1_x\times L^2_x}+C(k)\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}\right]\\ &+v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}+K\max_{j\in\{3,6\}}\left\Vert\overrightarrow{u}(t)\right\Vert^j_{H^1_x\times L^2_x},\end{aligned}$$

*where $C(k)>0$ is a constant depending only on $k$ and $n_k$ is the number defined in the statement of Theorem 4.1.7.*

*Proof of Theorem 4.3.1.* To simplify the proof of this theorem, we describe briefly the organization of our arguments. First, we denote $L(t)$ as

$$L(t) = L_1(t) + L_2(t) + L_3(t),$$

such that

$$L_1(t) = \int_{\mathbb{R}} \partial_t u(t,x)^2 + \partial_x u(t,x)^2 + U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)\right) u(t,x)^2\,dx, \tag{L1}$$

$$L_2(t) = 2\int_{\mathbb{R}} \partial_t u(t,x)\partial_x u(t,x)\left[\dot{x}_1(t)\chi_1(t,x) + \dot{x}_2(t)\left(1-\chi_1(t,x)\right)\right]dx, \tag{L2}$$

$$L_3(t) = -2\int_{\mathbb{R}} u(t,x)A(t,x)\,dx. \tag{L3}$$

Next, instead of estimating the size of $\left|\dot{L}(t)\right|$, we are going to estimate $\dot{L}_j(t)$ for each $j \in \{1,\,2,\,3\}$. Then, using these estimates, we can evaluate with high precision

$$\left|\dot{L}_1(t) + \dot{L}_2(t) + \dot{L}_3(t)\right|,$$

and obtain the second inequality of Theorem 4.3.1. The proof of the first inequality of Theorem 4.3.1 is short and it will be done later.

From identity (4.21), Remark 4.2.8 and equation (4.52) satisfied by $A(t,x)$, we deduce from the triangle inequality that

$$\|A(t,x)\|_{H^1_x} \lesssim \|\Lambda(\varphi_{k,v})(t,x)\|_{H^1_x} + v^2 e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)| + v\max_{j\in\{1,2\}}|\dot{y}_j(t)|.$$

Therefore, from Theorem 4.1.6 and Theorem 4.1.7, we obtain the existence of a value $C(k)>0$ depending only on $k$ such that if $v \ll 1$, then

$$\|A(t,x)\|_{H^1(\mathbb{R})} \lesssim C(k)v^{2k}\left(\ln\frac{1}{v} + |t|v\right)^{n_k} e^{-2\sqrt{2}|t|v} + v^2 e^{-2\sqrt{2}|t|v}\max_{j\in\{1,2\}}|y_j(t)| + v\max_{j\in\{1,2\}}|\dot{y}_j(t)|. \tag{4.56}$$

In conclusion, we obtain from (L3) and Cauchy-Schwartz inequality the existence of a value $C(k)>0$ depending only on $k$ satisfying

$$|L_3(t)| \lesssim \|u(t)\|_{L^2_x}\left[C(k)v^{2k}\left(\ln\frac{1}{v} + |t|v\right)^{n_k} e^{-2\sqrt{2}|t|v} + v^2 e^{-2\sqrt{2}|t|v}\max_{j\in\{1,2\}}|y_j(t)| + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v\right]. \tag{4.57}$$

Next, Lemmas 4.2.1, 4.2.2, Remark 4.2.8 and identity (4.52) satisfied by $A(t,x)$ imply the following inequality

$$\|\partial_t A(t,x)\|_{H^1_x} \lesssim \left\|\frac{\partial}{\partial t}\left[\Lambda(\phi_k)(v,t,x)\right]\right\|_{H^1_x} + \max_{j\in\{1,2\}}|y_j(t)|v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2 + \max_{j\in\{1,2\}}|\ddot{y}_j(t)|v,$$

from which with Theorem 4.1.7 we conclude the existence of a new value $C(k)$ depending only on $k$ satisfying

$$\|\partial_t A(t,x)\|_{H^1_x} \lesssim C(k)v^{2k+1}\left(\ln\frac{1}{v}+|t|v\right)^{n_k} e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}}|y_j(t)|v^3 e^{-2\sqrt{2}|t|v} + \max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2 + \max_{j\in\{1,2\}}|\ddot{y}_j(t)|v. \quad (4.58)$$

In conclusion, the identity (L3), estimate (4.58) and Cauchy-Schwartz inequality imply the existence of a new value $C(k)>0$ depending only on $k$, which satisfies

$$\begin{aligned}\left|\dot{L}_3(t)+2\int_{\mathbb{R}}\partial_t u(t,x)A(t,x)\,dx\right| &\lesssim \|u(t,x)\|_{L^2_x}\left[C(k)v^{2k+1}\left(\ln\frac{1}{v}+|t|v\right)^{n_k}e^{-2\sqrt{2}|t|v}+\max_{j\in\{1,2\}}|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\right] \\ &\quad + \|u(t,x)\|_{L^2_x}\left[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+\max_{j\in\{1,2\}}|\ddot{y}_j(t)|v\right]. \quad (4.59)\end{aligned}$$

Next, Theorem 4.1.7 implies that if $v\ll 1$, then

$$\begin{aligned}&\dot{L}_1(t)\\ =&2\int_{\mathbb{R}}\partial_t u(t,x)\left[\partial_t^2 u(t,x)-\partial_x^2 u(t,x)+U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t,x)\right]dx\\ &-\frac{\dot{d}(t)}{2\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)u(t,x)^2\,dx\\ &+\frac{\dot{d}(t)}{2\left(1-\frac{\dot{d}(t)^2}{4}\right)^{\frac{1}{2}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx\\ &+O\left(\frac{v}{\ln\frac{1}{v}}\|\overrightarrow{u}(t)\|^2_{H^1_x\times L^2_x}\right)\end{aligned} \quad (4.60)$$

Therefore, from Lemma 4.2.5, identity (4.52), Remark 4.2.6, hypothesis (4.55), estimates (4.59), (4.60) and orthogonality conditions (4.28), we obtain the existence of a value $C(k)>0$

depending only on $k$ such that if $v \ll 1$, then

$$
\begin{aligned}
&\dot{L}_1(t)+\dot{L}_3(t)\\
&=2\int_{\mathbb{R}}\partial_t u(t,x)\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(\varphi_{k,v}(t,x)\right)\right]u(t,x)\,dx\\
&\quad+\frac{\dot{d}(t)}{2\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx\\
&\quad-\frac{\dot{d}(t)}{2\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}(w_{k,v}(t,-x))\right)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)u(t,x)^2\,dx\\
&\quad+O\left(v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\,\|u(t)\|_{H^1_x}+\max_{j\in\{3,6\}}\|\overrightarrow{u}(t)\|^j_{H^1_x\times L^2_x}+\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\max_{j\in\{1,2\}}|y_j(t)|^2\right)\\
&\quad+O\left(\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\left[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\right]+\|\overrightarrow{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln\frac{1}{v}}\right)\\
&\quad+O\left(C(k)\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}\right).
\end{aligned}
\tag{4.61}
$$

Moreover, using estimates (4.22), Lemma 4.2.2 and identity $U(\phi)=\phi^2(1-\phi^2)^2$, we obtain from Theorem 4.1.7 that if $0<v\ll 1$ and $s\geq 0$, then

$$
\left\|\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(\phi_{k,v}(t,x)\right)\right]\right\|_{H^s_x}\lesssim_{s,k} v^2e^{-2\sqrt{2}|t|v}.
$$

Therefore, we deduce using Cauchy-Schwarz inequality that

$$
\begin{aligned}
&\left|2\int_{\mathbb{R}}\partial_t u(t,x)\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(\phi_{k,v}(t,x)\right)\right]u(t,x)\,dx\right|\\
&\quad\lesssim\left\|\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,x)\right)\right)-U^{(2)}\left(\phi_{k,v}(t,x)\right)\right]u(t,x)\right\|_{L^2_x}\|\partial_t u(t,x)\|_{L^2_x}\\
&\quad\lesssim\left\|\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(\phi_{k,v}(t,x)\right)\right]\right\|_{H^1_x}\|\overrightarrow{u}(t)\|^2_{H^1_x\times L^2_x}\\
&\quad\lesssim v^2\,\|\overrightarrow{u}(t)\|^2_{H^1_x\times L^2_x}.
\end{aligned}
$$

In conclusion,

$$
\begin{aligned}
&\dot{L}_1(t)+\dot{L}_3(t)\\
&=\frac{\dot{d}(t)}{2\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx\\
&\quad-\frac{\dot{d}(t)}{2\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}(w_{k,v}(t,-x))\right)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)u(t,x)^2\,dx\\
&\quad+O\left(v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|\,\|u(t)\|_{H^1_x}+\max_{j\in\{3,6\}}\|\overrightarrow{u}(t)\|^j_{H^1_x\times L^2_x}+\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\max_{j\in\{1,2\}}|y_j(t)|^2\right)\\
&\quad+O\left(\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\left[\max_{j\in\{1,2\}}|\dot{y}_j(t)|v^2+|y_j(t)|v^3e^{-2\sqrt{2}|t|v}\right]+\|\overrightarrow{u}(t)\|^2_{H^1_x\times L^2_x}\frac{v}{\ln\frac{1}{v}}\right)\\
&\quad+O\left(C(k)\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}\right).
\end{aligned}
\tag{4.62}
$$

Based on the arguments of [26] and Chapter 2, we are going to estimate the derivative of $L_2(t)$, for more accurate information see the third step of Lemma 4.2 in [26] or Theorem 2.4.1 from Chapter 2. Because of an argument of analogy, we only need to estimate the time derivative of

$$L_{2,1}(t) = 2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\partial_t u(t,x)\partial_x u(t,x)\,dx$$

to evaluate with high precision the derivative of $L_2(t)$. From the estimates (4.51), we can verify first that if $v \ll 1$, then

$$\dot{L}_{2,1}(t) = 2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\partial_t^2 u(t,x)\partial_x u(t,x)\,dx + 2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\partial_t u(t,x)\partial_{x,t}^2 u(t,x)\,dx \\ +O\left(\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right),$$

from which we deduce, using integration by parts and estimates (4.49), (4.51), that

$$\begin{aligned}\dot{L}_{2,1}(t) =& 2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\partial_t^2 u(t,x)\partial_x u(t,x)\,dx + O\left(\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right)\\ =& 2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\left[\partial_t^2 u(t,x) - \partial_x^2 u(t,x)\right]\partial_x u(t,x)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t,x)\partial_x u(t,x)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\partial_x^2 u(t,x)\partial_x u(t,x)\,dx\\ &-2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t,x)\partial_x u(t,x)\,dx\\ &+O\left(\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right),\end{aligned}$$

and, after using integration by parts again, we deduce from (4.51) that

$$\begin{aligned}\dot{L}_{2,1}(t) =& 2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t,x)\left[\partial_t^2 u(t) - \partial_x^2 u(t)\right]\partial_x u(t)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}} \chi_1(t)U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t)\partial_x u(t)\,dx\\ &+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}} \chi_1(t)U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)u(t)^2\,dx\\ &+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}} \chi_1(t)U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t)^2\,dx\\ &+O\left(\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right).\end{aligned}$$

Next, using estimates (4.2) satisfied by $H_{0,1}$, the definition of $\chi_1(t,x)$, Theorem 4.1.7 and identity (4.26), we deduce, for $v \ll 1$, the following inequality

$$\left|\chi_1(t,x)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)\right| + \left|(1-\chi_1(t,x))H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)\right| \lesssim e^{-\sqrt{2}\frac{49d(t)}{100}} \lesssim v^{\frac{98}{100}} \ll \frac{1}{\ln\frac{1}{v}},$$

from which we conclude that

$$\dot{L}_{2,1}(t) = 2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)\left[\partial_t^2 u(t,x)-\partial_x^2 u(t,x)\right]\partial_x u(t,x)\,dx$$
$$+2\dot{x}_1(t)\int_{\mathbb{R}}\chi_1(t)U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t,x)\partial_x u(t,x)\,dx$$
$$+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx$$
$$+O\left(\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right).$$

Furthermore, from Remark 4.2.6, estimate (4.56) of $A(t,x)$ and identity (4.53) satisfied by $u(t,x)$, we conclude the existence of a value $C(k)>0$ depending only on $k$ and satisfying, for any positive number $v\ll 1$,

$$\dot{L}_{2,1}(t) = \frac{\dot{x}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx$$
$$+O\left(\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}\left[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|+C(k)v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}+v\max_{j\in\{2,6\}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^j\right]\right)$$
$$+O\left(\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}\left[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)|+v^2|\dot{y}_j(t)|\right]+\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right).$$

Therefore, using an argument of analogy, we obtain, for any positive number $v\ll 1$, that

$$\dot{L}_2(t) = \frac{\dot{x}_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)u(t,x)^2\,dx$$
$$+\frac{\dot{x}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx$$
$$+O\left(\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}\left[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|+C(k)v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}\right]+v\max_{j\in\{3,6\}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^j\right)$$
$$+O\left(\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}\left[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)|+v^2|\dot{y}_j(t)|\right]+\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right), \tag{4.63}$$

where $C(k)>0$ is a parameter depending only on $k$. Moreover, using (4.48) and Theorem 4.1.7, we deduce from estimate (4.63) that

$$\dot{L}_2(t) = \frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,x)\right)u(t,x)^2\,dx$$
$$-\frac{\dot{d}(t)}{\sqrt{4-\dot{d}(t)^2}}\int_{\mathbb{R}}U^{(3)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)u(t,x)^2\,dx$$
$$+O\left(\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}\left[v\max_{j\in\{1,2\}}|\ddot{y}_j(t)|+C(k)v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}\right]+v\max_{j\in\{3,6\}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^j\right)$$
$$+O\left(\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}\left[v^3e^{-2\sqrt{2}v|t|}\max_{j\in\{1,2\}}|y_j(t)|+v^2|\dot{y}_j(t)|\right]+\frac{v}{\ln\frac{1}{v}}\left\Vert\overrightarrow{u}(t)\right\Vert_{H^1_x\times L^2_x}^2\right). \tag{4.64}$$

Finally, the estimate (4.64) and (4.61) imply, for any $k \in \mathbb{N}_{\geq 3}$, the existence of a parameter $C(k) > 0$, depending only on $k$, which satisfies for any positive number $v \ll 1$ the estimate

$$\begin{aligned}|\dot{L}(t)| =& O\left(v \max_{j\in\{1,2\}} |\ddot{y}_j(t)| \left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} + \max_{j\in\{3,6\}} \left\|\overrightarrow{u}(t)\right\|^j_{H^1_x\times L^2_x}\right)\\ &+O\left(\left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} \max_{j\in\{1,2\}} |y_j(t)|^2\right)\\ &+O\left(\left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x}\left[\max_{j\in\{1,2\}} |\dot{y}_j(t)| v^2 + |y_j(t)| v^3 e^{-2\sqrt{2}|t|v}\right]\right)\\ &+O\left(\left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x} \frac{v}{\ln\left(\frac{1}{v^2}\right)} + C(k)\left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k}\right), \qquad (4.65)\end{aligned}$$

from which we obtain the existence of a new constant $C(k) > 0$ satisfying the second inequality of Theorem 4.3.1 if the condition (4.55) is true and $v \ll 1$.

Now, it remains to prove the first inequality of Theorem 4.3.1. Using change of variables and Lemma 2.2.6, it is not difficult to verify that there exists $K > 0$ such that if $v \ll 1$, then

$$L_1(t) \geq K \left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x}.$$

Next, from the definition of $L_2(t)$ and estimates (4.49), we obtain that if $v \ll 1$, then

$$|L_2(t)| \ll v^{\frac{3}{4}} \left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x},$$

and while condition (4.55) is true, we deduce from Theorem 4.1.7 and estimate (4.56) the following inequality

$$|L_3(t)| \lesssim_k \left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} v^{2k}\left(\ln\frac{1}{v}\right)^{n_k}.$$

So, using Young inequality, we can find a parameter $C_1(k) > 0$ large enough depending only on $k$ such that

$$|L_3(t)| \leq \frac{K}{2}\left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x} + C_1(k) v^{4k}\left(\ln\frac{1}{v}\right)^{2n_k}.$$

In conclusion, all the estimates above imply the first inequality of Theorem 4.3.1 if $0 < v \ll 1$ and condition (4.55) is true. □

## 4.4 Proof of Theorem 4.2.3

From the information of Theorem 4.3.1 in the last section, we are ready to start the demonstration of Theorem 4.2.3.

*Proof of Theorem 4.2.3.* First, for any $(t, x) \in \mathbb{R}^2$, Lemma 4.2.5 implies that $\phi(t, x)$ has the following representation

$$\phi(t,x) = \varphi_{k,v}(t,x) + \frac{y_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H^{'}_{0,1}\left(w_{k,v}(t,x)\right) + \frac{y_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H^{'}_{0,1}\left(w_{k,v}(t,-x)\right) + u(t,x),$$

such that the function $u(t,x)$ satisfies the orthogonality conditions (4.28) and $y_1$, $y_2$ are functions in $C^2(\mathbb{R})$.

**Step** 1.(Ordinary differential system of $y_1(t)$, $y_2(t)$.) From Remarks 3.5.3, 4.2.6 and the definition of $A(t,x)$ in (4.52), we have that $u(t,x)$ is a solution of a partial differential equation of the form

$$\begin{aligned}
&\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right) - H_{0,1}\left(w_{k,v}(t,-x)\right)\right) u(t,x)\\
&\quad = -\frac{\ddot{y}_1(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H_{0,1}^{'}\left(w_{k,v}(t,x)\right) - \frac{\ddot{y}_2(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} H_{0,1}^{'}\left(w_{k,v}(t,-x)\right)\\
&\qquad + A(t,x) + \mathcal{P}_1(v,t,x), \quad (4.66)
\end{aligned}$$

where $\mathcal{P}_1(v,t,x)$ satisfies for any $0<v\ll 1$ and any $t\in\mathbb{R}$ the inequality

$$\begin{aligned}
\|\mathcal{P}_1(v,t,x)\|_{H^1_x} \lesssim{}& \|u(t)\|_{H^1_x}^2 + \max_{j\in\{1,2\}} |y_j(t)|^2 + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^3 \left(\ln\left(\frac{1}{v^2}\right) + |t|v\right) e^{-2\sqrt{2}|t|v}\\
&+ \|u(t)\|_{H^1_x}^6 + \max_{j\in\{1,2\}} |y_j(t)|^6 + \max_{j\in\{1,2\}} |y_j(t)| v^4 \left(\ln\left(\frac{1}{v^2}\right) + |t|v\right) e^{-2\sqrt{2}|t|v}.
\end{aligned}$$

With the objective of simplifying our computations, we denote

$$\begin{aligned}
NOL(t) ={}& \|u(t)\|_{H^1}^2 + \max_{j\in\{1,2\}} |y_j(t)|^2 + v^{2(k+1)} \left(|t|v + \ln\left(\frac{1}{v^2}\right)\right)^{n_k+1} e^{-2\sqrt{2}|t|v} \quad (4.67)\\
&+ \|u(t)\|_{H^1_x}^6 + \max_{j\in\{1,2\}} |y_j(t)|^6 + \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^3 \left(\ln\left(\frac{1}{v^2}\right) + |t|v\right) e^{-2\sqrt{2}|t|v}\\
&+ \max_{j\in\{1,2\}} |y_j(t)| v^4 \left(\ln\left(\frac{1}{v^2}\right) + |t|v\right)^{\max\{1,\eta_k\}} e^{-2\sqrt{2}|t|v},
\end{aligned}$$

where $\eta_k$ is the number denoted in Lemma 4.2.7. Also, from Theorem 4.1.7, Lemma 4.2.7 and identity (4.52), we deduce that

$$\begin{bmatrix} \left\langle A(t,x), \dot{H}_{0,1}\left(w_{k,v}(t,x)\right)\right\rangle \\ \left\langle A(t,x), \dot{H}_{0,1}\left(w_{k,v}(t,-x)\right)\right\rangle \end{bmatrix} = e^{-\sqrt{2}d(t)} \begin{bmatrix} -4\sqrt{2} & 4\sqrt{2} \\ 4\sqrt{2} & -4\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + Rest(t), \quad (4.68)$$

where, if $v\ll 1$, the real function $Rest(t)$ satisfies for any $t\in\mathbb{R}$

$$\begin{aligned}
e^{2\sqrt{2}|t|v}\,|Rest(t)| \lesssim_k{}& v^{2(k+1)} \left(|t|v + \ln\left(\frac{1}{v^2}\right)\right)^{n_k+1} + \max_{j\in\{1,2\}} |y_j(t)| v^4 \left(|t|v + \ln\left(\frac{1}{v^2}\right)\right)^{\max\{1,\eta_k\}}\\
&+ \max_{j\in\{1,2\}} |\dot{y}_j(t)| v^3 \left(|t|v + \ln\left(\frac{1}{v^2}\right)\right). \quad (4.69)
\end{aligned}$$

From the orthogonality conditions (4.28), Theorem 4.1.7 and Lemma 4.2.1, we obtain the following estimate

$$\begin{aligned}
\left\langle \partial_t^2 u(t,x),\, H_{0,1}^{'}\left(w_{k,v}(t,x)\right)\right\rangle ={}& \frac{\dot{d}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}} \left\langle \partial_t u(t,x),\, H_{0,1}^{(2)}\left(w_{k,v}(t,x)\right)\right\rangle_{L^2_x}\\
&+ O\left(\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x} v^2\right). \quad (4.70)
\end{aligned}$$

Also, using integration by parts, identity $-H_{0,1}^{(3)}(x)+U^{(2)}(H_{0,1}(x))H_{0,1}^{'}(x)=0$, Lemma 3.2.1 and Cauchy-Schwarz inequality, we deduce that if $0<v\ll 1$, then

$$\begin{aligned}
&\left\langle-\partial_x^2 u(t)+U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)u(t),\, H_{0,1}^{'}\left(w_{k,v}(t,x)\right)\right\rangle\\
=&\left\langle u(t),\,\left[U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)-H_{0,1}\left(w_{k,v}(t,-x)\right)\right)-U^{(2)}\left(H_{0,1}\left(w_{k,v}(t,x)\right)\right)\right]H_{0,1}^{'}\left(w_{k,v}(t,x)\right)\right\rangle\\
&+O\left(v^2\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\right)\\
=&O\left(v^2\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\right).
\end{aligned} \tag{4.71}$$

From now on, we denote any continuous function $f(t)$ as $O_k\left(NOL(t)\right)$, if and only if $f$ satisfies the following estimate

$$|f(t)|\lesssim_k NOL(t).$$

In conclusion, applying the scalar product of the equation (4.66) with $\dot{H}_{0,1}\left(w_{k,v}(t,x)\right)$ and $\dot{H}_{0,1}\left(w_{k,v}(t,-x)\right)$, we obtain using Lemma 3.2.1 and estimates (4.70), (4.71) that

$$\begin{aligned}
\begin{bmatrix}\left\|\dot{H}_{0,1}\right\|_{L^2_x}^2 & O\left(d(t)e^{-\sqrt{2}d(t)}\right)\\ O\left(d(t)e^{-\sqrt{2}d(t)}\right) & \left\|\dot{H}_{0,1}\right\|_{L^2_x}^2\end{bmatrix}\begin{bmatrix}\ddot{y}_1(t)\\ \ddot{y}_2(t)\end{bmatrix} =&e^{-\sqrt{2}d(t)}\begin{bmatrix}-4\sqrt{2} & 4\sqrt{2}\\ 4\sqrt{2} & -4\sqrt{2}\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}\\
&+\begin{bmatrix}O\left(v^2\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\right)\\ O\left(v^2\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\right)\end{bmatrix}\\
&-\begin{bmatrix}\frac{\dot{d}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left\langle\partial_t u(t,x),\, H_{0,1}^{(2)}\left(w_{k,v}(t,x)\right)\right\rangle_{L^2_x}\\ \frac{\dot{d}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left\langle\partial_t u(t,x),\, H_{0,1}^{(2)}\left(w_{k,v}(t,-x)\right)\right\rangle_{L^2_x}\end{bmatrix}\\
&+\begin{bmatrix}O_k\left(NOL(t)\right)\\ O_k\left(NOL(t)\right)\end{bmatrix}.
\end{aligned} \tag{4.72}$$

**Step 2.**(Refined ordinary differential system.) Motivated by equation (4.72), for $j\in\{1,\,2\}$ we define the functions

$$c_j(t)=y_j(t)-y_j(T_{0,k})+2\sqrt{2}\int_{T_{0,k}}^t\frac{\dot{d}(s)}{\sqrt{1-\frac{\dot{d}(s)^2}{4}}}\left\langle u(s),\,H_{0,1}^{(2)}\left(w_{k,v}(s,(-1)^{j+1}x)\right)\right\rangle_{L^2_x}\,ds.$$

Clearly, we can verify using (4.22), Lemma 4.2.1 and Cauchy-Schwarz inequality that

$$\begin{aligned}
\dot{c}_j(t)=&\dot{y}_j(t)+\frac{2\sqrt{2}\dot{d}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left\langle u(t,x),\,H_{0,1}^{(2)}\left(w_{k,v}(t,(-1)^{j+1}x)\right)\right\rangle_{L^2_x},\\
\ddot{c}_j(t)=&\ddot{y}_j(t)+\frac{2\sqrt{2}\dot{d}(t)}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\left\langle\partial_t u(t,x),\,\ddot{H}_{0,1}\left(w_{k,v}(t,(-1)^{j+1}x)\right)\right\rangle_{L^2_x}+O\left(v^2\,\|u(t)\|_{H^1_x}\right).
\end{aligned}$$

In conclusion, from the ordinary differential system of equations (4.72) we deduce that

$$\frac{d}{dt}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{c}_1(t) \\ \dot{c}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16e^{-\sqrt{2}d(t)} & 16e^{-\sqrt{2}d(t)} & 0 & 0 \\ 16e^{-\sqrt{2}d(t)} & -16e^{-\sqrt{2}d(t)} & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \dot{c}_1(t) \\ \dot{c}_2(t) \end{bmatrix}$$
$$+ \begin{bmatrix} O(v \left\lVert u(t) \right\rVert_{H^1_x}) \\ O(v \left\lVert u(t) \right\rVert_{H^1_x}) \\ O_k(NOL(t)) + O\left(v^2 \left\lVert \overrightarrow{u}(t) \right\rVert_{H^1_x \times L^2_x}\right) \\ O_k(NOL(t)) + O\left(v^2 \left\lVert \overrightarrow{u}(t) \right\rVert_{H^1_x \times L^2_x}\right) \end{bmatrix}.$$

Actually, using the following change of variables $e_1(t) = y_1(t) - y_2(t)$, $e_2(t) = y_1(t) + y_2(t)$, $\xi_1(t) = c_1(t) - c_2(t)$ and $\xi_2(t) = c_1(t) + c_2(t)$, we obtain from the ordinary differential system of equations above that

$$\frac{d}{dt}\begin{bmatrix} e_1(t) \\ e_2(t) \\ \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -32e^{-\sqrt{2}d(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{bmatrix} + \begin{bmatrix} O(v \left\lVert u(t) \right\rVert_{H^1_x}) \\ O(v \left\lVert u(t) \right\rVert_{H^1_x}) \\ O_k(NOL(t)) + O\left(v^2 \left\lVert \overrightarrow{u}(t) \right\rVert_{H^1_x \times L^2_x}\right) \\ O_k(NOL(t)) + O\left(v^2 \left\lVert \overrightarrow{u}(t) \right\rVert_{H^1_x \times L^2_x}\right) \end{bmatrix}. \tag{4.73}$$

To simplify our notation, we denote

$$M_1(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -32e^{-\sqrt{2}d(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.74}$$

It is not difficult to verify that all the solutions of linear ordinary differential equation

$$\dot{L}(t) = M_1(t)L(t) \text{ for } L(t) \in \mathbb{R}^4,$$

are the linear space generated by the following functions

$$L_1(t) = \begin{bmatrix} \tanh(\sqrt{2}vt) \\ 0 \\ \sqrt{2}v \operatorname{sech}(\sqrt{2}vt)^2 \\ 0 \end{bmatrix},\ L_2(t) = \begin{bmatrix} \sqrt{2}vt \tanh(\sqrt{2}vt) - 1 \\ 0 \\ 2v^2 t \operatorname{sech}(\sqrt{2}vt)^2 + \sqrt{2}v \tanh(\sqrt{2}vt) \\ 0 \end{bmatrix},$$
$$L_3(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ L_4(t) = \begin{bmatrix} 0 \\ t \\ 0 \\ 1 \end{bmatrix}.$$

Also, by elementary computation, we can verify for any $t \in \mathbb{R}$ that

$$\det\left[L_1(t), L_2(t), L_3(t), L_4(t)\right] = -\sqrt{2}v. \tag{4.75}$$

In conclusion, using the variation of parameters technique, we can write any $C^1$ solution of (4.73) as $L(t) = \sum_{i=1}^{4} a_i(t)L_i(t)$, such that $a_i(t) \in C^1(\mathbb{R})$ for all $1 \leq i \leq 4$ and

$$\begin{bmatrix} \tanh(\sqrt{2}vt) & \sqrt{2}vt\tanh(\sqrt{2}vt) - 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vt)^2 & 2v^2t\,\mathrm{sech}(\sqrt{2}vt)^2 + \sqrt{2}v\tanh(\sqrt{2}vt) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{a}_1(t) \\ \dot{a}_2(t) \\ \dot{a}_3(t) \\ \dot{a}_4(t) \end{bmatrix}$$
$$= \begin{bmatrix} O(v\,\|u(t)\|_{H^1_x}) \\ O(v\,\|u(t)\|_{H^1_x}) \\ O_k(NOL(t)) + O\left(v^2\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\right) \\ O_k(NOL(t)) + O\left(v^2\,\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}\right) \end{bmatrix}, \quad (4.76)$$

with

$$\begin{bmatrix} \tanh(\sqrt{2}vT_{0,k}) & \sqrt{2}vT_{0,k}\tanh(\sqrt{2}vT_{0,k}) - 1 & 0 & 0 \\ 0 & 0 & 1 & T_{0,k} \\ \sqrt{2}v\,\mathrm{sech}(\sqrt{2}vT_{0,k})^2 & 2v^2t\,\mathrm{sech}(\sqrt{2}vT_{0,k})^2 + \sqrt{2}v\tanh(\sqrt{2}vT_{0,k}) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1(T_{0,k}) \\ a_2(T_{0,k}) \\ a_3(T_{0,k}) \\ a_4(T_{0,k}) \end{bmatrix}$$
$$= \begin{bmatrix} y_1(T_{0,k}) - y_2(T_{0,k}) \\ y_1(T_{0,k}) + y_1(T_{0,k}) \\ \dot{c}_1(T_{0,k}) \\ \dot{c}_2(T_{0,k}) \end{bmatrix}. \quad (4.77)$$

**Step** 3.(Estimate of $\|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x}$.) From now on, for $C_1 > 1$, $C_2 > 0$ being fixed numbers to be chosen later, we consider the following set

$$B_{C_1,C_2} = \left\{t \in \mathbb{R}\Big|\ \max_{j\in\{1,2\}} |y_j(t)|v^2 + |\dot{y}_j(t)|v \leq C_1 v^{2(k+1)}\left(\ln\frac{1}{v}\right)^{n_k+3}\exp\left(\frac{C_2 v|t-T_{0,k}|}{\ln\frac{1}{v}}\right)\right\}.$$

We also consider the following set

$$D_{u,v} = \left\{t \in \mathbb{R}\Big|\ \|\overrightarrow{u}(t)\|_{H^1_x\times L^2_x} < v^2\right\}.$$

First, if $v^2|y(T_{0,k})| + v|\dot{y}(T_{0,k})| < v^{3k}$ and $v \ll 1$, then $T_{0,k} \in B_{C_1,C_2} \cap D_{u,v}$. Indeed, this happens when

$$\|(\varphi_{k,v}(T_{0,k}), \partial_t\varphi_{k,v}(T_{0,k})) - (\phi(T_{0,k}), \partial_t\phi(T_{0,k})))\|_{H^1_x\times L^2_x} < v^{4k},$$

because since $u(t,x)$ satisfies the orthogonality conditions (4.28), we can verify using Lemma 3.2.1 that

$$\|\varphi_{k,v}(T_{0,k}) - \phi(T_{0,k})\|^2_{H^1_x} \cong \max_{j\in\{1,2\}} y_j(T_{0,k})^2 + \|u\,(T_{0,k})\|^2_{H^1_x}. \quad (4.78)$$

By a similar reasoning but using now Lemma 4.2.1 and estimate (4.78), we can verify that if $0 < v \ll 1$, then

$$\max_{j\in\{1,2\}} \dot{y}_j(T_{0,k})^2 + \|\partial_t u\,(T_{0,k})\|^2_{L^2_x} \lesssim \|(\varphi_{k,v}(T_{0,k}), \partial_t\varphi_{k,}(T_{0,k})) - (\phi(T_{0,k}), \partial_t\phi(T_{0,k}))\|^2_{H^1_x\times L^2_x}, \quad (4.79)$$

where $T_{0,k}$ satisfies the hypothesis of Theorem 4.2.3, for more details see Appendix $B$ in [47]. Also, for any $\theta \in (0,1)$, if $v \ll 1$, then while

$$|t - T_{0,k}| < \frac{\ln \frac{1}{v}^{2-\theta}}{v},$$

and $t \in B_{C_1,C_2} \cap D_{u,v}$, we can verify the following estimate

$$\max_{j\in\{1,2\}} v^2|y_j(t)| + v|\dot{y}_j(t)| < v^{2k+1}\left(\ln \frac{1}{v}\right)^{n_k},$$

from which with estimate (4.72), the definition of $NOL(t)$ at (4.67), the definition of $D_{u,v}$ and the assumption of $k \geq 2$, we obtain that

$$\max_{j\in\{1,2\}} |\ddot{y}_j(t)| \lesssim_k v^{2k}\left(\ln \frac{1}{v}\right)^{n_k} + v\left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} + \left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x}.$$

In conclusion, if $v \ll 1$, from Theorem 4.3.1, we deduce that the functional $L(t)$ defined in last section satisfies, for a constant $C_0$ and a parameter $C(k)$ depending only on $k$, the estimates

$$\begin{aligned}
|\dot{L}(t)| \lesssim & v \max_{j\in\{1,2\}} |\ddot{y}_j(t)| \left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} + \left\|\overrightarrow{u}(t)\right\|^3_{H^1_x\times L^2_x} \\
& + C(k)\left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} v^{2k+1}\left(\ln \frac{1}{v}\right)^{n_k} + \left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x} \frac{v}{\ln\left(\frac{1}{v^2}\right)}, \\
C_0\left\|\overrightarrow{u}(t)\right\|^2_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})} \leq & L(t) + C(k)v^{4k}\left(\ln \frac{1}{v}\right)^{2n_k}.
\end{aligned}$$

Therefore, from the ordinary differential system of equations defined in (4.72), we conclude for $v \ll 1$ that if $t \in B_{C_1,C_2} \cap D_{u,v}$ and

$$|t - T_{0,k}| < \frac{\ln \frac{1}{v}^{2-\theta}}{v}, \tag{4.80}$$

then there exists a constant $C(k) > 0$ depending only on $k$ satisfying

$$|\dot{L}(t)| \lesssim C(k)\left\|\overrightarrow{u}(t)\right\|_{H^1_x\times L^2_x} v^{2k+1}\left(\ln \frac{1}{v}\right)^{n_k} + \left\|\overrightarrow{u}(t)\right\|^2_{H^1_x\times L^2_x} \frac{v}{\ln\left(\frac{1}{v^2}\right)}.$$

Therefore, by a similar argument to the proof of Theorem 4.5 in [47], we can verify from Theorem 4.3.1 and the Gronwall Lemma applied on $L(t)$ that there exists a constant $K > 1$ non depending on $k$ and $v$ such that if $t$ satisfies condition (4.80) and $t \in B_{C_1,C_2} \cap D_{u,v}$, then we have the following estimate

$$\left\|(u(t), \partial_t u(t))\right\|_{H^1_x\times L^2_x} \lesssim_k \max\left(\left\|\overrightarrow{u}\,(T_{0,k})\right\|_{H^1_x\times L^2_x}, v^{2k}\left(\ln \frac{1}{v}\right)^{n_k+1}\right) \exp\left(\frac{K|t - T_{0,k}|v}{\ln \frac{1}{v}}\right). \tag{4.81}$$

In conclusion, if $v \ll 1$, $t \in B_{C_1,C_2}$ and $t$ satisfies (4.80), then $t \in D_{u,v}$ and (4.81) is true.

**Step 4.**(Estimate of $y_1(t)$, $y_2(t)$.) Next, we are going to use the estimate (4.81) in the ordinary differential system of equations (4.73) to estimate the evolution of $y_1(t)$ and $y_2(t)$ while

$t \in B_{C_1,C_2}$ and $t$ satisfies condition (4.80). From (4.67), we have that if $t \in B_{C_1,C_2}$, $t$ satisfies condition (4.80) and $0 < v \ll 1$, then

$$NOL(t) \ll v^2 \max\left(\|\overrightarrow{u}(T_{0,k})\|_{H^1_x \times L^2_x}, v^{2k}\left(\ln\frac{1}{v}\right)^{n_k+1}\right)\exp\left(\frac{K|t-T_{0,k}|v}{\ln\frac{1}{v}}\right). \tag{4.82}$$

In conclusion, from the Cauchy problem (4.23) satisfied by $\phi$, identity (4.75) and estimates (4.78), (4.79), and (4.82), we deduce from the linear system (4.76) the following estimates

$$\begin{aligned}
|\dot{a}_1(t)| &\lesssim_k v^{2k+1}\left[|t|v+1\right]\left(\ln\frac{1}{v}\right)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln\frac{1}{v}}\right),\\
|\dot{a}_2(t)| &\lesssim_k v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln\frac{1}{v}}\right),\\
|\dot{a}_3(t)| &\lesssim_k v^{2k+1}\left[|t|v+1\right]\left(\ln\frac{1}{v}\right)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln\frac{1}{v}}\right),\\
|\dot{a}_4(t)| &\lesssim_k v^{2k+2}\left(\ln\frac{1}{v}\right)^{n_k+1}\exp\left(K\frac{v|t-T_{0,k}|}{\ln\frac{1}{v}}\right).
\end{aligned}$$

In conclusion, using the initial condition (4.77), we deduce from the fact that $T_{0,k}$ is in $B_{C_1,C_2}$,the Fundamental Theorem of Calculus and the elementary estimate

$$|t|v < \ln\frac{1}{v}\exp\left(\frac{v|t|}{\ln\frac{1}{v}}\right),$$

that if $\{\theta t + (1-\theta)T_{0,k} \,|\, 0<\theta<1\} \subset B_{C_1,c_2}$ and $t$ satisfies (4.80), then

$$\begin{aligned}
|a_1(t)| + |a_3(t)| &\lesssim_k v^{2k}\left(\ln\frac{1}{v}\right)^{n_k+3}\exp\left(\frac{(K+1)|t-T_{0,k}|v}{\ln\frac{1}{v}}\right),\\
v|a_2(t)| + |a_4(t)| &\lesssim_k v^{2k+1}\left(\ln\frac{1}{v}\right)^{n_k+2}\exp\left(\frac{K|t-T_{0,k}|v}{\ln\frac{1}{v}}\right).
\end{aligned}$$

In conclusion from the ordinary differential system of equations (4.73) satisfied by $e_j(t)$ for $j \in \{1,2,3,4\}$, the fact that $e_1(t) = y_1(t) - y_2(t)$, $e_2(t) = y_1(t) + y_2(t)$ and $\xi_1(t) = c_1(t) - c_2(t)$, $\xi_2(t) = c_1(t) + c_2(t)$, we can verify by triangle inequality and the identity

$$\begin{bmatrix} e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\end{bmatrix} = \sum_{j=1}^{4} a_j L_j(t)$$

the existence of $C_1(k) > 0$ depending on $k$ such that for $C_2 = K+2$ and $v \ll 1$ we have that if

$$|t - T_{0,k}| < \frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v},$$

then $t \in B_{C_1(k),C_2}$. □

**Remark 4.4.1.** *For any constants $\theta, \gamma \in (0,1)$, obviously*

$$\lim_{v \to +0} v^{\gamma}\exp\left(\left[\ln\frac{1}{v}\right]^{\theta}\right) = 0.$$

*In conclusion, for fixed $k \in \mathbb{N}$ large and $0 < \theta < \frac{1}{4}$, we can deduce from Theorem 4.2.3 that there is a $\Delta_{k,\theta} > 0$ such that if $0 < v < \Delta_{k,\theta}$, then*

$$\|(\phi(t,x), \partial_t \phi(t,x)) - (\phi_k(v,t,x), \partial_t \phi_k(v,t,x))\|_{H^1_x \times L^2_x} < v^{2k-\frac{1}{2}},$$

*for all $t$ satisfying*

$$|t - T_{0,k}| < \frac{\left(\ln \frac{1}{v}\right)^{2-\theta}}{v}.$$

## 4.5 Proof of Theorem 4.1.3

The main objective of this section is to prove Theorem 4.1.3.

**Remark 4.5.1.** *The importance of this theorem is to describe the dynamics of the two solitons before the collision instant, for all $t < 0$ and $|t| \gg 1$. More precisely, if two moving kinks are coming from an infinite distance with a sufficiently low speed $v$ satisfying $v \leq \delta(2k)$, then the inelasticity of the collision is going to be of order at most $O(v^k)$ and the kinks will move away each one with the speed of size in modulus $v + O(v^k)$ when $t$ goes to $-\infty$.*

The proof of Theorem 4.1.3 uses energy estimate techniques from the article [24]. Furthermore, the demonstration of Theorem 4.1.3 is quite similar to the proof of Theorem 1 of the article [31] and also uses modulation techniques inspired by [54] and [31].

From now on, we consider

$$P_+(\phi(t), \partial_t \phi(t)) = -\int_0^{+\infty} \partial_t \phi(t,x) \partial_x \phi(t,x)\, dx, \tag{4.83}$$

and since the solution $\phi(t,x)$ is an odd function in the variable $x$ for all $t \in \mathbb{R}$, we have that

$$E(\phi) = 2\left[\int_0^{+\infty} \frac{\partial_x \phi(t,x)^2 + \partial_t \phi(t,x)^2}{2} + U(\phi(t,x))\, dx\right] = 2E_+(\phi(t), \partial_t \phi(t)),$$

where

$$E_+(\phi(t), \partial_t \phi(t)) = \int_0^{+\infty} \frac{\partial_x \phi(t,x)^2 + \partial_t \phi(t,x)^2}{2} + U(\phi(t,x))\, dx \tag{4.84}$$

is a conserved quantity.

### 4.5.1 Modulation techniques

First, similarly to [31], we consider for any $0 < v < 1$, $y \in \mathbb{R}$ the following function on $x \in \mathbb{R}$

$$\overrightarrow{H_{0,1}}((v,y),x) = \begin{bmatrix} H_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) \\ \frac{-v}{\sqrt{1-v^2}} H'_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{4.85}$$

and $\overrightarrow{H_{-1,0}}((v,y),x) = -\overrightarrow{H_{0,1}}((v,y),-x)$, for all $x \in \mathbb{R}$.

Next, we consider the anti-symmetric map

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4.86}$$

and based on [31], we consider for any $0 < v < 1$ and any $y \in \mathbb{R}$ the following functions, which were defined in subsection 2.3 of [31],

$$C_{v,y}(x) = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} H'_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) \\ \frac{-v}{1-v^2} H^{(2)}_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{4.87}$$

$$D_{v,y}(x) = \begin{bmatrix} \frac{v}{1-v^2}\frac{x-y}{\sqrt{1-v^2}} H'_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) \\ \frac{-1}{(1-v^2)^{\frac{3}{2}}} H'_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) - \frac{v^2}{(1-v^2)^{\frac{3}{2}}}\frac{x-y}{\sqrt{1-v^2}} H^{(2)}_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{4.88}$$

see also the article [7].

The following identity is going to be useful for our next results.

**Lemma 4.5.2.** *For any $v \in (0,1)$, it holds*

$$\left\langle \partial_x \overrightarrow{H_{0,1}}\left((v,0),x\right), JD_{0,v}\right\rangle = -\left(1-v^2\right)^{-\frac{3}{2}} \left\Vert H'_{0,1}\right\Vert^2_{L^2_x}.$$

*Proof.* See the proof of Lemma 2.4 from the article [31]. □

Next, for any value $y_0 \gg 1$, we are going to modulate any odd function $(\phi_0, \phi_1)$ close to $\overrightarrow{H_{-1,0}}((v,y_0),x) + \overrightarrow{H_{0,1}}((v,y_0),x)$ in the energy norm in terms of an orthogonal condition.

**Lemma 4.5.3.** *There exist $K > 0$ and $\delta_0, \delta_1 \in (0,1)$ such that if $0 < v < \delta_1$, $y_0 > \frac{1}{\delta_1}$, $0 \leq \delta \leq \delta_0$ and $(\phi_1 - H_{0,1} - H_{-1,0}, \phi_2) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ is an odd function satisfying*

$$\left\Vert (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v,y_0),x) - \overrightarrow{H_{0,1}}((v,y_0),x)\right\Vert_{H^1_x \times L^2_x} \leq \delta v, \tag{4.89}$$

*then there exists a unique $\hat{y} > 1$ such that $|\hat{y} - y_0| \leq K\delta v$ and the function*

$$\overrightarrow{\kappa}(x) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v,\hat{y}),x) - \overrightarrow{H_{0,1}}((v,\hat{y}),x)$$

*satisfies*

$$\Vert \overrightarrow{\kappa} \Vert_{H^1_x \times L^2_x} \leq K\delta v, \tag{4.90}$$

*and $\langle \overrightarrow{\kappa}(x), J \circ D_{v,\hat{y}}(x)\rangle = 0$.*

*Proof of Lemma 4.5.3.* The proof is completely analogous to the proof of Lemma 2.1 of the article [31]. □

**Corollary 4.5.4.** *In the notation of Lemma 4.5.3, there exists a constant $C > 1$ such that if $v \in (0,1)$ is small enough, then there exists at most one number $y \geq 2\ln\frac{1}{v}$ satisfying with the function*

$$\overrightarrow{\kappa_0}(x) = (\phi_1(x), \phi_2(x)) - \overrightarrow{H_{-1,0}}((v,y),x) - \overrightarrow{H_{0,1}}((v,y),x)$$

*the estimate $\Vert \overrightarrow{\kappa_0} \Vert_{H^1_x \times L^2_x} \leq \min\{\delta_0 v, \frac{K}{3C}\delta_0 v\}$ and $\langle \overrightarrow{\kappa_0}(x), J \circ D_{v,y}(x)\rangle = 0$.*

*Proof of Corollary 4.5.4.* Let $y_1$, $y_2$ two real numbers satisfying the results of Corollary 4.5.4. We consider the following functions

$$\begin{aligned}\overrightarrow{\kappa_1}(x) =&(\kappa_{1,0}(x),\kappa_{1,1}(x)) = (\phi_1(x),\phi_2(x)) - \overrightarrow{H_{-1,0}}((v,y_1),x) - \overrightarrow{H_{0,1}}((v,y_1),x),\\ \overrightarrow{\kappa_2}(x) =&(\kappa_{2,0}(x),\kappa_{2,1}(x)) = (\phi_0(x),\phi_1(x)) - \overrightarrow{H_{-1,0}}((v,y_2),x) - \overrightarrow{H_{0,1}}((v,y_2),x).\end{aligned}$$

Choosing $x = y_1$, we obtain the following identity

$$H_{0,1}(0) - H_{0,1}\left(\frac{y_1-y_2}{\sqrt{1-v^2}}\right) = -H_{0,1}\left(\frac{-2y_1}{\sqrt{1-v^2}}\right) + H_{0,1}\left(\frac{-y_1-y_2}{\sqrt{1-v^2}}\right) + \kappa_{2,0}(y_1) - \kappa_{1,0}(y_1). \tag{4.91}$$

Since there exists a constant $c>0$ satisfying for any $f \in H^1_x(\mathbb{R})$ the inequality

$$\|f\|_{L^\infty_x(\mathbb{R})} \leq c\,\|f\|_{H^1_x},$$

we deduce from equation (4.91) and the hypotheses of Corollary 4.5.4 that

$$\left|H_{0,1}(0) - H_{0,1}\left(\frac{y_1-y_2}{\sqrt{1-v^2}}\right)\right| \leq \frac{2cK}{3C}\delta_0 v + \left|H_{0,1}\left(\frac{-2y_1}{\sqrt{1-v^2}}\right)\right| + \left|H_{0,1}\left(\frac{-y_1-y_2}{\sqrt{1-v^2}}\right)\right|,$$

from which we deduce the following estimate

$$\left|H_{0,1}(0) - H_{0,1}\left(\frac{y_1-y_2}{\sqrt{1-v^2}}\right)\right| \leq \frac{2cK}{3C}\delta_0 v + 2v^4.$$

Consequently, since $H_{0,1}$ is an increasing function and $H^{'}_{0,1}(0) = \frac{1}{2}$, we obtain that if $\delta_1 \ll 1$ and $0 < v < \delta_1$, then

$$|y_1 - y_2| \leq \frac{5Kc}{3C}\delta_0 v.$$

Therefore, choosing $C = 2c+1$, from Lemma 4.5.3, we shall have $y_1 = y_2$ if $v > 0$ is small enough. □

Finally, using Lemma 4.5.3 and repeating the argument of the demonstration of Lemma 2.11 in [31], we can verify the following result.

**Lemma 4.5.5.** *There exist $K>1$, $\delta_0>0$ and $\delta_1 \in (0,1)$ such that if $0<\delta_2<\delta_0$, $0<v<\delta_1$, $y_0 > \frac{7}{2}\ln\frac{1}{v}$ and the solution $(\phi(t,x),\partial_t\phi(t,x))$ of $(\phi^6)$ satisfies for a $T>0$*

$$\sup_{t\in[0,T]}\inf_{y\in\mathbb{R}_{\geq y_0}}\left\|(\phi(t,x),\partial_t\phi(t,x)) - \overrightarrow{H_{-1,0}}((v,y),x) - \overrightarrow{H_{0,1}}((v,y),x)\right\|_{H^1_x\times L^2_x} \leq \delta_2 v, \tag{4.92}$$

*then there exist a real function $y_1 : [0,T] \to \mathbb{R}_{\geq \frac{y_0}{2}}$ such that the solution $(\phi(t),\partial_t\phi(t))$ satisfies for any $0 \leq t \leq T$ :*

$$(\phi(t),\partial_t\phi(t)) = \overrightarrow{H_{-1,0}}((v,y_1(t)),x) + \overrightarrow{H_{0,1}}((v,y_1(t)),x) + (\psi_1(t),\psi_2(t)), \tag{4.93}$$

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} \leq K\delta_2 v, \tag{4.94}$$

*where $(\psi_1(t),\psi_2(t)) \in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ and $y_1(t)$ satisfy the orthogonality condition of Lemma 4.5.3, and $y_1(t)$ is a functions of class $C^1$ satisfying the following inequality:*

$$|\dot{y}_1(t) - v| \leq K\left[\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} + e^{-2\sqrt{2}y_1(t)}\right]. \tag{4.95}$$

*Proof.* First, from Lemma 4.5.3 and the fact that $\overrightarrow{\phi} \in C\left(\mathbb{R}; H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})\right)$, if $\delta_1$ is small enough, we can find a constant $K > 0$ and a function $\hat{y} : [0, T] \to \left(3 \ln \frac{1}{v}, +\infty\right)$ such that for

$$\overrightarrow{\kappa}(t, x) = (\phi(t, x), \partial_t \phi(t, x)) - \overrightarrow{H_{-1,0}}((v, \hat{y}(t)), x) - \overrightarrow{H_{0,1}}((v, \hat{y}(t)), x), \tag{4.96}$$

we have $\overrightarrow{\kappa}(t)$, $\hat{y}(t)$ satisfying the orthogonality condition of Lemma 4.5.3 and

$$\|\overrightarrow{\kappa}(t)\|_{H^1_x \times L^2_x} \leq K \delta_2 v, \tag{4.97}$$

for all $0 \leq t \leq T$.

Next, we are going to construct a linear ordinary differential system of equations with solution $y_1(t)$ and we are going to verify that if $y_1(0) = \hat{y}(0)$, then $y_1(t) = \hat{y}(t)$, for all $t \in [0, T]$.

**Step 1.**(Construction of the ordinary differential equation satisfied by $y_1$.)

The argument of the demonstration of the remaining part of Lemma 4.5.5 is completely analogous to the proof of Lemma 2.11 of [31]. More precisely, similarly to Lemma 2.11 of [31], we will construct an ordinary differential equation with solution $y_1(t)$, which, during their time of existence, preserves the following orthogonality conditions

$$\left\langle (\psi_1(t, x), \psi_2(t, x)), J D_{v, y_1(t)}(x) \right\rangle = 0, \tag{4.98}$$

where $J$ is defined in (4.86), and we are going to verify that if $y_1(0) = \hat{y}(0)$, then $y_1(t) = \hat{y}(t)$ for all $0 \leq t \leq T$. From the global well-posedness of the partial differential ($\phi^6$) in the energy space, we have for any $T_0 > 0$ that $\phi(t, x) - H_{0,1}(x) - H_{-1,0}(x) \in C\left([-T_0, T_0], H^1_x(\mathbb{R})\right)$ and $\partial_t \phi(t, x) \in C\left([-T_0, T_0], L^2_x(\mathbb{R})\right)$. Therefore, if there exists a interval $[0, T_1] \subset [0, T]$ such that $y_1 \in C^1([0, T_1])$ when restricted to this interval and

$$(\phi(t), \partial_t \phi(t)) = \overrightarrow{H_{-1,0}}\left((v, y_1(t)), x\right) + \overrightarrow{H_{0,1}}\left((v, y_1(t)), x\right) + (\psi_1(t), \psi_2(t)), \text{ for any } t \in [0, T_1], \tag{4.99}$$

then $(\psi_1(t), \psi_2(t)) = (\psi_1(t, x), \psi_2(t, x))$ satisfies, for any functions $h_1$, $h_2 \in \mathscr{S}(\mathbb{R})$, the following identity

$$\frac{d}{dt} \left\langle (\psi_1(t, x), \psi_2(t, x)), (h_1(x), h_2(x)) \right\rangle = \left\langle \partial_t (\psi_1(t, x), \psi_2(t, x)), (h_1(x), h_2(x)) \right\rangle,$$

if $t \in [0, T_1]$.

Consequently, if we derive the equation (4.98) in time, we obtain the following linear ordinary differential equation satisfied by $y_1(t)$

$$\dot{y}_1(t) \left\langle (\psi_1(t, x), \psi_2(t, x)), J \partial_{y_1} D_{v, y_1(t)}(x) \right\rangle + \left\langle \partial_t (\psi_1(t, x), \psi_2(t, x)), J D_{v, y_1(t)}(x) \right\rangle = 0. \tag{4.100}$$

Clearly, since $x^m H^{'}_{0,1}(x) \in \mathscr{S}(\mathbb{R})$ for all $m \in \mathbb{N} \cup \{0\}$, we have that the functions $\omega_1$, $\omega_2$ : $[0, T] \times (1, +\infty) \to \mathbb{R}$ defined by

$$\omega_1(t, y) = \left\langle (\psi_1(t, x), \psi_2(t, x)), J \partial_y D_{v,y}(x) \right\rangle, \; \omega_2(t, y) = \left\langle \partial_t (\psi_1(t, x), \psi_2(t, x)), J D_{v,y}(x) \right\rangle$$

are continuous and, for any $t \in [0,T]$, $\omega_1(t,\cdot)$, $\omega_2(t,\cdot) : (1,+\infty) \to \mathbb{R}$ are smooth.
**Step 2.**(Partial differential equation satisfied by $\overrightarrow{\psi}$.) First, we consider the following self-adjoint operator $\text{Hess}(y_1(t),x) : H^2_x(\mathbb{R}) \subset L^2_x(\mathbb{R}) \to \mathbb{R}$, which satisfies, for all $t \in [0,T]$,

$$\text{Hess}(y_1(t),x) = \begin{bmatrix} -\partial_x^2 + U^{(2)}\left(H_{0,1}\left(\frac{x-y_1(t)}{\sqrt{1-v^2}}\right) - H_{0,1}\left(\frac{-x-y_1(t)}{\sqrt{1-v^2}}\right)\right) & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.101}$$

and the self-adjoint operator $\text{Hess}_1(y_1(t),x) : H^2_x(\mathbb{R}) \subset L^2_x(\mathbb{R}) \to \mathbb{R}$ denoted by

$$\text{Hess}_1(y_1(t),x) = \begin{bmatrix} -\partial_x^2 + U^{(2)}\left(H_{0,1}\left(\frac{x-y_1(t)}{\sqrt{1-v^2}}\right)\right) & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.102}$$

Next, we consider the following maps $Int : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathcal{T} : \mathbb{R}^2 \times H^1_x(\mathbb{R}) \to \mathbb{R}^2$, which we denote by

$$\begin{aligned} Int(y,x) = {} & \begin{bmatrix} 0 \\ U'\left(-H_{0,1}\left(\frac{-x-y_1}{\sqrt{1-v^2}}\right)\right) + U'\left(H_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right)\right) \end{bmatrix} \\ & - \begin{bmatrix} 0 \\ U'\left(H_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) - H_{0,1}\left(\frac{-x-y}{\sqrt{1-v^2}}\right)\right) \end{bmatrix}, \end{aligned} \tag{4.103}$$

$$\mathcal{T}(y,x,\psi) = \begin{bmatrix} 0 \\ -\sum_{j=3}^{6} U^{(j)}\left(H_{0,1}\left(\frac{x-y}{\sqrt{1-v^2}}\right) - H_{0,1}\left(\frac{-x-y}{\sqrt{1-v^2}}\right)\right)\frac{\psi(x)^{j-1}}{(j-1)!} \end{bmatrix}, \tag{4.104}$$

for any $(y,x) \in \mathbb{R}^2$ and $\psi \in H^1_x(\mathbb{R})$. Therefore, if $[0,T_1] \subset [0,T]$, $y_1 \in C^1([0,T_1])$ and $y_1 \geq 1$, $0 < v_1 < 1$ then, from the partial differential equation ($\phi^6$) and identity (4.99), we deduce that $(\psi_1(t,x),\psi_2(t,x))$ is a solution in the space $C([0,T_1], H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R}))$ of the following partial differential equation

$$\begin{aligned} \partial_t(\psi_1(t,x),\psi_2(t,x)) = {} & (\dot{y}_1(t) - v)\left[C_{v,y_1(t)}(x) - C_{v,y_1(t)}(-x)\right] \\ & + J\,\text{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)) + Int(y_1(t),x) + \mathcal{T}(y_1(t),x,\psi_1(t)), \end{aligned} \tag{4.105}$$

where $J$ is the antissymetric operator defined in (4.86).

In the next step, we are going to assume the existence of $0 \leq T_1 \leq T$ such that $y_1$ is of class $C^1$ in the interval $[0,T_1]$, and $y_1 \geq 1$ for any $t \in [0,T_1]$. Moreover, we will prove that when this condition is true, then $|\dot{y}_1(t) - v|$ is sufficiently small for all $t \in [0,T_1]$.
**Step 3.**(Estimate of $|\dot{y}_1(t) - v|$.) Uniquely in this step, for any continuous non-negative function $f : [0,T_1]\times(0,1)\times(1,+\infty) \to \mathbb{R}$, we say that a function $g : [0,T_1]\times(0,1)\times(1,+\infty) \to \mathbb{R}$ is $O(f)$, if and only if, $g$ is a continuous function satisfying the following properties:

- there exists a constant $c > 0$ such that $|g(t,v,y)| < cf(t,v,y)$ for all $(t,v,y)$ in $[0,T_1] \times (0,1) \times (1,+\infty)$,
- $g(t,\cdot) : (0,1) \times (1,+\infty) \to \mathbb{R}$ is smooth for all $t \in [0,T_1]$.

We recall that $J$, $C_{v,y_1(t)}$ and $D_{v,y_1(t)}$ are defined, respectively, in (4.86), (4.87) and (4.88). Using Lemma 3.2.1, we obtain that if $y_1(t)\geq 1$ and $v\in(0,1)$ is small enough, then

$$\begin{aligned}\left|\left\langle C_{v,y_1(t)}(x), J\circ D_{v,y_1(t)}(-x)\right\rangle\right| + \left|\left\langle C_{v,y_1(t)}(x), JC_{v,y_1(t)}(-x)\right\rangle\right| \\ + \left|\left\langle D_{v,y_1(t)}(x), JD_{v,y_1(t)}(-x)\right\rangle\right| \lesssim y_1(t)^4 e^{-2\sqrt{2}y_1(t)}.\end{aligned} \tag{4.106}$$

Furthermore, using the partial differential equation (4.105) satisfied by $(\psi_1(t,x),\psi_2(t,x))$, we deduce for any $t\in[0,T_1]\subset[0,T]$ the following identity

$$\begin{aligned}\left\langle \partial_t(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle =& (\dot{y}_1(t)-v)\left\langle C_{v,y_1(t)}(x), JD_{v,y_1(t)}(x)\right\rangle\\ &-(\dot{y}_1(t)-v)\left\langle C_{v,y_1(t)}(-x), JD_{v,y_1(t)}(x)\right\rangle\\ &+\left\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle\\ &+\left\langle \mathcal{T}(y_1(t),x,\psi_1(t)) + Int(y_1(t),x), JD_{v,y_1(t)}(x)\right\rangle.\end{aligned} \tag{4.107}$$

Moreover, from Lemma 4.5.2 and identity $J^*=-J$, we have

$$\left\langle JD_{v,y_1(t)}(x), C_{v,y_1(t)}(x)\right\rangle = -\left\langle D_{v,y_1(t)}(x), JC_{v,y_1(t)}(x)\right\rangle = \left(1-v^2\right)^{-\frac{3}{2}}\left\|H_{0,1}^{'}\right\|_{L^2_x}^2. \tag{4.108}$$

Therefore, using equation (4.107), estimates (4.106) and Lemma 3.2.1, we deduce the following estimate

$$\begin{aligned}\left\langle \partial_t(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle =& (\dot{y}_1(t)-v)\left[\left(1-v^2\right)^{-\frac{3}{2}}\left\|H_{0,1}^{'}\right\|_{L^2_x}^2 + O\left(y_1(t)^4 e^{-2\sqrt{2}y_1(t)}\right)\right]\\ &+\left\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}\right\rangle\\ &+\left\langle \mathcal{T}(y_1(t),x,\psi_1(t)), JD_{v,y_1(t)}(x)\right\rangle\\ &+\left\langle Int(y_1(t),x), JD_{v,y_1(t)}(x)\right\rangle.\end{aligned}$$

Furthermore, since, for any $\zeta\in\mathbb{R}$, we have the following identity

$$\begin{aligned}&U^{'}\left(H_{0,1}^{\zeta}(x)+H_{-1,0}(x)\right) - U^{'}\left(H_{0,1}^{\zeta}(x)\right) - U^{'}\left(H_{-1,0}(x)\right)\\ &\quad= -24H_{-1,0}(x)H_{0,1}^{\zeta}(x)\left(H_{-1,0}(x)+H_{0,1}^{\zeta}(x)\right) + \sum_{j=1}^{4}\binom{5}{j} H_{-1,0}(x)^j H_{0,1}^{\zeta}(x)^{5-j},\end{aligned}$$

we deduce from Lemma 3.2.1 and the definition of function $Int$ that $\|Int(y_1(t),x,\psi(t))\|_{L^2_x}\lesssim e^{-2\sqrt{2}y_1(t)}$. Next, since $\left\|U^{(l)}\right\|_{L^\infty[-1,1]}<+\infty$ for any $l\in\mathbb{N}\cup\{0\}$, we deduce using Lemma 4.2.2 and the definition of function $\mathcal{T}$ that

$$\|\mathcal{T}(y_1(t),x,\psi_1(t))\|_{L^2_x} \leq \|\mathcal{T}(y_1(t),x,\psi_1(t))\|_{H^1_x} \lesssim \|\psi_1(t,x)\|_{H^1_x}^2.$$

As a consequence,

$$\begin{aligned}\left\langle \partial_t(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle =& (\dot{y}_1(t)-v)\left[\left(1-v^2\right)^{-\frac{3}{2}}\left\|H_{0,1}^{'}\right\|_{L^2_x}^2 + O\left(y_1(t)^4 e^{-2\sqrt{2}y_1(t)}\right)\right]\\ &+\left\langle J\operatorname{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v_1(t),y_1(t)}(x)\right\rangle\end{aligned} \tag{4.109}$$

$$+O\left(e^{-2\sqrt{2}y_1(t)} + \left\|\overrightarrow{\psi}(t)\right\|_{H^1_x\times L^2_x}^2\right), \tag{4.110}$$

for any $t \in [0, T_1]$.

Furthermore, using identities (4.101), (4.102), the formula of $D_{v,y}$ in (4.88) and Lemma 3.2.1, we can deduce the following estimate

$$\left\|\left[\mathrm{Hess}(y_1(t),x) - \mathrm{Hess}_1(y_1(t),x)\right] D_{v,y_1(t)}(x)\right\|_{L^2_x(\mathbb{R};\mathbb{R}^2)} \lesssim e^{-2\sqrt{2}y_1(t)},$$

for all $t \in [0, T_1]$. Thus, after using integration by parts and Cauchy-Schwarz inequality, we deduce for all $t \in [0, T_1]$ that

$$\left|\left\langle J\left[\mathrm{Hess}(y_1(t),x) - \mathrm{Hess}_1(y_1(t),x)\right] \overrightarrow{\psi}(t), JD_{v_1(t),y_1(t)}(x)\right\rangle\right| \lesssim \left\|\overrightarrow{\psi}(t)\right\|_{H^1_x \times L^2_x} e^{-2\sqrt{2}y_1(t)}.$$

Consequently, since $\langle j(a) : a\rangle = 0$ for all $a \in \mathbb{R}^2$, we obtain that if $y_1$ is a function of class $C^1$ in the interval $[0, T_1]$ and $v \in (0,1)$ is small enough, then

$$\begin{aligned}\left\langle \partial_t(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle = &(\dot{y}_1(t) - v)\left[-\frac{\left\|H'_{0,1}\right\|^2_{L^2_x}}{(1-v^2)^{\frac{3}{2}}} + O\left(y_1(t)^4 e^{-2\sqrt{2}y_1(t)}\right)\right] \\ &+ \left\langle J\,\mathrm{Hess}_1(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle \\ &+ O\left(e^{-2\sqrt{2}y_1(t)} + \left\|\overrightarrow{\psi}(t)\right\|^2_{H^1_x\times L^2_x}\right),\end{aligned} \tag{4.111}$$

for any $t \in [0, T_1]$.

Next, using (4.102), it is not difficult to verify the following identity

$$\mathrm{Hess}_1(y_1(t),x) D_{v,y_1(t)}(x) - vJ\left[\partial_x D_{v,y_1(t)}(x)\right] = JC_{v,y_1(t)}(x),$$

see Lemma 2.4 of [31] for the proof. Consequently, we have for any $t \in [0, T_1]$ that

$$\begin{aligned}\left\langle J\,\mathrm{Hess}_1(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\right\rangle =& -v\left\langle (\psi_1(t,x),\psi_2(t,x)), J\partial_{y_1} D_{v,y_1(t)}(x)\right\rangle \\ &+ \left\langle (\psi_1(t,x),\psi_2(t,x)), JC_{v,y_1(t)}(x)\right\rangle.\end{aligned}$$

In conclusion, estimate (4.111) and identity (4.100) imply that

$$\begin{aligned}(\dot{y}_1(t) - v)&\left[\frac{-\left\|H'_{0,1}\right\|^2_{L^2_x}}{(1-v^2)^{\frac{3}{2}}} + O\left(\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} + y_1(t)^4 e^{-2\sqrt{2}y_1(t)}\right)\right] \\ &= O\left(e^{-2\sqrt{2}y_1(t)} + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}\right),\end{aligned} \tag{4.112}$$

for all $t \in [0, T_1]$.

**Step 4.**(Proof that $y_1 \in C^1$.) Furthermore, the equations (4.100) and (4.107) imply that $y_1$

shall satisfy the following ordinary differential equation

$$\begin{aligned}
(\dot{y}_1(t)-v)\Big[&\Big\langle C_{v,y_1(t)}(x), JD_{v,y_1(t)}(x)\Big\rangle - \Big\langle C_{v,y_1(t)}(-x), JD_{v,y_1(t)}(x)\Big\rangle \\
&+\Big\langle (\psi_1(t),\psi_2(t)), J\partial_{y_1} D_{v,y_1(t)}(x)\Big\rangle\Big] \\
=&-v\Big\langle (\psi_1(t,x),\psi_2(t,x)), J\partial_{y_1} D_{v,y_1(t)}(x)\Big\rangle \\
&-\Big\langle J\,\mathrm{Hess}(y_1(t),x)(\psi_1(t,x),\psi_2(t,x)) + \mathcal{T}(y_1(t),x,\psi_1(t)) + Int(y_1(t),x), JD_{v,y_1(t)}(x)\Big\rangle,
\end{aligned} \tag{4.113}$$

which is a first-order non-autonomous differential system of the form

$$(\dot{y}_1(t)-v)\,\alpha_v\,(t,y_1(t)) = \beta_v\,(t,y_1(t)),$$

where the functions the functions $\alpha_v$, $\beta_v : [0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous when $v\in(0,1)$.

Moreover, from the hypotheses of Lemma 4.5.5, Lemma 3.2.1 and identities (4.101), (4.103), (4.104), we can deduce for any $t\in[0,T]$ that the restrictions of $\alpha_v(t,\cdot)$ and $\beta_v(t,\cdot)$ in the set $\left(3\ln\frac{1}{v},+\infty\right)$ are locally Lipschitz when $v$ is small enough.

Furthermore, from the first step, we have $y_1(0)=\hat{y}(0)>3\ln\frac{1}{v}$ which implies $y_1(0)^4e^{-2\sqrt{2}y_1(0)}<v^3$, if $v$ is small enough. Moreover, we deduce from (4.96) and (4.97) that $\|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x}\leq K\delta_2 v$ and we also have

$$\alpha_v(0,y_1(0)) = \frac{-\left\|H_{0,1}^{'}\right\|_{L^2_x}^2}{(1-v^2)^{\frac{3}{2}}} + O(v) > 0,$$

because of the estimate (4.112) when $v$ is small enough.

Consequently, Picard-Lindelöf Theorem implies the existence of an interval $[0,T_1]\subset[0,T]$ such that $y_1:[0,T_1]\to\mathbb{R}_{>2\ln\frac{1}{v}}$ is a $C^1$ function and since $y_1$ satisfies (4.100), we have for any $t\in[0,T_1]$ that

$$\Big\langle (\psi_1(t,x),\psi_2(t,x)), JD_{v,y_1(t)}(x)\Big\rangle = \Big\langle \overrightarrow{\psi}(0,x), JD_{v,y_1(0)}(x)\Big\rangle = 0. \tag{4.114}$$

Furthermore, since $\hat{y}(t)\geq 3\ln\frac{1}{v}$, we can deduce from the continuity of function $y_1$, Lemma 4.5.3 and Corollary 4.5.4 the identity $y_1(t)=\hat{y}(t)$ for all $t\in[0,T_1]$. As a consequence, $y_1(t)\geq 3\ln\frac{1}{v}$ for all $t\in[0,T_1]$ and

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} = \left\|\overrightarrow{\phi}(t,x) - \overrightarrow{H_{-1,0}}((v,y_1(t)),x) - \overrightarrow{H_{0,1}}((v,y_1(t)),x)\right\|_{H^1_x\times L^2_x} \leq K\delta_2 v \tag{4.115}$$

for all $t\in[0,T_1]$, because of estimate (4.96) and identity (4.97).

Therefore, using a bootstrap argument and estimate (4.112), we can conclude that the function $y_1$ is in $C^1[0,T]$ and satisfies (4.114) for all $t\in[0,T]$. Finally, estimate (4.95) is a direct consequence of (4.112), (4.115) and the fact that $y_1\geq 3\ln\frac{1}{v}$. □

### 4.5.2 Orbital stability of the parameter $y$

In this subsection, we consider $\phi(t,x)$ as a solution of $(\phi^6)$ having finite energy and with an initial data $(u_1(x),u_2(x))$ satisfying the hypotheses of Theorem 4.1.3. Moreover, if $v$ is small enough, from the local well-posedness of the partial differential equation $(\phi^6)$ in the space of solutions with finite energy, we can deduce from Lemma 4.5.3 the existence of a constant $C>0$ and a positive number $\epsilon$ such that for all $t\in[0,\epsilon]$

$$(\phi(t,x),\partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v,y(t)),x)+\overrightarrow{H_{0,1}}((v,y(t)),x)+(\psi_1(t,x),\psi_2(t,x)),$$

where $(\psi_1(t,x),\psi_2(t,x))$ is an odd function in $x$, and $y(t)$, $(\psi_1(t,x),\psi_2(t,x))$ satisfy the orthogonality conditions in Lemma 4.5.3 and the following inequality

$$|y(t)-y_0|+\|(\psi_1(t,x),\psi_2(t,x))\|_{H^1_x\times L^2_x}\leq 2C\,\|(u_1,u_2)\|_{H^1_x\times L^2_x}\,. \tag{4.116}$$

Finally, we are ready to start the proof of Theorem 4.1.3

**Remark 4.5.6** (Main argument)**.** *The main techniques of the demonstration of Theorem 4.1.3 are inspired by the proof of Theorem* 1 *of [31].*

*More precisely, recalling the functions $E_+$ and $P_+$ from* (4.84) *and* (4.83)*, we will analyze the function*

$$M(\phi(t)) = E_+(\phi(t)) - vP_+(\phi(t)). \tag{4.117}$$

*First, from the local well-posedness of the partial differential equation $(\phi^6)$ in the energy space, it is enough to verify Theorem 4.1.3 to the case where $(u_1(x),u_2(x))$ is a smooth odd function because the estimate* (4.14) *and the density of smooth functions in Sobolev spaces would imply that* (4.14) *would be true for any $(u_1(x),u_2(x))\in H^1_x\times L^2_x$ satisfying the hypothesis of Theorem 4.1.3.*

*Since $P_+(t)$ is not necessarily a conserved quantity, $M(t)$ is not necessarily a constant function given any smooth initial initial data of $(\phi(0,x),\partial_t\phi(0,x))$ satisfying the hypotheses of Theorem 4.14.*

*However, $P_+(t)$ is a non-increasing function in time, more precisely, for smooth solutions $\phi(t,x)$ of* (4.12)*, we can verify using integration by parts, from the fact that $\phi(t,x)$ is an odd function in $x$ for any $t\in\mathbb{R}$, the estimate*

$$\frac{d}{dt}\left[-\int_0^{+\infty}\partial_t\phi(t,x)\partial_x\phi(t,x)\,dx\right]=\frac{1}{2}\phi(t,0)^2\geq 0. \tag{4.118}$$

*In conclusion, since it was verified before that $E_+(t)$ is a conserved quantity, we have that*

$$M(\phi(t))\leq M(\phi(0)) \text{ for any } t\geq 0,$$

*and using Lemma 4.5.3, we are going to verify that $M(0)-M(t)$ satisfies a coercive inequality, from which we will deduce* (4.14)*.*

*Proof of Theorem 4.1.3.* From the observations in Remark 4.5.6, it is enough to prove Theorem 4.1.3 for the case where $\overrightarrow{\psi}_0(x)$ is a smooth odd function. To simplify our proof, we separate the argument into different steps.

**Step 1.**(Local description of solution $\phi(t,x)$.)From the observation of inequality (4.116) and from the Lemma 4.5.3, we can verify the existence of an interval $[0,\epsilon]$ such that if $t\in[0,\epsilon]$, then

$$(\phi(t,x),\partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v,y(t)),x)+\overrightarrow{H_{0,1}}((v,y(t)),x)+(\psi_1(t,x),\psi_2(t,x)), \qquad (4.119)$$

with $v(t)$, $y(t)$, $(\psi_1(t,x),\psi_2(t,x))$ satisfying all the conditions of Lemma 4.5.3.

**Step 2.**(Estimate of $E_+(\phi(t),\partial_t\phi(t))$ around the kinks.) We recall the definition of $E_+(\phi(t),\partial_t\phi(t))$ in (4.84) given by

$$E_+(\phi(t),\partial_t\phi(t)) = \int_0^{+\infty}\frac{\partial_x\phi(t,x)^2+\partial_t\phi(t,x)^2}{2}+U(\phi(t,x))\,dx.$$

Next, we substitute $\phi(t,x)$ and $\partial_t\phi(t,x)$ in the equation above by the formula of $(\phi(t,x),\partial_t\phi(t,x))$ in Step 1. Using (4.3), (4.2) and since $y(t)>1$ for $0\leq t\leq\epsilon$, we obtain for all $x\geq 0$ that

$$\left|\frac{\partial^l}{\partial x^l}H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right|\lesssim_l (1-v^2)^{-\frac{l}{2}}e^{-\sqrt{2}(y(t)+x)} \text{ for any } l\in\mathbb{N}\cup\{0\}, \qquad (4.120)$$

from which we also deduce, using Lemma 3.2.1, the following estimate

$$\int_{\mathbb{R}}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)H_{-1,0}^{'}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\lesssim\left(1-v^2\right)^{\frac{1}{2}}y(t)e^{-2\sqrt{2}y(t)}. \qquad (4.121)$$

In addition, since $\left\|U^{(l)}\right\|_{L^\infty[-1,1]}<+\infty$ for any $l\in\mathbb{N}$, we can deduce using Lemma 4.2.2 the following inequality

$$\left\|U^{(l)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^l\right\|_{H^1_x}\lesssim_l\|\psi_1(t,x)\|_{H^1_x}^l.$$

In conclusion, since

$$\phi(t,x)=H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi_1(t,x), \qquad (4.122)$$

$$\partial_t\phi(t,x)=-\frac{v}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)+\frac{v}{\sqrt{1-v^2}}H_{-1,0}^{'}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)+\psi_2(t,x), \qquad (4.123)$$

we deduce from the formula (4.84), estimates (4.120), (4.121) and Taylor's Expansion Theorem that

$$E_{+}\left(\phi(t),\partial_{t}\phi(t)\right)=\int_{0}^{+\infty}\frac{1+v^{2}}{2(1-v^{2})}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)^{2}+U\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\,dx$$
$$-\frac{1}{\sqrt{1-v^{2}}}\int_{0}^{+\infty}vH_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\psi_{2}(t,x)\,dx-H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\partial_{x}\psi_{1}(t,x)$$
$$+\int_{0}^{+\infty}U^{'}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\psi_{1}(t,x)\,dx$$
$$+\frac{1}{2}\left[\int_{0}^{+\infty}\partial_{x}\psi_{1}(t,x)^{2}+U^{(2)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\psi_{1}(t,x)^{2}+\psi_{2}(t,x)^{2}\right]dx$$
$$+O\left(\left(1-v^{2}\right)^{-\frac{1}{2}}y(t)e^{-2\sqrt{2}y(t)}\right)$$
$$+O\left(\left\Vert\overrightarrow{\psi}(t)\right\Vert_{H^{1}_{x}\times L^{2}_{x}}e^{-\sqrt{2}y(t)}+\left\Vert\psi_{1}(t,x)\right\Vert^{3}_{H^{1}_{x}(\mathbb{R})}\right),\tag{4.124}$$

while $(\phi(t,x),\partial_{t}\phi(t,x))$ satisfies identities (4.122) and (4.123). Moreover, from (4.122), we can obtain from (4.124), while $(\phi(t),\partial_{t}\phi(t))$ satisfies (4.122) and (4.123), that

$$E_{+}\left(\phi(t),\partial_{t}\phi(t)\right)=\int_{-\infty}^{+\infty}\frac{1+v^{2}}{2(1-v^{2})}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)^{2}+U\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\,dx$$
$$\frac{-1}{\sqrt{1-v^{2}}}\int_{-\infty}^{+\infty}vH_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\psi_{2}(t,x)-H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\partial_{x}\psi_{1}(t,x)$$
$$+\int_{-\infty}^{+\infty}U^{'}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\psi_{1}(t.x)\,dx$$
$$+\frac{1}{2}\left[\int_{0}^{+\infty}\partial_{x}\psi_{1}(t,x)^{2}+U^{(2)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\psi_{1}(t,x)^{2}+\psi_{2}(t,x)^{2}\,dx\right]$$
$$+O\left(\left(1-v^{2}\right)^{-\frac{1}{2}}y(t)e^{-2\sqrt{2}y(t)}\right)$$
$$+O\left(\left\Vert\overrightarrow{\psi}(t)\right\Vert_{H^{1}_{x}\times L^{2}_{x}}e^{-\sqrt{2}y(t)}+\left\Vert\psi_{1}(t,x)\right\Vert^{3}_{H^{1}_{x}(\mathbb{R})}\right),\tag{4.125}$$

We also recall the Bogomolny identity $H_{0,1}^{'}(x)=\sqrt{2U(H_{0,1}(x))}$, from which we deduce with change of variables that

$$\frac{1}{2}\int_{\mathbb{R}}H_{0,1}^{'}\left(\frac{x}{\sqrt{1-v^{2}}}\right)^{2}\,dx=\int_{\mathbb{R}}U\left(H_{0,1}\left(\frac{x}{\sqrt{1-v^{2}}}\right)\right)\,dx=\sqrt{1-v^{2}}\frac{\left\Vert H_{0,1}^{'}\right\Vert^{2}_{L^{2}_{x}}}{2}.\tag{4.126}$$

**Step 3.**$\Big($Conclusion of the estimate of $E_{+}(t)$. $\Big)$
Since $\overrightarrow{H_{0,1}}((v,y(t)),x))$ is defined by

$$\overrightarrow{H_{0,1}}((v,y(t)),x)=\begin{bmatrix}H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v(t)^{2}}}\right)\\-\frac{v}{\sqrt{1-v^{2}}}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\end{bmatrix},$$

and we can verify by similar reasoning to (4.124) the identity

$$E\left(\overrightarrow{H_{0,1}}((v,y(t)),x)\right)=\int_{-\infty}^{+\infty}\frac{1+v^{2}}{2(1-v^{2})}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)^{2}+U\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^{2}}}\right)\right)\,dx,$$

we conclude that $E\left(\overrightarrow{H_{0,1}}((v,y(t)),x)\right) = \frac{1}{\sqrt{1-v^2}}\left\Vert H_{0,1}^{'}\right\Vert_{L_x^2}^2$. In conclusion, using (4.125), we obtain that

$$\begin{aligned}E_+(\phi(t),\partial_t\phi(t)) =& \frac{1}{\sqrt{1-v^2}}\left\Vert H_{0,1}^{'}\right\Vert_{L_x^2}^2 - \int_{-\infty}^{+\infty}\frac{v}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\psi_2(t,x)\,dx\\ &+\int_{-\infty}^{+\infty}\frac{1}{\sqrt{1-v^2}}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\partial_x\psi_1(t,x)\\ &+\int_{-\infty}^{+\infty}U^{'}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)\,dx\\ &+\frac{1}{2}\left[\int_0^{+\infty}\partial_x\psi_1(t,x)^2+U^{(2)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2+\psi_2(t,x)^2\right]\\ &+O\left(\left(1-v^2\right)^{-\frac{1}{2}}y(t)e^{-2\sqrt{2}y(t)}+\Vert(\psi_1(t),\psi_2(t))\Vert_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)}\right)\\ &+O\left(\Vert\psi_1(t)\Vert_{H_x^1(\mathbb{R})}^3\right),\end{aligned}$$

from this using integration by parts we conclude that

$$\begin{aligned}E_+(\phi(t),\partial_t\phi(t)) =& \frac{1}{\sqrt{1-v^2}}\left\Vert H_{0,1}^{'}\right\Vert_{L_x^2}^2 + v\left\langle J\circ C_{v,y(t)},\overrightarrow{\psi(t)}\right\rangle\\ &+\frac{1}{2}\left[\int_0^{+\infty}\psi_2(t,x)^2+\partial_x\psi_1(t,x)^2+U^{(2)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2\right]\\ &+O\left(\left(1-v^2\right)^{-\frac{1}{2}}y(t)e^{-2\sqrt{2}y(t)}\right)\\ &+O\left(\Vert(\psi_1(t),\psi_2(t))\Vert_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)}+\Vert\psi_1(t)\Vert_{H_x^1}^3\right),\end{aligned} \tag{4.127}$$

where the function $C_{v,y}(x)$ is defined in (4.87).

**Step 4.**(Estimate of $-vP_+(\phi(t),\partial_t\phi(t))$.)

First, we recall from (4.83) that $P_+(\phi(t),\partial_t\phi(t))$ is given by

$$P_+(\phi(t),\partial_t\phi(t)) = -\int_0^{+\infty}\partial_t\phi(t,x)\partial_x\phi(t,x)\,dx.$$

Then, while $(\phi(t,x),\partial_t\phi(t,x))$ satisfies the formula

$$(\phi(t,x),\partial_t\phi(t,x)) = \overrightarrow{H_{-1,0}}((v,y(t)),x)+\overrightarrow{H_{0,1}}((v,y(t)),x)+(\psi_1(t,x),\psi_2(t,x)),$$

using the estimates (4.120) and (4.121), we obtain by similar reasoning to the estimate of (2.12) of Lemma 2.3 in [31] that

$$\begin{aligned}-vP_+(\phi(t),\partial_t\phi(t)) =& -\frac{v^2}{\sqrt{1-v^2}}\left\Vert H_{0,1}^{'}\right\Vert_{L_x^2}^2 - v\left\langle J\circ C_{v,y(t)},\overrightarrow{\psi}(t)\right\rangle\\ &+v\int_0^{+\infty}\partial_x\psi_1(t,x)\psi_2(t,x)\,dx+O\left(\frac{v^2}{(1-v^2)}y(t)e^{-2\sqrt{2}y(t)}\right)\\ &+O\left(\frac{v}{\sqrt{1-v^2}}e^{-\sqrt{2}y(t)}\Vert(\psi_1(t),\psi_2(t))\Vert_{H_x^1\times L_x^2}\right),\end{aligned} \tag{4.128}$$

more precisely the errors in the estimate (4.128) above come from estimate (4.120) and Cauchy-Schwarz inequality applied in

$$\int_0^{+\infty}\left|H_{-1,0}^{'}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)\right|\left[\left|\partial_x\psi_1(t,x)\right|+\left|\psi_2(t,x)\right|\right]dx,$$

from Lemma 3.2.1 applied in the following integral

$$\int_0^{+\infty}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)H_{-1,0}^{'}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)dx,$$

and from the elementary estimate

$$\int_{-\infty}^{0}H_{0,1}^{'}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)^2dx+\int_0^{+\infty}H_{-1,0}^{'}\left(\frac{x+y(t)}{\sqrt{1-v^2}}\right)^2dx\lesssim e^{-2\sqrt{2}y(t)},$$

which can be obtained from (4.120).

**Step 5.**(Estimate and monotonicity of $M(\phi(t),\partial_t\phi(t))$.) From estimates (4.127) and (4.128), we deduce

$$\begin{aligned}M\left(\phi(t),\partial_t\phi(t)\right)=&E_{+}\left(\phi(t),\partial_t\phi(t)\right)-vP_{+}\left(\phi(t),\partial_t\phi(t)\right)\\=&\sqrt{1-v^2}\left\|\dot{H}_{0,1}\right\|_{L_x^2}^2\\&+\frac{1}{2}\left[\int_0^{+\infty}\psi_2(t,x)^2+\partial_x\psi_1(t,x)^2+U^{(2)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v(t)^2}}\right)\right)\psi_1(t,x)^2\,dx\right]\\&+O\left(v\left\|(\psi_1(t),\psi_2(t))\right\|_{H_x^1\times L_x^2}^2+\left\|(\psi_1(t),\psi_2(t))\right\|_{H_x^1\times L_x^2}e^{-\sqrt{2}y(t)}\right)\\&+O\left(\left\|\psi_1(t)\right\|_{H_x^1}^3++y(t)e^{-2\sqrt{2}y(t)}\right).\end{aligned}\tag{4.129}$$

Furthermore, using estimate (4.2) and Lemma 3.2.1, we can also verify the following estimates

$$\begin{aligned}E_{+}\left(\overrightarrow{H_{0,1}}(v,y(t))+\overrightarrow{H_{-1,0}}(v,y(t))\right)=&\frac{1}{\sqrt{1-v^2}}\left\|H_{0,1}^{'}\right\|_{L_x^2}^2+O\left(y(t)e^{-2\sqrt{2}y(t)}\right),\\P_{+}\left(\overrightarrow{H_{0,1}}(v,y(t))+\overrightarrow{H_{-1,0}}(v,y(t))\right)=&\frac{v}{\sqrt{1-v^2}}\left\|H_{0,1}^{'}\right\|_{L_x^2}^2+O\left(y(t)e^{-2\sqrt{2}y(t)}\right).\end{aligned}$$

Therefore, we obtain that

$$M\left(\overrightarrow{H_{0,1}}(v,y(t))+\overrightarrow{H_{-1,0}}(v,y(t))\right)=\sqrt{1-v^2}\left\|H_{0,1}^{'}\right\|_{L_x^2}^2+O\left(y(t)e^{-2\sqrt{2}y(t)}\right),\tag{4.130}$$

from which we deduce

$$\begin{aligned}M\left(\phi(t),\partial_t\phi(t)\right)=&M\left(\overrightarrow{H_{0,1}}(v,y(0))+\overrightarrow{H_{-1,0}}(v,y(0))\right)\\&+\frac{1}{2}\left[\int_0^{+\infty}\psi_2(t,x)^2+\partial_x\psi_1(t,x)^2+U^{(2)}\left(H_{0,1}\left(\frac{x-y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2\,dx\right]\\&+O\left(\max\left\{y(t)e^{-2\sqrt{2}y(t)},y(0)e^{-2\sqrt{2}y(0)}\right\}\right)\\&+O\left(v\left\|(\psi_1(t),\psi_2(t))\right\|_{H_x^1\times L_x^2}^2+\left\|(\psi_1(t),\psi_2(t))\right\|_{H_x^1\times L_x^2}^3\right).\end{aligned}$$

Consequently, since $M(\phi(0), \partial_t\phi(0)) \geq M(\phi(t), \partial_t\phi(t))$ for all $t \geq 0$ and

$$(\phi(0), \partial_t\phi(0)) = \overrightarrow{H_{0,1}}(v, y(0)) + \overrightarrow{H_{-1,0}}(v, y(0)) + (\psi_1(0), \psi_2(0)),$$

we have for every $t \geq 0$ the following estimate

$$\int_0^{+\infty} \psi_2(t,x)^2 + \partial_x\psi_1(t,x)^2 + U^{(2)}\left(H_{0,1}\left(\frac{x - y(t)}{\sqrt{1-v^2}}\right)\right)\psi_1(t,x)^2\,dx$$
$$\lesssim y(t)e^{-2\sqrt{2}y(t)} + y(0)e^{-2\sqrt{2}y(0)} + v\,\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^2 + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^3$$
$$+ \|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x},$$

from which with Lemma A.4.5 we deduce for all $t \geq 0$ that

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x}^2 \lesssim y(t)e^{-2\sqrt{2}y(t)} + y(0)e^{-2\sqrt{2}y(0)} + \|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x}, \tag{4.131}$$

if $v \ll 1$.

**Step 6.**(Final Argument.)

The last argument is to prove that the set denoted by

$$BO = \left\{t \in \mathbb{R}_{\geq 0} \middle|\ \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} \leq v^{1+\frac{\theta}{4}},\ y(t) \geq y(0) \text{ and (4.119) is true.}\right\}, \tag{4.132}$$

is the proper $\mathbb{R}_{\geq 0}$. From the hypotheses of Theorem 4.1.3 and Step 1, we can verify that $0 \in BO$.

Furthermore, from Step 1, we have obtained that there exists $\epsilon > 0$ such that if $0 \leq t \leq \epsilon$, then

$$(\phi(t,x), \partial_t\phi(t,x)) = \overrightarrow{H}_{-1,0}((v, y(t)), x) + \overrightarrow{H}_{0,1}((v, y(t)), x) + (\psi_1(t,x), \psi_2(t,x))$$

and

$$|y(t) - y_0| + \|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} \leq 2C\,\|(u_1,u_2)\|_{H^1_x\times L^2_x}. \tag{4.133}$$

Since $\|(u_1,u_2)\|_{H^1_x\times L^2_x} \leq v^{2+\theta}$ and Lemma 4.5.3 implies the estimate $\|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x} \lesssim \|(u_1,u_2)\|_{H^1_x\times L^2_x}$, from (4.133) and Lemma 4.5.5, we deduce the existence of a constant $0 < K$ independent of $\epsilon$ and $v$ such that $y(t)$ is a function of class $C^1$ in $[0,\epsilon]$ and for any $t \in [0,\epsilon]$, the inequality

$$|\dot{y}(t) - v| \leq K\left[\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} + e^{-2\sqrt{2}y(t)}\right] \tag{4.134}$$

is true. Therefore,

$$\dot{y}(t) \geq v - K\left[\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x} + e^{-2\sqrt{2}y(t)}\right], \tag{4.135}$$

while $t \in [0,\epsilon]$. Moreover, from inequality (4.133) and the observations done before, to prove that $[0,\epsilon] \subset BO$ it is only needed to verify that $y(t) \geq y(0)$ for all $t \in [0,\epsilon]$.

First, since $y(t)$ is continuous for $t \in [0,\epsilon]$, there exists $\epsilon_2 \in (0,\epsilon)$ such that if $0 \leq t \leq \epsilon_2$, then

$$y(t) \geq \frac{3y(0)}{4},$$

so (4.133), (4.135) and the estimate $\|(\psi_1(0),\psi_2(0))\|_{H^1_x\times L^2_x} \lesssim \|(u_1,u_2)\|_{H^1_x\times L^2_x} \leq v^{2+\theta}$ imply that if $0\leq t\leq \epsilon_2$ and $0<v\ll 1$, then

$$\dot{y}(t) \geq v - v^2 - Ke^{-\frac{3\sqrt{2}y(0)}{2}} \geq \frac{4v}{5}. \tag{4.136}$$

In conclusion, estimate (4.133), the hypothesis of $y_0 \geq 4\ln\frac{1}{v}$ and inequality (4.136) imply for $v\ll 1$ that if $0\leq t\leq \epsilon_2$, then $y(t)\geq y(0)+\frac{4v}{5}t$ and $[0,\epsilon_2]\subset BO$.

If $t\in[\epsilon_2,\epsilon]$, it is not difficult to verify that $y(t)\geq y(0)$ in this region. Indeed, the continuity of the function $y$ would imply otherwise the existence of $t_i$ satisfying $\epsilon_2<t_i\leq\epsilon$, $y(t_i)=y(0)$ and $y(s)>y(0)$ for any $\epsilon_2\leq s<t_i$, which implies that estimate (4.136) is true for $t\in[\epsilon_2,t_1]$. But, repeating the argument above, we would conclude that $y(t_i)\geq y(0)+\frac{4v}{5}t_i$, which is a contradiction. In conclusion, the interval $[0,\epsilon]$ is contained in the set $BO$.

Similarly, from Lemma 4.5.5, we can use inequality (4.135) to verify that $y(t)\geq y(0)+\frac{4v}{5}t$ always when $[0,t]\subset BO$. Therefore, estimate (4.131) implies

$$\|(\psi_1(t),\psi_2(t))\|_{H^1_x\times L^2_x(x)} \lesssim \|(u_1,u_2)\|^{\frac{1}{2}}_{H^1_x\times L^2_x} + y(0)^{\frac{1}{2}}e^{-\sqrt{2}y(0)} \ll v^{1+\frac{\theta}{4}}, \tag{4.137}$$

if $[0,t]\in BO$.

In conclusion, $BO=\mathbb{R}_{\geq 0}$ and estimates (4.134), (4.137) imply the result of Theorem 4.1.3 for all $t\geq 0$. □

## 4.6 Proof of Theorem 4.1.2

First, from Theorem 1.3 in the article [8], we know for any $0<v<1$ that there exist $\delta(v)>0$, $T(v)>0$ and a solution $\phi(t,x)$ of $(\phi^6)$ with finite energy satisfying the identity

$$\phi(t,x) = H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{-x-vt}{\sqrt{1-v^2}}\right) + \psi(t,x), \tag{4.138}$$

and the following decay estimate

$$\sup_{t\geq T}\|(\psi(t,x),\partial_t\psi(t,x))\|_{H^1_x\times L^2_x}\, e^{\delta t} < +\infty, \tag{4.139}$$

for any $T\geq T(v)$ and $\delta\leq\delta(v)$. Moreover, we can find $\delta(v)$, $T(v)>0$ such that

$$\sup_{t\geq T(v)}\|(\psi(t,x),\partial_t\psi(t,x))\|_{H^1_x\times L^2_x}\, e^{\delta(v)t} < 1, \tag{4.140}$$

indeed, in [8] it was proved using the Fixed point theorem that for any $0<v<1$ that there is a unique solution of $(\phi^6)$ that satisfies (4.139) for some $T$, $\delta>0$.

Next, if we restrict the argument of the proof of Proposition 3.6 of [8] to the traveling kink-kink of the $\phi^6$ model, we can find explicitly the values of $\delta(v)$ and $T(v)$. More precisely, we have:

**Theorem 4.6.1.** *There is $\delta_0 > 0$ such that if $0 < v < \delta_0$, then there exists a unique solution $\phi(t,x)$ of $(\phi^6)$ with*

$$h(t,x) = \phi(t,x) - H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right),$$

*satisfying* (4.139) *for some $0 < \delta < 1$ and $T > 0$. Furthermore, we have if $t \geq \frac{4\ln\frac{1}{v}}{v}$ that*

$$\|(h(t,x), \partial_t h(t,x))\|_{H^1_x \times L^2_x} \leq e^{-vt}. \tag{4.141}$$

*This solution is also an odd function on $x$.*

*Proof.* See Appendix Section $A.4$

□

Finally, we have obtained all the framework necessary to start the demonstration of Theorem 4.1.2.

*Proof of Theorem 4.1.2.* First, from Theorem 4.6.1, for any $k \in \mathbb{N}$ bigger than 2 and $0 < v \leq \delta_0$, we have that the traveling kink-kink with speed $v$ satisfies for $T_{0,k} = \frac{32k\ln\left(\frac{1}{v^2}\right)}{2\sqrt{2}v}$ the following estimate:

$$\|(h(T_{0,k}), \partial_t h(T_{0,k}))\|_{H^1_x \times L^2_x} \leq v^{16\sqrt{2}k}, \tag{4.142}$$

for $h(t,x)$ the function denoted in Theorem 4.6.1. Now, we start the proof of the second item of Theorem 4.1.2.

**Step 1.**(Proof of the second inequality of Theorem 4.1.2.)

First, in notation of Theorem 4.1.7, we consider

$$\phi_k(v,t,x) = \varphi_{k,v}(t, x+\tau_{k,v}).$$

For the $T_{0,k}$ given before, we can verify using Theorems 4.1.6, 4.1.7 that

$$\left\|\phi_k(v,T_{0,k},x) - H_{0,1}\left(\frac{x-vT_{0,k}}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x+vT_{0,k}}{\sqrt{1-v^2}}\right)\right\|_{H^1_x}$$
$$+\left\|\partial_t\phi_k(v,T_{0,k},x) + \frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x-vT_{0,k}}{\sqrt{1-v^2}}\right) - \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x+vT_{0,k}}{\sqrt{1-v^2}}\right)\right\|_{H^1_x} \leq v^{15k}.$$

In conclusion, Theorem 4.2.3 and Remark 4.4.1 imply that there is $\Delta_{k,\theta} > 0$ such that if also $v < \Delta_{k,\theta}$, then

$$\|(\phi(t,x), \partial_t\phi(t,x)) - (\phi_k(v,t,x), \partial_t\phi_k(v,t,x))\|_{H^1_x \times L^2_x} < v^{2k-\frac{1}{2}},$$

while

$$|t - T_{0,k}| < \frac{\left(\ln\frac{1}{v}\right)^{2-\frac{\theta}{2}}}{v}.$$

Also, Theorem 4.1.7 and Theorem 4.1.6 implies that if $v \ll 1$ and

$$-4\frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v} \leq t \leq -\frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v},$$

then there exist $e_{k,v}$ satisfying $\left|e_{v,k} - \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\right)\right| \ll 1$ such that

$$\left\|\phi_k(v,t,x) - H_{0,1}\left(\frac{x - e_{k,v} + vt}{\sqrt{1-v^2}}\right) - H_{-1,0}\left(\frac{x + e_{k,v} - vt}{\sqrt{1-v^2}}\right)\right\|_{H^1_x}$$
$$+\left\|\partial_t\phi_k(v,t,x) - \frac{v}{\sqrt{1-v^2}}H^{'}_{0,1}\left(\frac{x - e_{k,v} + vt}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}H^{'}_{-1,0}\left(\frac{x + e_{k,v} - vt}{\sqrt{1-v^2}}\right)\right\|_{L^2_x} \ll v^{2k-\frac{1}{2}}. \tag{4.143}$$

In conclusion, the second inequality of Theorem 4.1.2 follows from the observation above and Remark 4.4.1.

**Step 2.**(Proof of the first inequality of Theorem 4.1.2.)

From Step 1, for $t_0 = -\frac{\left(\ln\frac{1}{v}\right)^{2-\theta}}{v}$, it was obtained that $\phi(t_0,x)$ satisfies (4.143). Next, we are going to study the behavior of $\phi(t,x)$ for $t \leq t_0$, which is equivalent to studying the function $\phi_1(t,x) = \phi(-(t+t_0),x)$ for $t \geq 0$.

However, from the estimate (4.143), we can verify that $(\phi_1(0,x), \partial_t\phi_1(0,x))$ satisfies the hypotheses of Theorem 4.1.3, if we consider $y_0 = e_{k,v} - vt_0$ and $0 < v <\ll 1$. Therefore, using the result of Theorem 4.1.3 and the identity $\phi_1(t,x) = \phi(-(t+t_0),x)$, we obtain the first inequality of Theorem 4.1.2. $\square$

# Appendix A

## A.1 Auxiliary Results

We start the Appendix Section by presenting the following lemma:

**Lemma A.1.1.** *Let* $\left\|\overrightarrow{g(t)}\right\| = \|(g(t), \partial_t g(t))\|_{H^1_x \times L^2_x}$. *Assuming the same hypothesis as in Theorem 2.1.10 and using its notation, we have while* $\max_{j\in\{1,2\}} |d_j(t) - x_j(t)| < 1$ *that*

$$\max_{j\in\{1,2\}} \left|\ddot{d}_j(t) - \ddot{x}_j(t)\right| = O\Big(\max_{j\in\{1,2\}} |d_j(t) - x_j(t)|\,\epsilon + \epsilon z(t) e^{-\sqrt{2}z(t)} + \left\|\overrightarrow{g(t)}\right\| \epsilon^{\frac{1}{2}}\Big).$$

**Lemma A.1.2.** *For* $U(\phi) = \phi^2(1-\phi^2)^2$, *we have that*

$$\begin{aligned} U'\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right) - U'\left(H^{x_1(t)}_{-1,0}(x)\right) - U'\left(H^{x_2(t)}_{0,1}(x)\right) \\ = & 24 e^{-\sqrt{2}z(t)} \left(\frac{H^{x_1(t)}_{-1,0}(x)}{\sqrt{1+e^{-2\sqrt{2}(x-x_1(t))}}} + \frac{H^{x_2(t)}_{0,1}(x)}{\sqrt{1+e^{2\sqrt{2}(x-x_2(t))}}}\right) \\ & -30 e^{-\sqrt{2}z(t)} \left(\frac{H^{x_1(t)}_{-1,0}(x)^3}{\sqrt{1+e^{-2\sqrt{2}(x-x_1(t))}}} + \frac{H^{x_2(t)}_{0,1}(x)^3}{\sqrt{1+e^{2\sqrt{2}(x-x_2(t))}}}\right) + r(t,x), \end{aligned}$$

*such that* $\|r(t)\|_{L^2_x(\mathbb{R})} = O(e^{-2\sqrt{2}z(t)})$.

*Proof.* By direct computations, we verify that

$$\begin{aligned} U'\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) - U'\left(H^{x_1(t)}_{-1,0}\right) - U'\left(H^{x_2(t)}_{0,1}\right) = & -24 H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) \\ & +30 H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\left[\left(H^{x_1(t)}_{-1,0}\right)^3 + \left(H^{x_2(t)}_{0,1}\right)^3\right] \\ & +60\left(H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\right)^2 \left[H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right]. \end{aligned}$$

First, from the definition of $H_{0,1}(x)$, we verify that

$$\begin{aligned} 60\left(H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\right)^2 \left[H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right] = & \frac{60 e^{-2\sqrt{2}z(t)} H^{x_2(t)}_{0,1}}{(1+e^{2\sqrt{2}(x-x_2(t))})(1+e^{-2\sqrt{2}(x-x_1(t))})} \\ & + \frac{60 e^{-2\sqrt{2}z(t)} H^{x_1(t)}_{-1,0}}{(1+e^{-2\sqrt{2}(x-x_1(t))})(1+e^{2\sqrt{2}(x-x_2(t))})}. \end{aligned}$$

Using (2.4), we can verify using by induction for any $k \in \mathbb{N}$ that

$$\left|\frac{d^k}{dx^k}\left[\frac{1}{(1+e^{2\sqrt{2}x})}\right]\right| = \left|\frac{d^k}{dx^k}\left[1 - \frac{e^{2\sqrt{2}x}}{(1+e^{2\sqrt{2}x})}\right]\right| = \left|\frac{d^k}{dx^k}\left[H_{0,1}(x)^2\right]\right| = O(1), \tag{A.1}$$

and since $\frac{H_{0,1}(x)}{(1+e^{2\sqrt{2}x})} = \frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}$ is a Schwartz function, we deduce using Lemma 2.2.3 that $60(H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1})^2(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1})$ is in $H^k_x(\mathbb{R})$ and it satisfies for all $k>0$ the following estimate

$$\left\Vert\frac{\partial^k}{\partial x^k}\left[(H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1})^2(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1})\right]\right\Vert_{L^2_x} = O\left(e^{-2\sqrt{2}z(t)}\right). \tag{A.2}$$

Next, using the identity

$$H^{x_1(t)}_{-1,0}(x)H^{x_2(t)}_{0,1}(x) = -\frac{e^{-\sqrt{2}z(t)}}{\sqrt{(1+e^{2\sqrt{2}(x-x_2(t))})(1+e^{-2\sqrt{2}(x-x_1(t))})}}, \tag{A.3}$$

the identity

$$1-\frac{1}{\sqrt{1+e^{2\sqrt{2}x}}} = \frac{e^{2\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}+1+e^{2\sqrt{2}x}},$$

and Lemma 2.2.3, we deduce that

$$\left\Vert 24(H^{x_1(t)}_{-1,0})^2H^{x_2(t)}_{0,1}+24e^{-\sqrt{2}z(t)}\frac{H^{x_1(t)}_{-1,0}(x)}{\sqrt{1+e^{-2\sqrt{2}(x-x_1(t))}}}\right\Vert_{L^2_x} = O\left(e^{-2\sqrt{2}z(t)}\right), \tag{A.4}$$

$$\left\Vert 30(H^{x_1(t)}_{-1,0})^4H^{x_2(t)}_{0,1}+30e^{-\sqrt{2}z(t)}\frac{(H^{x_1(t)}_{-1,0}(x))^3}{\sqrt{1+e^{-2\sqrt{2}(x-x_1(t))}}}\right\Vert_{L^2_x} = O\left(e^{-3\sqrt{2}z(t)}\right). \tag{A.5}$$

The estimate of the remaining terms $-24H^{x_1(t)}_{-1,0}\left(H^{x_2(t)}_{0,1}\right)^2$, $30H^{x_1(t)}_{-1,0}\left(H^{x_2(t)}_{0,1}\right)^4$ is completely analogous to (A.4) and (A.5) respectively. In conclusion, all of the estimates above imply the estimate stated in the Lemma A.1.2. □

*Proof of Lemma A.1.1.* First, we recall the global estimate $e^{-\sqrt{2}z(t)} \lesssim \epsilon$. We also recall the identity (2.33)

$$\int_{\mathbb{R}}\left(8(H_{0,1}(x))^3-6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx = 2\sqrt{2},$$

which, by integration by parts, implies that

$$\int_{\mathbb{R}} 24\frac{H_{0,1}(x)\partial_x H_{0,1}(x)}{\sqrt{1+e^{2\sqrt{2}x}}}-30\frac{(H_{0,1}(x))^3\partial_x H_{0,1}(x)}{\sqrt{1+e^{2\sqrt{2}x}}}\,dx = 4. \tag{A.6}$$

We recall $d_1(t)$, $d_2(t)$ defined in (1.22) and (1.23) respectively and $d(t)=d_2(t)-d_1(t)$. Since $\ddot{d}_j(t)=(-1)^j8\sqrt{2}e^{-\sqrt{2}d(t)}$ for $j\in\{1,2\}$, we have $\ddot{d}(t)=16\sqrt{2}e^{-\sqrt{2}d(t)}$, which implies clearly with the identities

$$\Vert\partial_x H_{0,1}\Vert^2_{L^2_x} = \Vert\partial^2_x H_{0,1}\Vert^2_{L^2_x} = \tfrac{1}{2\sqrt{2}}$$

that $\ddot{d}_j(t)\Vert\partial_x H_{0,1}\Vert^2_{L^2_x} = (-1)^j4e^{-\sqrt{2}d(t)}$. We also recall the partial differential equation satisfied by the remainder $g(t,x)$ (II), which can be rewritten as

$$\begin{aligned}U'&\left(H^{x_2(t)}_{0,1}(x)+H^{x_1(t)}_{-1,0}(x)\right)-U'\left(H^{x_1(t)}_{-1,0}(x)\right)-U'\left(H^{x_2(t)}_{0,1}(x)\right)-\ddot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}(x)\\ &=-\left(\partial^2_t g(t,x)-\partial^2_x g(t,x)+U^{(2)}\left(H^{x_2(t)}_{0,1}(x)+H^{x_1(t)}_{-1,0}(x)\right)g(t,x)\right)\\ &\quad+\sum_{k=3}^{6}U^{(k)}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\frac{g(t)^{k-1}}{(k-1)!}-\dot{x}_1(t)^2\partial^2_x H^{x_1(t)}_{-1,0}(x)\\ &\quad-\dot{x}_2(t)^2\partial^2_x H^{x_2(t)}_{0,1}(x)+\ddot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0}(x).\end{aligned} \tag{A.7}$$

Furthermore, from the estimate (A.6), Lemma A.1.2 and Lemma 2.2.3, we obtain that

$$\begin{aligned}\Big\langle U'\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)-U'\left(H_{-1,0}^{x_1(t)}\right)-U'\left(H_{0,1}^{x_2(t)}\right),\,\partial_x H_{0,1}^{x_2(t)}\Big\rangle&\\ =&\ddot{x}_2(t)\left\|\partial_x H_{0,1}\right\|_{L^2_x}^2-\left(\ddot{x}_2(t)-\ddot{d}_2(t)\right)\left\|\partial_x H_{0,1}\right\|_{L^2_x}^2\\ &+O\Big(\left|\ddot{x}_1(t)\right|z(t)e^{-\sqrt{2}z(t)}\Big)\\ &+O\Big(e^{-\sqrt{2}z(t)}\max_{j\in\{1,\,2\}}\left|x_j(t)-d_j(t)\right|+e^{-2\sqrt{2}z(t)}z(t)\Big).\end{aligned}\tag{A.8}$$

We recall from the proof of Theorem 2.4.1 the following estimate

$$\begin{aligned}\Big|\int_{\mathbb{R}}\left[U^{(2)}\left(H_{0,1}^{x_2(t)}(x)\right)-U^{(2)}\left(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x)\right)\right]\partial_x H_{0,1}^{x_2(t)}(x)g(t,x)\,dx\Big|&\\ =O\Big(\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)}\Big).&\end{aligned}$$

Also, from the Modulation Lemma, we have that

$$\begin{aligned}\langle\partial_t^2 g(t),\partial_x H_{0,1}^{x_2(t)}\rangle&=\frac{d}{dt}\left[\langle\partial_t g(t),\partial_x H_{0,1}^{x_2(t)}\rangle\right]+\dot{x}_2(t)\langle\partial_t g(t),\partial_x H_{0,1}^{x_2(t)}\rangle\\ &=\frac{d}{dt}\left[\dot{x}_2(t)\langle g(t),\partial_x^2 H_{0,1}^{x_2(t)}\rangle\right]+\dot{x}_2(t)\langle\partial_t g(t),\partial_x H_{0,1}^{x_2(t)}\rangle\\ &=\ddot{x}_2(t)\langle g(t),\partial_x^2 H_{0,1}^{x_2(t)}\rangle+2\dot{x}_2(t)\langle\partial_t g(t),\partial_x H_{0,1}^{x_2(t)}\rangle.\end{aligned}$$

In conclusion, since $\partial_x H_{0,1}^{x_2(t)}\in\ker D^2E_{pot}\left(H_{0,1}^{x_2(t)}\right)$ and $e^{-\sqrt{2}z(t)}=O\left(\epsilon^{\frac{1}{2}}\right)$, we obtain from (A.8) and (A.7) that

$$\left|\ddot{x}_2(t)-\ddot{d}_2(t)\right|=O\Big(\max_{j\in\{1,\,2\}}\left|d_j(t)-x_j(t)\right|\epsilon+\epsilon z(t)e^{-\sqrt{2}z(t)}+\left\|\overrightarrow{g(t)}\right\|\epsilon^{\frac{1}{2}}\Big),$$

the estimate of $\left|\ddot{x}_1(t)-\ddot{d}_1(t)\right|$ is completely analogous, which finishes the proof of Lemma A.1.1. □

**Lemma A.1.3.** *For any $\delta>0$ there is a $\epsilon(\delta)>0$ such that if*

$$\left\|\phi(x)-H_{0,1}(x)\right\|_{H^1_x}<+\infty,\ 0<E_{pot}(\phi(x))-E_{pot}(H_{0,1})<\epsilon(\delta),\tag{A.9}$$

*then there is a real number $y$ such that*

$$\left\|\phi(x)-H_{0,1}(x-y)\right\|_{H^1_x}\leq\delta.$$

*Proof of Lemma A.1.3.* The proof of Lemma A.1.3 will follow by a contradiction argument. We assume the existence of a sequence of real functions $\left(\phi_n(x)\right)_n$ satisfying

$$\lim_{n\to+\infty}E_{pot}(\phi_n)=E_{pot}(H_{0,1}),\tag{A.10}$$

$$\left\|\phi_n(x)-H_{0,1}(x)\right\|_{H^1_x}<+\infty,\tag{A.11}$$

such that

$$\lim_{n\to+\infty}\inf_{y\in\mathbb{R}}\left\|\phi_n(x)-H_{0,1}(x+y)\right\|_{H^1_x}>0.\tag{A.12}$$

First, the condition (A.10) and the fact that $\lim_{\phi\to+\infty} U(\phi) = +\infty$ imply the existence of a positive constant $c$, which satisfies $\|\phi_n\|_{L^\infty} < c$ if $n \gg 1$.

Next, since $U(\phi) = \phi^2(1-\phi^2)^2$ and $|E_{pot}(\phi_n) - E_{pot}(H_{0,1})| \ll 1$ for $1 \ll n$, it is not difficult to verify from the definition of the potential energy functional $E_{pot}$ that if $1 \ll n$, then

$$\|\phi_n(x) - 1\|^2_{L^2(\{x|\phi_n(x)>1\})} + \left\|\frac{d\phi_n(x)}{dx}\right\|^2_{L^2(\{x|\phi_n(x)>1\})} \lesssim |E_{pot}(\phi_n) - E_{pot}(H_{0,1})|\,.$$

By an analogous argument, we can verify that

$$\|\phi_n(x)\|^2_{L^2(\{x|-\frac{1}{2}<\phi_n(x)<0\})} + \left\|\frac{d\phi_n(x)}{dx}\right\|^2_{L^2(\{x|-\frac{1}{2}<\phi_n(x)<0\})} \lesssim |E_{pot}(\phi_n) - E_{pot}(H_{0,1})|\,,$$

and if there is $x_0 \in \mathbb{R}$ such that $\phi_n(x_0) \leq -\frac{1}{2}$, we would obtain that

$$\begin{aligned}
&\int_{x_0}^{+\infty} \frac{1}{2}\frac{d\phi_n(x)}{dx}^2 + U(\phi_n(x))\,dx \\
&\quad = \int_{x_0}^{+\infty} \sqrt{2U(\phi_n(x))}\left|\frac{d\phi_n(x)}{dx}\right| dx + \frac{1}{2}\int_{x_0}^{+\infty}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx \\
&\qquad \geq \int_{-\frac{1}{2}}^{1} \sqrt{2U(\phi)}\,d\phi = E_{pot}(H_{0,1}) + \int_{-\frac{1}{2}}^{0}\sqrt{2U(\phi)}\,d\phi > E_{pot}(H_{0,1}),
\end{aligned}$$

which contradicts (A.10) if $n \gg 1$. Thus, if we consider the following function

$$\varphi_n(x) = \min\left(\max\left(\phi_n(x), 0\right), 1\right),$$

which satisfies $E_{pot}\left(\varphi_n\right) \geq E_{pot}\left(H_{0,1}\right)$ and

$$\frac{d\varphi_n(x)}{dx} = \begin{cases} \frac{d\phi_n(x)}{dx}, \text{ if } 0 < \phi_n(x) < 1, \\ 0, \text{ for almost every } x \in \mathbb{R} \text{ satisfying either } \phi_n(x) \leq 0 \text{ or } \phi_n(x) \geq 1, \end{cases}$$

we can deduce with the estimates above and inequality $\limsup_{n\to+\infty} \|\phi_n\|_{L^\infty} < c$ that if $n \gg 1$, then

$$\begin{aligned}
\|\phi_n(x) - \varphi_n(x)\|^2_{L^2_x} + \left\|\frac{d\phi_n(x)}{dx} - \frac{d\varphi_n(x)}{dx}\right\|^2_{L^2_x} &\lesssim \left|E_{pot}\left(\phi_n\right) - E_{pot}\left(H_{0,1}\right)\right|, \\
\left|E_{pot}\left(\phi_n\right) - E_{pot}\left(\varphi_n\right)\right| &\lesssim \left|E_{pot}\left(\phi_n\right) - E_{pot}\left(H_{0,1}\right)\right|.
\end{aligned}$$

Consequently, using triangle inequality and conditions (A.10), (A.12), we would obtain that

$$\lim_{n\to+\infty}\inf_{y\in\mathbb{R}} \|\varphi_n(x) - H_{0,1}(x+y)\|_{H^1_x} > 0.$$

In conclusion, we can restrict the proof to the case where $0 \leq \phi_n(x) \leq 1$ and $n \gg 1$.

Now, from the density of $H^2(\mathbb{R})$ in $H^1(\mathbb{R})$, we can also restrict the contradiction hypotheses to the situation where $\frac{d\phi_n}{dx}(x)$ is a continuous function for all $n \in \mathbb{N}$. Also, we have that if $\|\phi(x)-H_{0,1}(x)\|_{H^1_x}<+\infty$, then $E_{pot}(\phi(x))\geq E_{pot}(H_{0,1}(x))$. In conclusion, there is a sequence of positive numbers $(\epsilon_n)_n$ such that

$$E_{pot}(\phi_n)=E_{pot}(H_{0,1})+\epsilon_n,\ \lim_{n\to+\infty}\epsilon_n=0.$$

Also, $\tau_y\phi(x)=\phi(x-y)$ satisfies $E_{pot}(\phi(x))=E_{pot}(\tau_y\phi(x))$ for any $y\in\mathbb{R}$. In conclusion, since for all $n\in\mathbb{N}$, $\lim_{x\to+\infty}\phi_n(x)=1$ and $\lim_{x\to-\infty}\phi_n(x)=0$, we can restrict to the case where

$$\phi_n(0)=\frac{1}{\sqrt{2}},$$

for all $n\in\mathbb{N}$.

Next, we consider the notations $(v)_+=\max(v,0)$ and $(v)_-=-\left(v-(v)_+\right)$. Since $\frac{d\phi_n(x)}{dx}$ is a continuous function on $x$, we deduce that $\left(\frac{d\phi_n(x)}{dx}\right)_+$ and $\left(\frac{d\phi_n(x)}{dx}\right)_-$ are also continuous functions on $x$ for all $n\in\mathbb{N}$. In conclusion, for any $n\in\mathbb{N}$, we have that the set

$$\mathcal{U}_-=\left\{x\in\mathbb{R}|\ \frac{d\phi_n(x)}{dx}<0\right\} \tag{A.13}$$

is an enumerable union of disjoint open intervals $(a_{k,n},b_{k,n})_{k\in\mathbb{N}}$, which are bounded, since $\lim_{x\to+\infty}\phi_n(x)=1$, $\lim_{x\to-\infty}\phi_n(x)=0$ and $0\leq\phi_n(x)\leq 1$.

Now, let $E$ be a set of disjoint open bounded intervals $(h_{i,n},l_{i,n})\subset\mathbb{R}$ satisfying the conditions

$$\phi_n(h_{i,n})=\phi_n(l_{i,n}), \tag{A.14}$$

and $\{i|\,(h_{i,n},l_{i,n})\in E\}=I\subset\mathbb{Z}$. For any $i\in I$, the following function

$$f_{i,n}(x)=\begin{cases}\phi_n(x)\text{ if }x\leq h_{i,n},\\ \phi_n(x+l_{i,n}-h_{i,n})\text{ if }x>h_{i,n},\end{cases}$$

satisfies $E_{pot}(H_{0,1})\leq E_{pot}(f_{i,n})\leq E_{pot}(\phi_n)=E_{pot}(H_{0,1})+\epsilon_n$, which implies that

$$\int_{h_{i,n}}^{l_{i,n}}\frac{1}{2}\frac{d\phi_n(x)}{dx}^2+U(\phi_n(x))\leq\epsilon_n.$$

Furthermore, we can deduce from Lebesgue's dominated convergence theorem that

$$\sum_{i\in I}\int_{h_{i,n}}^{l_{i,n}}\frac{1}{2}\frac{d\phi_n(x)}{dx}^2+U(\phi_n(x))\leq\epsilon_n, \tag{A.15}$$

for every finite or enumerable collection $E$ of disjoint open bounded intervals $(h_{i,n},l_{i,n})\subset\mathbb{R}$, $i\in I\subset\mathbb{Z}$ such that $\phi_n(h_{i,n})=\phi_n(l_{i,n})$. In conclusion, we can deduce from (A.15) that

$$\int_{\mathbb{R}}\left(\frac{d\phi_n(x)}{dx}\right)_-^2dx\leq 2\epsilon_n, \tag{A.16}$$

and so for $1\ll n$ we have that

$$\left\|\frac{d\phi_n(x)}{dx}-\left|\frac{d\phi_n(x)}{dx}\right|\right\|_{L^2_x}^2\leq 8\epsilon_n,\ \phi_n(0)=\frac{1}{\sqrt{2}}. \tag{A.17}$$

Moreover, we can verify that

$$E_{pot}(\phi_n) = \frac{1}{2}\left[\int_{\mathbb{R}}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx\right] + \int_{\mathbb{R}} \sqrt{2U(\phi_n(x))}\left|\frac{d\phi_n(x)}{dx}\right| dx,$$

from which we deduce with $\lim_{x\to-\infty}\phi_n(x) = 0$ and $\lim_{x\to+\infty}\phi_n(x) = 1$ that

$$\begin{aligned} E_{pot}(H_{0,1}) + \epsilon_n \geq & \frac{1}{2}\left[\int_{\mathbb{R}}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx\right] + \int_0^1 \sqrt{2U(\phi)}\,d\phi \\ = & \frac{1}{2}\left[\int_{\mathbb{R}}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx\right] + E_{pot}(H_{0,1}). \end{aligned}$$

Then, from estimate (A.17), we have that

$$\frac{d\phi_n(x)}{dx} = \sqrt{2U(\phi_n(x))} + r_n(x),\ \phi_n(0) = \frac{1}{\sqrt{2}}, \tag{A.18}$$

with $\|r_n\|_{L^2_x}^2 \lesssim \epsilon_n$ for all $1 \ll n$.

We recall that $U(\phi) = \phi^2(1-\phi^2)^2$ is a Lipschitz function in the set $\{\phi|\,0 \leq \phi \leq 1\}$. Then, because $H_{0,1}(x)$ is the unique solution of the following ordinary differential equation

$$\begin{cases} \dfrac{d\phi(x)}{dx} = \sqrt{2U(\phi(x))}, \\ \phi(0) = \dfrac{1}{\sqrt{2}}, \end{cases}$$

we deduce from Gronwall Lemma that for any $K > 0$ we have

$$\lim_{n\to+\infty}\|\phi_n(x) - H_{0,1}(x)\|_{L^\infty[-K,K]} = 0,\ \lim_{n\to+\infty}\left\|\frac{d\phi_n(x)}{dx} - H^{'}_{0,1}(x)\right\|_{L^2[-K,K]} = 0. \tag{A.19}$$

Also, if $1 \ll n$, then $\left\|\frac{d\phi_n(x)}{dx}\right\|_{L^2_x}^2 < 2E_{pot}(H_{0,1}) + 1$, and so we obtain from Cauchy-Schwarz inequality that

$$|\phi_n(x) - \phi_n(y)| \leq |x-y|^{\frac{1}{2}}\left\|\frac{d\phi_n}{dx}\right\|_{L^2_x}^2 < M\,|x-y|^{\frac{1}{2}}, \tag{A.20}$$

for a constant $M > 0$. The inequality (A.20) implies that for any $1 > \omega > 0$ there is a number $h(\omega) \in \mathbb{N}$ such that if $n \geq h(\omega)$ then

$$\|\phi_n(x) - H_{0,1}(x)\|_{L^\infty\{x|\,\frac{1}{\omega}<|x|\}} < \omega, \tag{A.21}$$

otherwise we would obtain that there are $0 < \theta < \frac{1}{4}$, a subsequence $(m_n)_{n\in\mathbb{N}}$ and a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} m_n = +\infty$, $|x_n| > n+1$ such that

$$|\phi_{m_n}(x_n) - 1| > \theta \text{ if } x_n > 0, \tag{A.22}$$

$$|\phi_{m_n}(x_n)| > \theta \text{ if } x_n < 0. \tag{A.23}$$

However, since we are considering $\phi_n(x) \in C^1(\mathbb{R})$ and $0 \leq \phi_n \leq 1$, we would obtain from the intermediate value theorem that there would exist a sequence $(y_n)_n$ with $y_n > x_n > n+1$ or $y_n < x_n < -n-1$ such that

$$1-\theta \leq \phi_{m_n}(y_n) \leq 1+\theta, \text{ if } y_n > 0, \tag{A.24}$$

$$\phi_{m_n}(y_n) = \theta \text{ otherwise.} \tag{A.25}$$

But, estimates (A.20), (A.24), (A.25) and identity $U(\phi) = \phi^2(1-\phi^2)^2$ would imply that

$$1 \lesssim \int_{|x|\geq n-2} U(\phi_{m_n}(x))\,dx \text{ for all } n \gg 1, \tag{A.26}$$

and because of estimate (A.19) and the following identity

$$\lim_{K\to+\infty} \int_{-K}^{K} \frac{1}{2} H_{0,1}^{'}(x)^2 + U(H_{0,1}(x)) = E_{pot}(H_{0,1}(x)), \tag{A.27}$$

estimate (A.26) would imply that $\lim_{n\to+\infty} E_{pot}(\phi_{m_n}) > E_{pot}(H_{0,1})$ which contradicts our hypotheses.

In conclusion, for any $1 > \omega > 0$ there is a number $h(\omega)$ such that if $n \geq h(\omega)$ then (A.21) holds. So we deduce for any $0 < \omega < 1$ that there is a number $h_1(\omega)$ such that

$$\text{if } n \geq h_1(\omega), \text{ then } |\phi_n(x) - H_{0,1}(x)| \leq \omega \text{ for all } x \in \mathbb{R}. \tag{A.28}$$

Then, if $\omega \leq \frac{1}{100}$, $n \geq h(\omega)$ and $K \geq 200$, estimates (A.28) and (A.19) imply that

$$\int_{K}^{+\infty} U(\phi_n(x)) + \frac{1}{2}\frac{d\phi_n(x)}{dx}^2\,dx \geq \frac{1}{2}\int_{K}^{+\infty} (1-\phi_n(x))^2 + \frac{d\phi_n(x)}{dx}^2\,dx, \tag{A.29}$$

$$\int_{-\infty}^{-K} U(\phi_n(x)) + \frac{1}{2}\frac{d\phi_n(x)}{dx}^2\,dx \geq \frac{1}{2}\int_{-\infty}^{-K} \phi_n(x)^2 + \frac{d\phi_n(x)}{dx}^2\,dx. \tag{A.30}$$

In conclusion, from estimates (A.28), (A.29), (A.30) and

$$\lim_{K\to+\infty} \int_{|x|\geq K} \frac{1}{2} H_{0,1}^{'}(x)^2 + U(H_{0,1}(x))\,dx = 0,$$

we obtain that $\lim_{n\to+\infty} \|\phi_n(x) - H_{0,1}(x)\|_{L^2_x} = 0$ and, from the initial value problem (A.18) satisfied for each $\phi_n$, we conclude that $\lim_{n\to+\infty} \left\|\frac{d\phi_n}{dx}(x) - H_{0,1}^{'}(x)\right\|_{L^2_x} = 0$. In conclusion, inequality (A.12) is false. $\square$

From Lemma A.1.3, we obtain the following corollary:

**Corollary A.1.4.** *For any $\delta > 0$ there exists $\epsilon_0 > 0$ such that if* $0 < \epsilon \leq \epsilon_0$,

$$\|\phi(x) - H_{0,1}(x) - H_{-1,0}(x)\|_{H^1_x} < +\infty,$$

*and $E_{pot}(\phi) = 2E_{pot}(H_{0,1}) + \epsilon$, then there exist $x_2, x_1 \in \mathbb{R}$ such that*

$$x_2 - x_1 \geq \frac{1}{\delta},\ \|\phi(x) - H_{0,1}(x-x_2) + H_{-1,0}(x-x_1)\|_{H^1_x} \leq \delta. \tag{A.31}$$

*Proof of Corollary A.1.4.* First, from a similar reasoning to the proof of Lemma A.1.3 we can assume by density that $\frac{d\phi(x)}{dx} \in H^1_x(\mathbb{R})$. Next, from the hypothesis

$$\|\phi(x) - H_{0,1}(x) - H_{-1,0}(x)\|_{H^1(\mathbb{R})} < +\infty,$$

we deduce using the intermediate value theorem that there is a $y \in \mathbb{R}$ such that $\phi(y) = 0$. Now, we consider the functions

$$\phi_-(x) = \begin{cases} \phi(x) \text{ if } x \leq y, \\ 0 \text{ otherwise,} \end{cases}$$

and

$$\phi_+(x) = \begin{cases} 0 \text{ if } x \leq y, \\ \phi(x) \text{ otherwise.} \end{cases}$$

Clearly, $\phi(x) = \phi_-(x)$ for $x < y$ and $\phi(x) = \phi_+(x)$ for $x > y$. From identity $U(0) = 0$, we deduce that

$$E_{pot}(\phi) = E_{pot}(\phi_-) + E_{pot}(\phi_+),$$

also, we have that

$$E_{pot}(H_{-1,0}) < E_{pot}(\phi_-),\ E_{pot}(H_{0,1}) < E_{pot}(\phi_+).$$

In conclusion, since $E_{pot}(\phi) = 2E_{pot}(H_{0,1}) + \epsilon$, Lemma A.1.3 implies that if $\epsilon < \epsilon_0 \ll 1$, then there exist $x_2,\ x_1 \in \mathbb{R}$ such that

$$\begin{aligned} &\|\phi(x) - H_{0,1}(x - x_2) - H_{-1,0}(x - x_1)\|_{H^1_x} \\ &\quad \leq \|\phi_+ - H_{0,1}(x - x_2)\|_{H^1_x} + \|\phi_- - H_{-1,0}(x - x_1)\|_{H^1_x} \leq e^{-\frac{4}{\delta}} \ll \delta. \end{aligned} \tag{A.32}$$

So, to finish the proof of Corollary A.1.4, we need only to verify that we have $x_2 - x_1 \geq \frac{1}{\delta}$ if $0 < \epsilon_0 \ll 1$. But, we recall that $H_{0,1}(0) = \frac{1}{\sqrt{2}}$, from which with estimate (A.32) we deduce that

$$\left|\phi_+(x_2) - \frac{1}{\sqrt{2}}\right| \lesssim \delta, \left|\phi_-(x_1) + \frac{1}{\sqrt{2}}\right| \lesssim \delta, \tag{A.33}$$

so if $\epsilon_0 \ll 1$, then $x_1 < y < x_2$. Using the fact that $U$ is a smooth function, Lemma 2.2.7 and identity (2.35), we can verify the existence of a constant $C > 0$ satisfying the following inequality

$$|DE_{pot}\left(H_{0,1}(x - x_2) + H_{-1,0}(x - x_1) + u\right)(v)| \leq C \|v\|_{H^1_x}.$$

for any $u,\ v \in H^1(\mathbb{R})$ such that $\|u\|_{H^1_x} \leq 1$. Therefore, using estimate (A.32) and the Fundamental Theorem of Calculus, we deduce that if $0 < \epsilon_0 \ll 1$, then

$$|E_{pot}(\phi) - E_{pot}\left(H_{0,1}(x - x_2) + H_{-1,0}(x - x_1)\right)| < e^{-2\sqrt{2}\frac{1}{\delta}}. \tag{A.34}$$

Furthermore, since the function $A(z) = E_{pot}\left(H^z_{0,1}(x) + H_{-1,0}(x)\right)$ is a continuous function on $\mathbb{R}_{\geq 0}$ and $A(z) > 2E_{pot}\left(H_{0,1}\right)$ for any $z \geq 0$, we have for any $k > 0$ that there exists $\delta_k > 0$ satisfying

$$\sup_{\{z \in [0,k]\}} A(z) > 2E_{pot}\left(H_{0,1}\right) + \delta_k.$$

In conclusion, we obtain from Lemma 2.2.4 and the estimate (A.34) that $x_2 - x_1 \geq \frac{1}{\delta}$ if $0 < \epsilon_0 \ll 1$ and $\epsilon < \epsilon_0$. $\square$

Now, we complement our manuscript by presenting the proof of identity (2.33).

*Proof of Identity* (2.33). From the definition of the function $H_{0,1}(x)$, we have

$$\int_{\mathbb{R}} \left(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx = \int_{\mathbb{R}} \frac{8e^{2\sqrt{2}x} + 2e^{4\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{5}{2}}}\,dx,$$

by the change of variable $y(x) = (1 + e^{2\sqrt{2}x})$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} &\left(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx \\ &= \frac{1}{2\sqrt{2}}\int_1^{\infty} \frac{8}{y^{\frac{5}{2}}} + \frac{2(y-1)}{y^{\frac{5}{2}}}\,dy \\ &= \frac{1}{2\sqrt{2}}\int_1^{\infty} \frac{6}{y^{\frac{5}{2}}} + \frac{2}{y^{\frac{3}{2}}}\,dy, = \frac{1}{2\sqrt{2}}(-4y^{-\frac{3}{2}} - 4y^{-\frac{1}{2}})\Big|_1^{\infty} = 2\sqrt{2}.\end{aligned}$$

$\square$

## A.2 Proof of Theorem 2.1.7

*Proof of Theorem 2.1.7.* We use the notations of Theorem 2.1.10 and Theorem 2.4.1. Clearly, if the result of Theorem 2.1.7 is false, then by contradiction for any $N \gg 1$ the inequality

$$\left\|\overrightarrow{g(t)}\right\| \leq \frac{\epsilon}{N} \tag{A.35}$$

could be possible for all $0 \leq t \leq N\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}} = T$ if $\epsilon \ll 1$ enough.

From Modulation Lemma, we can denote the solution $\phi(t,x)$ as

$$\phi(t,x) = H_{-1,0}^{x_1(t)}(x) + H_{0,1}^{x_2(t)}(x) + g(t,x),$$

such that

$$\langle g(t,x),\, \partial_x H_{-1,0}^{x_1(t)}(x)\rangle = 0,\ \langle g(t,x),\, \partial_x H_{0,1}^{x_2(t)}(x)\rangle = 0.$$

Also, for all $t \geq 0$, we have that $g(t,x)$ has a unique representation as

$$g(t,x) = P_1(t)\partial_x^2 H_{-1,0}^{x_1(t)}(x) + P_2(t)\partial_x^2 H_{0,1}^{x_2(t)}(x) + r(t,x), \tag{A.36}$$

such that $r(t)$ satisfies the following new orthogonality conditions

$$\left\langle r(t),\, \partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle = 0,\ \left\langle r(t),\, \partial_x^2 H_{0,1}^{x_2(t)}\right\rangle = 0. \tag{A.37}$$

In conclusion, we deduce that

$$\|g(t)\|_{L_x^2}^2 = \left\|\partial_x^2 H_{0,1}\right\|_{L_x^2}^2 (P_1^2 + P_2^2) + \|r(t)\|_{L_x^2}^2 + 2P_1P_2\left\langle \partial_x^2 H_{0,1}^{z(t)},\, \partial_x^2 H_{-1,0}\right\rangle. \tag{A.38}$$

We recall from Theorem 2.2.8 that $\frac{1}{\sqrt{2}}\ln\frac{1}{\epsilon} < z(t)$ for all $t \geq 0$. Since, from Lemma 2.2.3, we have that $\left\langle \partial_x^2 H_{-1,0}^{x_1(t)},\, \partial_x^2 H_{0,1}^{x_2(t)} \right\rangle \lesssim z(t)e^{-\sqrt{2}z(t)}$ and $z(t)e^{-\sqrt{2}z(t)} \lesssim \epsilon \ln\frac{1}{\epsilon}$ if $0 < \epsilon \ll 1$, we deduce from the equation (A.38) that there is a uniform constant $K > 1$ such that for all $t \geq 0$ we have the following estimate

$$\frac{\|g(t)\|_{L^2_x}}{K} \leq |P_1(t)| + |P_2(t)| + \|r(t)\|_{L^2_x} \leq K \left\|\overrightarrow{g(t)}\right\|. \tag{A.39}$$

From Theorem 2.2.8 and the orthogonality conditions (A.37), we deduce that

$$\begin{aligned}
\left\langle \partial_t r(t),\, \partial_x^2 H_{0,1}^{x_2(t)} \right\rangle =& \dot{x}_2(t) \left\langle r(t),\, \partial_x^3 H_{0,1}^{x_2(t)} \right\rangle = O\Big( \|r(t)\|_{L^2_x} \epsilon^{\frac{1}{2}} \Big),\\
\left\langle \partial_t r(t),\, \partial_x^2 H_{-1,0}^{x_1(t)} \right\rangle =& \dot{x}_2(t) \left\langle r(t),\, \partial_x^3 H_{-1,0}^{x_1(t)} \right\rangle = O\Big( \|r(t)\|_{L^2_x} \epsilon^{\frac{1}{2}} \Big).
\end{aligned}$$

In conclusion, estimate (A.39) and Lemma 2.2.3 imply that there is a $K > 1$ such that

$$\left|\dot{P}_1(t)\right| + \left|\dot{P}_2(t)\right| + \|\partial_t r(t)\|_{L^2_x} \leq K \left\|\overrightarrow{g(t)}\right\| \tag{A.40}$$

for all $t \geq 0$. Finally, Minkowski inequality and estimate (A.39) imply that there is a uniform constant $K > 1$ such that

$$\|\partial_x r(t,x)\|_{L^2_x} \leq K \left\|\overrightarrow{g(t)}\right\|. \tag{A.41}$$

We recall from Theorem 2.2.9 the following estimate

$$\frac{\epsilon}{K} \leq \left\|\overrightarrow{g(t)}\right\|^2 + \dot{x}_1(t)^2 + \dot{x}_2(t)^2 + e^{-\sqrt{2}z(t)} \leq K\epsilon \tag{A.42}$$

for some uniform constant $K > 1$. Now, from hypothesis (A.35), we obtain from Theorem 2.1.10 and Corollary 2.1.13 that there are constants $M \in \mathbb{N}$ and $C > 0$ such that for all $t \geq 0$ the following inequalities are true

$$\max_{j\in\{1,2\}} |x_j(t) - d_j(t)| \leq \epsilon \left(\ln\frac{1}{\epsilon}\right)^{M+1} \exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{A.43}$$

$$\max_{j\in\{1,2\}} \left|\dot{x}_j(t) - \dot{d}_j(t)\right| \leq \epsilon^{\frac{3}{2}} \left(\ln\frac{1}{\epsilon}\right)^{M} \exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{A.44}$$

$$\max_{j\in\{1,2\}} \left|\ddot{x}_j(t) - \ddot{d}_j(t)\right| \leq \epsilon^{\frac{3}{2}} \left(\ln\frac{1}{\epsilon}\right) \exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\left(\frac{1}{\epsilon}\right)}\right), \tag{A.45}$$

for a uniform constant $C > 0$.

From the partial differential equation (2.1) satisfied by $\phi(t,x)$ and the representation

(A.36) of $g(t,x)$, we deduce in the distributional sense that for any $h(x) \in H^1(\mathbb{R})$ that

$$\begin{aligned}
&\left\langle h(x),\, (\ddot{P}_1(t) + \dot{x}_1(t)^2)\partial_x^2 H_{-1,0}^{x_1(t)} + (\ddot{P}_2(t) + \dot{x}_2(t)^2)\partial_x^2 H_{0,1}^{x_2(t)} \right\rangle \\
&\qquad = -\left\langle h(x), P_1(t)\Big[\Big(-\partial_x^2 + U^{(2)}(H_{-1,0}^{x_1(t)})\Big)\partial_x^2 H_{-1,0}^{x_1(t)}\Big]\right\rangle \\
&\qquad - \left\langle h(x), P_2(t)\Big[\Big(-\partial_x^2 + U^{(2)}(H_{0,1}^{x_2(t)})\Big)\partial_x^2 H_{0,1}^{x_2(t)}\Big]\right\rangle \\
&\qquad - \left\langle h(x),\, \Big[\partial_t^2 r(t) - \partial_x^2 r(t) + U^{(2)}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) r(t)\Big]\right\rangle \\
&\qquad - \left\langle h(x),\, \Big[U^{'}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) + U^{'}(H_{0,1}^{x_2(t)}) - U^{'}(H_{-1,0}^{x_1(t)})\Big]\right\rangle \\
&\qquad + \left\langle h(x),\, \ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x) + \ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x)\right\rangle \\
&\qquad - \left\langle h(x),\, P_1(t)\Big[\Big(U^{(2)}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) - U^{(2)}(H_{-1,0}^{x_1(t)})\Big)\partial_x^2 H_{-1,0}^{x_1(t)}\Big]\right\rangle \\
&\qquad - \left\langle h(x), P_2(t)\Big[\Big(U^{(2)}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) - U^{(2)}(H_{0,1}^{x_2(t)})\Big)\partial_x^2 H_{0,1}^{x_2(t)}\Big]\right\rangle \\
&\qquad + O\left(\|h\|_{L_x^2}\left[\|g(t)\|_{H_x^1}^2 + \max_{j\in\{1,2\}}|\ddot{x}_j(t)|\right]\right) \\
&\qquad + O\left(\|h\|_{L_x^2}\left[\max_{j\in\{1,2\}}\left|\dot{P}_j(t)\dot{x}_j(t)\right| + \max_{j\in\{1,2\}}|P_j(t)|\, e^{-\sqrt{2}z(t)}\right]\right) \\
&\qquad + O\left(|P_j(t)\ddot{x}_j(t)| + \left|P_j(t)\dot{x}_j(t)^2\right|\right).
\end{aligned} \tag{A.46}$$

From Lemma A.1.2 and estimates (A.43) and (A.45), we obtain from (A.46) that

$$\begin{aligned}
&\left\langle h(x),\, (\ddot{P}_1(t) + \dot{x}_1(t)^2)\partial_x^2 H_{-1,0}^{x_1(t)} + (\ddot{P}_2(t) + \dot{x}_2(t)^2)\partial_x^2 H_{0,1}^{x_2(t)} \right\rangle \\
&\qquad = -\left\langle h(x), P_1(t)\Big[\Big(-\partial_x^2 + U^{(2)}(H_{-1,0}^{x_1(t)})\Big)\partial_x^2 H_{-1,0}^{x_1(t)}\Big]\right\rangle \\
&\qquad - \left\langle h(x), P_2(t)\Big[\Big(-\partial_x^2 + U^{(2)}(H_{0,1}^{x_2(t)})\Big)\partial_x^2 H_{0,1}^{x_2(t)}\Big]\right\rangle \\
&\qquad - \left\langle h(x),\, \Big[\partial_t^2 r(t) - \partial_x^2 r(t) + U^{(2)}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) r(t)\Big]\right\rangle \\
&\qquad + O\left(\|h\|_{L_x^2}\left[\max_{j\in\{1,2\}}\left|\ddot{x}_j(t) - \ddot{d}_j(t)\right| + e^{-\sqrt{2}d(t)}\right]\right) \\
&\qquad + O\left(\|h\|_{L_x^2}\left[|z(t) - d(t)|\, e^{-\sqrt{2}z(t)} + e^{-2\sqrt{2}z(t)}\right]\right) \\
&\qquad + O\left(\|h\|_{L_x^2}\left[\|g(t)\|_{H_x^1}^2 + \max_{j\in\{1,2\}}|\ddot{x}_j(t)|\right]\right) \\
&\qquad + O\left(\|h\|_{L_x^2}\left[\max_{j\in\{1,2\}}\left|\dot{P}_j(t)\dot{x}_j(t)\right| + \max_{j\in\{1,2\}}|P_j(t)|\, e^{-\sqrt{2}z(t)} + |P_j(t)\ddot{x}_j(t)|\right]\right) \\
&\qquad + O\left(\|h\|_{L_x^2}\left|P_j(t)\dot{x}_j(t)^2\right|\right).
\end{aligned} \tag{A.47}$$

From the condition (A.37), we deduce that

$$\begin{aligned}
\left\langle \partial_t^2 r(t),\, \partial_x^2 H_{0,1}^{x_2(t)} \right\rangle &= \frac{d}{dt}\left[\dot{x}_2(t)\left\langle r(t),\, \partial_x^3 H_{0,1}^{x_2(t)}\right\rangle\right] + \dot{x}_2(t)\left\langle \partial_t r(t),\, \partial_x^3 H_{0,1}^{x_2(t)}\right\rangle, \\
\left\langle \partial_t^2 r(t),\, \partial_x^2 H_{-1,0}^{x_1(t)} \right\rangle &= \frac{d}{dt}\left[\dot{x}_1(t)\left\langle r(t),\, \partial_x^3 H_{-1,0}^{x_1(t)}\right\rangle\right] + \dot{x}_1(t)\left\langle \partial_t r(t),\, \partial_x^3 H_{-1,0}^{x_1(t)}\right\rangle,
\end{aligned}$$

which imply with Theorem 2.2.8 the existence of a uniform constant $C>0$ such that

$$\left|\left\langle \partial_t^2 r(t),\, \partial_x^2 H_{0,1}^{x_2(t)}\right\rangle\right| \leq C\epsilon^{\frac{1}{2}} \left\|\overrightarrow{r(t)}\right\|,\, \left|\left\langle \partial_t^2 r(t),\, \partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle\right| \leq C\epsilon^{\frac{1}{2}} \left\|\overrightarrow{r(t)}\right\|. \tag{A.48}$$

From (A.39), (A.40) and (A.41), we obtain that $\left\|\overrightarrow{r(t)}\right\| \lesssim \left\|\overrightarrow{g(t)}\right\|$.

In conclusion, after we apply the partial differential equation (A.47) in the distributional sense to $\partial_x^2 H_{0,1}^{x_2(t)}$, $\partial_x^2 H_{-1,0}^{x_1(t)}$, the estimates (A.39), (A.40), (A.41), (A.43), (A.45) and (A.48) imply that there is a uniform constant $K_1>0$ such that if $\epsilon \ll 1$ enough, then for $j\in\{1,\,2\}$ we have that for $0\leq t\leq \frac{N\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$

$$\left|\ddot{P}_j(t)+\dot{x}_j(t)^2\right| \leq K_1\left(e^{-\sqrt{2}d(t)}+\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)+\frac{\epsilon}{N}\right),$$

from which we deduce for all $0\leq t\leq N\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that

$$\left|\sum_{j=1}^{2}\ddot{P}_j(t)+\dot{x}_j(t)^2\right| \leq 2K_1\left(e^{-\sqrt{2}d(t)}+\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)+\frac{\epsilon}{N}\right). \tag{A.49}$$

Since $\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq -\left|\sum_{j=1}^{2}\ddot{P}_j(t)+\dot{x}_j(t)^2\right|+\sum_{j=1}^{2}\dot{x}_j(t)^2$, we deduce from the estimates (A.49) and (A.42) that

$$\begin{aligned}\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{K}-\left[e^{-\sqrt{2}z(t)}+\left\|\overrightarrow{g(t)}\right\|^2\right]\\ -2K_1\left[e^{-\sqrt{2}d(t)}+\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right]-\frac{2K_1\epsilon}{N}.\end{aligned} \tag{A.50}$$

We recall that from the statement of Theorem 2.1.10 that $e^{-\sqrt{2}d(t)}=\frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt+c\right)^2$, with $v=\left(\frac{\dot{z}(0)^2}{4}+8e^{-\sqrt{2}z(0)}\right)^{\frac{1}{2}}$, which implies that $v\lesssim\epsilon^{\frac{1}{2}}$. Since we have verified in Theorem 2.2.8 that $e^{-\sqrt{2}z(t)}\lesssim\epsilon$, the mean value theorem implies that $\left|e^{-\sqrt{2}z(t)}-e^{-\sqrt{2}d(t)}\right| = O(\epsilon\left|z(t)-d(t)\right|)$, from which we deduce from (A.43) that

$$\left|e^{-\sqrt{2}z(t)}-e^{-\sqrt{2}d(t)}\right| = O\left(\epsilon^2\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right).$$

In conclusion, if $\epsilon\ll 1$ enough, we obtain for $0\leq t\leq\frac{N\ln\left(\frac{1}{\epsilon}\right)}{\epsilon^{\frac{1}{2}}}$ from (A.50) that

$$\begin{aligned}\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{K}-\left[e^{-\sqrt{2}d(t)}+\left\|\overrightarrow{g(t)}\right\|^2\right]\\ -4K_1\left[e^{-\sqrt{2}d(t)}+\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right]-\frac{2K_1\epsilon}{N}.\end{aligned} \tag{A.51}$$

The conclusion of the demonstration will follow from studying separate cases in the choice of $v>0$, $c$. We also observe that $K$, $K_1$ are uniform constants and the value of $N\in\mathbb{N}_{>0}$ can

be chosen at the beginning of the proof to be as much large as we need.
**Case 1.**($v^2 \leq \frac{8\epsilon}{(1+4K_1)2K}$.) From inequality (A.51), we deduce that

$$\left|\sum_{j=1}^{2} \ddot{P}_j(t)\right| \geq \frac{\epsilon}{2K} - \left\|\overrightarrow{g(t)}\right\|^2 - 4K_1\left(\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right) - \frac{2K_1\epsilon}{N},$$

then, from (A.35) we deduce for $0 \leq t \leq \frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that if $\epsilon$ is small enough and $N > 10KK_1$, then $\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{4K}$, and so,

$$\left|\sum_{j=1}^{2} \dot{P}_j(t)\right| \geq \frac{\epsilon t}{4K} - \left|\sum_{j=1}^{2} \dot{P}_j(0)\right|,$$

which contradicts the fact that (A.40) and (A.35) should be true for $\epsilon \ll 1$.
**Case 2.**($v^2 \geq \frac{8\epsilon}{(1+4K_1)2K}$, $|c| > 2\ln\left(\frac{1}{\epsilon}\right)$.) It is not difficult to verify that for $0 \leq t \leq \min\left(\frac{|c|}{2\sqrt{2}v}, N\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}\right)$, we have that $e^{-\sqrt{2}d(t)} \leq \frac{v^2}{8}\operatorname{sech}\left(\frac{c}{2}\right)^2 \lesssim \epsilon^3$. Therefore, if $N > 10KK_1$ and $\epsilon > 0$ is small enough, estimate (A.51) would imply that $\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{4K}$ is true in this time interval. Also, since now $v \cong \epsilon^{\frac{1}{2}}$, we have that

$$\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}} \lesssim \frac{|c|}{2\sqrt{2}v},$$

so we obtain a contradiction by a similar argument to the Case 1.
**Case 3.**($v^2 \geq \frac{8\epsilon}{(1+4K_1)2K}$ and $|c| \leq 2\ln\frac{1}{\epsilon}$.) For $N \gg 1$ and $t_0 = \frac{(1+4K_1)^{\frac{1}{2}}K^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, we have during the time interval $\left\{t_0 \leq t \leq 2\frac{(1+4K_1)^{\frac{1}{2}}K^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}\right\}$ that $e^{-\sqrt{2}d(t)} \leq \frac{v^2}{8}\operatorname{sech}\left(2\ln\frac{1}{\epsilon}\right)^2 \lesssim \epsilon^5$ and $\frac{\epsilon}{N} < \frac{\epsilon}{20K}$. In conclusion, estimate (A.50) implies that $\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{4K}$ is true in this time interval. From the Fundamental Calculus Theorem, we have that

$$\left|\sum_{j=1}^{2}\dot{P}_j(t)\right| \geq \frac{\epsilon(t-t_0)}{4K} - \left|\sum_{j=1}^{2}\dot{P}_j(t_0)\right|.$$

In conclusion, hypothesis (A.35) and estimate (A.40) imply for $T = 2\frac{(1+2K_1)^{\frac{1}{2}}K^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ and $N \gg 1$ that

$$\left|\sum_{j=1}^{2}\dot{P}_j(T)\right| \geq \frac{\epsilon^{\frac{1}{2}}(1+2K_1)^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{8K^{\frac{1}{2}}},$$

which contradicts the fact that (A.35) and (A.40) should be true, which finishes our proof. □

## A.3 Linear properties of $-\frac{d^2}{dx^2} + U^{(2)}(H_{0,1}(x))$

**Lemma A.3.1.** *The function $\xi : \mathbb{R} \to \mathbb{R}$ denoted by $\xi(x) = \left(\frac{x}{4\sqrt{2}} - \frac{1}{16e^{2\sqrt{2}x}}\right)$ satisfies*

$$\left[-\frac{d^2}{dx^2} + U^{(2)}(H_{0,1}(x))\right]\xi(x)H_{0,1}^{'}(x) = H_{0,1}^{'}(x).$$

*Proof of Lemma A.3.1.* Clearly, we have that $\xi^{'}(x) = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}e^{2\sqrt{2}x}}$, so using identity $H_{0,1}^{'}(x) = \sqrt{2}e^{\sqrt{2}x}\left(1+e^{2\sqrt{2}x}\right)^{-\frac{3}{2}}$, we obtain that

$$-\frac{d}{dx}\left[\xi^{'}(x)H_{0,1}^{'}(x)^2\right] = H_{0,1}^{'}(x)^2.$$

Therefore, since $0 < H'_{0,1} \in \ker\left(-\frac{d^2}{dx^2} + U^{(2)}(H_{0,1})\right)$, we conclude that

$$\left[-\frac{d^2}{dx^2} + U^{(2)}(H_{0,1})\right]\xi(x)H'_{0,1}(x) = H'_{0,1}(x).$$

□

**Remark A.3.2.** *From the identity*

$$U^{(2)}\left(H_{0,1}(x)\right) = 2 - 24H_{0,1}(x)^2 + 30H_{0,1}(x)^4,$$

*we deduce that*

$$\left[-\frac{d^2}{dx^2} + U^{(2)}(H_{0,1}(x))\right]e^{-\sqrt{2}x} = \left(30H_{0,1}(x)^4 - 24H_{0,1}(x)^2\right)e^{-\sqrt{2}x}.$$

*In conclusion, Lemma A.3.1 implies that*

$$\left[-\frac{d^2}{dx^2} + U^{(2)}(H_{0,1}(x))\right]\left(e^{-\sqrt{2}x} + 8\sqrt{2}\xi(x)H'_{0,1}(x)\right) = \left(30H_{0,1}(x)^4 - 24H_{0,1}(x)^2\right)e^{-\sqrt{2}x} + 8\sqrt{2}H'_{0,1}(x),$$

*so,*

$$-\frac{d^2}{dx^2}\mathcal{G}(x) + U^{(2)}(H_{0,1}(x))\mathcal{G}(x) = \left(30H_{0,1}(x)^4 - 24H_{0,1}(x)^2\right)e^{-\sqrt{2}x} + 8\sqrt{2}H'_{0,1}(x),$$

*for all $x \in \mathbb{R}$.*

**Lemma A.3.3.** *In notation of Lemma 3.2.23, if $g(x) \in \mathscr{S}(\mathbb{R})$ and $\left\langle g(x),\, H'_{0,1}(x)\right\rangle = 0$, then we have that $L_1(g)(x) \in \mathscr{S}(\mathbb{R})$.*

*Proof of Lemma A.3.3.* **Step 1.**($f(x) \in \cap_{k\geq 1} H^k_x(\mathbb{R})$.) Following Lemma 3.2.23, we have the existence of the unique function $f = L_1(g) \in H^1_x(\mathbb{R})$ such that $\left\langle f(x),\, H'_{0,1}(x)\right\rangle = 0$ and

$$-f^{(2)}(x) + U^{(2)}(H_{0,1}(x))f(x) = g(x). \tag{A.52}$$

The identity (A.52) above implies that $f \in H^2_x(\mathbb{R})$. Moreover, since $H_{0,1} \in L^\infty_x(\mathbb{R})$ and

$$H'_{0,1}(x) = \sqrt{2}\frac{e^{\sqrt{2}x}}{\left(1 + e^{2\sqrt{2}x}\right)^{\frac{3}{2}}} \in \mathscr{S}(\mathbb{R}),$$

we obtain that $\frac{d^l}{dx^l}U^{(2)}(H_{0,1}(x)) \in \mathscr{S}(\mathbb{R})$ for all natural $l \geq 1$. So, we obtain that if $f(x) \in H^k_x(\mathbb{R})$ for $k \geq 1$, then, since $H^k_x(\mathbb{R})$ is an algebra for $k \geq 1$, $g(x) - U^{(2)}(H_{0,1}(x))f(x) \in H^k(\mathbb{R})$.

Then, from equation (A.52), if $f \in H^k_x(\mathbb{R})$, then $f^{(2)}(x) \in H^k_x(\mathbb{R})$, which would imply that $f^{(k+2)}(x)$ is in $L^2_x(\mathbb{R})$, and by elementary Fourier analysis theory or interpolation theory we would verify obtain $f^{(l)}(x) \in L^2_x(\mathbb{R})$ for any natural $l$ satisfying $0 \leq l \leq k+2$. In conclusion, by a standard argument of induction, we obtain that, for any natural $k$, $f(x) \in H^k_x(\mathbb{R})$, and as a consequence $f(x) \in C^\infty(\mathbb{R})$.

**Step 2.**($f(x) \in \mathscr{S}(\mathbb{R})$.) Since $U^{(2)}(\phi) = 2 - 24\phi^2 + 30\phi^4$, we have $\lim_{x\to+\infty} U^{(2)}(H_{0,1}(x)) = 8$ and $\lim_{x\to-\infty} U^{(2)}(H_{0,1}(x)) = 2$. From equation (A.52), we have the following identities

$$-f^{(2)}(x) + 2f(x) = g(x) + \left[2 - U^{(2)}(H_{0,1}(x))\right] f(x), \tag{A.53}$$

$$-f^{(2)}(x) + 8f(x) = g(x) + \left[8 - U^{(2)}(H_{0,1}(x))\right] f(x). \tag{A.54}$$

Next, we consider a smooth cut function $\chi : \mathbb{R} \to \mathbb{R}$ satisfying $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 0, \text{ if } x \leq 4, \\ 1, \text{ if } x \geq 5. \end{cases}$$

Identity (A.54) implies that $h(x) = \chi(x)f(x)$ satisfies

$$-h^{(2)}(x) + 8h(x) = \chi(x)g(x) + \left[8 - U^{(2)}\left(H_{0,1}(x)\right)\right] \chi(x)f(x) - 2\chi^{'}(x)f^{'}(x) - \chi^{(2)}(x)f(x). \tag{A.55}$$

From the definition of $\chi$, $\chi^{'}$ is a smooth function with compact support, so both functions $\chi^{'}$, $\chi^{(2)} \in \mathscr{S}(\mathbb{R})$. In conclusion, since $f \in C^{\infty}(\mathbb{R})$ from first step, we deduce that $\chi^{'}f^{'}$, $\chi^{(2)}f \in \mathscr{S}(\mathbb{R})$. Also, using estimate (3.3) for $k = 1$

$$\left|H^{'}_{0,1}(x)\right| \lesssim \min\left(e^{\sqrt{2}x}, e^{-2\sqrt{2}x}\right),$$

we conclude from the Fundamental theorem of calculus the following estimate

$$\left|8 - U^{(2)}\left(H_{0,1}(x)\right)\right| \lesssim e^{-2\sqrt{2}x} \text{ for all } x > 1.$$

So, $f$ being in $C^{\infty}\left(\mathbb{R}\right)$, the definition of $\chi$ and estimate (3.3) imply

$$\left[8 - U^{(2)}\left(H_{0,1}(x)\right)\right] \chi(x)f(x) \in \mathscr{S}(\mathbb{R}).$$

In conclusion, since $f(x)\chi(x) \in H^k_x\left(\mathbb{R}\right)$ for any $k \geq 0$, identity (A.55) implies that $\chi(x)f(x) \in \mathscr{S}(\mathbb{R})$. By analogy, using (A.53) and the function $h_1 = (1-\chi)f$, we conclude that $(1-\chi)f \in \mathscr{S}(\mathbb{R})$, so $f \in \mathscr{S}(\mathbb{R})$. □

## A.4 Complementary Estimates

In this Appendix section, we complement our article by demonstrating complementary estimates.

**Lemma A.4.1.** *For*

$$\mathcal{G}(x) = e^{-\sqrt{2}x} - \frac{e^{-\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}} + x\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}} + k_1\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}},$$

*we have that*

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H^{'}_{0,1}(x)^2\mathcal{G}(x)\,dx = \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H^{'}_{0,1}(x)^2 e^{-\sqrt{2}x}\,dx - \sqrt{2}\int_{\mathbb{R}}\left[U^{(2)}(H_{0,1}(x)) - 2\right] H^{'}_{0,1}(x)e^{-\sqrt{2}x}\,dx.$$

**Remark A.4.2.** *Indeed, the value $k_1$ in Lemma A.4.1 can be replaced by zero, since*

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}^{'}(x)^3\,dx = 0.$$

*proof of Lemma A.4.1.* First, from identity $H_{0,1}^{(2)}(x) = U^{'}(H_{0,1}(x))$ and integration by parts, we have the following identity

$$\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}(x)\right) H_{0,1}^{'}(x)^2\mathcal{G}(x)\,dx = \int_{\mathbb{R}} U^{'}(H_{0,1}(x))\left[\mathcal{G}^{(2)}(x) - U^{(2)}(H_{0,1})\mathcal{G}(x)\right]\,dx.$$

Moreover, since $-\mathcal{G}^{(2)}(x) + U^{(2)}(H_{0,1}(x))\mathcal{G}(x) = \left[U^{(2)}(H_{0,1}(x)) - 2\right]e^{-\sqrt{2}x} + 8\sqrt{2}H_{0,1}^{'}(x)$ and $\left\langle H_{0,1}^{'},\, U^{'}(H_{0,1})\right\rangle = 0$, we conclude using integration by parts that

$$\begin{aligned}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}(x)\right) H_{0,1}^{'}(x)^2\mathcal{G}(x)\,dx =& -\int_{\mathbb{R}} U^{'}\left(H_{0,1}(x)\right)\left[U^{(2)}(H_{0,1}(x)) - 2\right]e^{-\sqrt{2}x}\,dx\\ =& -\int_{\mathbb{R}} H_{0,1}^{(2)}(x)\left[U^{(2)}\left(H_{0,1}(x)\right) - 2\right]e^{-\sqrt{2}x}\,dx,\\ =& \int_{\mathbb{R}} U^{(3)}\left(H_{0,1}(x)\right) H_{0,1}^{'}(x)^2 e^{-\sqrt{2}x}\,dx\\ & -\sqrt{2}\int_{\mathbb{R}}\left[U^{(2)}\left(H_{0,1}(x)\right) - 2\right]H_{0,1}^{'}(x)e^{-\sqrt{2}x}\,dx.\end{aligned}$$

□

Now, using integration by parts and identity (27) of [47], we have that

$$-\sqrt{2}\int_{\mathbb{R}}\left[U^{(2)}\left(H_{0,1}(x)\right) - 2\right]e^{-\sqrt{2}x}H_{0,1}^{'}(x)\,dx = -2\int_{\mathbb{R}}\left[6H_{0,1}(x)^5 - 8H_{0,1}(x)^3\right]e^{-\sqrt{2}x}\,dx = 4\sqrt{2}, \tag{A.56}$$

from which we deduce the following Lemma.

**Lemma A.4.3.**

$$\int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}^{'}(x)^2\mathcal{G}(x)\,dx - \int_{\mathbb{R}} U^{(3)}(H_{0,1}(x))H_{0,1}^{'}(x)^2 e^{-\sqrt{2}x}\,dx = 4\sqrt{2}.$$

**Lemma A.4.4.** *There exist $\delta > 0$, $c > 0$ such that if $0 < v < \delta$, $d(t) = \frac{1}{\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt\right)^2\right)$, then for*

$$H_{0,1}^{+}(x,t) = H_{0,1}\left(\frac{x - \frac{d(t)}{2}}{\sqrt{1 - \frac{\dot{d}(t)^2]}{4}}}\right),$$

$$H_{0,1}^{-}(x,t) = H_{-1,0}\left(\frac{x + \frac{d(t)}{2}}{\sqrt{1 - \frac{\dot{d}(t)^2]}{4}}}\right),$$

*and any $g \in H_x^1(\mathbb{R})$ such that*

$$\left\langle g(x), \partial_x H_{0,1}^{+}(x,t)\right\rangle = 0,\ \left\langle g(x), \partial_x H_{0,1}^{-}(x,t)\right\rangle = 0,$$

*we have*

$$c\left\|g\right\|_{H_x^1}^2 \leq \left\langle -\partial_x^2 g(x) + U^{(2)}\left(H_{0,1}^{+}(x,t) + H_{0,1}^{-}(x,t)\right)g(x),\, g(x)\right\rangle. \tag{A.57}$$

*Proof of Lemma A.4.4.* First, to simplify our computations we denote

$$\gamma_{d(t)} = \frac{1}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}.$$

Next, we can verify using a change of variables that

$$\left\langle U^{(2)}\left(H^{+}_{0,1}(x,t)\right)g(x),\,g(x)\right\rangle = \sqrt{1-\frac{\dot{d}(t)^2}{4}}\int_{\mathbb{R}} U^{(2)}\left(H_{0,1}(y)\right)\left[g\left(\left(y+\frac{d(t)}{2}\gamma_{d(t)}\right)\gamma_{d(t)}^{-1}\right)\right]^2\,dy,$$

and

$$\int_{\mathbb{R}} \frac{dg(x)}{dx}^2\,dx = \frac{1}{\sqrt{1-\frac{\dot{d}(t)^2}{4}}}\int_{\mathbb{R}}\left[\frac{d}{dy}\left[g\left(y\gamma_{d(t)}^{-1}\right)\right]\right]^2\,dy. \tag{A.58}$$

We denote now

$$g_1(t,y) = g\left(y\sqrt{1-\frac{\dot{d}(t)^2}{4}}\right) = g(y\gamma_{d(t)}^{-1}).$$

Moreover, $L = -\partial_x^2 + U^{(2)}(H_{0,1}(x))$ is a positive operator in $L^2(\mathbb{R})$ when it is restricted to the orthogonal complement of $H^{'}_{0,1}(x)$ in $L^2_x(\mathbb{R})$, see [26] or [47] for the proof. In conclusion, we deduce that there is a constant $C>0$ independent of $v>0$ such that

$$\left\langle -\frac{d^2}{dx^2}g(x) + U^{(2)}\left(H^{+}_{0,1}(x,t)\right)g(x),\,g(x)\right\rangle \geq C\sqrt{1-\frac{\dot{d}(t)^2}{4}}\,\|g_1(t,y)\|^2_{H^1_y(\mathbb{R})}, \tag{A.59}$$

so, from $\dot{d}(t) = v\tanh(\sqrt{2}vt)$ and identity (A.58), we deduce that there is a constant $C_1>0$ such that if $v\ll 1$, then

$$\left\langle -\frac{d^2}{dx^2}g(x) + U^{(2)}\left(H^{+}_{0,1}(x,t)\right)g(x),\,g(x)\right\rangle \geq C_1\,\|g(x)\|^2_{H^1(\mathbb{R})}. \tag{A.60}$$

Similarly, we can verify for the same constant $C_1>0$ that if $\left\langle g(x),\,\partial_x H^{-}_{-1,0}(x,t)\right\rangle = 0$ and $v\ll 1$, then

$$\left\langle -\frac{d^2}{dx^2}g(x) + U^{(2)}\left(H^{-}_{0,1}(x,t)\right)g(x),\,g(x)\right\rangle \geq C_1\,\|g(x)\|^2_{H^1(\mathbb{R})}. \tag{A.61}$$

The remaining part of the proof proceeds exactly as the proof of Lemma 2.6 of [47]. □

**Lemma A.4.5.** *There exist $C>1$, $c>0\,\delta>0$ such that if $0<v<\delta$, then we have for any $(\varphi_1,\varphi_2)\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$ that*

$$\int_{\mathbb{R}} \varphi_2^2+\partial_x\varphi_1^2+U^{(2)}\left(H_{0,1}\left(\frac{x}{\sqrt{1-v^2}}\right)\right)\varphi_1(x)^2\,dx \geq c\,\|(\varphi_1,\varphi_2)\|^2_{H^1_x\times L^2_x} - C\left\langle(\varphi_1,\varphi_2),JD_{v,0}(x)\right\rangle^2.$$

*Proof.* The proof is completely analogous to the proof of property (2) of Lemma 2.8 in the article [31]. □

## A.5 Proof of Theorem 4.6.1

We start by denoting

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and we consider for $x \in \mathbb{R}$ and $-1 < v < 1$ the following functions

$$\psi^0_{-1,0}(x,v) = J \begin{bmatrix} H'_{-1,0}(\frac{x}{\sqrt{1-v^2}}) \\ \frac{v}{1-v^2} H^{(2)}_{-1,0}\left(\frac{x}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{A.62}$$

$$\psi^1_{-1,0}(x,v) = J \begin{bmatrix} vxH'_{-1,0}\left(\frac{x}{\sqrt{1-v^2}}\right) \\ \frac{1}{\sqrt{1-v^2}} H'_{-1,0}\left(\frac{x}{\sqrt{1-v^2}}\right) + \frac{v^2 x}{1-v^2} H^{(2)}_{-1,0}\left(\frac{x}{\sqrt{1-v^2}}\right) \end{bmatrix}, \tag{A.63}$$

and we denote, for $j \in \{0,1\}$, $\psi^j_{0,1}(x,v) = \psi^j_{-1,0}(-x,-v)$.

Next, we will use Lemma 2.6 of [8].

**Lemma A.5.1.** *The functions*

$$Y^0_{-1,0}(v;x,t) = -J\psi^0_{-1,0}(x+vt,v), \tag{A.64}$$

$$Y^1_{-1,0}(v;x,t) = -J\psi^1_{-1,0}(x+vt,v) + t\sqrt{1-v^2}\,Y^0_{-1,0}(v;x+vt,t) \tag{A.65}$$

*are solutions of the linear differential system*

$$\frac{d}{dt}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = J \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \tag{A.66}$$

*and the functions*

$$Y^0_{0,1}(v;x,t) = -J\psi^0_{0,1}(x-vt,v), \tag{A.67}$$

$$Y^1_{0,1}(v;x,t) = -J\psi^1_{0,1}(x-vt,v) + t\sqrt{1-v^2}\,Y^0_{0,1}(v;x-vt,t) \tag{A.68}$$

*are solutions of the linear differential system*

$$\frac{d}{dt}\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = J \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\right) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}. \tag{A.69}$$

Now, similarly to [8], we consider the linear operator $L_{+,-}(v,t)$ defined by

$$L_{+,-}(v,t) = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) & 0 \\ 0 & 1 \end{bmatrix}. \tag{A.70}$$

We recall that

$$H_{0,1}(x) = \frac{e^{\sqrt{2}x}}{\sqrt{1+e^{2\sqrt{2}x}}},$$

and

$$\left|\frac{d^l}{dx^l} H_{0,1}(x)\right| \lesssim \min\left(e^{\sqrt{2}x}, e^{-2\sqrt{2}x}\right),$$

for any $l \in \mathbb{N}$.

From now on, we denote $\psi^j_{-1,0}(v;t,x) = \psi^j_{-1,0}(x+vt,v)$ and $\psi^j_{0,1}(v;t,x) = \psi^j_{-1,0}(x-vt,v)$ for any $j \in \{0,1\}$. Furthermore, using Lemma 3.2.1, we can verify similarly to the proof of Proposition 2.8 of [8] the following result.

**Lemma A.5.2.** *There exists $C > 0$, such that for any $0 < v < 1$, we have for all $t \in \mathbb{R}_{\geq 1}$ that*

$$\left\| \frac{\partial}{\partial t} \psi^{0}_{0,1}(v;t,x) - L_{+,-} J \psi^{0}_{0,1}(v;t,x) \right\|_{L^2_x} \leq C \exp\left( \frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}} \right), \tag{A.71}$$

$$\left\| \frac{\partial}{\partial t} \psi^{0}_{-1,0}(v;t,x) - L_{+,-} J \psi^{0}_{-1,0}(v;t,x) \right\|_{L^2_x} \leq C \exp\left( \frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}} \right), \tag{A.72}$$

$$\left\| \frac{\partial}{\partial t} \psi^{1}_{0,1}(v;t,x) - L_{+,-} J \psi^{1}_{0,1}(v;t,x) + \sqrt{1-v^2}\psi^{0}_{0,1}(v;t,x) \right\|_{L^2_x} \leq C(|t|v+1)v \exp\left( \frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}} \right), \tag{A.73}$$

$$\left\| \frac{\partial}{\partial t} \psi^{1}_{-1,0}(v;t,x) - L_{+,-} J \psi^{1}_{-1,0}(v;t,x) + \sqrt{1-v^2}\psi^{0}_{-1,0}(v;t,x) \right\|_{L^2_x} \leq C(|t|v+1)v \exp\left( \frac{-2\sqrt{2}v|t|}{\sqrt{1-v^2}} \right). \tag{A.74}$$

Next, we consider a smooth cut function $0 \leq \chi(x) \leq 1$ that satisfies

$$\chi(x) = \begin{cases} 1, \text{if } x \leq 2(1-10^{-3}), \\ 0, \text{if } x \geq 2. \end{cases}$$

From now on, for each $0 < v < 1$, we consider $p(v) = \frac{v}{2}(1-10^{-3})$ and we also denote

$$\chi_1(v;t,x) = \chi\left(\frac{x+vt}{p(v)t}\right), \; \chi_2(v;t,x) = 1 - \chi\left(\frac{x+vt}{p(v)t}\right).$$

**Lemma A.5.3.** *There is $c, \delta_0 > 0$ such that if $0 < v < \delta_0$, then*

$$Q(t,r) = \frac{1}{2}\left[\int_{\mathbb{R}} \partial_t r(t,x)^2 + \partial_x r(t,x)^2 + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) r(t,x)^2 \, dx\right] + \sum_{j=1}^{2} v \int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j \partial_t r(t,x) \partial_x r(t,x) \, dx,$$

*satisfies for any $t \geq \frac{\ln\left(\frac{1}{v}\right)}{v}$*

$$Q(t,r) \geq c \left\| (r(t), \partial_t r(t)) \right\|^2_{H^1_x \times L^2_x} - \frac{1}{c}\left[\sum_{j=0}^{1} \left\langle (r(t), \partial_t r(t)), \psi^{j}_{-1,0}(v;t) \right\rangle^2 + \left\langle (r(t), \partial_t r(t)), \psi^{j}_{0,1}(v;t) \right\rangle^2\right].$$

*Proof.* From definition of $\psi^{1}_{-1,0}$ and $\psi^{1}_{0,1}$, we can verify that there is a constant $C > 0$ such that if $v \ll 1$, then

$$\left| \left\langle r(t), H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) \right\rangle^2 \right| \leq C\left[\langle (r(t), \partial_t r(t)), \psi^{1}_{0,1}(v;t) \rangle^2 + v^2 \left\| (r(t), \partial_t r(t)) \right\|^2_{H^1_x \times L^2_x}\right], \tag{A.75}$$

$$\left| \left\langle r(t), H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right) \right\rangle^2 \right| \leq C\left[\langle (r(t), \partial_t r(t)), \psi^{1}_{-1,0}(v;t) \rangle^2 + v^2 \left\| (r(t), \partial_t r(t)) \right\|^2_{H^1_x \times L^2_x}\right]. \tag{A.76}$$

Then, using the estimates (2.13) and (A.76), the proof of Lemma A.5.3 is analogous to the demonstration of Lemma 2.3 of [26] or the proof of Lemma 2.5 in [47] or the demonstration of Lemma A.4.4 in the section Appendix $A$. $\square$

**Remark A.5.4.** *Indeed, Proposition* 2.10 *of [8] implies that for any* $0 < v < 1$*, there is* $T_v$ *and* $c_v$*, such that Lemma A.5.3 holds with* $c_v$ *in the place of* $c$ *for all* $t \geq T_v$*.*

**Lemma A.5.5.** *There is* $C > 0$*, such that, for any* $0 < v < 1$*, if* $f(t,x) \in L^{\infty}_t(\mathbb{R}; H^1_x(\mathbb{R}))$ *and* $h(t,x) \in L^{\infty}_t(\mathbb{R}_{\geq 1}; H^1_x(\mathbb{R})) \cap C^1_t(\mathbb{R}_{\geq 1}; L^2_x(\mathbb{R}))$ *is a solution of the integral equation associated to the following partial differential equation*

$$\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) h(t,x) = f(t,x),$$

*for some boundary condition* $(h(t_0), \partial_t h(t_0)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$*, then*

$$Q(t,h) = \frac{1}{2}\left[\int_{\mathbb{R}} \partial_t h(t,x)^2 + \partial_x h(t,x)^2 + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) h(t,x)^2\,dx\right] + \sum_{j=1}^{2} v \int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j \partial_t h(t,x)\partial_x h(t,x)\,dx,$$

*satisfies*

$$\left|\frac{\partial}{\partial t} Q(t,h)\right| \leq C\Bigg[\|f(t)\|_{L^2_x(\mathbb{R})} \|(h(t),\partial_t h(t))\|_{H^1_x \times L^2_x} + \|(h(t),\partial_t h(t))\|^2_{H^1_x\times L^2_x}\left(v\exp\left(\frac{-\sqrt{2}vt\,(1-10^{-3})^2}{\sqrt{1-v^2}}\right) + \frac{1}{t}\right)\Bigg]$$

*for all* $t \geq 1$.

*Proof.* First, from the equation satisfied by $h(t,x)$, we obtain that

$$\int_{\mathbb{R}}\left[\partial_t^2 h(t,x) - \partial_x^2 h(t,x) + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) h(t,x)^2\right]\partial_t h(t,x)\,dx = \int_{\mathbb{R}} f(t,x)\partial_t h(t,x)\,dx. \quad \text{(A.77)}$$

As a consequence, using integration by parts, we deduce that

$$\begin{aligned}
\frac{d}{dt}&\left[\int_{\mathbb{R}} \partial_t h(t)^2 + \partial_x h(t)^2 + U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) h(t)^2\,dx\right] \\
=&-\frac{v}{\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) h(t)^2\,dx \\
&+\frac{v}{\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right) + H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right) H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right) h(t)^2\,dx \\
&+2\int_{\mathbb{R}} f(t,x)h(t,x)\,dx.
\end{aligned} \quad \text{(A.78)}$$

Next, from the definition of $\chi_1(v;t,x)$ and $\chi_2(v;t,x)$, we can verify for each $j\in\{1,2\}$ that

$$\begin{aligned}\frac{d}{dt}\left[v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_x h(t,x)\,dx\right]=&v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\&+v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_{t,x}^2 h(t,x)\,dx\\&+O\left(\left\Vert\chi^{'}\right\Vert_{L^\infty_x(\mathbb{R})}\frac{v}{t}\left\Vert(h(t),\partial_t h(t))\right\Vert^2_{H^1_x\times L^2_x}\right),\end{aligned}$$

from which we deduce using integration by parts that

$$\begin{aligned}\frac{d}{dt}\left[v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t h(t,x)\partial_x h(t,x)\,dx\right]=&v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x r(t,x)\,dx\\&+O\left(\left\Vert\chi^{'}\right\Vert_{L^\infty_x(\mathbb{R})}\frac{1}{t}\left\Vert(h(t),\partial_t h(t))\right\Vert^2_{H^1_x\times L^2_x}\right).\end{aligned}\tag{A.79}$$

From the equation satisfied by $h(t,x)$, we have that

$$\begin{aligned}&v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\&=v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j f(t,x)\partial_x h(t,x)\,dx\\&\quad+v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j\partial_x^2 h(t,x)\partial_x h(t,x)\,dx\\&\quad-v\int_{\mathbb{R}}\chi_j(v;t,x)(-1)^j U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t,x)\partial_x h(t,x)\,dx.\end{aligned}$$

So, using integration by parts, we obtain for any $j\in\{1,\,2\}$ that

$$\begin{aligned}&2\sqrt{1-v^2}\int_{\mathbb{R}}\chi_j(v;t,x)\partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\&=\int_{\mathbb{R}}\chi_j(v;t,x)U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)h(t,x)^2\,dx\\&\quad+\int_{\mathbb{R}}\chi_j(v;t,x)U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)h(t,x)^2\,dx\\&\quad+O\left(\left\Vert\chi^{'}\right\Vert_{L^\infty_x(\mathbb{R})}\frac{1}{vt}\left\Vert(h(t),\partial_t h(t))\right\Vert^2_{H^1_x\times L^2_x}+\left\Vert f(t)\right\Vert_{L^2_x}\left\Vert(h(t),\partial_t h(t))\right\Vert_{H^1_x\times L^2_x}\right).\end{aligned}$$

From the definitions of $\chi_1(v;t,x)$ and $\chi_2(v;t,x)$, we can verify for all $t>1$ that

$$\begin{aligned}H^{'}_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)\chi_1(v;t,x)<&\sqrt{2}\exp\left(-\frac{\sqrt{2}vt(1+2\times10^{-3})}{\sqrt{1-v^2}}\right),\\H^{'}_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\chi_2(v;t,x)<&\sqrt{2}\exp\left(-\frac{\sqrt{2}vt(1-10^{-3})^2}{\sqrt{1-v^2}}\right),\end{aligned}$$

In conclusion, we obtain that

$$\begin{aligned}
&\sum_{j=1}^{2} v \int_{\mathbb{R}} \chi_j(v;t,x)(-1)^j \partial_t^2 h(t,x)\partial_x h(t,x)\,dx\\
=&\frac{v}{2\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H_{0,1}^{'}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)h(t,x)^2\,dx\\
&-\frac{v}{2\sqrt{1-v^2}}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)H_{-1,0}^{'}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)h(t,x)^2\,dx\\
&+O\left(\left\Vert\chi^{'}\right\Vert_{L^{\infty}_x(\mathbb{R})}\frac{1}{t}\left\Vert(h(t),\partial_t h(t))\right\Vert_{H^1_x\times L^2_x}^2+v\left\Vert f(t)\right\Vert_{L^2_x}\left\Vert(h(t),\partial_t h(t))\right\Vert_{H^1_x\times L^2_x}\right)\\
&+O\left(v\exp\left(-\frac{\sqrt{2}vt(1-10^{-3})^2}{(1-v^2)^{\frac{1}{2}}}\right)\Vert h(t,x)\Vert_{H^1_x(\mathbb{R})}^2\right).
\end{aligned} \tag{A.80}$$

So, using estimate (A.80), Lemma A.5.5 will follow from the sum of (A.78) and (A.79). □

**Lemma A.5.6.** *There is $C>0$, such that, for any $0<v<1$, if $f(t,x)\in L^{\infty}_t(\mathbb{R};H^1_x(\mathbb{R}))$ and $h(t,x)\in L^{\infty}_t(\mathbb{R}_{\geq 1};H^1_x(\mathbb{R}))\cap C^1_t(\mathbb{R}_{\geq 1};L^2_x(\mathbb{R}))$ is a solution of the integral equation associated to the following partial differential equation*

$$\partial_t^2 h(t,x)-\partial_x^2 h(t,x)+U^{(2)}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^2}}\right)+H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^2}}\right)\right)h(t,x)=f(t,x),$$

*for some boundary condition $(h(t_0),\partial_t h(t_0))\in H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})$, then for $\overrightarrow{h}(t)=(h(t,x),\partial_t h(t,x))$ we have*

$$\left|\frac{d}{dt}\left\langle\overrightarrow{h}(t),\,\psi^0_{-1,0}(v;t)\right\rangle\right|\leq C\left[\Vert f(t)\Vert_{L^2_x(\mathbb{R})}+\left\Vert\overrightarrow{h}(t)\right\Vert_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\exp\left(\frac{-2\sqrt{2}vt}{(1-v^2)^{\frac{1}{2}}}\right)\right],$$

$$\left|\frac{d}{dt}\left\langle\overrightarrow{h}(t),\,\psi^0_{0,1}(v;t)\right\rangle\right|\leq C\left[\Vert f(t)\Vert_{L^2_x(\mathbb{R})}+\left\Vert\overrightarrow{h}(t)\right\Vert_{H^1_x(\mathbb{R})\times L^2_x(\mathbb{R})}\exp\left(\frac{-2\sqrt{2}vt}{(1-v^2)^{\frac{1}{2}}}\right)\right],$$

*and,*

$$\begin{aligned}
&\left|\frac{d}{dt}\left\langle\overrightarrow{h}(t),\,\psi^1_{-1,0}(v;t)\right\rangle+(1-v^2)^{\frac{1}{2}}\left\langle\overrightarrow{h}(t),\,\psi^0_{-1,0}(v;t)\right\rangle\right|\\
&\qquad\leq C\left[\Vert f(t)\Vert_{L^2_x}+\left\Vert\overrightarrow{h}(t)\right\Vert_{H^1_x\times L^2_x}(|t|v+1)\exp\left(\frac{-2\sqrt{2}vt}{(1-v^2)^{\frac{1}{2}}}\right)\right],
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{d}{dt}\left\langle\overrightarrow{h}(t),\,\psi^1_{0,1}(v;t)\right\rangle+(1-v^2)^{\frac{1}{2}}\left\langle\overrightarrow{h}(t),\,\psi^0_{0,1}(v;t)\right\rangle\right|\\
&\qquad\leq C\left[\Vert f(t)\Vert_{L^2_x}+\left\Vert\overrightarrow{h}(t)\right\Vert_{H^1_x\times L^2_x}(|t|v+1)\exp\left(\frac{-2\sqrt{2}vt}{(1-v^2)^{\frac{1}{2}}}\right)\right],
\end{aligned}$$

*Proof of Lemma A.5.6.* It follows directly from the identity

$$\frac{d}{dt}\overrightarrow{h}(t)=JL_{+,-}\overrightarrow{h}(t)+\begin{bmatrix}0\\ f(t,x)\end{bmatrix}, \tag{A.81}$$

and from Lemma A.5.2. □

*Proof of Theorem 4.6.1.* For $T_0 \geq \frac{4\ln\left(\frac{1}{v}\right)}{v}$, we consider similarly to [8] the following norms denoted by

$$\|u\|_{L^2_{v,T_0}} = \sup_{t\geq T_0} e^{vt}\,\|u(t,x)\|_{L^2_x(\mathbb{R})}\,,\ \|u\|_{H^1_{v,T_0}} = \sup_{t\geq T_0} e^{vt}\left[\|u(t,x)\|^2_{H^1_x(\mathbb{R})} + \|\partial_t u(t,x)\|^2_{L^2_x(\mathbb{R})}\right]^{\frac{1}{2}}.$$

Next, from Lemma A.5.6, we can verify using the Fundamental Theorem of Calculus that there is a constant $C > 1$ such that if $v \ll 1$, then for any $t \geq T_0$ we have that

$$\left|\left\langle \overrightarrow{h}(t),\, \psi^0_{-1,0}(v;t)\right\rangle\right| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v}\right], \tag{A.82}$$

$$\left|\left\langle \overrightarrow{h}(t),\, \psi^1_{-1,0}(v;t)\right\rangle\right| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v^2} + \|h\|_{H^1_{v,T_0}}\, t e^{-\left(2\sqrt{2}+1\right)vt} + \|h\|_{H^1_{v,T_0}}\frac{e^{-\left(2\sqrt{2}+1\right)vt}}{v^2}\right], \tag{A.83}$$

and that

$$\left|\left\langle \overrightarrow{h}(t),\, \psi^0_{0,1}(v;t)\right\rangle\right| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v} + \|h\|_{H^1_{v,T_0}}\frac{e^{-(2\sqrt{2}+1)vt}}{v}\right], \tag{A.84}$$

$$\left|\left\langle \overrightarrow{h}(t),\, \psi^1_{0,1}(v;t)\right\rangle\right| \leq C\left[\|f\|_{L^2_{v,T_0}}\frac{e^{-vt}}{v^2} + \|h\|_{H^1_{v,T_0}}\, t e^{-\left(2\sqrt{2}+1\right)vt} + \|h\|_{H^1_{v,T_0}}\frac{e^{-\left(2\sqrt{2}+1\right)vt}}{v^2}\right]. \tag{A.85}$$

Also, from Lemma A.5.5, we can verify using the Fundamental Theorem of Calculus for any $t \geq T_0$ that there is a constant $K \geq 1$ such that if $v \ll 1$, then

$$\int_t^{+\infty}\left|\frac{d}{ds}Q(s,h)\right| ds \leq K\left[\frac{e^{-2vt}}{v}\|f\|_{L^2_{v,T_0}}\|h\|_{H^1_{v,T_0}} + \|h\|^2_{H^1_{v,T_0}}\left(\frac{e^{-2vt}}{vt} + e^{-t(2v+\sqrt{2}v(1-10^{-3})^2)}\right)\right] \tag{A.86}$$

In conclusion, similarly Step 1 in the proof of Lemma 3.1 of [8], we deduce using the estimates (A.82), (A.84), (A.83), (A.85) with Lemma A.5.3 and the estimate above (A.86) that there exists a new constant $C > 1$ such that for any $t \geq T_0$ and $v \ll 1$ we have

$$\|h\|^2_{H^1_{v,T_0}} \leq \frac{C}{v^4}\|f\|^2_{L^2_{v,T_0}}. \tag{A.87}$$

The fact that the constant $C$ in (A.87) is independent of $v$ follows from $T_0 \geq \frac{4\ln\left(\frac{1}{v}\right)}{v}$, which implies that

$$\frac{e^{-2vt}}{v^4} + \frac{e^{-2vt}}{vt} \ll v^4.$$

We also observe that if $(g_1(t,x), \partial_t g_1(t,x))$ and $(g_2(t,x), \partial_t g_2(t,x))$ are in the space $(g(t), \partial_t g(t)) \in H^1_x(\mathbb{R}) \times L^2_x(\mathbb{R})$ such that

$$\|(g(t), \partial_t g(t))\|_{L^\infty([T_0,+\infty], H^1_x \times L^2_x)} \leq 1, \tag{A.88}$$

then, since $U\in C^{\infty}$, we can verify that the following function

$$\begin{aligned} &N(v,\overrightarrow{g})(t,x)\\ =&U^{'}\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^{2}}}\right)+H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^{2}}}\right)+g(t,x)\right)-U^{'}\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^{2}}}\right)\right)\\ &-U^{'}\left(H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^{2}}}\right)\right)-U^{(2)}\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^{2}}}\right)+H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^{2}}}\right)\right)g(t,x) \end{aligned} \tag{A.89}$$

satisfies for some new constant $C\geq 1$ and any $v\ll 1$

$$\|N(v,\overrightarrow{g_{1}}(t))-N(v,\overrightarrow{g_{2}}(t))\|_{H^{1}_{x}}\leq C\left[\|g_{1}(t)\|_{H^{1}_{x}}+\|g_{2}(t)\|_{H^{1}_{x}}\right]\|g_{1}(t)-g_{2}(t)\|_{H^{1}_{x}},$$

which implies the following estimate given by

$$\|N(v,\overrightarrow{g_{1}}(t))-N(v,\overrightarrow{g_{2}}(t))\|_{H^{1}_{v,T_{0}}}\leq Ce^{-vt}\left[\|g_{1}\|_{H^{1}_{v,T_{0}}}+\|g_{2}\|_{H^{1}_{v,T_{0}}}\right]\|g_{1}-g_{2}\|_{H^{1}_{v,T_{0}}}. \tag{A.90}$$

In conclusion, by repeating the argument of the proof of proposition 3.6 of [8], we can verify using the Lipschtiz estimate of (A.90) and estimate (A.87) that if $T_{0}\geq\frac{4\ln\left(\frac{1}{v}\right)}{v}$ and $v\ll 1$, then there exists a map

$$S:\{u\in H^{1}_{v,T_{0}}|\,\|u\|_{H^{1}_{v,T_{0}}}\leq 1\}\to\{u\in H^{1}_{v,T_{0}}|\,\|u\|_{H^{1}_{v,T_{0}}}\leq 1\} \tag{A.91}$$

such that $\mu(t,x)=S(u)(t,x)$ is the unique solution of the equation

$$\partial_{t}^{2}\mu(t,x)-\partial_{x}^{2}\mu(t,x)+U^{(2)}\left(H_{-1,0}\left(\frac{x+vt}{\sqrt{1-v^{2}}}\right)+H_{0,1}\left(\frac{x-vt}{\sqrt{1-v^{2}}}\right)\right)\mu(t,x)=N(v,\overrightarrow{\mu})(t,x), \tag{A.92}$$

such that $\mu\in H^{1}_{v,T_{0}}$. Indeed, the uniqueness is guaranteed by estimate (A.87) and from estimates (A.87) and (A.90) we have that the map $S$ is a contraction in the set

$$B=\{u\in H^{1}_{v,T_{0}}|\,\|u\|_{H^{1}_{v,T_{0}}}\leq 1\},$$

and so, Theorem 4.6.1 follows similarly to the proof of Proposition 3.6 of [8] by using the Banach's fixed point theorem. $\square$

# Bibliography


[1] Miguel A. Alejo, Claudio Muñoz, and José M. Palacios. On the asymptotic stability of the sine-Gordon kink in the energy space. 2020. arXiv:2003.09358.

[2] Fabrice Bethuel, Giandomenico Orlandi, and Didier Smets. Dynamics of multiple degree Ginzburg-Landau vortices. *Comptes Rendus Mathematique*, 342:837–842, 2006.

[3] Alan R. Bishop and Toni Schneider. *Solitons and Condensed Matter Physics*. Springer-Verlag, 1978.

[4] Vladimir Buslaev and Catherine Sulem. On asymptotic stability of solitary waves for nonlinear Schrödinger equations. *Annales de l'Institut Henri Poincaré Analyse non linéaire*, 20:419–475, 2003.

[5] Vladimir S. Buslaev and Galina Perelman. On the stability of solitary waves for nonlinear Schrödinger equation. *Nonlinear Evolution Equations*, 164:75–98, 1995.

[6] Fabrice Béthuel, Philippe Gravejat, Jean-Claude Saut, and Didier Smets. Orbital stability of the black soliton to the Gross-Pitaevskii equation. *Indiana University Mathematics Journal*, 57(6):2611–2642, 2008.

[7] Gong Chen and Jacek Jendrej. *Trans. Amer. Math. Soc.*, 372:7461–7496, 2019.

[8] Gong Chen and Jacek Jendrej. Kink networks for scalar fields in dimension $1 + 1$. *Nonlinear Analysis*, 215, 2022.

[9] Peter Clarkson and Mark J. Ablowitz. *Solitons, nonlinear evolution equations and inverse scattering*, volume 149 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1991.

[10] Earl Coddington and Norman Levinson. *Theory of Ordinary Differential Equations*. McGraw Hill Education, 1955.

[11] James E. Colliander and Robert L. Jerrard. Vortex dynamics for the Ginzburg-Landau-Schrödinger equation. *International Mathematical Reseach Notices*, 1998:333–358, 1998.

[12] Vianney Combet. Multi-soliton solutions for the supercritical $gKdV$ equations. *Communications in Partial Differential Equations*, 36(3):380–419, 2010.

[13] Jean-Marc Delort and Nader Masmoudi. On the stability of kink solutions of the $\phi^4$ model in $1+1$ space time dimensions. 2021. Hal:2862414.

[14] Patrick Dorey, Kieran Mersh, Tomasz Romanczukiewicz, and Yasha Shnir. Kink-antikink collisions in the $\phi^6$ model. *Physical Review Letters*, 107, 2011.

[15] Maciej Dunajski and Nicholas S. Manton. Reduced dynamics of Ward solitons. *Nonlinearity*, 18(4):1677–1689, 2005.

[16] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate studies in mathematics*. AMS.

[17] Vakhid A. Gani, Alexander E. Kudryavtsev, and Mariya A. Lizunova. Kink interactions in the (1+1)-dimensional $\phi^6$ model. *Physical Review D: Particles and Fields*, 89, 2014.

[18] Dan-Andrei Geba and Manoussos G. Grillakis. *An introduction to the theory of wave maps and related geometric problems*. World Scientific, 2016.

[19] Pierre Germain and Fabio Pusateri. Quadratic Klein-Gordon equations with a potential in one dimension. *Forum of Mathematics, Pi*, 10:1–172, 2022.

[20] John T. Giblin, Lam Hui, Eugene A. Lim, and I-Sheng Yang. How to run through walls: Dynamics of bubble and soliton collisions. *Physical Review D: Particles and Fields*, 82, 2010.

[21] Konstantin A. Gorshkov and Lev A. Ostrovsky. Interactions of solitons in nonintegrable systems: Direct perturbation method and applications. *Physica D: Nonlinear Phenomena*, 3:428–438, 1981.

[22] Stephen Gustafson and Israel M. Sigal. Effective dynamics of magnetic vortices. *Advances in Mathematics*, 199:448–498, 2006.

[23] Stephen W. Hawking, Ian G. Moss, and John M. Stewart. Bubble collisions in the very early universe. *Physical Review D*, 26:2681–2713, 1982.

[24] Daniel B. Henry, José Fernando Perez, and Walter Wreszinski. Stability theory for solitary-wave solutions of scalar field equations. *Communications in mathematical physics*, 85:351–361, 1982.

[25] Ryogo Hirota. Exact solution of the korteweg-de vries equation for multiple collisions of solitons. *Physical Review Letters*, 27(4):1192–1194, 1971.

[26] Jacek Jendrej, Michał Kowalczyk, and Andrew Lawrie. Dynamics of strongly interacting kink-antikink pairs for scalar fields on a line. *Duke Mathematical Journal*, 2019.

[27] Robert Leon Jerrard and Halil Mete Soner. Dynamics of Ginzburg-Landau vortices. *Archive for Rational Mechanics and Analysis*, 142:99–125, 1998.

[28] Robert Leon Jerrard and Daniel Spirn. Refined jacobian estimates and Gross–Pitaevsky vortex dynamics. *Archive for Rational Mechanics and Analysis*, 190:425–475, 2008.

[29] Michał Kowalczyk, Yvan Martel, and Claudio Muñoz. Kink dynamics in the $\phi^4$ model: asymptotic stability for odd perturbations in the energy space. *Journal of the American Mathematical Society*, 30:769–798, 2017.

[30] Michał Kowalczyk, Yvan Martel, and Claudio Muñoz. Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes. *Journal of the European Mathematical Society*, 24(6):2133–2167, 2021.

[31] Michał Kowalczyk, Yvan Martel, Claudio Muñoz, and Hanne Van Den Bosch. A sufficient condition for asymptotic stability of kinks in general (1+1)-scalar field models. *Annals of PDE*, 7:1–98, 2021.

[32] Joachim Krieger, Kenji Nakanishi, and Wilhelm Schlag. Global dynamics above the ground state energy for the one-dimensional NLKG equation. *Mathematische Zeitschrift*, 272:297–316, 2012.

[33] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Communications on Pure and Applied Mathematics*, 21:467–490, 1968.

[34] Max A. Lohe. Soliton structures in $p(\phi)_2$. *Physical Review D*, 20(12), 1979.

[35] Nicholas Manton. Forces between kinks and antikinks with long-range tails. *Journal of Physics A: Mathematical and Theoretical*, 52(6), 2019.

[36] Nicholas Manton and Paul Sutcliffe. *Topological Solitons*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004.

[37] Nicholas S. Manton and James Martin Speight. Asymptotic interactions of critically coupled vortices. *Communications in Mathematical Physics*, 236:535–555, 2003.

[38] Yvan Martel. Asymptotic $N$-Soliton-like-Solutions of the subcritical and critical generalized Korteweg-de Vries equations. *American Journal of Mathematics*, 127(5):1103–1140, 2005.

[39] Yvan Martel and Frank Merle. Stability of two soliton colision for non-integrable $gKdV$ equations. *Communications in Mathematical Physics*, 286:39–79, 2009.

[40] Yvan Martel and Frank Merle. Description of two soliton collision for the quartic $gKdV$ equation. *Annals of Mathematics*, 174:757–857, 2011.

[41] Yvan Martel and Frank Merle. Inelastic interaction of nearly equal solitons for the quartic $gKdV$ equation. *Inventiones Mathematicae*, 183(3):563–648, 2011.

[42] Yvan Martel and Frank Merle. Inelasticity of soliton collisions for the 5D energy critical wave equation. *Inventiones Mathematicae*, 214(3):1267–1363, 2018.

[43] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. *Communications in Mathematical Physics*, 231:347–373, 2002.

[44] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability in $H^1$ of the sum of $K$ solitary waves for some nonlinear Schrödinger equations. *Duke Mathematical Journal*, 133(3):405–466, 2006.

[45] Robert M. Miura. The korteweg–de vries equation: a survey of results. *SIAM Review*, 18:412–459, 1976.

[46] Abdon Moutinho. Approximate kink-kink solutions for the $\phi^6$ model in the low-speed limit. 2022. arXiv:2211.09714 .

[47] Abdon Moutinho. Dynamics of two interacting kinks for the $\phi^6$ model. 2022. arXiv:2205.04301(Accepted in Comm. in Math. Phys.).

[48] Abdon Moutinho. On the collision problem of two kinks for the $\phi^6$ model with low speed. 2022. arXiv:2211.09749 .

[49] Claudio Muñoz. On the inelastic 2-soliton collision for generalized *KdV* equations. *International Mathematical Research Notices*, 9:1624–1719, 2010.

[50] Claudio Muñoz. Inelastic character of solitons of slowly varying *gKdV* equations. *Communications in Mathematical Physics*, 314(3):817–852, 2012.

[51] Claudio Muñoz. The Gardner equation and the stability of multi-kink solutions of the *mKdV* equation. *Discrete and Continuous Dynamical Systems*, 16(7):3811–3843, 2016.

[52] Yurii N. Ovchinnikov and Israel M. Sigal. The Ginzburg-Landau equation III. vortices dynamics. *Nonlinearity*, 11(5):1277–1294, 1998.

[53] Galina Perelman. Two soliton collision for nonlinear Schrödinger equations in dimension 1. *Annales de l'IHP Anal. Non Linéaire*, 28(3):357–384, 2011.

[54] Pierre Raphaël and Jeremy Szeftel. Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical *NLS*. *Journal of the American Mathematical Society*, 24(2):471–546, 2010.

[55] Etienne Sandier and Sylvia Serfaty. Gamma-convergence of gradient flows and applications to Ginzburg-Landau vortex dynamics. *Communications on Pure and Applied Mathematics*, 57:1627 – 1672, 2004.

[56] Wilhem Schlag and Jonas Lührmann. Asymptotic stability of the sine-Gordon kink under odd perturbations. 2021. arXiv:2106.09605.

[57] Jalal Shatah and Michael Struwe. *Geometric Wave Equations*, volume 2 of *Courant Lecture Notes*. AMS, 2000.

[58] David Stuart. Dynamics of Abelian Higgs vortices in the near Bogolmony regime regime. *Communications in Mathematical Physics*, 159:51–91, 1994.

[59] David Stuart. The geodesic approximation for Yang-Mills-Higgs equations. *Communications in Mathematical Physics*, 166:149–190, 1994.

[60] David Stuart. Analysis of the adiabatic limit for solitons in classical field theory. *Proceedings of the Royal Society A*, 463:2753–2781, 2007.

[61] Terence Tao. *Nonlinear dispersive equations: local and global analysis.* AMS, 2006.

[62] Alexander Vilekin and Paul Shellard. *Cosmic Strings and Other Topological Defects.* Cambridge University Press, 1994.