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Théorèmes combinatoires et probabilistes sur certaines familles de polytopes.

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RÉSUMÉ

Cette thèse porte sur plusieurs aspects combinatoires et probabilistes des polytopes convexes et autour d'une famille particulière de polytopes, les zonotopes.
La première partie permet de calculer l'équivalent asymptotique du nombre de zonotopes entiers inscrits dans un hypercube de côté n et de dimension d, pour d fixé et pour n tendant vers l'infini. Les zonotopes entiers peuvent être vus comme une généralisation multidimensionnelle des partitions d'entiers. L'équivalent trouvé fait intervenir les zéros non triviaux de la fonction ζ de Riemann. Sont aussi calculés les premiers moments de paramètres tels que la longueur d'une arête aléatoire ou le diamètre du graphe d'un zonotope pour la distribution uniforme.
Dans la deuxième partie, nous considérons les zonotopes entiers dont les générateurs sont dans un cône et dont les extrémités sont fixées. En éloignant les extrémités l'une de l'autre, un tel zonotope aléatoire uniforme satisfait, en tout point tangent à un hyperplan, non seulement un théorème central limite mais aussi un théorème de limite local. En dimension 2, nous prouvons un théorème de limite fonctionnel pour les fluctuations de la frontière d'un polygone aléatoire uniforme dans un carré dont la longueur du côté tend vers l'infini. Cette limite fait apparaître un terme de dérive identifié comme une courbe cubique.
Finalement, la troisième partie porte sur les polytopes k-équiprojectifs, c'est-à-dire les polytopes de dimension 3 dont toute projection orthogonale non parallèle à une face est un k-gone. Nous donnons une caractérisation des polytopes équiprojectifs par leur cônes normaux. Cette caractérisation permet de construire des polytopes équiprojectifs par somme de Minkowski et de calculer le paramètre k. Grâce à cela et à des bornes sur le nombre de types combinatoires des zonotopes, nous donnons une borne inférieure sur le nombre de polytopes k-équiprojectifs.

Mots clés : Zonotopes, somme de Minkowski, polytopes entiers, énumération, combinatoire analytique, théorème de Donsker, pont Brownien, polytopes équiprojectifs, types d'ordres.

This thesis deals with several combinatorial and probabilistic aspects of convex polytopes, giving a particular focus on a family of polytopes, the zonotopes.
The first part is dedicated to calculating the asymptotic equivalent of the number of integer zonotopes inscribed in a hypercube of side n and dimension d, for fixed d and for n going. Lattice zonotopes can be seen as a multidimensional generalization of integer partitions. The equivalent found involves the non-trivial zeros of the Riemann ζ function. Also, we compute the first moments of parameters such as the length of a random edge or the diameter of the graph of a zonotope for the uniform distribution.
In the second part, we consider the entire zonotopes whose generators are in a cone and whose extremities are fixed. By moving the endpoints away from each other, such a uniform random zonotope satisfies, at any point tangent to a hyperplane, not only a central limit theorem but also a local limit theorem. In dimension 2, we prove a functional limit theorem for the boundary fluctuations of a uniform random polygon in a square whose side length tends to infinity. This limit shows a drift term that is a cubic curve.
Finally, the third part deals with k-equiprojective polytopes, which are polytopes of dimension 3 of which any orthogonal projection not parallel to a face is a k-gon. We prove a characterization of equiprojective polytopes by their normal cones. This characterization makes it possible to construct equiprojective polytopes by Minkowski sum, and to calculate the parameter k. Thanks to this and to bounds on the number of combinatorial types of zonotopes, we give a lower bound on the number of k-equiprojective polytopes.

Keywords: Zonotopes, Minkowski sum, lattice polytopes, enumeration, analytic combinatorics, Donsker's theorem, Brownian bridge, equiprojective polytopes, order types.

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CHAPITRE 1

MA THÈSE EN BREF

Les polytopes convexes sont des objets géométriques élémentaires, pourtant leur étude combinatoire est pavée de conjectures plus célèbres les unes que les autres à cause de leur apparente simplicité. Cette thèse porte sur des questions combinatoires sur les polytopes convexes qui peuvent être regroupées en deux parties : l'une porte sur l'étude des zonotopes entiers (les chapitres 4 et 5) et l'autre sur la caractérisation des polytopes équiprojectifs (le chapitre 6). Ces deux parties ont en commun la présence des zonotopes et leur énumération, ce qui sera le fil rouge de cette thèse.

Dans l'espace complet normé $\mathbb{R}^d$ (dans toute cette thèse, la lettre d désignera la dimension) , un zonotope entier est défini comme la somme de Minkowski de segments de $\mathbb{R}^d$ dont les extrémités sont des points de coordonnées entières. Explicitement, pour $k \in \mathbb{N}$, si $s_1, ..., s_k$ sont de tels segments, le zonotope Z défini à partir de $s_1, ..., s_k$ est

$$Z = \left\{ \sum_{i=1}^{k} x_i, \quad (x_1, ..., x_k) \in s_1 \times ... \times s_k \right\}. \tag{1.0.1}$$

Z peut être vu comme une somme de Minkowski de vecteurs, appelés ses générateurs, à une translation près. En 1980, Vladimir Arnold pose la question d'énumérer les polytopes entiers, c'est-à-dire les polytopes dont les sommets ont des coordonnées entières, dans $\mathbb{R}^d$ à volume donné, à translation et rotation près [7]. La limite asymptotique de ce nombre n'est toujours pas connue à ce jour, ni même sa limite logarithmique, excepté en deux dimensions. En suivant les résultats et reformulations successives de cette question, nous nous sommes interrogés sur l'énumération des zonotopes entiers dans un hypercube de dimension d et de côté n, pour un entier n qui tend vers $+\infty$. En étudiant la fonction génératrice des zonotopes entiers et en utilisant la méthode du col en dimension d, nous avons obtenu [33] :

Théorème 1.0.1. *On pose $\boldsymbol{n} = (n, ..., n) \in \mathbb{R}^d$. Soit $z_d(\boldsymbol{n})$ le nombre de zonotopes entiers* *inscrits dans l'hypercube* $[0,n]^d$*. Il existe un polynôme* Q_d *de degré* d*, des constantes* α_d *et* β_d *et une fonction* $I_d : (0,1) \to \mathbb{R}$ *qui ne dépendent que de la dimension* d*, définis explicitement dans le Théorème 4.3.1, tels que*

$$z_d(\boldsymbol{n}) \sim \alpha_d n^{\beta_d} \exp\left(Q_d\left(n^{\frac{1}{d+1}}\right) + I_d\left(n^{\frac{1}{d+1}}\right)\right), \quad \text{quand } n \to +\infty. \tag{1.0.2}$$

Cet équivalent fait intervenir les zéros non-triviaux de la fonction ζ de Riemann, conséquence des liens avec la théorie des nombres puisque les sommets des objets considérés sont des éléments de $\mathbb{Z}^d$. Ce résultat nous permet également d'obtenir des statistiques sur des paramètres des zonotopes, tels le diamètre du graphe du zonotope et la longueur d'un côté.

Théorème 1.0.2. *Soit* Z_n *un zonotope entier aléatoire uniforme inscrit dans l'hypercube* $[0,n]^d$ *et* D_{Z_n} *le diamètre du graphe de* Z_n*. Alors*

$$\mathbb{E}\left[D_{Z_n}\right] = \frac{\sqrt[d+1]{\zeta(d+1)/\zeta(d)}}{\zeta(d+1)} n^{\frac{d}{d+1}}(1+o(1)), \quad \text{quand } n \to +\infty. \tag{1.0.3}$$

De plus, notons $\{g_1^n, \ldots, g_k^n\}$ *l'ensemble des générateurs de* Z_n *non colinéaires deux à deux, alors nous avons*

$$\mathbb{E}\left[||g_1^n||_1\right] = \frac{n^{\frac{1}{d+1}}}{\sqrt[d+1]{\zeta(d+1)/\zeta(d)}}(1+o(1)), \quad \text{quand } n \to +\infty. \tag{1.0.4}$$

$$\operatorname{var}\left[||g_1^n||_1\right] = \left(\frac{n^{\frac{1}{d+1}}}{\sqrt[d+1]{\zeta(d+1)/\zeta(d)}}\right)^2 (1+o(1)), \quad \text{quand } n \to +\infty. \tag{1.0.5}$$

Pour étudier les zonotopes entiers à translation près, nous nous plaçons dans un cône pointé saillant convexe $\mathcal{C}$ et nous nous réduisons à étudier les zonotopes obtenus par somme de Minkowski de vecteurs $\mathcal{C}$ dont la somme (vectorielle) est le point $n\boldsymbol{k}$ dans $\mathcal{C} \cap \mathbb{Z}^d$. Un tel zonotope Z tiré aléatoirement uniformément converge en loi vers une forme limite après renormalisation, ce qui a été récemment prouvé par Bárány, Bureaux et Lund [14]. Dans une publication en collaboration avec Philippe Marchal [34], nous avons démontré que chaque point de la frontière de Z tangent à un hyperplan H satisfait un théorème central limite. Cela implique notamment le théorème suivant :

Théorème 1.0.3. *Soit* Z *un zonotope entier uniformément aléatoire parmi les zonotopes dont les générateurs sont des vecteurs de* $\mathbb{N}^d$ *dont la somme fait* $(n, n, \ldots, n)$*. Si* $\boldsymbol{u}$ *est un vecteur non nul de* $\mathbb{R}^d$ *et* $\mathbf{X}_{\boldsymbol{u}}^n$ *le point de la frontière de* Z *tangent à l'hyperplan normal à* $\boldsymbol{u}$*, alors il existe une matrice symétrique* $\Gamma_{\boldsymbol{u}}$ *explicite telle que*

$$\left(n^{\frac{d+2}{d+1}}\Gamma_{\boldsymbol{u}}\right)^{-1/2}(\mathbf{X}_{\boldsymbol{u}}^n - \mathbb{E}[\mathbf{X}_{\boldsymbol{u}}^n]) \underset{n\to+\infty}{\overset{(d)}{\to}} \mathcal{N}, \tag{1.0.6}$$

où $\mathcal{N}$ est une variable gaussienne centrée réduite de dimension d.

En dimension 2, les polygones peuvent être entièrement caractérisés par les zonogones. Bárány et Vershik ont montré que les polygones entiers contenus dans le carré de côté n, renormalisés, tendent vers une forme limite $\mathcal{P}_0$ [10, 151]. Soit un polygone entier $\mathcal{P}_n$ contenu dans le carré $[-n, n]^2$ tiré aléatoirement de manière uniforme. On définit les points A_n, B_n, C_n et D_n comme le milieu des segments respectivement le plus au sud, à l'est, au nord et à l'ouest de la frontière de $\mathcal{P}$. De même, on définit S_n, E_n, N_n et W_n les milieux respectifs des côtés sud, est, nord et ouest du carré $[-n, n]^2$ (e.g. le point N_n a pour coordonnées $(0, n)$). Nous avons explicité la limite de $n^{-2/3}(\mathcal{P}_n - n\mathcal{P}_0)$:

Théorème 1.0.4. *Avec les notation ci-dessus et en notant $x(X)$, respectivement $y(X)$, la première, resp. la deuxième coordonnée du point X :*

- *Le quadruplet $n^{-2/3}(y(A_n - S_n), x(B_n - E_n), y(C_n - N_n), x(D_n - W_n))$ converge en probabilité vers la masse de Dirac $\delta_{(0,0,0,0)}$.*
- *Le quadruplet $n^{-2/3}(x(A_n - S_n), y(B_n - E_n), -x(C_n - N_n), -y(D_n - W_n))$ converge en distribution vers une variable aléatoire gaussienne (R, S, T, U) de densité*

$$C_1 \exp\left(-\frac{1}{18}[(r-s)^2 + (s-t)^2 + (t-u)^2 + (u-r)^2] - \frac{1}{3}[r^2 + s^2 + t^2 + u^2]\right) \tag{1.0.7}$$

  où C_1 est une constante de renormalisation.
- *Pour tout $t \in [0, 1]$, soit $\mathbf{X}_n(t)$ le point de la frontière de $\mathcal{P}_n$ tangent au vecteur $(t, 1-t)$ dont la coordonnée y est la plus petite. On note sa moyenne $\overline{\mathbf{X}}_n(t) = \mathbb{E}(\mathbf{X}_n(t))$. Pour tous $r, s \in \mathbb{R}$, on désigne par $\mathcal{E}_n(r, s)$ l'évènement*

$$\mathcal{E}_n(r, s) = \{\lfloor n^{-2/3}x(A_n - S_n)\rfloor = r,\ \lfloor n^{-2/3}y(B_n - E_n)\rfloor = s\}, \tag{1.0.8}$$

  c'est-à-dire l'évènement dans lequel $n^{-2/3}(B_n - A_n) = (-r, s) + \epsilon_n$. Alors il existe une famille continue de matrices non singulières $(Q(t), 0 \leq t \leq 1)$ telle que nous ayons la convergence de processus conditionnés suivante :

$$\left(n^{-2/3}(\mathbf{X}_n(t) - \overline{\mathbf{X}}_n(t)) \,\middle|\, \mathcal{E}_n(r, s)),\ t \in [0, 1]\right) \overset{(d)}{\to} \begin{pmatrix} r \\ 0 \end{pmatrix} + (\boldsymbol{\mu}_{r,s}(t) + Q(t)\beta_t,\ t \in [0, 1]) \tag{1.0.9}$$

où (β_t) *est un pont brownien de dimension 2 et* $\boldsymbol{\mu}_{r,s}$ *est la courbe paramétrée suivante*

$$\boldsymbol{\mu}_{r,s}(t) = \begin{pmatrix} -2t(t-1)^2 & t(2t^2-5t+4) \\ t^2(2t-1) & -2t^2(t-1) \end{pmatrix} \begin{pmatrix} s \\ -r \end{pmatrix} \tag{1.0.10}$$

Le terme de dérive $\boldsymbol{\mu}_{r,s}$ est courbe algébrique cubique, ce que nous n'avons pas vu auparavant dans la littérature des convergences de processus stochastiques.

La deuxième partie de nos travaux porte sur les polytopes équiprojectifs. En dimension 3, pour un entier positif k quelconque, un polytope convexe P est dit k-équiprojectif si toute projection orthogonale de P sur un plan est un k-gone, sauf pour les projections sur des plans orthogonaux à une face de P. Dans sa célèbre série de problèmes de géométrie de 1968, Geoffrey Colin Shephard dans [138, 137] demande s'il est possible de trouver une méthode pour construire tous les polytopes équiprojectifs. Avec Lionel Pournin [35], nous avons établi une borne inférieure sur le nombre de types combinatoires différents de polytopes équiprojectifs (au sens où deux polytopes ont le même type combinatoire si leurs treillis des faces sont isomorphes). La démonstration de cette borne s'articule en deux étapes. D'abord, nous avons calculé une borne inférieure du nombre de types combinatoires de zonotope de dimension 3, obtenue en faisant le lien entre zonotope et matroïde orienté réalisable d'une part et matroïde orienté réalisable et type d'ordre d'autre part. Cette borne inférieure sur le nombre de types d'ordres avait été calculée par Jacob E. Goodman et Richard Pollack [76], puis reprise par Noga Alon [4]. On démontre aisément qu'un zonotope de dimension 3 obtenu à partir de k générateurs non colinéaires deux à deux est un polytope $2k$-équiprojectif. La deuxième étape est de trouver des constructions de polytopes équiprojectifs, pour étendre la borne inférieure du nombre de zonotopes aux cas où k est impair. Nous obtenons :

Théorème 1.0.5. *Le nombre de type combinatoire différents de polytopes k-équiprojectifs est au minimum*

$$k^{k\left(\frac{3}{2}+O\left(\frac{1}{\log k}\right)\right)}, \quad \text{quand } k \to +\infty. \tag{1.0.11}$$

Pour démontrer ce résultat pour k impair, nous avons étudié les constructions de polytopes équiprojectifs par somme de Minkowski de deux polytopes convexes. Le théorème qui nous permet d'obtenir des constructions générales est une caractérisation des polytopes équiprojectifs basée sur les cônes agrégés dans une direction. Soit P un polytope de dimension 3 et $\boldsymbol{u}$ un vecteur non nul de $\mathbb{R}^3$. Le cône agrégé de P dans la direction de $\boldsymbol{u}$ est l'union des cônes normaux de dimension 2 de P qui sont contenus dans l'hyperplan linéaire normal à u. Cette caracterisation est construite à partir de la caractérisation de Masud Hasan et Anna Lubiw [84]. Nous avons montré :

Théorème 1.0.6. *Un polytope P de dimension 3 est équiprojectif si et seulement si, pour toute arête e de P, dont la direction est normale à l'hyperplan linéaire* $\mathcal{H}_e$*, une des deux conditions suivante est vérifiée :*

- *le cône agrégé de P dans la direction de e est égal à $\mathcal{H}_e$.*
- *le cône agrégé de P dans la direction de e et l'intérieur relatif de l'opposé de ce cône forment une partition de $\mathcal{H}_e$.*

Ce résultat nous permet d'établir des conditions sur les polytopes P et Q pour que la somme de Minkowski $P + Q$ soit équiprojective. Cette caractérisation permet de préciser quelques questions ouvertes intermédiaires dont la résolution permettraient de résoudre le problème de Shephard.

Chapitre 2

Introduction

Dans ce chapitre, nous allons introduire les notions géométriques et combinatoires abordées dans les différentes parties de cette thèse.

Les objets centraux sont les polytopes convexes, c'est pourquoi la première partie de ce chapitre sera consacrée à leur présentation. Ces objets sont un pilier fondamental de la géométrie et peu nombreux sont les domaines des mathématiques où les polytopes convexes n'ont aucune application. Il nous semblait important d'introduire les polytopes de la manière la plus large possible, pour donner au lecteur, sans prétendre à un panorama exhaustif, une vision de ces objets plus large que leur stricte apparition dans nos travaux.

Le deuxième objet principal de ce manuscrit est le zonotope. Pour l'introduire, nous avons choisi de commencer par les partitions, pour définir les zonotopes comme une généralisation multidimensionnelle des partitions d'entier. Nous retracerons l'histoire de l'étude asymptotique du dénombrement des partitions, pour faire écho au résultat d'énumération de zonotopes du chapitre 4. Ensuite, après avoir donné plusieurs définitions équivalentes des zonotopes, nous rappellerons le lien entre zonotopes et matroïdes orientés, pour appuyer les résultats donnés dans le chapitre 5.

En plus de ces deux objets, il est important de remettre en contexte les résultats des chapitres 4 et 5. Nous finirons donc par rappeler les résultats énumératifs sur les polytopes entiers convexes depuis la question liminaire de Vladimir Arnold en 1980.

2.1 Les polytopes

Cette section donne une description générale des polytopes convexes et quelques résultats notoires dans ce domaine. Nous allons donc commencer par donner la majorité du vocabulaire et des notations géométriques dont nous aurons besoin, puis nous donnerons successivement un bref aperçu de l'étude de la combinatoire des faces des polytopes convexes, des polytopes entiers et de leurs applications et de l'utilisation des polytopes en optimisation linéaire. Ces parties sont inégalement reliées aux sujets des chapitres suivants, mais ce tour d'horizon permettra au lecteur de contextualiser les résultats qui suivront.

2.1.1 Généralités

Les polygones et les polyèdres ont une histoire pluri-millénaire. Leur origine est indissociable de la géométrie et remonte à au moins 4500 ans [93]. Nous restent de l'Antiquité, entre autres les pyramides égyptiennes, les théorèmes des géomètres grecs Pythagore et Thalès, les solides de Platon, etc. Parallèlement, pour la majorité des gens les polygones ont aussi une longue histoire personnelle ; ils ont sûrement marqué les premiers souvenirs de mathématiques de chaque écolier et sont même présents dans les dessins des tout-petits. Ce qu'un géomètre pourrait appeler pompeusement simplexe de dimension 2 (le triangle) est la représentation universelle des robes et des dents pour tout enfant de moins de 5 ans.

Pour généraliser la notion de polygone en dimension d, nous allons d'abord rappeler quelques définitions. Dans $\mathbb{R}^d$, l'***enveloppe convexe*** d'un ensemble fini S est le plus petit ensemble convexe $\text{conv}(S)$ qui contienne S, c'est-à-dire le plus petit ensemble qui contient S tel que pour tout élément x et y de $\text{conv}(S)$, le segment $[x, y] = \{tx + (1-t)y, t \in [0,1]\}$ soit dans $\text{conv}(S)$. De manière équivalente, on la définit aussi comme l'ensemble des barycentres à coefficient positif de S, c'est-à-dire:

$$\text{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i, x_i \in S, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{2.1.1}$$

Nous rappelons aussi que dans $\mathbb{R}^d$, un ***demi-espace affine*** est l'ensemble des vecteurs x qui vérifient $g(x) \geq 0$ pour une certaine forme linéaire affine $g : \mathbb{R}^d \to \mathbb{R}$, définie par

$$g(x_1, ..., x_d) = a_0 + a_1 x_1 + ... + a_d x_d, \text{ pour } a_0, ..., a_d \in \mathbb{R}. \tag{2.1.2}$$

telle que $a_1, ..., a_d$ ne soient pas tous nuls. L'hyperplan affine de forme linéaire affine g sépare donc $\mathbb{R}^d$ en deux demi-espaces affines. Nous pouvons maintenant donner les deux définitions équivalentes d'un polytope dans $\mathbb{R}^d$:

Définition 2.1.1. *Un* ***polytope (convexe)*** *est un ensemble de $\mathbb{R}^d$ défini par une de ces deux définitions équivalentes:*

- *l'enveloppe convexe d'un nombre fini de vecteurs de $\mathbb{R}^d$.*
- *L'intersection d'un nombre fini de demi-espaces affines tels que cet ensemble est borné.*

A l'instar des principales références du domaine ([78, 155]), nous écrirons simplement polytope au lieu de polytope convexe, les polytopes non convexes n'étant pas abordés dans ce manuscrit. Un polytope P est de dimension d et on le notera un ***d-polytope*** dans la suite, si l'espace affine engendré par P est de dimension d. Cet espace affine engendré par P est défini de manière rigoureuse par

$$\mathrm{aff}(P) = \left\{ \sum_{0 \leq i \leq n} \lambda_i p_i, \lambda_i \in \mathbb{R}, p_i \in P, \sum_{0 \leq i \leq n} \lambda_i = 1 \right\}. \tag{2.1.3}$$

Un sous-ensemble f de P est une ***face*** s'il existe un hyperplan affine $\mathcal{H}$ tel que $P \cap \mathcal{H} = f$ et que P est dans un seul des deux demi-espaces affines engendrés par $\mathcal{H}$. Cette définition étend à toutes les dimensions la notion usuelle de face en 3 dimensions. En particulier, $\emptyset$ est une face de P (quand P est aussi considéré comme une face, $\emptyset$ et P sont appelées faces ***impropres*** de P, par opposition à toutes les autres, qui sont ***propres***). Ainsi une face de dimension 0 est appelé ***sommet***, une face de dimension 1 est une ***arête*** et une face de dimension $d-1$ est appelée une ***facette***. Une k-face de P est aussi un k-polytope. L'***intérieur*** de P, noté $\mathring{P}$ est l'intérieur topologique de P dans l'espace affine qu'il engendre. De manière analogue, l'ensemble $\partial P = P \setminus \mathring{P}$ est la ***frontière*** de P. L'ensemble des faces du polytope P est noté $L(P)$, et, en munissant cet ensemble de la relation d'inclusion, on obtient l'ensemble partiellement ordonné $(L(P), \subset)$, qui se trouve être un treillis, appelé le ***treillis des faces*** de P.

Les deux définitions équivalentes (cette équivalence connue sous le nom de théorème de Minkowski–Weyl, appelé par G. Ziegler le "Théorème principal des polytopes" [155]) peuvent être vues comme des définitions "duales": le polytope est défini par ses sommets dans la première, alors qu'il est défini par ses facettes dans la seconde. Le même type de définition s'applique aux ***cônes polyédraux***, qui sont l'intersection d'un nombre fini de demi-espaces linéaires passant par l'origine ou l'ensemble engendré par les combinaisons coniques d'un nombre fini de vecteurs de $\mathbb{R}^d$.

Pour un d-polytope P, nous notons $f_k(P)$ le nombre de k-faces de P. Le vecteur qui encode le nombre de faces de P, $(f_0(P), f_1(P), ..., f_{d-1}(P))$, est appelé le ***f-vecteur*** de P. On ajoute parfois pour la face vide $f_{-1}(P) = 1$ dans le f-vecteur. L'étude de ce f-vecteur, et donc de la combinatoire des faces d'un polytopes est un domaine de recherche essentiel, nous en reparlerons dans la section suivante.

Si on s'intéresse aux arêtes d'un polytope P, on peut regarder le graphe (V, E) de P ou V est l'ensemble des sommets et E l'ensemble des arêtes de P. Le ***diamètre du***

graphe de P, $\delta(P)$ est défini comme le plus petit entier tel que tous les sommets de P soient reliés deux à deux par au plus $\delta(P)$ arêtes. Cette notion fait l'objet de l'une des plus célèbres conjectures polytopales, décrite plus loin.

Voici quelques familles de polytopes très connues : tout d'abord, pour un d-polytope B et un point x_0 affinement indépendant, la ***pyramide*** de base B et de sommet x_0 est l'enveloppe convexe de $B \cup \{x_0\}$. Les ***simplexes*** sont définis comme l'enveloppe convexe de points affinement indépendants. Le d-simplexe a donc $d+1$ sommets. Le 2-simplexe est le triangle, le 3-simplexe est le tétraèdre et on peut simplement construire par récurrence le d-simplexe comme une pyramide de base le $(d-1)$-simplexe et d'un point qui n'est pas contenu dans l'enveloppe affine de cette base. Enfin, l'***hypercube*** de dimension d (ou d-cube) est le polytope $[0,1]^d$ (ou l'image de celui-ci par la composition d'une translation, d'une rotation et d'une dilatation).

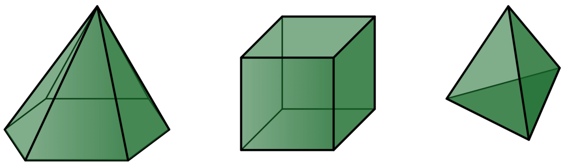


Figure 2.1: Une pyramide à base hexagonale, un cube et un tétraèdre.

Revenons à un d-polytope P quelconque dans $\mathbb{R}^d$. Nous allons introduire la notion d'éventail normal de P, notion très utilisée en combinatoire des polytopes [155] et dans le chapitre 6. Un ***éventail*** est un ensemble de cônes polyédraux tel que toute intersection entre deux cônes soit une face de chacun et que pour tout cône dans l'éventail, toutes ses faces non vides sont dans l'éventail. Pour une face F de P, on défini le ***cône normal*** N_F de F comme l'ensemble des vecteurs de $\mathbb{R}^d$ dont la forme linéaire associée est maximale en F, c'est-à-dire

$$N_F = \left\{ \boldsymbol{c} \in \mathbb{R}^d, F \subset \{\boldsymbol{x} \in P, \boldsymbol{c} \cdot \boldsymbol{x} = \max_{y \in P} \boldsymbol{c} \cdot \boldsymbol{y}\} \right\}. \tag{2.1.4}$$

Et on définit ainsi l'***éventail normal*** de P:

$$\mathcal{N}(P) := \{N_F, F \text{ face de } P\}. \tag{2.1.5}$$

Dans les chapitres 4 et 5, nous étudions des polytopes dans le cas où leurs sommets sont dans $\mathbb{Z}^d$. Ces polytopes sont appelés ***polytopes entiers***. Quand on regarde les polytopes entiers, il est naturel de s'intéresser aux éléments de $\mathbb{Z}^d$ qui appartiennent à un polytope P et à ses faces. Dès lors, nous parlerons de ***vecteur primitif*** si le pgcd de ses coordonnées est égal à 1 (un segment primitif est donc un segment de $\mathbb{R}^d$ qui ne contient que deux points entiers, ses deux extrémités).

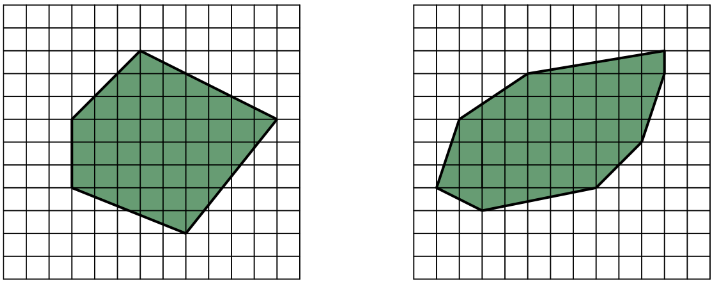


Figure 2.2: Deux polygones entiers.

2.1.2 La combinatoire des faces

L'étude combinatoire des faces des polytopes (caractérisation de polytopes en fonction de leur nombre de k-faces, étude du nombre de k-faces d'un polytope en fonction de k...) est un domaine central dans l'étude des polytopes, qui recèle des conjectures parmi les plus célèbres. Nous allons ébaucher un rapide tour d'horizon des résultats marquants dans ce domaine, ainsi que quelques conjectures célèbres.

En 2 dimensions, la combinatoire des polygones est triviale, nous ne nous attarderons pas dessus avant les questions énumératives de la Section 2.4.1. En 3 dimensions, le célèbre théorème de Steinitz caractérise entièrement les polyèdres par le graphe induit par leurs sommets et arêtes :

Théorème 2.1.1 (Steinitz, 1922 [145])**.** *G est le graphe d'un 3-polytope si et seulement s'il est simple, planaire et 3-sommets-connexe.*

Ziegler consacre un chapitre entier de [155] à ce théorème qui revêt une importance majeure pour plusieurs raisons. Premièrement, malgré la simplicité de ce théorème, il n'existe pas de preuve "simple". De plus, nous ne connaissons aucun équivalent de ce théorème pour les dimensions supérieures à ce jour et il ne semble pas y en avoir [80, Section 13.1]. Enfin, la preuve de ce théorème permet de démontrer plusieurs corollaires et versions plus fortes de ce théorème, qui ne sont pas vrais en dimension supérieures. Quatre-vingts ans après Steinitz, J. Mihalisin and V. Klee ont démontré une caractérisation des graphes orientés de 3-polytopes [115]. Le lecteur est invité à lire les chapitres consacrés à ce théorème dans [80, 155] pour des démonstrations détaillées et une liste des corollaires.

En dimension d, les intérêts combinatoires portent sur le f-vecteur défini dans la section précédente, qui encode le nombre de k-faces pour tous les entiers k entre -1 et d, la face de dimension -1 étant une convention pour la face vide. La question ouverte principale est la caractérisation d'un polytope par un f-vecteur. Quels vecteurs de $\mathbb{N}^{d+1}$ sont le f-vecteur d'un polytope ? Ce problème est encore loin d'être résolu, mais nous exposerons ici des résultats remarquables, dont certains très récents. Le premier résultat remarquable est la formule d'Euler–Poincaré :

$$-f_{-1} + f_0 - f_1 + ... + (-1)^d f_d = 0. \quad (2.1.6)$$

Cette formule est connue notamment en dimension 3, pour les graphes planaires comme la formule d'Euler. Soit G un graphe planaire à n sommets, e arêtes, et qui lorsqu'il est plongé dans le plan, le décompose en f portions de dimensions 2, alors

$$n - e + f = 2. \quad (2.1.7)$$

Outre cette formule, plusieurs résultats notoires donnent des conditions sur les f-vecteurs des polytopes. Quel est le nombre maximal de k-faces d'un d-polytopes à n sommets ? La réponse à cette question a été conjecturée par Motzkin en 1957 [117] et démontrée par McMullen en 1970 [111]. Elle est connue sous le nom de théorème de la borne supérieure et elle borne les faces des polytopes à n sommets par les ***polytopes cycliques***. Un d-polytope cyclique à n sommets est l'enveloppe convexe de n points de la courbe des moments :

$$\{(t, t^2, ..., t^d), t \in \mathbb{R}\}. \quad (2.1.8)$$

Les polytopes construits de cette façon sont combinatoirement équivalents (les treillis de leurs faces sont isomorphes).

Théorème 2.1.2 (de la borne supérieure, McMullen, 1970)**.** *Si P est un d-polytope à $n = f_0$ sommets, alors pour tout k, le nombre de ses k-faces est au plus le nombre de k-faces du polytope cyclique correspondant $C_d(n)$:*

$$f_k(P) \leq f_k(C_d(n)). \quad (2.1.9)$$

On peut démontrer ce théorème avec l'opération d'écaillage (qui correspond en anglais à la notion de "shellability"), qui est une façon de décomposer un complexe polytopal en un agrégat de facettes avec des règles de recollement particulières. Ce résultat montre, plus qu'une borne sur le nombre de k-faces, qu'un polytope borne tous les nombres de k-faces, ce qui peut paraître contre intuitif. Récemment, K. Adiprasito et R. Sanyal ont pouvé un théorème de la borne supérieure sur les sommes de Minkowski de polytopes [2] (la somme de Minkowski est définie dans la Section 2.3.1). Pour la borne inférieure, les résultats sont plus compliqués et font intervenir le h-vecteur, un dérivé du f-vecteur qui respecte les égalités de Dehn-Sommerville [118]. Nous allons dans quelques paragraphes énoncer deux autres résultats, très récents, de borne inférieure des f_k, qui tenaient lieu de conjectures majeures pendant plusieurs dizaines d'années.

Une autre question légitime est : les coordonées du f-vecteur forment-elles une suite unimodale ? Une suite de n nombres $(a_1, ..., a_n)$ est unimodale si

$$a_1 \leq a_2 \leq .... \leq a_k \geq a_{k+1} \geq ... \geq a_n, \quad (2.1.10)$$

pour un certain k situé entre 1 et n. La question se pose pour les polytopes simpliciaux, c'est-à-dire les polytopes dont toutes les facettes sont des simplexes. Les premiers exemples laissent penser que la réponse est positive, pourtant, en 1981, A. Björner trouve un contre-exemple en dimension 24 [22], mais montre qu'il y a tout de même une sorte d'unimodularité, au sens où le f-vecteur d'un d-polytope simplicial satisfait la double chaîne d'inégalité

$$\begin{cases} f_0 < f_1 < \dots < f_{\lfloor \frac{d}{2} \rfloor} \\ f_{\lfloor \frac{3(d-1)}{4} \rfloor} > \dots > f_{d-1}. \end{cases} \tag{2.1.11}$$

Pour les polytopes simpliciaux, le g-théorème donne une caractérisation totale des f-vecteurs. Ce théorème a été prouvé par L. Billera et C. Lee pour la suffisance [19] et par R. Stanley pour la nécessité [143].

Finalement, nous voudrions donner quatre conjectures célèbres rassemblées par G. Kalai dans [95], dont deux ont été récemment démontrées. La première est la conjecture énoncée en 1967 par B. Grünbaum (voir [79] et la version plus récente [80]) prouvée en 2021 par L. Xue [153], concernant une borne inférieure sur le nombre de k-faces des polytopes qui ont "peu" de sommets. Formellement, Xue a montré qu'un d-polytope convexe à $d + S \leq 2d$ sommets a au moins

$$\phi_k(d+s,d) = \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1} \tag{2.1.12}$$

k-faces, grâce à une double induction astucieuse. Pour un d-polytope P quelconque, I. Bárány posa la question suivante [20, Problem 17.6.5]: peut-on trouver une constante c_d, par exemple $c_d = 1$, telle que

$$f_k(P) \geq c_d \min(f_0(P), f_{d-1}(P))? \tag{2.1.13}$$

En 2022, J. Hinman prouve que le réponse est oui pour $c_d = 1$, en s'appuyant sur la notion d'angle multidimensionnel développée en 1970 par M. Perles et G. Shephard [122]. Ces deux résultats extrêmement récents, sur deux conjectures de premier plan datant du siècle dernier, montrent à la fois la difficulté des résultats sur la combinatoire des faces des polytopes et l'avancée scientifique significative de la recherche contemporaine.

Les deux conjectures qui suivent ne sont toujours pas résolues au moment où nous écrivons ces lignes et datent respectivement de la fin des années 90 et de 1989. La première concerne les 4-polytopes et la proportion des faces de dimensions intermédiaires par rapport au nombre de sommets et de facettes, à savoir existe-t-il une constante qui majore la quantité $(f_1(P) + f_2(P))/(f_0(P) + f_3(P))$ pour tout 4-polytope P [60] ?

La seconde, peut-être la conjecture la plus connue avec la conjecture de Hirsch, énoncée dans la section 2.1.4, est due à G. Kalai dans [96]:

Conjecture 2.1.1 (La conjecture 3^d de G. Kalai)**.** *Un d-polytope centralement symétrique a au minimum* 3^d *faces non-vides.*

Cette conjecture est démontrée pour $d \leq 4$ [131] et pour certaines classes de polytopes, comme les polytopes de Hansen [67].

2.1.3 Les polytopes entiers

Les chapitres 4 et 5 portent exclusivement sur des polytopes entiers, ou une sous-famille, mais avant de décrire dans les prochaines sections de cette introduction les motivations combinatoires de nos résultats, voici un tour d'horizon rapide de la recherche sur ces objets fascinants et leurs applications.

Les polytopes de $\mathbb{R}^d$ dont les sommets sont à coordonnées entières sont un objet d'étude à part entière et la majorité des attentes, des travaux et des questions sur les polytopes entiers s'articulent autour des questions difficiles [49] suivantes: combien y a-t-il de points entiers dans un polytope entier ? Quelle est la relation entre ce nombre et le volume du polytope et combien de points y a-t-il dans l'intérieur relatif du polytope ?

Dans les années 70, M. Demazure [51], G. Kempf, F. Knudsen, D. Mumford et B. Saint-Donat [98] et d'autres créent la théorie des variétés toriques et rendent indispensable l'étude des polytopes entiers. En effet la géométrie torique repose sur une association entre les variétés algébriques et les polytopes entiers, ce qui ramène les questions sur les variétés à des propriétés de polytopes entiers [63]. Depuis, ce dictionnaire est utilisé dans tous les domaines applicatifs des variétés toriques, notamment en physique, où les variétés de Calabi-Yau sont un élément de base de la théorie des cordes [156].

Le théorème de George Pick [123] est le premier résultat d'ampleur pour répondre à ces questions, dans le cas le plus simple des polygones.

Théorème 2.1.3 (Pick, 1900)**.** *Soit* $P \subset \mathbb{R}^2$ *un polygone. Si P est entier,*

$$\mathrm{vol}(P) = \mathrm{Card}(P \cap \mathbb{Z}^2) - \frac{1}{2}\,\mathrm{Card}(\partial P \cap \mathbb{Z}^2) - 1. \tag{2.1.14}$$

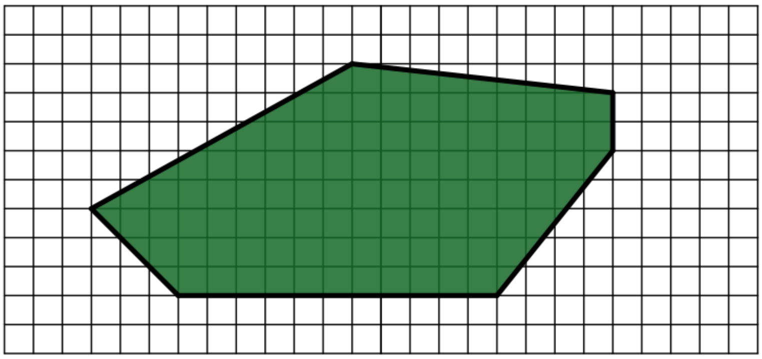


Figure 2.3: La surface de ce polygone est 102.5, ce qu'on retrouve avec le théorème de Pick.

Par la suite H. Blichfeldt [24] puis D. Hensley [87] et d'autres donneront une borne sur le volume de P en toute dimension dépendant des points entiers intérieurs du polytope. Mais l'un des plus beaux résultats du domaine est le résultat suivant [59], de Eugène Ehrhart, qui caractérise le nombre de points entiers dans une dilatation d'un polytope.

Théorème 2.1.4 (Ehrhart, 1967)**.** *Soit P un polytope entier dans $\mathbb{R}^d$. On note $\phi_P : \mathbb{N} \to \mathbb{N}^*$ la fonction telle que $\phi_P(m) = \mathrm{Card}(mP \cap \mathbb{Z}^d)$. Si P est de dimension n, alors ϕ_P est polynomiale de degré n, son coefficient dominant est le volume normalisé de P, noté $V(P)$, et son coefficient constant est 1. De plus $\phi_P(0) = 1$ et*

$$\phi_P(-m) = (-1)^d \,\mathrm{Card}(m\mathring{P} \cap \mathbb{Z}^d) \tag{2.1.15}$$

où $\mathring{P}$ désigne l'intérieur de P.

Ces polynômes (dits de Ehrhart) ont été l'objet de nombreuses études, dont la caractérisation de ces polynômes par des transformées de Fourier, par Ricardo Diaz and Sinai Robins [57]. Récemment, Manecke et Sanyal ont donné une famille de polynômes similaires pour compter les points dont les coordonnées sont premières entres elles dans la dilatation de polytopes [108]. Ces polynômes sont typiquement utilisés dans les domaines dont nous avons parlé précédemment.

Les autres domaines principaux d'applications sont les problèmes d'optimisation. En optimisation linéaire, le problème de savoir s'il y a un ou plusieurs points entiers dans un polytope, puis de les caractériser, est primordial. Pour les polytopes qui n'ont que très peu de points entiers intérieurs, il est naturel de regarder les cas extrêmes de "taille" de polytope. Pour cela on définit la ***largeur*** d'un polytope P, comme le minimum

$$\min_f (\max_{p,q \in P} |f(p) - f(q)|), \tag{2.1.16}$$

parmi les formes linéaires non constantes f à coefficients entiers. Parmi les polytopes entiers qui ont très peu de points entiers, les familles les plus connues sont les

polytopes creux [44] qui n'ont aucun point entier dans leur intérieur relatif, dont le théorème de planéité [100] borne la largeur, ou encore les ***polytopes vides***, qui n'ont d'autres points entiers que leurs sommets, dont font partie des célèbres hypersimplexes.

Parmi les polytopes qui ont exactement un point entier dans leur intérieur relatif, on compte les d-polytopes de Fano, dont chaque facette contient exactement d points entiers et les polytopes réflexifs, qui sont les polytopes entiers pour lesquels il y a un point entier à l'intérieur tel que toutes les facettes sont à distance 1 de ce point. Ces deux familles sont très importantes en géométrie algébrique.

2.1.4 Optimisation linéaire

Le dernier thème abordé dans cette section est peut-être le domaine d'utilisation des polytopes le plus connu : l'optimisation sous contrainte. Le problème le plus simple est le problème linéaire qui revient à trouver un certain vecteur $x \in \mathbb{R}^n$ et qui s'énonce

$$\begin{aligned} &\text{maximiser } c^{\mathsf{T}} x \\ &\text{sous la contrainte } Ax \leq b \end{aligned} \tag{2.1.17}$$

où c est un vecteur de $\mathbb{R}^n$ et b et A sont respectivement un vecteur dans $\mathbb{R}^m$ et une matrice dans $\mathbb{R}^{m \times n}$. La fonction à maximiser est appelée la fonction objectif. On rappelle que l'inégalité $Ax \leq b$ définit un polyèdre, qui n'est pas forcément borné. Il y a naturellement deux étapes pour résoudre ce problème. D'abord, savoir s'il existe un x qui vérifie les contraintes (nommé ***point réalisable***), puis s'il y en a, trouver l'optimum ou démontrer qu'on ne peut l'atteindre. Nous ne nous attarderons pas sur les problèmes d'optimisation pour revenir aux polytopes qui nous intéressent, mais les algorithmes qui résolvent ce problème peuvent être regroupés en trois groupes: la méthode du simplexe, la méthode du point intérieur et la méthode des ellipsoïdes. Cette dernière est particulièrement intéressante en théorie car Khachyian a démontré qu'elle résout les problèmes linéaire en temps polynomial [99], mais c'est la méthode du simplexe qui est à l'origine d'une des plus grandes conjectures du $\text{XX}^{\text{ème}}$ siècle sur les polytopes.

L'algorithme du simplexe, inventé par G. Dantzig en 1947 [48], est la première méthode de résolution des problème linéaires. Le concept est de trouver d'abord une base $B \subset \{1, ..., m\}$ pour laquelle la matrice composée des lignes d'indices B de A, A_B, est inversible et telle que $x_B = A_B^{-1} b_B$ est un point réalisable et un sommet du polyèdre défini par les contraintes. Partant de celui-ci, on construit un nouveau x_B qui a une plus grande valeur objectif en utilisant une règle de pivot. On peut interpréter géométriquement cette construction comme un chemin sur les sommets du polyèdres des contraintes. Les règles de pivot déterminent donc la façon de choisir le prochain sommet parmi tous les voisins de x_B, la plus connue étant la règle de Dantzig qui choisit l'arête qui maximise la fonction objectif après renormalisation (donc l'arête dont la "pente" est la plus proche de la direction de c). Une fois l'arête choisie, on remplace la condition $A_i^\intercal x \leq b_i$ qui n'est plus à l'égalité par la nouvelle condition vérifiée à l'autre bout de l'arête.

Cette méthode dépend donc de la taille des chemins entre deux sommets d'un polyèdre et donc du diamètre du graphe de ce polyèdre. En effet le diamètre du graphe du polytope est une borne inférieure du nombre d'étapes de l'algorithme du simplexe dans le pire des cas. Or la question de savoir si il existe une règle de pivot pour laquelle l'algorithme du simplexe termine en temps polynomial n'est toujours pas résolue. Le neuvième problème du XXIème siècle de S. Smale porte d'ailleurs sur la question générale de la complexité de la programmation linéaire [140]. Cela justifie directement l'étude de bornes supérieures sur le diamètre des graphes de polytope, et à cet égard, W. Hirsch donne, dans une lettre à Dantzig, sa fameuse conjecture:

Conjecture 2.1.2 (Hirsch, 1957, réfuté en 2010)**.** *Le graphe d'un d-polytope ayant n sommets a un diamètre inférieur ou égal à $n - d$.*

Plus de 50 ans plus tard, F. Santos Leal [130] trouve un contre-exemple en dimension 43 (!) en utilisant le résultat de V. Klee et W. Walkup qui établit l'équivalence entre la conjecture de Hirsch et la même conjecture pour les seuls d-polytopes à $2d$ arêtes [101]. La conjecture actuelle est donc la suivante:

Conjecture 2.1.3 (Conjecture polynomiale de Hirsch)**.** *Il existe un fonction polynomiale f telle que pour tout polytope P à n sommets, le graphe de P a un diamètre d'au plus $f(n)$.*

En raison de la difficulté à la résoudre et de sa longévité, cette conjecture est une des questions phares de l'étude des polytopes (voir les chapitres consacrés dans [155, Chapitre 3], [80, Chapitre 16]).

2.2 Les partitions

Les travaux réalisés dans les chapitres 4 et 5 portent essentiellement sur une famille de polytopes particuliers, ***zonotopes*** que nous avons déjà mentionné plus haut, qui peuvent être vus comme une généralisation des partitions. Les résultats de ces chapitres s'inscrivent donc dans la lignée des travaux combinatoires sur les partitions, que nous allons introduire dans cette section.

Prenons un nombre entier positif n. Une ***partition*** de n est une écriture de n comme somme d'entiers positifs. Dans les faits, c'est équivalent à partitionner n objets indiscernables en groupes de plus petite taille. Combien de possibilités différentes a-t-on de partitionner n ? Cette question va être le fil rouge de cette section.

Commençons par un exemple pour clarifier les notations. A gauche nous décomposons 5 en une somme d'entiers et à droite nous notons dans une liste le nombre d'apparitions de chaque nombre non nul (cette liste est infinie mais dite presque nulle, car il n'y a que des 0 à partir d'un certain rang).

$$5 = 5 \qquad (0,0,0,0,1,0,...) \tag{2.2.1}$$
$$5 = 4+1 \qquad (1,0,0,1,0,0,...) \tag{2.2.2}$$
$$5 = 3+2 \qquad (0,1,1,0,0,0,...) \tag{2.2.3}$$
$$5 = 3+1+1 \qquad (2,0,1,0,0,0,...) \tag{2.2.4}$$
$$5 = 2+2+1 \qquad (1,2,0,0,0,0,...) \tag{2.2.5}$$
$$5 = 2+1+1+1 \qquad (3,1,0,0,0,0,...) \tag{2.2.6}$$
$$5 = 1+1+1+1+1 \qquad (5,0,0,0,0,0,...) \tag{2.2.7}$$

Remarquons que la formalisation de droite est tout-à-fait adéquate pour noter une partition : il y a une bijection entre les décompositions d'un nombre en somme et ces suites de nombres presque nulles. En voyant ces listes comme des fonctions, nous garderons la notation suivante:

Définition 2.2.1. *Une partition ω de n est une application de $\mathbb{N}^*$ dans $\mathbb{N}$ telle que*

$$n = \omega(1) + 2\omega(2) + 3\omega(3) + ... \tag{2.2.8}$$

.

Nous noterons $p(n)$ le nombre de partitions de n, pour tout $n \in \mathbb{N}^*$. Nous avons donc d'après la liste précédente $p(5) = 7$, voici quelques valeurs suivantes: $p(6) = 11$, $p(7) = 15$, $p(8) = 22$, puis $p(20) = 627$ et $p(100) = 19056992$. Les questions sont nombreuses sur la suite des $p(n)$: à quelle vitesse croit elle ? Y a-t-il un moyen de les calculer efficacement ? Quelles propriétés arithmétiques a-t-elle ? Peut-on faire des liens avec d'autres pans de la théorie des nombres, par exemple les nombres premiers ? Au fil des siècles, de nombreuses propriétés ont été trouvées, nous en donnerons deux des plus connues :

Proposition 2.2.1. *Le nombre de partitions de n en exactement m parties est égal au nombre de partitions de n en parties de taille au plus m.*

Proposition 2.2.2. *Le nombre de partitions de n en parties de tailles distinctes est égal au nombre de partitions de n en parties de taille impaire.*

La première proposition est graphiquement évidente avec les diagrammes de Young, la seconde sera détaillée plus bas.

L'étude des $p(n)$, qui font déjà leur apparition dans des correspondances entre G. Leibniz et J. Bernoulli, ne débute vraiment qu'avec Euler, à la suite d'une fameuse lettre de P. Naudé le Jeune, en 1740 où il pose les questions suivantes : de combien de façons différentes le nombre 50 peut-il s'écrire

- comme la somme de 7 entiers distincts ?
- comme la somme de 7 entiers égaux ou distincts ?

L. Euler présente, en 1741, sa solution révolutionnaire [62, 105] : il introduit des produits infinis et des sommes infinies. Il vient de créer les ***séries génératrices***. Pour répondre à la première question, il écrit d'abord le produit infini suivant :

$$(1+xz)(1+x^2z)(1+x^3z)... \tag{2.2.9}$$

En développant tous les facteurs, on retrouve la réponse à la première question comme le coefficient de $x^{50}z^7$ dans la série obtenue. Il utilise le fait que $\frac{1}{1-x}$ se développe en série en $1+x+x^2+x^3+...$, pour en déduire que la deuxième question se traite de façon similaire en observant les coefficients du développement en série de

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)...}. \tag{2.2.10}$$

Dans ce produit, le $k^{\text{ème}}$ facteur encode le nombre de parties de tailles k dans la partition de n. La série génératrice du nombre de partitions d'un nombre n suit directement ; et pour tout entier naturel n, $p(n)$ est le coefficient de z^n dans le développement en série de

$$\prod_{k=1}^{+\infty} \frac{1}{1-z^k}. \tag{2.2.11}$$

Pour retrouver l'égalité de la Proposition 2.2.2, remarquons l'égalité entre les séries génératrices respectives des partitions à parties de tailles distinctes et des partitions à parties de tailles impaires :

$$\prod_{k=1}^{+\infty}(1+x^k) = \prod_{k=1}^{+\infty}\frac{1-x^{2k}}{1-x^k} = \prod_{k=0}^{+\infty}\frac{1}{1-x^{2k+1}} \tag{2.2.12}$$

2.2.1 Résultats asymptotiques

> " The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities – many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formluæ, one may fairly say, there are none. "[82]

> « Depuis l'époque d'Euler jusqu'à ce jour, la théorie des partitions a été développée presque exclusivement du point de vue algébrique. Cela

consiste en un ensemble d'identités formelles qui sont, c'est peu de le dire, d'une formidable ingéniosité d'une grande beauté. En ce qui concerne les formules *asymptotiques*, il faut avouer qu'il n'y en a aucune. » [82]

Comme s'en étonnent G. H. Hardy et S. Ramanujan, dans leur introduction, la question d'une étude asymptotique de cette série génératrice n'est pas explorée pendant plus de 150 ans, exceptée une formule de P. A. MacMahon [107]. En particulier, il n'y a pas de résultat asymptotique dans l'étude des $p(n)$, avant le célèbre résultat de Hardy et Ramanujan [82]

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \tag{2.2.13}$$

Les techniques de Hardy et Ramanujan ont un impact majeur sur les outils du XX$^{\text{ème}}$ en théorie des nombres. Pour calculer cet équivalent, ils utilisent la *méthode du cercle de Hardy-Littlewood* basée sur la formule de Cauchy du $n^{\text{ème}}$ coefficient d'une série. Au delà de la formule trouvée, Ramanujan apporta l'intuition d'une formule plus précise, en $O(1)$, qualifiée par G. Andrews [6, p.69] de "his most important contribution; it was both absolutely essential and most extraordinary".

En 1937, Rademacher reprend les idées et intuitions de Ramanujan et Hardy pour établir une formule explicite de $p(n)$ comme série convergente:

Proposition 2.2.3 (Rademacher)**.**

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{+\infty} A_k(n) k^{\frac{1}{2}} \left\{ \frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x-\frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x-\frac{1}{24}\right)^{\frac{1}{2}}} \right\} \tag{2.2.14}$$

où, en notant $a \wedge b = PGCD(a,b)$,

$$A_k(n) = \sum_{\substack{0 \le h < k \\ h \wedge k = 1}} e^{i\pi s(h,k) - 2in\frac{h}{k}}, \tag{2.2.15}$$

et où $s(h,k)$ est une somme de Dedekind, à savoir

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right) \tag{2.2.16}$$

La série est un peu indigeste mais les termes de la série décroissent très vite, rendant les calculs de $p(n)$ assez rapides : si l'on prend $p(200) = 3972999029388$, la somme des 5 premiers termes donne $3972999029387,89$ et la somme des 8 premiers termes donnent approche $p(200)$ à 0.004 près. Les méthodes se sont ensuite améliorées, avec l'apport significatif de G. Meinardus [112], qui identifie une stratégie générale pour obtenir des formules asymptotiques en partant d'une fonction de partition générale :

$$f(x) = \prod_{k=1}^{+\infty} (1 - x^n)^{-a_n}, \quad a_n \geq 0, \tag{2.2.17}$$

stratégie dont nous retrouverons les principes généraux dans le chapitre 4. Il se base notamment sur l'étude de la série de Dirichlet associée à la suite (a_n). En 1962, Newman [119] donne une preuve très écourtée de l'équivalent de Hardy et Ramanujan, en réduisant les difficultés d'intégration, mais la méthode du cercle utilisée reste sensiblement la même. Pourtant, Erdős [61] réussit en 1942 à trouver l'équivalent à une constante près, à l'aide d'une astucieuse réécriture récursive de $p(n)$!

Même si Meinardus est sûrement celui qui le plus marqué les méthodes de combinatoire asymptotique, nous ne voulons pas oublier les travaux de Ingham [90], Brigham [32], Wright [152] et Roth et Szekeres [129] qui ont tous été significatifs. La section 72 du livre de Leveque en fait une revue plus exhaustive [106].

A part la preuve d'Erdős, toutes les autres se basent sur des techniques analytiques qui seront plus tard regroupées au sein de théorèmes généraux de la combinatoire analytique [66]. Dans les années 90, une nouvelle approche probabiliste, basée sur les modèles de physique statistique, voit le jour avec les travaux de B. Fristedt [68] et Sinai [139]. Ils utilisent la compatibilité de la fontion génératrice des partitions avec la statistique de Maxwell-Boltzman, dans lequel précisément la grandeur associée à l'équilibre thermodynamique est appelée fonction de partition. Dans ce cadre, Baez-Duarte redémontre l'équivalent de Hardy-Ramanujan avec un théorème central limite [9].

2.2.2 La forme limite

Depuis ce rapprochement plus large entre probabilités et combinatoire de la fin du XX$^{\text{ème}}$ siècle, un grand nombre de questions stochastiques se posent sur les objets combinatoires, la question principale étant souvent : quelle est le comportement limite de tel objet combinatoire quand il devient très grand ?

Les partitions étant à la fois un objet central et simple en combinatoire (peut-être même le plus simple et central après le groupe des permutations), les études dans ce domaine ont été précurseurs d'autres objets plus complexes. Avant d'introduire le principal résultat concernant les partitions, que nous devons à A. M. Vershik

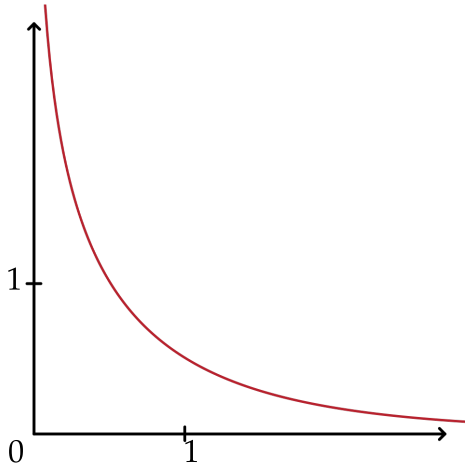


Figure 2.4: La forme limite d'une partition aléatoire uniforme de n renormalisée, quand $n \to +\infty$.

[150], nous nous permettons un rapide tour d'horizon des résultats de combinatoire asymptotiques.

Dans ce paragraphe, la notion de "convergence" ne sera pas explicité ; nous renvoyons les lecteurs vers les travaux correspondants pour une formalisation des convergences, des renormalisations et des espaces de convergence idoines. Le premier processus obtenu comme limite d'un objet combinatoire est sûrement le mouvement brownien, qui est la limite d'une marche aléatoire renormalisée. Les fameux ***chemins de Dyck***, (dénombrés par les non moins fameux ***nombres de Catalan***), convergent vers l'excursion brownienne [146]. Nous ne pouvons pas parler de limites d'objets combinatoires sans citer l'arbre brownien (*Brownian continuum random tree*) de D. Aldous [3], limite des arbres de Galton-Watson. Les triangulations, quadrangulations et d'autres types de cartes convergent vers la carte brownienne [109]. L'importance capitale de ces travaux tient au caractère universel de ces limites : l'arbre brownien par exemple est la forme limite d'une quantité de familles d'arbres très différents.

Revenons au comportement limite d'une partition aléatoire. Vershik démontre en 1996 le résultat suivant.

Théorème 2.2.1 (Vershik)**.** *Soit Γ la courbe d'équation cartésienne $e^{-x} + e^{-y} = 1$. Pour toute partition ω de n, on note E_ω l'ensemble de points suivant*

$$E_\omega = \left\{ \left(\frac{\pi i}{\sqrt{6n}}, \frac{\pi \omega(i)}{\sqrt{6n}} \right), i \geq 1 \right\}. \tag{2.2.18}$$

La partition ω de n suivant la mesure uniforme sur les partitions de n convergent en probabilité vers la courbe Γ, dans le sens où pour tout $\epsilon > 0$:

$$\mathbb{P}\left[\sup_{x \in E_\omega} d(x, \Gamma) \leq \epsilon \right] \underset{n \to +\infty}{\longrightarrow} 1. \tag{2.2.19}$$

La preuve de cette convergence était cherchée depuis longtemps par les physiciens et la première trace de cette courbe remonte à 1952 et les travaux de Temperley qui cherchait à comprendre avec un modèle d'entropie la formation de "dents" à la surface des crystaux [147]. Les résultats de I. Bárány, J. Bureaux et B. Lund [14] sur la convergence des zonotopes vers une forme limite, décrits dans la prochaine section, sont une généralisation en dimension supérieure de ce résultat.

2.3 Les zonotopes

Les ***zonotopes*** sont une famille de polytopes dont toutes les faces sont centralement symétriques, mais, nous verrons qu'ils sont aussi totalement analogues aux partitions et dans ce sens que nos travaux s'inscrivent dans la lignée des travaux de combinatoire asymptotique de la section précédente. Dans cette section nous allons d'abord introduire ces objets, puis aborder leurs principales applications intrinsèques et leur utilisation en combinatoire algébrique, pour finalement introduire les matroïdes orientés, objets dont le lien avec les zonotopes est essentiel dans le chapitre 6.

2.3.1 Partitions multipartites

Au lieu de prendre un entier n comme dans la section précédente, prenons un vecteur dans $\mathbb{R}^d$. On peut faire une décomposition analogue à celle de la partition de la section précédente, mais avec des vecteurs. Il y a alors plusieurs généralisations possibles.

Une première généralisation vient de la même question que celle des partitions : combien de façon y a-t-il pour décomposer un vecteur en une somme de vecteurs positifs non nuls ? La partition d'un vecteur d dimensionnel v de $(\mathbb{R}_+)^d$ en vecteurs à coordonnées positives tels que leur somme donne v s'appelle une ***partition multipartite***. Prenons à nouveau un exemple pour illustrer. En 2 dimensions, combien y a-t-il de partitions du vecteur $(2,2)$? Il y a 9 partitions différentes:

$$\begin{aligned}(2,2) &= (2,1)+(0,1) = (2,0)+(0,2) = (2,0)+(0,1)+(0,1) && (2.3.1)\\ &= (1,2)+(1,0) = (1,1)+(1,1) = (1,1)+(1,0)+(0,1) && (2.3.2)\\ &= (1,0)+(0,1)+(0,2) = (1,0)+(1,0)+(0,1)+(0,1). && (2.3.3)\end{aligned}$$

Si on regarde ces partitions d'un point de vue géométrique, on peut associer une chaine de segments à chaque partition. Comme l'ordre des parties de la partition ne compte pas, on peut arbitrairement ordonner les segments selon leur pente et ainsi associer à chaque partition une chaîne polygonale convexe dans le plan $\mathbb{R}^2$. On peut alors représenter la partition $(3,3) = (2,2)+(1,0)+(0,1)$ avec la chaîne $[(0,0),(1,0)] \cup [(1,0),(3,2)] \cup [(3,2),(3,3)]$ où chaque segment correspond à une partie de la partition:

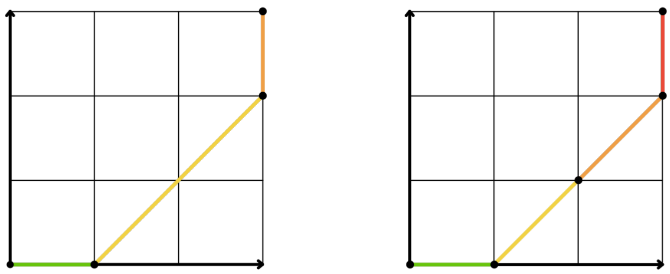


Figure 2.5: Deux partitions 2-partites différentes de $(3,3)$.

En fait, cette chaîne polygonale encode 2 partitions différentes,

$$(2,2)+(1,0)+(0,1) \text{ et } (1,1)+(1,1)+(1,0)+(0,1),$$

comme le montre la figure 2.5. Aussi, géométriquement, une autre généralisation des partitions en dimensions supérieures est intéressante : les ***partitions (d'entier) strictes multipartites***, qui sont les partitions multipartites dont toutes les parties sont des vecteurs primitifs (dont le PGCD des coordonnées est 1). En deux dimensions, on a une correspondance entre les chaînes polygonales convexes (ou concaves) à sommets entiers entre 0 et un point $\boldsymbol{n}$ et les partitions strictes bipartites de $\boldsymbol{n}$.

En dimension 3 ou plus, on peut aussi trouver une correspondance entre les partitions strictes multipartites et des polytopes convexes, en prenant la somme de Minkowski des vecteurs de la partition (voir le lemme 2.3.1). La somme de Minkowski (ou somme vectorielle) de deux polytopes P et P' est définie par

$$P+P' = \{\boldsymbol{x}+\boldsymbol{x}', \boldsymbol{x}\in P, \boldsymbol{x}'\in P'\}. \tag{2.3.4}$$

La somme de Minkowski des vecteurs $\boldsymbol{v}_1, ..., \boldsymbol{v}_k \in \mathbb{Z}^d$ est donc l'enveloppe convexe des vecteurs $\sum_{1\leq i\leq k} \epsilon_i \boldsymbol{v}_i$ avec $\epsilon_i \in \{0,1\}$, soit :

$$Z = \left\{ \sum_{1\leq i\leq k} \lambda_i \boldsymbol{v}_i, \;\; \lambda_i \in [0,1] \right\}. \tag{2.3.5}$$

Ces polytopes s'appellent des zonotopes et c'est dans le sens de cette généralisation que les travaux des chapitres 4 et 5 sont des extensions des résultats asymptotiques sur les partitions présentés dans la section précédente.

Définition 2.3.1 (Zonotope 1)**.** *Un zonotope est un polytope qui est une somme de Minkowski de segments.*

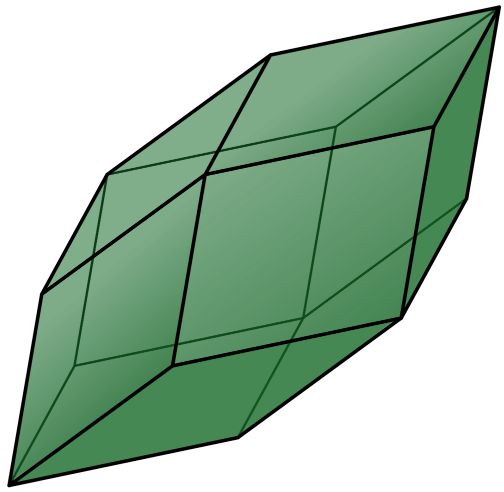


Figure 2.6: Un zonotope à 4 générateurs non colinéaires entre eux en trois dimensions

Prenons $Z \subset \mathbb{R}^d$ un zonotope défini comme la somme des segments $s_1, ..., s_k$ de $\mathbb{R}^d$. Par translation, nous pouvons construire des vecteurs $\boldsymbol{z}$ et $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ de tels que

$$Z = \left\{ \boldsymbol{z} + \sum_{1 \leq i \leq k} \lambda_i \boldsymbol{v}_i, \ \ \lambda_i \in [-1, 1] \right\}. \tag{2.3.6}$$

Habituellement, les générateurs sont supposés non colinéaires entre eux, pour éviter les cas dégénérés, comme dans le chapitre 6, ou alors on ne considère que les générateurs primitifs, en leur associant une multiplicité, comme dans les chapitres 4 et 5. Les vecteurs dont la somme de Minkowski donne le zonotope Z à une translation près s'appellent les ***générateurs*** de Z. Les zonotopes ont beaucoup de propriétés. Toute d'-face f d'un d-zonotope Z est un d'-zonotope. Dans l'écriture ci-dessus du zonotope Z, cette face est définie par d' variables λ_i dans $[-1, 1]$, tous les autres coefficients étant fixés à -1 ou 1. De plus, tous les zonotopes sont centralement symétriques (le centre de symmétrie de Z est z). En 2 dimensions, tous les polygones centralement symétriques sont zonotopes (comme le montre la Figure 2.6), pour les dimensions supérieures, cette condition nécessaire n'est plus suffisante, il existe cependant plusieurs caractérisations des zonotopes par leur symétrie. En voici trois exemples en guise de seconde définition [28] :

Définition 2.3.2 (Zonotope 2)**.** *Un d-zonotope est un d-polytope qui satisfait une des conditions équivalentes suivantes:*

- *toutes ses faces sont centralement symétriques.*
- *toutes ses 2-faces sont centralement symétriques.*
- *toutes ses k-faces sont centralement symétriques, pour au moins un* $k \in \{2, ..., d-2\}$.

Ces définitions montrent le profond côté géométrique des zonotopes. La dernière définition, au contraire, est la première étape pour montrer la nature combinatoire des zonotopes, développée dans la prochaine section. Toutes ces définitions expriment la grande richesse des domaines de la géométrie dans lesquelles les zonotopes apparaissent [155, Chapitre 7].

Définition 2.3.3 (Zonotope 3). *Un zonotope est un polytope obtenu en faisant une projection affine d'un hypercube.*

Pour voir l'équivalence des première et troisième définitions, reprenons le zonotope Z défini en (2.3.6) et prenons l'hypercube de dimension k, $C_k = [-1,1]^k$. Notons $V = (\boldsymbol{v}_1, ..., \boldsymbol{v}_k)$ le vecteur des générateurs de Z, Z est obtenu par la projection affine $V \cdot C_k + \boldsymbol{z}$, qui envoie la $i^{\text{ème}}$ dimension de l'hypercube sur $[-\boldsymbol{v}_i, \boldsymbol{v}_i]$ (les coordonnées x_i sont les poids λ_i de chaque vecteur $\boldsymbol{v}_i$ dans la somme).

Venons-en au dénombrement des ***zonotopes entiers***, zonotopes dont les sommets ont des coordonnées entières. Naturellement, le dénombrement des zonotopes implique deux restrictions : regarder ces objets à une translation près et se restreindre à compter des ensembles de générateurs primitifs pour ne pas compter deux ensembles qui engendrent le même zonotope. La première restriction peut être modifiée en fonction du problème considéré, mais elle est naturelle du point de vue des partitions multipartites.

Deux zonotopes entiers sont donc équivalents s'ils ont les mêmes générateurs primitifs à une translation près. Ainsi les générateurs $-\boldsymbol{v}$ et $\boldsymbol{v}$ sont équivalents. Pour dénombrer la totalité des zonotopes entiers dans un ensemble convexe borné donné, on peut donc réduire l'ensemble des zonotopes entiers à ceux engendrés par des vecteurs primitifs, dont la première coordonnée non nulle est positive, ensemble que l'on retrouve dans les travaux de A. Deza L. Pournin [53] et dans le chapitre 4.

Restreignons-nous au cas où les générateurs ne peuvent être que des vecteurs de l'orthant $\mathbb{N}^d$. Notons $\mathcal{Z}(\boldsymbol{k})$ l'ensemble des zonotopes dont les générateurs sont des éléments de $\mathbb{N}^d$ dont la somme des générateurs est égale à $\boldsymbol{k}$ et l'ensemble des vecteurs primitifs de $\mathbb{N}^d$ est noté $\mathbb{P}_{d+}$ (nous préférons cette notation à $\mathbb{P}^d_+$ car en notant $\mathbb{P}_1$ les nombres premiers relatifs, $\mathbb{P}_{d+} \neq (\mathbb{P}_{1+})^d$). On a le résultat suivant:

Lemme 2.3.1. *Il y a une bijection naturelle entre les partitions strictes multipartites de* $\boldsymbol{k}$ *et* $\mathcal{Z}(\boldsymbol{k})$.

Le résultat sera démontré dans la section 5.1 en utilisant la définition 2.2.1 des partitions: une partition stricte multipartite de $\mathbb{N}^d$ est une fonction $\omega : \mathbb{P}_{d+} \to \mathbb{N}$ à support fini.

Ce lemme donne la fonction génératrice de $\mathcal{Z} = \bigcup_{\boldsymbol{k} \in \mathbb{N}^d} \mathcal{Z}(\boldsymbol{k})$, qui est la série formelle d-dimensionnelle $f_{\mathcal{Z}} : \mathbb{R}^d \to \mathbb{R}$ telle que le coefficient de $x_1^{k_1} x_2^{k_2} ... x_d^{k_d}$ est le nombre de zonotopes de $\mathcal{Z}$ dont la somme des générateurs est $(k_1, k_2, ..., k_d)$. Ce nombre correspond donc au nombre de partitions strictes multipartites de $(k_1, k_2, ..., k_d)$.

$$f_{\mathcal{Z}}(x) = \prod_{\boldsymbol{p} \in \mathbb{P}_{d+}} \frac{1}{1 - \boldsymbol{x}^{\boldsymbol{p}}} \tag{2.3.7}$$

où $\boldsymbol{x}^{\boldsymbol{p}} = x_1^{p_1} x_2^{p_2} ... x_d^{p_d}$. La combinatoire des zonotopes entiers repose donc sur les points primitifs de $\mathbb{R}^d$, desquels nous allons donner quelques résultats principaux.

Ensemble de points primitifs

Pour cela, rappelons la ***fonction ζ de Riemann*** définie sur $[1, +\infty)$ par

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}}, \tag{2.3.8}$$

$\mathcal{P}$ étant l'ensemble des nombre premiers positifs ($\mathcal{P} = \mathbb{P}_{1+}$). Tout d'abord, il est connu depuis longtemps que la densité des points primitifs dans un sous ensemble de $\mathbb{Z}^d$ est de $\frac{1}{\zeta(d)}$. Le cas de la dimension 2 remonte à Mertens et Cesàro à la fin du XIX$^{\text{ème}}$ siècle [64]. Plus précisément, on peut donner le théorème suivant, conséquence de la formule d'inversion de Mobiüs [64] ([83, Théorème 459] dans le cas $d = 2$):

Théorème 2.3.1. *Pour $d \geq 2$ et F un sous-ensemble borné de $\mathbb{R}^d$. Pour tout $r \in (0, +\infty)$, on note $F_r = \{\boldsymbol{x} \in Z^d, \boldsymbol{x}r^{-1} \in F\}$. Si $\frac{|F_r|}{r^d}$ converge vers une limite non nulle quand r tend vers $+\infty$, alors*

$$\lim_{r \to +\infty} \frac{|F_r \cap \mathcal{P}_d\}|}{|F_r|} = \frac{1}{\zeta(d)}. \tag{2.3.9}$$

Le corollaire probabiliste de ce résultat est tout aussi beau [69]:

Corollaire 2.3.1. *Sous les mêmes hypothèses, notons Y_r un élément aléatoire de F_r choisi uniformément. Alors le PGCD de ses coordonnées* $\mathrm{PGCD}(Y_r)$ *converge en distribution vers la loi zêta de paramètre d, quand $r \to +\infty$.*

D'autres lois limites probabilistes sont connues concernant ces points primitifs, aussi appelés points visibles, en raison de l'absence de points entiers entre ceux-ci et l'origine. Nous voulons donner un dernier résultat marquant sur les points primitifs de $\mathbb{Z}^2$. Nous pouvons nous demander à propos des points primitifs quels types d'ensembles de points nous pouvons voir, ou ne pas voir depuis l'origine. Existe-t-il un carré de côté $c \in \mathbb{N}^*$ de sommets entiers dont tous les points sur les arêtes sont tous visibles, ou tous non visibles ? Un théorème de F. Herzog et B. M. Stewart répond à cette question [88, 104], mais il nécessite quelques définitions préalables. Un *motif* P est un sous-ensemble de $\mathbb{Z}^2$ et il est dit *réalisable* si P peut être translaté d'un vecteur de $\mathbb{Z}^2$ tel que tous les éléments de P sont visibles depuis l'origine. De plus, appelons *carré complet modulo m*, un ensemble de m^2 éléments de $\mathbb{Z}^2$:

$$\{(x_i, y_i), x_i \in \mathbb{Z}, y_i \in \mathbb{Z}, 1 \leq i \leq m^2\}, \tag{2.3.10}$$

tel que $\{(x_i \mod m, y_i \mod m)\} = \{(a,b), 0 \leq a, b \leq m-1\}$. Par exemple, les points $(1,3), (2,2), (3,4)$ et $(4,1)$ forment un carré complet modulo 2.

Proposition 2.3.1 (Théorème de Herzog et Stewart (1971))**.** *Un motif P est réalisable si et seulement si P ne contient aucun carré complet modulo p, pour tout p premier.*

Le résultat est aussi élégant que la preuve est intuitive; si P contient un carré complet modulo p, alors il contiendra toujours, peu importe la translation, un point dont les coordonnées sont toutes congrues à 0 modulo p. Inversement, en utilisant le lemme des restes chinois avec tous les nombres premiers inférieurs à la taille de P, nous pouvons construire un point qui soit "visible" de tous les points de P.

Enfin, récemment, A. Deza et L. Pournin ont calculé le nombre exact de points primitifs qu'il y a dans une boule L_1 centrée en l'origine [55], pour des questions de diamètre maximal des graphes de polytope, question évoquée dans la section 2.1.4. En résumé, retenons de cette brève énumération de résultats que les points primitifs ont été étudiés de nombreuses façons dans les domaines de combinatoire et de probabilité et que c'est bien leur présence dans la construction combinatoire des zonotopes qui entraîne l'apparition de la fonction ζ de Riemann dans ce manuscrit.

2.3.2 Point de vue algébrique et applications

Dans cette section sont présentées les applications et l'intérêt des zonotopes dans plusieurs domaines de recherche, à commencer par le lien entre les zonotopes et les configurations de vecteurs, grâce à des vecteurs de signes. Ce lien, décrit dans la section suivante en termes de matroïdes orientés, sera notamment utilisé dans le chapitre 6. Nous allons l'exposer en nous basant sur le chapitre 7 de [155].

Le but est d'associer à chaque face un *vecteur de signe* (donc un vecteur dont les coordonnées sont soit $+$, soit $-$ soit 0); tel que le treillis des faces d'un zonotope donné corresponde au poset des vecteurs de signe.

Dans la suite considérons l'exemple suivant : le zonotope $Z_1 \subset \mathbb{R}^2$ est défini par les générateurs $v_1 = (1,0), v_2 = (0,1), v_3 = (1,1)$. On note $V = (v_1, v_2, v_3) \subset \mathbb{R}^{2\times 3}$ et pour tout polytope P, rappelons que $L(P)$ est l'ensemble des faces de P, c'est-à-dire $L(P) = \{F,\ F \text{ face de } P\}$. En utilisant la relation d'inclusion, on peut donc dessiner le *treillis des faces* de Z_1, donné par $(L(Z_1), \subset)$:

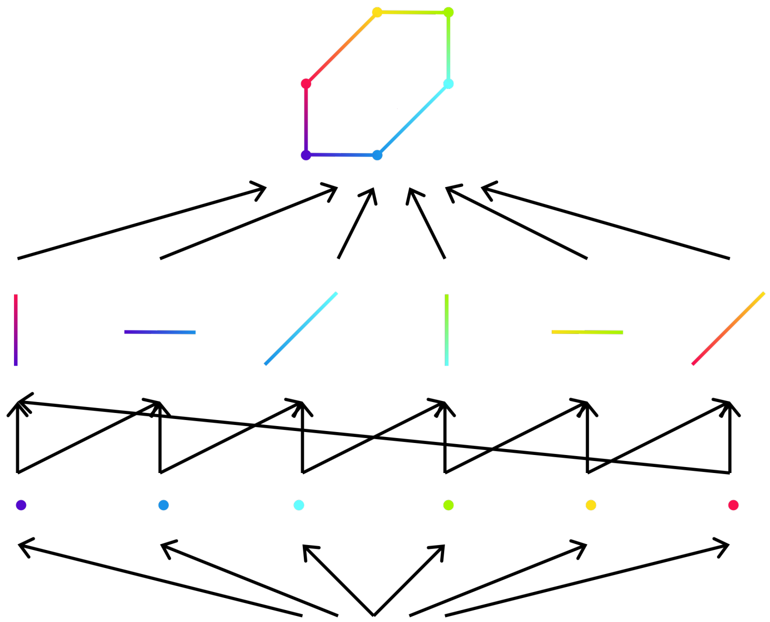


Figure 2.7: Le treillis des faces de Z_1.

L'association des faces d'un zonotope à un vecteur de signes se fait comme suit. Reprenons le cube $C_k = \{\boldsymbol{x} \in \mathbb{R}^k, -1 \leq x_i \leq 1\}$ et associons chaque face F non vide à un k-uplet de signe $\sigma \in \{+,-,0\}^k$ de la manière suivante : soit F une face de C_k, $\sigma = \sigma(F)$ est l'unique vecteur de $\{+,-,0\}^k$ tel que

$$F = \{ \sum_{1\leq i\leq k} \lambda_i \boldsymbol{e}_i, \ \lambda_i = +1 \text{ si } \sigma_i = +, \quad \lambda_i = -1 \text{ si } \sigma_i = -, \quad -1 \leq \lambda_i \leq +1 \text{ si } \sigma_i = 0\}. \tag{2.3.11}$$

On peut alors induire un ordre partiel sur les vecteurs de signe en définissant l'ordre "$\leq$" donné par $0 \leq -$ et $0 \leq +$. On a alors une correspondance entre l'ordre d'inclusion dans l'ensemble des face de l'hypercube et l'ordre des vecteurs de signe : plus une face est grande au sens de l'inclusion, plus son vecteur de signe est petit. Pour avoir la correspondance complète entre le treillis des faces et le treillis des signes, il faut ajouter un élément maximal $\hat{0}$, supérieur à tout vecteur de signe σ, correspondant à la face vide. Nous avons alors

$$(L(C_k), \subset) \cong \left(\{\hat{0}\} \cup \{+,-,0\}^k, \geq\right). \tag{2.3.12}$$

Grâce à la définition 2.3.3, nous pouvons exprimer Z_1 comme projection de C_3 et donc associer un vecteur de signe aux faces de Z_1. En notant π la projection qui définit Z_1 à partir de C_3, l'image réciproque $\pi^{-1}(G)$ d'une face G de Z_1 est une face de C_3. Nous définissons donc naturellement $\sigma(G) = \sigma(\pi^{-1}(G))$, ce qui conserve la propriété que $G \subset G'$ si et seulement si $\sigma(G) \geq \sigma(G')$ et

$$(L(Z_1), \subset) \cong \left(\{\hat{0}\} \cup \{\sigma(G), G \in L(Z_1) \setminus \{\emptyset\}\}, \geq\right). \qquad (2.3.13)$$

On note dans les figures 2.7 et 2.8 l'équivalence entre les treillis de face et les ensembles de vecteurs de signes partiellement ordonnés.

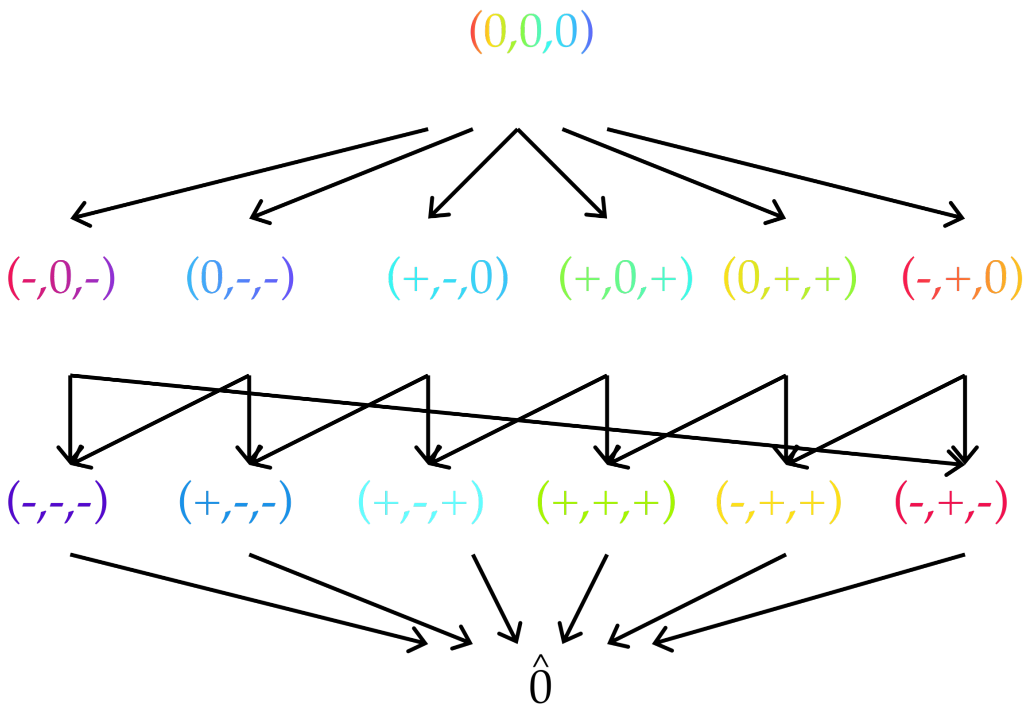


Figure 2.8: Le treillis des vecteurs de signes relatifs à Z_1, avec l'ordre $+ \geq 0$ et $- \geq 0$.

Laissons de côté le cube C_3 et regardons l'éventail normal de Z_1. Par définition, pour tous polytopes P et Q, l'éventail normal de la somme de Minkowski $P + Q$ est l'ensemble des intersections des cônes normaux de P avec ceux de Q, c'est-à-dire

$$\mathcal{N}(P + Q) = \{C \cap C', \ \ C \in \mathcal{N}(P), C' \in \mathcal{N}(Q)\}. \qquad (2.3.14)$$

Par construction de Z_1 comme somme de Minkowski de segments (ici les vecteurs colonnes de V), l'éventail normal $\mathcal{N}(Z_1)$ est donc l'éventail engendré par la famille d'hyperplans

$$\mathcal{A}_V = \{H_1, H_2, H_3\} \tag{2.3.15}$$

dans $\mathbb{R}^d$, où $H_i = \{\boldsymbol{c} \in \mathbb{R}^2, \ \ \boldsymbol{c} \cdot \boldsymbol{v}_i = 0\}$. $\mathcal{A}_V$ est aussi appelé un ***arrangement d'hyperplans***. Définissons le *demi-espace positif* $H_i^+ = \{\boldsymbol{c} \in \mathbb{R}^d, \ \ \boldsymbol{c} \cdot \boldsymbol{v}_i \geq 0\}$ et le demi-espace négatif de la même façon. La position de tout vecteur $\boldsymbol{c}$ par rapport à H_i est donnée par le signe de $\boldsymbol{c} \cdot \boldsymbol{v}_i$: c'est 0 si $\boldsymbol{c}$ appartient à H_i, $+$ si $\boldsymbol{c}$ est dans l'intérieur de H_i^+ et $-$ s'il est dans celui de H_i^-.

Ainsi pour tout vecteur $\boldsymbol{c}$ de $\mathbb{R}^d$, sa position dans l'éventail normal de Z est donné par $\text{sign}(V^\intercal \cdot \boldsymbol{c}) \in \{+, -, 0\}^3$ et donne donc le schéma suivant

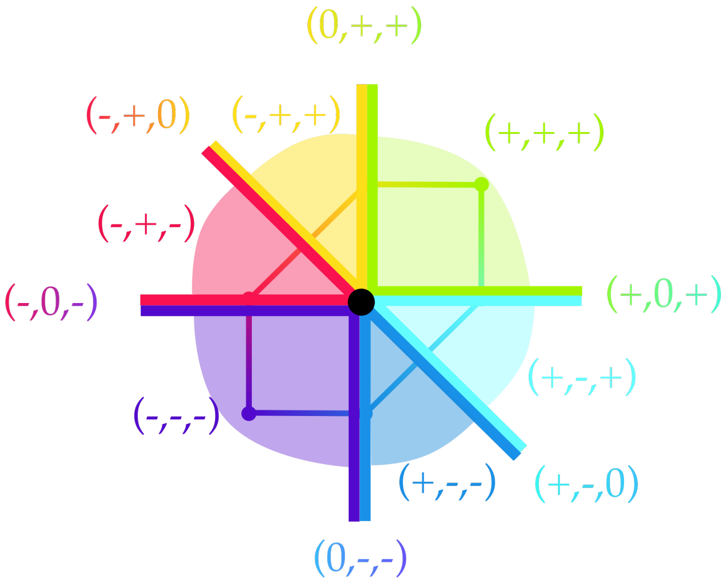


Figure 2.9: L'éventail normal de Z_1.

L'équivalence (2.3.13) relie donc le treillis des faces d'un zonotope au treillis des cônes engendrés par les hyperplans linéaires $\mathcal{A}_V$. Enfin, définissons un dernier concept : les ***configurations de vecteurs***. Une configuration de vecteurs de $\mathbb{R}^d$ est est un k-uplet de vecteurs $X = (\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_k) \subset \mathbb{R}^d$, dont on étudie l'agencement. L'agencement des vecteurs de X est directement lié aux signes des covecteurs $\boldsymbol{c}^\intercal X$ pour toute forme linéaire $\boldsymbol{c}^\intercal$. On définit donc l'ensemble des covecteurs de signes de la configuration X comme

$$\mathcal{V}^*(X) = \{\text{sign}(\boldsymbol{c}^\intercal X), \ \boldsymbol{c} \in \mathbb{R}^k\}. \tag{2.3.16}$$

On peut donc établir de la même façon que précédemment une équivalence entre le treillis des faces d'un zonotope et le treillis des covecteurs de signes de la configuration de vecteurs de ses générateurs non-colinéaires entre eux (on exclut ici le cas dégénéré de deux générateurs colinéaires). Ceci nous permet de conclure le résultat suivant :

Proposition 2.3.2 (Corollaire 7.17 de [155])**.** *Soit $V \in \mathbb{R}^{d\times k}$ un k-uplet de vecteurs non-colineaires deux à deux de $\mathbb{R}^d$. Alors il existe une bijection naturelle entre les trois familles suivantes:*

- *Les (vecteurs de signe des) faces non vides du zonotope $Z(V) \subset \mathbb{R}^d$.*
- *Les (vecteurs de signe des) faces de l'arrangement d'hyperplans $\mathcal{A}_V$.*
- *Les covecteurs de signe de la configuration V.*

Ce lien entre les zonotopes et les configurations de vecteurs est le point de départ de nombreux résultats géométriques et combinatoires [134, 8, 89, 124, 144, 108]. Dans la section suivante, nous allons développer la notion de matroïde orienté [155, 23] que nous utiliserons dans le chapitre 6.

En plus de leur importance combinatoire, les zonotopes sont aussi utiles pour leur simplicité combinatoire comparée aux polytopes. Sur la question du diamètre des polytopes, abordée précédemment dans la section 2.1.4, A. Deza, G. Manoussakis et S. Onn conjecturent que le diamètre maximal d'un polytope entier dans l'hypercube $[0, n]^d$ est atteint par un zonotope [52, 56], conjecture importante pour l'algorithme du simplexe sur des polytopes entiers [50]. C'est un des cas où les zonotopes captureraient (conjecturalement) le comportement des polytopes. Ces questions ont mené au diamètre maximal des zonotopes calculé A. Deza et L. Pournin à partir d'une étude du nombre de points primitifs dans dans les multiples du simplexe standard de R^d et dans les boules pour la norme L^1 [55].

Deux applications concrètes des zonotopes dans la littérature retiennent notre attention, toute deux utilisant la simplicité de l'encodage d'un zonotope : la première, par L. Guibas, A. Nguyen et L. Zhang, les utilise dans la détection de collision, pour borner les objets par des zonotopes et ainsi accélérer les temps de calcul [81]. La seconde, par A. Girard, pour résoudre les problèmes d'accessibilité à la solution d'équations linéaires incertaines [71]. Les zonotopes sont enfin très étudiés en théorie des pavages, de par leur aspect hautement symétrique [110, 135, 45, 46].

2.3.3 Matroïdes orientés

Les ***matroïdes*** sont des structures combinatoires introduites en 1935 par Whitney (voir [103, 120]), pour abstraire le concept d'indépendance linéaire d'un ensemble de vecteurs. La connaissance des matroïdes en dehors des spécialistes s'est faite par le bien connu algorithme glouton, en optimisation combinatoire, algorithme qui ne marche précisément que sur ces structures. La théorie des matroïdes reprend les notions d'algèbre linéaire d'indépendance, de base ou encore de rang, mais aussi les notions de théorie des graphes comme les circuits et les cycles.

Les matroïdes peuvent être définis d'un grand nombre de façons équivalentes, nous allons donner leur définition par les bases. Un matroïde est un couple $M_E =$

$(E, \mathcal{B}_E)$, où E est un ensemble fini et $\mathcal{B}_E$ est une collection non vide de parties de E, qui vérifie l'axiome d'échange de Steinitz (SEA)

$$\begin{aligned} &\text{Pour tous } B_1, B_2 \in \mathcal{B}_E, \text{ et tout élément } e \in B_1 \setminus B_2, \\ &\text{il existe } f \in B_2 \setminus B_1 \text{ tel que } (B_1 \setminus \{e\}) \cup \{f\} \in \mathcal{B}_E. \end{aligned} \tag{SEA}$$

Les éléments de $\mathcal{B}_E$ sont appelés ***bases*** de E. Tous les éléments de $\mathcal{B}_E$ ont le même cardinal, appelé ***rang*** du matroïde. On appelle ***ensemble indépendant*** tout sous-ensemble d'un élément de $\mathcal{B}_E$ (les bases sont donc les ensembles indépendants maximaux au sens de l'inclusion) et on note l'ensemble des ensembles indépendants $\mathcal{I}_E$. Intuitivement, les notions de base et d'indépendance gardent donc le même sens qu'en algèbre linéaire. Enfin, les éléments minimaux au sens de l'inclusion de $\mathcal{P}(E) \setminus \mathcal{I}_E$ sont appelés ***circuits*** de E et l'ensemble des circuits de E est noté $\mathcal{C}_E$. Chacune des trois notions de base, de circuit ou d'ensemble indépendant suffit à définir les matroïdes, à travers l'axiome d'échange (SEA) pour les bases, l'axiome d'augmentation (AA) pour les ensembles indépendants et l'axiome d'élimination (EA) pour les circuits; ces deux derniers étant:

$$\begin{aligned} &\text{Pour tous } I_1, I_2 \in \mathcal{I}_E \text{ tels que } |I_1| < |I_2|, \\ &\text{il existe } e \in I_2 \setminus I_1 \text{ tel que } I_0 \cup \{e\} \in \mathcal{I}_E. \end{aligned} \tag{AA}$$

$$\begin{aligned} &\text{Pour tous circuits distincts } C_1, C_2 \in \mathcal{C}_E \text{ tels que } e \in C_1 \cap C_2, \\ &\text{il existe un circuit } C_3 \subset C_1 \cup C_2 \setminus \{e\}. \end{aligned} \tag{EA}$$

L'exemple le plus simple est sans doute le matroïde naturellement associé à un n-uplet de vecteurs de $\mathbb{R}^d$, où un k-uplet est indépendant si le k-uplet est libre dans $\mathbb{R}^d$. On peut aussi associer un matroïde à tout ensemble d'arête E d'un graphe fini non orienté $G = (V, E)$. L'ensemble $E' \subset E$ est indépendant si le graphe (V, E') n'a pas de cycle. Les circuits au sens de la théorie des matroïdes coïncident alors avec les circuits au sens de la théorie des graphes.

Remarquons dans ces exemples que ces matroïdes reposent sur des ensembles non orientés, ou pour lesquels nous n'avons pas pris en compte une quelconque notion d'orientation. Les ***matroïdes orientés***, au contraire, permettent de rendre compte des structures de dépendance dans des espaces vectoriels sur des corps ordonnés et permettent donc de rendre compte de la structure du treillis des vecteurs de signes de la section précédente.

Soient E un ensemble fini et $\mathcal{V}$ un sous-ensemble de $\{+, -, 0\}^E$, l'ensemble des vecteurs de signe. On note $\mathbf{0}$ le vecteur de n zéros et pour tout $u \in \mathcal{V}$, le *support* de

u est l'ensemble des indices des coordonnées non nulles de u, à savoir $\text{supp}(u) = \{i, u_i \neq 0\}$. Sur les vecteurs de signe, la *composition* des vecteurs u et v est une loi de composition interne définie coordonnée par coordonnée par

$$(u \circ v)_i = \begin{cases} u_i & \text{si } u_i \neq 0, \\ v_i & \text{sinon.} \end{cases} \tag{2.3.17}$$

L'*ensemble de séparation* de u et v est l'ensemble $S(u,v) = \{i, u_i = -v_i \neq 0\}$, et finalement, si $j \in S(u,v)$, on dit que le vecteur w *élimine* j *entre* u *et* v si $w_j = 0$ et $w_i = (u \circ v)_i$ pour tout $i \notin S(u,v)$. Rappelons également l'ordre sur les signes, vu dans la section précédente: $+ > 0$ et $- > 0$. Toutes ces notions prendrons sens dans la définition des matroïdes orientés et dans l'exemple qui suit:

Définition 2.3.4 (Matroïdes orientés)**.** *Soient E un ensemble fini et la collection $\mathcal{V} \in \{+,-,0\}^E$. Le couple $M = (E, \mathcal{V})$ est un matroïde orienté si elle satisfait*

$$\mathbf{0} \in \mathcal{V}, \tag{V0}$$

$$u \in \mathcal{V} \Rightarrow -u \in \mathcal{V}, \tag{V1}$$

$$u, v \in \mathcal{V} \Rightarrow u \circ v \in \mathcal{V}, \tag{V2}$$

$$u, v \in \mathcal{V}, j \in S(u,v) \Rightarrow \exists w \in \mathcal{V} : w \text{ élimine } j \text{ entre } u \text{ et } v. \tag{V3}$$

De plus, le rang de M est le plus grand entier r tel qu'il existe r éléments $u_1, ..., u_r \in \mathcal{V}$ tels que $u_1 < u_2 < ... < u_r$.

Pour expliquer cette définition, montrons en exemple le résultat suivant, qui prouvera la proposition 2.3.3 : soit $U \subset \mathbb{R}^n$ un sous-espace linéaire de dimension r et $\text{SIGN}(U) = \{\text{sign}(u), u \in U\}$, le couple $(\{1, .., n\}, \text{SIGN}(U))$ est un matroïde orienté de rang r.

U étant un sous-espace vectoriel, les axiomes (V0) et (V1) sont immédiats. Pour démontrer (V2), prenons $\boldsymbol{x}, \boldsymbol{y} \in U$ tels que $\text{sign}(\boldsymbol{x}) = u$ et $\text{sign}(\boldsymbol{y}) = v$. Avec la définition de la composition, nous pouvons montrer qu'il existe $\epsilon > 0$ tel que

$$\text{sign}(\boldsymbol{x} + \epsilon \boldsymbol{y}) = u \circ v. \tag{2.3.18}$$

En effet, nous pouvons prendre ϵ assez petit pour que pour chaque coordonnée i, le signe de $u_i + \epsilon v_i$ soit celui de u_i lorsque ce nombre n'est pas nul et celui de v_i quand u_i est nul (voir la figure 2.10). Comme $\boldsymbol{x} + \epsilon \boldsymbol{y} \in U$, (V2) est vérifié.

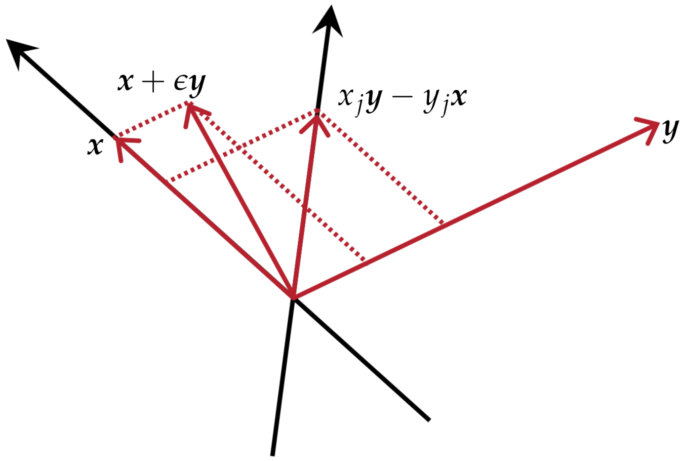


Figure 2.10: Exemple de construction d'éléments vérifiant (V2) et (V3)

Enfin, prenons $u, v \in \text{SIGN}(U)$, $j \in S(u, v)$ et $(\boldsymbol{x}, \boldsymbol{y}) \in U^2$ tels que $\text{sign}(\boldsymbol{x}) = u$ et $\text{sign}(\boldsymbol{y}) = v$. Supposons que $u_j = +$, on définit alors le vecteur de signe

$$w = \text{sign}(x_j \boldsymbol{y} - y_j \boldsymbol{x}). \tag{2.3.19}$$

La coordonnée j de w est 0 et pour toute coordonnée $i \notin S(u, v)$, $w_i = (u \circ v)_i$, comme illustré ci-dessus. Ainsi w élimine, par définition, j entre u et v et $w \in \text{SIGN}(U)$, ce qui nous donne (V3).

De ce résultat, nous pouvons donc déduire la proposition suivante:

Proposition 2.3.3 (Matroïdes orientés réalisables). *Soient $V \in \mathbb{R}^{n \times k}$ un k-uplet de vecteurs de $\mathbb{R}^n$ et $\mathcal{V}_V$ la collection des vecteurs de signe de V,*

$$\mathcal{V}_V = \{\text{sign}(V^\intercal \boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^n\}. \tag{2.3.20}$$

Le couple $(\{1, \ldots, k\}, \mathcal{V}_V)$ est un matroïde orienté. Un tel matroïde orienté est appelé ***matroïde orienté réalisable****.*

Les propositions 2.3.2 et 2.3.3 donnent donc une bijection naturelle entre les types combinatoires de zonotopes et les matroïdes orientés réalisables, qui sera crucial pour obtenir un encadrement du nombre de types combinatoires de zonotopes dans le chapitre 6.

Les matroïdes orientés non réalisables correspondent à des arrangements de pseudo-lignes, c'est-à-dire une courbe dans le plan projectif $\mathbb{R}P^2$ qui est topologiquement équivalente à un droite. Les matroïdes orientés ont donc permis des avancées majeures sur tous ces modèles (arrangement d'hyperplans ou configuration de vecteurs, arrangement d'hypersphères et arrangement de pseudo-lignes), tels la notion d'universalité [116, 1] et les polytopes matroïdes [27]. Le lecteur peut lire le chapitre de J. Richter-Gebert and G. Ziegler [128] ou les livres [23, 155] pour une description plus détaillée.

2.4 Étude probabiliste et combinatoire des polytopes entiers.

Nous allons ici nous intéresser à l'histoire de l'étude probabiliste des polytopes entiers, mais nous devons au préalable mentionner la variété des travaux probabilistes sur les polytopes (voir Section 5.0.2 pour plus de détails). Le cas le plus connu est le ***polytope aléatoire***, défini comme l'enveloppe convexe d'un ensemble de points obtenus par un processus ponctuel de Poisson dans un ensemble E. Des résultats de convergences peuvent être trouvés dans [126, 41] pour E un ensemble convexe borné. Une autre approche, par P. Calka, Y. Demichel et N. Enriquez consiste à étudier les polygones formés par la mosaïque de Voronoï engendrée par un processus ponctuel de Poisson [40]. Plus précisément, pour un corps convexe $K \subset \mathbb{R}^2$, on considère la mosaïque de Voronoï engendrée par un processus ponctuel de Poisson d'intensité λ conditionné à l'existence d'une cellule K_λ qui contient K; les auteurs étudient le comportement asymptotique de K_λ quand λ tend vers $+\infty$. Mais ces résultats sont de nature très différente des nôtres, car les polytopes entiers sont intrinsèquement liés au réseau $\mathbb{Z}^d$ et à ses propriétés arithmétiques (voir Remarque 2.4.1) et nous redirigeons le lecteur vers le chapitre de M. Reitzner [127] pour un exposé approfondi. Revenons donc aux polytopes entiers.

Quelle est l'aire minimale d'un polygone convexe entier à exactement n sommets ? Plus généralement quel est le volume minimal d'un d-polytope convexe entier à n sommets ? Ou quel est le nombre maximal de sommets que peut avoir un polytope dans un ensemble convexe borné donné ? Ces trois questions font partie des multiples problèmes liés aux polytopes entiers et à leur combinatoire [13]. Dans la section 2.1, nous avons fait un tour d'horizon des différentes thématiques autour des polytopes, en rappelant notamment certaines questions combinatoires encore ouvertes d'une part, dans la section 2.1.2 et sur les utilisations de polytopes entiers dans la section 2.1.3 d'autre part. Nous allons ici présenter les questions combinatoires sur les polytopes entiers qui ont mené aux travaux réalisés dans les chapitres 4 et 5. Nous allons notamment voir que la question du dénombrement des polytopes entiers dans un ensemble compact convexe est à la frontière entre la théorie des nombres et la géométrie.

La question du volume minimal d'un d-polytope est une question centrale car elle est utile dans la plupart des autres questions liées aux volumes des polytopes entiers [13]. En 1926, V. Jarník [92] pose la question suivante : soit $\gamma \subset \mathbb{R}^2$ une courbe fermée convexe de longueur au plus l et dont le rayon de courbure est majoré par $7l$, quel est le nombre maximum de points de Z^2 contenus dans γ ? Sa réponse est

$$\max_{\gamma} |\gamma \cap \mathbb{Z}^2| = \frac{3}{\sqrt[3]{2\pi}} l^{2/3} + O\left(l^{1/3}\right). \tag{2.4.1}$$

Le passage de cette question aux polygones convexes entiers est immédiat : le polygone $P = \mathrm{conv}(\gamma \cap \mathbb{Z}^2)$ est un polygone convexe entier non vide dès que γ contient 3 points entiers ou plus. La formule de V. Jarník donne donc que le périmètre minimum l d'un polygone à n sommets est

$$l = \frac{\sqrt{6\pi}}{9} n^{3/2} + O\left(n^{3/4}\right). \tag{2.4.2}$$

La question est immédiatement étendue au volume d'un polytope en n'importe quelle dimension. Le premier résultat pionnier pour cette question est un théorème de G. E. Andrews qui donne une borne inférieure sur le volume, du bon ordre de grandeur [5] (rappelons que le volume d'un polytope P est la mesure de Lebesgue de P dans l'espace affine engendré par P, noté $\mathrm{vol}(P)$):

Théorème 2.4.1 (Andrews, 1963)**.** *Si P est un d-polytope convexe entier à n sommets, alors il existe une constante $\kappa_d > 0$ ne dépendant que de d telle que le volume* $\mathrm{vol}(P)$ *satisfait*

$$\mathrm{vol}(P) > \kappa_d n^{(d+1)/(d-1)}. \tag{2.4.3}$$

En plus de son lien avec les résultats que nous allons énoncer ci-dessous, ce résultat donne une borne sur le nombre maximal de sommets que peut avoir un polytope entier contenu dans un ensemble convexe compact d-dimensionnel $K \subset \mathbb{R}^d$ par $(\mathrm{vol}(K)/\kappa_d)^{(d-1)/(d+1)}$, puis par l'inégalité isopérimétrique, il permet de majorer le nombre de sommets par $S(K)^{d/(d+1)}$ où $S(K)$ est la surface de K, c'est a dire le volume $d-1$-dimensionnel de la frontière de K. On retrouve l'ordre de grandeur du résultat de Jarník.

2.4.1 La question d'Arnold

En 1980, V. Arnold pose le problème suivant [7] : considérons que deux polytopes convexes entiers P et Q sont *équivalents* s'il existe une transformation affine qui préserve le réseau des points entiers par laquelle l'image de P est Q. Dans ce cas, P et Q ont le même volume. Quel est le nombre $n_d(V)$ de classes d'équivalence de d-polytopes convexes entiers dont le volume est V et notamment quel est l'ordre de grandeur de $\log n_d(V)$? Ce problème est lié à l'étude de certains diagrammes de Newton, à l'étude des polynômes à d variables près de l'origine et à la question du volume minimum d'un polytope convexe entier énoncé en introduction de cette section. Dans son papier, V. Arnold prouve, dans le cas bi-dimensionnel, l'existence de deux constantes $c_1, c_2 > 0$ telles qu'on a l'encadrement suivant pour V suffisamment grand :

$$c_1 V^{1/3} \leq \log n_2(V) \leq c_2 V^{1/3} \log V, \quad \text{pour } V \to +\infty. \tag{2.4.4}$$

Sa démonstration est basée sur l'inégalité d'Andrews ci-dessus, qu'il redémontre indépendamment dans le cas $d = 2$. Réciproquement, le théorème d'Andrews s'obtient dans le cas $d = 2$ à partir de la minoration d'Arnold. Cet exemple illustre parfaitement la proximité entre les différentes questions relatives aux polytopes entiers mentionnée plus haut.

En 1984, Konyagin et Sevast'yanov généralisent cet encadrement à tout $d \in \mathbb{N}^*$ [102], prouvant qu'il existe deux constantes $c_1, c_2 > 0$ telles que

$$c_1 V^{\frac{d-1}{d+1}} \leq \log n_d(V) \leq c_2 V^{\frac{d-1}{d+1}} \log V, \quad \text{pour } V \to +\infty. \tag{2.4.5}$$

I. Bárány et J. Pach pour $d = 2$ [15], puis I. Bánány et A. M. Vershik la même année pour le cas général [11], ont montré que le terme $\log V$ dans le majorant est superflu. Il existe donc deux constantes $c_1, c_2 > 0$ telles que, pour V suffisamment grand, $(\log n_d(V))/V^{\frac{d-1}{d+1}}$ appartient à l'intervalle $[c_1, c_2]$. Naturellement, on peut se demander s'il existe $\alpha > 0$ tel que

$$\log n_d(V) \sim \alpha V^{\frac{d-1}{d+1}}, \quad \text{pour } V \to +\infty. \tag{2.4.6}$$

La question reste ouverte depuis plus de 30 ans, sans réelle avancée sur ce sujet [13]. Dans l'esprit de la démonstration du majorant d'Arnold, Vershik pose la question intermédiaire du nombre de polytopes convexes entiers dans un hypercube de côté n. Nous pouvons le reformuler ainsi en 2 dimensions : combien y a-t-il de polygones convexes dont les sommets sont sur la grille $\frac{1}{n}\mathbb{Z}^2$ dans le carré $[-1,1] \times [-1,1]$ quand n tend vers l'infini et à quoi ressemblent-ils ? Le sens de cette dernière question est probabiliste.

La réponse est trouvée par Bárány [10], Vershik [151] et Y. G. Sinai [139] indépendamment. Notons $\mathcal{P}_n(K)$ l'ensemble des polygones convexes à sommets dans $\frac{1}{n}\mathbb{Z}^2$ qui sont dans l'ensemble convexe borné K et

$$p(n) = |\mathcal{P}_n([-1,1] \times [-1,1])|. \tag{2.4.7}$$

Ils montrent l'équivalence suivante :

$$\log p(n) \sim 12 \left(\frac{\zeta(3)}{\zeta(2)} \right)^{1/3} n^{2/3}, \quad \text{pour } n \to +\infty, \tag{2.4.8}$$

où ζ est la fonction ζ de Riemann. Outre ce résultat, les trois publications prouvent aussi que les polygones admettent une forme limite, à savoir qu'un polygone tiré uniformément dans $\mathcal{P}_n([-1,1] \times [-1,1])$ a une très grande probabilité d'être très proche d'un convexe K_0 (illustré dans la figure 2.11), au sens de la distance de Hausdorff. Nous noterons la distance de Hausdorff entre deux ensembles $\delta(\cdot,\cdot)$.

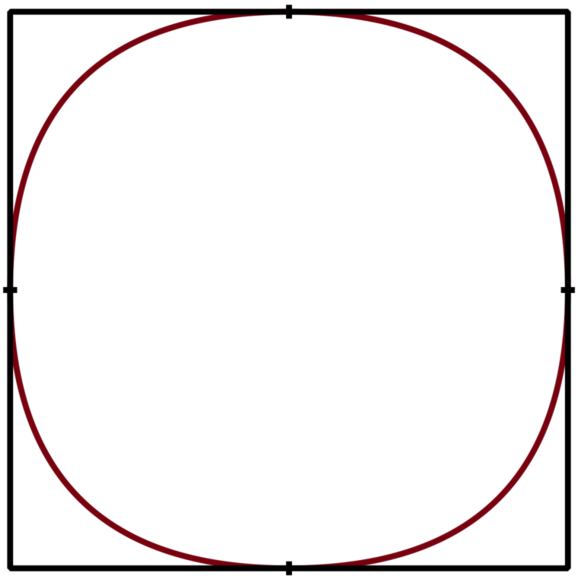


Figure 2.11: Forme limite K_0 des polygones tirés uniformément dans $\mathcal{P}_n([-1,1] \times [-1,1])$.

Théorème 2.4.2 (Bárány, Vershik). *Il existe* $K_0 \subset [-1,1] \times [-1,1]$, *tel que pour tout* $\epsilon > 0$, *on a*

$$\lim_{n \to +\infty} \frac{|\{P \in \mathcal{P}_n([-1,1] \times [-1,1]), \delta(P, K_0) < \epsilon\}|}{p(n)} = 1. \tag{2.4.9}$$

Plus tard, Bárány généralise ces résultat à n'importe quel convexe borné K et prouve que le nombre de polygones convexes à sommets dans $\frac{1}{n}\mathbb{Z}^2$ dans un ensemble convexe borné K de $\mathbb{R}^2$ est équivalent à $3(\zeta(3)/4\zeta(2))^{1/3} A_K n^{2/3}$, où A_k est le périmètre affine de K (voir [12] pour une définition du périmètre affine). Il montre aussi qu'un polygone tiré aléatoirement dans $\mathcal{P}_n(K)$ converge vers une forme limite K_0 quand n tend vers $+\infty$ [12]. Dans le cas du carré $K = [-1,1] \times [-1,1]$, la forme limite K_0, illustré dans la figure 2.11 est la réunion de 4 arc de paraboles qui relient les milieux des côtés adjacents de K deux à deux.

Remarque 2.4.1. *Que se passe-t-il dans les dimensions supérieures ? De manière surprenante au premier abord, nous n'avons aucune généralisation de l'expression* (2.4.8) *ni du théorème 2.4.2 plus précise que l'encadrement donné précédemment. Cette résistance étonnante, peut être pressentie en observant la formule* (2.4.8)*. Cette formule, qui donne un résultat purement géométrique, fait intervenir la fonction* ζ *de Riemann, objet fondamental de la théorie des nombres. Cette fonction apparaît à cause de la structure du réseaux* $\mathbb{Z}^2$*. Les résultats en dimension 2 sont en fait le fruit de la simplicité combinatoire des polygones, comparée aux dimensions supérieures.*

La question du dénombrement est étroitement liée à la génération aléatoire. Ainsi, nous ne savons pas faire génération aléatoire de polytopes entiers dans un hypercube de côté de *longueur 10 en dimension $d \geq 3$, contrairement à de nombreux modèles de combinatoire, probabilistes, ou de physique statistique où la génération aléatoire permet de tirer des conjectures, ou de les réfuter par des exemples.*

Remarque 2.4.2. *Rappelons que les zonogones sont les polygones centralement symétriques. Pour un polygone P, dont le diamètre est atteint aux sommets A et B, on peut donc associer deux zonogones qui correspondent aux deux arcs de P reliant A et B. L'étude des polygones convexes entiers peut donc passer par l'étude des zonotopes convexes.*

Nous allons maintenant énoncer rigoureusement la dernière phrase de la Remarque 2.4.2 et rappeler les résultats récents d'énumération et de forme limite des zonotopes.

2.4.2 Polygones, zonogones et forme limite des zonotopes entiers.

Prenons $P \subset \mathbb{R}^2$ un polygone convexe entier. On distingue quatre segments extrémaux de ∂P, $[A, A']$, $[B, B']$, $[C, C']$ et $[D, D']$ (possiblement de longueur nulle), les segments respectivement les plus à sud, à l'est, au nord et à l'ouest de ∂P, où les points cardinaux représentent les directions du réseaux $\mathbb{Z}^2$.

P peut alors être décomposé comme l'enveloppe convexe de quatre arcs polygonaux convexes, qui relient les quatre segments extrémaux (comme illustré dans la figure 2.12).

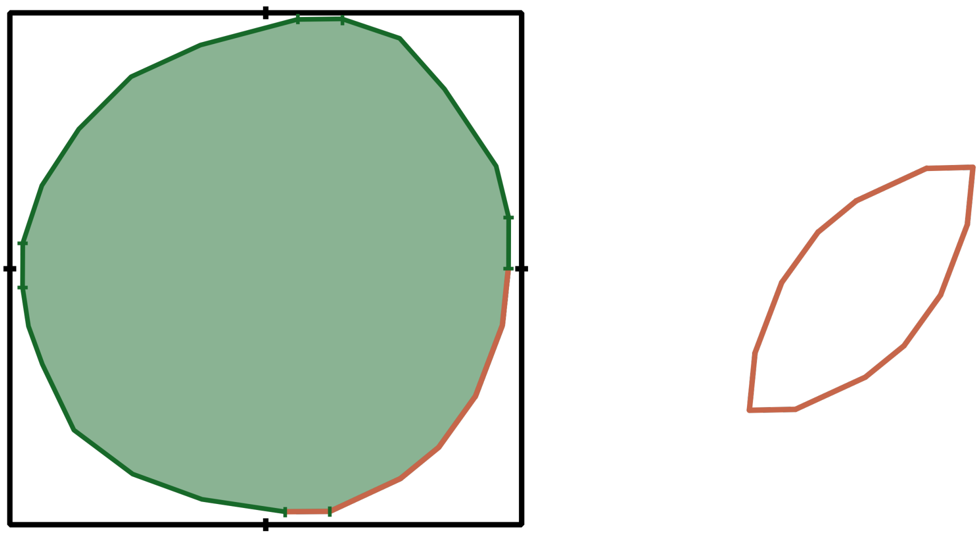


Figure 2.12: Exemple de polygone entier tiré uniformément dans $\mathcal{P}_n([-1, 1] \times [-1, 1])$, dont les segments extrémaux sont marqués à gauche et le zonogone correspondant à son arc sud-est à droite.

L'étude des polygones peut donc être ramenée à l'étude des arcs polygonaux convexes, comme ce fut déjà le cas pour Vershik dans [151], pour établir la forme limite des polygones énoncée précédemment. Or un arc polygonal convexe est équivalent au zonogone composé de cet arc et de son symétrique par rapport au milieu du segment reliant ses deux extrémités.

Formellement, soient k points $(x_i, y_i)_{1\le i\le k}$ de $\mathbb{Z}^2$ tels que $0 = x_0 < x_1 < ... < x_{k-1} \le x_k$, $0 = y_0 \le y_1 < y_2 < ... < y_k$ et

$$0 \le \frac{y_1 - y_0}{x_1 - x_0} < ... < \frac{y_k - y_{k-1}}{x_k - x_{k-1}} \le +\infty. \tag{2.4.10}$$

Un ***arc convexe polygonal*** de k segments est la réunion des segments $[(x_i, y_i), (x_{i+1}, y_{i+1})]$ pour $i \in \{0, 1, ..., k-1\}$. Le point (x_0, y_0) est l'origine de l'arc et (x_k, y_k) est la fin. Du point de vue combinatoire, cet arc convexe croissant est entièrement défini par les vecteurs $(x_{i+1} - x_i, y_{i+1} - y_i) \in \mathbb{N}^2$ pour $0 \le i \le k-1$ et il correspond au zonogone dont les générateurs sont $((x_{i+1} - x_i, y_{i+1} - y_i))_{0\le i\le k-1}$ dans la bijection naturelle entre les zonogones et les arcs polygonaux convexes.

L'étude des zonogones est donc directement liée à celle des polygones. Cela n'est plus du tout le cas pour les dimensions supérieures. Cependant, à défaut de résultats sur les polytopes en toute dimension, les résultats sur les zonogones peuvent être généralisés en toute dimension, comme on le verra dans les chapitres 4 et 5.

En 2013 et 2016, O. Bodini, P. Duchon, A. Jacquot et L. Mutafchiev [25] puis J. Bureaux et N. Enriquez [38] ont calculé l'équivalent asymptotique du nombre d'arcs polygonaux convexes entre $(0, 0)$ et (n, n). Les premiers étudiaient les polyominos convexes et utilisaient la même décomposition cardinale en appliquant la méthode du point-col de manière semblable à la méthode utilisée dans le chapitre 4, les seconds prennent un modèle probabiliste de physique statistique, introduit par Sinai, que nous ré-emploierons dans le chapitre 5 et utilisent un théorème de limite locale de S. V. Bogachev et S. M. Zarbaliev [26]. En considérant $z_2(n)$ le nombre d'arcs polygonaux convexes entre $(0, 0)$ et (n, n), ils obtiennent, en notant $\kappa = \frac{\zeta(3)}{\zeta(2)}$,

$$z_2(n) \underset{n\to+\infty}{\sim} \frac{e^{-2\zeta'(-1)}}{(2\pi)^{7/6}\sqrt{3}\kappa^{1/18}n^{17/18}} \exp\left(3\kappa^{1/3}n^{2/3} + I\left(\left(\frac{\kappa}{n}\right)^{1/3}\right)\right), \tag{2.4.11}$$

où $I(x)$ est une intégrale de la fonction $s \mapsto \frac{\Gamma(s)\zeta(s+1)(\zeta(s-1)+\zeta(s))}{\zeta(s)x^s}$ sur un contour entourant les zéros non triviaux de la fonction ζ de Riemann. Nous retrouvons ce lien avec la théorie des nombres qui a fait l'objet d'une remarque précédente. La présence de l'intégrale $I(x)$ permet à Bureaux et Enriquez de démontrer que l'équivalent asymptotique des arcs convexes est lié à l'hypothèse de Riemann :

Proposition 2.4.1 (Bureaux et Enriquez)**.** *Avec les notations précédentes, L'hypothèse de Riemann est valide si et seulement si pour tout $\epsilon > 0$,*

$$I\left(\left(\frac{\kappa}{n}\right)^{1/3}\right) = o\left(n^{\frac{1}{6}+\epsilon}\right). \tag{2.4.12}$$

La question est alors de généraliser ces résultats en dimensions supérieures. La formule du nombre de zonotopes dans un d-hypercube de côté n est l'objet du chapitre 4. Toutefois, dans un papier majeur de 2018 [14], J. Bureaux, I. Bárány et B. Lund ont calculé l'équivalent logarithmique de ce nombre, ainsi que démontré la convergence de ces zonotopes vers une forme limite.

Dans leur modèle, ils étudient les zonotopes dans des cônes convexes saillants et pointés, nous en reparlerons dans le chapitre 5. En dimension 2, en prenant le cône $(\mathbb{R}_+)^2$, les zonogones considérés sont exactement les arcs convexes polygonaux. Notons le vecteur $\mathbf{1} = (1, ..., 1)$ et par $z_d\left((\mathbb{R}_+)^d, n\mathbf{1}\right)$ le nombre de zonotopes dont les générateurs sont des vecteurs entiers de $\mathbb{R}^d$ dont la somme fait $(n, ..., n)$, alors

$$\log z_d\left((\mathbb{R}_+)^d, n\mathbf{1}\right) \underset{n\to+\infty}{\sim} (d+1)\kappa_d^{\frac{1}{d+1}} n^{\frac{d}{d+1}}, \quad \text{avec } \kappa_d = \frac{\zeta(d+1)}{\zeta(d)}. \tag{2.4.13}$$

Considérons maintenant un cône convexe saillant pointé $\mathcal{C}$ de dimensions $d \geq 2$. Pour un vecteur k contenu dans l'intérieur de $\text{int}(\mathcal{C})$, on définit $\mathcal{Z}(\mathcal{C}, n\boldsymbol{k})$ l'ensemble des zonotopes dont les générateurs sont des vecteurs de $\mathcal{C}$ tels que la somme des générateurs est égale à $n\boldsymbol{k}$. On a alors :

Théorème 2.4.3 (Bureaux, Bárány, Lund)**.** *Il existe un ensemble convexe Z_0 tel que, pour tout $\epsilon > 0$,*

$$\lim_{n\to+\infty} \frac{\left|\{Z \in \mathcal{Z}(\mathcal{C}, n\boldsymbol{k}), \delta(\frac{1}{n}Z, Z_0) \leq \epsilon\}\right|}{z_d(\mathcal{C}, n\boldsymbol{k})} = 1. \tag{2.4.14}$$

Ce résultat s'appuie sur la démonstration d'un théorème central limite de la somme des générateurs $\mathbf{X}(Z)$ et sur une inégalité de Chernoff, pour la distribution introduite par Sinai, dite "de Boltzmann" que nous rappellerons dans le chapitre 5. La question naturelle après un tel résultat est de regarder les fluctuations des zonotopes autour de cette forme limite Z_0. L'un des résultats principaux de cette thèse (voir le théorème 5.4.1) est la preuve que ces fluctuations sont gaussiennes. Nous en calculons aussi explicitement la covariance.

Chapitre 3

Introduction (English)

In this chapter, we will introduce the geometric and combinatorial notions discussed in the various chapters of this thesis.

The central objects are the convex polytopes, which is why the first part of this chapter will be devoted to their presentation. These objects are fundamental in geometry, and there are few areas of mathematics where convex polytopes have no application. It seemed essential to us to introduce the polytopes in the broadest possible way, to give the reader, without claiming an exhaustive overview, a vision of these objects broader than their strict appearance in our work.

The second primary object of this manuscript is the zonotope. To introduce it, we will start with partitions to define zonotopes as a multidimensional generalization of integer partitions. We will retrace the history of the asymptotic study of the enumeration of partitions to echo the result of the enumeration of zonotopes of Chapter 4. Then, after having given some possible definitions of zonotopes, we will recall the link between zonotopes and oriented matroids to support the results given in Chapter 5.

In addition to these two objects, putting the results of Chapters 4 and 5 into context is essential. We will thus finish by recalling the enumerative results on the convex lattice polytopes since the introductory question of Vladimir Arnold in 1980.

3.1 Polytopes

This section gives a general description of convex polytopes and some notable results in this area. We will therefore start by giving the majority of the vocabulary and geometric notations that we will need. Subsequently, we will successively give a brief overview of the study of the combinatorics of the faces of convex polytopes, of lattice polytopes and their applications, and the use of polytopes in linear optimization. These parts are unequally related to the topics of the following chapters, but this overview will allow the reader to contextualize the following results.

3.1.1 Generalities

Polygons and polyhedra have a multi-millennial history. Their origin is inseparable from geometry and dates back at least 4500 years [93]. We have left from Antiquity, among others, the Egyptian pyramids, the theorems of the Greek geometers Pythagoras and Thales, and the Platonic solids. At the same time, polygons also have a long personal history with most people; they have surely marked the first memories of mathematics of every pupil and are even present in the drawings of toddlers. What an academic might pompously call a 2-dimensional simplex (the triangle) is the universal representation of dresses and teeth for any child under 5.

To generalize the notion of polygon in dimension d, we will first recall some definitions. In $\mathbb{R}^d$, the ***convex hull*** of a finite set S is the smallest convex set $\text{conv}(S)$ that contains S, that is, the smallest set that contains S such that for any element x and y of $\text{conv}(S)$, the segment $[x, y] = \{tx + (1-t)y, t \in [0,1]\}$ is in $\text{conv}(S)$. Equivalently, a polytope can be defined as the set of barycenters with positive coefficients of S, that is:

$$\text{conv}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i, x_i \in S, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{3.1.1}$$

We also recall that in $\mathbb{R}^d$, an ***affine half-space*** is the set of vectors x satisfying $g(x) \geq 0$ for some linear form affine $g : \mathbb{R}^d \to \mathbb{R}$, defined by

$$g(x_1, ..., x_d) = a_0 + a_1 x_1 + ... + a_d x_d, \text{ pour } a_0, ..., a_d \in \mathbb{R}. \tag{3.1.2}$$

such that $a_1, ..., a_d$ are not all zero. The affine hyperplane of affine linear form g therefore separates $\mathbb{R}^d$ into two affine half-spaces. We can now give the two equivalent definitions of a polytope in $\mathbb{R}^d$:

Definition 3.1.1. *A **(convex) polytope** is a set of $\mathbb{R}^d$ defined by one of these two equivalent definitions:*

- *the convex hull of a finite number of vectors of* $\mathbb{R}^d$.
- *The intersection of a finite number of affine half-spaces such that this set is bounded.*

Following the example of the main references in the field ([78, 155]), we will simply write polytope instead of convex polytope, non-convex polytopes not being addressed in this manuscript. A polytope P has dimension d and we will denote it a ***d-polytope*** in the following if the affine space generated by P is of dimension d. This affine space generated by P is rigorously defined by

$$\text{aff}(P) = \left\{ \sum_{0 \leq i \leq n} \lambda_i p_i, \lambda_i \in \mathbb{R}, p_i \in P, \sum_{0 \leq i \leq n} \lambda_i = 1 \right\}. \tag{3.1.3}$$

A subset f of P is a ***face*** if there exists an affine hyperplane $\mathcal{H}$ such that $P \cap \mathcal{H} = f$ and that P is contained in only one of the two affine half-spaces generated by $\mathcal{H}$. This definition extends the usual notion of face in 3 dimensions to all dimensions. In particular, $\emptyset$ is a face of P (when P is also considered a face, $\emptyset$ and P are called ***improper*** faces of P , as opposed to all the others, which are called ***proper***). Thus a face of dimension 0 is called ***vertex***, a face of dimension 1 is a ***edge***, and a face of dimension $d-1$ is called a ***facet***. A k-face of P is also a k-polytope. The ***interior*** of P, denoted $\mathring{P}$ is the topological interior of P in the affine space it generates. Analogously, the set $\partial P = P \setminus \mathring{P}$ is the ***boundary*** of P. The set of faces of the polytope P is denoted $L(P)$, and, ordering this set with the inclusion relation, we obtain the partially ordered set $(L(P), \subset)$, which happens to be a poset, called the ***lattice of faces*** of P.

The two equivalent definitions (this equivalence is known as the Minkowski–Weyl theorem, called by G. Ziegler the "Main polytope Theorem" [155]) can be seen as "dual" definitions: the polytope is defined by its vertices in the first, whereas its facets in the second define it. The same type of definition applies to ***polyhedral cones***, which are the intersection of a finite number of linear half-spaces passing through the origin or the set generated by the conic combinations of a finite number of vectors of $\mathbb{R}^d$.

For a d-polytope P, we denote $f_k(P)$ the number of k-faces of P. The vector that encodes the number of faces of P, $(f_0(P), f_1(P), ..., f_{d-1}(P))$, is called the ***f-vector*** of P. We sometimes add for the empty face $f_{-1}(P) = 1$ in the f-vector (by convention, the dimension of $\emptyset$ is -1). The study of this f-vector, and therefore of the combinatorics of the faces of a polytope is an essential field of research; we will discuss it again in the next section.

If we are interested in the edges of a polytope P, we can look at the graph (V, E) of P where V is the set of vertices and E the set of edges of P. The ***diameter of the graph*** of P, $\delta(P)$ is defined as the smallest integer such that all the vertices of P are connected two by two by at most this number. This notion is the subject of one of the most famous polytopal conjecture, described later.

Here are some well-known families of polytopes: first, for a d-polytope B and a point x_0 affinely independent, the ***pyramid*** with base B and vertex x_0 is the convex hull of $B \cup \{x_0\}$. The ***simplices*** are defined as the convex hull of affinely independent points. The d-simplex therefore has $d+1$ vertices. The 2-simplex is the triangle, the 3-simplex is the tetrahedron, and we can simply construct by induction the d-simplex as a pyramid with a base of the $(d-1)$-simplex and a point that is not contained in the affine space of this base. Finally, the ***hypercube*** of dimension d (or d-cube) is the Cartesian product of the segment $[0,1] \subset \mathbb{R}$, that is the polytope $[0,1]^d$ (or any translation, rotation and dilation of this one).

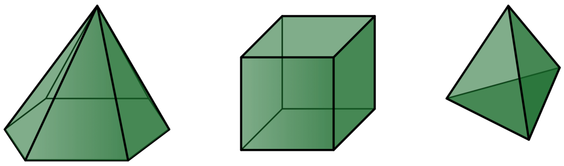


Figure 3.1: A pyramid over an hexagon, a cube, and a 3-simplexe.

Let us return to any d-polytope P in $\mathbb{R}^d$. We will introduce the notion of normal fan of P, a notion widely used in polytope combinatorics [155] that will be instrumental in Chapter 6. A ***fan*** is a set of polyhedral cones such that any intersection between two cones is a face of each, and for any cone in the fan, all its nonempty faces are in the fan. For a face F of P, we define the ***normal cone*** N_F of F as the set of vectors of $\mathbb{R}^d$ whose associated linear form is maximum in F, i.e.

$$N_F = \left\{ \boldsymbol{c} \in \mathbb{R}^d, F \subset \{\boldsymbol{x} \in P, \boldsymbol{c} \cdot \boldsymbol{x} = \max_{\boldsymbol{y} \in P} \boldsymbol{c} \cdot \boldsymbol{y}\} \right\}. \tag{3.1.4}$$

And we define the ***normal fan*** of P as follows:

$$\mathcal{N}(P) := \{N_F, F \text{ face of } P\}. \tag{3.1.5}$$

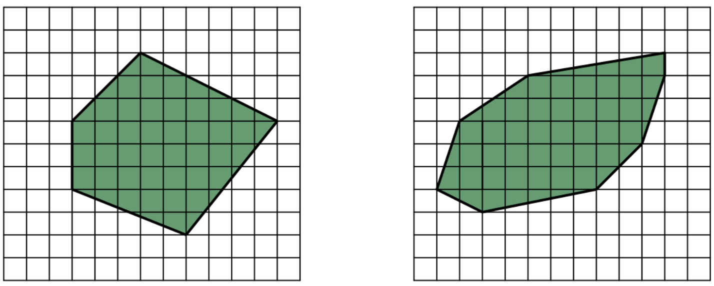


Figure 3.2: Two lattice polygons.

In Chapters 4 and 5, we study polytopes in the case where their vertices are in $\mathbb{Z}^d$. These polytopes are called ***lattice polytopes***. When we look at lattice polytopes, it is natural to be interested in the elements of $\mathbb{Z}^d$ which belong to a polytope P and to its faces. Henceforth, we will speak of ***primitive vector*** (or primitive point) if the gcd of its coordinates is equal to 1. A primitive segment is then precisely, up to translation, a segment between 0 and a primitive point. In other words, a segment is primitive when the only lattice points it contains are its extremities.

3.1.2 Combinatorics of faces

The combinatorial study of the faces of polytopes (e.g. the characterization of polytopes according to their number of k-faces, or the study of the number of faces of a polytope according to their dimension) is a central area in the study of polytopes, which harbors some of the most famous conjectures. We will give a quick overview of the salient results in this field, as well as some famous conjectures.

In 2 dimensions, the combinatorics of polygons is trivial, we will not dwell on it here, but note that enumerative questions are very interesting as we shall see in Section 3.4.1. In 3 dimensions, Steinitz's famous theorem fully characterizes polyhedra by the graph induced by their vertices and edges:

Theorem 3.1.1 (Steinitz, 1922 [145]). *G is the graph of a 3-polytope if and only if it is simple, planar and 3-vertex-connected.*

Ziegler devotes an entire chapter of [155] to this theorem which is of seminal importance for several reasons. First, despite the simplicity of this theorem, there is no "simple" proof. Also, we have yet to find any equivalent of this theorem for higher dimensions so far, and there doesn't seem to be any [80, Section 13.1]. Finally, the proof of this theorem allows one to demonstrate several corollaries and stronger versions of this theorem, which are not true in higher dimensions. Eighty years after Steinitz, J. Mihalisin and V. Klee demonstrated a characterization of the directed 3-polytopal graphs, that is a graph that is isomorphic to the directed graph resulting from the orientation of the graph of a 3-polytope P by some affine function on P [115]. The reader is invited to read the chapters dedicated to this theorem in [80, 155] for detailed proofs, and a list of corollaries.

In dimension d, the combinatorial interests relate to the f-vector defined in the previous section, which encodes the number of k-faces for all the integers k between -1 and d. The main open question is the characterization of polytopes according to their f-vector. In particular, which vectors of $\mathbb{N}^{d+1}$ are the f-vector of a polytope? This problem is still far from being solved, but we will present a selection of remarkable results, some of which are very recent. The first of these results is the Euler–Poincaré formula:

$$-f_{-1} + f_0 - f_1 + ... + (-1)^d f_d = 0. \tag{3.1.6}$$

This formula is known in particular in dimension 3, for planar graphs as Euler's formula. Let G be a planar graph with n vertices, e edges, whose embedding in the plane breaks it down into f connected components of dimension 2, then

$$n - e + f = 2. \tag{3.1.7}$$

Besides this formula, several well-known results give conditions for which vectors can be the f-vectors of a polytope. What is the maximum number of k-faces of a d-polytope with n vertices? The answer to this question was conjectured by Motzkin in 1957 [117] and demonstrated by McMullen in 1970 [111]. It is known as the upper bound theorem, and it bounds the number of faces of a polytope at n vertices using the ***cyclic polytopes***. A cyclic d-polytope with n vertices is the convex hull of n points on the d-dimensional moment curve

$$\{(t, t^2, ..., t^d), t \in \mathbb{R}\}. \tag{3.1.8}$$

All the polytopes resulting from this construction are combinatorially equivalent (their face lattices are isomorphic) and we will denote any d-dimensionnal one of them with n vertices by $C_d(n)$.

Theorem 3.1.2 (of the upper bound, McMullen, 1970)**.** *If P is a d-polytope with $n = f_0$ vertices, then for all k, the number of its k-faces is at most the number of k-faces of the corresponding cyclic polytope $C_d(n)$:*

$$f_k(P) \leq f_k(C_d(n)). \tag{3.1.9}$$

This theorem can be demonstrated with the notion of shellability, which is a way of decomposing a polytopal complex into an aggregate of facets with special gluing rules. This result is much stronger than it looks as it tells that there is a single polytope (or more precisely a single combinatorial type) that achieves the maximal value of the number of faces of all dimensions, which may seems counter-inuitive. Recently, K. Adiprasito and R. Sanyal came up with an upper bound theorem on Minkowski sums of polytopes [2]). Minkowski sum will be properly defined in Section 3.3.1. For the lower bound, the results are more complicated and involve the h-vector, a derivative of the f-vector which respects Dehn-Sommerville equalities [118]. In a few paragraphs, we will state two other, very recent results of the lower bound of the f_k, which stood for significant conjectures for several decades.

Another legitimate question is: do the coordinates of the f-vector form a unimodal sequence? A sequence of n numbers $(a_1, ..., a_n)$ is unimodal if

$$a_1 \leq a_2 \leq .... \leq a_k \geq a_{k+1} \geq ... \geq a_n, \tag{3.1.10}$$

for some k between 1 and n. The question arises for simplicial polytopes, i.e. polytopes whose all the facets are simplices. The first examples may mislead to think that the answer is positive; however, in 1981, A. Björner finds a counterexample in dimension 24 [22], but shows that there still is a partial unimodularity, in the sense that the f-vector of a simplicial d-polytope satisfies the double-chain inequality

$$\begin{cases} f_0 < f_1 < \ldots < f_{\lfloor \frac{d}{2} \rfloor} \\ f_{\lfloor \frac{3(d-1)}{4} \rfloor} > \ldots > f_{d-1}. \end{cases} \tag{3.1.11}$$

For simplicial polytopes, the g-theorem fully characterizes f-vectors. This theorem has been proved by L. Billera and C. Lee for sufficiency [19] and by R. Stanley for necessity [143].

Finally, we would like to review four famous problems highlighted by G. Kalai in [95], two of which have recently been proven. The first is a conjecture stated in 1967 by B. Grünbaum (see [79] and the more recent version [80]) proved in 2021 by L. Xue [153], concerning a lower bound on the number of k-faces of polytopes which have "few" vertices. Formally, Xue showed that a convex d-polytope with $d+s \leq 2d$ vertices has at least

$$\phi_k(d+s,d) = \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1} \tag{3.1.12}$$

k-faces, thanks to a clever double induction. As for the second problem, which was first stated as a question, I. Bárány asked the following question [20, Problem 17.6.5]: for any d-polytope P, can we find a constant c_d, for example $c_d = 1$, such that

$$f_k(P) \geq c_d \min(f_0(P), f_{d-1}(P))? \tag{3.1.13}$$

In 2022, J. Hinman proved that the answer is yes for $c_d = 1$, based on the notion of multidimensional angle developed in 1970 by M. Perles and G. Shephard [122]. These two extremely recent results, on two leading conjectures dating from the last century, show both the difficulty of establishing results on the combinatorics of polytopes and the significant scientific advance of contemporary research.

The other two problems are still unsolved when we write these words and date respectively from the late 90s and the late 80s. The first concerns the 4-polytopes and the proportion of faces of intermediate dimensions compared to the number of vertices and facets, namely does there exist a constant which bounds the quantity $(f_1(P) + f_2(P))/(f_0(P) + f_3(P))$ for any 4-polytope P [60]?

The last problem, perhaps the best-known conjecture in polyhedral combinatorics along with the Hirsch conjecture, that was already mentioned in Section 3.1.4, is due to G. Kalai in [96]:

Conjecture 3.1.1 (G. Kalai's conjecture 3^d). *A centrally symmetric d-polytope has at least 3^d nonempty faces.*

This conjecture is proved for $d \leq 4$ [131] and for some classes of polytopes [67], such as the Hanner polytopes, that can be constructed by induction on the dimension by using the operations of taking the dual of a polytope or the cartesian product of two of them.

3.1.3 Lattice polytopes

Chapters 4 and 5 focus exclusively on lattice polytopes (polytopes whose coordinates of vertices belong to the lattice $\mathbb{Z}^d$), or a subfamily. Before describing in the next sections the combinatorial motivations of our results, here is a brief overview on these fascinating objects and their applica- tions.

Lattice polytopes are a subject in their own right in the scientific community, and the majority of expectations, works and questions about lattice polytopes revolve around the difficult following problems (see for instance [49]):

- how many integer points are there in a lattice polytope?
- what is the relationship between this number and the volume of the polytope?
- how many points are there in the relative interior of the polytope?

In the 1970s, M. Demazure [51], G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat [98] and others created the theory of toric varieties where the study of lattice polytopes is essential. Indeed, toric geometry is based on an association between algebraic varieties and lattice polytopes, and naturally associates questions about varieties with the properties of lattice polytopes [63]. Since then, this dictionary has been used in all fields of application of toric manifolds, notably in physics, where Calabi-Yau manifolds are a basic element of string theory [156].

George Pick's theorem [123] is the first result of magnitude to answer these questions, in the simplest case of polygons.

Theorem 3.1.3 (Pick, 1900). *Let $P \subset \mathbb{R}^2$ be a polygon. If P is integer,*

$$\mathrm{vol}(P) = \left|P \cap \mathbb{Z}^2\right| - \frac{1}{2}\left|\partial P \cap \mathbb{Z}^2\right| - 1. \tag{3.1.14}$$

Subsequently, H. Blichfeldt [24], then D. Hensley [87], and others will give a bound on the volume of P in any dimension depending on the interior integer points of the polytope. One of the most beautiful results in the field is the following result [59], by Eugène Ehrhart, which provides a key property of the number of integer points in the dilations of lattice polytopes.

Theorem 3.1.4 (Ehrhart, 1967). *Let P be a lattice polytope in $\mathbb{R}^d$. We denote $\phi_P : \mathbb{N} \to \mathbb{N}^*$ the function such that $\phi_P(m) = \left|mP \cap \mathbb{Z}^d\right|$. If P has dimension n, then ϕ_P is a polynomial*

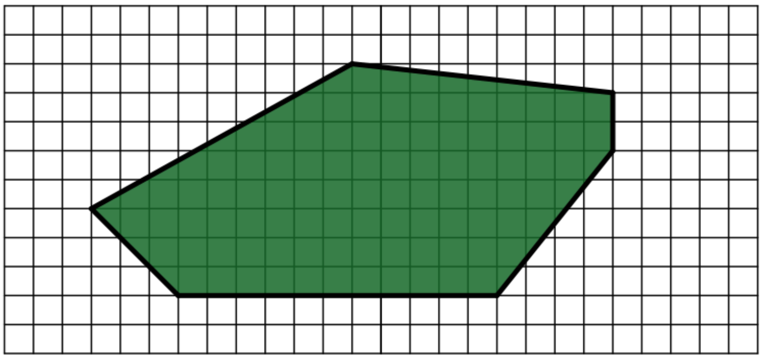


Figure 3.3: TThe area of this lattice polygon is 102.5, which can be computed with Pick's theorem.

of degree n, its leading coefficient is the normalized volume of P, denoted $V(P)$, and its constant coefficient is 1. Moreover $\phi_P(0) = 1$ and

$$\phi_P(-m) = (-1)^d \left| m\mathring{P} \cap \mathbb{Z}^d \right| \tag{3.1.15}$$

where $\mathring{P}$ denotes the interior of P.

These polynomials (called Ehrhart's polynomials) have been the subject of numerous studies, including the characterization of these polynomials by Fourier transforms, by Ricardo Diaz and Sinai Robins [57]. Recently, Manecke and Sanyal gintroduced a similar polynomials that count the points whose coordinates are relatively prime in the dilation of polytopes [108]. These polynomialswere computed explicitly in the special case of cross polytopes in [55].

The other main area of application where lattice polytopes appear is optimization. In linear optimization, the problem of knowing if there are one or more integer points in a polytope, and then of characterizing them, is primordial. For polytopes whose interior contains very few integer points, it is natural to look at extreme cases of polytope "size". To do this, we define the ***width*** of a polytope P, as the minimum

$$\min_f (\max_{p,q \in P} |f(p) - f(q)|), \tag{3.1.16}$$

among the non-constant linear forms f with integer coefficients. Among the lattice polytopes which have very few integer points, the best-known families are the ***hollow polytopes*** [44] which have no integer points in their relative interior, of which the flatness Theorem [100] bounds the width, or even the ***empty polytopes***, which have no other integer points than their vertices, which include the famous hypersimplices.

Among the polytopes which have precisely one integer point in their relative interior, we count ***Fano d-polytopes***, whose each facet contains exactly d integer points, and the ***reflexive polytopes***, which are the lattice polytopes with exactly one inner point, such that every facet is at distance 1 from this point. These two families are very important in algebraic geometry.

3.1.4 Linear optimization

The final topic covered in this section is perhaps the most well-known area of use of polytopes: constrained optimization. The simplest problem is the linear problem which amounts to finding a certain vector $x \in \mathbb{R}^n$, and it can be expressed as

$$\begin{aligned} &\text{maximize } c^{\intercal} x \\ &\text{subject to } Ax \leq b, \end{aligned} \tag{3.1.17}$$

where c is a vector of $\mathbb{R}^n$ and b and A are respectively a vector in $\mathbb{R}^m$ and a matrix in $\mathbb{R}^{m\times n}$. The function to be maximized is called the objective function. We recall that the inequality $Ax \leq b$ defines a polyhedron, which is not necessarily bounded. There are of course two steps to solve this problem. First, find out if there is a x that satisfies the constraints (named ***feasible point***), then, if there is, find the optimum or prove that we can't reach it. We will not dwell on optimization problems to return to the polytopes that interest us, but the algorithms that solve this problem can be grouped into three groups: the simplex method, the interior point method, and the ellipsoid method. The latter is particularly interesting in theory because Khachyian demonstrated that it solves linear problems in polynomial time [99], but it is the simplex method that is at the origin of one of the greatest conjectures of the XX th century on polytopes.

The simplex algorithm, invented by G. Dantzig in 1947 [48], is the first method for solving linear problems. The concept is first to find a basis $B \subset \{1, ..., m\}$ for which the matrix composed of rows of A whose index belongs to B, A_B, is invertible and such that $x_B = A_B^{-1} b_B$ is a feasible point, and a vertex of the polyhedron defined by the matrix inequality. Starting from this, we build a new x_B which has a higher objective value using a pivot rule. This construction can be interpreted geometrically as a path on the vertex set of the constraint polyhedra. The pivot rules therefore determine the way to choose the next vertex among all the neighbors of x_B, the best known being Danzig's rule which chooses the edge which maximizes the objective function after renormalization (therefore, the edge whose " slope" is closest to the direction of c). Once the edge has been chosen, we replace the condition $A_i^{\intercal} x \leq b_i$ where equality doesn't stand anymore with the new condition verified at the other end of the edge.

This method therefore depends on the size of the paths between two vertices of a polyhedron, and, therefore on the diameter of the graph of this polyhedron. Indeed the diameter of the polytope graph is a lower bound of the number of steps of the simplex algorithm in the worst case. However, the question of knowing if there exists a pivot rule for which the simplex terminates in polynomial time is still not resolved. Besides, the ninth Problem of the XXI$^{\text{th}}$ century of S. Smale deals with the broaden question of complexity in Linear Programming [140]. This directly justifies the study of upper bounds on the diameter of polytope graphs, and in this regard, W. Hirsch gives, in a letter to Danzig, his famous conjecture:

Conjecture 3.1.2 (Hirsch, 1957, disproved in 2010)**.** *The graph of a d-polytope with n vertices has a diameter less than or equal to $n - d$.*

More than 50 years later, F. Santos Leal [130] finds a counterexample in dimension 43 (!) using the result of V. Klee and W. Walkup which establishes the equivalence between Hirsch's conjecture and the same conjecture for only d-polytopes with $2d$ edges [101]. The current conjecture is therefore:

Conjecture 3.1.3 (Hirsch's polynomial conjecture)**.** *There exists a polynomial function f such that for any polytope P with n vertices, the graph of P has a diameter of at most $f(n)$.*

Because of the difficulty in solving it and its longevity, this conjecture is one of the leading questions in the study of polytopes (see the dedicated chapters in [155, Chapter 3], [80, Chapter 16]).

3.2 Partitions

The work carried out in Chapters 4 and 5 essentially concerns a family of particular polytopes called ***zonotopes***. Since zonotopes can be seen as a generalization of partitions, the asymptotic results of these chapters are in line with combinatorial work on partitions, which we will introduce in this section.

Take a positive integer n. A ***partition*** of n is a writing of n as a sum of positive integers. In fact, it's equivalent to partitioning n indistinguishable objects into smaller groups. How many different possibilities are there to partition n? This question will be our common thread throughout this section.

Let's start with an example to clarify the notations. On the left, we decompose 5 into a sum of integers, and on the right we denote in a list the number of appearances of each nonzero number (this list is infinite but said to be almost zero, because there are only 0s to above a certain rank).

$$5 = 5 \qquad (0,0,0,0,1,0,\ldots) \tag{3.2.1}$$
$$5 = 4+1 \qquad (1,0,0,1,0,0,\ldots) \tag{3.2.2}$$
$$5 = 3+2 \qquad (0,1,1,0,0,0,\ldots) \tag{3.2.3}$$
$$5 = 3+1+1 \qquad (2,0,1,0,0,0,\ldots) \tag{3.2.4}$$
$$5 = 2+2+1 \qquad (1,2,0,0,0,0,\ldots) \tag{3.2.5}$$
$$5 = 2+1+1+1 \qquad (3,1,0,0,0,0,\ldots) \tag{3.2.6}$$
$$5 = 1+1+1+1+1 \qquad (5,0,0,0,0,0,\ldots) \tag{3.2.7}$$

Note that the formalization on the right is quite adequate for denoting a partition: there is a bijection between the decompositions of a number as a sum and these sequences of numbers that are almost zero. Seeing these lists as functions, we will keep the following notation:

Definition 3.2.1. *A partition ω of n is an application of $\mathbb{N}^*$ in $\mathbb{N}$ such that*

$$n = \omega(1) + 2\omega(2) + 3\omega(3) + ... \tag{3.2.8}$$

.

We will denote $p(n)$ the number of partitions of n, for all $n \in \mathbb{N}^*$. So we have from the previous list $p(5) = 7$, and here are some following values: $p(6) = 11$, $p(7) = 15$, $p(8) = 22$, then $p(20) = 627$ and $p(100) = 19056992$. There are many questions about the sequence of $p(n)$: how fast does it grow? Is there a way to calculate them efficiently? What arithmetic properties does it have? Can we make links with other parts of number theory, for example prime numbers? Over the centuries, many properties have been found, we will give two of the best-known:

Proposition 3.2.1. *The number of partitions of n into exactly m parts is equal to the number of partitions of n into parts of size at most m.*

Proposition 3.2.2. *The number of partitions of n into parts of distinct sizes is equal to the number of partitions of n into parts of odd size.*

The first proposition is graphically obvious with the Young diagrams, the second will be detailed below.

The study of $p(n)$, which already appeared in correspondence between G. Leibniz and J. Bernoulli, only really began with Euler, following a famous letter by P. Naudé le Jeune, in 1740 where he asks the following questions: how many different ways can the number 50 be written

- as the sum of 7 distinct integers?
- as the sum of 7 equal or distinct integers?

L. Euler presents, in 1741, his revolutionary solution [62, 105]: he introduces infinite products and infinite sums. He has just created the ***generating series***. To answer the first question, he first writes the following infinite product:

$$(1 + xz)(1 + x^2 z)(1 + x^3 z)... \tag{3.2.9}$$

By expanding all the factors, we find the answer to the first question as the coefficient of $x^{50}z^7$ in the series obtained. He uses the fact that $\frac{1}{1-x}$ expands serially to $1 + x + x^2 + x^3 + ...$, to deduce that the second question is handled in a way similar by observing the coefficients of the series expansion of

$$\frac{1}{(1-xz)(1-x^2z)(1-x^3z)...}. \tag{3.2.10}$$

In this product, the k^{th} factor encodes the number of parts of sizes k in the partition of n. The generating series of the number of partitions of a number n follows directly; and for any natural number n, $p(n)$ is the coefficient of z^n in the series expansion of

$$\prod_{k=1}^{+\infty} \frac{1}{1-z^k}. \tag{3.2.11}$$

To find the equality of Proposition 3.2.2, denote the equality between the respective generating series of partitions with parts of distinct sizes and partitions with parts of odd sizes:

$$\prod_{k=1}^{+\infty}(1+x^k) = \prod_{k=1}^{+\infty} \frac{1-x^{2k}}{1-x^k} = \prod_{k=0}^{+\infty} \frac{1}{1-x^{2k+1}} \tag{3.2.12}$$

3.2.1 Asymptotic results

> " The theory of partitions has, from the time of Euler onwards, been developed from an almost exclusively algebraical point of view. It consists of an assemblage of formal identities – many of them, it need hardly be said, of an exceedingly ingenious and beautiful character. Of *asymptotic* formluæ, one may fairly say, there are none. "[82]

As G. H. Hardy and S. Ramanujan mention with surprise in their introduction, the question of an asymptotic study of this generating series has not been explored for more than 150 years, except for a formula by P. A. MacMahon [107]. In particular, there is no asymptotic result in the study of $p(n)$, before the famous result of Hardy and Ramanujan [82]

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \tag{3.2.13}$$

Hardy and Ramanujan's techniques have a major impact on XX$^{\text{th}}$ tools in number theory. To calculate this equivalent, they use the *Hardy-Littlewood circle method* based on the Cauchy formula of the n^{th} coefficient of a series. Beyond the asymptotic formula, Ramanujan brought the intuition of a more precise formula, in $O(1)$, qualified by G. Andrews [6, p.69] of "his most important contribution; it was both absolutely essential and most extraordinary."

In 1937, Rademacher took up the ideas and intuitions of Ramanujan and Hardy to establish an explicit formula for $p(n)$ as a convergent series:

Proposition 3.2.3 (Rademacher)**.**

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{+\infty} A_k(n) k^{\frac{1}{2}} \left\{ \frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\left(\frac{2}{3}\left(x - \frac{1}{24}\right)\right)^{\frac{1}{2}}\right)}{\left(x - \frac{1}{24}\right)^{\frac{1}{2}}} \right\} \tag{3.2.14}$$

where, denoting $a \wedge b = gcd(a,b)$,

$$A_k(n) = \sum_{\substack{0 \le h < k \\ h \wedge k = 1}} e^{i\pi s(h,k) - 2in\frac{h}{k}}, \tag{3.2.15}$$

and where $s(h,k)$ is a Dedekind sum, namely

$$s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right). \tag{3.2.16}$$

The series may look indigestible, but the terms of the series decrease very quickly, making the calculations of $p(n)$ quite fast: if we take $p(200) = 3,972,999,029,388$, the sum of the five first terms gives $3,972,999,029,387.89$ and the sum of the first eight terms gives approach $p(200)$ to 0.004. The methods were then improved, with the significant contribution of G. Meinardus [112], who identifies a general strategy for obtaining asymptotic formulas starting from a general partition function:

$$f(x) = \prod_{k=1}^{+\infty} (1 - x^n)^{-a_n}, \quad a_n \ge 0, \tag{3.2.17}$$

this strategy being roughly the same as the one conducted in Chapter 4. It is based in particular on the study of the Dirichlet series associated with the sequence (a_n). In 1962, Newman [119] gives a very shortened proof of the equivalent of Hardy and Ramanujan, reducing the difficulties of integration, but the circle method used remains substantially the same. However, Erdős [61] managed in 1942 to find the equivalent up to a constant, using a clever recursive rewrite of $p(n)$!

Even if Meinardus is surely the one who marked the most asymptotic combinatorics methods, we do not want to forget the works of Ingham [90], Brigham [32], Wright [152] and Roth and Szekeres [129] all of which were significant. Section 72 of Leveque's book makes it a more exhaustive review [106].

Apart from Erdős' proof, all the others are based on analytic techniques which will later be grouped into general theorems of analytic combinatorics [66]. In the 1990s, a new probabilistic approach, based on statistical physics models, emerged with the work of B. Fristedt [68] and Sinai [139]. They use the compatibility of the partition generating function with the Maxwell-Boltzman statistics, in which precisely the quantity associated with thermodynamic equilibrium is called the partition function. In this framework, Baez-Duarte redefines the Hardy-Ramanujan equivalent with a central limit theorem [9].

3.2.2 Limit shape

Beginning in the second half of the XX$^{\text{th}}$ century, with the study of limit laws and processes in probabilities, a large number of asymptotic stochastic questions arise about combinatorial objects, the main question often being: what is the limit behavior of such a combinatorial object when it becomes very large?

Since partitions are both a central and simple object in combinatorics (perhaps even the most simple and central after the group of permutations), studies in this field have been precursors of other more complex objects. Before introducing the main result concerning partitions, which we owe to A. M. Vershik [150], we allow ourselves a quick overview of asymptotic combinatorics results.

In this paragraph, the notion of "convergence" will not be explained; we refer readers to the corresponding works for a formalization of convergences, renormalizations, and suitable convergence spaces. The first process obtained as a limit of a combinatorial object is surely Brownian motion, which is the limit of a renormalized random walk. The famous ***Dyck paths***, (enumerated by the no less famous ***Catalan numbers***), converge on the brownian excursion [146]. We cannot talk about limits of combinatorial objects without citing the Brownian tree (*Brownian continuum random tree*) of D. Aldous [3], limit of Galton-Watson trees. Triangulations, quadrangulations and other types of maps converge to the Brownian map [109]. The capital importance of this work lies in the universal character of these limits: the Brownian tree, for example, is the limit form of a number of very different families of trees.

Let's return to the limiting behavior of a random partition. In 1996, Vershik proved the following result:

Theorem 3.2.1 (Vershik)**.** *Let Γ be the curve with Cartesian equation $e^{-x} + e^{-y} = 1$. For any partition ω of n, we denote E_ω the following set of points*

$$E_\omega = \left\{ \left(\frac{\pi i}{\sqrt{6n}}, \frac{\pi \omega(i)}{\sqrt{6n}} \right), i \geq 1 \right\}. \tag{3.2.18}$$

The partition ω of n according to the uniform measure on the partitions of n converge in probability towards the curve Γ, in the sense that for all $\epsilon > 0$:

$$\mathbb{P}\left[\sup_{x\in E_\omega} d(x,\Gamma) \leq \epsilon\right] \underset{n\to+\infty}{\longrightarrow} 1. \tag{3.2.19}$$

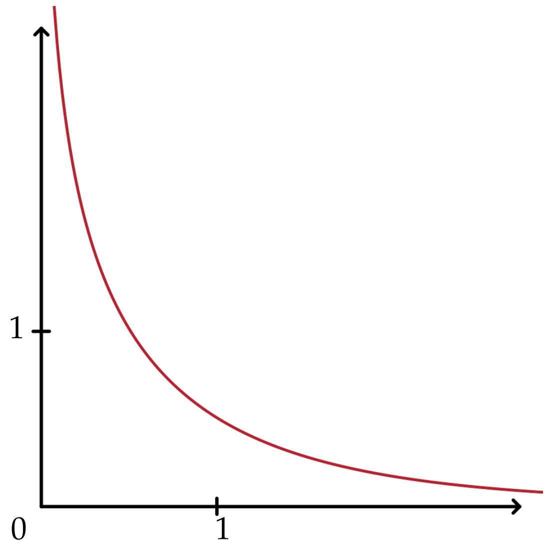


Figure 3.4: The limit shape of a uniformly random partition of n, after renormalization, when $n \to +\infty$.

Proof of this convergence had long been sought by physicists, and the first trace of this curve dates back to 1952 and the work of Temperley who wanted to understand the formation of "teeth" on the surface of crystals using an entropy model [147]. The results of I. Bárány, J. Bureaux and B. Lund [14] on the convergence of zonotopes to a limit form, developed in the next section, are a higher-dimensional generalization of this result.

3.3 Zonotopes

Zonotopes are a family of polytopes all of whose faces are centrally symmetric, but we will see that they can be thought of as generalization of partitions to higher dimensions, and this is why our results are in line with the works on asymptotic combinatorics of the previous section. In this section we will first introduce these objects, then discuss their main intrinsic applications and their use in algebraic combinatorics, which will finally lead us to introduce the notion of oriented matroids in order to precise the link between zonotopes and oriented matroids. This characterization will be essential in Chapter 6.

3.3.1 Multipartite partitions

Instead of taking an integer n as in the previous section, let's take a vector in $\mathbb{R}^d$. We can make a decomposition analogously to that of the integer partitions in the previous section, but with vectors. There are then several possible generalizations.

A first generalization comes from the same question as that of partitions: how many ways are there to decompose a vector into a sum of nonzero positive vectors? The partition of a vector d dimensional v of $(\mathbb{R}_+)^d$ into vectors with positive coordinates such that their sum gives v is called a ***multipartite partition*** . Let's take an example again to illustrate. In 2 dimensions, how many partitions of the vector $(2,2)$ are there? There are nine different partitions:

$$(2,2) = (2,1)+(0,1) = (2,0)+(0,2) = (2,0)+(0,1)+(0,1) \quad (3.3.1)$$
$$= (1,2)+(1,0) = (1,1)+(1,1) = (1,1)+(1,0)+(0,1) \quad (3.3.2)$$
$$= (1,0)+(0,1)+(0,2) = (1,0)+(1,0)+(0,1)+(0,1). \quad (3.3.3)$$

If we look at these partitions from a geometric point of view, we can associate them with chains of segments. As the order of the vectors in the partition does not count, we can arbitrarily order them with respect to their slope and associate to each partition a convex polygonal chain in the plane $\mathbb{R}^2$. Thus we can represent the partition $(3,3) = (2,2)+(1,0)+(0,1)$ with the chain $[(0,0),(1,0)] \cup [(1,0),(3,2)] \cup [(3,2),(3,3)]$ where each segment corresponds to a part of the partition:

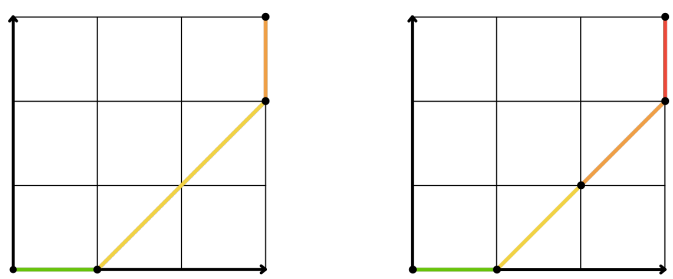


Figure 3.5: Two different 2-partite partitions of $(3,3)$.

In fact, this polygonal string encodes two different partitions, $(2,2)+(1,0)+(0,1)$ and $(1,1)+(1,1)+(1,0)+(0.1)$. This is why an other generalization of partitions in higher dimensions is interesting: the ***strict integer partitions***, which are the multipartite partitions of which all the parts are primitive vectors (that means the GCD of their coordinates is 1). In two dimensions, there is a correspondence between the convex (or concave) polygonal chains with integer vertices between 0 and a point (n,n) and the strict bipartite partitions of (n,n).

In dimension 3 or more, one can also find a correspondence between strict multipartite partitions and convex polytopes, by taking the Minkowski sum of the vectors of the partition (see Lemma 3.3.1). The Minkowski sum (or vector sum) of two polytopes P and P' is defined by

$$P + P' = \{\boldsymbol{x} + \boldsymbol{x}', \boldsymbol{x} \in P, \boldsymbol{x}' \in P'\}. \tag{3.3.4}$$

The Minkowski sum of the vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_k \in \mathbb{Z}^d$ is therefore the convex hull of the vectors $\sum_{1 \le i \le k} \epsilon_i \boldsymbol{v}_i$ with $\epsilon_i \in \{0, 1\}$, that is:

$$Z = \left\{ \sum_{1 \le i \le k} \lambda_i \boldsymbol{v}_i, \;\; \lambda_i \in [0, 1] \right\}. \tag{3.3.5}$$

These polytopes are called zonotopes, and it is in the sense of this generalization that the work of Chapters 4 and 5 are extensions of the asymptotic results on partitions presented in the previous section.

Definition 3.3.1 (Zonotope 1)**.** *A zonotope is a polytope which is a Minkowski sum of segments.*

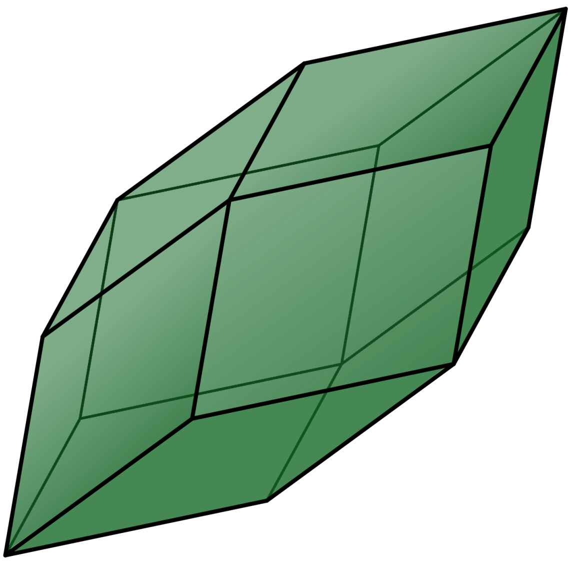


Figure 3.6: A 3-dimensional zonotope with 4 pairwise noncollinear generators

Let $Z \subset \mathbb{R}^d$ be a zonotope defined as the sum of the segments $s_1, ..., s_k \subset \mathbb{R}^d$. By translation, we can construct some vectors $\boldsymbol{z}$ and $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ such that

$$Z = \left\{ \boldsymbol{z} + \sum_{1 \le i \le k} \lambda_i \boldsymbol{v}_i, \;\; \lambda_i \in [-1, 1] \right\}. \tag{3.3.6}$$

Generally, the generators are assumed to be pairwise non-collinear to avoid degenerate cases, like we will do in Chapter 6. In order cases, we only consider primitive generators, and we add a multiplicity parameter, like in Chapters 4 and 5. The vectors whose Minkowski sum gives the zonotope Z up to a translation are called the ***generators*** of Z. Zonotopes have many properties. Any d'-face f of a d-zonotope Z is a d'-zonotope. In the above writing of the Z zonotope, f is defined by d' variables λ_i in $[-1,1]$, all the other coefficients being set to -1 or 1. Moreover, all zonotopes are centrally symmetric (the center of symmetry of Z is z). In 2 dimensions, all centrally symmetric polygons are zonotopes, named zonogons, but in higher dimensions it is no longer a sufficient condition. Yet there are several characterizations of zonotopes by their symmetry [28]. Let us write some:

Definition 3.3.2 (Zonotope 2)**.** *A d-zonotope is a d-polytope that satisfies one of the following equivalent conditions:*

- *all of its faces are centrally symmetric.*
- *all its 2-faces are centrally symmetric.*
- *all its k-faces are centrally symmetric, for all* $k \in \{2, ..., d-2\}$.

These definitions show the profound geometric side of zonotopes. The last definition, on the contrary, is the first step to show the combinatorial nature of zonotopes, developed in the next section. All these definitions express the high diversity of the fields of geometry in which zonotopes appear [155, Chapter 7].

Definition 3.3.3 (Zonotope 3)**.** *A zonotope is a polytope obtained by making an affine projection of a hypercube.*

To see the equivalence between the first and third definitions, let's take the zonotope Z defined in (3.3.6), and take the hypercube of dimension k, $C_k = [-1,1]^k$. Let $V = (\boldsymbol{v}_1, ..., \boldsymbol{v}_k)$ be the vector of the generators of Z, Z is obtained by the affine projection $V \cdot C_k + \boldsymbol{z}$, which sends the i^{th} dimension of the hypercube on the segment $[-\boldsymbol{v}_i, \boldsymbol{v}_i]$ (the coordinates x_i are the weights λ_i of each vector $\boldsymbol{v}_i$ in the sum).

Let's come to the enumeration of ***lattice zonotopes***, which are zonotopes whose vertices have integer coordinates. Naturally, the counting of zonotopes implies two restrictions: looking at these objects up to a translation, and restricting ourselves to primitive generators as we count the sets of generators (the set of generators encoding the strict integer partition). The former restriction can be modified depending on the problem under consideration, but it is natural from the point of view of multipartite partitions.

Two lattice zonotopes are therefore equivalent if they are the same up to a translation. Thus the generators $-\boldsymbol{v}$ and $\boldsymbol{v}$ have the same weight in a zonotope. To count all the lattice zonotopes in a given bounded convex set, we can therefore reduce the set of lattice zonotopes to those generated by primitive vectors, whose first nonzero coordinate is positive, a set that we find in the article of A. Deza and L. Pournin [53], and in Chapter 4.

Let us restrict ourselves to the case where the generators can only be vectors of the orthant $\mathbb{N}^d$. Let $\mathcal{Z}(\boldsymbol{k})$ be the set of zonotopes whose generators are elements of $\mathbb{N}^d$ whose sum of generators is equal to $\boldsymbol{k}$, and the set of vectors primitives of $\mathbb{N}^d$ is denoted $\mathbb{P}_{d+}$ (we prefer the notation $\mathbb{P}_{d+}$ to $\mathbb{P}_+^d$ because if $\mathbb{P}_1$ is the set of prime numbers, $\mathbb{P}_{d+} \neq (\mathbb{P}_{1+})^d$). We have the following result:

Lemme 3.3.1. *There is a natural bijection between the strict integer partitions of* $\boldsymbol{k}$ *and* $\mathcal{Z}(\boldsymbol{k})$.

The result will be demonstrated in Section 5.1 using the notation of Definition 3.2.1 for strict integer partition : it is a $\omega function$: $\mathbb{P}_{d+} \to \mathbb{N}$ with finite support.

This lemma gives the generating function of $\mathcal{Z} = \bigcup_{\boldsymbol{k}\in\mathbb{N}^d} \mathcal{Z}(\boldsymbol{k})$, which is the d-dimensional formal series $f_{\mathcal{Z}} : \mathbb{R}^d \to \mathbb{R}$ such that the coefficient of $x_1^{k_1}x_2^{k_2}...x_d^{k_d}$ is the number of zonotopes of $\mathcal{Z}$ whose sum of generators is $(k_1, k_2, ..., k_d)$. This number therefore corresponds to the number of strict multipartite partitions of $(k_1, k_2, ..., k_d)$.

$$f_{\mathcal{Z}}(x) = \prod_{\boldsymbol{p}\in\mathbb{P}_{d+}} \frac{1}{1-\boldsymbol{x}^{\boldsymbol{p}}} \tag{3.3.7}$$

where $\boldsymbol{x}^{\boldsymbol{p}} = x_1^{p_1}x_2^{p_2}...x_d^{p_d}$. The combinatorics of integer zonotopes is therefore based on the primitive points of $\mathbb{R}^d$, from which we will give some main results.

Set of primitive points

For this, recall the ***function*** ζ ***of Riemann*** defined on $[1, +\infty)$ by

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}, \tag{3.3.8}$$

$\mathcal{P}$ being the set of positive prime numbers ($\mathcal{P} = \mathcal{P}_{1+}$). First of all, it has long been known that the density of primitive points in a subset of $\mathbb{Z}^d$ is $\frac{1}{\zeta(d)}$. The case of dimension 2 goes back to Mertens and Cesàro at the end of the XIX$^{\text{th}}$ century [64]. More precisely, we can give the following theorem, a consequence of the Mobiüs inversion formula [64] ([83, Theorem 459] in the case $d = 2$):

Theorem 3.3.1. *For* $d \geq 2$, *and* F *a bounded subset of* $\mathbb{R}^d$. *For all* $r \in (0, +\infty)$, *we denote* $F_r = \{\boldsymbol{x} \in Z^d, \boldsymbol{x}r^{-1} \in F\}$. *If* $\frac{|F_r|}{r^d}$ *converges to a nonzero limit as* r *tends to* $+\infty$, *then*

$$\lim_{r\to+\infty} \frac{|F_r \cap \mathcal{P}_d\}|}{|F_r|} = \frac{1}{\zeta(d)}. \tag{3.3.9}$$

The probabilistic corollary of this result is just as beautiful [69]:

Corollary 3.3.1. *Under the same assumptions, let Y_r be a random element of F_r chosen uniformly. Then the gcd of its coordinates* $\gcd(Y_r)$ *converges in distribution to the zeta law with parameter d, when* $r \to +\infty$.

Other probabilistic limit laws are known concerning these primitive points. They are also known as visible points, because there is no integer point between themselves and the origin. Let us give a last striking result on the primitive points of $\mathbb{Z}^2$. One may wonder what kinds of primitive point sets we can see, or not see, from the origin. For example, does there exist a square of side $c \in \mathbb{N}^*$ of integer vertices all of whose points on the edges are all visible, or all not visible? A theorem by F. Herzog and B. M. Stewart answers this question [88, 104], but it requires some preliminary definitions. A *pattern P* is a subset of $\mathbb{Z}^2$, and it is said *feasible* if P can be translated by a vector of $\mathbb{Z}^2$ such that all elements of P are visible from the origin. Moreover, let's call *complete square modulo m*, a set of m^2 elements of $\mathbb{Z}^2$:

$$\{(x_i, y_i), x_i \in \mathbb{Z}, y_i \in \mathbb{Z}, 1 \leq i \leq m^2\}, \tag{3.3.10}$$

such that $\{(x_i \mod m, y_i \mod m)\} = \{(a, b), 0 \leq a, b \leq m-1\}$. For example, the points $(1,3), (2,2), (3,4)$ and $(4,1)$ form a complete square modulo 2.

Proposition 3.3.1 (Herzog and Stewart's theorem (1971))**.** *A pattern P is realizable if and only if P does not contain any complete square modulo p, for any prime p.*

The result is as elegant as the proof is intuitive; if P contains a complete square modulo p, then it will always contain, regardless of the translation, a point whose coordinates are all congruent to 0 modulo p. Conversely, using the Chinese remainder lemma with all primes less than the size of P, we can construct a point that is "visible" from all points of P.

Finally, recently, A. Deza and L. Pournin calculated the exact number of primitive points that there are in a L_1-ball centered at the origin [55], for questions of maximum diameter of polytope graphs, question mentioned in Section 3.1.4. In summary, let us retain from this brief enumeration of results that primitive points have been studied in many ways in the fields of combinatorics and probability, and that it is indeed their presence in the combinatorial construction of zonotopes that leads to the appearance of the Riemann's ζ function in this manuscript.

3.3.2 Algebraic point of view and applications

This section gives an overview of the applications and the interest of zonotopes in several fields of research, starting with the link between zonotopes and vector configurations, thanks to sign vectors. This link is the basis for zonotope usage in combinatorics. It will be described in the following section in terms of oriented matroids, and will be used in particular in Chapter 6. The following is based on Chapter 7 of [155].

The goal is to associate a ***sign vector*** to each face (therefore a vector whose coordinates are either $+$, or $-$ or 0); such that the lattice of the faces of a given zonotope corresponds to the poset of the sign vectors. In the sequel consider the following example: the zonotope $Z_1 \subset \mathbb{R}^2$ is defined by the generators $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1,1)$. We denote $V = (v_1, v_2, v_3) \in \mathbb{R}^{2\times 3}$, and for any polytope P, recall that $L(P)$ theis the set of faces of P, namely $L(P) = \{F,\ F \text{ face of } P\}$. Using the inclusion relation, we can therefore draw the ***face lattice*** of Z_1, given by $(L(Z_1), \subset)$:

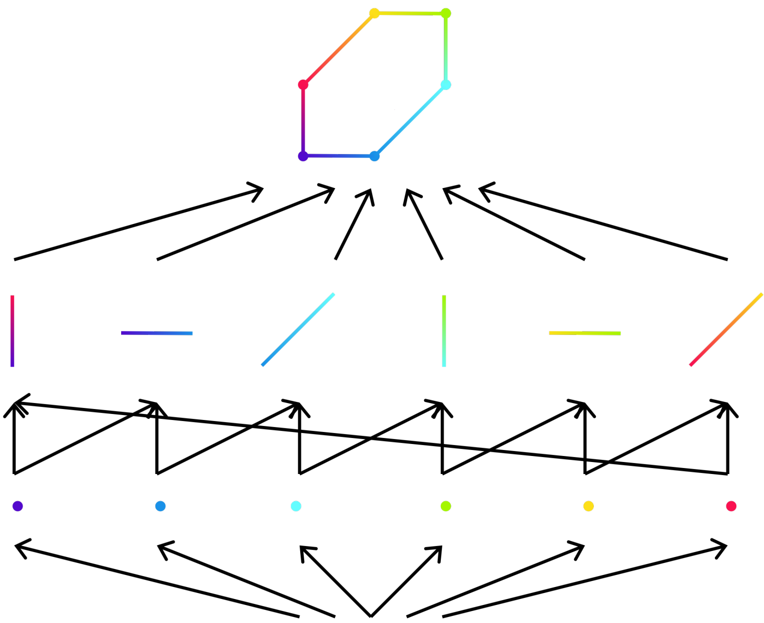


Figure 3.7: The face lattice of Z_1.

Now, let us detail the vector sign mapping process. Take the cube $C_k = \{\boldsymbol{x} \in \mathbb{R}^k, -1 \leq x_i \leq 1\}$, and associate each non-empty face F with a k-tuple of sign $\sigma \in \{+,-,0\}^k$ as follows: let F be a face of C_k, $\sigma = \sigma(F)$ is the unique vector of $\{+,-,0\}^k$ such that

$$F = \{ \sum_{1\leq i\leq k} \lambda_i e_i,\ \ \lambda_i = +1 \text{ si } \sigma_i = +, \tag{3.3.11}$$

$$\lambda_i = -1 \text{ si } \sigma_i = -, \tag{3.3.12}$$

$$-1 \leq \lambda_i \leq +1 \text{ si } \sigma_i = 0\}. \tag{3.3.13}$$

We can then induce a partial order on the sign vectors by defining the order "$\leq$" given by $0 \leq -$ and $0 \leq +$. We then have a correspondence between the order of inclusion in the set of faces of the hypercube, and the order of the sign vectors: the larger a face is in the sense of inclusion, the smaller its sign vector is. In order to have the complete correspondence between the face lattice and the sign vectors lattice, it is necessary to add a maximal element $\hat{0}$, greater than any sign vector σ, corresponding to the empty face. We then have

$$(L(C_k), \subset) \cong \left(\{\hat{0}\} \cup \{+, -, 0\}^k, \geq\right). \tag{3.3.14}$$

Using Definition 3.3.3, we can express Z_1 as a projection of C_3, and therefore associate a sign vector to the faces of Z_1. Denoting π the projection which defines Z_1 from C_3, the inverse image $\pi^{-1}(G)$ of a face G of Z_1 is a face of C_3. We therefore naturally define $\sigma(G) = \sigma(\pi^{-1}(G))$, so that $G \subset G'$ if and only if $\sigma(G) \geq \sigma(G')$, and

$$(L(Z_1), \subset) \cong (\{\hat{0}\} \cup \{\sigma(G), G \in L(Z_1) \setminus \{\emptyset\}\}, \geq). \tag{3.3.15}$$

Notice the equivalence between the face lattice in Figure 3.7 and the sign vectors lattice in Figure 3.8.

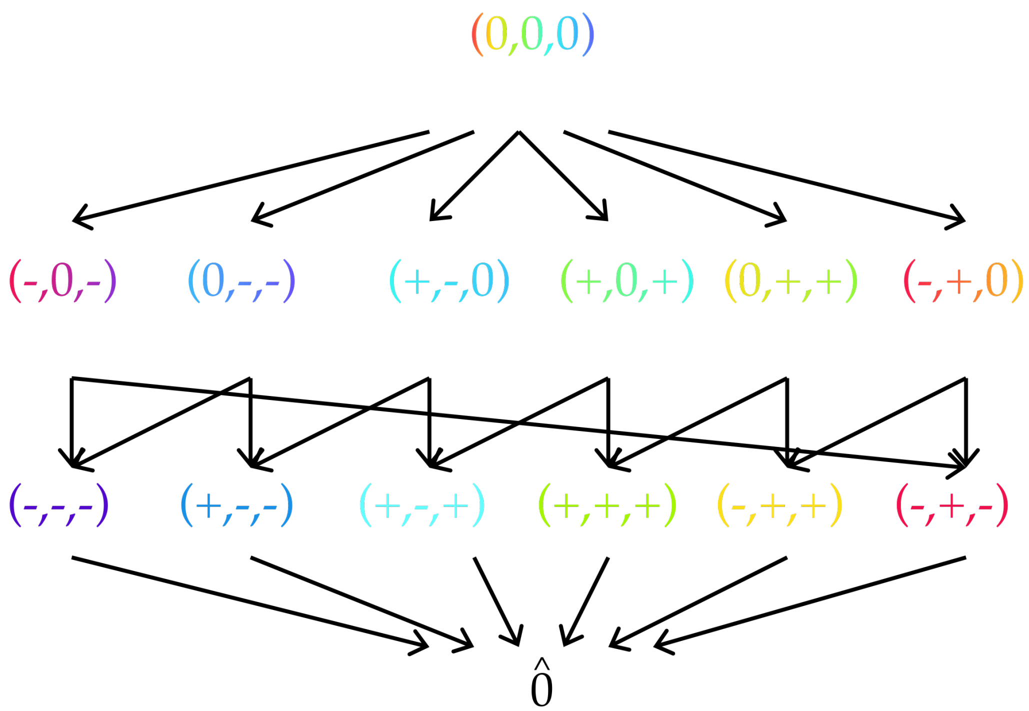


Figure 3.8: Lattice of the sign vectors of Z_1, with the order $+ \geq 0$ and $- \geq 0$.

Let's leave cube C_3 aside and look at the normal fan of Z_1. By definition, for all polytopes P and Q, the normal fan of the Minkowski sum $P + Q$ is the set of intersections of the normal cones of P with those of Q, that is to say

$$\mathcal{N}(P+Q) = \{C \cap C', \ \ C \in \mathcal{N}(P), C' \in \mathcal{N}(Q)\}. \tag{3.3.16}$$

By construction of Z_1 as a Minkowski sum of segments (here the column vectors of V) , the normal fan $\mathcal{N}(Z_1)$ is therefore the fan generated by the family of hyperplanes

$$\mathcal{A}_V = \{H_1, H_2, H_3\} \tag{3.3.17}$$

in $\mathbb{R}^d$, where $H_i = \{\boldsymbol{c} \in \mathbb{R}^2, \ \ \boldsymbol{c} \cdot \boldsymbol{v}_i = 0\}$. $\mathcal{A}_V$ is also called an ***arrangement of hyperplanes***. Let's define the *positive half-space* $H_i^+ = \{\boldsymbol{c} \in \mathbb{R}^d, \ \ \boldsymbol{c} \cdot \boldsymbol{v}_i \geq 0\}$, respectively the half-negative space $H_i^- = \{\boldsymbol{c} \in \mathbb{R}^d, \ \ \boldsymbol{c} \cdot \boldsymbol{v}_i \leq 0\}$. The position of any vector $\boldsymbol{c}$ with respect to H_i is given by the sign of $\boldsymbol{c} \cdot \boldsymbol{v}_i$: it is 0 if $\boldsymbol{c}$ belongs to H_i, $+$ if $\boldsymbol{c}$ is in the interior of H_i^+ and $-$ if it is in the interior of H_i^-.

Thus for any vector $\boldsymbol{c}$ of $\mathbb{R}^d$, its position in the normal fan of Z is given by the sign vector of its position with respect to the hyperplanes $\text{sign}(V^\intercal \cdot \boldsymbol{c}) \in \{+,-,0\}^3$ and therefore gives for Z_1 the following diagram:

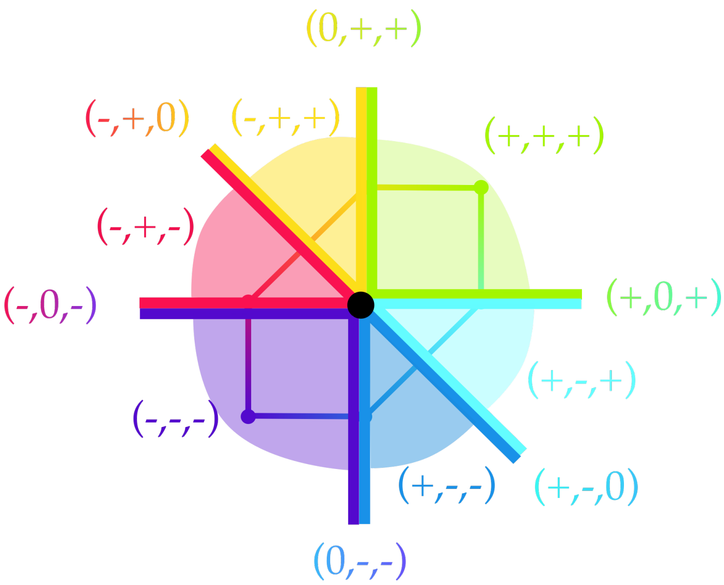


Figure 3.9: The normal fan of Z_1.

The equivalence (3.3.15) therefore maps the face lattice of a zonotope on the lattice of the cones generated by the linear hyperplanes $\mathcal{A}_V$, ordered by inclusion. Finally, one last concept is introduced: ***vector configurations***. A configuration of vectors of $\mathbb{R}^d$ is a k-tuple $X = (\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_k)$ of d-dimensionnal vectors, of which we study the layout. The arrangement of the vectors of X is directly related to the signs of the covectors $\boldsymbol{c}^{\intercal}X$ for any linear form $\boldsymbol{c}^{\intercal}$. Therefore, define the set of sign covectors of the configuration X as

$$\mathcal{V}^*(X) = \{\text{sign}(\boldsymbol{c}^{\intercal}X),\, \boldsymbol{c} \in \mathbb{R}^k\}. \tag{3.3.18}$$

Analogously to what has been done before, there is an equivalence between the face lattice of a zonotope and the lattice of the sign covectors of the vector configuration defined by the pairwise non-collinear generator set (we exclude here the degenerated case of two collinear generators). This allows us to conclude the following result:

Proposition 3.3.2 (Corollary 7.17 of [155])**.** *Let $V \in \mathbb{R}^{d\times k}$ be a k-tuple of pairwise non-collinear vectors of $\mathbb{R}^d$. Then there is a natural bijection between the following three families:*

- *The (sign vectors of) nonempty faces of the zonotope $Z(V) \subset \mathbb{R}^d$.*
- *The (sign vectors of) faces of the arrangement of hyperplanes $\mathcal{A}_V$*
- *The sign covectors of the configuration V.*

This matching between zonotopes and vector configurations is the starting point for many geometric and combinatorial results [134, 8, 89, 124, 144, 108]. In the following section, we will develop the notion of oriented matroid [155, 23] that we will use in Chapter 6.

Apart from their combinatorial aspect, zonotopes are also useful for their combinatorial simplicity compared to polytopes. On the question of the diameter of polytopes, discussed previously in Section 3.1.4, A. Deza, G. Manoussakis and S. Onn conjectured that the maximum diameter of a lattice polytope in the hypercube $[0, n]^d$ is reached by a zonotope [52, 56], which is an important conjecture for the simplex algorithm on lattice polytopes [50]. These questions led to the maximum zonotope diameter calculated by A. Deza and L. Pournin based on the study of the number of primitive points in the multiples of the standard simplex of $\mathbb{R}^d$ in the balls for the L^1-norm [55].

Two concrete applications of zonotopes in the literature hold our attention, both using the simplicity of encoding a zonotope: the first, by L. Guibas, A. Nguyen and L. Zhang, uses them in collision detection, to approximate objects with zonotopes and thus speed up the computation time [81]. The second, by A. Girard, is to solve the problems of accessibility to the solution of uncertain linear equations [71]. Finally, zonotopes are widely studied in tiling theory, due to their highly symmetrical appearance [110, 135, 45, 46].

3.3.3 Oriented matroids

Matroids are combinatorial structures introduced in 1935 by Whitney (see [103, 120]), to abstract the concept of linear independence in a set of vectors. They are especially broadly known thanks to the well-known greedy algorithm, in combinatorial optimization, an algorithm which works precisely only on these structures. The theory of matroids takes up the notions of linear algebra of independence, base or rank, but also the notions of graph theory such as circuits and cycles.

Matroids can be defined in a large number of equivalent ways, we will define them with their bases. A matroid is a pair $M_E = (E, \mathcal{B}_E)$, where E is a finite set and $\mathcal{B}_E$ is a nonempty collection of parts of E, which verifies Steinitz's exchange axiom (SEA)

$$\text{For all } B_1, B_2 \in \mathcal{B}_E, \text{ and any element } e \in B_1 \setminus B_2, \text{ there is } f \in B_2 \setminus B_1 \text{ such that } B_1 \setminus \{e\} \cap \{f\} \in \mathcal{B}_E. \tag{SEA}$$

The elements of $\mathcal{B}_E$ are called ***bases*** of E. All elements of $\mathcal{B}_E$ have the same cardinal, called ***rank*** of the matroid. We call ***independent set*** any subset of any element of $\mathcal{B}_E$ (the bases are therefore the maximal independent sets in the sense of inclusion) and we consider the set of independent sets $\mathcal{I}_E$. Intuitively, the notions of base and independence therefore keep the same meaning as in linear algebra. Finally, the minimal elements in the sense of the inclusion of $\mathcal{P}(E) \setminus \mathcal{I}_E$ are called ***circuits*** of E, and the set circuits of E is denoted $\mathcal{C}_E$. Each of the three notions of base, circuit or independent sets is sufficient to define matroids, through the axiom of exchange (SEA) for bases, the axiom of augmentation (AA) for independent sets and the axiom of elimination (EA) for circuits; the latter two being:

$$\text{For any } I_1, I_2 \in \mathcal{I}_E \text{ such that } |I_1| < |I_2|, \text{ there exists } e \in I_2 \setminus I_1 \text{ such that } I_0 \cup \{e\} \in \mathcal{I}_E. \tag{AA}$$

$$\text{For any distinct circuits } C_1, C_2 \in \mathcal{C}_E \text{ such that } e \in C_1 \cap C_2, \text{ there is a circuit } C_3 \subset C_1 \cup C_2 \setminus \{e\}. \tag{EA}$$

The simplest example is probably the matroid naturally associated with an n-tuple of vectors of $\mathbb{R}^d$, where a k-tuple is independent if it is independent in $\mathbb{R}^d$ (in the algebraic sense). We can also associate a matroid to any set of edges E of an undirected finite graph $G = (V, E)$. The set $E' \subset E$ is independent if the graph (V, E') has no cycle. Circuits in the sense of matroid theory then coincide with circuits in the sense of graph theory.

Note in these examples that these matroids are based on non-oriented sets, or for which we have not taken into account any notion of orientation. The ***oriented matroids***, on the contrary, make it possible to account for dependence structures in vector spaces over ordered fields, thus making it possible to account for the lattice structure of the sign vectors of the section former.

Let E be a finite set and $\mathcal{V}$ a subset of $\{+,-,0\}^E$, the set of sign vectors. We denote $\mathbf{0}$ the vector of n zeros and for all $u \in \mathcal{V}$. The ***support*** of u is the set of indices of the nonzero coordinates of u, namely $\text{supp}(u) = \{i, u_i \neq 0\}$. On sign vectors, the ***composition*** of vectors u and v is an internal composition law defined coordinate by coordinate by

$$(u \circ v)_i = \begin{cases} u_i & \text{if } u_i \neq 0, \\ v_i & \text{otherwise.} \end{cases} \tag{3.3.19}$$

The ***separation set*** of u and v is the set $S(u,v) = \{i, u_i = -v_i \neq 0\}$, and finally, if $j \in S(u,v)$, we say that the vector *w eliminates j between u and v* if $w_j = 0$ and $w_i = (u \circ v)_i$ for all $i \notin S(u,v)$. Let us also recall the order on the signs, seen in the previous section: $+ \geq 0$ and $- \geq 0$. All these notions will make sense in the definition of oriented matroids and in the following example:

Definition 3.3.4 (Oriented matroids). *Let E be a finite set and the collection $\mathcal{V} \in \{+,-,0\}^E$. The pair $M = (E, \mathcal{V})$ is an oriented matroid if it satisfies*

$$\mathbf{0} \in \mathcal{V}, \tag{V0}$$

$$u \in \mathcal{V} \Rightarrow -u \in \mathcal{V}, \tag{V1}$$

$$u, v \in \mathcal{V} \Rightarrow u \circ v \in \mathcal{V}, \tag{V2}$$

$$u, v \in \mathcal{V}, j \in S(u,v) \Rightarrow \exists w \in \mathcal{V} : w \text{ eliminate } j \text{ between } u \text{ and } v. \tag{V3}$$

Moreover, the rank of M is the largest integer r such that there are r elements $u_1, ..., u_r \in \mathcal{V}$ such that $u_1 < u_2 < ... < u_r$.

To explain this definition, let us show as an example the following result, that will serve as a proof for Proposition 3.3.3: let $E \subset \mathbb{R}^n$ be a linear subspace of dimension r and $\text{SIGN}(E) = \{\text{sign}(x), x \in E\}$, the pair $(\{1,..,n\}, \text{SIGN}(E))$ is an oriented matroid of rank r.

E being a vector subspace, the axioms (V0) and (V1) are immediate. To demonstrate (V2), take $\boldsymbol{x}, \boldsymbol{y} \in E$ such that $\text{sign}(\boldsymbol{x}) = u$ and $\text{sign}(\boldsymbol{y}) = v$. With the definition of composition, we can show that there exists $\epsilon > 0$ such that

$$\text{sign}(\boldsymbol{x} + \epsilon \boldsymbol{y}) = u \circ v. \tag{3.3.20}$$

Indeed, we can take ϵ small enough so that for each coordinate i, the sign of $u_i + \epsilon v_i$ is that of u_i when this number is not zero (see Figure 3.10). As $\boldsymbol{x} + \epsilon \boldsymbol{y} \in E$, (V2) is checked.

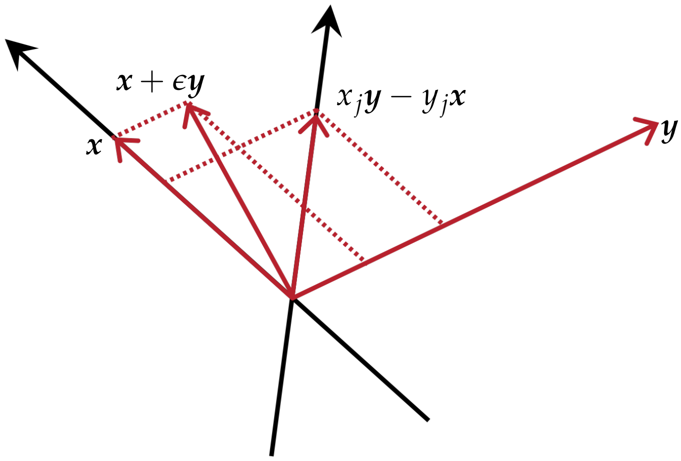


Figure 3.10: Example of the constructions of (V2) and (V3)

Finally, take $u, v \in \text{SIGN}(E)$, $j \in S(u,v)$ and $(x,y) \in E^2$ such that $\text{sign}(x) = u$ and $\text{sign}(y) = v$. Suppose $u_j = +$, then the sign w of vector $x_j y - y_j x \in E$ for coordinate j is 0, and for any coordinate $i \notin S(u,v)$, $w_i = (u \circ v)_i$, as shown above. Thus w eliminates, by definition, j between u and v and $w \in \text{SIGN}(E)$, which gives us (V3).

From this result, we can deduce the following proposition:

Proposition 3.3.3 (Realizable oriented matroids)**.** *Let $V \in \mathbb{R}^{n \times k}$ be a k-tuple of vectors of $\mathbb{R}^n$ and $\mathcal{V}_V$ the collection of sign vectors of V,*

$$\mathcal{V}_V = \{\text{sign}(V^\intercal x), x \in \mathbb{R}^n\}. \tag{3.3.21}$$

*The pair $(\{1, ..., k\}, \mathcal{V}_V)$ is an oriented matroid. Such an oriented matroid is called **realizable oriented matroid**.*

The propositions 3.3.2 and 3.3.3 thus give a natural bijection between the combinatorial types of zonotopes and the oriented realizable matroids, which is crucial in finding bounding inequalities for the number of combinatorial types of zonotopes in Chapter 6.

Non-realizable oriented matroids correspond to arrangements of pseudo-lines, that is a curve in the projective plane $\mathbb{R}P^2$ which is topologically equivalent to a straight line. Oriented matroids have therefore allowed major advances on all these models (arrangement of hyperplanes or configuration of vectors, arrangement of hyperspheres, and arrangement of pseudo-lines), such as the notion of universality [116, 1], and the matroid polytopes [27]. The reader can read the chapter of J. Richter-Gebert and G. Ziegler [128] or the books [23, 155] for a more detailed description.

3.4 Probabilistic and combinatorial study of lattice polytopes.

We hereby focus on the history of the probabilistic study of lattice polytopes. Still, we must first mention the variety of probabilistic work on polytopes (see Section 5.0.2 for more details). The best-known case is the ***random polytope***, defined as the convex hull of a set of points obtained by a Poisson point process in a convex body E. Convergence results can be found in [126, 17, 41] for E a bounded convex set. Another approach by P. Calka, Y. Demichel, and N. Enriquez consists in studying the polygons formed by the Voronoï tessellation generated by a Poisson point process [40]. More precisely, for a convex body $K \subset \mathbb{R}^2$, we consider the Voronoi tessellation generated by a Poisson point process of intensity λ conditioned on the existence of a cell K_λ which contains K; the authors study the asymptotic behavior of K_λ when λ tends to $+\infty$. Yet these results are of a very different nature from ours because the lattice polytopes are intrinsically linked to the lattice $\mathbb{Z}^d$ and its arithmetic properties (see Remark 2.4.1). We redirect the reader to the chapter of M Reitzner [127] for an in-depth survey. So let's come back to integer polytopes.

What is the minimum area of a lattice convex polygon with exactly n vertices? More generally, what is the minimal volume of a lattice convex d-polytope with n vertices? Or what is the maximum number of vertices a polytope can have in a given bounded convex set? These three questions are part of the multiple problems related to lattice polytopes and their combinatorics [13]. In Section 3.1, we gave an overview of the different themes around polytopes, focusing in particular in Section 3.1.2 on the combinatorial questions still open on the one hand, and on the uses of lattice polytopes in Section 3.1.3 on the other hand. We will present here the combinatorial questions on lattice polytopes which led to the work done in Chapters 4 and 5. In particular, we will see how the question of counting lattice polytopes in a convex compact set is at the boundary between number theory and geometry.

The question of the minimal volume of a d- polytope is a central question because the results of this problem are useful in most other questions related to the volumes of lattice polytopes [13]. In 1926, V. Jarník [92] asks the following question: let $\gamma \subset \mathbb{R}^2$ be a closed convex curve of length at most l, and whose radius of curvature is upper bounded by $7l$, what is the maximum number of points of Z^2 contained in γ? His answer is

$$\max_{\gamma} |\gamma \cap \mathbb{Z}^2| = \frac{3}{\sqrt[3]{2\pi}} l^{2/3} + O\left(l^{1/3}\right). \tag{3.4.1}$$

The transition from this question to lattice convex polygons is immediate: the polygon $P = \text{conv}(\gamma \cap \mathbb{Z}^2)$ is a nonempty lattice convex polygon as soon as γ contains three integer points or more. The formula of V. Jarník therefore gives that the minimum perimeter l of a polygon with n vertices is

$$l = \frac{\sqrt{6\pi}}{9} n^{3/2} + O\left(n^{3/4}\right). \tag{3.4.2}$$

The question is naturally extended to the volume of a polytope in higher dimensions. The first pioneering result for this question is a theorem by G. E. Andrews which gives a lower bound of the volume, of the right order of magnitude [5] (remind that the volume of a polytope P, denoted $\mathrm{vol}(P)$ is the Lebesgue measure of P in the the affine space generated by P):

Theorem 3.4.1 (Andrews, 1963)**.** *If P is a lattice convex d-polytope with n vertices, then there exists a constant $\kappa_d > 0$ depending only on d such that the volume $V(P)$ satisfies*

$$\mathrm{vol}(P) > \kappa_d n^{(d+1)/(d-1)}. \tag{3.4.3}$$

In addition to its link with the results that we are going to state below, this result gives a bound on the maximum number of vertices that a lattice d-polytope can have in a convex compact $K \subset \mathbb{R}^d$ that is $(\mathrm{vol}(K)/\kappa_d)^{(d-1)/(d+1)}$. Then, by the isoperimetric inequality, it allows to upper bound the number of vertices with $S(K)^{d/(d+1)}$ where $S(K)$ is the surface of K, that is the $d-1$-dimensional volume of the boundary of S. We find the order of magnitude of Jarník's result.

3.4.1 Arnold's question

In 1980, V. Arnold posed the following problem [7]: consider that two lattice convex polytopes P and Q are *equivalent* if there exists an affine transformation which preserves the lattice of integer points by which the image of P is Q. In this case, P and Q have the same volume. What is the equivalence class number $n_d(V)$ of lattice convex d-polytopes whose volume is V, and in particular what is the order of magnitude of $\log n_d(V)$? This problem is linked to the study of certain Newton diagrams, to the study of polynomials with variable d near the origin and to the question of the minimum volume of a lattice convex polytope stated in the introduction to this section. In his paper, V. Arnold proves, in the two-dimensional case, the existence of two constants $c_1, c_2 > 0$ such that we have the following bounding for V large enough:

$$c_1 V^{1/3} \leq \log n_2(V) \leq c_2 V^{1/3} \log V, \quad \text{for } V \to +\infty. \tag{3.4.4}$$

His proof is based on Andrews' inequality above, which he proves independently in the case $d = 2$. Conversely, Andrews' theorem is obtained in the case $d = 2$ from Arnold's lower bound. This example perfectly illustrates the proximity between the different questions relating to lattice polytopes mentioned above.

In 1984, Konyagin and Sevast'yanov generalize these inequalities to all $d \in \mathbb{N}^*$ [102], proving that there exist two constants $c_1, c_2 > 0$ such that

$$c_1 V^{\frac{d-1}{d+1}} \leq \log n_d(V) \leq c_2 V^{\frac{d-1}{d+1}} \log V, \quad \text{for } V \to +\infty. \tag{3.4.5}$$

I. Bárány and J. Pach for $d = 2$ [15], then I. Bánány and A. M. Vershik the same year for any d [11], showed that the term $\log V$ in the upper bound is superfluous. There therefore exist two constants $c_1, c_2 > 0$ such that, for sufficiently large V, $(\log n_d(V))/V^{\frac{d-1}{d+1}}$ belongs at the interval $[c_1, c_2]$. Naturally, one can ask whether there exists $\alpha > 0$ such that

$$\log n_d(V) \sim \alpha V^{\frac{d-1}{d+1}}, \quad \text{pour } V \to +\infty. \tag{3.4.6}$$

The question has remained open for more than 30 years, without any real progress on this subject [13]. In the spirit of Arnold's upper bound proof, Vershik asks the intermediate question of the number of lattice convex polytopes in a hypercube of side length n. We can rephrase it like this in 2 dimensions: how many convex polygons whose vertices are on the grid $\frac{1}{n}\mathbb{Z}^2$, are there in the square $[-1, 1] \times [-1, 1]$ as n tends to infinity and what do they look like? The meaning of this last question is probabilistic.

The answer is found by Bárány [10], Vershik [151] and Y. G. Sinai [139] independently. Let $\mathcal{P}_n(K)$ be the set of convex polygons with vertices in $\frac{1}{n}\mathbb{Z}^2$ which are in the bounded convex set K, and

$$p(n) = |\mathcal{P}_n([-1, 1] \times [-1, 1])|. \tag{3.4.7}$$

They show the following equivalence:

$$\log p(n) \sim 12 \left(\frac{\zeta(3)}{\zeta(2)} \right)^{1/3} n^{2/3}, \quad \text{pour } n \to +\infty, \tag{3.4.8}$$

where ζ is the Riemann function ζ. Besides this result, the three publications also prove that polygons admit a limiting form, namely that a polygon drawn uniformly in $\mathcal{P}_n([-1, 1] \times [-1, 1])$ has a very high probability of being very close to a convex K_0, in the sense of the Hausdorff distance between two sets $\delta(\cdot, \cdot)$.

Theorem 3.4.2 (Bárány, Vershik). *There exists $K_0 \subset [-1, 1] \times [-1, 1]$, such that for all $\epsilon > 0$, we have*

$$\lim_{n \to +\infty} \frac{|\{P \in \mathcal{P}_n([-1, 1] \times [-1, 1]), \delta(P, K_0) < \epsilon\}|}{p(n)} = 1. \tag{3.4.9}$$

Later, Bárány generalizes these results to any bounded convex set K, and proves that the number of convex polygons with vertices in $\frac{1}{n}\mathbb{Z}^2$ in a bounded convex set K of $\mathbb{R}^2$ is equivalent to $3(\zeta(3)/4\zeta(2))^{1/3}A_K n^{2/3}$, where A_k is the affine perimeter of K (see [12] for a definition of the affine perimeter). He also shows that a polygon uniformly drawn in $\mathcal{P}_n(K)$ converges to a limit form K_0 when n grows large [12]. In the case of the square $K = [-1,1] \times [-1,1]$, the limiting form K_0 is the union of 4 arcs of parabolas that connect the midpoints of the adjacent sides of K two together, as shown in Figure 3.11.

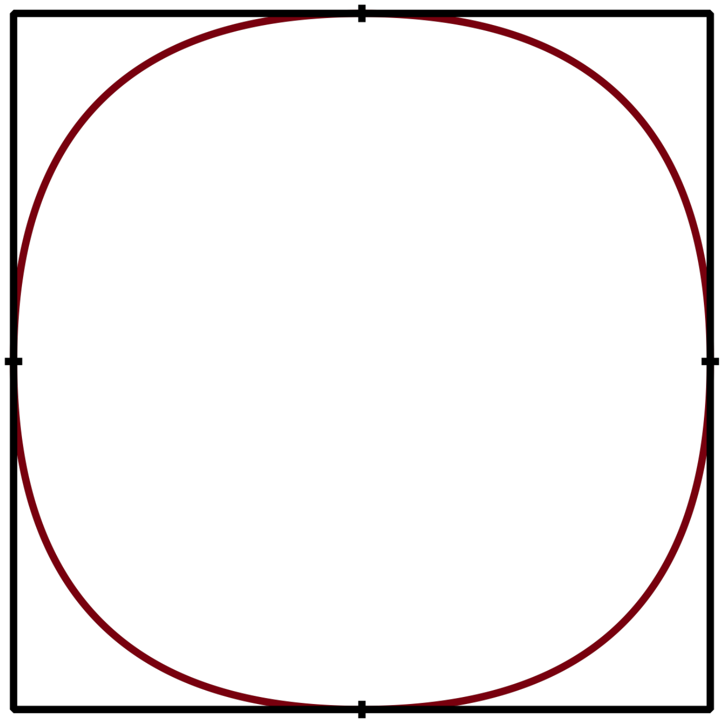


Figure 3.11: Limit shape K_0 of the uniform lattice polygons drawn in $\mathcal{P}_n([-1,1] \times [-1,1])$.

Remark 3.4.1. *What happens in the higher dimensions? Surprisingly at first sight, we have no generalization of the expression* (3.4.8) *nor of Theorem 3.4.2 more precise than the framework given previously. This surprising resistance can be intuitively send by observing the formula* (3.4.8)*. This formula, which gives a purely geometric result, involves the Riemann ζ function, a fundamental object in number theory. This function appears because of the network structure of $\mathbb{Z}^2$. The results in dimension 2 are in fact the fruit of the combinatorial simplicity of polygons, compared to higher dimensions.*

The question of enumeration is closely linked to random generation. For example, we do not know how to randomly generate lattice polytopes in a hypercube of side length 10 in dimension $d \geq 3$. This is unlike many combinatorial, probabilistic or statistical physics models where random generation allows conjectures to be drawn, or disproved by examples.

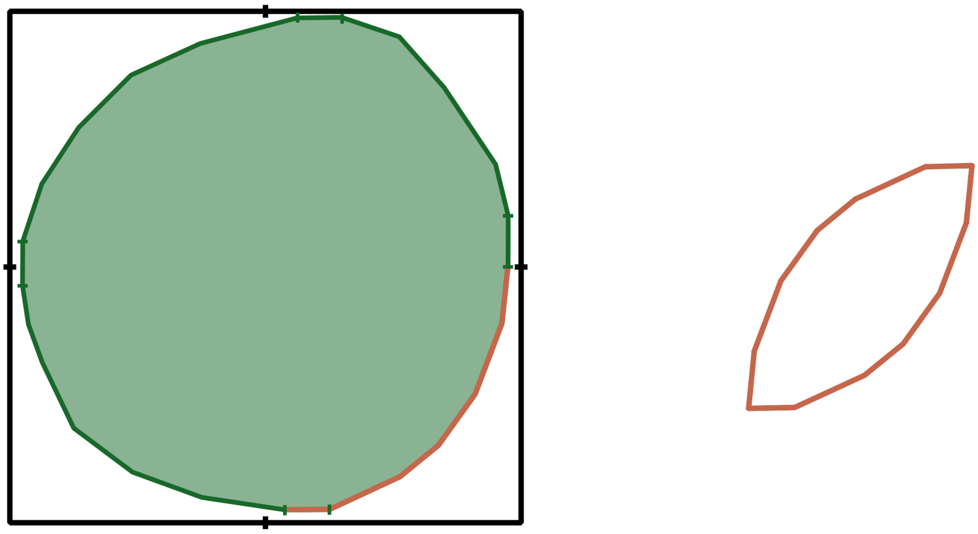


Figure 3.12: Example of uniform lattice polygon in $\mathcal{P}_n([-1,1] \times [-1,1])$, with extremal segments on the left, and the zonogon corresponding to its south-est arc on the right.

Remark 3.4.2. *Recall that the zonogons are the centrally symmetric polygons. For a polygon P, whose diameter is reached at the vertices A and B, we can therefore associate two zonogons that correspond to the two arcs of P connecting A and B. The study of lattice convex polygons can therefore go through the study of convex zonotopes.*

We will now rigorously state the last sentence of Remark 3.4.2 this last sentence, and display the recent results of enumeration and limit form of zonotopes.

3.4.2 Polygons, zonogons, and limit form of lattice zonotopes.

Let $P \subset \mathbb{R}^2$ be a lattice convex polygon. We distinguish four extremal segments of ∂P, $[A, A']$, $[B, B']$, $[C, C']$ and $[D, D']$ (that may be of length 0), the respectively southernmost, easternmost, northernmost and westernmost segments of ∂P, where the cardinal points represent the directions of the networks $\mathbb{Z}^2$.

P can then be decomposed as the convex hull of four convex polygonal arcs, which connect the four extremal segments (as illustrated in Figure 3.12).

The study of polygons can therefore be reduced to the study of convex polygonal arcs, as Vershik already mentioned in [151] to establish the limit form of polygons stated previously. However, a convex polygonal arc is equivalent to the zonogon composed of this arc and its symmetry around the middle of the segment connecting its two extremities.

Formally, let k points $(x_i, y_i)_{1 \leq i \leq k}$ of $\mathbb{Z}^2$ such that $0 = x_0 < x_1 < ... < x_{k-1} \leq x_k$,

$0 = y_0 \leq y_1 < y_2 < ... < y_k$ and

$$0 \leq \frac{y_1 - y_0}{x_1 - x_0} < ... < \frac{y_k - y_{k-1}}{x_k - x_{k-1}} \leq +\infty. \tag{3.4.10}$$

A ***polygonal convex arc*** of k segments is the union of the segments $[(x_i, y_i), (x_{i+1}, y_{i+1})]$ for $i \in \{0, 1, ..., k-1\}$. The point (x_0, y_0) is the origin of the arc and (x_k, y_k) is the end. From the combinatorial point of view, this increasing convex arc is entirely defined by the vectors $(x_{i+1} - x_i, y_{i+1} - y_i) \in \mathbb{N}^d$ for $0 \leq i \leq k-1$, and it corresponds to the zonogone whose generators are $((x_{i+1} - x_i, y_{i+1} - y_i))_{0 \leq i \leq k-1}$ in the natural bijection between zonogons and convex polygonal arcs.

The study of zonogons is therefore directly linked to that of polygons. This is no longer the case at all for higher dimensions. However, in the absence of results on polytopes in all dimensions, one can still generalize a lot of results on zonogons in all dimensions, as we shall see in Chapter 4 and 5.

In 2013 and 2016, O. Bodini, P. Duchon, A. Jacquot and L. Mutafchiev [25] then J. Bureaux and N. Enriquez [38] calculated the asymptotic equivalent of the number of polygonal arcs convex between $(0, 0)$ and (n, n). The former studied convex polyominoes, using the same cardinal decomposition as the one presented above, and by applying the saddle-point method similarly to the method used in Chapter 4, the latter used a probabilistic model of statistical physics, introduced by Sinai, which we will introduce in Chapter 5, and they used a local limit theorem of S. V. Bogachev and S. M. Zarbaliev [26]. Considering $z_2(n)$ the number of convex polygonal arcs between $(0, 0)$, and (n, n), they get

$$z_2(n) \underset{n \to +\infty}{\sim} \frac{e^{-2\zeta'(-1)}}{(2\pi)^{7/6}\sqrt{3}\kappa^{1/18}n^{17/18}} \exp\left(3\kappa^{1/3}n^{2/3} + I\left(\left(\frac{\kappa}{n}\right)^{1/3}\right)\right), \quad \text{avec } \kappa = \frac{\zeta(3)}{\zeta(2)}, \tag{3.4.11}$$

where $I(x)$ is an integral of the function $s \mapsto \frac{\Gamma(s)\zeta(s+1)(\zeta(s-1)+\zeta(s))}{\zeta(s)x^s}$ on a contour surrounding the non-trivial zeros of the Riemann function ζ. Here, we see the connection with the theory of numbers which was the subject of a previous remark. The presence of the integral $I(x)$ allows Bureaux and Enriquez to demonstrate that the asymptotic equivalent of convex arcs is related to the Riemann hypothesis:

Proposition 3.4.1 (Bureaux, Enriquez)**.** *With the previous notations, the Riemann hypothesis is valid if and only if for all $\epsilon > 0$,*

$$I\left(\left(\frac{\kappa}{n}\right)^{1/3}\right) = o\left(n^{\frac{1}{6}+\epsilon}\right). \tag{3.4.12}$$

The question is then to generalize these results in higher dimensions. The formula for the number of zonotopes in a d-hypercube of side n is the subject of Chapter 4. However, in a seminal paper [14] in 2018, J. Bureaux, I. Bárány and B. Lund calculated the logarithmic equivalent of this number, as well as demonstrated the convergence of these zonotopes towards a limiting form.

In their model, they study zonotopes in salient and pointed convex cones, we shall talk aboout it in Chapter 5. In dimension 2, taking the cone $(\mathbb{R}_+)^2$, the considered zonogons are exactly the polygonal convex arcs. Denote the vector $\mathbf{1} = (1, ..., 1)$ and $z_d\left((\mathbb{R}_+)^d, n\mathbf{1}\right)$ the number of zonotopes whose generators are integer vectors of $\mathbb{R}^d$ whose sum is $(n, ..., n)$, then

$$\log z_d \left((\mathbb{R}_+)^d, n\mathbf{1} \right) \underset{n\to+\infty}{\sim} (d+1)\kappa_d^{\frac{1}{d+1}} n^{\frac{d}{d+1}}, \quad \text{with } \kappa_d = \frac{\zeta(d+1)}{\zeta(d)}. \tag{3.4.13}$$

Now consider a pointed salient convex cone $\mathcal{C}$ of dimensions $d \geq 2$. For a vector $\boldsymbol{k}$ contained in the interior of int$\mathcal{C}$, we define $\mathcal{Z}(\mathcal{C}, n\boldsymbol{k})$ the set of zonotopes whose generators are vectors of $\mathcal{C}$ such that the sum of the generators is equal to $n\boldsymbol{k}$. We obtain:

Theorem 3.4.3 (Bureaux, Bárány, Lund)**.** *There exists a convex set Z_0 such that, for all $\epsilon > 0$,*

$$\underset{n\to+\infty}{lim} \frac{\left|\{Z \in \mathcal{Z}(\mathcal{C}, n\boldsymbol{k}), \delta(\frac{1}{n}Z, Z_0) \leq \epsilon\}\right|}{z_d(\mathcal{C}, n\boldsymbol{k})} = 1. \tag{3.4.14}$$

This result is based on the proof of a central limit theorem of the sum of the generators $\mathbf{X}(Z)$, and on a Chernoff inequality, for the distribution introduced by Sinai, called "Boltzmann's" that we will recall in Chapter 5. The natural question after such a result is to look at the fluctuations of the zonotopes around this limit form Z_0. We show in Theorem 5.4.1 that these fluctuations are Gaussian, and we explicitly compute their covariance.

Chapitre 4

Combinatorics: Asymptotic enumeration of Zonotopes

4.0.1 Introduction

This chapter is dedicated to the enumeration of the lattice zonotopes inscribed in the hypercube $[0,n]^d$, for two integers n and d. We are interested in estimating the number of zonotopes as n grows large, following the question of V. Arnold, as detailed in the previous chapter. We stress the difference between this model and the models of J. Bureaux, N. Enriquez [38], I. Bárány, and B. Lund [14] as we don't restrict generators to be in a given cone other than $\mathbb{R}^d$.

This result is achieved using tools of analytic combinatorics. As we shall see later, though, we could have taken a probabilistic point of view (see Remark 4.3.1 and Chapter 5) instead of the combinatorial one. In order for this chapter to be the most consistent possible we will start with a few preliminary results and a quick overview of analytic combinatorics, based on the famous book of P. Flajolet and R. Sedgewick [66], particularly the symbolic method and the saddle-point method. We will then apply the combinatorial characterization on the zonotopes to obtain a generating function. This generating function is a partition function, as explained in the previous chapter.

Once the description of the generating function of the lattice zonotopes is done for our model, the enumeration problem becomes an analytic one over the generating function : the saddle-point integral method in multi-dimension. The second section will deal with the analytic properties that will be necessary in the saddle-point method strictly speaking. At the end of it, we will show that the enumeration of zonotopes is linked to Eulerian polynomials. This result was found in collaboration with O. Bodini.

The saddle-point integral method will be conducted in the third section. In order to do so, we will show that the generating function is part of a class of functions called H-admissible. The saddle-point method applies to such functions, which gives the enumeration of the lattice zonotopes inscribed in $[0, n]^d$ in Theorem 4.3.1. In fact, more can be said for this function. Partition functions are related to probabilistic Boltzmann distributions that come from statistical physics. This is the key point of the duality between the combinatorial analysis of partition function in analytic combinatorics (that is led in this chapter, and e.g., in [25]) and the probabilistic approach lead in Chapter 5 and in [38, 14]. We will show that the H-admissibility of a partition function leads to a local limit theorem of rate 1 for the associated Boltzmann distribution, and we will link the multidimensional H-admissibility criteria of B. Gittenberger and J.Mandlburger [73], and the framework for local limit theorem of J. Bureaux [36].

After the enumeration of lattice zonotopes, we will compute the first moment of two parameters of a lattice zonotope. If we draw a random lattice zonotope inscribed in $[0, n]^k$ under the uniform distribution, Theorem 4.3.2 and Proposition 4.3.2 will respectively give the mean of its number of generators, and the average length of a generator.

4.1 Some tools of Analytic combinatorics

In combinatorics, each new object under study raises the same questions: how many are they, when we restrict to a finite family? How much does this number grow when enlarging the family? What does an object drawn at random look like? Analytic combinatorics is the gathering of the analytic methods that are used to answer these questions. These tools and methods were classified in [66], which establish a mapping between the combinatorial properties of objects and the analytic properties of their generating function. We will try to stick to their notation as long as possible.

This section aims to briefly give the tools and ideas to enumerate with analytic combinatorics to keep the overall consistency of the chapter. The last part of this section is the application of the first one for the zonotopes inscribed in the hypercube $[0, n]^d$.

4.1.1 The symbolic method

The symbolic method is the gathering of systematic rules that match combinatorial specific structures on discrete sets of objects with specific operations on the generating functions. With this method, one can get an equation verified by the generating function of a certain combinatorial class, which is the first step to the enumeration of this class, based on the combinatorial structure of this class. We will start with defining properly the words we used in this short introduction.

Let $\mathcal{A}$ be a finite or denumerable set. $\mathcal{A}$ a *combinatorial class* is there exists a function $s : \mathcal{A} \to \mathbb{N}$ such that for any non-negative integer k, $s^{-1}(\{k\})$ is finite. This function is called the *size function* on the class $\mathcal{A}$, and we can denote the size of an element $a \in \mathcal{A}$ by $|a| = s(a)$.

We define $\mathcal{A}_n = s^{-1}(\{n\})$ as the set of the elements of $\mathcal{A}$ of size n. Naturally, the *counting sequence* $(A_n)_{n \geq 0}$ of the combinatorial class $\mathcal{A}$ derives from the family $(\mathcal{A}_n)_{n \geq 0}$, where $A_n = \text{card}(\mathcal{A}_n)$, for all $n \geq 0$.

Example 4.1.1. *Let's take the most famous combinatorial example: the binary trees. Let $\mathcal{T}$ be the class of binary trees where the size function assigns to each binary tree its number of internal nodes. Therefore the counting sequence of the n-internal-node binary trees starts with $A_0 = 1$, $A_1 = 1$, $A_2 = 2$, and then 5, 14, 42, 132, 429...*

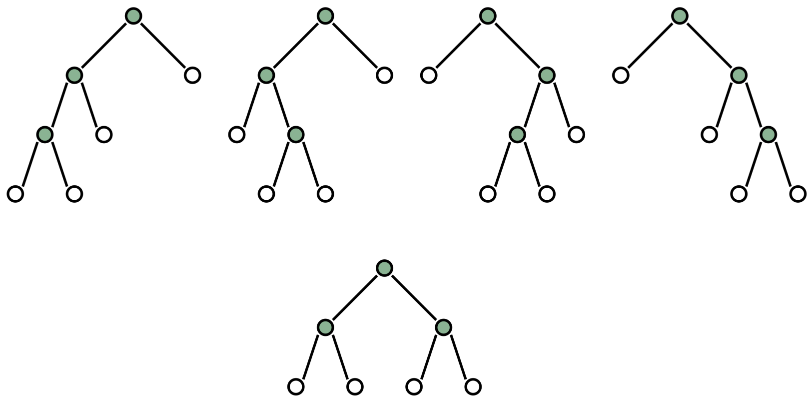


Figure 4.1: The five binary trees of length 3.

If we take the class $\mathcal{D}$ of Dyck words, where a Dyck word is a word made up of the letters a and b such that there is as many a's as b's in it and that there is no prefix with more bs than as. The size of a Dyck word is the number of as. The counting sequence of Dyck words starts with $1, 1, 2, 5, 14, 32$ etc. The two combinatorial classes $\mathcal{T}$ and $\mathcal{D}$ will be called combinatorially isomorphic if they share the same counting sequence. It is the case in this example, as we will see. This sequence is famously known as the Catalan numbers, $T_n = \frac{1}{n+1}\binom{2n}{n}$ (sequence A000108 in the OEIS).

We can define the *ordinary generating function* of $\mathcal{A}$ (versus exponential generating function, that will not be discussed here) as the formal power series

$$A(z) = \sum_{n \geq 0} A_n z^n. \tag{4.1.1}$$

The ordinary generating function will be shortened to generating function in the following, and we will keep the convention to keep the same letter for a class, its counting sequence and its generating function, for instance the binary trees $\mathcal{T}$ will be associated with the counting sequence (T_n) and the generating function $T(z)$. The variable z encodes the size of the elements of $\mathcal{A}$. It is important to see that we can rewrite the generating function as

$$A(z) = \sum_{a \in \mathcal{A}} z^{|a|}. \tag{4.1.2}$$

The key point is that *we can convert operations on classes to operations on generating functions*. For two distinct classes $\mathcal{A}$ and $\mathcal{B}$, of generating function respectively $A(z)$ and $B(z)$, and for $\times$ being the Cartesian product, we can see that

$$\mathcal{S} = \mathcal{A} \cup \mathcal{B} \qquad S_n = A_n + B_n \qquad S(z) = A(z) + B(z).$$
$$\mathcal{P} = \mathcal{A} \times \mathcal{B} \qquad P_n = \sum_{i=1}^{n} A_i B_{n-i} \qquad P(z) = A(z) \cdot B(z).$$

Here are the two basic operations on classes, namely the distinct union and the Cartesian product of a class.

Example 4.1.1 (continued)**.** *Let $\mathcal{X}$, resp. $\mathcal{E}$ be the class composed of respectively one element x of size 1 and one element e of size 0. Then, we can write the binary tree class recursive construction:*

$$\mathcal{T} = \mathcal{E} \cup (\mathcal{T} \times \mathcal{X} \times \mathcal{T}). \tag{4.1.3}$$

Which translates into generating functions:

$$T(z) = 1 + zT(z)^2. \tag{4.1.4}$$

The same goes for Dyck words, which we can see with the standard decomposition of a Dyck word W into either e, either the word aW_1bW_2 where W_1 and W_2 are two Dyck words.

From this equation, we deduct that

$$T(z) = D(z) = \frac{1 - \sqrt{1-4z}}{2z} = 1 + z + 2z^2 + 5z^3 + 14z^4 + o\left(z^4\right). \tag{4.1.5}$$

The counting sequence (T_n) is entirely encoded in the formal power series of $z \mapsto \frac{1-\sqrt{1-4z}}{2z}$.

We can therefore construct a whole dictionary of matches between combinatorial operations and function operations. Here are the main combinatorial operations, given a class $\mathcal{A}$ with no element of size 0:

- The *sequence class* $\mathrm{SEQ}(\mathcal{A})$ is the class of the sequences of elements of $\mathcal{A}$ is defined by $\mathrm{SEQ}(\mathcal{A}) = \{e\} \cup \mathcal{A} \cup \mathcal{A} \times \mathcal{A} \cup \mathcal{A} \times \mathcal{A} \times \mathcal{A} \cup \ldots$
- The *multiset class* $\mathrm{MSET}(\mathcal{A})$ is the class of lists of elements of $\mathcal{A}$ with arbitrary repetitions allowed but where the order does not count.
- The *cycle class* $\mathrm{CYC}(\mathcal{A})$ is the class of sequences of elements of $\mathcal{A}$ taken up to a cyclic permutation of the elements.

Here is a table to summarize the combinatorial operations and the derived operations over the generating function from [66], where we denote ϕ the Euler totient function:

Name	Operation	Generating function
Distinct union	$\mathcal{C} = \mathcal{A} \cup \mathcal{B}$	$C(z) = A(z) + B(z)$
Product	$\mathcal{C} = \mathcal{A} \times \mathcal{B}$	$C(z) = A(z) \cdot B(z)$
Sequence	$\mathcal{B} = \textsc{Seq}(\mathcal{A})$	$B(z) = \dfrac{1}{1 - A(z)}$
Multiset	$\mathcal{B} = \textsc{MSet}(\mathcal{A})$	$B(z) = \prod_{n \geq 1} (1 - z^n)^{-B_n}$
Cycle	$\mathcal{B} = \textsc{Cyc}(\mathcal{A})$	$B(z) = \sum_{k=1}^{+\infty} \dfrac{\phi(k)}{k} \log \dfrac{1}{1 - B\left(z^k\right)}$

With this dictionary, one can access equations satisfied by the generating function of the studied class. Then the next step is to extract coefficients from the generating function. For any formal series A, we denote the n^{th} coefficient extractor $[z^n]A(z) = A_n$. One way to do so is to use the residue Theorem formula. Let $A(z)$ be the formal series under study, and U an open set of $\mathbb{C}$ such that A is defined on U and $0 \in U$. For any closed path γ surrounding 0 and oriented counterclockwise:

$$[z^n]A(z) = \frac{1}{2i\pi} \oint_\gamma \frac{A(z)}{z^{n+1}} dz. \tag{4.1.6}$$

4.1.2 Asymptotics with saddle point method

The problem of extracting the n^{th} coefficient of a generating function f is a purely analytic problem, which is why this field of enumerative combinatorics is named analytic combinatorics. A lot of general theorems were proved to match specific conditions on f to an asymptotic equivalent of $[z^n]f(z)$, that one can find in [66, 113]. The basic principle is that there exists a general correspondence between the asymptotic expansion of f near its singularity nearest to 0, and the asymptotic expansion of the function's coefficients. More precisely, we can state two major rules:

- The location of the singularity dictates the asymptotic exponential growth c^n of the n^{th} coefficient.
- The nature of the singularity dictates the subexponential factor $\theta(n)$.

We will not go further into the detail of this general correspondence. Here we investigate a routinely used method to extract coefficients of a generating function f, called the saddle-point method.

The saddle-point method is a method to approximate an integral based on Laplace's method, which means that we will approximate a Gaussian integral so that the integral becomes computable. Its name comes from the saddle point, which is a critical point of the surface of the graph of a function defined on $\mathbb{C}$, such that we can find non-colinear complex number x and y so that the point is a local maximum in the direction of x, and a local minimum in the direction of y. The two sets of surface points at each side of the saddle-point in the direction of x are called *valleys*.

For $n > 0$, let $F_n : U \subset \mathbb{C} \to \mathbb{R}$ be a meromorphic function (typically defined as $\frac{A(z)}{z^{n+1}}$ in 4.1.6). We denote $f_n = \log F_n$, and we assume that the integral

$$\int_\gamma F_n(z)dz = \int_\gamma e^{f_n(z)}dz \tag{4.1.7}$$

exists for a contour γ that comes through a saddle point ζ (i. e. $f_n'(\zeta) = 0$, this equality is called the *saddle equation*), and that lies in the two opposite valleys. We split γ as $\gamma = \gamma_1 \cup \gamma_2$ such that $\zeta \in \gamma_2$, see Figure 4.2. Then, one has to verify the following:

Condition 1, Tails pruning: The tail integral on γ_1 is negligible,

$$\int_{\gamma_1} F_n(z)dz = o\left(\int_\gamma F_n(z)dz\right). \tag{4.1.8}$$

Condition 2, Central approximation: On γ_2, we have the following approximation

$$f_n(z) = f_n(\zeta) + \frac{f_n''(\zeta)}{2}(z-\zeta)^2 + o(\eta_n). \tag{4.1.9}$$

uniformly on γ_2, and $\eta_n \underset{n\to+\infty}{\to} 0$. This step implies

$$\int_{\gamma_2} F_n(z)dz \sim \int_{\gamma_2} e^{f_n(\zeta)+\frac{f_n''(\zeta)}{2}(z-\zeta)^2}dz, \quad \text{for } n \to +\infty. \tag{4.1.10}$$

Condition 3, Tails completions: We rewrite the term $f_n''(\zeta) = \epsilon|f_n''(\zeta)|$, and even if it means reorienting γ_2, we assume that z is increasing along γ_2. The integral on the right-hand side of the asymptotic equivalence above can be completed into a full Gaussian integral:

$$\int_{\gamma_2} e^{\frac{f_n''(\zeta)}{2}(z-\zeta)^2}dz \sim e^{-\epsilon/2}\int_{-\infty}^{+\infty} e^{-\frac{|f_n''(\zeta)|}{2}x^2}dx, \quad \text{for } n \to +\infty. \tag{4.1.11}$$

Saddle-point approximation: If the three conditions are verified, then one has

$$\frac{1}{2\pi i}\int_{\gamma_1} F_n(z)dz \sim e^{f(\zeta)+\epsilon/2}\int_{-\infty}^{+\infty} e^{-\frac{|f''(\zeta)|}{2}x^2}dx = \frac{e^{f(\zeta)}}{\sqrt{2\pi f''(\zeta)}}, \quad \text{for } n \to +\infty. \tag{4.1.12}$$

The Cauchy formula of the residue Theorem mentioned in (4.1.6) typically satisfies the initial conditions, as we can see in the following figure.

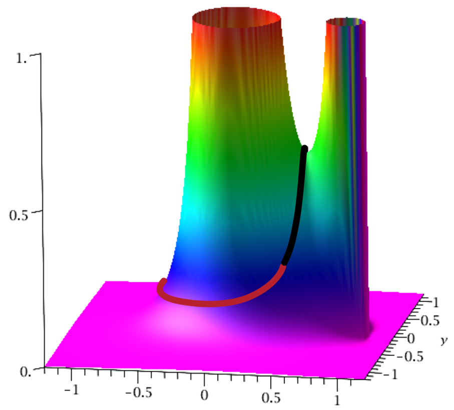


Figure 4.2: Example of the saddle contour of a Cauchy formula: the main contour γ_2 is in black and the negligible contour γ_1 in red.

In this chapter, we will use the saddle-point method in dimension d, which complicates the three convergence conditions. To handle the calculation, classes of function were determined in order to use the saddle-point method directly on them. The most famous class is the H-admissible functions, named after the seminal work of W. K. Hayman [86]. We shall see in a subsequent section a notion of d-dimensional H-admissibility.

4.1.3 The generating function of Zonotopes

Remind from the introduction that a *lattice zonotope* Z is a polytope for whom there exists $k \in \mathbb{N}$ and $v_1, ..., v_k \in \mathbb{Z}^d$ such that, up to translation, Z is:

$$Z = \left\{ \sum_{i=1}^{k} \alpha_i v_i \mid \alpha_1, ..., \alpha_k \in [0,1] \right\}. \tag{4.1.13}$$

As we study inscribed zonotopes in a hypercube, we say that two zonotopes are *equivalent* if there is an affine translation carrying one to the other we consider lattice zonotopes up to translation thereafter. We denote $\mathcal{Z}_d$ the set of class of equivalence of d-dimensional lattice zonotopes and will use the L^1 norm of its generators,

$\sum_{i=0}^{k} |\boldsymbol{v}_i|$, as the size of a lattice zonotope Z. $\mathcal{Z}_d$ is a combinatorial class in the sense of Section 4.1.1. Therefore we will combinatorially characterize $\mathcal{Z}_d$ with a set of generators.

Given a multiset (an unordered finite set of elements with repetition allowed) $V = \{\boldsymbol{v}_1, .., \boldsymbol{v}_k\} \in (\mathbb{Z}^d \setminus \{\boldsymbol{0}\})^k$ and the integral zonotope Z determined by V, the set of generators uniquely defines a zonotope but the converse is not true. Let $\mathbb{P}_d$ be the set of primitive vectors of $\mathbb{Z}^d$, and $\mathbb{P}_{d+}$ the set of primitive vectors of $\mathbb{N}^d$. A vector $(z_1, ...z_d) \in \mathbb{Z}^d$ is *primitive* if $\gcd(z_1, ..., z_d) = 1$, and therefore notice that $0 \notin \mathbb{P}_d$ (we prefer the notation $\mathbb{P}_d$ rather than $\mathbb{P}^d$ because $\mathbb{P}_d$ is very different from $(\mathbb{P}_1)^d$). Finally, let $\mathcal{P}_d\star$ denote the set of primitive vectors whose first non-zero coordinate is positive, following the convention from [52].

Lemma 4.1.1. *There is a one-to-one correspondence between $\mathcal{Z}_d$ and the finite (multi)sets of $\mathcal{P}_d\star$.*

Proof. We need to prove that for any zonotope Z there is a unique multiset $W = \{\boldsymbol{w}_1, ..., \boldsymbol{w}_l\} \in \mathcal{P}_d\star^l$ of elements of $\mathcal{P}_d\star$ that determines the class of equivalence of Z, in the sense that it is the only set of elements of $\mathcal{P}_d\star$ that generates an affine translation of Z. W is constructed that way: given a generator $\boldsymbol{v} \subset \mathbb{R}^d$ of Z, there is a unique affine translation $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(\boldsymbol{v})$ has 0 as one of its extremities, and that its first coordinate is positive. Then $f(v)$ can be uniquely written as $m_i\boldsymbol{w}$, with $\boldsymbol{w} \in \mathcal{P}_d\star$. Then add m_i copies of $\boldsymbol{w}$ in W. For two different multisets U and V that generate affine translations of Z, U and V give the same W with the construction above. □

A multiset W of $\mathcal{P}_d\star$ is called a *strict integer partition* of the vector $\sum_{w\in W} w$ from $(\mathbb{R}^d)_+$, where $(\mathbb{R}^d)_+$ is the set of all vectors whose first coordinate is positive, as mentioned in the Section 2.3.1. This can be written in terms of function with finite support. Let Ω be the space of functions $\omega : \mathcal{P}_d\star \to \mathbb{N}$ with finite support. The function associated with W is the function of multiplicities ω_W defined by $\omega_W(\boldsymbol{x}) = \mathrm{card}\{\boldsymbol{w} \in W \mid \boldsymbol{w} = \boldsymbol{x}\}$ for any $\boldsymbol{x} \in \mathcal{P}_d\star$. To recap

- The function ω associated with a zonotope Z up to translation is defined for all $\boldsymbol{v} \in \mathcal{P}_d\star$ as the number of occurrences of $\boldsymbol{v}$ in an edge of Z that is colinear to $\boldsymbol{v}$ if there is such an edge, and 0 otherwise.
- Given a function ω, the list of generator which defines Z is $\{\omega(\boldsymbol{v})\boldsymbol{v}, \boldsymbol{v} \in \mathrm{supp}(\omega)\}$.

Now, we associate to each class of equivalence in $\mathcal{Z}_d$ the zonotope Z generated by elements of $\mathcal{P}_d\star$, and we will directly write $Z \in \mathcal{Z}_d$ thereafter. We denote the *endpoint* $\boldsymbol{k}$ of Z (and we say that Z ends at $\boldsymbol{k}$) as

$$\boldsymbol{k} = \sum_{i=1}^{l} \boldsymbol{w}_i. \qquad (4.1.14)$$

Analogously, we say that 0 is the *origin* of Z, and that Z begins at 0.

This combinatorial structure can be summed up in a finite set of repeated elements without taking care of the order, which coincides with the notion of a multiset in the symbolic method in the previous section. This gives us the possibility, to apply well-known analytical tools on the generating function of the multiset of elements of $\mathcal{P}_d\star$.

Let Zon_d encode the multivariate generating function of the class $\mathcal{Z}_d$. We will show in (4.1.17) that this function is defined on the open centered disk of radius 1, hence for $\boldsymbol{x} = (x_1, x_2, ..., x_d) \in (-1,1)^d$, the generating function is defined as

$$Zon_d(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{N}^d} z_{\boldsymbol{n}} x_1^{n_1} ... x_d^{n_d}, \tag{4.1.15}$$

where $z_{\boldsymbol{n}}$ is the number of lattice zonotopes inscribed in a box of size $n_1 \times n_2 \times ... \times n_d$, for any $\boldsymbol{n} \in \mathbb{N}^d$. In other words, the orthogonal projection of those zonotopes on the i^{th} coordinate is the segment $[0, n_i]$. The variable x_i then encodes the length of the projection of a zonotope on the ith axis of coordinates. We classically denote $\boldsymbol{x}^{\mathbf{n}} = x_1^{n_1} ... x_d^{n_d}$, $\mathbf{1}$ the vector $(1, 1, ..., 1)$, and $\boldsymbol{x} \cdot \boldsymbol{y}$ as the canonical scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ in the sequel. We also introduce the notation $[\boldsymbol{v}]$ to designate the vector of the absolute values of $\boldsymbol{v}$, $(|v_1|, ..., |v_d|)$.

If we do the Minkowski sum of a given generator $\omega(\boldsymbol{v})\boldsymbol{v}$ and a zonotope (which means adding this generator to the set of generators of the zonotope), the size of the hypercube in which is inscribed the zonotope increases in each dimension i by $\omega(\boldsymbol{v})|v_i|$. Then we can rewrite these functions using the finite support function as follows:

$$Zon_d(\boldsymbol{x}) = \sum_{\omega \in \Omega} \prod_{\boldsymbol{v} \in \mathcal{P}_d\star} \boldsymbol{x}^{\omega(\boldsymbol{v})[\boldsymbol{v}]}, \tag{4.1.16}$$

which can be transformed into the generating function of a multiset by factorization and elementary operations:

$$Zon_d(\boldsymbol{x}) = \prod_{\boldsymbol{v} \in \mathcal{P}_d\star} \left(1 - \boldsymbol{x}^{[\boldsymbol{v}]}\right)^{-1}. \tag{4.1.17}$$

This function is defined on the open complex d-dimensional disk of radius 1 and centered at 0, with a singularity at 1. In the following, we use the change of variables $x_i = e^{-\theta_i}$ to study the function as $\boldsymbol{\theta}$ goes to 0. The resulting form is a partition function over the set $\mathcal{P}_d\star$ of primitive vectors whose first non-zero coordinate is positive. This function is analogous to the partition function used in the Boltzmann probabilistic point of view in [38], that is, for $\boldsymbol{\theta} \in (0, +\infty)^d$:

$$\prod_{v \in \mathcal{P}_{d\star}} \left(1 - e^{-\theta \cdot [v]}\right)^{-1}. \tag{4.1.18}$$

Denoting $d(v)$ the number of non-zero coordinates of v, the product can be simplified by gathering in the same factor the vectors whose absolute value of coordinates coincide (the first non-zero coordinate shall be positive), that is

$$Zon_d\left(e^{-\theta}\right) = \prod_{v \in \mathbb{P}_{d+}} \left(1 - e^{-\theta \cdot v}\right)^{-2^{d(v)-1}}. \tag{4.1.19}$$

4.2 Asymptotics of the generating function

The extraction of the coefficient $[x^n]Zon_d(x)$ is done in three steps:

- We asymptotically estimate $Zon_d\left(e^{-\theta}\right)$ when $\theta \to \mathbf{0}$ (Proposition 4.2.1). This is carried out in the second part of this section
- We verify that Zon_d satisfies the conditions of the saddle-point method (Proposition 4.3.1, Section 4.3).
- Finally, we compute the saddle equation to compute the asymptotic equivalent of z_n (Lemma 4.3.4, Section 4.3).

Before these three steps, we start the analysis of the generating function by giving an integral formula and an equivalent of the logarithm of Zon_d and of its partial derivative. The second part of this section focuses on the study of the univariate generating function defined by $Zon_d(x\mathbf{1})$. We compute the asymptotic equivalent of this function. Those two parts are fundamental in the convergence analysis conducted in the next section.

4.2.1 Integral formula and asymptotic equivalent of partial derivative.

In order to asymptotically study the generating function, we will mainly use in the sequel an integral formula, deriving from the Mellin inversion formula:

Lemma 4.2.1. *Taking $c > d$, for all $\theta \in (0, +\infty)^d$, we have*

$$\log\left(Zon_d(e^{-\theta})\right) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s+1)\Gamma(s)}{\zeta(s)} \sum_{v \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}} \frac{2^{d(v)-1}}{(\theta \cdot v)^s} ds. \tag{4.2.1}$$

Proof. Let $\theta \in (0, +\infty)^d$, we first compute the logarithm of the equation (4.1.17), depending on the number $d(x)$ of non-zero coordinate of v, and write the Taylor series expansion of the logarithm:

$$\log\left(Zon_d(e^{-\theta})\right) = -\sum_{v\in\mathbb{P}_{d+}} 2^{d(v)-1}\log\left(1-e^{-\theta\cdot v}\right) = \sum_{v\in\mathbb{P}_{d+}}\sum_{m\geq 1}\frac{2^{d(v)-1}}{m}e^{-m\theta\cdot v} \tag{4.2.2}$$

Now, we can compute the Mellin transform of this function by replacing the exponential terms with their transform through the Mellin inversion formula. Recall that the Mellin transform of the real function $y \mapsto e^{-y}$ is the Γ function, hence by the inverse of the Mellin transform, for every real numbers $y > 0$ and $c > 0$:

$$e^{-y} = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\Gamma(s)y^{-s}ds. \tag{4.2.3}$$

Using this relation, we replace each $e^{-m\theta\cdot v}$. Then, for any real number c such that $c > d$, we have $\sup_{s\in c+i\mathbb{R}}\left(\sum_{v\in\mathbb{P}_{d+}}\frac{1}{(\theta\cdot v)^s}\right) < +\infty$ and $\sup_{s\in c+i\mathbb{R}}\left(\sum_{m\geq 1}\frac{1}{m^{s+1}}\right) < +\infty$. Therefore the Fubini-Tonelli theorem leads to:

$$\log\left(Zon_d(e^{-\theta})\right) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\sum_{m\geq 1}\sum_{v\in\mathbb{P}_{d+}}\frac{2^{d(v)-1}}{m}\frac{\Gamma(s)}{(m\theta\cdot v)^s}ds. \tag{4.2.4}$$

We can extract the variable m from the sums and bring out the Riemann ζ function. We obtain

$$\log\left(Zon_d(e^{-\theta})\right) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\zeta(s+1)\sum_{v\in\mathbb{P}_{d+}}\frac{2^{d(v)-1}\Gamma(s)}{(\theta\cdot v)^s}ds. \tag{4.2.5}$$

Finally, the sum over the primitive vectors can be completed to a sum over $\mathbb{Z}_+^d$ the using the partition $\mathbb{Z}_+^d = \cup_{k\geq 1}k\mathbb{P}_{d+}$,

$$\sum_{v\in\mathbb{Z}_+^d\setminus\{\mathbf{0}\}}\frac{2^{d(v)-1}}{(\theta\cdot v)^s} = \sum_{k\geq 1}\frac{1}{k^s}\sum_{v\in\mathbb{P}_{d+}}\frac{2^{d(v)-1}}{(\theta\cdot v)^s} = \zeta(s)\sum_{v\in\mathbb{P}_{d+}}\frac{2^{d(v)-1}}{(\theta\cdot v)^s}. \tag{4.2.6}$$

For $\theta \in (0,+\infty)^d$, the integral becomes, after introducing a term $\zeta(s)$ to use (4.2.6),

$$\log\left(Zon_d(e^{-\boldsymbol{\theta}})\right) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s+1)\Gamma(s)}{\zeta(s)}\sum_{\boldsymbol{v}\in\mathbb{Z}_+^d\setminus\{\mathbf{0}\}}\frac{2^{d(\boldsymbol{v})-1}}{(\boldsymbol{\theta}\cdot\boldsymbol{v})^s}ds. \tag{4.2.7}$$

□

The following lemma establishes the asymptotic equivalent of the partial derivatives of $\log\left(Zon_d(e^{-\boldsymbol{\theta}})\right)$. These asymptotic equivalents of partial derivative are crucial in the proof of Proposition 4.3.1 that gives an asymptotic equivalent of the coefficients of Zon_d.

Lemma 4.2.2. *For* $(k_1, ..., k_d) \in \mathbb{N}^d$ *and for all* $\epsilon > 1$*, for* $\boldsymbol{\theta} \in (0, +\infty)^d$ *such that for* $1 \le i, j \le d$ *and* $i \ne j$*,* $\frac{\theta_i}{\theta_j} \in (\frac{1}{\epsilon}, \epsilon)$*,*

$$\frac{\partial^{k_1+k_2+...+k_d}}{\partial\theta_1^{k_1}\partial\theta_2^{k_2}...\partial\theta_d^{k_d}}\log\left(Zon_d(e^{-\boldsymbol{\theta}})\right) \underset{\boldsymbol{\theta}\to\mathbf{0}}{\sim} (-1)^{k_1+...+k_d}\frac{2^{d-1}\zeta(d+1)}{\zeta(d)}\frac{\prod_{k=1}^d k_i!}{\theta_1^{k_1+1}...\theta_d^{k_d+1}}. \tag{4.2.8}$$

Proof. We introduce the Barnes zeta function in d dimensions $\zeta_d(s, w, \boldsymbol{\theta})$, where s, w and $\{\theta_i, 1 \le i \le d\}$ are complex numbers such that $\Re(s) > d$, $\Re(w) > 0$ and for $1 \le i \le d$, $\Re(\theta_i) > 0$ as:

$$\zeta_d(s, w, \boldsymbol{\theta}) = \sum_{\boldsymbol{v}\in\mathbb{N}^d}\frac{1}{(w + \boldsymbol{\theta}\cdot\boldsymbol{v})^s}. \tag{4.2.9}$$

This function can be meromorphically continued to all complex s, with simple poles at $1, 2, ..., d$. Remark that all the vectors $\boldsymbol{v}$ for which $d(\boldsymbol{v}) = d$ are the vectors without any zero coordinate, that is vectors belonging to $\mathbb{N}^d \cap (0, +\infty)^d$. Additionally, we can write

$$\sum_{\boldsymbol{v}\in\mathbb{N}^d\cap(0,+\infty)^d}\frac{2^{d(\boldsymbol{v})-1}}{(\boldsymbol{\theta}\cdot\boldsymbol{v})^s} = 2^{d-1}\zeta_d\left(s, \sum_{i=1}^d\theta_i, \boldsymbol{\theta}\right). \tag{4.2.10}$$

This leads to rewriting the sum in the integrand of the right term of (4.2.1) in Lemma 4.2.1 as a sum of Barnes zeta functions. For $s > d$, that is:

$$\sum_{\boldsymbol{v}\in\mathbb{Z}_+^d\setminus\{\mathbf{0}\}}\frac{2^{d(\boldsymbol{v})-1}}{(\boldsymbol{\theta}\cdot\boldsymbol{v})^s} = \sum_{j=1}^d\sum_{1\le i_1<...<i_j\le d}2^{j-1}\zeta_i\left(s, \sum_{k=1}^j\theta_{i_k}, (\theta_{i_1}, ..., \theta_{i_2})\right). \tag{4.2.11}$$

The Barnes ζ function in δ dimensions has a simple pole at $s \in \{1, 2, ..., \delta\}$, hence the only pole in the integrand comes from the d-dimensional Barnes ζ function. We translate the domain of integration $(c - i\infty, c + i\infty)$ by 1 to the left by using the residue theorem at the pole at $s = d$. For $d - 1 < c < d$, we have:

$$\log\left(Zon_d(e^{-\boldsymbol{\theta}})\right) = \frac{2^{d-1}\zeta(d+1)\Gamma(d)}{\zeta(d)\prod_{i=1}^{d}\theta_i} + \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s+1)\Gamma(s)}{\zeta(s)}\sum_{\boldsymbol{v}\in\mathbb{Z}_+^d\setminus\{\boldsymbol{0}\}}\frac{2^{d(\boldsymbol{v})-1}}{(\boldsymbol{\theta}\cdot\boldsymbol{v})^s}ds. \tag{4.2.12}$$

With the bound established in Lemma 4.2.3 (below), the integral in the right-hand term of this equation is differentiable, hence the right term is differentiable, and therefore we can differentiate it. The derivative of the integral is negligible in front of the leading term, so we get the wanted result.

□

Lemma 4.2.3. *For all $\boldsymbol{k} \in \mathbb{N}^d$, $\delta \in (0,1)$, and $\epsilon > 0$, there exists a constant $C > 0$ such that all $\boldsymbol{\theta}$ with their coordinates (θ_i) respecting $\epsilon < \theta_i < \frac{1}{\epsilon}$ satisfy, for all s such that $\Re(s) \in (d - 1 + \delta, d + 1)$*

$$\left|\frac{\partial^{\boldsymbol{k}\cdot\boldsymbol{1}}}{(\partial\boldsymbol{\theta})^{\boldsymbol{k}}}\zeta_d\left(s, \boldsymbol{\theta}\cdot\boldsymbol{1}, \boldsymbol{\theta}\right)\right| \leq \frac{C|s|^C}{|\boldsymbol{\theta}|^{\boldsymbol{k}\cdot\boldsymbol{1}+\Re(s)}|s-d|}. \tag{4.2.13}$$

Proof. Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $x \in \mathbb{R}_+$, consider the function defined by $x \mapsto F(x) = \sum_{n_2,...,n_d \geq 0}(w + \theta_1 x + \theta_2 n_2 + ... + \theta_d n_d)^{-s}$. We apply the Euler–Maclaurin formula to the d-dimensional Barnes ζ function using F, leading to

$$\zeta_d(s, w, \boldsymbol{\theta}) = \sum_{n_1 \geq 0} F(n_1) = \int_0^\infty F(x)dx - \frac{F(0)}{2} + \int_0^\infty \left(\{x\} - \frac{1}{2}\right)F'(x)dx. \tag{4.2.14}$$

We rewrite the right term as the following expression because

$$F(x) = \zeta_{d-1}(s, w + \theta_1 x, (\theta_2, ...\theta_d)), \tag{4.2.15}$$

for $\Re(w) > 0$ and $\Re(\theta_i) > 0$. For each $1 \leq i \leq d$:

$$\begin{aligned}\zeta_d(s, w, \boldsymbol{\theta}) = &\int_0^\infty \zeta_{d-1}(s, w + \theta_1 x, (\theta_2, ...\theta_d))dx - \frac{\zeta_{d-1}(s, w, (\theta_2, ...\theta_d))}{2} \\ &- s\theta_1\int_0^\infty\left(\{x\} - \frac{1}{2}\right)\zeta_{d-1}(s+1, w + \theta_1 x, (\theta_2, ...\theta_d))dx.\end{aligned} \tag{4.2.16}$$

As mentioned before, the d-dimensional Barnes ζ function of parameters s, w, and $\boldsymbol{\theta}$ has a meromorphic continuation to all complex s whose only singularities are simple poles at $1, 2, ...,$ and d. This implies that only the first term in the previous expression has a pole at $s = d$ while the other two terms have poles at $s = d-1, d-2, ..., 1$.

We can use again Euler–Maclaurin formula for each of ζ_{d-1}. We recall the case $d = 2$, which is given in the Lemma A.1 in [38]:

$$\begin{aligned}\zeta_2(s, \omega, (\theta_1, \theta_2)) =& \frac{1}{\theta_1\theta_2}\frac{\omega^{-s+2}}{(s-1)(s-2)} + \frac{(\theta_1+\theta_2)\omega^{-s+1}}{2\theta_1\theta_2(s-1)} + \frac{\omega^{-s}}{4} \\ &- \frac{\theta_2}{\theta_1}\int_0^{+\infty}\frac{\{y\}-\frac{1}{2}}{\omega+\theta_2 y)^s}dy - \frac{\theta_1}{\theta_2}\int_0^{+\infty}\frac{\{x\}-\frac{1}{2}}{\omega+\theta_1 x)^s}dx \\ &- s\frac{\theta_2}{2}\int_0^{+\infty}\frac{\{y\}-\frac{1}{2}}{\omega+\theta_2 y)^{s+1}}dy - s\frac{\theta_1}{2}\int_0^{+\infty}\frac{\{x\}-\frac{1}{2}}{(\omega+\theta_1 x)^{s+1}}dx \\ &+ s(s+1)\theta_1\theta_2\int_0^{+\infty}\int_0^{+\infty}\frac{(\{x\}-\frac{1}{2})(\{y\}-\frac{1}{2})}{(\omega+\theta_1 x+\theta_2 y)^{s+2}}dxdy\end{aligned} \tag{4.2.17}$$

Therefore, with recursive use of the Euler–Maclaurin formula on each of the three terms of (4.2.16) and so after, we obtain a formula with a finite number of terms, linearly depending on the (θ_i). The first term is

$$\frac{1}{\theta_1...\theta_d}\frac{w^{-s+d}}{(s-1)...(s-d)}. \tag{4.2.18}$$

All the other terms of the development are of the form of the terms of (4.2.17), with quotients of θ_i and θ_j. All these terms are differentiable, and therefore so is the right term of (4.2.17). By assumptions, all ratios θ_i/θ_j range between ϵ and $\frac{1}{\epsilon}$, hence we can find a constant C such that all terms are upper bounded by

$$\frac{C|s|^C}{|\boldsymbol{\theta}|^{\boldsymbol{k}\cdot\mathbf{1}+\Re(s)}|s-d|}. \tag{4.2.19}$$

□

4.2.2 Asymptotic equivalent of the univariate generating function.

We now can compute the asymptotic equivalent of the generating function when all variables θ_i are equal and tend to 1. It will be used in the proof of the main theorem to compute the equivalent of the coefficient of the generating function given in Proposition 4.3.1. In the process, we introduce the following notation:

Definition 4.2.1. *Let* $\mathrm{M}(\mathbb{C})$ *be the field of meromorphic functions in* $\mathbb{C}$. *We define the polynomial* $P_d(X)$ *as*

$$P_d(X) = \sum_{\delta=1}^{d} \binom{d}{\delta} 2^{\delta-1} \frac{\prod_{k=1}^{\delta-1}(X-k)}{(\delta-1)!} = p_{d,d-1}X^{d-1} + ... + p_{d,1}X + p_{d,0}, \tag{4.2.20}$$

with the convention that $\prod_{k=1}^{\delta-1}(X-k) = 1$ *if* $\delta = 1$, *and the operator* $\mathbf{\Pi}_d$ *as*

$$\begin{aligned} \mathbf{\Pi}_d \colon \mathrm{M}(\mathbb{C}) &\to \mathrm{M}(\mathbb{C}) \\ \phi &\mapsto p_{d,d-1}\phi(\cdot\, - (d-1)) + ... + p_{d,1}\phi(\cdot\, - 1) + p_{d,0}\phi. \end{aligned}$$

Remark 4.2.1. *The family of polynomials* $(P_d(X))_{d\geq 1}$ *is recursively defined by* $P_{d+2}(X) = \frac{2X}{d+1}P_{d+1}(X) + P_d(X)$, *with* $P_1(X) = 1$ *and* $P_2(X) = 2X$. *Which results in alternating odd and even polynomials, the even ones having 1 as a constant term. Moreover, the leading term of* P_d *is of degree* $d-1$ *and its coefficient is* $\frac{2^{d-1}}{(d-1)!}$.

P_1	1	$\Pi_1(\phi)$	ϕ
P_2	$2X$	$\Pi_2(\phi)$	$2\phi(\cdot - 1)$
P_3	$\frac{4}{3}X^2 + 1$	$\Pi_3(\phi)$	$\frac{4}{3}\phi(\cdot - 2) + \phi$
P_4	$\frac{4}{3}X^3 + \frac{8}{3}X$	$\Pi_4(\phi)$	$\frac{4}{3}\phi(\cdot - 3) + \frac{8}{3}\phi(\cdot - 1)$

Table 4.1: First polynomials P_d.

The operator naturally results from the Mellin transform, as for $s \neq 1$, $\mathbf{\Pi}_d(\zeta)(s) = \sum_{k\geq 1} \frac{P_d(k)}{k^s}$. With this notation, we present the following proposition, where the polynomial in $\frac{1}{\theta}$ is the origin of the polynomial in the exponential part of Theorem 4.3.1:

Proposition 4.2.1. *Let* γ *denote the contour defined as the union of the two oriented paths, depending on an integer* $A > 0$ *chosen such that it surrounds all the zeros of the Riemann* ζ *function: on the right, the curve* $\gamma_r(t) = 1 - \frac{A}{\log(2+|t|)} + it$ *for* t *going from* $-\infty$ *to* $+\infty$, *and on the left the curve* $\gamma_l(t) = \frac{A}{\log(2+|t|)} + it$ *for* t *going from* $+\infty$ *to* $-\infty$ *(see Figure 4.3).*

We define the function $I_{crit,d}$ *by*

$$I_{crit,d}(\theta) = \frac{1}{2i\pi}\int_\gamma \frac{\mathbf{\Pi}_d(\zeta(s))}{\zeta(s)}\zeta(s+1)\Gamma(s)\theta^{-s}ds \tag{4.2.21}$$

For all $\theta > 0$, the asymptotic equivalent of Zon $\left(e^{-\theta}\right)$ when $\theta \to 0$ is:

$$\log\left(Zon\left(e^{-\theta}\right)\right) = \sum_{\delta=1}^{d-1}\left(p_{d,\delta}\frac{\zeta(\delta+2)\Gamma(\delta+1)}{\zeta(\delta+1)\theta^{\delta+1}}\right) + I_{crit,d}(\theta) + C + O\left(n^{-\frac{1}{d+1}}\right) \tag{4.2.22}$$

with $C = 2\mathbf{\Pi}_d\left(\log(2\pi)\zeta - \zeta'\right)(0) + 2\mathbf{\Pi}_d\left(\zeta\right)(0)\log(\theta)$.

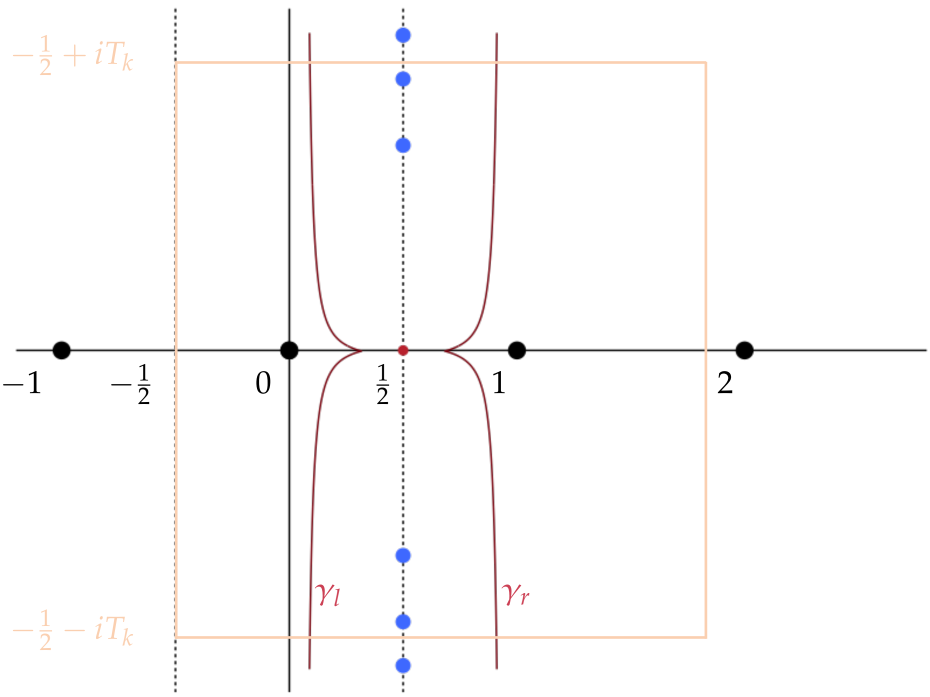


Figure 4.3: The contour γ surrounding the critical strip. The blue points stand for the non-trivial zeros of the Riemann ζ function.

Proof. For any real number c such that $c > d$, and $\theta \in (0, +\infty)$, we fix $\boldsymbol{\theta} = \theta\mathbf{1}$ and express the integral expression (4.2.1) in Lemma 4.2.1 as:

$$\log\left(Zon_d(e^{-\theta}\mathbf{1})\right) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\zeta(s+1)\Gamma(s)}{\zeta(s)}\sum_{v\in\mathbb{Z}_+^d\setminus\{\mathbf{0}\}}\frac{2^{d(v)-1}}{(\theta\mathbf{1}\cdot v)^s}ds. \tag{4.2.23}$$

The vector v can be clustered with the $\binom{d}{d(v)} - 1$ other vectors with the same non-zero ordered coordinates, as the scalar product $\mathbf{1}\cdot v$ is equal for all these vectors. We deduce that

$$\sum_{v \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}} \frac{2^{d(v)-1}}{(\mathbf{1}\cdot v)^s} = \sum_{\delta=1}^{d} 2^{\delta-1} \binom{d}{\delta} \sum_{v \in \mathbb{Z}_+^\delta \cap (0,+\infty)^\delta} \frac{1}{(\mathbf{1}\cdot v)^s}. \tag{4.2.24}$$

Then, we can turn the sum over $\mathbb{Z}_+^\delta \cap (0,+\infty)^\delta$ into a sum labeled by the scalar product value, denoted $n \in \mathbb{N}$:

$$\sum_{v \in \mathbb{Z}_+^\delta \cap (0,+\infty)^\delta} \frac{1}{(\mathbf{1}\cdot v)^s} = \sum_{n \geq 1} \frac{\binom{n-1}{\delta-1}}{n^s}. \tag{4.2.25}$$

Hence, with the operator $\mathbf{\Pi}_d$ defined in Definition 4.2.1, we can write, for $c > d$:

$$\log\left(Zon_d(e^{-\theta}\mathbf{1})\right) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\mathbf{\Pi}_d(\zeta(s))}{\zeta(s)\theta^s} \zeta(s+1)\Gamma(s) ds, \tag{4.2.26}$$

We observe that (4.2.26) is the expression of an inverse Mellin transform. In other words, the Mellin transform of the left-hand side of (4.2.26) is θ^s times the integrand in the right-hand side: for $s \in \mathbb{C}$ and $\Re(s) > d$,

$$\begin{aligned}\mathcal{M}\left[\log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\right](s) &= \int_0^{+\infty} \log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\theta^{s-1} d\theta \\ &= \frac{\mathbf{\Pi}_d(\zeta(s))}{\zeta(s)} \zeta(s+1)\Gamma(s).\end{aligned} \tag{4.2.27}$$

In order to give some concrete idea of this formula, here are the Mellin transform of the 2, 3, and 4-dimensional cases:

$$\begin{aligned}\mathcal{M}\left[\log\left(Zon_2\left(e^{-\theta}\mathbf{1}\right)\right)\right](s) &= \frac{2\zeta(s-1)}{\zeta(s)} \zeta(s+1)\Gamma(s), \\ \mathcal{M}\left[\log\left(Zon_3\left(e^{-\theta}\mathbf{1}\right)\right)\right](s) &= \frac{2\zeta(s-2)+\zeta(s)}{\zeta(s)} \zeta(s+1)\Gamma(s), \\ \mathcal{M}\left[\log\left(Zon_4\left(e^{-\theta}\mathbf{1}\right)\right)\right](s) &= \frac{\frac{4}{3}\zeta(s-3)+\frac{8}{3}\zeta(s-1)}{\zeta(s)} \zeta(s+1)\Gamma(s).\end{aligned}$$

All the function ζ, Γ, and $1/\zeta$ that compose $\mathcal{M}\left[\log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\right]$ can be continued into meromorphic ones on $\mathbb{C}$, and so do the Mellin transform. To obtain the successive orders of the right term of (4.2.22), we shift the vertical line to the left (in the complex plane) and use the residue theorems on the poles of the integrand of (4.2.26) to switch the line of integration from the right to the left of each pole of the Mellin transform. This method is widely known as utmost-left propagation of the integration contour in a transform inversion formula with the residue theorem around the poles (see [66, p. 765]). The poles of $\mathcal{M}\left[\log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\right]$ are located at each real number δ, δ being an integer between 0 and d, and at Riemann's ζ function's non-trivial zeros due to the denominator.

The integral of $\mathcal{M}\left[\log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\right]\theta^{-s}$ is well defined on the vertical lines with real part in $(i, i+1)$ for $i \in [\![1, d-1]\!]$ because of exponential decrease of $\Gamma(s)$ when $\Im(s)$ goes to $\pm\infty$. Starting from (4.2.26), we use the residue theorem around the pole at d to shift the line of integration of equation $\Re(s) = c$, with $c > d$, to the line of equation $\Re(s) = c_1$, with $c_1 \in (d-1, d)$. It follows that, for this $c_1 \in (d-1, d)$:

$$\log(Zon_d(e^{-\theta}\mathbf{1})) = \frac{p_{d,d-1}\zeta(d+1)\Gamma(d)}{\zeta(d)\theta^d} + \frac{1}{2i\pi}\int_{c_1-i\infty}^{c_1+i\infty}\frac{\mathbf{\Pi}_d(\zeta(s))}{\zeta(s)}\zeta(s+1)\Gamma(s)\theta^{-s}ds. \tag{4.2.28}$$

We can repeat the process until $1 < c_2 < 2$. The last step is analogous to the proof of Lemma 2.2 in [38]. Recall that γ is the contour defined in Proposition 4.2.1, parameterized by A. The existence of such A is proven in [148, Theorem 3.8] (we only need the existence of A here, as we don't need it when we discuss the value of $I_{\text{crit},d}$ in Section 4.3.3).

Hereafter, we prove that the integral of $\mathcal{M}\left[\log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\right]$ on the line of equation $\Re(s) = c_2$ is the sum of the residue of the integrand around $s = 1$, $I_{\text{crit},d}$, and the integral of $\mathcal{M}\left[\log\left(Zon_d\left(e^{-\theta}\mathbf{1}\right)\right)\right]$ on the line of equation $\Re(s) = -\frac{1}{2}$.

The latter integral is convergent because the term $\Gamma(s)$ exponentially decreases when $\Im(s) \to \pm\infty$ and $1/\zeta(s)$ is bounded on the line of equation $\Re(s) = -\frac{1}{2}$. With this bound, the integrand divided by $\theta^{\frac{1}{2}}$ is dominated, and we obtain that the integral on the domain of equation $\Re(s) = -\frac{1}{2}$ is of order $O\left(\theta^{\frac{1}{2}}\right)$ when $\theta \to 0$. $I_{\text{crit},d}$ is also convergent, because $1/\zeta(s) = O(\log(\Im(s))$ by formula (3.11.8) in [148].

We use the following result of Valiron [148, Theorem 9.7]:

Theorem 4.2.1 (Valiron). *There exists $\alpha > 0$ and a sequence (T_k) such that for all $k \in \mathbb{N}$, we have $k < T_k < k+1$ and $|\zeta(s)| > T_k^{-\alpha}$ for all s that satisfies $-1 \leq \Re(s) \leq 2$ and $|\Im(s)| = T_k$*

To get a visual illustration, see the yellow rectangle in Figure 4.3. Then, if we apply the residue theorem with the positively oriented rectangle of vertices $\frac{3}{2} \mp iT_k$ and $-\frac{1}{2} \pm iT_k$ on the inverse Mellin transform integral, and let k grow to $+\infty$, the contribution of the horizontal segments tends to 0. As a result, the integral on the line of equation $\Re(s) = c_2$ of the Mellin transform of $\log(Zon_d(e^{-\theta}\mathbf{1}))$ is equal to the sum (in the descending order regarding the real part) of the residue at 1, I_{crit}, the residue at 0, and the integral on the line made up of the complex numbers s such that $\Re(s) = -\frac{1}{2}$.

$$\begin{aligned}\log(Zon_d(e^{-\theta}\mathbf{1})) = \sum_{\delta=1}^{d-1} \left(p_{d,\delta} \frac{\zeta(\delta+2)\Gamma(\delta+1)}{\zeta(\delta+1)\theta^{\delta+1}} \right) + I_{\text{crit},d}(\theta) + C \\ + \frac{1}{2i\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\mathbf{\Pi}_d(\zeta)(s)}{\zeta(s)} \zeta(s+1)\Gamma(s)\theta^{-s} ds,\end{aligned} \tag{4.2.29}$$

with $C = 2\mathbf{\Pi}_d \left(\log(2\pi)\zeta - \zeta'\right)(0) + 2\mathbf{\Pi}_d\left(\zeta\right)(0)\log(\theta)$.

□

Remark 4.2.2 (Eulerian polynomials). *Come back to the series expansion of the logarithm of the generating function in equation* (4.2.2) *:*

$$\log\left(Zon_d(e^{-\boldsymbol{\theta}})\right) = \sum_{\boldsymbol{v}\in\mathbb{P}_{d+}} \sum_{m\geq 1} \frac{2^{d(\boldsymbol{v})-1}}{m} e^{-m\boldsymbol{\theta}\cdot\boldsymbol{v}}. \tag{4.2.30}$$

We can write this sum as a new sum over the number of 0*-coordinates of the vector* $\boldsymbol{v}$ *:*

$$\log\left(Zon_d(e^{-\boldsymbol{\theta}})\right) = \sum_{\delta=1}^{d} \sum_{m\geq 1} \frac{2^{\delta-1}}{m} \sum_{\substack{I\subset\{1,\ldots,d\} \\ |I|=\delta}} \sum_{\substack{\boldsymbol{v}\in\mathbb{P}_{d+} \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} e^{-m\boldsymbol{\theta}\cdot\boldsymbol{v}}. \tag{4.2.31}$$

When we investigate the partial derivative of the function $\log Zon_d$ *like in the Lemma 4.2.2, we look at the following function:*

$$\frac{\partial^{k_1+\ldots+k_d}}{\partial\theta_1^{k_1}\ldots\partial\theta_d^{k_d}} \log\left(Zon_d(e^{-\boldsymbol{\theta}})\right) = \sum_{\delta=1}^{d} \sum_{m\geq 1} \frac{2^{\delta-1}}{m} \sum_{\substack{I\subset\{1,\ldots,d\} \\ |I|=\delta}} \sum_{\substack{\boldsymbol{v}\in\mathbb{P}_{d+} \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} \left(\prod_{i=1}^{d} v_i^{k_i}\right) e^{-m\boldsymbol{\theta}\cdot\boldsymbol{v}}. \tag{4.2.32}$$

Therefore, the study of the partial derivative of $\log \mathit{Zon}_d$ *is depending on the study of the sum of the type of the last sum in the right-hand term of the last equation. When we carry the study for the mono-variable* $\boldsymbol{\theta} = \theta\mathbf{1}$*, for* $I \subset \{1, ..., d\}$*, we can expressed with the Mellin formula used in previous proof :*

$$\int_0^{+\infty} \sum_{\substack{\boldsymbol{v}\in\cap\mathbb{P}_{d+} \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} \left(\prod_{i=1}^{d} v_i^{k_i}\right) e^{-m\theta(\mathbf{1}\cdot\boldsymbol{v})}\theta^{s-1}ds = \sum_{\substack{\boldsymbol{v}\in\mathbb{P}_{d+} \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} \left(\prod_{i=1}^{d} v_i^{k_i}\right) \frac{\Gamma(s)}{(m\mathbf{1}\cdot\boldsymbol{v})^s}. \tag{4.2.33}$$

Then, we used the same partition of the set of positive integer coordinate vectors, $\mathbb{N}^d = \cup_{k\geq 1} k\mathbb{P}_{d+}$*, as in equation* (4.2.6)*:*

$$\zeta(s - \Sigma_{i=1}^{d} k_i) \sum_{\substack{\boldsymbol{v}\in\cap\mathbb{P}_{d+} \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} \left(\prod_{i=1}^{d} v_i^{k_i}\right) \frac{\Gamma(s)}{(m\mathbf{1}\cdot\boldsymbol{v})^s} = \sum_{\substack{\boldsymbol{v}\in\mathbb{N}^d \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} \left(\prod_{i=1}^{d} v_i^{k_i}\right) \frac{\Gamma(s)}{(m\mathbf{1}\cdot\boldsymbol{v})^s}. \tag{4.2.34}$$

Finally, see that the right-hand term is the Mellin transform of

$$\sum_{\substack{\boldsymbol{v}\in\mathbb{N}^d \\ \forall i\in I,\, v_i=0 \\ \forall i\notin I,\, v_i\neq 0}} \left(\prod_{i=1}^{d} v_i^{k_i}\right) e^{-(m\mathbf{1}\cdot\boldsymbol{v})\theta} = \prod_{i=1}^{d} \left(\sum_{v\geq 1} v_i^{k} e^{-m v_i \theta}\right) = \frac{\prod_{i=1}^{d} A_{k_i}(e^{-m\theta})}{(1 - e^{-m\theta})^{d+\Sigma_i k_i}}, \tag{4.2.35}$$

where A_k *is the* k^{th} *Eulerian polynomial. L. Euler introduced eulerian polynomials in* [105] *while studying the series expansion of* $(x, y) \mapsto \frac{y\exp(x)}{1-y\exp(x)}$*. We can define them recursively by*

$$\begin{cases} A_0(y) = y, \text{ and} \\ A_n(y) = \sum_{m=0}^{n} A(n, m) y^{n-m}. \end{cases} \tag{4.2.36}$$

This remark is made after Olivier Bodini noticed the appearance of the Eulerian polynomials in the study of the derivative of the function $\log \mathit{Zon}_d$ *in an early joint attempt to compute the results of this chapter.*

4.3 Asymptotic number of zonotopes in a hypercube

The proof of the first theorem relies on the saddle point method in d dimensions, which follows roughly the same framework as the 1-dimensional case display in

Section 4.1.2. Coming back to the formula (4.1.17), we extend the function Zon_d on the open unit disk of $\mathbb{C}$ centered at 0 for each variable x_i. Then for any integer i, $1 \leq i \leq d$, given a positively oriented circle $\mathcal{C}(r)$ centered at 0, of radius $r < 1$, we have:

$$[x_i^n]Zon_d(\boldsymbol{x}) = \frac{1}{2i\pi}\int_{\mathcal{C}(r)} Zon_d(\boldsymbol{x})x_i^{-n-1}dx_i. \tag{4.3.1}$$

The whole point of the saddle point method is to find r (Lemma 4.3.4) such that this Cauchy's integral is asymptotically equivalent to a Gaussian integral that can be computed (Proposition 4.3.1). To draw a parallel between the probabilistic approach of [38] and the analytic combinatorics approach of [25], the Gaussian approximation written in Proposition 4.3.1 is actually a local limit theorem with rate 1 (see Theorem 5.2.1), in the probabilistic approach of [38] and [26]. This will be discussed in Remark 4.3.1.

4.3.1 H-admissibility of a multivariate saddle point integral method

The generating function is studied on the cartesian product of d open unit disks contained in the complex plane, therefore for i between 1 and d, x_i is a complex number with absolute value less than 1, and we denote $x_i = e^{-\theta_i + i\alpha_i}$, with $\theta_i > 0$ and $\alpha_i \in]-\pi, \pi]$. The vector notation is then $\boldsymbol{x} = e^{-\boldsymbol{\theta}}e^{i\boldsymbol{\alpha}}$. Zon_d is defined on this domain, and we rewrite the integrand of (4.3.1) as:

$$Zon_d(\boldsymbol{x})x_i^{-n-1} = \frac{1}{e^{-n\theta_i}}\exp\left(\log(Zon_d(e^{-\boldsymbol{\theta}}e^{i\boldsymbol{\alpha}})) - (n+1)i\alpha_i\right). \tag{4.3.2}$$

In the following the notation $\boldsymbol{x} \to \mathbf{l}$ means that each component x_i tends to l_i and there exists a constant $\epsilon > 1$, such that $\frac{x_i}{x_j} \in (1/\epsilon, \epsilon)$ for all $i \neq j$.

To ease the calculation, we introduce notations for the partial derivative of Zon_d. The existence of such derivatives was proved in the proof of Lemma 4.2.2. We denote in the rest of the section $\boldsymbol{a}(\boldsymbol{x}) = (a_i(\boldsymbol{x}))_{1\leq i\leq d}$, $\mathbf{B}(\boldsymbol{x}) = \left(B_{i,j}(\boldsymbol{x})\right)_{1\leq i,j\leq d}$, and $\mathbf{C}(\boldsymbol{x}) = \left(C_{i,j,k}(\boldsymbol{x})\right)_{1\leq i,j,k\leq d}$ respectively for the sets of the first, second, and third order partial derivative of the logarithm of Zon_d at $\boldsymbol{x}$. E.g. for the first order, for $1 \leq j \leq d$, we have

$$a_j(\boldsymbol{x}) = \frac{x_j \frac{\partial}{\partial x_j} Zon_d(\boldsymbol{x})}{Zon_d(\boldsymbol{x})}. \tag{4.3.3}$$

The goal of this subsection is to prove the following proposition that gives the asymptotic equivalent of the coefficient of the generating function. In analytic combinatorics terms, that proposition is the *H-admissibility* (see the introduction of [73]), but we will make no use of that terminology here.

Proposition 4.3.1. *Let $\boldsymbol{r} = e^{-\boldsymbol{\theta}}$ be a vector in $[0,1[^d$. Then, as $e^{-\boldsymbol{\theta}} \to \mathbf{1}$, we have:*

$$[\boldsymbol{r}^{\boldsymbol{n}}]Zon_d(\boldsymbol{r}) = \frac{Zon_d(e^{-\boldsymbol{\theta}})}{e^{-\boldsymbol{n}\cdot\boldsymbol{\theta}}\sqrt{(2\pi)^d \det \mathbf{B}(e^{-\boldsymbol{\theta}})}} \left(\exp\left(-\frac{1}{2}(\boldsymbol{a}(e^{-\boldsymbol{\theta}}) - \boldsymbol{n})\mathbf{B}(e^{-\boldsymbol{\theta}})^{-1}(\boldsymbol{a}(e^{-\boldsymbol{\theta}}) - \boldsymbol{n}) \right) + o(1) \right), \tag{4.3.4}$$

uniformly on $\mathbb{N}^d$.

Remark 4.3.1 (Saddle-point method and local limit theorem)**.** *We will see in Chapter 5, that we can naturally associate to the generating function Zon_d a probability distribution. Roughly, for any $Z \in \mathcal{Z}_d$, we associate the finite support function $\omega_Z \in \Omega$, then we define the Boltzmann probability distribution $\mathbb{P}_n$ over $\mathcal{Z}_d$ by*

$$\mathbb{P}_n(Z) = \frac{1}{Zon_d(e^{-\boldsymbol{\theta}})} \prod_{v \in \mathcal{P}_d{}^\star} e^{-\boldsymbol{\theta}\cdot(\sum_{v \in \mathcal{P}_d{}^\star} \omega_Z(\boldsymbol{v})\boldsymbol{v})}. \tag{4.3.5}$$

We can express the coefficient extractor in terms of probability, namely

$$\mathbb{P}_n\left[\sum_{\boldsymbol{v} \in \mathcal{P}_d{}^\star} \omega_Z(\boldsymbol{v})\boldsymbol{v} = \boldsymbol{n}\right] = \frac{[\boldsymbol{r}^{\boldsymbol{n}}]Zon_d(\boldsymbol{r})}{Zon_d(e^{-\boldsymbol{\theta}})}. \tag{4.3.6}$$

In terms of the local limit theorem, Proposition 4.3.1 is exactly a local limit theorem of rate 1*, as defined in [37] and stated in Section 5.2. Therefore, for any partition function, the H-admissibility in the sense of B. Gittenberger and J. Mandlburger in [73] implies in a local limit theorem of rate 1 for the corresponding Boltzmann probability distribution.*

More generally, the framework for the local limit theorem of J. Bureaux in [37] is similar in many ways to the H-admissibility of B. Gittenberger and J. Mandlburger in [73] on many points. Lemma 4.3.1 implies the Lyapunov condition (Lemma 5.2.1 in Chapter 5), and Lemma 4.3.2 is really close to the characteristic function upper-bound condition (Lemma 5.2.3). Those remarks, and the similarity of their respective proof of their framework, shed light on the high proximity between probabilistic limit theorems and analytic combinatorics asymptotic theorems. We are glad that this thesis can be an illustration of this high proximity.

This proposition relies on a few technical lemmas about the asymptotic behavior of the generating partition function, following the 5 conditions of Definition 2 in [73] labeled by (**I**) to (**V**) therein and in the lemmas below.

Lemma 4.3.1. *(**I**) Let β be a real number in the interval $(1+\frac{d}{3}, 1+\frac{d}{2})$, and define the cuboid $\Delta(e^{-\boldsymbol{\theta}})$ in $]-\pi,\pi]^d$ as*

$$\Delta(e^{-\boldsymbol{\theta}}) = \left\{ \boldsymbol{\alpha} \in \mathbb{R}^d,\ \text{for } 1 \le i \le d,\ |\alpha_i| < \left(\max_{1\le i\le d}(\theta_i) \right)^{\beta} \right\}. \tag{4.3.7}$$

$\mathbf{B}(e^{-\boldsymbol{\theta}})$ is positive definite, and, for all $\epsilon > 1$, for all $\boldsymbol{\theta}$ such that $\frac{1}{\epsilon} < \frac{\theta_i}{\theta_j} < \epsilon$, we have, uniformly for $\boldsymbol{\alpha} \in \Delta(e^{-\boldsymbol{\theta}})$,

$$Zon_d(e^{-\boldsymbol{\theta}} e^{i\boldsymbol{\alpha}}) = Zon_d(e^{-\boldsymbol{\theta}}) \exp\left(i\boldsymbol{\alpha}^\intercal \boldsymbol{a}(e^{-\boldsymbol{\theta}}) - \frac{\boldsymbol{\alpha}^\intercal \mathbf{B}(e^{-\boldsymbol{\theta}})\boldsymbol{\alpha}}{2} \right) (1 + o(1)), \text{ as } e^{-\boldsymbol{\theta}} \to \mathbf{1}. \tag{4.3.8}$$

Proof. Let $\boldsymbol{\theta} \in (0, +\infty)^d$, we compute the equivalent of the values of $\boldsymbol{a}(e^{-\boldsymbol{\theta}})$, $\mathbf{B}(e^{-\boldsymbol{\theta}})$, and $\mathbf{C}(e^{-\boldsymbol{\theta}})$, with Lemma 4.2.2, using Kronecker's δ_{ij} notation (and for three parameters, we write $\delta_{i,j,k} = 1$ if $i = j = k$ and 0 otherwise):

$$\boldsymbol{a}_i(e^{-\boldsymbol{\theta}}) \underset{\boldsymbol{\theta}\to\mathbf{0}}{\sim} \frac{2^{d-1}\zeta(d+1)}{\zeta(d)} \frac{1}{\theta_i \prod_{j=1}^d \theta_j}, \tag{4.3.9}$$

$$\mathbf{B}_{i,j}(e^{-\boldsymbol{\theta}}) \underset{\boldsymbol{\theta}\to\mathbf{0}}{\sim} (1 + \delta_{i,j}) \frac{2^{d-1}\zeta(d+1)}{\zeta(d)} \frac{1}{\theta_i\theta_j \prod_{k=1}^d \theta_k}, \tag{4.3.10}$$

$$\mathbf{C}_{i,j,k}(e^{-\boldsymbol{\theta}}) \underset{\boldsymbol{\theta}\to\mathbf{0}}{\sim} (1 + \delta_{i,j} + \delta_{j,k} + \delta_{k,i} + 2\delta_{i,j,k}) \frac{2^{d-1}\zeta(d+1)}{\zeta(d)} \frac{1}{\theta_i\theta_j\theta_k \prod_{l=1}^d \theta_l}. \tag{4.3.11}$$

The asymptotic equivalents (4.3.9) and (4.3.10) will be useful later in the following lemma and to determine the solution of the saddle equation (Lemma 4.3.4).
$\mathbf{B}(e^{-\boldsymbol{\theta}})$ is a symmetric matrix, and the matrix of the asymptotic equivalents (4.3.10) is positive definite. Therefore for $\boldsymbol{\theta}$ small enough, $\mathbf{B}(e^{-\boldsymbol{\theta}})$ is positive definite. To conclude with the Lagrange form of Taylor's expansion theorem, we ensure that for any $t \in (0, 1)$, we have

$$\mathbf{C}_{i,j,k}(e^{-\boldsymbol{\theta}+it\boldsymbol{\alpha}})\alpha_i\alpha_j\alpha_k \underset{\boldsymbol{\theta}\to\mathbf{0}}{\to} 0, \text{ uniformly for } \boldsymbol{\alpha} \in \Delta(\boldsymbol{\theta}). \tag{4.3.12}$$

This is the case because $\beta \in (1 + \frac{d}{3}, 1 + \frac{d}{2})$. □

Lemma 4.3.2. *(II)* $|Zon_d(e^{-\boldsymbol{\theta}+i\boldsymbol{\alpha}})| = o\left(\frac{Zon_d(e^{-\boldsymbol{\theta}})}{\sqrt{\det \boldsymbol{B}(e^{-\boldsymbol{\theta}})}}\right)$ *as $\boldsymbol{\theta} \to \mathbf{0}$, holds uniformly for $\boldsymbol{\alpha} \notin \Delta(\boldsymbol{\theta})$*

Proof. Let $\boldsymbol{\theta} \in (0, +\infty)^d$, and take $\boldsymbol{\alpha} \notin \Delta(\boldsymbol{\theta})$, and start from the following equality:

$$\log\left(\frac{|Zon_d(e^{-\boldsymbol{\theta}+i\boldsymbol{\alpha}})|}{Zon_d(e^{-\boldsymbol{\theta}})}\right) = -\sum_{v\in\mathbb{P}_{d+}} 2^{d(v)-1}\log\left(\frac{|1-e^{-\boldsymbol{\theta}\cdot v+i\boldsymbol{\alpha}\cdot v}|}{1-e^{-\boldsymbol{\theta}\cdot v}}\right). \tag{4.3.13}$$

Using $\frac{|1-xe^{iy}|}{1-x} = \sqrt{1+\frac{4xsin^2(y/2)}{(1-x)^2}}$ for $x\in]0,1[$, we have

$$\log\left(\frac{|Zon_d(e^{-\boldsymbol{\theta}+i\boldsymbol{\alpha}})|}{Zon_d(e^{-\boldsymbol{\theta}})}\right) = -\sum_{v\in\mathbb{P}_{d+}} 2^{d(v)-2}\log\left(1+\frac{4e^{-\boldsymbol{\theta}\cdot v}\sin^2\left(\frac{\boldsymbol{\alpha}\cdot v}{2}\right)}{(1-e^{-\boldsymbol{\theta}\cdot v})^2}\right). \tag{4.3.14}$$

We can upper bound the quotient within the logarithm in the right-hand side, whose denominator is smaller than 1, and using the fact that for $0\le x\le 4$, we have $\frac{\log(5)}{4}x \le \log(1+x)$. We obtain:

$$\log\left(\frac{|Zon_d(e^{-\boldsymbol{\theta}+i\boldsymbol{\alpha}})|}{Zon_d(e^{-\boldsymbol{\theta}})}\right) \le -\log(5)\sum_{v\in\mathbb{P}_{d+}} 2^{d(v)-2}\left(e^{-\boldsymbol{\theta}\cdot v}\sin^2\left(\frac{\boldsymbol{\alpha}\cdot v}{2}\right)\right). \tag{4.3.15}$$

We denote $U_{\boldsymbol{\theta},\boldsymbol{\alpha}}$ as

$$U_{\boldsymbol{\theta},\boldsymbol{\alpha}} = \sum_{v\in\mathbb{P}_{d+}} 2^{d(v)-2}\left(e^{-\boldsymbol{\theta}\cdot v}\sin^2\left(\frac{\boldsymbol{\alpha}\cdot v}{2}\right)\right) \tag{4.3.16}$$

To obtain the little-o of the lemma, it is sufficient to lower bound $U_{\boldsymbol{\theta},\boldsymbol{\alpha}}$ with a polynomial bound. Since $\boldsymbol{\alpha}\notin\Delta(\boldsymbol{\theta})$, there is an integer $1\le k\le d$ such that $\alpha_k \ge \max_{1\le i\le d}(\theta_i)$. Without loss of generality, we can suppose that $k=1$. We then consider the family of primitive vectors $(\mathbf{v}_i = (i,1,\mathbf{0}))_{i\ge 1}$:

$$\sin^2(\boldsymbol{\alpha}\cdot v_i) = \sin^2\left(\frac{i\alpha_1+\alpha_2}{2}\right) \tag{4.3.17}$$

We focus on the function $x\mapsto sin^2(\frac{x\alpha_1+\alpha_2}{2})$ which is $\frac{\pi}{\alpha_1}$-periodic. By construction, $\alpha_1\le\pi$, so we have $\frac{\pi}{\alpha_1}\ge 1$. This final inequality guarantees that the k-th element of the sequence (v_i) satisfying $sin^2(\frac{i\alpha_1+\alpha_2}{2})\ge\frac{1}{4}$ is lower that $2k$. Thus

$$U_{\boldsymbol{\theta},\boldsymbol{\alpha}} \ge \sum_{i=1}^{\infty} e^{-\boldsymbol{\theta}\cdot v_i}\sin^2\left(\frac{\boldsymbol{\alpha}\cdot v}{2}\right) \ge \frac{1}{4}\sum_{i=1}^{\infty} e^{-2i\max\limits_{1\le j\le d}(\theta_j)-1}. \tag{4.3.18}$$

Finally, elementary operations in the right term, give

$$U_{\boldsymbol{\theta},\boldsymbol{\alpha}} \geq \frac{e^{-\max\limits_{1\leq j\leq d}(\theta_j)-1}}{4(1-e^{-2\max\limits_{1\leq j\leq d}(\theta_j)})} \sim \frac{e^{-1}}{8\max\limits_{1\leq j\leq d}(\theta_j)} \tag{4.3.19}$$

which gives, for a real number $c > 0$:

$$|Zon_d(e^{-\boldsymbol{\theta}+i\boldsymbol{\alpha}})| = O\left(Zon_d(e^{-\boldsymbol{\theta}})e^{-c\max\limits_{1\leq j\leq d}(\theta_j)^{-1}}\right) \tag{4.3.20}$$

The asymptotic equivalent of the determinant of $\mathbf{B}(e^{-\boldsymbol{\theta}})$ derives directly from the expression (4.3.10), and is

$$\det \mathbf{B}(e^{-\boldsymbol{\theta}}) \underset{\boldsymbol{\theta}\to\mathbf{0}}{\sim} \frac{2^{d-1}(d+1)\zeta(d+1)}{\zeta(d)} \frac{1}{\prod_{k=1}^{d}\theta_k^3}. \tag{4.3.21}$$

The big O of the inverse of the square root of the determinant follows:

$$\frac{1}{\sqrt{\det \mathbf{B}(e^{-\boldsymbol{\theta}})}} = O\left(\max_{1\leq j\leq d}(\theta_j)^{\frac{3d}{2}}\right), \tag{4.3.22}$$

and we get the uniform convergence of the lemma. □

Lemma 4.3.3. *The following properties hold:*

- *(**III**) The eigenvalues $\lambda_1(\boldsymbol{\theta}), ..., \lambda_d(\boldsymbol{\theta})$ of $\mathbf{B}(e^{-\boldsymbol{\theta}})$ all tend to $+\infty$ as $\boldsymbol{\theta} \to \mathbf{0}$.*
- *(**IV**) $\mathbf{B}_{ii}(e^{-\boldsymbol{\theta}}) = o(a_i(e^{-\boldsymbol{\theta}})^2)$ as $\boldsymbol{\theta} \to \mathbf{0}$.*
- *(**V**) For $\boldsymbol{\theta} \to \mathbf{0}$, and $\boldsymbol{\alpha} \in [-\pi,\pi]^d \setminus \{\mathbf{0}\}$, we have $|Zon_d(e^{-\boldsymbol{\theta}+i\boldsymbol{\alpha}})| < Zon_d(e^{-\boldsymbol{\theta}})$.*

Proof. Property (**III**) follows from the equivalence (4.3.10), Property (**IV**) directly comes from equivalences (4.3.9) and (4.3.10), and Property (**V**) is a direct consequence of the inequality (4.3.15) that stands true for $\boldsymbol{\alpha} \in [-\pi,\pi]^d \setminus \{\mathbf{0}\}$. □

proof of Proposition 4.3.1. With Lemmas 4.3.1, 4.3.2, 4.3.3, the d-dimensional function Zon_d satisfies all 5 conditions of Definition 2 in [73], therefore it satisfies Theorem 4 in [73] which gives Proposition 4.3.1.

□

4.3.2 Saddle-point equation and enumeration

Proposition 4.3.1 gives an asymptotic equivalent for the coefficient of Zon_d. Let's consider $\boldsymbol{n} \in (\mathbb{N}^*)^d$ the vector of the dimensions of the box. In order to compute $z_d(\boldsymbol{n})$ the number of lattice zonotopes in this box, we determine $\boldsymbol{\theta}$ as the solution of (4.3.24) which is often called the saddle point equation. This solution is the $\boldsymbol{\theta}$ that cancels the exponential term in Proposition 4.3.1:

Lemma 4.3.4. *For all $\epsilon > 0$, the vector $\widetilde{\boldsymbol{\theta}}_{\boldsymbol{n}}$ defined by*

$$\widetilde{\theta}_{\boldsymbol{n},i} = \left(\frac{2^{d-1}\zeta(d+1)}{\zeta(d)}\right)^{\frac{1}{d+1}} \frac{\left(\prod_{j=1}^{d} n_i\right)^{\frac{1}{d+1}}}{n_i} \tag{4.3.23}$$

satisfies, as $\boldsymbol{n}$ goes to ∞ such that for all $1 \leq i, j \leq d$ we have $1/\epsilon < \frac{n_i}{n_j} < \epsilon$:

$$\boldsymbol{a}(e^{-\widetilde{\boldsymbol{\theta}}_{\boldsymbol{n}}}) = \boldsymbol{n}(1 + o(1)) \tag{4.3.24}$$

Proof. Let $\boldsymbol{\theta} \in (0, +\infty)^d$, and consider the asymptotic equivalence coming from (4.3.9):

$$\boldsymbol{a}_i(e^{-\boldsymbol{\theta}}) - n_i \underset{\boldsymbol{\theta}\to\boldsymbol{0}}{=} \frac{2^{d-1}\zeta(d+1)}{\zeta(d)} \frac{1}{\theta_i \prod_{j=1}^{d} \theta_j} - n_i + o\left(\frac{1}{\theta_i \prod_{j=1}^{d} \theta_j}\right), \text{ for } 1 \leq i \leq n. \tag{4.3.25}$$

We set each $\boldsymbol{a}_i(e^{-\boldsymbol{\theta}}) - n_i$ to 0, and we obtain the result by computing the product of the (θ_j):

$$\prod_{j=1}^{d} \theta_j = \sqrt[d+1]{\left(\frac{2^{d-1}\zeta(d+1)}{\zeta(d)}\right)^{d} \frac{1}{\prod_{j=1}^{d} n_j}}. \tag{4.3.26}$$

Then we obtain the wanted expression by replacing the product $\prod_{j=1}^{d} \theta_j$ in each equation $\boldsymbol{a}_i(e^{-\boldsymbol{\theta}}) - n_i = 0$. □

Proposition 4.3.1 and Lemma 4.3.4 imply that the number of lattice zonotopes inscribed in a box of dimensions $n\boldsymbol{k}$ with $\boldsymbol{k} \in (\mathbb{N}^*)^d$ is

$$z_d(n\boldsymbol{k}) \underset{n\to+\infty}{\sim} \frac{Zon_d\left(e^{-\widetilde{\theta}_{nk}}\right)}{e^{-n\boldsymbol{k}\cdot\widetilde{\theta}_{nk}}\sqrt{(2\pi)^d \det \mathbf{B}\left(e^{-\widetilde{\theta}_{nk}}\right)}} \tag{4.3.27}$$

Ultimately, as Sinai did for the two-dimensional case [139], this final asymptotic equivalence can lead to the estimate of the number of lattice zonotopes in any box $(c_1 n, ..., c_d n)$ (with positive constants (c_i)). Yet, a more in-depth work (analogous of what has be conducted in Section 4.2.2) is needed to obtain an equivalent of $Zon_d\left(e^{-\widetilde{\theta}_{nk}}\right)$, so we limit our scope to the box $[0, +\infty]^d$, which leads to the parameters $\theta_1 = ... = \theta_d = \left(\frac{2^{d-1}\zeta(d+1)}{\zeta(d)n}\right)^{\frac{1}{d+1}}$. Theorem 4.3.1 follows, and we state it hereafter with detailed notations.

Theorem 4.3.1. *Let $z_d(n\mathbf{1})$ be the number of lattice zonotopes inscribed in $[0,n]^d$. We denote $\kappa_d = \frac{2^{d-1}\zeta(d+1)}{\zeta(d)}$, and $\mathbf{\Pi}_d$ and $(p_{d,\delta})_{1\le\delta<d}$ respectively the operator and the coefficients defined in Definition 4.2.1. With γ the contour defined in Proposition 4.2.1, we define the polynomial Q_d and the function $I_{crit,d}$ respectively by:*

$$Q_d(X) = (d+1)\kappa_d^{\frac{1}{d+1}} X^d + \sum_{\delta=2}^{d-1} p_{d,\delta-1} \frac{\zeta(\delta+1)(\delta-1)!}{\zeta(\delta)} \kappa_d^{-\frac{\delta}{d+1}} X^\delta, \tag{4.3.28}$$

and

$$I_{crit,d}(\theta) = \frac{1}{2i\pi}\int_\gamma \frac{\mathbf{\Pi}_d(\zeta(s))}{\zeta(s)} \zeta(s+1)\theta^{-s}\Gamma(s)ds. \tag{4.3.29}$$

As n grows to $+\infty$, we have

$$z_d(n\mathbf{1}) \sim \alpha_d n^{\beta_d} \exp\left(Q_d(n^{\frac{1}{d+1}}) + I_{crit,d}\left(\left(\frac{\kappa_d}{n}\right)^{\frac{1}{d+1}}\right)\right) \tag{4.3.30}$$

with $\alpha_d = \frac{\kappa_d^{\frac{d}{2(d+1)}+\frac{2}{d+1}\mathbf{\Pi}_d[\zeta](0)} \exp(2\mathbf{\Pi}_d[\log(2\pi)\zeta - \zeta'](0))}{(2\pi)^{d/2}\sqrt{d+1}}$, *and* $\beta_d = \frac{-1}{2(d+1)}\left(d(d+2) + 4\mathbf{\Pi}_d\left[\zeta\right](0)\right)$.

Moreover, under the hypothesis that all zeros of the Riemann ζ function in the critic stripe are simple poles, $I_{crit,d}$ can be rewritten to a sum over the set Z of non-trivial zeros (named hypothesis H1), that is:

$$I_{crit,d}\left(\left(\frac{\kappa_d}{n}\right)^{\frac{1}{d+1}}\right) = \sum_{r\in Z} Res\left(\frac{\mathbf{\Pi}_d[\zeta](r)}{\zeta(r)}\right)\left(\frac{n}{\kappa_d}\right)^{\frac{r}{d+1}}\zeta(r+1)\Gamma(r). \tag{4.3.31}$$

Proof. We rewrite (4.3.27) using the equivalent of the asymptotic equivalent of the univariate generating function of Proposition 4.2.1. □

To illustrate this theorem, we give the following table:

	α_d	β_d	$Q_d(X)$
2D	$\frac{2^{1/9}3^{13/18}\zeta(3)^{2/9}e^{-4\zeta'(-1)}}{6\pi^{16/9}}$	$-\frac{11}{9}$	$\frac{2^{2/3}3^{4/3}\zeta(3)^{1/3}}{\pi^{2/3}}X^2$
3D	$\frac{2^{5/8}3^{3/4}5^{8/9}e^{-4\zeta'(-2)}}{120\zeta(3)^{1/8}\pi}$	$-\frac{13}{8}$	$\frac{2^{9/4}\pi}{3^{1/2}5^{1/3}\zeta(3)^{1/4}}X^3$
4D	$\left(\frac{2^{189}3^{142}\zeta(5)^{71}}{5^{83}\pi^{379}}\right)^{\frac{1}{225}} e^{-\frac{8}{3}\zeta'(-3)-\frac{16}{3}\zeta'(-1)}$	$-\frac{521}{225}$	$\frac{(5720)^{1/5}\zeta(5)^{1/5}}{\pi^{4/5}}X^4 + \frac{(16720)^{2/5}\zeta(5)^{2/5}\zeta(3)}{\pi^{18/5}}X^2$

Table 4.2: Parameters of the asymptotic equivalent of the number of lattice zonotope in a hypercube of dimension 2, 3, and 4

Remark 4.3.2. *Unexpectedly, the generalization of the exponential part is not a monomial as is in 2 and 3 dimensions (see below) but a polynomial of the $d+1$-th root of the box size.*

Remark 4.3.3. *As was already observed in dimension 2, the exact estimate contains a term depending on the set of non-trivial zeros of the ζ function. In Section 4.3.3, we discuss the value of this term and extend the results of [25, 38] that show that $I_{crit,2}$ is negligible in dimension 2 for any "computably" large n (typically $< 10^{20}$).*

We make two numerical remarks about Theorem 4.3.1 in order to ease the understanding of the behavior of $z_d(n\mathbf{1})$ when d and n grow large.

4.3.3 Numerical discussion

Discussion on I_{crit}

Under hypothesis H1, $I_{\text{crit},d}$ is the sum (4.3.31). In fact, even without any assumption on the poles, it can still be expressed as a sum, but the terms for each zero would be more complex. In the critical strip, the two first zeros of the ζ function are at $r_1 = \frac{1}{2} + i14.1347...$ and at $r_2 = \frac{1}{2} + i21.0220....$ Due to the exponential decrease of the function Γ when deviating from the real line, the term $\mathbf{\Pi}_d[\zeta](r_1)\left(\frac{n}{\kappa_d}\right)^{\frac{r_1}{d+1}}\zeta(r_1+1)\Gamma(r_1)$ is about 10^4 greater than the term involving r_2.

Eventually, the bounds on the density of poles r given by Selberg [133] leads to consider that all the weight of all zeros but r_1 is negligible in $I_{\mathrm{crit},d}$. We compute the approximation for the 2, 3, and 4-dimensional cases:

$$\begin{aligned} I_{\mathrm{crit},2}\left(\left(\frac{\kappa_2}{n}\right)^{\frac{1}{3}}\right) &\approx -1.3579\times 10^{-10} n^{1/6}\cos\left(4.7116\ln(0.6842n)\right) \\ &\qquad - 1.4236\times 10^{-9} n^{1/6}\sin\left(4.7116\ln(0.6842n)\right), \\ I_{\mathrm{crit},3}\left(\left(\frac{\kappa_3}{n}\right)^{\frac{1}{4}}\right) &\approx -1.2325\times 10^{-10} n^{1/8}\cos\left(3.5337\ln(0.2777n)\right) \\ &\qquad - 1.2921\times 10^{-9} n^{1/8}\sin\left(3.5337\ln(0.2777n)\right), \\ I_{\mathrm{crit},4}\left(\left(\frac{\kappa_4}{n}\right)^{\frac{1}{5}}\right) &\approx -3.1764\times 10^{-9} n^{1/10}\cos\left(2.8269\ln(0.1305n)\right) \\ &\qquad - 0.0628\times 10^{-9} n^{1/10}\sin(2.8269\ln(0.1305n)). \end{aligned} \tag{4.3.32}$$

To provide an order of magnitude, $I_{\mathrm{crit},d}\left(\left(\frac{\kappa_d}{n}\right)^{\frac{1}{d+1}}\right)$ is smaller that 10^{-6} when $n < 10^{20}$ (for the first dimensions tried).

About moving up in dimension

For a more down-to-earth analysis of the logarithmic equivalent, i.e., the leading term of the exponential term, we can use the expansion of $\zeta(d) = 1 + 2^d + o(2^d)$ as d grows large. One can see that:

$$\left(\frac{2^{d-1}\zeta(d+1)}{\zeta(d)}\right)^{\frac{1}{d+1}} = 2 + O\left(\frac{1}{d}\right), \qquad \text{as } d \to +\infty. \tag{4.3.33}$$

Therefore, when we get that, in higher dimensions, the logarithm of $z_d(n\mathbf{1})$ is nearly equivalent to $2(d+1)n^{\frac{d}{d+1}}$.

We wish to draw attention to the fact that the approach in [38] and in [14] focuses on polygonal lines (respectively zonotopes) beginning at $\mathbf{0}$ and ending at a given point whereas in this chapter, we enumerate the number of lattice zonotopes in a hypercube. Given a hypercube $[-1,1]^d$, we can split it into 2^d hypercubes centered at $(\pm\frac{1}{2}, ..., \pm\frac{1}{2})$, and view a lattice zonotope in the hypercube as the sum of lattice zonotopes with generators in each of the square with positive first coordinate. This short explanation, also described in [14, Theorem 6.2], explains the additional 2^{d-1} in the leading term of the exponential part.

4.3.4 Number of generators and multiplicity of primitive generators

Once the combinatorial description of zonotopes and Theorem 4.3.1 have been established, one can compute moments of combinatorial parameters of zonotopes. In this section, we are interested in two parameters: the diameter of the graph of a lattice zonotope and the number of occurrences of a generator in a random lattice zonotope.

The diameter (in the sense of the diameter of the graph) of a polytope of a given size is a key combinatorial parameter on which few things are known. Along with its own scientific interest (the famously now disproved Hirsch conjecture), it is connected to the complexity of the simplex algorithm. This makes upper bounds of this quantity actively looked for (see [50] for more references). In this perspective, zonotopes have been conjectured to be a class that reaches the largest possible diameter among all the lattice polytopes contained in $[0,n]^d$ ([52, Conjecture 3.3]). The largest possible diameter of a lattice zonotope contained $[0,n]^d$ is given in [55], while its exact asymptotic behavior is estimated in [54] when d is fixed and $n \to +\infty$.

We hereby compute the asymptotic estimates of the mean of the distribution of the diameter of lattice zonotopes inscribed in $[0,n]^d$.

Theorem 4.3.2. *Let μ^n_{diam} be the mean of the distribution of the diameter of a lattice zonotope inscribed in $[0,n]^d$. Then, as n grows large, we have:*

$$\mu^n_{diam} \underset{n\to+\infty}{=} \frac{\sqrt[d+1]{\kappa_d}}{\zeta(d+1)} \; n^{\frac{d}{d+1}}(1+o(1)). \tag{4.3.34}$$

In this section, we establish the asymptotic behavior of the first moments of parameters which can be computed with our approach. To do that, we add a variable u (that we will name a parameter variable in the sequel) that acts as a counting variable for the parameter. Then the partial derivative along this variable gives us the average value of the quantity under study.

We begin by giving a result similar to Lemma 4.3 of [14] about the number of generators. That lemma gives the average number of generators of a lattice zonotope contained in a given cone and ending at a given point. This average can also be computed when the lattice zonotope is contained in a hypercube. As said before, the number of generators of a zonotope is the diameter of its graph, which gives Theorem 4.3.2.

Proof of Theorem 4.3.2. Recall that $[v]$ denotes the vector of the absolute values of the coordinate of $\boldsymbol{v}$. The structure of the generating function (4.1.17) is well known as it is a partition function, and each term $\left(1-e^{-\boldsymbol{\theta}\cdot[v]}\right)^{-1}$ yields the contribution of the generator $\boldsymbol{v}$. When we expand it in series, the k-th term represents the possibility of having k times the generator v:

$$\left(1-e^{-\boldsymbol{\theta}\cdot[v]}\right)^{-1}=\sum_{k=0}^{+\infty}e^{-k\boldsymbol{\theta}\cdot[v]} \tag{4.3.35}$$

Therefore, we make the following modification of each term to use the variable u that encodes the number of generators in a lattice zonotope:

$$1+\sum_{k=1}^{+\infty}ue^{-k\boldsymbol{\theta}\cdot[v]}. \tag{4.3.36}$$

We recall the notation $e^{-\boldsymbol{\theta}}=\left(e^{-\theta_1},...,e^{-\theta_d}\right)$. We call $Zon_{d,\text{gen}}$ the modified generating function defined as

$$Zon_{d,\text{gen}}\left(e^{-\boldsymbol{\theta}},u\right)=\prod_{v\in\mathbb{P}_{d+}}\left(1+\sum_{k=1}^{+\infty}ue^{-k\boldsymbol{\theta}\cdot\boldsymbol{v}}\right)^{2^{d(v)-1}}. \tag{4.3.37}$$

The main idea of the proof lies in the following definition of the average number of generators for lattice zonotope inscribed in $[0,n]^d$, μ^n_{gen} from the generating function (see for instance [66, Chapter 3]):

$$\mu^n_{gen}=\frac{[\boldsymbol{x}^{\boldsymbol{n}}]\ \frac{\partial}{\partial u}Zon_{d,\text{gen}}(\boldsymbol{x},u)\Big|_{u=1}}{[\boldsymbol{x}^{\boldsymbol{n}}]Zon_d(\boldsymbol{x})}, \tag{4.3.38}$$

where $[x^n]F(x)$ (resp. $[\boldsymbol{x}^{\boldsymbol{n}}]F(\boldsymbol{x})$) denotes the coefficient of x^n (resp. $\boldsymbol{x}^{\boldsymbol{n}}$) in the series expansion of $F(x)$ (resp. $F(\boldsymbol{x})$). All that remains is to compute the equivalent of Proposition 4.3.1 and Lemma 4.3.4 for $\frac{\partial}{\partial u}Zon_{d,\text{gen}}\Big|_{u=1}$. Therefore we compute the partial derivative along u

$$\frac{\partial}{\partial u}Zon_{d,\text{gen}}\left(e^{-\boldsymbol{\theta}},u\right)\Big|_{u=1}=\left(\sum_{v\in\mathbb{P}_{d+}}\left(2^{d(v)-1}\right)e^{-\boldsymbol{\theta}\cdot\boldsymbol{v}}\right)Zon_d\left(e^{-\boldsymbol{\theta}}\right). \tag{4.3.39}$$

It naturally leads to the asymptotic equivalence between the logarithm of $\frac{\partial}{\partial u}Zon_{d,\text{gen}}\Big|_{u=1}$ and the one of Zon_d when $\boldsymbol{\theta}\to\boldsymbol{0}$, and therefore all the framework of Section 4.3.1 can be applied to $\frac{\partial}{\partial u}Zon_{d,\text{gen}}\Big|_{u=1}$. Indeed, this function satisfies Lemmas 4.3.1, 4.3.2, and 4.3.3 as well. Hence we obtain a statement analog to that of Proposition 4.3.1, and the same parameter given by Lemma 4.3.4 in Section 4.3.2 4.3. Denoting $\tilde{\theta}_n = \left(\frac{2^{d-1}\zeta(d+1)}{\zeta(d)n}\right)^{\frac{1}{d+1}}$, the result is

$$\frac{[\boldsymbol{x}^{\boldsymbol{n}}]\ \frac{\partial}{\partial u} Zon_{d,\text{gen}}(\boldsymbol{x},u)\Big|_{u=1}}{[\boldsymbol{x}^{\boldsymbol{n}}]Zon_d(\boldsymbol{x})} \underset{n\to+\infty}{=} \left(\sum_{\boldsymbol{v}\in\mathbb{P}_{d+}} \left(2^{d(\boldsymbol{v})-1}\right) e^{-\tilde{\theta}_n \mathbf{1}\cdot\boldsymbol{v}}\right)(1+o(1)). \tag{4.3.40}$$

As $d(\boldsymbol{v})$ denotes the number of the nonnull coordinates of $\boldsymbol{v}$, we rewrite this sum using the Mellin inversion formula (same process as in the proof of Lemma 4.2.1 and of Proposition 4.2.1), with $c > d$:

$$\begin{aligned}
\sum_{\boldsymbol{v}\in\mathbb{P}_{d+}} \left(2^{d(\boldsymbol{v})-1}\right) e^{-\tilde{\theta}_n \mathbf{1}\cdot\boldsymbol{v}} &= \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \sum_{\boldsymbol{v}\in\mathbb{P}_{d+}} \left(2^{d(\boldsymbol{v})-1}\right) \frac{\Gamma(s)}{\left(\tilde{\theta}_n \mathbf{1}\cdot\boldsymbol{v}\right)^s} ds \\
&= \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \sum_{k=1}^{d} 2^{k-1}\binom{d}{k}\left(\sum_{n\geq 1}\frac{\binom{n-1}{k-1}}{n^s}\right)\frac{\Gamma(s)}{\zeta(s)\tilde{\theta}_n^s} ds.
\end{aligned}$$

With the left propagation of the integration contour used twice in this chapter, we obtain

$$\sum_{\boldsymbol{v}\in\mathbb{P}_{d+}} \left(2^{d(\boldsymbol{v})-1}\right) e^{-\tilde{\theta}_n \mathbf{1}\cdot\boldsymbol{v}} \underset{n\to+\infty}{=} \frac{2^{d-1}}{\zeta(d)\tilde{\theta}_n^d} + O\left(\frac{1}{\tilde{\theta}_n^{d-1}}\right) = \frac{\sqrt[d+1]{\kappa_d}}{\zeta(d+1)} n^{\frac{d}{d+1}} + O\left(n^{\frac{d-1}{d+1}}\right), \tag{4.3.41}$$

with $\kappa_d = \frac{2^{d-1}\zeta(d+1)}{\zeta(d)}$, which concludes the proof. Finally, the diameter of the graph of a zonotope is equal to its number of generators by construction, which is to say

$$\mu_{\text{gen}}^n = \mu_{\text{diam}}^n. \tag{4.3.42}$$

□

We can also determine another interesting property about lattice zonotopes, the estimated size of an edge of a random lattice zonotope. Depending on ω the multiplicity function of the randomly drawn zonotope defined in Section 4.1.3, an edge $\boldsymbol{e}$ is a translation of $\omega(\boldsymbol{v})\boldsymbol{v}$ for a given primitive vector $\boldsymbol{v}$. In the following proposition, we give the estimated value of $\omega(\boldsymbol{v})$ and its estimated variance.

Proposition 4.3.2. *The number of occurrences of a primitive generator v_0 in a lattice zonotope inscribed in $[0,n]^d$ is distributed with mean μ^n_{occ} and variance $(\sigma^2)^n_{occ}$ such as:*

$$\mu^n_{occ} \underset{n\to+\infty}{\sim} \frac{n^{\frac{1}{d+1}}}{\kappa_d^{\frac{1}{d+1}}||v_0||_1}, \quad (\sigma^2)^n_{occ} \underset{n\to+\infty}{\sim} \left(\frac{n^{\frac{1}{d+1}}}{\kappa_d^{\frac{1}{d+1}}||v_0||_1}\right)^2. \tag{4.3.43}$$

Proof. Without loss of generality, we can choose $v_0 \in \mathbb{P}_{d+}$. As for the previous parameter, we insert a parameter variable u counting for the number of occurrences of v_0; it is substituted for:

$$\sum_{k=0}^{+\infty} e^{-k\theta\cdot v_0} \longrightarrow \sum_{k=0}^{+\infty} u^k e^{-k\theta\cdot v_0}. \tag{4.3.44}$$

Let $Zon_{d,\mathrm{occ}}$ be the modified generating function, that is the function that takes $\theta \in (0,+\infty)^d$, and returns $Zon_d\left(e^{-\theta}\right)\frac{1-e^{-\theta\cdot v_0}}{1-ue^{-\theta\cdot v_0}}$.

The mean and variance are asymptotically estimated like in the previous proof, respectively

$$\mu^n_{\mathrm{occ}} = \frac{[x^n]\ \frac{\partial}{\partial u} Zon_{d,\mathrm{occ}}(x,u)\Big|_{u=1}}{[x^n]Zon_d(x)} \quad \text{and} \quad (\sigma^2)^n_{\mathrm{occ}} = \frac{[x^n]\ \frac{\partial}{\partial u} u \frac{\partial}{\partial u} Zon_{d,\mathrm{occ}}(x,u)\Big|_{u=1}}{[x^n]Zon_d(x)} - {\mu^n_{\mathrm{occ}}}^2 \tag{4.3.45}$$

□

4.3.5 Zonotope enumeration and the Riemann ζ function

The precision of the asymptotic number of zonotopes in a hypercube in Theorem 4.3.1 suffers from the lack of knowledge about Riemann's hypothesis. The oscillatory term $I_{\mathrm{crit},d}$ depends on the non-trivial zeros of the Riemann ζ function. In the 2-dimensional case, Bureaux and Enriquez proved that the order of magnitude of $I_{\mathrm{crit},d}$ is a sufficient and necessary condition for Riemann's Hypothesis to hold. We can't resist the urge to generalize their result to any dimension.

Proposition 4.3.3. *Riemann's hypothesis holds if and only if, for any $\epsilon > 0$, we have*

$$I_{crit,d} = O\left(n^{\frac{1}{2(d+1)}+\epsilon}\right). \tag{4.3.46}$$

Proof. Remember that $I_{\mathrm{crit},d}$ is defined by equation (4.3.29), for γ the path defined in Proposition 4.2.1:

$$I_{\text{crit},d} = \frac{1}{2i\pi} \int_{\gamma} \frac{\Pi_d(\zeta(s))}{\zeta(s)} \zeta(s+1)\theta^{-s}\Gamma(s)ds, \tag{4.3.47}$$

Suppose Riemann's hypothesis holds, then, for all $\epsilon > 0$ when can replace γ_r with the vertical line of x-coordinate $\frac{1}{2} + \epsilon$, and we use the same argument as in the proof of Proposition 4.2.1: on γ_r, the integrand divided by $\theta^{1/2+\epsilon}$ is bounded, because $1/\zeta(s) = O(\log(\Im(s))$ by formula (3.11.8) in [148]. This gives the desired result.

Conversely, suppose that Riemann's hypothesis does not hold. Therefore, there exists a complex r in the critical stripes such that $\Re(r) \neq \frac{1}{2}$ such that $\zeta(r) = 0$. It is well known that the distribution of non-trivial zeros is symmetric with respect to the critical line $\Re(s)$, due to the seminal functional equation [148, Equation (2.1.1)] satisfied for all $s \in \mathbb{C} \setminus \{1\}$:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{4.3.48}$$

Therefore, we can assume that $\Re(r) > \frac{1}{2}$. Let $I = \{z, \zeta(z) = 0, \Re(z) = \Re(r), \Im(z) > 0\}$, we also assume that $r = \underset{z \in I}{\operatorname{argmin}}(\Im(z))$. Then we use the residue theorem on r, while γ' surrounds every other non-trivial zeros but r, and derives from γ. Then

$$I_{\text{crit},d} = \operatorname{Res}\left(\frac{\Pi_d[\zeta](r)}{\zeta(r)}\right)\left(\frac{n}{\kappa_d}\right)^{\frac{r}{d+1}} \zeta(r+1)\Gamma(r) + \frac{1}{2i\pi}\int_{\gamma'} \frac{\Pi_d(\zeta(s))}{\zeta(s)}\zeta(s+1)\theta^{-s}\Gamma(s)ds \tag{4.3.49}$$

The decrease of the Γ function and Selberg's bound on the density of the poles z in [133] ensure that for $\epsilon < \Re(r) - \frac{1}{2}$,

$$\frac{1}{n^{\frac{1}{2(d+1)}+\epsilon'}} = o\left(I_{\text{crit},d}\right). \tag{4.3.50}$$

□

CHAPITRE 5

PROBABILITY: FLUCTUATIONS OF RANDOM LATTICE ZONOTOPES AND POLYGONS.

5.0.1 Introduction

This chapter is based on an article co-authored with Philippe Marchal [34]. In the previous chapter, we studied lattice zonotope enumeration in a hypercube and the statistics of a couple of their combinatorial parameters. Yet, as stated in the introduction, there is a thin line between Arnold's question and Vershik's. Given a bounded convex body K, and nK the dilatation of K by a factor n, the question "how many zonotopes are there in nK, with n growing large" is related to the natural question of the geometric characterization "What is the shape of a random zonotope in nK ?".

In 2018, I. Bárány, J. Bureaux and B. Lund proved a limit shape result for a certain type of K [14]. Let $\mathcal{C}$ be a closed convex salient and pointed cone in $\mathbb{R}^d$ and let $\boldsymbol{k}$ be an integer point in the interior of $\mathcal{C}$. Then the shape of a random uniform lattice zonotope, whose generators are in $\mathcal{C}$ and such that their sum is $n\boldsymbol{k}$, converges towards a deterministic limit after rescaling . This can be thought of as the first-order limit of such a random zonotope, and it is natural to ask for the second-order asymptotics. In this chapter, we will answer the following:

Question 5.0.1. *What are the fluctuations of a random, uniform lattice zonotope in $\mathcal{C}$, whose sum of generators is $n\boldsymbol{k}$ around its limit shape?*

Let's leave the zonotopes and go to the polytopes; we will come back to them later. As for the enumeration, there is almost nothing known about polytope limit shapes in dimensions larger than 3. Even the asymptotic shape of a random polytope drawn in $[0, n]^3$ is not known. In the 2-dimensional case, though, zonogons are just centrally symmetric convex lattice polygons composed of two polygonal arcs that are symmetrical with respect to the point that is half of the sum of its generators. Alternatively, a lattice polygon can be viewed as the union of four arcs of zonogons. This simple characterization of polytopes in dimension 2 explains that we have asymptotic results for lattice polygons: in the mid-90s, A. Vershik [151], Y. Sinai [139] and I. Bárány [10] showed that if we pick a random convex lattice polygon in a large square, each of the four arcs converges into an arc of parabola. Our answer to Question 5.0.1 can be refined in a functional Donsker-like limit theorem that solves:

Question 5.0.2. *What is the limit of the fluctuations of a random, uniform lattice convex polygon in the square $[0, n]^2$ around its limit shape?*

The answer to these questions are respectively given in Theorems 5.2.2 and 5.4.1. As we shall see, the limit theorem obtained for lattice polygons is very far from the other models studied in the literature and summarised below. In particular, the limit fluctuation process brings up a cubic drift, of which we have no knowledge of other instances in the literature.

This chapter is organized as follows. In Section 5.1, we recall and refine results about cones from [14] and [70] that will be used in the sequel, and we describe the probabilistic model. This Boltzmann-like model is highly connected to the generating function of lattice zonotopes in the previous chapter. As a matter of fact, Questions 5.0.1 and 5.0.2 deal with random uniform distribution, but the Boltzmann distribution will be a key tool to compute results and to spread those to the uniform distribution.

Section 5.2 is dedicated to the limits theorems, a central limit theorem, and a local limit theorem towards Gaussian limits with a renormalizing factor $n^{(d+2)/(2d+2)}$.

The former is just a refinement of the central limit theorem proved in [14], while the latter is proved using the framework of J. Bureaux [36] discussed in the previous section. We conclude with the convergence in the distribution of a finite set of renormalized and recentered tangential points of the boundary of a random zonotope towards a Gaussian process with a Brownian sheet structure.

Subsequently, we prove in Section 5.3 the functional limit theorem of the renormalized recentered fluctuations process of the shape of a random lattice zonotope, using the notion of Tightness. After a few general results about function limit theorems, we will extend a criterion of convergence of Billingsley's book [21] to any dimension.

Finally, the last section is devoted to proving the convergence of the fluctuations of a random lattice polygon in a square: we first show the convergence of "cardinal" extremal points towards a Gaussian 4-tuple, then we extend the Donsker-like theorem of the previous section to drifted random variables.

5.0.2 Overview of other random polytope limits

Other models of random distributions over polytopes have been studied, and limit theorems have been established, but there is a difference by nature between those models and ours.

Let K be a convex subset of $\mathbb{R}^d$ with non-empty interior, and for $\lambda \in [1, +\infty)$, let $\mathcal{P}_\lambda$ be a homogeneous Poisson point process of intensity λ on K. The ***random polytope*** K_λ is the convex hull of $\mathcal{P}_\lambda$, and its scientific interest goes back as far as 150 years ago. See the chapter of M. Reitzner [127] for more historical details. The asymptotic study of this model as $\lambda \to +\infty$ is very different from ours as the convex hull nearly fills all K when the number of points grows. Therefore their results focus on the scaling limit and the fluctuations of the boundary.

Two cases were mostly studied depending on K; either the boundary δK is smooth, or either K is a polytope. In both cases, seminal works from M. Reitzner [126] for the former, and I. Barany and M. Reitzner [16] for the latter established central limit theorems for the number of k-faces $f_k(K_\lambda)$ and the volume $\mathrm{Vol}(K_\lambda)$. More recently, second-order properties and a scaling limit of δK_λ were established by P. Calka, J. Yukich, and T. Schreiber in a series of papers [41, 42, 43, 132].

I. Barany and V. Vu also established a central limit theorem for the number of faces and the volume of Gaussian polytopes, that are the convex hull of n random independent points in $\mathbb{R}^d$ following a standard normal distribution [17].

Another approach comes from Linear Programming and the simplex method that was stated in the introduction. A linear problem can be expressed as follows:

$$\begin{aligned} &\text{maximise } c^\intercal x \\ &\text{subject to } Ax \le \mathbf{1}, \end{aligned} \tag{5.0.1}$$

where the known data is the vectors $c \in \mathbb{R}^d$, and $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^n$ and the matrix $A \in \mathbb{R}^{n \times d}$. The constraints of this linear problem are inequalities that correspond to half-spaces. Thus, they refer to a polyhedron (that is, the Minkowski sum of a cone and a polytope, can be unbounded). Polytopes randomly drawn from distributions over A have been used in order to test the expected running time of algorithms under slight random perturbations of an input, called smoothed complexity and introduced by D. Spielman and S.-H. Teng [142], for linear programming algorithms. In [30, 31], K. Borgward studies the diameter of the graph of random polytopes where the rows of A are distributed under a rotational symmetric distribution, that is, if $a^\intercal$ is a row of A :

$$\mathrm{P}\left[||a||_2 \le r\right] = \frac{\int_0^r (1-t)^\beta t^{n-1} dt}{\int_0^1 (1-t)^\beta t^{n-1} dt}, \tag{5.0.2}$$

where $r \in [0, 1]$ and $\beta \in (-1, +\infty)$. When $\beta \to -1$, it corresponds to drawing the rows of A uniformly on the sphere, which was studied by G. Bonnet, D. Dadush, U. Grupel, S. Huiberts, and G. Livshyts in [29]. In both of these models, authors come up with asymptotic bounds of the mean of the diameter of the graph. These studies do not study limit shapes because the construction makes it clear that it is a sphere, but these asymptotic studies give interesting asymptotic information such as the number of vertices (see [29]).

5.1 Cones and probabilistic model

Let d be a strictly positive integer. The sequel is about cones and zonotopes. Let $\mathcal{C}$ be a cone; we denote $\operatorname{int}\mathcal{C}$ its topological interior. Thereafter, $\mathcal{C}$ is a closed convex salient and pointed cone, which means that $0 \in \mathcal{C}$, there is no pair $(\boldsymbol{x}, -\boldsymbol{x})$ lying in $\mathcal{C}$ for some non zero vector $\boldsymbol{x}$, and is not empty. Let $\boldsymbol{k}$ be vector of $\mathbb{Z}^d \cap \mathcal{C}$.

As in the previous chapter, the only zonotopes considered here are *integral zonotopes*, which are lattice zonotopes whose generators are vectors of $\mathbb{R}^d$. Let Z be such integral zonotope for which there exist $k \in \mathbb{N}$ and $\boldsymbol{v}_1, ..., \boldsymbol{v}_k \in \mathbb{Z}^d$ such that the Minkowski sum of $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ gives Z. We define $\mathcal{Z}(\mathcal{C}, \boldsymbol{k})$ as the set of integral zonotopes in $\mathcal{C}$ that end at $\boldsymbol{k}$. This set is clearly finite, since there is a finite number of vectorwise non-null-integer families $(\boldsymbol{v}_i)_{i\leq k}$ such that $\boldsymbol{v}_1 + ... + \boldsymbol{v}_k = \boldsymbol{k}$ and since $\mathcal{C}$ is closed and salient.

Recall from Section 4.1.3 that we call $\sum_{i=1}^{k} \boldsymbol{v}_i$ the endpoint of Z in $\mathcal{C}$ and we say that Z "ends" at $\sum_{i=1}^{k} \boldsymbol{v}_i$. Similarly, the origin is the starting point of Z. The same reasoning as in the proof of Lemma 4.1.1 shows that there is a one-to-one correspondence between the set $\mathcal{Z}(\mathcal{C}, \boldsymbol{k})$ of integral zonotopes in $\mathcal{C}$ and the space $\Omega(\mathcal{C})$ of finite support functions $\omega : \mathbb{P}_d \cap \mathcal{C} \to \mathbb{Z}_+$ that encode strict integer partitions. Given such a function ω, the endpoint of the associated zonotopes, denoted $X(\omega)$, is

$$\mathbf{X}(\omega) = \sum_{\boldsymbol{x} \in \mathbb{P}_d \cap \mathcal{C}} \omega(\boldsymbol{x})\boldsymbol{x}. \tag{5.1.1}$$

Picking a random zonotope in $\mathcal{Z}(\mathcal{C}, \boldsymbol{k})$ is equivalent to picking $\omega \in \Omega(\mathcal{C})$ such that $\mathbf{X}(\omega) = \boldsymbol{k}$. In the sequel, $Z(\omega)$ denotes the zonotope that corresponds to ω, and $\omega(Z)$ the element of $\Omega(\mathcal{C})$ that corresponds to Z.

5.1.1 The probabilistic model

Fix $\boldsymbol{k} \in \operatorname{int}\mathcal{C} \cap \mathbb{Z}^d$. The following probability distribution stems from the correspondence between integral zonotopes and strict integer partitions described above. For any $\omega \in \Omega(\mathcal{C})$, we denote $Z(\omega)$ the associated zonotope.

We define the probability distribution P_n for all $\omega \in \Omega(\mathcal{C})$, depending on the parameters $\beta_n \in \mathbb{R}_+$ and $\boldsymbol{a} \in \mathbb{Z}^d \cap \operatorname{int}\mathcal{C}$ (respectively made explicit in (5.1.3) and in (5.1.9)) by

$$\mathrm{P}_n(\omega) = \frac{1}{Z_n(\boldsymbol{a})} e^{-\beta_n \boldsymbol{a}\cdot\mathbf{X}(\omega)}, \quad \text{where } Z_n(\boldsymbol{a}) = \sum_{\omega \in \Omega(\mathcal{C})} e^{-\beta_n \boldsymbol{a}\cdot\mathbf{X}(\omega)}. \tag{5.1.2}$$

This probability distribution is known as the Boltzmann probability distribution, as it is directly inspired by the Boltzmann distribution in statistical physics. $Z_n(\boldsymbol{a})$ is called the *partition function* of the model. If the name of $Z_n(\boldsymbol{a})$ is not explicit enough, we shall stress that this distribution is strongly connected to the partitions structures (and, more generally, all the combinatorial structures of the set). We notice that $Z_n(\boldsymbol{a})$ is exactly the generating function of the previous chapter for a change of variables $\boldsymbol{x} = e^{-\beta_n \boldsymbol{a}}$. For the distribution to be centered in $n\boldsymbol{k}$, fix β_n throughout this chapter to be

$$\beta_n = \sqrt[d+1]{\frac{\zeta(d+1)}{\zeta(d)n}} \tag{5.1.3}$$

The key point is that the only parameters on which $\mathrm{P}_n(\omega)$ relies are n and the endpoint $\mathbf{X}(\omega)$, hence given $n > 0$, two zonotopes ending at the same point have the same probability. This leads to focus on endpoints that depend on n, in particular, we define the uniform distribution $\mathrm{Q}_{n\boldsymbol{k}}$ on the elements $\Omega(\mathcal{C})$ ending at $n\boldsymbol{k}$ by

$$\mathrm{Q}_{n\boldsymbol{k}}(\omega) = \mathrm{P}_n\left(\omega \mid \mathbf{X}(\omega) = n\boldsymbol{k}\right) \tag{5.1.4}$$

The parameters $\boldsymbol{a}$ and β_n are determined in order for P_n to be "close to" $\mathrm{Q}_{n\boldsymbol{k}}$ when n grows large, in the sense that it satisfies a local limit theorem of mean $n\boldsymbol{k}$. The point of using P_n to approximate $\mathrm{Q}_{n\boldsymbol{k}}$ is that P_n has a much simpler structure. Remind the definition of $\mathbf{X}(\omega)$ in 5.1.1 as a sum over $\mathbb{P}_d \cap \mathcal{C}$; therefore the exponential becomes

$$e^{-\beta_n \boldsymbol{a}\cdot\mathbf{X}(\omega)} = \prod_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}} e^{-\beta_n \boldsymbol{a}\cdot\omega(\boldsymbol{x})\boldsymbol{x}}. \tag{5.1.5}$$

This product structure is passed along to $Z_n(\boldsymbol{a})$ and P_n:

$$Z_n(\boldsymbol{a}) = \prod_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}} \frac{1}{1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}}, \quad \text{and } \mathrm{P}_n(\omega) = \prod_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}} e^{-\beta_n \boldsymbol{a}\cdot\omega(\boldsymbol{x})\boldsymbol{x}}\left(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right) \tag{5.1.6}$$

Remark 5.1.1. *We deduce that* $(\omega(\boldsymbol{x}))_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}}$ *is a mutually independent set under* P_n*, and that the variable* $\omega(\boldsymbol{x})$ *has a geometric distribution of parameter* $1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}$*. The simplicity of* P_n *lies in the fact that under* P_n*, a random zonotope is a product of geometric independent variables for each possible primitive generator.*

5.1.2 Useful formulas about cones.

This part is dedicated to recalling a few results from I. Bárány, J. Bureau and B. Lund [14] about cones, more precisely Theorem 3.1 and Proposition A.1 that are needed in the sequel. We keep the d-dimensional cone $\mathcal{C}$ and $\boldsymbol{k} \in \mathbb{Z}^d \cap \mathcal{C}$ from above. For a given $\boldsymbol{a} \in \mathbb{Z}^d$, we denote the section $\mathcal{C}(\boldsymbol{b} = t)$ and the cap $\mathcal{C}(\boldsymbol{b} \leq t)$ by

$$\mathcal{C}(\boldsymbol{b} = t) = \{\boldsymbol{x} \in \mathcal{C} | \boldsymbol{x} \cdot \boldsymbol{b} = t\},$$
$$\mathcal{C}(\boldsymbol{b} \leq t) = \{\boldsymbol{x} \in \mathcal{C} | \boldsymbol{x} \cdot \boldsymbol{b} \leq t\}.$$

And define the open dual cone $\mathcal{C}^\circ$ of $\mathcal{C}$ as

$$\mathcal{C}^\circ = \{\boldsymbol{a} \in \mathbb{R}^d, \ \forall \boldsymbol{x} \in \mathcal{C}, \ \boldsymbol{a} \cdot \boldsymbol{x} > 0\}. \tag{5.1.7}$$

The following proposition fixes $\boldsymbol{a}$ in the distribution P_n:

Proposition 5.1.1 (Theorem 3.3, [70])**.** *Given $\mathcal{C}$ and a vector $\boldsymbol{v} \in \operatorname{int} \mathcal{C}$, there is a unique $\boldsymbol{a}' = \boldsymbol{a}'(\mathcal{C}, \boldsymbol{v})$ such that $\boldsymbol{a}'$ is the center of gravity of the section $\mathcal{C}(\boldsymbol{v} = 1)$ and that the cap $\mathcal{C}(\boldsymbol{a}')$ is the unique cap that has minimal volume among all caps of $\mathcal{C}$ that contains $\boldsymbol{v}$.*

It immediately follows that $\frac{d}{d+1}\boldsymbol{a}'$ is the center of gravity of $\mathcal{C}(\boldsymbol{v} \leq 1)$, and therefore

$$\frac{d}{d+1}\boldsymbol{a}' = \frac{1}{\mathrm{Vol}\mathcal{C}(\boldsymbol{v} \leq 1)} \int_{\mathcal{C}(\boldsymbol{v} \leq 1)} \boldsymbol{x} d\boldsymbol{x}. \tag{5.1.8}$$

In the probability distribution P_n, given $\boldsymbol{a}'(\mathcal{C}, \boldsymbol{k})$ obtained by Proposition 5.1.1, we define $\boldsymbol{a} \in \mathbb{Z}^d \cap \mathcal{C}$ the unique vector $\boldsymbol{a} = \lambda \boldsymbol{a}'(\mathcal{C}, \boldsymbol{k})$ for some $\lambda > 0$ such that

$$(d+1)! \int_{\mathcal{C}(\boldsymbol{a} \leq 1)} \boldsymbol{x} d\boldsymbol{x} = \boldsymbol{k}. \tag{5.1.9}$$

This value will make sense in the proof of Theorem 5.2.1. The other results of this section aim to approximate sums such as $\mathbb{Z}_n(\boldsymbol{a})$ and its derivatives, with an integral over $\mathcal{C}$.

Proposition 5.1.2 (Proposition A.1, [14])**.** *For $d > 1$, let $f : \mathcal{C} \to \mathbb{R}$ be a continuously differentiable homogeneous function of degree h, i.e. $f(\lambda \boldsymbol{x}) = \lambda^h f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathcal{C}$ and $\lambda \geq 0$. For every $\boldsymbol{a} \in \mathbb{R}^d$ such that $\boldsymbol{a} \in \mathcal{C}^\circ$, that is $\boldsymbol{a} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathcal{C}$:*

$$\beta^{d+h} \sum_{\boldsymbol{x} \in \mathcal{C} \cap \mathbb{P}_d} f(\boldsymbol{x}) e^{-\beta \boldsymbol{a} \cdot \boldsymbol{x}} = \frac{1}{\zeta(d)} \int_{\mathcal{C}} f(\boldsymbol{x}) e^{-\boldsymbol{a} \cdot \boldsymbol{x}} d\boldsymbol{x} + O(\beta) \tag{5.1.10}$$

It is written under the assumption $d \geq 3$, but their proof still stands for $d = 2$. At some point hereafter, we will need to highlight that the term $O(\beta)$ is independent of the subcone chosen in $\mathcal{C}$. For this purpose, we highlight some elements of the proof of Proposition 5.1.2 that give the following corollary: S

Corollary 5.1.1. *For $d > 1$, let $\mathcal{C}_1$ be a subset of $\mathcal{C}$, and let $f : \mathcal{C} \to \mathbb{R}$ be a continuously differentiable homogeneous function of degree h. For every $\boldsymbol{a} \in \mathbb{R}^d$ such that $\boldsymbol{a} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathcal{C}$:*

$$\beta^{d+h} \sum_{\boldsymbol{x} \in \mathcal{C}_1 \cap \mathbb{P}_d} f(\boldsymbol{x}) e^{-\beta \boldsymbol{a} \cdot \boldsymbol{x}} \leq \frac{1}{\zeta(d)} \int_{\mathcal{C}_1} f(\boldsymbol{x}) e^{-\beta \boldsymbol{a} \cdot \boldsymbol{x}} d\boldsymbol{x} + \epsilon(\boldsymbol{a}, \mathcal{C}) \beta \tag{5.1.11}$$

where $\epsilon(\boldsymbol{a}, \mathcal{C})$ does not depends on the choice of $\mathcal{C}_1$.

Proof. Let A be a compact subset of the dual cone $\mathcal{C}^\circ$ of $\mathcal{C}$, the set $\{\boldsymbol{v} \in \mathbb{R}^d, \forall \boldsymbol{x} \in \mathcal{C}, \boldsymbol{v} \cdot \boldsymbol{x} > 0\}$. Since A is compact and since $\mathcal{C}_1(\boldsymbol{u} \leq 1) \subset \mathcal{C}(u \leq 1)$, there exists $L > 0$ such that $\mathcal{C}(\boldsymbol{u} \leq 1)$ is contained in $[-L, L]^d$. It follows that, for $t > 0$, we have $t\mathcal{C}_1 \subset t\mathcal{C} \subset [-tL, tL]^d$.

Here is a simple bound of the approximation of a sum with an integral that we will not prove here; the reader can find it in the Appendix of [14]:

Lemma 5.1.1 (Lemma A.2, [14])**.** *Let $L > 0$, $M > 0$, and let K a compact convex subset of the hypercube $[-L, L]^d$. Then, for every M-Lipschitz continuous function $f : K \to \mathbb{R}$,*

$$\left| \sum_{\boldsymbol{x} \in K \cap \mathbb{Z}^d} f(\boldsymbol{x}) - \int_K f(\boldsymbol{x}) d\boldsymbol{x} \right| \leq M \frac{\sqrt{d}}{2} (2L)^d + 4d!(2L+1)^{d-1} \sup_k |f| \tag{5.1.12}$$

We use Lemma 5.1.1 with the compact convex subset $t\mathcal{C}_1$ of $[-tL, tL]^d$, to obtain a constant $\gamma_{L,f}$ that only depends on L and the function f such that

$$\sup_{\boldsymbol{a} \in A} \left| \sum_{\boldsymbol{x} \in t\mathcal{C}_1(\boldsymbol{a} \leq 1) \cap \mathbb{Z}^d} f(\boldsymbol{x}) - \int_{t\mathcal{C}_1(\boldsymbol{a} \leq 1)} f(\boldsymbol{x}) d\boldsymbol{x} \right| \leq \gamma_{L,f} (1+t)^{d+h-1}. \tag{5.1.13}$$

Then, for any $\beta > 0$, the integration over t of this inequality multiplied by $\beta e^{-\beta t}$ yields

$$\sup_{\boldsymbol{a} \in A} \int_0^{+\infty} \left| \sum_{\boldsymbol{x} \in t\mathcal{C}_1(\boldsymbol{a} \leq 1) \cap \mathbb{Z}^d} f(\boldsymbol{x}) - \int_{t\mathcal{C}_1(\boldsymbol{a} \leq 1)} f(\boldsymbol{x}) d\boldsymbol{x} \right| \beta e^{-\beta t} dt \leq \frac{\gamma_{L,f}}{\beta^{d+h-1}}. \tag{5.1.14}$$

The sum in the left-hand term of the inequality can be rewritten, and using the Fubini-Tonelli theorem, one can check that

$$\int_0^{+\infty} \sum_{x \in t\mathcal{C}_1 \cap \mathbb{Z}^d} f(\boldsymbol{x}) \mathbb{1}_{\boldsymbol{x} \cdot \boldsymbol{a} \leq t} \beta e^{-\beta t} dt = \sum_{x \in \mathcal{C}_1 \cap \mathbb{Z}^d} f(\boldsymbol{x}) e^{-\beta \boldsymbol{a} \cdot \boldsymbol{x}}, \tag{5.1.15}$$

and similarly for the integral. Reversing the integral and the absolute value in the left term in (5.1.14), we obtain

$$\sup_{\boldsymbol{a} \in A} \left| \sum_{x \in \mathcal{C}_1 \cap \mathbb{Z}^d} f(\boldsymbol{x}) e^{-\beta \boldsymbol{a} \cdot \boldsymbol{x}} - \int_{\mathcal{C}_1} f(\boldsymbol{x}) e^{-\beta \boldsymbol{a} \cdot \boldsymbol{x}} d\boldsymbol{x} \right| \leq \frac{\gamma_{L,f}}{\beta^{d+h-1}}. \tag{5.1.16}$$

Finally the prime numbers $\mathbb{P}_d$ in $\mathbb{Z}^d$ has asymptotic density of $\frac{1}{\zeta(d)}$, which allows to conclude. □

5.2 Limit theorems

A central limit theorem of the endpoint of a random zonotope drawn under P_n was computed with respect to the limit in n in [14]. Here, we first extend the central limit theorem to any point of the boundary of the zonotope tangent to a given hyperplane. Then we prove a local limit theorem for these points.

Let Z be a zonotope in $\mathcal{Z}(\mathcal{C}, n\boldsymbol{k})$ drawn under P_n, and let $\boldsymbol{u}$ be a non-null vector in $\mathbb{R}^d$. The linear hyperplane of vector normal $\boldsymbol{u}$ is denoted $\mathcal{P}_{\boldsymbol{u}}$ and the two halfspaces generated by $\boldsymbol{u}$ are respectively ${\mathcal{P}_{\boldsymbol{u}}}^+$ and ${\mathcal{P}_{\boldsymbol{u}}}^-$. Z has two distinct convex set $A^1_{\boldsymbol{u}}$ and $A^2_{\boldsymbol{u}}$ tangent to translations of $\mathcal{P}_{\boldsymbol{u}}$. In concrete terms, if $\mathcal{P}_{\boldsymbol{u}}$ does not contain any generators of Z, then $A^1_{\boldsymbol{u}}$ and $A^2_{\boldsymbol{u}}$ are two points of the boundary, yet if there are generators that belong to $\mathcal{P}_{\boldsymbol{u}}$, then $A^1_{\boldsymbol{u}}$ and $A^2_{\boldsymbol{u}}$ are two faces of Z. These two sets are centrally symmetric to each other with respect to $\frac{n\boldsymbol{k}}{2}$. In order to associated one tangential vector to $\boldsymbol{u}$ instead of two convex sets, we define the vector $\mathbf{X}^n_{\boldsymbol{u}}$

$$\mathbf{X}^n_{\boldsymbol{u}}(\omega) = \sum_{x \in \mathcal{C}(\boldsymbol{u} \geq 0)} \omega(\boldsymbol{x})\boldsymbol{x}, \quad \text{where } \mathcal{C}(\boldsymbol{u} \geq 0) = \mathcal{C} \cap \mathcal{P}^+_{\boldsymbol{u}}. \tag{5.2.1}$$

This point is the endpoint of the zonotope generated by the generators of Z that verify $\boldsymbol{v} \cdot \boldsymbol{u}$, and it belongs to either $A^1_{\boldsymbol{u}}$, either $A^2_{\boldsymbol{u}}$. If $\mathbf{X}^n_{\boldsymbol{u}} \in A^1_{\boldsymbol{u}}$, then $\mathbf{X}^n_{-\boldsymbol{u}} \in A^1_{\boldsymbol{u}}$.

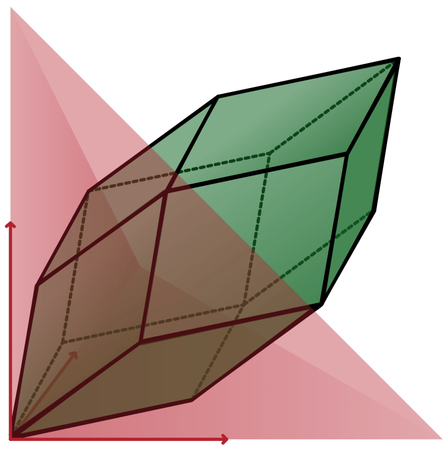


Figure 5.1: Zonotope with 4 different primitive generators in the cone $(\mathbb{R}_+)^3$

We generalise $\mathbf{X}_{\boldsymbol{u}}^n$ to any closed convex salient and pointed cone $\mathcal{C}_1 \subset \mathcal{C}$, defining the random variable $\mathbf{X}_{\mathcal{C}_1}^n(\omega) = \sum_{\boldsymbol{x}\in\mathcal{C}_1} \omega(\boldsymbol{x})\boldsymbol{x}$.

Remark 5.2.1. *We point out that while the boundary of a d-dimensional zonotope is a $d-1$-dimensional object, the fluctuations of the tangent point $\mathbf{X}_{\boldsymbol{u}}^n$ are d-dimensional.*

5.2.1 Central Limit Theorem for $\mathbf{X}_u$

Remark 5.1.1 about the independence between the generators under P_n yields that the point $\mathbf{X}_{\mathcal{C}_1}^n(\omega)$ is distributed according to the following probability distribution:

$$\mathrm{P}_{n,\mathcal{C}_1}(\omega) = \prod_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} e^{-\beta_n \boldsymbol{a}\cdot\omega(\boldsymbol{x})\boldsymbol{x}}(1-e^{-\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}) = \frac{1}{Z_{n,\mathcal{C}_1}(\boldsymbol{a})} e^{-\beta_n \boldsymbol{a}\cdot\mathbf{X}_{\mathcal{C}_1}^n(\omega)}, \tag{5.2.2}$$

where $Z_{n,\mathcal{C}_1}(\boldsymbol{a})$ is the partition function of $\mathrm{P}_{n,\mathcal{C}_1}$. For exponential distributions like $\mathrm{P}_{n,\mathcal{C}_1}$ and P_n, the expectation and the covariance matrix are known to be written in terms of the derivative of the logarithm of this function, that is for the expectation $\mu_{\mathcal{C}_1}^n$ and the covariance $\Gamma_{\mathcal{C}_1}^n$ of $\mathbf{X}_{\mathcal{C}_1}^n$:

$$\mu_{\mathcal{C}_1}^n = \mathbb{E}_n\left[\mathbf{X}_{\mathcal{C}_1}^n\right] = -\nabla \log Z_{n,\mathcal{C}_1}(\boldsymbol{a}) \quad \text{and} \quad \Gamma_{\mathcal{C}_1}^n = \mathrm{cov}(\mathbf{X}_{\mathcal{C}_1}^n) = \nabla^2 \log Z_{n,\mathcal{C}_1}(\boldsymbol{a}).$$

In particular, we have $\boldsymbol{\mu}_{\mathcal{C}(\boldsymbol{u}\geq 0)}^n = \boldsymbol{\mu}_{\boldsymbol{u}}^n$ and $\Gamma_{\mathcal{C}(\boldsymbol{u}\geq 0)}^n = \Gamma_{\boldsymbol{u}}^n$ respectively the mean and covariance matrix of $\mathbf{X}_{\boldsymbol{u}}^n$. Denote also the rescaled limits $\boldsymbol{\mu}_{\mathcal{C}_1}$ (with $\boldsymbol{\mu}_{\mathcal{C}(\boldsymbol{u}\geq 0)} = \boldsymbol{\mu}_{\boldsymbol{u}}$) and $\Gamma_{\mathcal{C}_1}$ (with $\Gamma_{\mathcal{C}(\boldsymbol{u}\geq 0)} = \Gamma_{\boldsymbol{u}}$) respectively as:

$$\boldsymbol{\mu}_{\mathcal{C}_1} = (d+1)! \int_{\mathcal{C}(\boldsymbol{a}\leq 1)\cap\mathcal{C}_1} \boldsymbol{x} d\boldsymbol{x}, \quad \Gamma_{\mathcal{C}_1} = \left(\frac{\zeta(d)}{\zeta(d+1)}\right)^{\frac{1}{d+1}} (d+1)! \int_{\mathcal{C}(\boldsymbol{a}\leq 1)\cap\mathcal{C}_1} \boldsymbol{x}\boldsymbol{x}^{\intercal} d\boldsymbol{x}, \tag{5.2.3}$$

where $d\boldsymbol{x} = d\boldsymbol{x}_1 d\boldsymbol{x}_2 ... d\boldsymbol{x}_d$.

Proposition 5.2.1 (Central limit theorem). *Let $Z \in \mathcal{Z}(\mathcal{C}, n\boldsymbol{k})$ be a random zonotope drawn under P_n, and $\mathcal{C}_1 \subset \mathcal{C}$ be a d-dimensional closed convex cone with $0 \in \mathcal{C}_1$. Then $\mathbf{X}^n_{\mathcal{C}_1}$ satisfies a central limit theorem in the sense that*

$$(\Gamma^n_{\mathcal{C}_1})^{-1/2}\left(\mathbf{X}^n_{\mathcal{C}_1} - \boldsymbol{\mu}^n_{\mathcal{C}_1}\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(\mathbf{0}, I_d), \tag{5.2.4}$$

with $\lim_{n\to\infty} \frac{1}{n}\boldsymbol{\mu}^n_{\mathcal{C}_1} = \boldsymbol{\mu}_{\mathcal{C}_1}$ *and* $\lim_{n\to\infty} n^{-\frac{d+2}{d+1}}\Gamma^n_{\mathcal{C}_1} = \Gamma_{\mathcal{C}_1}$.

In particular, this central limit theorem gives the asymptotic rescaled mean and variance of $\mathbf{X}^n_{\boldsymbol{u}}$, which are continuous with respect to $\boldsymbol{u}$ according to their definition in (5.2.3).

Proof. We start with writing the expectation. After the expansion of the quotient as a series, the Fubini-Tonelli theorem yields:

$$\boldsymbol{\mu}^n_{\mathcal{C}_1} = \sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} \frac{\boldsymbol{x}e^{-\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}}{1-e^{-\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}} = \sum_{i\geq 1}\sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} \boldsymbol{x}e^{-i\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}. \tag{5.2.5}$$

For $i \leq 1/\beta_n$, we use Proposition 5.1.2 to approximate the i-th summation over $\mathbb{P}_d \cap \mathcal{C}_1$ into a d-dimensional integral. Denoting i_0 the first integer such that $i_0 > 1/\beta_n$, there exists $a > 0$ independent of $\boldsymbol{x}$ such that

$$\sum_{i>1/\beta_n} \boldsymbol{x}e^{-i\beta_n\boldsymbol{a}\cdot\boldsymbol{x}} \leq \boldsymbol{x}e^{-i_0\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}\frac{a}{\beta_n} \tag{5.2.6}$$

Therefore, using again Proposition 5.1.2 on the summation over $\mathbb{P}_d \cap \mathcal{C}_1$ of the right-end term of 5.2.6, we obtain that the terms with $i > 1/\beta_n$ only contribute for $O(1/\beta_n)$. We obtain, as n goes to $+\infty$,

$$\boldsymbol{\mu}^n_{\mathcal{C}_1} = \frac{\zeta(d+1)}{\beta_n^{d+1}\zeta(d)}\int_{\mathcal{C}_1} \boldsymbol{x}e^{\boldsymbol{a}\cdot\boldsymbol{x}}d\boldsymbol{x} + O\left(\frac{1}{\beta_n^d}\right). \tag{5.2.7}$$

After simplification of the prefactor of the integral, the change of variable $\boldsymbol{x} = t\boldsymbol{x}'$ with $t = \boldsymbol{a} \cdot \boldsymbol{x}$ gives

$$\frac{1}{n}\boldsymbol{\mu}^n_{\mathcal{C}_1} = (d+1)! \int_{\mathcal{C}(\boldsymbol{a}\leq 1)\cap\mathcal{C}_1} \boldsymbol{x} d\boldsymbol{x} + O\left(n^{\frac{-1}{d}}\right). \tag{5.2.8}$$

The details of the calculation of the asymptotic behavior of the variance are exactly the same, namely

$$\Gamma^n_{\mathcal{C}_1} = \sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} \boldsymbol{x}\boldsymbol{x}^{\intercal} \frac{e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}}{(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}})^2} = n^{\frac{d+2}{d+1}} \left(\frac{\zeta(d)}{\zeta(d+1)}\right)^{\frac{1}{d+1}} (d+1)! \int_{\mathcal{C}(\boldsymbol{a}\leq 1)\cap\mathcal{C}_1} \boldsymbol{x}\boldsymbol{x}^{\intercal} d\boldsymbol{x} + O(n). \tag{5.2.9}$$

The central limit theorem is obtained by ensuring that the Lyapunov ratio $L_{n,\mathcal{C}_1}$, defined just below, tends to 0 as n grows large:

$$L_{n,\mathcal{C}_1} = \sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} \mathbb{E}\left[\left\|(\Gamma^n_{\mathcal{C}_1})^{-1/2}\left(\omega(\boldsymbol{x}) - \mathbb{E}[\omega(\boldsymbol{x})]\right)\boldsymbol{x}\right\|^3\right]. \tag{5.2.10}$$

□

Theorem 5.2.1 only needs that $L_{n,\mathcal{C}_1}$ tends to 0, but in order to state a local limit theorem for $\mathbf{X}^n_{\mathcal{C}_1}$ (Theorem 5.2.1), we compute the order of approximation of the Lyapunov ratio:

Lemma 5.2.1.

$$L_{n,\mathcal{C}_1} = O\left(n^{-\frac{d}{2(d+1)}}\right) \tag{5.2.11}$$

Proof. We start by bounding the Lyapunov ratio with the operating norm of $(\Gamma^n_{\mathcal{C}_1})^{-1/2}$, denoted $||(\Gamma^n_{\mathcal{C}_1})^{-1/2}||$:

$$\sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} \mathbb{E}\left[\left\|\Gamma^{n\ -1/2}_{\mathcal{C}_1}\left(\omega(\boldsymbol{x}) - \mathbb{E}[\omega(\boldsymbol{x})]\right)\boldsymbol{x}\right\|^3\right] \leq ||(\Gamma^n_{\mathcal{C}_1})^{-1/2}||^3 \sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} ||\boldsymbol{x}||^3 \mathbb{E}\left[\left|\left(\omega(\boldsymbol{x}) - \mathbb{E}[\omega(\boldsymbol{x})]\right)\right|^3\right] \tag{5.2.12}$$

We respectively compute the second and fourth moments of $\overline{\omega}(\boldsymbol{x}) = \omega(\boldsymbol{x}) - \mathbb{E}[\omega(\boldsymbol{x})]$:

$$\mathbb{E}\left[|\overline{\omega}(\boldsymbol{x})|^2\right] = \frac{e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}}{\left(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right)^2}$$

$$\mathbb{E}\left[|\overline{\omega}(\boldsymbol{x})|^4\right] = \frac{e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\left(1+7e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}+e^{-2\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right)}{\left(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right)^4} \leq \frac{9e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}}{\left(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right)^4}$$

Therefore by applying the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}\left[|(\omega(\boldsymbol{x})-\mathbb{E}[\omega(\boldsymbol{x})])|^3\right] \leq \frac{3e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}}{\left(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right)^3} \tag{5.2.13}$$

By equation 5.2.9, it is already known that $||\Gamma_{\mathcal{C}_1}^{n\ -1/2}||$ is of order $n^{-\frac{d+2}{2(d+1)}}$, thus the end of the proof is obtained using the same arguments as in the computation of the asymptotic mean in the proof of Proposition 5.2.1, namely

$$\sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} ||\boldsymbol{x}||^3 \frac{3e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}}{\left(1-e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}}\right)^3} = O\left(n^{\frac{d+3}{d+1}}\right). \tag{5.2.14}$$

□

Form Proposition 5.2.1, one can easily extend the central limit theorem to the weak convergence of a k-tuple of rescaled tangent points to a Gaussian k-tuple. Let us introduce the rescaled vector $\widetilde{\mathbf{X}}_{\boldsymbol{u}}^n$ of $\mathbf{X}_{\boldsymbol{u}}^n$:

$$\widetilde{\mathbf{X}}_{\boldsymbol{u}}^n = n^{-\frac{d+2}{2(d+1)}}\left(\mathbf{X}_{\boldsymbol{u}}^n - \mathbb{E}[\mathbf{X}_{\boldsymbol{u}}^n]\right) \tag{5.2.15}$$

Likewise, for any closed convex pointed and salient cone $\mathcal{C}_1 \subset \mathcal{C}$, we define the random variable $\widetilde{\mathbf{X}}^n(\mathcal{C}_1) = n^{-\frac{d+2}{2(d+1)}}\left(\mathbf{X}^n(\mathcal{C}_1) - \mathbb{E}[\mathbf{X}^n(\mathcal{C}_1)]\right)$.

Corollary 5.2.1 (Limit of a k-tuple of tangent points.). *Let Z be a zonotope drawn under P_n, and let $(\boldsymbol{u}_1, ..., \boldsymbol{u}_m)$, respectively $\left(\widetilde{\mathbf{X}}_{\boldsymbol{u}_1}^n, ..., \widetilde{\mathbf{X}}_{\boldsymbol{u}_m}^n\right)$, be an m-tuple of vectors of $\mathbb{R}^d \setminus \{\mathbf{0}\}$, respectively the rescaled m-tuple of the points where between Z and the hyperplanes $\mathcal{P}_{\boldsymbol{u}_i}$, for $1 \leq i \leq m$, are tangent as defined in (5.2.1).*

Then

$$\left(\widetilde{\mathbf{X}}_{\boldsymbol{u}_1}^n, ..., \widetilde{\mathbf{X}}_{\boldsymbol{u}_m}^n\right) \xrightarrow[n\to+\infty]{\mathrm{P}_n} \left(\widetilde{\mathbf{X}}_{\boldsymbol{u}_1}, ..., \widetilde{\mathbf{X}}_{\boldsymbol{u}_m}\right) \tag{5.2.16}$$

where $\left(\widetilde{\mathbf{X}}_{\boldsymbol{u}_1}, ..., \widetilde{\mathbf{X}}_{\boldsymbol{u}_m}\right)$ is a centred Gaussian vector with covariance structure given by

$$cov(\widetilde{\mathbf{X}}_{\boldsymbol{u}_i}, \widetilde{\mathbf{X}}_{\boldsymbol{u}_j}) = \Gamma_{\mathcal{C}(\boldsymbol{u}_i\geq 0, \boldsymbol{u}_j\geq 0)}, \text{ for } 1 \leq i, j \leq m. \tag{5.2.17}$$

Proof. For $1 \leq i \leq k$, we write $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}$ as the sum of all the generators contributing, that is:

$$\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i} = n^{-\frac{d+2}{2(d+1)}} \sum_{\boldsymbol{v} \in \mathbb{P}_d \cap \mathcal{C}(\boldsymbol{u}_i \geq 0)} (\omega(\boldsymbol{v}) - \mathbb{E}[\omega(\boldsymbol{v})])\, \boldsymbol{v}. \tag{5.2.18}$$

We introduce the set of cones $(\mathcal{C}_I)_{I \subset [\![1,m]\!]}$ and the set of random variables $(\mathbf{A}^n_I)_{I \subset [\![1,m]\!]}$, respectively defined by, for $I \subset [\![1,m]\!]$:

$$\mathcal{C}_I = \bigcap_{i \in I} \mathcal{C}(\boldsymbol{u}_i \geq 0) \bigcap_{j \notin I} \mathcal{C}(\boldsymbol{u}_j < 0), \quad \text{and } \mathbf{A}^n_I = n^{-\frac{d+2}{2(d+1)}} \sum_{\boldsymbol{v} \in \mathbb{P}_d \cap \mathcal{C}_I} (\omega(\boldsymbol{v}) - \mathbb{E}[\omega(\boldsymbol{v})])\, \boldsymbol{v}. \tag{5.2.19}$$

The cones $(\mathcal{C}_I)_{I \subset [\![1,m]\!]}$ are partitioning the cone $\mathcal{C}$, in particular they are mutually disjoint. As a consequence, the product form of P_n given in (5.1.6) implies that the random variables $(\mathbf{A}^n_I)$ are independent. Therefore, using Proposition 5.2.1,

$$(\mathbf{A}^n_I)_{I \subset [\![1,m]\!]} \xrightarrow[n \to +\infty]{\mathrm{P}_n} (\mathbf{A}_I)_{I \subset [\![1,m]\!]}\,, \tag{5.2.20}$$

where $(\mathbf{A}_I)_I$ are independent Gaussian variables of covariance $\Gamma(\mathcal{C}_I)$. For $1 \leq i \leq m$, notice that we can write $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}$ as a sum of $\mathbf{A}^n_I$:

$$\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i} = \sum_{\substack{I \subset [\![1,m]\!] \\ i \in I}} \mathbf{A}^n_I. \tag{5.2.21}$$

Hence, for any i, j between 1 and m, the covariance of $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}$ and $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_j}$ is the variance of $\sum_{\substack{I \subset [\![1,m]\!] \\ \{i,j\} \subset I}} \mathbf{A}^n_I$. The weak convergence of the $\left(\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}\right)_i$ follows.

□

5.2.2 Local limit theorem

The central limit theorem is enough to obtain the limit shape of uniformly distributed zonotopes ([14]) but not to describe the fluctuations around this limit shape. The local limit theorem below refines the approximation of the asymptotic behavior of the endpoint of generators in a d-dimensional subcone under P_n.

Theorem 5.2.1 (Local Limit Theorem)**.** *Let Z be a random integral zonotope drawn under the law P_n. Let $\mathcal{C}_1 \subset \mathcal{C}$ be a d-dimensional cone with $0 \in \mathcal{C}_1$, and $\mathbf{X}^n_{\mathcal{C}_1}$ be the endpoint of the generators of Z in $\mathcal{C}_1$. Then the random variable $\mathbf{X}^n_{\mathcal{C}_1}$ satisfies a local limit theorem of rate $n^{-\frac{d}{2(d+1)}}$. Formally:*

$$\limsup_{n\to+\infty} \sup_{\boldsymbol{x}\in\mathbb{Z}^d_+} n^{\frac{d}{2(d+1)}} \left| \mathrm{P}_n(\mathbf{X}^n_{\mathcal{C}_1} = \boldsymbol{x}) - \frac{g_d\left((\boldsymbol{x}-\boldsymbol{\mu}^n_{\mathcal{C}_1})^\top (\Gamma^n_{\mathcal{C}_1})^{-1}(\boldsymbol{x}-\boldsymbol{\mu}^n_{\mathcal{C}_1})\right)}{\sqrt{\det \Gamma^n_{\mathcal{C}_1}}} \right| < +\infty, \tag{5.2.22}$$

where g_d is the density of a standard normal d-dimensional variable.

This theorem is proved using the framework developed by J. Bureaux in [37]. The idea of this framework is to use the inversion formula of the characteristic function on the probability $\mathrm{P}_n(\mathbf{X}^n_{\mathcal{C}_1} = \boldsymbol{x})$, and decompose the difference onto three different domains, which involves satisfying three different conditions (among which is Lemma 5.2.1). Additionally to the previous notation, we denote $\sigma^2_{n,\mathcal{C}_1}$ the smallest eigenvalue of $\Gamma^n_{\mathcal{C}_1}$.

Lemma 5.2.2. *With the notation above, the inverse of the minimal eigenvalue of the covariance matrix satisfies*

$$\frac{1}{\sigma_{n,\mathcal{C}_1}\sqrt{\det \Gamma^n_{\mathcal{C}_1}}} = O\left(n^{-\frac{3(d+2)}{2(d+1)}}\right) \tag{5.2.23}$$

Proof. This lemma directly comes from the asymptotic estimate of $\Gamma^n_{\mathcal{C}_1}$. As seen in the central limit theorem, the covariance matrix estimate is

$$\Gamma^n_{\mathcal{C}_1} = n^{\frac{d+2}{d+1}} \left(\frac{\zeta(d)}{\zeta(d+1)}\right)^{\frac{1}{d+1}} (d+1)! \int_{\mathcal{C}(\boldsymbol{a}\le 1, u\ge 0)} \boldsymbol{x}^\intercal \boldsymbol{x} d\boldsymbol{x} + O(n). \tag{5.2.24}$$

Hence, with a diagonalization of $n^{-\frac{d+1}{d+2}}\Gamma^n_{\mathcal{C}_1}$, we obtain $\sigma^2_{n,\mathcal{C}_1} \asymp n^{\frac{d+2}{d+1}}$, it follows that:

$$\sigma_{n,\mathcal{C}_1}\sqrt{\det(\Gamma^n_{\mathcal{C}_1})} = O\left(n^{\frac{3(d+2)}{2(d+1)}}\right). \tag{5.2.25}$$

□

The last condition of the local limit theorem consists in bounding the characteristic function out of an ellipsoid denoted $\epsilon_{n,\mathcal{C}_1}$ and defined as:

$$\epsilon_{n,\mathcal{C}_1} = \left\{ \boldsymbol{t} \in \mathbb{R}^d \;:\; ||(\Gamma^n_{\mathcal{C}_1})^{1/2}\boldsymbol{t}|| \le \frac{1}{4L_{n,\mathcal{C}_1}} \right\} \tag{5.2.26}$$

Lemma 5.2.3. *If the cone $\mathcal{C}_1$ has dimension 2 or more,*

$$\sup_{t\in[-\pi,\pi]^d\setminus\epsilon_n} \left|\mathbb{E}\left[e^{i\boldsymbol{t}\cdot\mathbf{X}^n_{\mathcal{C}_1}}\right]\right| = O(n^{-1}) \tag{5.2.27}$$

Proof. The outline of the proof is quite standard and can be found in [37, 26], yet some obstacles come from the fact that $\mathcal{C}_1$ is not fixed. For any complex number z, the following inequality holds:

$$\left|\frac{1-|z|}{1-z}\right| \le \exp\left(\Re(z) - |z|\right) \tag{5.2.28}$$

We apply it to the characteristic function:

$$\left|\mathbb{E}\left[e^{i\boldsymbol{t}\cdot\mathbf{X}^n_{\mathcal{C}_1}}\right]\right| = \prod_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1}\left|\frac{1-e^{-\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}}{1-e^{-(\beta_n\boldsymbol{a}-i\boldsymbol{t})\cdot\boldsymbol{x}}}\right| \le \exp\left(\Re\left(\sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} e^{-(\beta_n\boldsymbol{a}-i\boldsymbol{t})\cdot\boldsymbol{x}}\right) - \sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} e^{-\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}\right). \tag{5.2.29}$$

This can be rewritten using the cosine as

$$\left|\mathbb{E}\left[e^{i\boldsymbol{t}\cdot\mathbf{X}^n_{\mathcal{C}_1}}\right]\right| \le \exp\left(\sum_{\boldsymbol{x}\in\mathbb{P}_d\cap\mathcal{C}_1} e^{-\beta_n\boldsymbol{a}\cdot\boldsymbol{x}}(\cos(\boldsymbol{t}\cdot\boldsymbol{x})-1)\right). \tag{5.2.30}$$

To bound the summation of the right-hand side of (5.2.30), we construct a sequence of $\boldsymbol{x}$ such that $\boldsymbol{x}\cdot\boldsymbol{t}$ is small enough to be well approximated. Using the diagonalization of $(\Gamma^n_{\mathcal{C}_1})^{1/2}$, there exists a positive constant c_1 such that $||(\Gamma^n_{\mathcal{C}_1})^{1/2}\boldsymbol{t}|| \le c_1 n^{\frac{d+2}{2(d+1)}}||\boldsymbol{t}||$. Furthermore, since $L_{n,\mathcal{C}_1} = O\left(n^{-\frac{d}{2(d+2)}}\right)$, for every $\boldsymbol{t}\notin\epsilon_n$, there exists a second positive constant c_2 such that $||(\Gamma^n_{\mathcal{C}_1})^{1/2}\boldsymbol{t}|| \ge c_2 n^{\frac{d}{2(d+1)}}$. We deduce that there is a constant $A>0$ such that

$$\max_{1\le i\le d}(|t_i|) \ge An^{-\frac{1}{d+1}} \tag{5.2.31}$$

In the sequel, we denote $\mathbf{e}_i$ the canonical standard basis vector of the i^{th} coordinate. Using the symmetry, we may assume that $|t_1| \ge an^{-\frac{1}{d+1}}$, which means $t_1\in[-\pi,-An^{-\frac{1}{d+1}}]\cup[An^{-\frac{1}{d+1}},\pi]$. Notice that such t_1 with the condition $\mathbf{e}_1\notin\mathcal{C}_1^{\perp}$

exists because the characteristic function would not depend on t_1 otherwise. The rest of the proof consists in finding two arithmetic sequences of primitive vectors whose common differences are far enough from each other to ensure the convergence of the scalar product of one sequence with $\boldsymbol{t}$ towards a polynomial limit.

$\mathcal{C}_1$ is at least of dimension 2, so we state that $\mathbf{e}_2 \notin \mathcal{C}_1^{\perp}$ without loss of generality. Therefore, there exists in the interior of the cone $\mathcal{C}_1$ a primitive vector $\boldsymbol{x}_1$ such that, denoting $p \wedge q$ the greatest common divisor of p and q:

$$\begin{cases} \boldsymbol{x}_1 \cdot \boldsymbol{e}_1 = p, \boldsymbol{x}_1 \cdot \boldsymbol{e}_2 = q, \text{ with } p \wedge q = 1 \text{ and } (p+1) \wedge q = 1 \\ \boldsymbol{x}_2 = \boldsymbol{x}_1 + \boldsymbol{e}_1 \in \mathbb{P}^d \cap \mathcal{C}_1 \\ \boldsymbol{x}_1 + 2\boldsymbol{e}_1 \in \mathcal{C}_1 \end{cases} \tag{5.2.32}$$

The arithmetic sequences $(\boldsymbol{x}_{1,i})_{i\geq 1}$ and $(\boldsymbol{x}_{2,i})_{i\geq 1}$ defined by $\boldsymbol{x}_{\alpha,i} = iq\boldsymbol{x}_{\epsilon} + \boldsymbol{e}_1$ (for $\alpha \in \{1,2\}$), are both sequences of primitive vectors, due to the coprimality of p and q on one side, and $p+1$ and q on the other. The term $\cos(\boldsymbol{t} \cdot (iq\boldsymbol{x}_{\epsilon} + \boldsymbol{e}_1))$ is periodic with respect to i, and its period is $\frac{2\pi}{|\boldsymbol{t}\cdot(q\boldsymbol{x}_\alpha)|}$. We compute a lower bound for the difference between the two periods t_1 and t_2 of respectively $(\boldsymbol{x}_{1,i})$ and $(\boldsymbol{x}_{2,i})$. We have

$$\frac{2\pi}{\boldsymbol{t} \cdot (q\boldsymbol{x}_1)} - \frac{2\pi}{\boldsymbol{t} \cdot (q\boldsymbol{x}_2)} = \frac{2\pi(qt_1)}{q^2(\boldsymbol{t} \cdot \boldsymbol{x}_1)(\boldsymbol{t} \cdot \boldsymbol{x}_2)\boldsymbol{x}_1} \geq A' \frac{n^{-\frac{1}{d+1}}}{q^2||\boldsymbol{x}_2|| \times ||\boldsymbol{x}_1||}, \tag{5.2.33}$$

with $A' \geq 0$. Therefore we have a constant $A_{x_1} \geq 0$, depending only on the choice of $\boldsymbol{x}_1$, such that at least one of these sequences has a period that differs from 1 by $\frac{A_{x_1}}{2} n^{-\frac{1}{d+1}}$ or more. Similarly, both periods cannot be greater than $\frac{4\pi}{A} n^{\frac{1}{d+1}}$ at the same time. For if we suppose that $|\boldsymbol{t} \cdot \boldsymbol{x}_1| \leq \frac{A}{2} n^{\frac{1}{d+1}}$, then

$$|\boldsymbol{t} \cdot \boldsymbol{x}_2| \geq q|t_1| - |\boldsymbol{t} \cdot \boldsymbol{x}_1| \geq \frac{A}{2} n^{\frac{1}{d+1}}. \tag{5.2.34}$$

Suppose $\boldsymbol{x}_1$ verifies these conditions. We can finally compute an upper bound for the argument of the exponential in 5.2.30:

$$\sum_{x \in \mathbb{P}^d \cap \mathcal{C}_1} e^{-\beta_n \boldsymbol{a} \cdot \boldsymbol{x}} (\cos(\boldsymbol{t} \cdot \boldsymbol{x}) - 1) \leq \sum_{i \geq 1} e^{-\beta_n \boldsymbol{a} \cdot \boldsymbol{x}_{1,i}} (\cos(\boldsymbol{x}_{1,i} \cdot \boldsymbol{t}) - 1). \tag{5.2.35}$$

Denote $A_{\max} = \max(\frac{2}{A_x}, \frac{4\pi}{A})$. The inequality $\cos(i\boldsymbol{t} \cdot q\boldsymbol{x}_1 + t_1) \leq \frac{1}{2}$ stands in the window of length three-quarters of the length of the period. The condition on the upper bound and the condition on the difference to 1 implies that the k^{th} term of $(\boldsymbol{x}_{1,i})_{i\geq 1}$ that verifies $\cos(\boldsymbol{t}\cdot\boldsymbol{x}_{1,i}) \leq \frac{1}{2}$ is in the first $2k$ terms, for $2k \geq A_{\max} n^{\frac{1}{d+1}}$. This leads to:

$$\sum_{i\geq 1} e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}_{1,i}}(\cos(\boldsymbol{x}_{1,i}\cdot\boldsymbol{t})-1) \leq \left(-\frac{1}{2}\right) \sum_{i\geq A_{\max} n^{\frac{1}{d+1}}} e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}_{1,2i}}. \tag{5.2.36}$$

Ultimately, manipulation of the indicial notation gives:

$$-\frac{1}{2} \sum_{i\geq A_{\max} n^{\frac{1}{d+1}}} e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}_{1,2i}} \leq -\frac{1}{2}\exp\left(-\beta_n \boldsymbol{a}\cdot\left(A_{\max} n^{\frac{1}{d+1}} q\boldsymbol{x}_1 + \boldsymbol{e}_1\right)\right) \sum_{i\geq 0} e^{-\beta_n i q \boldsymbol{a}\cdot\boldsymbol{x}_1}. \tag{5.2.37}$$

We recall that $\beta_n = \left(\frac{\zeta(d+1)}{\zeta(d)n}\right)^{\frac{1}{d+1}}$, hence the first exponential asymptotically converges to a constant. A quick asymptotic analysis of the sum gives

$$\sum_{i\geq 0} e^{-\beta_n i q \boldsymbol{a}\cdot\boldsymbol{x}_1} = \frac{1}{1-e^{-\beta_n q \boldsymbol{a}\cdot\boldsymbol{x}_1}} \asymp n^{\frac{1}{d+1}}. \tag{5.2.38}$$

Thus the biggest $\boldsymbol{x}_1$ over all possible combinations of coordinates gives a constant $\gamma > 0$, depending only on $\mathcal{C}_1$, such that

$$\left|\mathbb{E}\left[e^{i\boldsymbol{t}\cdot\mathbf{X}^n_{\mathcal{C}_1}}\right]\right| \leq \exp\left(-\gamma n^{\frac{1}{d+1}}\right), \tag{5.2.39}$$

which concludes the proof.

□

Proof of the Theorem. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence given by $a_n = n^{-\frac{d}{2(d+1)}}$. Then, with Lemma 5.2.1, 5.2.2, and 5.2.3, the assumptions for Proposition 7.1 from [37] are satisfied, and there is a local limit theorem of rate (a_n) for the variable $\mathbf{X}^n_{\mathcal{C}_1}$ under P_n.

□

5.2.3 Weak convergence of finite-dimensional marginals

The following proposition gives the weak convergence of finite-dimensional distribution of the process of tangent points under $Q_{n\boldsymbol{k}}$, leading to Theorem 5.2.2 and the Donsker-like theorem (Theorem 5.3.2). Recall that $Q_{n\boldsymbol{k}}$ is the uniform distribution over integral zonotopes ending at $n\boldsymbol{k} \in \mathbb{Z}_+^d$. The connection between $Q_{n\boldsymbol{k}}$ and P_n, for any $A \subset \Omega$:

$$Q_{n\boldsymbol{k}}[A] = P_n\left[A \mid \mathbf{X}(\omega) = n\boldsymbol{k},\ \omega \in A\right] = \frac{P_n\left[A \cap \{\mathbf{X}(\omega) = n\boldsymbol{k},\ \omega \in A\}\right]}{P_n\left[\mathbf{X}(\omega) = n\boldsymbol{k},\ \omega \in A\right]}. \quad (5.2.40)$$

Proposition 5.2.2 (Weak convergence of finite-dimensional marginals). *For $\boldsymbol{k} \in \operatorname{int} \mathcal{C} \cap \mathbb{Z}^d$, let $(\boldsymbol{u}_1, ..., \boldsymbol{u}_m)$ an m-tuple of $\mathbb{R}^d \setminus \{\mathbf{0}\}$, let Z be a random zonotope drawn under $Q_{n\boldsymbol{k}}$, and $\left(\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_1}, ..., \widetilde{\mathbf{X}}^n_{\boldsymbol{u}_m}\right)$ be the rescaled m-tuple of points where Z is tangent to respectively $\mathcal{P}_{\boldsymbol{u}_1}$, ..., $\mathcal{P}_{\boldsymbol{u}_m}$ as defined in (5.2.1). Then there is an independent family of Gaussian-centered random variables $(\mathbf{G}_I)_{I \subset [\![1,m]\!]}$ such that*

$$\left(\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_1}, ..., \widetilde{\mathbf{X}}^n_{\boldsymbol{u}_m}\right) \xrightarrow[n \to +\infty]{(d)} (\mathbf{N}_{\boldsymbol{u}_1}, ..., \mathbf{N}_{\boldsymbol{u}_m}), \quad (5.2.41)$$

where $\mathbf{N}_{\boldsymbol{u}_i} = \sum_{\substack{I \subset [\![1,m]\!] \\ i \in I}} \mathbf{G}_I$ and

$$\operatorname{cov}(\mathbf{G}_I) = \left(\Gamma_{\bigcap\limits_{i \in I} \mathcal{C}(\boldsymbol{u}_i \geq 0) \bigcap\limits_{j \notin I} \mathcal{C}(\boldsymbol{u}_j < 0)}^{\;-1} + \Gamma_{\bigcap\limits_{i \in I} \mathcal{C}(\boldsymbol{u}_i < 0) \bigcap\limits_{j \notin I} \mathcal{C}(\boldsymbol{u}_j \geq 0)}^{\;-1} \right)^{-1}. \quad (5.2.42)$$

When $\mathcal{C}(\boldsymbol{u}_i \geq 0)$ is d-dimensional, $\mathbf{N}_{\boldsymbol{u}_i}$ is a centered Gaussian variable of variance $\left(\Gamma_{\boldsymbol{u}_i}^{-1} + \Gamma_{-\boldsymbol{u}_i}^{-1}\right)^{-1}$.

Proof. We still denote ω as the function of multiplicities of Z. Starting with $\boldsymbol{u}_1, ..., \boldsymbol{u}_m$ vectors of $\mathbb{R}^d$, we denote $\boldsymbol{u}_{m+1}$ a vector in $\mathcal{C}^\circ$ (that is a vector $\boldsymbol{v}$ such that $\boldsymbol{v} \cdot \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathcal{C}$), hence $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_{m+1}}$ is the rescaled endpoint of the zonotope Z.

Using the same argument as in Proposition 5.2.1, the probability measure P_n is constructed as a product of geometric distributions of the primitive vectors in $\mathcal{C}$, and the occurrences of the primitive vectors (that is the set $\{\omega(\boldsymbol{x})\}_{\boldsymbol{x}}$) are mutually independent. Therefore we introduce again the family of variables $(\mathbf{A}_I)_{I \subset [\![1,m]\!]}$ defined in (5.2.19).

$(\mathbf{A}_I)_{I \subset [\![1,m]\!]}$ denotes the vertices of Z at the end of a path starting at the origin and composed of all generators contributing to the elements of I, but not contributing to the others, after recentering and renormalizing by $n^{-(d+2)/(2d+2)}$. The family of cones $(\mathcal{C}_I)_{I\subset[\![1,m]\!]}$ is mutually disjoint, hence the variables $(\mathbf{A}_I)_{I\subset[\![1,m]\!]}$ are mutually independent under P_n.

The family $\left(\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}\right)$ is generated by the family $(\mathbf{A}_I)_{I\subset[\![1,m]\!],\, I\neq\emptyset}$, as $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i} = \sum_{I\subset[\![1,m]\!], i\in I} \mathbf{A}_I$. Under P_n, the probability for the (2^m-1)-tuple of variables $\mathbf{A}_I$, with $I \neq \emptyset$ to be equal to $(\boldsymbol{x}_I)_{I\subset[\![1,m]\!],\, I\neq\emptyset}$ is, based on Bayes' theorem:

$$\mathrm{Q}_{n\boldsymbol{k}}\left[(\mathbf{A}_I)_{I\neq\emptyset} = (\boldsymbol{x}_I)_{I\neq\emptyset}\right] = \frac{\mathrm{P}_n\left[(\mathbf{A}_I)_I = (\boldsymbol{x}_I)_I \cap \widetilde{\mathbf{X}}^n_{\boldsymbol{u}_{m+1}} = 0\right]}{\mathrm{P}_n\left[\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_{m+1}} = 0\right]}. \tag{5.2.43}$$

Under the distribution $\mathrm{Q}_{n\boldsymbol{k}}$, the condition that the endpoint of the zonotope Z is at $n\boldsymbol{k}$ can be considered as a condition on $\mathbf{A}_\emptyset$, that is $\mathbf{A}_\emptyset = \boldsymbol{x}_\emptyset$ with $\boldsymbol{x}_\emptyset = -\sum_{I\subset[\![1,m]\!],\, I\neq\emptyset} \boldsymbol{x}_I$. Therefore, denoting $\mathcal{P}(E)$ the set of all subsets of E, it follows that

$$\begin{aligned}\mathrm{P}_n\left[(\mathbf{A}_I)_{I\subset[\![1,m]\!],\, I\neq\emptyset} = (\boldsymbol{x}_I)_{I\subset[\![1,m]\!],\, I\neq\emptyset} \cap \widetilde{\mathbf{X}}^n_{\boldsymbol{u}_{m+1}} = 0\right] &= \mathrm{P}_n\left[\mathbf{A}_I = \boldsymbol{x}_I,\ I \in \mathcal{P}\left([\![1,m]\!]\right)\right] \\ &= \prod_{I\in\mathcal{P}([\![1,m]\!])} \mathrm{P}_n(\mathbf{A}_I = \boldsymbol{x}_I).\end{aligned} \tag{5.2.44}$$

All the variables $\mathbf{A}_I$ satisfy Theorem 5.2.1 with mean 0 and covariance $\Gamma(\mathcal{C}_I)$, and so does $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_{m+1}}$ with covariance $\Gamma_{\mathcal{C}}$. Hence, (5.2.43), (5.2.44), and the local limit theorem lead to

$$\sup_{\substack{(\boldsymbol{x}_I)_{I\subset[\![1,m]\!]}\in\left(\mathbb{R}^d\right)^{2^m-1} \\ I\neq\emptyset}} \left| \mathrm{Q}_{n\boldsymbol{k}}\left[(\mathbf{A}_I)_{I\neq\emptyset} = (\boldsymbol{x}_I)_{I\neq\emptyset}\right] - \frac{\prod_{I\subset[\![1,m]\!]} \frac{g_d\left(\boldsymbol{x}_I^\top(\Gamma(\mathcal{C}_I))^{-1}\boldsymbol{x}_I\right)}{\det\Gamma(\mathcal{C}_I)^{1/2}}}{\frac{1}{(2\pi)^{d/2}\det\Gamma(\mathcal{C})^{1/2}}} \right| \underset{n\to+\infty}{\longrightarrow} 0, \tag{5.2.45}$$

where g_d is the density of the d-dimensional standard Gaussian variable. $(\mathbf{A}_I)_{I\subset[\![1,m]\!],\, I\neq\emptyset}$ satisfies a local limit theorem of rate 1 to a Gaussian family of variables named $(\mathbf{G}_I)_{I\subset[\![1,m]\!],\, I\neq\emptyset}$. The weak convergence follows. The covariance of the $(\mathbf{G}_I)$ is given by inverting the matrix of the quadratic form in the exponential.

Since $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i} = \sum_{I\subset[\![1,m]\!], i\in I} \mathbf{X}_I$, $\left(\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}\right)_{1\leq i\leq m}$ weakly converges to $\left(\sum_{\substack{I_i\subset[\![1,m]\!] \\ i\in I_i}} \mathbf{G}_{I_i}\right)_{1\leq i\leq m}$. When considering this limit with only one variable $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}_i}$, we deduce that it weakly converges to the Gaussian random variable $\mathbf{N}_{\boldsymbol{u}_i} = \sum_{\substack{I_i \subset [\![1,m]\!] \\ i \in I_i}} \mathbf{G}_{I_i}$ with mean 0 and covariance

$$\operatorname{cov}(\mathbf{N}_{\boldsymbol{u}_i}) = \left(\Gamma_{\mathcal{C}(\boldsymbol{u}_i \geq 0)}{}^{-1} + \Gamma_{\mathcal{C} \setminus \mathcal{C}(\boldsymbol{u}_i \geq 0)}{}^{-1}\right)^{-1} = \left(\Gamma_{\boldsymbol{u}_i}^{-1} + \Gamma_{-\boldsymbol{u}_i}^{-1}\right)^{-1}. \tag{5.2.46}$$

□

With Proposition 5.2.2, one can deduce that for any vector $\boldsymbol{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, and Z a random zonotope that ends at $n\boldsymbol{k}$ uniformly drawn, the vector $\mathbf{X}_{\boldsymbol{u}}^n$ satisfies a CLT as n grows large, namely

Theorem 5.2.2. *Let Z be a random, uniform lattice zonotope starting at the origin, ending at $n\boldsymbol{k}$, and with generators in $\mathcal{C}$. Let $\boldsymbol{u} \in \mathbb{R}^d \setminus \{0\}$ and let $\mathbf{X}_{\boldsymbol{u}}^n$ be a point of the boundary of Z tangent to the hyperplane with normal vector $\boldsymbol{u}$, as defined by* (5.2.1)*. Then there exists a symmetric matrix $\Gamma_{\boldsymbol{u}}$, given by* (5.2.3) *, such that*

$$(n^{-\frac{d+2}{d+1}}\Gamma_{\boldsymbol{u}})^{-1/2}\left(\mathbf{X}_{\boldsymbol{u}}^n - \mathbb{E}(\mathbf{X}_{\boldsymbol{u}}^n)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N} \tag{5.2.47}$$

where $\mathcal{N}$ is a standard, d-dimensional Gaussian variable.

Concretely, every vertex of a random, uniform lattice zonotope, whose generators are vectors of $\mathbb{N}^d$ and that ends at $[n, ..., n]$, properly recentred and rescaled, satisfies this CLT.

5.3 A Donsker-like theorem for zonotope fluctuation

For $\boldsymbol{u} \in \mathbb{S}^{d-1}$, and a lattice zonotope $Z \in \mathcal{Z}(\mathcal{C}, n\boldsymbol{k})$, Theorem 5.2.2 asserts that $\widetilde{\mathbf{X}}_{\boldsymbol{u}}^n$ asymptotically converges in distribution to a Gaussian variable. Therefore the next logical step is to prove that the random process $(\widetilde{\mathbf{X}}_{\boldsymbol{u}}^n)_{\boldsymbol{u}\in I}$, for some $I \subset \mathbb{S}^{d-1}$, converges in distribution towards the corresponding Gaussian process in the space of càdlàg functions with index in $I \subset \mathbb{S}^{d-1}$.

We will prove the convergence of this process only in the 2-dimensional case. Recall that the results of the previous section were true in any dimension. On the other hand, extending the functional results of this section to a higher dimension would require more involved tightness estimates, such as those appearing in Theorem 4 of [18]. We will not handle this question here due to the topological structure of $\mathbb{S}^{d-1}$.

5.3.1 Introduction to functional limit theorem

We start with some preliminaries over random processes' convergence. Let C be the space of real-values continuous functions on $\mathbb{R}^+$ and let $\mathfrak{C}$ be the Borel σ-algebra on C.

A continuous random process on $\mathbb{R}^+$ is thus a random variable with values in $(C[0,1], \mathfrak{C})$. The most famous random process is the Brownian motion; we recall its definition:

Definition 5.3.1. *The real-valued continuous-time random process $(W_t)_{t\geq 0}$ such that*

- *(W_t) is almost surely continuous in t,*
- *for every $t, s > 0$, $W_t - W_s$ is independent of W_u for all $0 \leq u < s$,*
- *for every $t, u \leq 0$, $W_{u+t} - W_t$ is normally distributed with mean 0 and variance u,*

*is a Brownian motion. If $W_0 = 0$ almost surely, (W_t) is a **standard Brownian motion**. Additionally, if $W_0 = W_1 = 0$, (W_t) is a **Brownian bridge**.*

Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables on some probability space, having mean 0 and a finite positive variance σ^s. Let $S_0 = 0$, and for each $n > 0$, let $S_n = \xi_1 + ... + \xi_n$. We define the random process X_t^n as $X^n(\omega) \in C[0,1]$, and for each $t \in \mathbb{R}^+$,

$$X_t^n(\omega) = \frac{1}{\sigma\sqrt{n}} \left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\xi_{\lfloor nt \rfloor + 1}(\omega) \right) \tag{5.3.1}$$

Theorem 5.3.1 (Donsker's Theorem, [58])**.** *Let $(\xi_i)_{i\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables, having mean 0 and variance σ^s. If X_n is defined by 5.3.1, then $X^n \overset{(d)}{\Rightarrow} W$.*

The weak convergence of a sequence of random processes can be decomposed into two pieces, named the convergence of finite-dimensional marginals and the tightness. Consequently, the notion of ***tightness*** is fundamental. A probability measure P on a space $(S, \mathfrak{S})$ is tight if, for each $\epsilon > 0$, there is a compact set $K \subset S$ such that $P(K) > 1 - \epsilon$. For example, if S is separable and complete, then every probability measure on $(S, \mathfrak{S})$ is tight, which is why the tightness property is not at stake when proving the weak convergence of a real-valued random variable (see [21]).

This explains the difficulty of proving the tightness in our situation: the process $(\widetilde{\mathbf{X}}_{\boldsymbol{u}}^n)$ is indexed on the sphere, so the tightness is to be proved on a set of functions $f : \mathbb{S}^{d-1} \to \mathbb{R}^k$. Yet the topology in the sphere (and more precisely, the set of homeomorphisms of the sphere used in a Skorohod-like norm) makes tightness harder to prove than a mere generalization.

5.3.2 Tightness for zonogon Boltzmann distribution.

Let $d = 2$ and let Z be a random zonogon in $\mathcal{Z}(\mathcal{C}, n\boldsymbol{k})$ drawn under the distribution Q_{nk}. Then the random process $(\widetilde{\mathbf{X}}^n_{\boldsymbol{u}})_{\boldsymbol{u}\in\mathbb{S}^1}$ is defined as the vertex of Z where Z and the hyperplane $\mathcal{P}_u$ are tangent defined by (5.2.1). We can cut the circle $\mathbb{S}^1$ into 4 parts: $S_1 = \mathbb{S}^1 \cap \mathcal{C}^\circ$, $S_2 = \mathbb{S}^1 \cap -\mathcal{C}^\circ$, and the 2 remaining arcs, where $\mathcal{C}^\circ$ is the open dual cone of $\mathcal{C}$. For every element $\boldsymbol{u}$ of S_1, $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}} = 0$, and similarly, for every element $\boldsymbol{u} \in S_2$, $\widetilde{\mathbf{X}}^n_{\boldsymbol{u}} = 0$. Therefore, the 2 remaining arcs are respectively called S^+ and S^- with the convention that S^+ is the upper arc when rotating $\mathcal{C}$ such that the endpoint is on the horizontal axis (we take the minimal rotation, that is less than $\pi/2$).

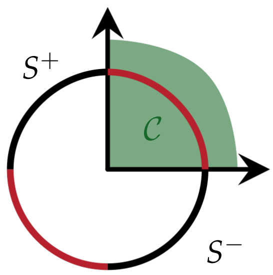


Figure 5.2: The arcs S^+ and S^- when $\mathcal{C} = \mathbb{R}^2_+$

Let f be a continuous bijection from $[0,1]$ to S^+. We define $(B^n_t)_{t\in[0,1]}$ the random process of the upper arc's fluctuation of Z under Q_{nk} as

$$B^n_t = \widetilde{\mathbf{X}}^n_{f(t)} \tag{5.3.2}$$

Let $\mathbb{E}_Q$, resp. $\mathbb{E}_P$ denote an expectation under the distribution Q_{nk}, resp. P_n. Now, we can state the following proposition:

Proposition 5.3.1 (Tightness). *Let $||\cdot||$ stand for norm 1. For $0 \le r < s < t \le 1$, and $\alpha > \frac{1}{2}$, and $\beta > 0$*

$$\mathbb{E}_Q\left[||B^n_s - B^n_r||^{2\beta}\,||B^n_t - B^n_s||^{2\beta}\right] \le |F(t) - F(r)|^{2\alpha}, \tag{5.3.3}$$

with F is a non-decreasing, continuous function on $[0,1]$.

Proof. Let $0 \le r \le s \le t \le 1$, and . It is sufficient to prove that there exists a constant $C > 0$ such that

$$\mathbb{E}_Q\left[||B^n_s - B^n_r||^2\,||B^n_t - B^n_s||^2\right] \le C\left(t^3 - r^3\right)^2. \tag{5.3.4}$$

We denote $\mathcal{E}(r,s,t)$ the left term of this inequality. Since (B_t^n) is a variable on a random zonotope that ends at $n\boldsymbol{k}$, the first step is to rewrite B_t^n for random zonotopes drawn under P_n. We have

$$\mathcal{E}(r,s,t) = \mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n_{f(s)} - \widetilde{\mathbf{X}}^n_{f(r)}\right|\right|^2 \left|\left|\widetilde{\mathbf{X}}^n_{f(t)} - \widetilde{\mathbf{X}}^n_{f(s)}\right|\right|^2 \middle| Z \in \mathcal{Z}\left(\mathcal{C}, n\boldsymbol{k}\right)\right]. \tag{5.3.5}$$

Denoting $\mathcal{C}_{r,s}$ (resp. $\mathcal{C}_{s,t}$) the cone of vectors contributing to $\mathbf{X}^n_{f(s)} - \mathbf{X}^n_{f(r)}$ (resp. $\mathbf{X}^n_{f(t)} - \mathbf{X}^n_{f(s)}$). The first cone formally is $\{\boldsymbol{x} \in \mathbb{R}, \boldsymbol{x}\cdot(f(r)-1, f(r)) \leq 0 \leq \boldsymbol{x}\cdot(f(s)-1, f(s))\}$. We can write the right term of the equation above as:

$$\mathcal{E}(r,s,t) = \mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s})\right|\right|^2 \left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{s,t})\right|\right|^2 \;|Z \in \mathcal{Z}\left(\mathcal{C}, n\boldsymbol{k}\right)\right]. \tag{5.3.6}$$

After writing each variable as a sum over the generators and expanding the product of the norms, one can notice that this expectancy will result in a sum over quadruplets primitive generators $\{\boldsymbol{x}_1, \boldsymbol{x}_2\boldsymbol{x}_3, \boldsymbol{x}_4\}$. Given such a quadruplet $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$, the probability for the generators to respectively occur k_1, k_2, k_3, and k_4 is

$$\mathrm{Q}_{n\boldsymbol{k}}(k_1,k_2,k_3,k_4) = \prod_{i=1}^{4} \mathrm{P}_n\left(\omega(\boldsymbol{x}_i) = k_i\right) \frac{\mathrm{P}_n\left(\mathbf{X}_1^n \setminus \{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4\} = (n,n) - \sum_{i=1}^4 k_i\boldsymbol{x}_i\right)}{\mathrm{P}_n\left(\mathbf{X}_1^n = (n,n)\right)}, \tag{5.3.7}$$

Notice the inequality on the upper term of the quotient of the right-hand term:

$$\begin{aligned}
\mathrm{P}_n\left(\mathbf{X}_1^n \setminus \{\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4\} = (n,n) - \sum_{i=1}^4 k_i\boldsymbol{x}_i\right) &= \frac{\mathrm{P}_n\left(\mathbf{X}_1^n = (n,n) - \sum_{i=1}^4 k_i\boldsymbol{x}_i \;\; \bigcap_{i=1}^4 k_i = 0\right)}{\mathrm{P}_n\left(\bigcap_{i=1}^4 k_i = 0\right)} \\
&\leq \frac{\mathrm{P}_n(\mathbf{X}_1^n = (n,n) - \sum_{i=1}^4 k_i\boldsymbol{x}_i)}{\mathrm{P}_n\left(\bigcap_{i=1}^4 k_i = 0\right)}.
\end{aligned}$$

The definition of β_n ensures that for $n_1 \in \mathbb{N}$, for all $n \geq n_1$ for any $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$,

$$\mathrm{P}_n\left(\bigcap_{i=1}^4 k_i = 0\right) = \prod_{i=1}^4 (1 - e^{-\beta_n \boldsymbol{a}\cdot\boldsymbol{x}_i}) \geq \frac{1}{2}. \tag{5.3.8}$$

Moreover, the local limit theorem ensures that there exists n_2 large enough such that for $n > n_2 \geq n_1$, for any $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$, the ratio is bounded :

$$\frac{\mathrm{P}_n(\mathbf{X}_1^n = (n,n) - \sum_{i=1}^4 k_i \boldsymbol{x}_i)}{\mathrm{P}_n\left(\mathbf{X}_1^n = (n,n)\right)} \leq 2. \tag{5.3.9}$$

Therefore, using (5.3.8) and (5.3.9) in (5.3.7), one can obtain upper bound $\mathcal{E}(r,s,t)$ with a nonconditional expectation, that one can split due to independence:

$$\mathcal{E}(r,s,t) \leq 4\,\mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s})\right|\right|_1^2 \left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{s,t})\right|\right|^2\right] = 4\,\mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s})\right|\right|^2\right] \mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{s,t})\right|\right|_1^2\right]. \tag{5.3.10}$$

In order to expand the 1-norm, let a and b be real numbers such that $\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s}) = (a,b)$. Therefore, using the inequality $|a+b| \leq 2a^2 + 2b^2$, one obtains $||\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s})||_1^2 \leq 2a^2 + 2b^2$. Hence, if we write $\boldsymbol{x} = (x_1, x_2)$, we have (recall that ω is the finite support function encoding the multiplicity of the generators of Z):

$$\mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s})\right|\right|^2\right] \leq \sum_{i=1}^2 \frac{2}{n^{4/3}}\,\mathbb{E}_P\left[\left(\sum_{\boldsymbol{x}\in\mathbb{P}^2\cap\mathcal{C}_{r,s}} x_i(\omega(\boldsymbol{x}) - \mathbb{E}_P[\omega(\boldsymbol{x})])\right)^2\right]. \tag{5.3.11}$$

The 2 summands on the right-hand side are the diagonal terms of $\Gamma^n(\mathcal{C}_{r,s})$. Using the bounding given by Corollary 5.1.1 in the same way we previously explicitly calculated in the proof in Proposition 5.2.1, we have

$$\mathbb{E}_P\left[\left(\sum_{\boldsymbol{x}\in\mathbb{P}^2\cap\mathcal{C}_{r,s}} x_1(\omega(\boldsymbol{x}) - \mathbb{E}_P[\omega(\boldsymbol{x})])\right)^2\right] \leq 6n^{4/3}\left(\frac{\zeta(2)}{\zeta(3)}\right)^{\frac{1}{3}} \int_{\substack{\mathbb{P}_2\cap\mathcal{C}_{r,s}\\ \boldsymbol{x}\cdot\boldsymbol{a}\leq 1}} x_1^2 d\boldsymbol{x} + \epsilon(n)n, \tag{5.3.12}$$

with $\epsilon(n)$ tending to 0 as n grows to $+\infty$, and $\epsilon(n)$ is independent of $\mathcal{C}_{r,s}$ Therefore for n_3 large enough, and $n \geq n_3$, there exists a constant $A > 0$ such that

$$\mathbb{E}_P\left[\left|\left|\widetilde{\mathbf{X}}^n(\mathcal{C}_{r,s})\right|\right|_1^2\right] \leq A(s^3 - r^3) \tag{5.3.13}$$

Finally, the inequality $(s^3 - r^3)(t^3 - s^3) \leq (t^3 - r^3)^2$ concludes the proof.

□

5.3.3 Random zonotope's Gaussian fluctuations

With Proposition 5.3.1 and Proposition 5.2.2, one can finally prove the weak convergence of the rescaled fluctuations around the limit shape of a random zonogon to Gaussian limit process, in the same vein as Donsker's theorem. Even if we do not establish the weak convergence for higher dimensions, remember that in dimension ≥ 3, the only results we could obtain using our method would be valid for zonotopes and not for polytopes, whereas in dimension 2, the study of polygons follows readily from the case of zonogons.

Here, we fix $\mathcal{C} = \mathbb{R}_+^2$, and $\boldsymbol{k} = (1,1)$, as these are the parameters used for the study of random polygons. The dual cone of $\mathbb{R}_+^2$ is $\mathbb{R}_-^2$, so the four parts of the circle $\mathbb{S}^1$ defined in the previous subsection are the four quarter circles corresponding to the quadrants of $\mathbb{R}^2$, with $S_+ = \mathbb{S}^1 \cap (\mathbb{R}_- \times \mathbb{R}_+$, and $S_- = \mathbb{S}^1 \cap (\mathbb{R}_+ \times \mathbb{R}_-$. With the norm 1, S_+ becomes $\{(t-1,t), t \in [0,1]\}$. The random process $(B_t^n) = (\widetilde{\mathbf{X}}^n_{(t-1,t)})_{0 \leq t \leq 1}$ describes the rescaled fluctuations of the upper arc of the random zonotope Z under $\mathrm{Q}_{(n,n)}$. The random process on S_- is centrally symmetrical to the one S_+ with respect to $(n/2, n/2)$, so the computations are the same.

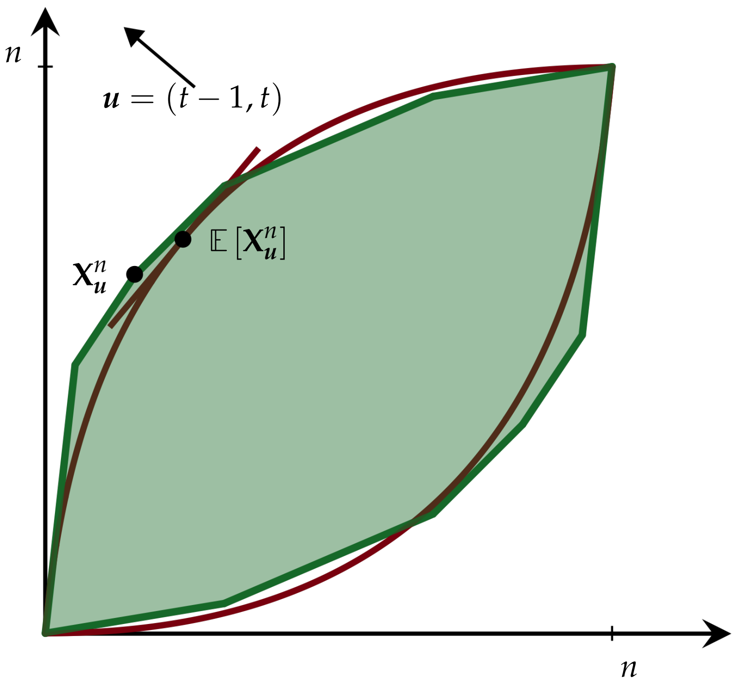


Figure 5.3: A zonogon in $\mathcal{Z}(\mathbb{R}_+^2, (n,n))$, its limit shape and the in the square $[0,n]^2$

Theorem 5.3.2. *Let Z be a random, uniform zonogon in $\mathcal{Z}(\mathbb{R}_+^2, (n,n))$. For $t \in [0,1]$, put $B_t^n = \widetilde{\mathbf{X}}^n_{(t-1,t)}$ with $\widetilde{\mathbf{X}}^n_{(t-1,t)}$ as defined in* (5.2.15)*. Then*

$$(B_t^n)_{t\in[0,1]} \overset{(d)}{\Rightarrow} (\mathbf{P}_t) \tag{5.3.14}$$

on the space $D[0,1]$ of càdlàg functions on $[0,1]$ equipped with the Skorokhod topology, where $(\mathbf{P}_t)$ is a Gaussian processus with covariance matrix given by (5.3.15)*. Alternatively, $(\mathbf{P}_t)$ can be written as $\mathbf{P}_t = P(t)D(t)\beta_t$, where $D(t)$ is a non-degenerate diagonal matrix, $P(t)$ is an orthogonal matrix and $(\beta_t)_{t\in[0,1]}$ is the 2-dimensional standard Brownian bridge.*

Proof. The weak convergence of the finite-dimensional distributions of B_t^n to those of $(\mathbf{P}_t)$ is given by Proposition 5.2.2, for $d = 2$. For $0 < t < 1$, the covariance of $\mathbf{P}_t$ is $(\Gamma^{-1}_{(t-1,t)} + \Gamma^{-1}_{(1-t,-t)})^{-1}$, that is

$$\mathrm{cov}(\mathbf{P}_t) = \left(\frac{\zeta(2)}{\zeta(3)}\right)^{1/3} \begin{pmatrix} -2\left(\left(4t^2-2t+1\right)t^3(t-1)\right. & \left(8t^2-8t+3\right)(t-1)^2t^2 \\ \left(8t^2-8t+3\right)(t-1)^2t^2 & -2\left(4t^2-6t+3\right)(t-1)^3t \end{pmatrix}. \tag{5.3.15}$$

Using the spectral theorem, we compute the orthogonal matrix $P(t)$ such that $P(t)\mathrm{cov}(\mathbf{P}_t)P(t)^\intercal$ is diagonal. We define the two polynomials $f(x) = -8x^2 - 8x - 1/2$ and $g(x) = 64x^4 + 128x^3 + 69x^2 + 19/2x + 1/16$, and we obtain

$$P(t)\mathrm{cov}(\mathbf{P}_t)P(t)^\intercal = t(1-t)\begin{pmatrix} Q_-(t) & 0 \\ 0 & Q_+(t)\end{pmatrix}, \tag{5.3.16}$$

where $Q_\pm(t) = f\left((t-\frac{1}{2})^2\right) \pm \sqrt{g\left((t-\frac{1}{2})^2\right)}$.

The tightness of $(B_t^n)_{0\le t\le 1}$ is a consequence of Proposition 5.3.1, using Theorem 13.5 in the book of Billingsley [21].

□

Remarkably, $Q_-(t)Q_+(t) = 3t(1-t)$. These functions are both symmetrical to $t = 1/2$, and Q_1 cancels out at 0 and 1 while Q_2 cancels out at $1/2$. We don't have any interpretation yet for these terms. We shall not give the explicit formula of the orthogonal matrix $P(t)$ here, but we display the asymptotic behavior at $t = 0$ and $t = 1$:

$$P(t) = \begin{pmatrix} -1+\frac{1}{8}t^2 & \frac{1}{2}t+\frac{1}{6}t^2 \\ \frac{1}{2}t+\frac{1}{6}t^2 & 1-\frac{1}{8}t^2 \end{pmatrix} + o(t^2), \tag{5.3.17}$$

$$P(t) = \begin{pmatrix} \frac{1}{2}(t-1)+\frac{1}{6}(t-1)^2 & -1+\frac{1}{8}(t-1)^2 \\ 1-\frac{1}{8}(t-1)^2 & \frac{1}{2}(t-1)+\frac{1}{6}(t-1)^2 \end{pmatrix} + o\left((t-1)^2\right).$$

5.3.4 Discussion on the shape around the limit

In the proof of the convergence of the variations of a polygon uniformly drawn in a square, we need to extend Theorem 5.3.2 to zonogons ending at $(n + rn^{2/3}, n + sn^{2/3})$. The following proposition handles this.

Proposition 5.3.2. *Let r, s be 2 real numbers and let Z be a random, uniform lattice zonogon in $\mathcal{Z}\left(\mathbb{R}^2_+, \left(\lfloor n + rn^{2/3} \rfloor \ , \lfloor n + sn^{2/3} \rfloor\right)\right)$. Let B^n_t be, as in Theorem 3, the rescaled position of the point of Z tangent to the hyperplane normal to $(t-1, t)$. Then*

$$(B^n_t)_{t\in[0,1]} \overset{(d)}{\Rightarrow} (\mathbf{P}_t + \boldsymbol{\nu}_{r,s}(t)) \tag{5.3.18}$$

in the space $D[0,1]$ of càdlàg functions on $[0,1]$ equipped with the Skorokhod topology, where $(\mathbf{P}_t)$ is the same process as in Theorem 5.3.2 and $\boldsymbol{\nu}_{r,s}(t)$ is a drift term given by (5.3.20).
Denoting $a = r - s$ and $b = r - 2s$, $(\boldsymbol{\nu}_{r,s}(t))_{0\le t\le 1}$ is a parametric cubic curve starting at $(0,0)$, ending at (r,s) and satisfying the equation:

$$4a(X+Y)^3 = 28b^3X + 7b^2(X+Y)^2 + 54abX(X+Y) + 27a^2X^2 \tag{5.3.19}$$

Proof. Both finite-dimensional distribution and tightness proofs are totally analogous to the proofs of Proposition 5.2.2 and 5.3.1. Given $\Gamma_{(t-1,t)}$ and $\Gamma_{-(t-1,t)}$ the covariances of $\widetilde{\mathbf{X}}_{(t-1,t)}$ and $\widetilde{\mathbf{X}}_{-(t-1,t)}$, the local limit theorem gives the following drift $\boldsymbol{\nu}_{r,s}(t)$:

$$\begin{aligned}\boldsymbol{\nu}_{r,s}(t) &= \left(\Gamma^{-1}_{(t-1,t)} + \Gamma^{-1}_{-(t-1,t)}\right)^{-1} \Gamma^{-1}_{-(t-1,t)} \begin{pmatrix} r \\ s \end{pmatrix} \\ &= \begin{pmatrix} t^2(2t-1) & -2t^2(t-1) \\ -2t(t-1)^2 & t(2t^2-5t+4) \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}\end{aligned} \tag{5.3.20}$$

All that remains is to analyze to find the equation of the curve $\boldsymbol{\nu}_{r,s}$. Notice that the coordinate of $\boldsymbol{\nu}_{r,s}$, $X(\boldsymbol{\nu}_{r,s})$ and $Y(\boldsymbol{\nu}_{r,s})$ are polynomials of t of degree 3, but their sum is a polynomial of degree 2, that is $X(\boldsymbol{\nu}_{r,s})(t) + Y(\boldsymbol{\nu}_{r,s})(t) = 3at^2 - 2bt$ where $a = r - s$ and $b = r - 2s$.

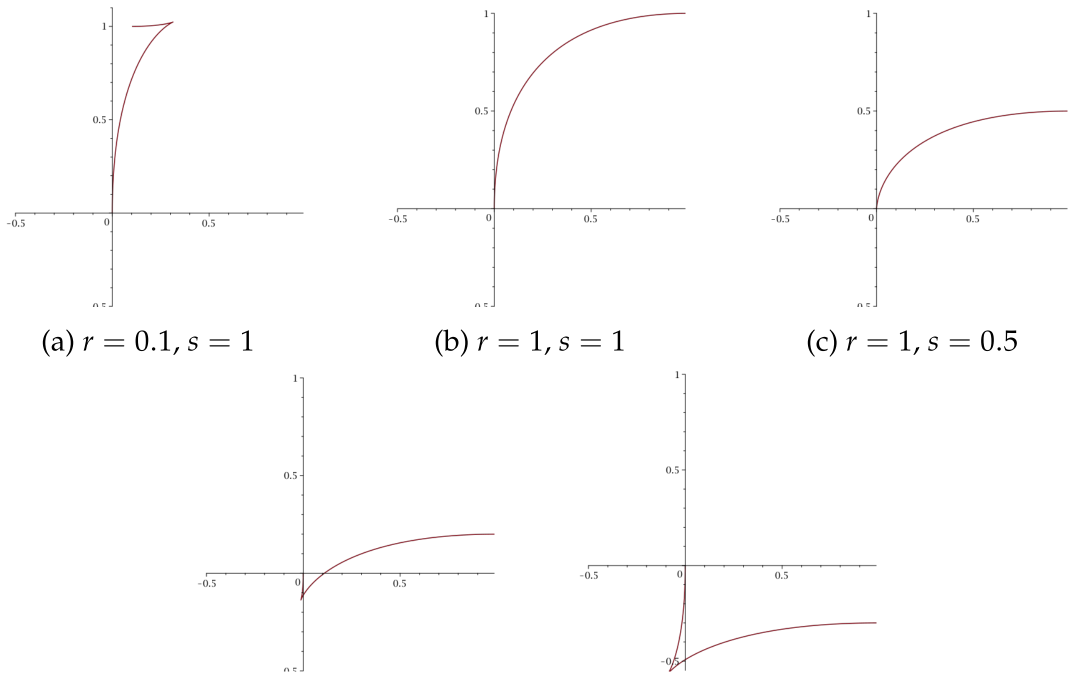
In order to get an equation of the curve, one can resolve the polynomial of degree 2 and inject the solution in $X(\boldsymbol{\nu}_{r,s})(t)$. After simplification, we obtain the equation (5.3.19)

$$4a(X+Y)^3 = 28b^3X + 7b^2(X+Y)^2 + 54abX(X+Y) + 27a^2X^2 \tag{5.3.21}$$

which is verified by $(X(\boldsymbol{\nu}_{r,s}), Y(\boldsymbol{\nu}_{r,s}))$. When $r = s$, the curve is the parabola that is the limit shape of a uniform integral zonogon ending at (r,s); otherwise, $\boldsymbol{\nu}_{r,s}(t)$ is a cubic curve. There are 2 different shapes, depending on $\frac{r}{s}$ belonging to $[1/2, 2]$ or not. The derivatives of $X(\boldsymbol{\nu}_{r,s})$ and $Y(\boldsymbol{\nu}_{r,s})$ with respect to t are:

$$X(\boldsymbol{\nu}_{r,s})'(t) = 6at^2 - 2bt, \quad \text{and } Y(\boldsymbol{\nu}_{r,s})'(t) = -6at^2 + (6a+2b)t - 2b \qquad (5.3.22)$$

If $\frac{r}{s} \in [1/2, 2]$, there is a cusp of multiplicity 2 at $t_0 = \frac{b}{3a} = \frac{r-2s}{3(r-s)}$. See Figure 5.4 for some examples of (r,s). □



(a) $r = 0.1, s = 1$ (b) $r = 1, s = 1$ (c) $r = 1, s = 0.5$

(d) $r = 1, s = 0.2$ (e) $r = 1, s = -0.4$

Figure 5.4: The curve $\boldsymbol{\nu}_{r,s}(t)$ for $0 \le t \le 1$, for different values of (r,s).

Remark 5.3.1. *The curve $\mu_{r,s}$ has a cusp if $-r/s \notin [1/2, 2]$. In particular, suppose that $r < 0$, $s > 0$ and $-r/s > 2$. Then $\boldsymbol{\mu}_{r,s}$ starts at $(0,0)$, ends in the positive quadrant, namely at $(-r, s)$, and yet for $t < (-r-2s)/3(-r-s)$, both coordinates of the speed $\boldsymbol{\mu}'_{r,s}(t)$ are negative. This may seem counter-intuitive.*

5.4 Brownian fluctuation of Large polygons

In the 2-dimensional case, zonogons are just centrally symmetric convex lattice polygons. Alternatively, any convex lattice polygon can be viewed as the union of four arcs of zonogons. Moreover, if we pick a uniform random convex lattice polygon contained in a large square, each of these arcs converges into an arc of parabola, as shown in the seminal papers of Barany [10], Sinai [139], and Vershik [151]. In this setting, our result on lattice zonotopes' fluctuations can be extended to convex lattice polygons' fluctuations.

In order to state a rigorous result, let us introduce the following notation. Consider a random, uniform, convex lattice polygon $\mathcal{P}_n$ contained in the square $[-n, n]^2$. Let $\mathcal{A}_n = (A_n, A'_n)$ be the southern-most segment of $\mathcal{P}_n$ and S_n be the "south pole", that is, $S_n = (0, -n)$. Both A'_n and A_n should be close to S_n and we would like to quantify this more precisely. Likewise, we can define $B_n, B'n, C_n, C'_n D_n, D'_n$, which should be close respectively to E_n, N_n, W_n where $E = (n, 0)$ etc. See Figure 5.5. Finally, we denote by $X : \mathbb{R}^2 \to \mathbb{R}$, resp. Y, the projection on the first (resp. second) coordinate.

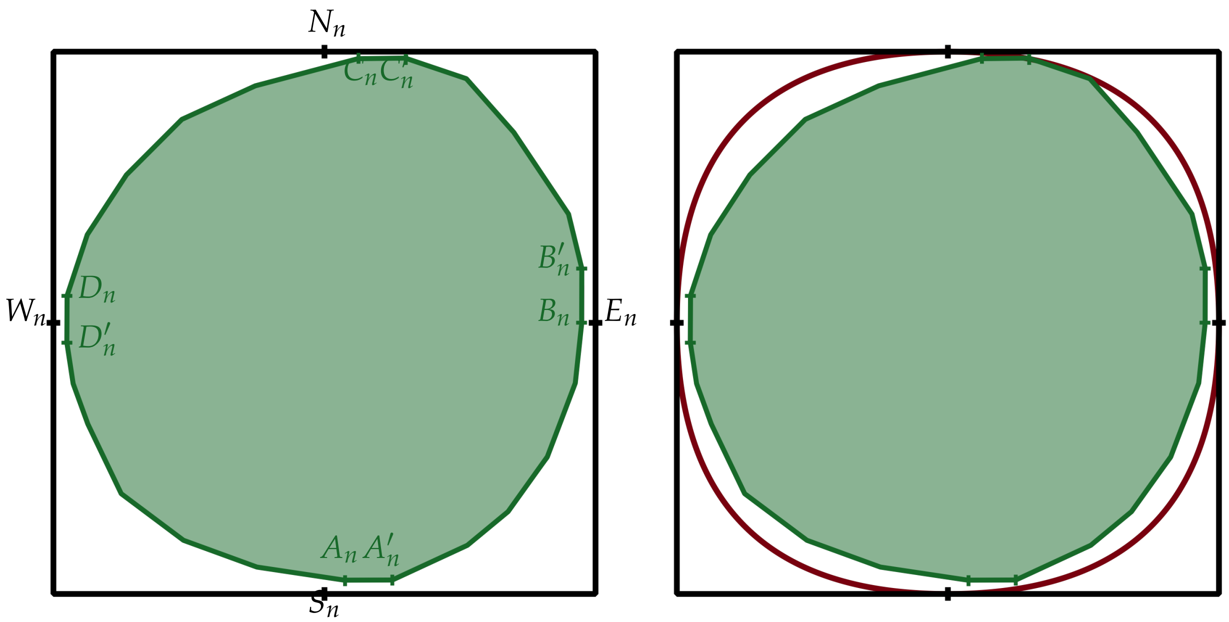


Figure 5.5: A random polygon in the square $[-n, n]^2$, where the extremal segments are labeled by their extremities, and the limit shape shown by Barany [10], Sinai [139], and Vershik [151]

Our result can be divided into two parts, the first one is dedicated to the distribution of the extremal points (A_n, B_n, C_n, D_n) with respect to (S_n, E_n, N_n, W_n), rescaled by factor $n^{2/3}$, and the second one is to use Proposition 5.3.2 for each polygonal arc. In the study of the distribution of extremal points, a few is to be said.

We first recall that at the first-order level, the polygon $\mathcal{P}_n$ converges to a deterministic shape, namely

- The distance $d(A_n, S_n)/n$ converges in distribution to 0 (and likewise for $d(B_n, E_n)/n$ etc.)
- The part of the boundary of $\mathcal{P}_n$ between A_n and B_n, converges, after renormalization to an arc of a parabola (and likewise between B_n and C_n, etc.)

A formal statement can be found in Barany [10].

Theorem 2 gives the second-order asymptotics. We shall need an estimate on the number of convex lattice chains from $(0,0)$ to $(n+\delta_n, n+\delta'_n)$, contained in $[0, n+\delta_n] \times [0, n+\delta'_n]$. The logarithm of this number is given by (see the proof of Lemma 2.1 in Bureaux-Enriquez [39])

$$\frac{\zeta(3)}{\zeta(2)}\frac{1}{\beta_1\beta_2} + \frac{1}{2i\pi}\int_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{\Gamma(s)\zeta(s+1)}{\zeta(s)}\chi(s;\boldsymbol{\beta})\, ds. \tag{5.4.1}$$

where

$$\chi(s;\boldsymbol{\beta}) := \sum_{\mathbf{v}\in\mathbb{Z}_+^2\setminus\{0\}} \frac{1}{(\boldsymbol{\beta}\cdot\mathbf{v})^s}, \qquad \Re(s) > 2. \tag{5.4.2}$$

and $\boldsymbol{\beta} = (\beta_1, \beta_2) = (1/(n+\delta_n), 1/(n+\delta'_n))$. Moreover, by the same lemma, for all nonnegative integers k_1, k_2, all $\epsilon > 0$ and all $\boldsymbol{\beta} = (\beta_1, \beta_2) \in (0, +\infty)^2$, such that $\epsilon < \frac{\beta_1}{\beta_2} < \frac{1}{\epsilon}$,

$$\frac{\partial^{k_1+k_2}}{\partial\beta_1^{k_1}\partial\beta_2^{k_2}} \log Z(\beta_1, \beta_2) \underset{\boldsymbol{\beta}\to 0}{\sim} (-1)^{k_1+k_2}\frac{\zeta(3)}{\zeta(2)}\frac{k_1!k_2!}{\beta_1^{k_1+1}\beta_2^{k_2+1}}. \tag{5.4.3}$$

Bureaux and Enriquez use these estimates to study the function χ near $(0,0)$ along the diagonal, that is, in the case when $\delta_n = \delta'_n = 0$. However, they point out that this can be extended to a more general setting. The case when δ_n and δ'_n are $o(n)$ corresponds to studying χ in the neighborhood of $(0,0)$ and near the diagonal. Using exactly the same arguments as in Lemma 2.2 leads to the following estimate. There exist $C, K > 0$ such that for all sequences $(\delta_n), (\delta'_n)$ satisfying $\delta_n = o(n)$, $\delta'_n = o(n)$, the number of convex lattice chains from $(0,0)$ to $(n+\delta_n, n+\delta'_n)$, contained in $[0, n+\delta_n] \times [0, n+\delta'_n]$ and without vertical steps (it is easy to see that this last additional condition on vertical steps does not change the form of the estimate) is given by

$$\exp(C(n+\delta_n)^{1/3}(n+\delta'_n)^{1/3} + H(n+\delta_n, n+\delta'_n) + K\log n + o(1)) \tag{5.4.4}$$

where H is a corrective term such that, uniformly over all $s, t \in \mathbb{R}$,

$$|H(n+sn^{2/3}, n+tn^{2/3})| = o(n^{1/3}) \tag{5.4.5}$$

On the other hand, by the same arguments, for fixed reals $s, t > 0$, the number of convex lattice chains from $(0,0)$ to (sn, tn), contained in $[0, sn] \times [0, tn]$ and such that (sn, tn) is the only point in the chain whose x-coordinate sn is given by

$$\exp(C(sn)^{1/3}(nt)^{1/3} + o(n^{1/3}) \tag{5.4.6}$$

We are now ready to prove the following lemma:

Lemma 5.4.1. *Let $\mathcal{P}_n$ be a uniform, random convex lattice polygon in $[-n,n]^2$, and (A_n, B_n, C_n, D_n), resp. (S_n, E_n, N_n, W_n) as defined above. Then* *(i) the quadruple*

$$n^{-2/3}(Y(A_n - S_n), X(B_n - E_n), Y(C_n - N_n), X(D_n - W'_n)) \tag{5.4.7}$$

converges in probability to $\delta_{(0,0,0,0)}$.

(ii) The quadruple

$$n^{-2/3}(X(A_n - S_n), Y(B_n - E_n), -X(C_n - N_n) - Y(D_n - W_n)) \tag{5.4.8}$$

converges in distribution to a Gaussian random variable (R, S, T, U) *with density*

$$C_1 \exp\left(-\frac{1}{18}[(r-s)^2 + (s-t)^2 + (t-u)^2 + (u-r)^2] - \frac{1}{3}[r^2 + s^2 + t^2 + u^2]\right) \tag{5.4.9}$$

for some normalizing constant $C_1 > 0$.

Proof. (i) We use the same arguments as in [10]. A convex lattice polygon can be seen as the union of 4 lattice convex chains. In particular, the number of polygons of this kind contained in $[-n, n]^2$ is at least the number of such polygons satisfying $A_n = S_n, B_n = E_n$ etc. which, according to (5.4.4), has the form

$$\exp(4Cn^{2/3} + 4K \log n + 4H(n, n) + o(1)) \tag{5.4.10}$$

If $c > 0$, if we want to count the number of convex lattice polygons contained in the rectangle $R_n := [-n, n] \times [-(n - cn^{(1/3)+\delta}), n]$, we can choose the points A_n, B_n, C_n, D_n and count the lattice chains in-between. Comparing (5.4.4) and (5.4.6), we see that this number is maximized when A_n, B_n, C_n, D_n lie near the middle of the segments of R_n. For such a choice of A_n, B_n, C_n, D_n, this number of convex lattice polygons is bounded above by

$$\exp(4Cn^{1/3}(n - (c/2)n^{2/3})^{1/3} + 4H(n, n - (c/2)n^{2/3}) + 4K \log n + o(1)) \tag{5.4.11}$$

On the other hand, the number of choices of (A_n, B_n, C_n, D_n) is bounded $(2n)^4$, so that the number of convex lattice polygons contained in R_n is bounded by

$$\exp(4Cn^{2/3} - \frac{2}{3}Ccn^{1/3} + 4H(n, n - (c/2)n^{2/3}) + (4K + 16) \log n + o(1)) \tag{5.4.12}$$

Using (5.4.5), we obtain

$$\frac{4H(n, n)}{-\frac{2}{3}Ccn^{1/3} + 4H(n, n - (c/2)n^{2/3})} \to \infty \tag{5.4.13}$$

as $n \to \infty$, which entails that the probability that a random, uniform convex lattice polygon contained $[-n, n]^2$ lies in R_n tends to 0 as $n \to \infty$. So for every ε, with probability going to 1, $n^{-2/3}|Y(A_n - S_n)| < \varepsilon$ and we have the same estimates for B_n, C_n, D_n.

(ii) From (i), we know that there exists a sequence (D_n) of integers such that $D_n = o(n^{2/3})$ and that with probability going to 1, $|Y(A_n) + n| < D_n$, $|X(B_n) - n| < D_n$ etc. Let now $(\delta_n, \delta'_n, \delta''_n, \delta'''_n)$ be sequences of integers such that for each n, $0 < \delta_n < D_n$ etc. We want to study the law of

$$n^{-2/3}(X(A_n - S_n), Y(B_n - E_n), -X(C_n - N_n) - Y(D_n - W_n)) \tag{5.4.14}$$

conditionally on the event

$$\mathbf{E}_n = \{Y(A_n - S_n) = \delta_n, X(B_n - E_n) = \delta'_n, Y(C_n - N_n) = \delta''_n, X(D_n - W'_n) = \delta'''_n\} \tag{5.4.15}$$

Fix $r, s, t, u \in \mathbb{R}$. For simplicity, we shall omit the integer part notation in the sequel. We want to estimate the number of 4-tuple of convex chains such that

$$X(A_n - S_n) = rn^{2/3}, Y(A_n - S_n) = \delta_n$$
$$Y(B_n - E_n) = sn^{2/3}, X(B_n - E_n) = \delta'_n$$

etc. Let $L_n(r, s, t, u)$ denote the logarithm of the number of such 4-tuples of chains. The number of chains going from A_n to B_n is the same as the number of chains from $(0, 0)$ to $(n - \delta'_n - rn^{2/3}, n - \delta_n + sn^{2/3})$. According to (5.4.4), this leads to

$$\begin{aligned} L_n(r,s,t,u) \;=\; & Cn^{2/3}\left(\left(1 - \frac{r}{3n^{1/3}} - \frac{2r^2}{9n^{2/3}} - \frac{\delta'_n}{n} + O(1/n)\right)\right. \\ & \times \left.\left(1 + \frac{s}{3n^{1/3}} - \frac{2s^2}{9n^{2/3}} - \frac{\delta_n}{n} + O(1/n)\right)\right) \\ & + H\left(n - rn^{2/3} - \delta'_n, n + sn^{2/3} - \delta_n\right) + k\log n + \dots \end{aligned} \tag{5.4.16}$$

where we only wrote the term corresponding to the chain from A_n to B_n, but of course, there are three other terms. Summing up, we see that terms of the form $r/n^{1/3}$ cancel out and we get

$$\begin{aligned} L_n(r,s,t,u) \;=\; & Cn^{2/3}\left(4 + 2\left(\frac{\delta_n}{n} + \frac{\delta'_n}{n} + \frac{\delta''_n}{n} + \frac{\delta'''_n}{n}\right) - \frac{4}{9}\left(\frac{r^2+s^2+t^2+u^2}{n^{2/3}}\right)\right. \\ & \left. - \frac{1}{9}\left(\frac{rs+st+tu+ur}{n^{2/3}}\right)\right) \\ & + 4K\log n + H\left(n - rn^{2/3} - \delta'_n, n + sn^{2/3} - \delta_n\right) + \dots + O(1/n) \end{aligned} \tag{5.4.17}$$

where again, in the last line, we have 3 additional terms involving the function H. In particular, we get

$$\frac{L_n(r,s,t,u)}{Cn^{2/3}} = \frac{L_n(0,0,0,0)}{Cn^{2/3}} - \frac{1}{18}[(r-s)^2 + (s-t)^2 + (t-u)^2 + (u-r)^2] - \frac{1}{3}[r^2+s^2+t^2+u^2] + \tag{5.4.18}$$

using the following fact that follows from (5.4.5):

$$|H\left(n - rn^{2/3} - \delta'_n, n + sn^{2/3} - \delta_n\right) - H\left(n - \delta'_n, n - \delta_n\right)| = o(n^{2/3}) \tag{5.4.19}$$

etc. This is true for all sequences $(\delta_n, \delta'_n, \delta''_n, \delta'''_n)$, which completes the proof.

□

Finally, we are ready to state Theorem 1.0.4:

Theorem 5.4.1. *Let $\mathcal{P}_n$ be a uniform, random convex lattice polygon in $[-n,n]^2$, and (A_n, B_n, C_n, D_n), resp. (S_n, E_n, N_n, W_n) as defined in the introduction of this section. Then*

(i) The quadruple

$$n^{-2/3}(Y(A_n - S_n), X(B_n - E_n), Y(C_n - N_n), X(D_n - W'_n)) \tag{5.4.20}$$

converges in probability to $\delta_{(0,0,0,0)}$.

(ii) The quadruple

$$n^{-2/3}(X(A_n - S_n), Y(B_n - E_n), -X(C_n - N_n) - Y(D_n - W_n)) \tag{5.4.21}$$

converges in distribution to a Gaussian random variable (R, S, T, U) with density

$$C_1 \exp\left(-\frac{1}{18}[(r-s)^2 + (s-t)^2 + (t-u)^2 + (u-r)^2] - \frac{1}{3}[r^2 + s^2 + t^2 + u^2]\right) \tag{5.4.22}$$

for some normalizing constant $C_1 > 0$.

(iii) For every $t \in [0,1]$, let $\mathbf{X}_n(t)$ be the point of the boundary of $\mathcal{P}_n$ with negative y-coordinate and tangent to the vector $(t, 1-t)$. Let $\overline{\mathbf{X}}_n(t) = \mathbb{E}(\mathbf{X}_n(t))$. Denote $\mathcal{E}_n(r,s)$, for each n, and for $r, s \in \mathbb{R}$, the event

$$\mathcal{E}_n(r,s) = \{\lfloor n^{-2/3}(X(A_n - S_n) \rfloor = r,\ \lfloor n^{-2/3}(Y(B_n - E_n) \rfloor = s\}. \tag{5.4.23}$$

Then there exists a continuous family of nonsingular matrices $(Q(t), 0 \le t \le 1)$ such that we have the convergence of conditional processes

$$\left(n^{-2/3}(\mathbf{X}_n(t) - \overline{\mathbf{X}}_n(t))|\mathcal{E}_n(r,s)), t \in [0,1]\right) \overset{(d)}{\to} \begin{pmatrix} r \\ 0 \end{pmatrix} + \left(\boldsymbol{\mu}_{r,s}(t) + Q(t)\beta_t, t \in [0,1]\right) \tag{5.4.24}$$

where (β_t) is a standard 2-dimensional Brownian bridge and $\boldsymbol{\mu}_{r,s}$ is a cubic curve parameterized by

$$\boldsymbol{\mu}_{r,s}(t) = \begin{pmatrix} -2t(t-1)^2 & t(2t^2 - 5t + 4) \\ t^2(2t-1) & -2t^2(t-1) \end{pmatrix} \begin{pmatrix} s \\ -r \end{pmatrix} \tag{5.4.25}$$

Proof. All that remains is to prove (iii). This part is a reformulation of Proposition 5.3.2 in Section 5.3. In (iii), We are looking at the southeast arc of the polygon, whereas Proposition 5.3.2 deals with what would be the northwest arc. Hence the identification

$$\boldsymbol{\mu}_{r,s}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\nu}_{-r,s}(t) \tag{5.4.26}$$

The matrix $Q(t)$ in Theorem 5.4.1 (iii) corresponds to

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P(t)D(t) \tag{5.4.27}$$

where $P(t)$ is the orthogonal matrix and $D(t)$ the diagonal matrix from Theorem 5.3.2. □

Remark 5.4.1. *A logical question raised by this result is to ask for the nature of the drift in higher dimensions. The drift should remain an algebraic surface, but we do not know the evolution of the degree of the surface with respect to the dimension d.*

Of course, (iii) is only stated for one of the four arcs of the polygon, but the result is true for each arc. Note that $\mathbf{X}_n(0) = A_n$ and $\mathbf{X}_n(1) = B_n$. The tangent point in Theorem 5.4.1 (iii) is defined by (5.2.1). Anyway, it follows from [33] that the edges of $\mathcal{P}_n$ have length of order $n^{\frac{1}{3}}$. Therefore, even if the set of tangent points is a whole edge, we could choose any point on this edge as $\overline{\mathbf{X}}_n(t)$ and because of the renormalizing factor $n^{-2/3}$, this would not change the result. For the same reason, we could replace (A_n, B_n, C_n, D_n) with (A'_n, B'_n, C'_n, D'_n) in the statement of the theorem.

We could re-express (iii) by saying that $(Q(t)\beta_t)$ is a 2-dimensional Gaussian process with a continuous family of covariance matrices (cov_t), defined by

$$\mathrm{cov}_t = \left(\frac{\zeta(2)}{\zeta(3)}\right)^{1/3} \begin{pmatrix} -2\left(4t^2-6t+3\right)(t-1)^3 t & \left(8t^2-8t+3\right)(t-1)^2 t^2 \\ \left(8t^2-8t+3\right)(t-1)^2 t^2 & -2\left(\left(4t^2-2t+1\right)t^3(t-1\right) \end{pmatrix} \tag{5.4.28}$$

In particular, for small t, the fluctuations of the process are of order $t^{1/2}$ in the x-coordinate and $t^{3/2}$ in the y-coordinate. We have similar estimates for t close to 1.

As a consequence, we could write informally that if $t \asymp n^{-\frac{1}{3}}$,

$$n\overline{\mathbf{X}}_n(t) \asymp \begin{pmatrix} n^{\frac{2}{3}} \\ n^{\frac{1}{3}} \end{pmatrix}, \quad n^{\frac{2}{3}}\mathbb{E}|Q(t)\beta_t| \asymp \begin{pmatrix} n^{\frac{1}{2}} \\ n^{\frac{1}{6}} \end{pmatrix}, \quad n^{\frac{2}{3}}\mu_{r,s}(t) \asymp \begin{pmatrix} n^{\frac{1}{3}} \\ 1 \end{pmatrix} \tag{5.4.29}$$

where $a_n \asymp b_n$ means that there exist two positive constants $c < C$ such that for each n, $cb_n < a_n < Cb_n$.

Note that in (iii), we state a conditional result. If we average over the law of $X(A_n - S_n), Y(B_n - E_n)$, we find that the mean of the asymptotic cubic curve is zero. That is, if we choose (R, S, T, U) according to the Gaussian distribution given in (ii), then for every $t \in [0, 1]$, $\mathbb{E}(\boldsymbol{\mu}_{R,S}(t)) = 0$.

The fact that for fixed r, s, the curve $\boldsymbol{\mu}_{r,s}$ is cubic is in sharp contrast with the usual situation where the drift is linear: if $(B_t, 0 \leq t \leq 1)$ is a Brownian motion started at 0 and conditioned to end at a, then (B_t) has the form $B_t = at + \beta_t$ where (β_t) is a Brownian bridge. In dimension 1, Brownian motion with a parabolic drift has been widely studied in connection with various problems such as statistical estimators, random partitions, epidemics models, Burgers turbulence etc. See for instance [91, 94, 149, 72, 77]. On the other hand, we are not aware of other instances of a cubic drift in the literature.

CHAPITRE 6

GEOMETRY: EQUIPROJECTIVE POLYHEDRA AND MINKOWSKI SUMS.

6.0.1 Introduction

The twenty problems gathered by G. Shephard may surprise some with their simplicity at first sight. These problems, along other simple problems of geometry, don't need much mathematical material to be stated, and a lot of them have not been solved yet. This mathematic area is often named intuitive geometry. In particular, these twenty problems were related to the combinatorics of (convex) polytopes [138, 137]. Among them, Question IX asks for a method to construct every *equiprojective polytope*, recalled by H. Croft, K. Falconer, and R. Guy in [47, Problem B10]. A 3-dimensional polytope P is k-equiprojective when its orthogonal projection on any plane is a polygon with k vertices, except for the planes that are orthogonal to a facet of P. To illustrate, the cube is a 6-equiprojective polytope, whereas a tetrahedron is not equiprojective as it can be projected onto a triangle or a quadrilateral. More generally, one can distinguish a first class of equiprojective polytopes in the prisms: a prism over a polygon with $k-2$ vertices is k-equiprojective.

About the construction of every equiprojective polytope, asked by Shephard, little is known. In 2008, M. Hasan and A. Lubiw figured out a characterization of equiprojective polytopes, along with an algorithm to recognize them [84]. More recently, some non-trivial equiprojective polytopes were computed by truncating Jonhson solids and by gluing them along a well-chosen facet [85]. Still, there is no clue about a general construction method. The present chapter presents the result of a work co-authored with Lionel Pournin [35], where we investigate the potential constructions of equiprojective polytopes using Minkowski sums.

The investigated problem is quite different from the studies carried out in the two previous chapters. In Chapters 4 and 5, the polytopes were lattice polytopes inscribed in a convex subset of $\mathbb{R}^d$, so parameters such as edge length or renormalization factor were essential. Here, translation, dilatation, and any kind of metric do not enter into consideration. Therefore, in this chapter, we will not consider the degenerate case in sets of generators: all the sets of generators will be considered pairwise non-collinear vector sets. Given a zonotope Z, the set of generators is still not unique because one can replace any generator with its opposite. Let $k \in \mathbb{N}^*$, and consider the set V of generators of Z*, such that $V = \{v_1, ..., v_k\}$. If V' is another set of generators of Z, then there exists $(\epsilon_1, ..., \epsilon_k) \in \{-1, 1\}^k$ such that

$$V' = \{\epsilon_1 v_1, ..., \epsilon_k v_k\}. \tag{6.0.1}$$

The aim of this chapter is two-fold: firstly, when is a Minkowski sum of polytopes equiprojective? Secondly, can these constructions give a lower bound for the number of "different" equiprojective polytopes? Here "different" can have numerous meanings, but we can at least investigate the number of combinatorial types of k-equiprojective polytopes, that is the number of k-equiprojective polytopes that do not have the same face lattice, which is the most meaningful differentiation criterion one can make about polytopes. Along the way, we establish a characterization of the equiprojective polytopes based on the notion of ***aggregated cones*** of a polytope along one of its edges direction.

In order to address these questions, the present chapter is divided into four parts.

In the first section, we will compute bounds for the number of combinatorial types of zonotopes that have n pairwise non-collinear generators. These bounds were computed by [76] and [4] in terms of order types. After introducing order types, we will show the connection between order types and zonotopes via oriented matroids. 3-dimensional zonotopes are $2k$-equiprojective polytopes, so this section gives a lower bound for the number of different combinatorial types of k-equiprojective polytopes for k even.

Then, we introduce the aggregated cone $C_P(u)$ of a polytope P given a vector u collinear to some edges of P. Roughly, an edge direction of a polytope P contained in $\mathbb{R}^3$ is just a vector u in the unit sphere $\mathbb{S}^2$ that is parallel to an edge of P and the aggregated cone of P at u is the union of the 2-dimensional normal cones of P that are contained in the orthogonal complement $u^\perp$ of u. We establish a characterization of equiprojective polytopes with their aggregated cones.

In a third section, we show that this characterization is particularly useful for computing the parameter k of a k-equiprojective polytope. Additionally, it also allows one to construct new equiprojective polytopes as the Minkowski sum of two polytopes, as the normal fan of a Minkowski sum is the common refinement of the normal fans of the summands.

Finally, for any odd number $k > 3$, we show a construction of a k-equiprojective polytope as a sum of Minkowski of a zonotope and a triangle to propagate the following theorem for odd k:

Theorem 6.0.1. *There are at least*

$$k^{k\left(\frac{3}{2}+O\left(\frac{1}{\log k}\right)\right)} \tag{6.0.2}$$

different combinatorial types of k-equiprojective polytopes.

We will conclude with Section 6.5, where a few questions in the spirit of Shephard's are stated about the decomposability of equiprojective polytopes.

6.1 The combinatorial types of zonotopes

Up to this point in the thesis, zonotopes are quite common objects. In previous chapters, we were concerned with lattice zonotopes which is not the case anymore. Here, the point of this section is to determine the number of combinatorial types of zonotopes in dimension 3, starting with recalling the definition of combinatorial types. Given a d-polytope P, remember that we denote $L(P)$ the set of its faces and that $(L(P), \subset)$ is a partially ordered set (poset) named the face lattice of P.

Definition 6.1.1. *Two polytopes are said to be* ***combinatorially equivalent*** *if their face lattices are isomorphic to each other (in terms of unlabeled partially ordered sets).*

An equivalence class under combinatorial equivalence is called a ***combinatorial type****.*

The combinatorial equivalence is trivially an equivalence relation. Concerning the zonotopes, the fact that we restrict the definition of a generator set to paiwise non-colinear vectors is strongly related to this equivalence relation. Given a zonotope Z, the set of pairwise non-collinear generators is not uniquely determined by Z, as we can change the sign of the generators arbitrarily. Yet all these sets share the same number of elements, and it will be important for the parameter of equiprojectivness.

The goal of this section is to estimate the number of combinatorial types of zonotopes in terms of their number of pairwise non-collinear generators. This will allow us to prove Theorem 6.0.1 when k is even thanks to the following statement, already noticed in [84], and trivial with their characterization of equiprojective polytopes, which we provide an alternative proof for.

Proposition 6.1.1. *A 3-dimensional zonotope Z is a k-equiprojective polytope, where k is twice the number of pairwise non-collinear generators of Z.*

Proof. Consider a 3-dimensional zonotope $Z \subset \mathbb{R}^d$, the set $V = \{v_1, ..., v_k\}$ of generators of Z, and a plane $\mathcal{H}$ that is not orthogonal to any face of Z. Let $\pi_{\mathcal{H}} : \mathbb{R}^3 \to \mathcal{H}$ be the orthogonal projection of $\mathbb{R}^3$ on the plane H.

Z is the Minkowski sum of the elements of V, so $\pi_{\mathcal{H}}(Z)$ is the Minkowski sum of the elements of $\pi_{\mathcal{H}}(V)$. Because $\mathcal{H}$ is not orthogonal to any face of Z, the vectors $\pi_{\mathcal{H}}(v_i)$ are pairwise non-collinear, and non-null. Therefore $\pi_{\mathcal{H}}(Z)$ is a $2k$-gone (in fact it is a zonogone with $2k$ edges).

□

6.1.1 Zonotopes and oriented matoids

In the introduction, we recalled in Proposition 2.3.2 that the combinatorial types of zonotopes are in bijections with realizable oriented matroids, by associating the oriented matroid derived from the configuration of the pairwise non-collinear vector set V to the combinatorial type of the zonotope Z generated by V [23, 155].

Recall from Proposition 2.3.3 that for $n \in \mathbb{N}^*$, a realizable oriented matroid is a pair $(E, \mathcal{V})$ where $E = \{1, ..., n\}$ and $\mathcal{V} \subset \{+, -, 0\}^n$, such that there exists a set of n vectors $X = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\} \in (\mathbb{R}^d)^n \setminus \{0\}$ for which

$$\mathcal{V} = \{\mathrm{sign}(X^\intercal\, z),\ z \in \mathbb{R}^d\}. \tag{6.1.1}$$

Conversely, given a set of n vectors $X = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\} \in (\mathbb{R}^d \setminus \{0\})^n$, we denote $(\{1, ..., n\}, \mathcal{V}_X)$ the oriented matroid naturally derived from the vector configuration X. Analogously to the combinatorial equivalence, we can define the *oriented matroid equivalence* between two sets of vectors. Two finite sets X and X' of vectors of $\mathbb{R}^d \setminus \{0\}$ are said to be oriented matroid equivalent if there is a permutation $\pi \in \mathfrak{S}_n$ such that, for $X_\pi = \{\boldsymbol{x}_{\pi^{-1}(1)}, ..., \boldsymbol{x}_{\pi^{-1}(n)}\}$,

$$\mathcal{V}_{X_\pi} = \mathcal{V}_{X'}. \tag{6.1.2}$$

The following statement is proven in [23] (see Corollary 2.2.3 therein). It provides the announced correspondence between the combinatorial type of a zonotope and the oriented matroids of its sets of generators.

Proposition 6.1.2. *Denote Z and Z' two zonotopes, then the three following statements are equivalent:*

- *Z and Z' have the same combinatorial type.*
- *For every set V of generators of Z, there exists a set V' of generators of Z' such that V and V' are oriented matroid equivalent.*
- *For one set V of generators of Z, there exists a set V' of generators of Z' such that V and V' are oriented matroid equivalent.*

Proof. The equivalence of the two last statements comes from the construction of the sets of generators. Remember that for two sets $V_1 = (\boldsymbol{u}_1, ...\boldsymbol{u}_n)$ and $V_2 = (\boldsymbol{v}_1, ..., \boldsymbol{v}_n)$ that are sets of generators of the zonotope Z, then there exists a permutation $\pi\mathfrak{S}_n$ such that for any $\boldsymbol{u}_i \in V_1$, $\boldsymbol{u}_i = \epsilon_i \boldsymbol{v}_i$ with $\epsilon \in \{-1, 1\}$. □

6.1.2 Order types

According to Proposition 6.1.2, the number of combinatorial types of d-dimensional zonotopes with n generators is exactly the number of oriented matroids of a d-dimensional configuration of n vectors from $\mathbb{R}^d \setminus \{0\}$. In order to compute estimates of this number, one more step is needed as estimates have been given by J. Goodman and R. Pollack in [76] and by N. Alon in [4] in for order types. The following is

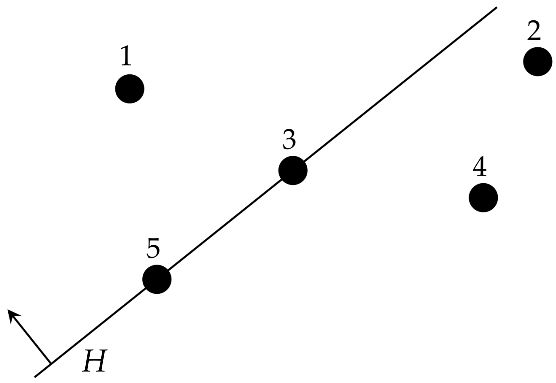


Figure 6.1: For five points crossed by a hyperplane H, the sign vector associated with H is $(-,+,0,+,0)$.

a quick introduction to order types in order to exhibit the natural bijection between order types and realizable oriented matroids introduced in the previous section.

The order types were introduced by J. E. Goodman and R. Pollack in [75] in 1983. In an ordered set, for example $\mathbb{R}$, the order type of a set S of n elements is the relative position between each other, that is the (commonly known) order of the elements of S. The order type of a vector configuration $X = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ in $\mathbb{R}^d$ generalizes this notion of the relative position of each element with respect to the others. Let H be an oriented hyperplane in $\mathbb{R}^d$, in the sense that there is a positive side and a negative side of H. Therefore H defines a sign vector σ in $\{+,-,0\}^n$, where $\sigma_i = +$ if $\boldsymbol{x}_i$ is on the positive side of H, $\sigma_i = 0$ if $\boldsymbol{x}_i$ is in H, and $\sigma_i = +$ if $\boldsymbol{x}_i$ is on the negative side of H (see Figure 6.1). The collection of the possible sign vectors given by the hyperplanes in $\mathbb{R}^d$ defines the order type of X [75, 65, 74].

Let's define it formally: let $X = \{\boldsymbol{x}_1, ..., \boldsymbol{x}_n\}$ be a vector configuration in $\mathbb{R}^d$ and consider a vector $\boldsymbol{c}_d \in \mathbb{R}^d \setminus \{0\}$, a number c_{d+1}, and the affine hyperplane $H = \{\boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \cdot \boldsymbol{c}_d + c_{d+1} = 0\}$. The sign vector defined by H with respect to X is therefore given by $\text{sign}(\boldsymbol{x}_i \cdot \boldsymbol{c}_d + c_{d+1})$ for each coordinate i. Remark that we can simplify the notation by considering $d+1$-dimensional vectors and linear hyperplanes. With $\boldsymbol{v}_i = (\boldsymbol{x}_i, 1) \in \mathbb{R}^{d+1}$, the configuration $V = (\boldsymbol{v}_1, ..., \boldsymbol{v}_n) \in (\mathbb{R}^{d+1} \setminus \{0\})^n$ is a set of n $(d+1)$-dimensional vectors, and $\boldsymbol{c} = (\boldsymbol{c}_d, c_{d+1})$, and the collection of sign vectors $\mathcal{V}(X)$ of the configuration X is

$$\mathcal{V}(X) = \{\text{sign}(V^\intercal \boldsymbol{c}),\ \boldsymbol{c} \in \mathbb{R}^{d+1}\}. \tag{6.1.3}$$

Definition 6.1.2. *With the above notations, the configurations X and Y of n vectors are order equivalent if there is a permutation $\pi \in \mathfrak{S}_n$ and an bijection $f : \mathbb{R}^d \to \mathbb{R}^d$ such that, for $X' = (X_{\pi(i)})_i$, $X' = f(Y)$ and $\mathcal{V}(X') = \mathcal{V}(Y)$.*

*The (unlabeled) **order type** of X is the equivalence class of X for the order equivalence.*

The two equations (6.1.1) and (6.1.3) show that the pair $(\{1, .., n\}, \mathcal{V}(X))$ is a oriented matroid, where the collection $\mathcal{V}(X)$ of sign vectors coincides with the collection of sign covectors of the configuration V. The configuration V is called *acyclic*, because there exists a vector $\boldsymbol{c} \in \mathbb{R}^{d+1}$ such that $\boldsymbol{c}^\intercal \boldsymbol{v}_i > 0$ for every i (here the last coordinate of $\boldsymbol{v}_i$ is 1 so we can choose $\boldsymbol{c} = (0, ..., 0, 1)$). By extension, the oriented matroid derived from the acyclic configuration V is named an acyclic oriented matroid. It is well known (and maybe straightforward from both definitions) that there exists a one-to-one correspondence between order types and equivalent classes of acyclic realizable oriented matroids, which are the oriented matroids defined from an acyclic vector configuration.

Lemma 6.1.1. *There is a natural bijection between order types of n vectors in $\mathbb{R}^d$ and equivalence classes of acyclic oriented matroids on a set of n $(d+1)$-dimensional vectors.*

Proof. With $\mathcal{M}_{\{1,...,n\}}$ being the class of acyclic oriented matroids on $\{1, ..., n\}$, denote $F : \{\mathcal{V}(X), X \in \mathbb{R}^{n\times d}\} \to \mathcal{M}_{\{1,...,n\}}$ the function that associates the order type of a vector configuration X with the oriented matroid equivalent class of $(\{1, ..., n\}, \mathcal{V}(X))$. The injectivity is a direct consequence of Definition 6.1.2.

Given an acyclic configuration V of n vectors in $\mathbb{R}^{d+1}$, there exists $\boldsymbol{c} \in \mathbb{R}^{d+1}$ such that for all $\boldsymbol{v} \in V$, $\boldsymbol{c}^\intercal \boldsymbol{v} > 0$. We rewrite V in an orthonormal basis whose last element is $-\frac{\boldsymbol{c}}{||\boldsymbol{c}||}$. For any $\boldsymbol{v}_i = (v_i^1, ..., v_i^{d+1}) \in V$, we define $\boldsymbol{x}_i = (x_i^1, ..., x_i^d) \in \mathbb{R}^d$ such that $x_i^j = \frac{\boldsymbol{v}_i^j}{\boldsymbol{v}_i^{d+1}}$, and $X = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n)$. Then the collections $\mathcal{V}(X)$ and $\mathcal{V}_V$ respectively defined by equations (6.1.3) and (2.3.16) satisfy $\mathcal{V}(X) = \mathcal{V}_V$. We can conclude that the equivalent class of the oriented matroid derived from V is the image of the order type of X by F, which gives the surjectivity.

□

As a consequence of Lemma 6.1.1, Theorem 4.1 from [4] can be rephrased as follows in terms of equivalent classes of oriented matroids.

Theorem 6.1.1. *The number $t(n, d)$ of equivalent classes of acyclic oriented matroids of sets of n vectors that span $\mathbb{R}^d$ satisfies*

$$\left(\frac{n}{d-1}\right)^{(d^2-2d)n\left(1+O\left(\frac{\log(d-1)}{\log n}\right)\right)} \leq t(n,d) \leq \left(\frac{n}{d-1}\right)^{(d^2-2d)n\left(1+O\left(\frac{\log\log(n/(d-1))}{d\log(n/(d-1))}\right)\right)} \tag{6.1.4}$$

when n/d goes to infinity.

Observe that the lower bound in that statement is established in [76, Section 5], but that the theorems in [4] and [76] are stated in terms of *labbeled* order types, that is in correspondence with oriented matroids. In order to have Theorem 6.1.1, it is sufficient to divide by $n!$. Its proof is surprisingly intuitive as it is a direct consequence of a formula given by T. Zaslavsky [154] that counts the number of regions determined by hyperplane arrangements. We can't resist stating it here:

Proof of the lower bound. Consider a configuration of n points of $\mathbb{R}^d$ for which there is no hyperplane containing more than d points. Therefore there are $\binom{n}{d}$ hyperplanes determined by this configuration. Zaslavsky proved that these hyperplanes "cut" $\mathbb{R}^d$ into

$$\binom{\binom{n}{d}}{d} + \binom{\binom{n}{d}}{d-1} + ... + \binom{\binom{n}{d}}{0} - n\binom{\binom{n-1}{d-1} - 1}{d} \tag{6.1.5}$$

connected components [154, p.65], which correspond to the number of possibilities of adding a new point in order to extend the configuration into a $n+1$-point configuration. Each of these configurations has a different order type because each of these connected component have a different relative position with respect to the n points.

Asymptotically, the sum (6.1.5) is in the order of its first term when n/d grows large, and the latter is $O(n^{d^2}/(k!)^{d+1})$. Any n-tuple of points can be constructed by recursively choosing a connected component of the hyperplane arrangement of k first points. Then, the number of order types is the number $N_{n,d}$ of possible recursive constructions, divided by the number of labeling possibilities, namely $n!$. We can conclude with the computation of $N_{n,d}$ with Stirling's formula:

$$N_{n,d} = O\left(\frac{((n-1)!)^{d^2}}{(d!)^{(d+1)(n-1)}}\right) = O\left(\left(\frac{n}{d}\right)^{d^2 n + O\left(d^2 n \frac{\log(d-1)}{\log(n)}\right)}\right). \tag{6.1.6}$$

□

Remember that the set of pairwise non-collinear generators that generates a zonotope Z is not unique because we can replace each generator with its opposite, hence there is always an acyclic configuration that generates Z, for any zonotope Z. Proposition 6.1.2 and Theorem 6.1.1 make it possible for one to estimate the number of combinatorial types of zonotopes depending on their dimension and their number of pairwise non-collinear generators. In the context of equiprojective polytopes, only the dimension $d = 3$ is needed; hence the following theorem is stated for d fixed, compared to Theorem 6.1.1 that is stated for n/d growing large.

Theorem 6.1.2. *The number $z(n, d)$ of combinatorial types of d-dimensional zonotopes with n generators satisfies*

$$n^{(d^2-2d)n\left(1+O\left(\frac{1}{\log n}\right)\right)} \leq z(n, d) \leq n^{(d^2-2d)n\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)}. \tag{6.1.7}$$

when d is fixed and n goes to infinity.

Proof. Theorem 6.1.1 states that

$$n^{(d^2-2d)n\left(1+O\left(\frac{1}{\log n}\right)\right)} \leq t(n, d) \leq n^{(d^2-2d)n\left(1+O\left(\frac{\log\log n}{\log n}\right)\right)}. \tag{6.1.8}$$

when d is fixed and n goes to infinity. Hence, it suffices to show that

$$\frac{t(n,d)}{2^n} \leq z(n,d) \leq t(n,d), \tag{6.1.9}$$

From Proposition 6.1.2, it comes that given a zonotope Z and its set of generators V, the oriented matroids induced by V can not be associated with another combinatorial type than that of Z. The inequality $z(n,d) \leq t(n,d)$ follows with Lemma 6.1.1.

Let Z be a zonotope with n pairwise non-collinear generators. Then Z admits 2^n sets of generators depending, in each direction parallel to an edge of Z, which vector we take between the two that have the size of the aforementioned edge. Therefore there are at most 2^n equivalent classes of oriented matroids associated with the combinatorial type of Z, which gives the left-hand side inequality $t(n,d) \leq 2^n z(n,d)$. □

Theorem 6.0.1 in the special case of the k-equiprojective polytopes such that k is even follows from Proposition 6.1.1 and from Theorem 6.1.2.

6.2 Aggregated cones

In order to prove Theorem 6.0.1 when k is odd, we will exhibit a construction of equiprojective polytopes using Minkowski sum. The first step to achieve so, is a characterization of the equiprojective polytopes compatible with Minkowski sum, stated in the introduction as Theorem 6.2.2. This characterization is based on another one by M. Hasan and A. Lubiw.

6.2.1 The theorem of M. Hasan and A. Lubiw

The starting point of our study of equiprojective polytopes is the article by Masud Hasan and Anna Lubiw [84]. We recall here their result to In the following definition, an *edge-facet incidence* of a 3-dimensional polytope P is a pair (e, F) such that F is a facet of P and e an edge of F.

Definition 6.2.1. *Two edge-facet incidences (e, F) and (e', F') of a 3-dimensional polytope P* compensate *when e and e' are parallel, and either*

(i) F and F' coincide or

(ii) F and F' are distinct but parallel facets of P whose relative interiors are on the same side of the plane that contains e and e'.

With these definitions, they prove a characterization of the equiprojective polytopes with edge-facet incidence. The following theorem is Theorem 1 from [84].

Theorem 6.2.1. *A 3-dimensional polytope is equiprojective if and only if the set of its edge-facet incidences can be partitioned into compensating pairs.*

The proof of this theorem relies on the detailed study of the evolution of the projection when changing the projection direction. More precisely, when looking at the variations of the projection direction around a direction parallel to a face, the fixed number of edges of the projected polygon implies this matching. Based on this characterization, we now look at the consequences on the normal cones of P.

6.2.2 Aggregated cones and equiprojectiveness

Now consider a polytope P contained in $\mathbb{R}^d$ of dimension possibly less than d. Recall from the introduction that, given a face F of P, the *normal cone of P at F* is defined as

$$N_P(F) = \{u \in \mathbb{R}^d : \forall (x,y) \in P{\times}F,\ u{\cdot}x \leq u{\cdot}y\}. \tag{6.2.1}$$

The normal cone of a j-dimensional face of P is a $(d-j)$-dimensional closed polyhedral cone. Here, P is a 3-dimensional polytope and it is embedded in $\mathbb{R}^3$; hence the normal cones of P at its edges are 2-dimensional. Likewise, the normal cone of a 2-dimensional face is a ray normal to the face.

The normal fan of P, which is the set of all normal cones of P, is denoted $\mathcal{N}(P)$. Given a vector $u \neq 0$, we define here the aggregated cone of P at a vector u that is the union of all normal cones of $\mathcal{N}(P)$ that are contained in the plane orthogonal to u. The linear plane orthogonal to u is denoted $u^\perp$.

Definition 6.2.2. *Consider a polytope P and a non-zero vector u, both contained in* $\mathbb{R}^3$. *The* aggregated cone $C_P(u)$ *of P at u is the union of the* 2*-dimensional normal cones of P contained in the plane* $u^\perp$.

In the above definition, notice that if there is no edge of P parallel to u, then $C_P(u)$ is empty. For this reason, we are only really interested in the vectors parallel to edges of a polytope and, more precisely, in having a set of pairwise non-collinear vectors such that any edge e of P is parallel to one of these vectors. Therefore we define a unique vector for each direction as follows

Definition 6.2.3. *Consider a polytope P contained in* $\mathbb{R}^3$ *(of dimension possibly less than* 3*). A vector u from* $\mathbb{S}^2$ *is an* edge direction *of P when*

(i) the first non-zero coordinate of u is positive and
(ii) there is an edge of P whose difference of vertices is a multiple of u.

Remark 6.2.1. *The aggregated cone is purposely chosen as a union of 2-dimensional cones. Theorem 6.2.2 would still be correct if the aggregated cone was the union of any normal cone of P contained in* $u^\perp$. *Yet with Definition 6.2.2 and Definition 6.2.3, any polytope P has a finite number of edge direction u and a finite number of non-empty aggregated cones* $C_P(u)$, *which gives better clarity for the following proofs and results.*

Note that edge directions could be defined indifferently as a finite subset of the real projective plane $\mathbb{RP}^2$. To illustrate aggregated cones, let us take a dodecahedron P in Figure 6.2. For each edge e, there is an opposite edge e' that corresponds to the only edge parallel to e. Hence e and e' share the same edge-direction u.

The notion of an aggregated cone at an edge direction is illustrated in Figure 6.2. This figure shows a regular dodecahedron P and two of its opposite edges e and e' that correspond to a single edge direction u.

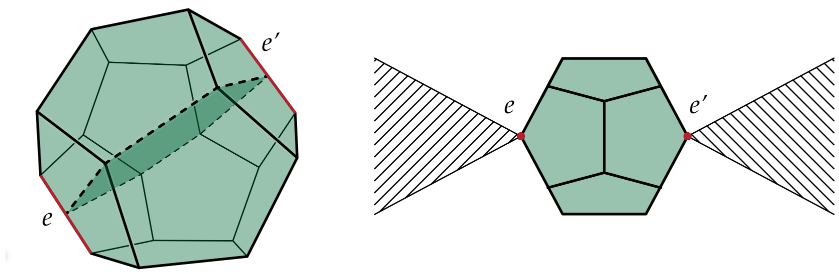


Figure 6.2: A regular dodecahedron P (left) and the aggregated cone (striped on the right) at the edge direction corresponding to the opposite edges e and e' of P.

On the left of the figure, we show the intersection of P with $u^\perp$ outlined with dashed lines if P is thought of as centered at the origin of $\mathbb{R}^3$. On the right of the figure, P is viewed from the edge direction u and the normal cone of P at e and at e' are drawn translated such that their apices and the projection of e and e' in $u^\perp$ coincide. $C_P(u)$ is the 2-dimensional cone obtained by translating the striped cones back so that their apices coincide with the origin of $\mathbb{R}^3$. In particular, the dodecahedron illustrates that $C_P(u)$ is not always a convex cone. However, by definition, it is always the union of finitely-many closed convex cones.

Let $\mathcal{E}_P(u)$ be the smallest set of closed convex cones whose union is $C_P(u)$. Each of these cones corresponds to one connected component of $C_P(u) \cap \mathbb{S}^2$. Let E be an element of $\mathcal{E}_P(u)$. It is important to remember that E may be the union of several normal cones of P. In fact, if two 2-dimensional normal cones in $C_P(u)$ share the same half-line boundary, then they are normal cones of P at edges of edge direction u that belong to the same facet. Hence the half-lines that bound normal cones of P and that do not bound E are the normal cones of P at facets that have two edges of edge direction u. The half-lines bounding E are the two normal cones of P at facets that admit exactly one edge of normal direction u.

Conversely, the closure of $u^\perp \backslash C_P(u)$ is also the union of finitely-many 2-dimensional closed convex cones contained in $u^\perp$ and we denote by $\mathcal{V}_P(u)$ the smallest such set of cones.

In order to prove Theorem 6.2.2 that characterizes equiprojectivity in terms of aggregated cones, we first shall prove the following lemma. The observations previously made about $\mathcal{E}_P(u)$, together with Theorem 6.2.1 lead to:

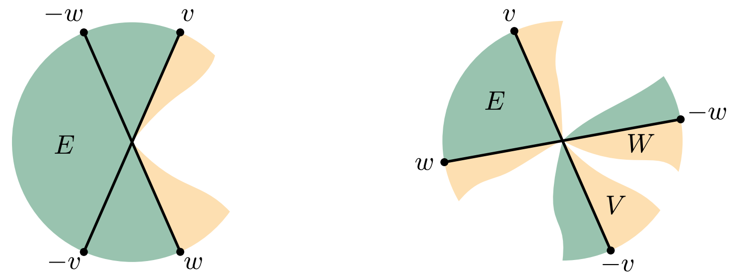


Figure 6.3: An illustration of the proof of Lemma 6.2.1.

Lemma 6.2.1. *Consider a 3-dimensional equiprojective polytope P and an edge direction u. If there exist two edges e and e' of edge direction u such that the intersection of the relative interior of $N_P(e)$ and $-N_P(e')$ is not null, then $C_P(u) = u^\perp$.*

Proof. Assume that P is equiprojective and consider an edge direction u of P and two edges e and e' of P of edge direction u such that the intersection $\mathring{N}_P(e) \cap -\mathring{N}_P(e')$ is not the empty set. Let $E \in \mathcal{E}_P(u)$ be the cone of $\mathcal{E}_P(u)$ that contain $N_P(e)$. Similarly, define $E' \in \mathcal{E}_P(u)$ for e'.

Suppose that $E \neq u^\perp$. Then E is bounded by two half-lines whose vertex is the origin (it may be that the union of these two half-lines is a straight line through the origin in the case when E is a half-plane). Consider the two unit vectors v and w that span these half lines. As discussed above, v and w are normal vectors to two facets F and G of P, respectively, such that both F and G admit a unique edge whose difference of vertices is a multiple of u. Denote these edges of F and G by e_F and e_G, respectively.

Since P is equiprojective, it follows from Theorem 6.2.1 that the edge-facet incidence (e_F, F) must be compensated, but since F doesn't have another edge parallel to e_F, there must exist a facet F' of P distinct from but parallel to F and an edge $e_{F'}$ of F' parallel to e_F such that the relative interiors of F and F' are on the same side of the plane that contains these two edges. Now observe that F' cannot have another edge parallel to $e_{F'}$. Indeed the edge-facet incidence formed by such an edge with F' couldn't be compensated by any other edge-facet incidence than $(e_{F'}, F')$ but this one already compensates (e_F, F). It follows that the half-line spanned by $-v$ is the boundary between a cone V contained in $\mathcal{V}_P(u)$ and a cone in $\mathcal{E}_P(u)$. In particular, E must be contained in a half-plane. Indeed, otherwise, $-v$ would be in the relative interior of E and it couldn't belong to the boundary of V as shown on the left of Figure 6.3. The cone E is then as shown on the right of the figure.

Since the relative interiors of F and F' are on the same side of the plane that contains e_F and $e_{F'}$, the cone in $\mathcal{E}_P(u)$ that contains $-v$ in its relative boundary must be on the same side than E of the line spanned by v and $-v$ while V must be on the opposite side, as shown on the right of Figure 6.3 (where the cones that belong to $\mathcal{E}_P(u)$

are colored green and the ones that belong to $\mathcal{V}_P(u)$ yellow). As a consequence, $-V$ either contains E or is contained in it. By the same argument, $-w$ is contained in the relative boundary of a cone W from $\mathcal{V}_P(u)$ that lies on the opposite side of the line spanned by w and $-w$ than E.

Let us come back to E'. The opposite of the relative interior of E' has a non-null intersection with E. It can be seen in Figure 6.3 that E' lies between V and W. However, $-E'$ would be contained in E, which is impossible because we proved that such a cone must contain or be contained in the opposite of a cone from $\mathcal{V}_P(u)$. This proves that $E = u^{\perp}$.

□

We can do the exact same reasoning to obtain the following result:

Lemma 6.2.2. *If P is a 3-dimensional equiprojective polytope and u an edge direction such that $\mathcal{V}_P(u)$ is not empty, then for N_1 and N_2 two elements of $\mathcal{V}_P(u)$ (they can be equal), the relative interior of the intersection between N_1 and $-N_2$ is null.*

The consequence of Lemma 6.2.1 and Lemma 6.2.2 is the following: given an 3-dimensional equiprojective polytope P and an edge direction u, then either $C_P(u)$ is $u^{\perp}$, either $C_P(u)$ is null, either $-C_P(u)$ is equal to the reunion of the elements of $\mathcal{V}_P(u)$.

6.2.3 Characterization of equiprojective polytopes

With Lemma 6.2.1, we can prove a characterization of equiprojective polytopes with their aggregated cones:

Theorem 6.2.2. *A 3-dimensional polytope P is equiprojective if and only if, for any edge direction u of P, either*

(i) the aggregated cone of P at u is equal to $u^{\perp}$ or
(ii) the aggregated cone of P at u and the relative interior of the opposite of that cone form a partition of $u^{\perp}$.

Proof. Let us prove the two implications separately. Assume that P is an equiprojective polytope, and consider an edge direction u. The sufficiency stems from Lemma 6.2.1 and 6.2.2.

The necessity is proven in the following lemma. It will be proven via Theorem 6.2.1 by constructing an explicit partition of the set of all edge-facet incidences into compensating pairs. □

Lemma 6.2.3. *If, for any edge direction u of a 3-dimensional polytope P,*

(i) the aggregated cone of P at u is equal to $u^{\perp}$ or
(ii) the aggregated cone of P at u and the relative interior of the opposite of that cone form a partition of $u^{\perp}$,

then P is equiprojective.

Proof. Assume that the condition in the statement of the lemma holds for any edge direction u of a 3-dimensional polytope P. We will partition the edge-facet incidences of P into compensating pairs and the desired result will follow from Theorem 6.2.1. Consider a facet F of P and an edge e of F. Denote by u the edge direction of P that is a multiple of the difference between the vertices of e. We consider two mutually exclusive cases.

First, assume that e is the only edge of F whose difference of vertices is a multiple of u. In that case, $C_P(u)$ is not equal to $u^{\perp}$ and $N_P(F)$ is one of the half-lines bounding $C_P(u)$. As $C_P(u)$ and the relative interior of $-C_P(u)$ form a partition of $u^{\perp}$, the half-line $-N_P(F)$ is also one of the half-lines bounding $C_P(u)$. Therefore, P has a facet F' parallel to and distinct from F that has a unique edge e' whose difference of vertices is a multiple of u. Moreover, the cones in $\mathcal{E}_P(u)$ bounded by $N_P(F)$ and $-N_P(F)$ lie on the same side of the line $N_P(F) \cup [-N_P(F)]$. As a consequence, the relative interiors of F and F' lie on the same side of the plane that contains e and e'.

This shows that (e', F') compensates (e, F) and we assign these two edge-facet incidences to a pair in the announced partition. Observe that the same process starting with (e', F') instead of (e, F) would have resulted in the same pair of compensating edge-facet incidences of P.

Now assume that F has an edge e' other than e whose difference of vertices is a multiple of u. In that case, (e, F) and (e', F) compensate and we assign these two edge-facet incidences to a pair of the announced partition. Again, if we had started with (e', F') instead of (e, F), this would have resulted in the same pair of edge-facet incidences of P. Hence, repeating this process for all the edge-facet incidences of P allows to form a partition of the edge-facet incidences of P into compensating pairs, as desired. □

Remark 6.2.2. *This theorem gives two new perspectives compared to Theorem 6.2.1. Since in practice, computing the faces of a Minkowski sum of polytopes is done via their normal fans, Theorem 6.2.2 provides a way to prove Theorem 6.0.1 in the case when k is odd. More generally, Theorem 6.2.2 allows to construct new classes of equiprojective polytopes. Moreover, the value k of a k-equiprojective polytope directly stems from the characterization with aggregated cones. The following section will address these two points.*

In Figure 6.4, both equiprojective polytopes are drawn with one color for each edge direction. The black, the aggregated cones of P at the purple and blue edges are equal to the whole plane, while the aggregated cones of P at the other colors are half-planes.

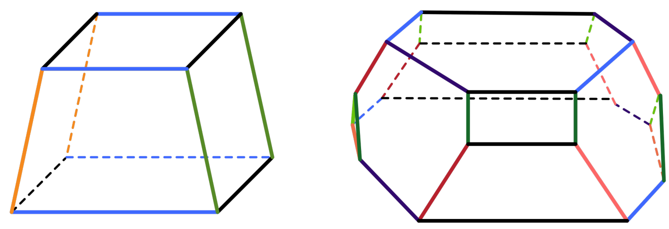


Figure 6.4: A prism and a equitruncated triangular cupola from [85].

6.3 Equiprojectivity and Minkowski sums

In this section, we exploit Theorem 6.2.2 in order to study the equiprojectivity of Minkowski sums. We will also focus on the value of k for which a polytope or a Minkowski sum of polytopes is k-equiprojective.

Definition 6.3.1. *Consider an edge direction u of a polytope P contained in $\mathbb{R}^3$. The* multiplicity *$\mu_P(u)$ of u as an edge direction of P is equal to 2 when $C_P(u)$ coincides with $u^\perp$ and to 1 when it does not.*

Note that, in Definition 6.3.1, P can be a 3-dimensional polytope, a polygon or a line segment. From now on, we denote by $\kappa(P)$ the value of k such that an equiprojective polytope P is k-equiprojective.

Theorem 6.3.1. *If P is an equiprojective polytope, then*

$$\kappa(P) = \sum \mu_P(u) \tag{6.3.1}$$

where the sum is over the edge directions u of P.

Proof. Consider an equiprojective polytope P contained in $\mathbb{R}^3$ and a hyperplane H that is not orthogonal to any of the facets of P. We can assume without loss of generality that H is through the origin of $\mathbb{R}^3$ by translating it if needed. Denote by $\pi : \mathbb{R}^3 \to H$ the orthogonal projection on H.

Since H is not orthogonal to a facet of P, the edges of $\pi(P)$ are the image by π of edges of P. Moreover, the edges of P whose orthogonal projection on H are edges of $\pi(P)$ are precisely the edges e of P such that the relative interior of $N_P(e)$ intersects H. Now consider an edge direction u of P. According to Theorem 6.2.2, two cases are possible. In the first case, $C_P(u)$ is equal to $u^\perp$ and, in that case, exactly two of the normal cones of P contained in $C_P(u)$ intersect H in their relative interiors. In the second case, $C_P(u)$ and the relative interior of $-C_P(u)$ form a partition of $u^\perp$ and exactly one of the normal cones of P contained in $C_P(u)$ intersects H in its relative interior. By the definition of the multiplicity of an edge direction, this proves the theorem. □

Now recall that the faces of the Minkowski sum of two polytopes P and Q can be recovered from the normal cones of these polytopes (see for instance Proposition 7.12 in [155]): the faces of $P+Q$ are precisely the Minkowski sums of a face F of P with a face G of Q such that the relative interiors of $N_P(F)$ and $N_Q(G)$ are non-disjoint. For this reason, Theorem 6.2.2 provides a convenient way to determine how equiprojectivity behaves under Minkowski sums.

Theorem 6.3.2. *Let P and Q each be an equiprojective polytope, a polygon, or a line segment contained in $\mathbb{R}^3$. The Minkowski sum $P+Q$ is equiprojective if and only if for each edge direction u shared by P and Q, either*

(i) $C_P(u)$ or $C_Q(u)$ is equal to $u^\perp$ or
(ii) $C_P(u)$ coincides with $C_Q(u)$ or with $-C_Q(u)$.

Proof. Pick an edge direction u of $P+Q$. Note that u must then be an edge direction of P or Q. According to Proposition 7.12 in [155],

$$C_{P+Q}(u) = C_P(u) \cup C_Q(u). \tag{6.3.2}$$

It will be important to keep in mind that the aggregated cones of a polygon or a line segment at their edge directions always are planes or half-planes.

If u is not an edge direction for both P and Q, then by (6.3.2), $C_{P+Q}(u)$ is equal to $C_P(u)$ or to $C_Q(u)$. Theorem 6.2.2 and the above remark on polygons and line segments then imply that $C_{P+Q}(u)$ is either equal to the plane $u^\perp$ or forms a partition of that plane with the relative interior of $-C_{P+Q}(u)$. Hence, if one of the assertions (i) or (ii) holds for every edge direction u shared by P and Q, then $P+Q$ must be equiprojective by Theorem 6.2.2.

Now assume that $P+Q$ is equiprojective and assume that u is an edge direction shared by P and Q such that neither $C_P(u)$ or $C_Q(u)$ is equal to $u^\perp$. According to Theorem 6.2.2, $C_{P+Q}(u)$ is either equal to $u^\perp$ or, together with the relative interior of $-C_{P+Q}(u)$, it forms a partition of $u^\perp$. By the same theorem and the above remark on polygons and line segments, the same holds for the aggregated cones of P and Q at u. As $C_P(u)$ and $C_Q(u)$ are not equal to $u^\perp$, the sum of the arc lengths of the circular arcs contained in $C_P(u) \cap \mathbb{S}^2$ and $C_Q(u) \cap \mathbb{S}^2$ is the circumference of the unit circle $u^\perp \cap \mathbb{S}^2$. Moreover, the sum of the arc lengths of the circular arcs contained in $C_{P+Q}(u) \cap \mathbb{S}^2$ is either equal to the circumference of $u^\perp \cap \mathbb{S}^2$ or to half of it. Hence according to (6.3.2), $C_P(u)$ necessarily coincides with or $C_Q(u)$ or with $-C_Q(u)$, as desired. □

Recall that the multiplicity of an edge direction is defined indifferently for a 3-dimensional polytope, a polygon, or a line segment. From now on, if P is a polygon or a line segment contained in $\mathbb{R}^3$, we denote

$$\kappa(P) = \sum \mu_P(u) \tag{6.3.3}$$

in analogy to Theorem 6.3.1, where the sum is over the edge directions u of P. It should be noted that when P is a polygon, $\kappa(P)$ is equal to the number of edges of P and when P is a line segment, $\kappa(P)$ is always equal to 2.

If P and Q are two polytopes contained in $\mathbb{R}^3$ that each can be an equiprojective polytope, a polygon, or a line segment, then we further denote

$$\lambda(P, Q) = k + 2k' \tag{6.3.4}$$

where k' is the number of edge directions u common to P and Q such that both $C_P(u)$ and $C_Q(u)$ are equal to $u^\perp$ while k is the number of edge directions u common to P and Q such that at least one of the cones $C_P(u)$ or $C_Q(u)$ is distinct from $u^\perp$ and is contained in the other one.

Theorem 6.3.3. *Consider two polytopes P and Q contained in $\mathbb{R}^3$, that each can be an equiprojective polytope, a polygon, or a line segment. If the Minkowski sum $P + Q$ is an equiprojective polytope, then*

$$\kappa(P + Q) = \kappa(P) + \kappa(Q) - \lambda(P, Q). \tag{6.3.5}$$

Proof. Assume that $P + Q$ is equiprojective and consider an edge direction u of $P + Q$. Note that u is then an edge direction of P or an edge direction of Q. As already mentioned in the proof of Theorem 6.3.2,

$$C_{P+Q}(u) = C_P(u) \cup C_Q(u). \tag{6.3.6}$$

Hence, if u is an edge direction of P but not one of Q, then

$$\mu_{P+Q}(u) = \mu_P(u) \tag{6.3.7}$$

and if u is an edge direction of Q but not P, then

$$\mu_{P+Q}(u) = \mu_Q(u). \tag{6.3.8}$$

If P and Q share u as an edge direction, we review the different possibilities given by Theorem 6.3.2 for how $\mu_{P+Q}(u)$, $\mu_P(u)$ and $\mu_Q(u)$ relate to one another. If both the aggregated cones $C_P(u)$ and $C_Q(u)$ are equal to $u^\perp$, then $\mu_P(u)$, $\mu_Q(u)$ and $\mu_{P+Q}(u)$ are all equal to 2. Hence,

$$\mu_{P+Q}(u) = \mu_P(u) + \mu_Q(u) - 2. \tag{6.3.9}$$

If $C_P(u)$ and $C_Q(u)$ are not both equal to $u^\perp$, but one of these aggregated cones is contained in the other, then

$$\mu_{P+Q}(u) = \max\{\mu_P(u), \mu_Q(u)\}. \tag{6.3.10}$$

Moreover, the smallest value between $\mu_P(u)$ and $\mu_Q(u)$ is equal to 1. Hence,

$$\mu_{P+Q}(u) = \mu_P(u) + \mu_Q(u) - 1. \tag{6.3.11}$$

Finally, if $C_P(u)$ and $C_Q(u)$ are opposite and not equal to $u^\perp$, then $\mu_P(u)$ and $\mu_Q(u)$ are both equal to 1 while $\mu_{P+Q}(u)$ is equal to 2. Therefore,

$$\mu_{P+Q}(u) = \mu_P(u) + \mu_Q(u). \tag{6.3.12}$$

The result is then obtained from Theorem 6.3.1 by summing (6.3.7), (6.3.8), (6.3.9), (6.3.11), and (6.3.12) when u ranges over the corresponding edge directions of $P + Q$. □

Observe that when two polytopes P and Q do not share an edge direction, $\lambda(P, Q)$ vanishes. Hence, we obtain the following corollary from Theorems 6.3.2 and 6.3.3. In turn, that corollary immediately implies Corollary 6.3.2.

Corollary 6.3.1. *Consider two polytopes P and Q contained in $\mathbb{R}^3$, that each can be an equiprojective polytope, a polygon, or a line segment. If P and Q do not share an edge direction, then $P + Q$ is equiprojective and*

$$\kappa(P + Q) = \kappa(P) + \kappa(Q). \tag{6.3.13}$$

Corollary 6.3.2. *Consider a 3-dimensional polytope P obtained as a Minkowski sum of finitely many polygons. If each set of edge directions of those polygons is distinct from one another, then P is equiprojective.*

6.4 Many k-equiprojective polytopes when k is odd

The goal of this section is to build many k-equiprojective polytopes when k is a large enough odd integer. This will be done using the Minkowski sum between a zonotope with $(k-3)/2$ generators and a well-chosen triangle. In this section, all the considered polytopes are contained in $\mathbb{R}^3$.

For any 3-dimensional zonotope Z, we will consider a triangle t_Z whose edge directions do not belong to any of the planes spanned by two edge directions of Z and such that the plane spanned by the edge directions of t_Z does not contain an edge direction of Z. Note that such a triangle always exists because Z and t_Z have only finitely many edge directions. A consequence of our requirements on t_Z is that this triangle does not share any edge direction with Z. Moreover, the normal cones of Z at its facets are never contained in the normal cone of t_Z at itself or in the plane spanned by the normal cone of t_Z at one of its edges. Inversely, the normal cone of t_Z at itself is not contained in the plane spanned by the normal cone of Z at any of its edges. It should be noted that there are many possible choices for t_Z but we fix that choice for each zonotope Z so that $Z \mapsto t_Z$ defines a map that sends a zonotope to a triangle.

It follows from the results of Section 6.3 that the Minkowski sum of Z with t_Z is always an equiprojective polytope.

Lemma 6.4.1. *If Z is a 3-dimensional zonotope with n generators, then its Minkowski sum with t_Z is a $(2n+3)$-equiprojective polytope.*

Proof. Recall that the edge directions of $Z + t_Z$ are precisely the edge directions of Z and the edge directions of t_Z. Since Z is a zonotope, the aggregated cones at all of its edge directions are planes. By Proposition 6.1.1, Z is $2n$-equiprojective where n is the number of generators of Z.

Now recall that when P is a polygon, $\kappa(P)$ is equal to the number of edges of P. As a consequence, $\kappa(t_Z)$ is equal to 3 and since Z does not share an edge direction with t_Z, it follows from Corollary 6.3.1 that the Minkowski sum of Z with t_Z is a $(2n+3)$-equiprojective polytope. □

Let us recall that the set of the normal cones of a polytope P ordered by reverse inclusion form a lattice $\mathcal{N}(P)$, called the *normal fan of P* and that

$$N_P : \mathcal{F}(P) \to \mathcal{N}(P) \tag{6.4.1}$$

is an isomorphism. This correspondence between the face lattice and the normal fan of a polytope will be useful in order to determine the combinatorial type of the Minkowski sum between a 3-dimensional zonotope Z and its associated triangle t_Z. Recall that a face F of $Z + t_Z$ can be uniquely written as the Minkowski sum of a face of Z with a face of t_Z. We will denote by $\tau_Z(F)$ the face of t_Z that appears in this Minkowski sum.

Note that τ_Z defines a morphism from the face lattice of $Z + t_Z$ to that of t_Z in the sense that it preserves face inclusion. This is a consequence, for instance of Proposition 7.12 from [155]. In this context, an *isomorphism* refers to a bijective morphism between the face lattice of two polytopes.

Lemma 6.4.2. *Consider two 3-dimensional zonotopes Z and Z'. If*

$$\psi : \mathcal{F}(Z + t_Z) \to \mathcal{F}(Z' + t_{Z'}) \tag{6.4.2}$$

is an isomorphism, then there exists an isomorphism

$$\phi : \mathcal{F}(t_Z) \to \mathcal{F}(t_{Z'}) \tag{6.4.3}$$

such that $\phi \circ \tau_Z$ is equal to $\tau_{Z'} \circ \psi$.

Proof. Assume that ψ is an isomorphism from the face lattice of $\mathcal{F}(t_Z)$ to that of $\mathcal{F}(t_{Z'})$. First observe that $Z + t_Z$ has exactly two parallel triangular facets each of which is a translate of t_Z. Further observe that all the other facets of $Z + t_Z$ are centrally-symmetric, since they are the Minkowski sum of a centrally-symmetric polygon with a line segment or a point. The same observations hold for the Minkowski sum of Z' with $t_{Z'}$: that Minkowski sum has exactly two parallel triangular facets, each a translate of $t_{Z'}$ and its other facets all are centrally-symmetric. As ψ induces an isomorphism between the face lattices of any facet F of $Z + t_Z$ and the face lattice of $\psi(F)$, this shows that ψ sends the two triangular facets of $Z + t_Z$ to the two triangular facets of $Z' + t_{Z'}$. Moreover, two parallel edges of a centrally-symmetric facet of $Z + t_Z$ are sent by ψ to two parallel edges of a centrally-symmetric facet of $Z' + t_{Z'}$.

Recall that all the aggregated cones of t_Z are half-planes. As Z and t_Z do not have edge direction in common, the aggregated cones of $Z + t_Z$ at the edge directions of t_Z are still half-planes. Note that the two half-lines that bound these half-planes are precisely the normal cones of $Z + t_Z$ at its triangular facets. Similarly, the aggregated cones $Z' + t_{Z'}$ at the edge directions of $t_{Z'}$ are half-planes bounded by the normal cones of $Z' + t_{Z'}$ at its triangular facets. Since two parallel edges of a centrally-symmetric facet of $Z + t_Z$ are sent by ψ to two parallel edges of a centrally-symmetric facet of $Z' + t_{Z'}$, this implies that for every edge e of t_Z, there exists an edge $\phi(e)$ of $t_{Z'}$ such that any face of $Z + t_Z$ contained in $\tau_Z^{-1}(\{e\})$ is sent to $\phi(e)$ by $\tau_{Z'} \circ \psi$.

Through the correspondence between face lattice and normal fans,

$$\overline{\psi} = N_{Z'+t_{Z'}} \circ \psi \circ N^{-1}_{Z+t_Z} \tag{6.4.4}$$

provides an isomorphism from $\mathcal{N}(Z + t_Z)$ to $\mathcal{N}(Z' + t_{Z'})$. Consider two edges e and f of t_Z and denote by x the vertex they share. Let $\mathcal{P}$ be the set of the normal cones of $Z + t_Z$ at its two triangular faces and at the faces contained in $\tau_Z^{-1}(\{e\})$ and in $\tau_Z^{-1}(\{f\})$. Similarly, let $\mathcal{P}'$ denote the set made up of the normal cones of $Z' + t_{Z'}$ at its triangular faces and at the faces from $\tau_{Z'}^{-1}(\{\phi(e)\})$ and $\tau_{Z'}^{-1}(\{\phi(f)\})$. By the above, $\overline{\psi}(\mathcal{P})$ is equal to $\mathcal{P}'$. Moreover, it follows from Proposition 7.12 in [155] that

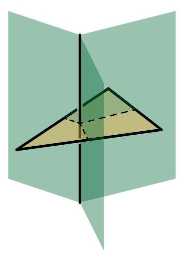


Figure 6.5: A triangle in $\mathbb{R}^3$ (colored yellow) and its normal cones at edges (colored green) and at itself (vertical line).

$\mathcal{P}$ and $\mathcal{P}'$ are polyhedral decompositions of the boundaries of the normal cone of t_Z at x and of the normal cone of $t_{Z'}$ at the vertex $\phi(x)$ shared by $\phi(e)$ and $\phi(f)$. As $\overline{\psi}$ is an isomorphism from $\mathcal{N}(Z+t_Z)$ to $\mathcal{N}(Z'+t_{Z'})$, it must then send the normal cones of $Z+t_Z$ at the faces from $\tau_Z^{-1}(\{x\})$ to the normal cones of $Z'+t_{Z'}$ at the faces from $\tau_{Z'}^{-1}(\{\phi(x)\})$. In other words, any face of $Z+t_Z$ contained in $\tau_Z^{-1}(\{x\})$ is sent to $\phi(x)$ by $\tau_Z \circ \psi$. Hence, setting $\phi(t_Z)$ to $t_{Z'}$ results in the desired isomorphism ϕ. □

Lemma 6.4.2 states, in other words, that if there exists an isomorphism

$$\psi : \mathcal{F}(Z+t_Z) \to \mathcal{F}(Z'+t_{Z'}) \tag{6.4.5}$$

then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(Z+t_Z) & \xrightarrow{\ \psi\ } & \mathcal{F}(Z'+t_{Z'}) \\ \downarrow{\scriptstyle \tau_Z} & & \downarrow{\scriptstyle \tau_{Z'}} \\ \mathcal{F}(t_Z) & \xrightarrow{\ \phi\ } & \mathcal{F}(t_{Z'}) \end{array}$$

where ϕ is an isomorphism. Recall that a face F of $Z+t_Z$, where Z is an arbitrary 3-dimensional zonotope is the Minkowski sum of a unique face of Z with $\tau_Z(F)$. We will denote by $\zeta_Z(F)$ the face of Z that appears in this sum. This defines a morphism ζ_Z from $\mathcal{F}(Z+t_Z)$ to $\mathcal{F}(Z)$. We derive the following from our requirements for the choice of t_Z.

Proposition 6.4.1. *Consider a 3-dimensional zonotope Z and two proper faces F and G of $Z+t_Z$. If F is a facet of G, then either*

(i) $\zeta_Z(G)$ coincides with $\zeta_Z(F)$ or
(ii) $\tau_Z(G)$ coincides with $\tau_Z(F)$.

Proof. Let us first assume that G is a facet of $Z+t_Z$ and F an edge of G. By our choice for t_Z, all the polygonal faces of $Z+t_Z$ are the Minkowski sum of a facet of Z with a vertex of t_Z, of a vertex of Z with t_Z itself, or of an edge of Z with an edge of t_Z. If $\zeta_Z(G)$ is a polygon and $\tau_Z(G)$ a vertex of t_Z then it is immediate that F is the Minkowski sum of an edge of $\zeta_Z(G)$ with $\tau_Z(G)$ and it follows that $\tau_Z(F)$ coincides with $\tau_Z(G)$. Similarly, if $\zeta_Z(G)$ is a vertex of Z and $\tau_Z(G)$ is equal to t_Z, then $\tau_Z(F)$ must be an edge of t_Z and $\zeta_Z(F)$ is equal to $\zeta_Z(G)$. Now if $\zeta_Z(G)$ is an edge of Z and $\tau_Z(G)$ an edge of t_Z, then observe that G is a parallelogram. As F is an edge of G, it must be a translate of either $\zeta_Z(G)$ or $\tau_Z(G)$. In the former case, $\zeta_Z(F)$ is equal to $\zeta_Z(G)$ and in the latter, $\tau_Z(F)$ is equal to $\tau_Z(G)$, as desired.

Finally, assume that G is an edge of $Z+t_Z$ and F is a vertex of G. Recall that Z and t_Z do not share an edge direction. As a consequence, either $\zeta_Z(G)$ is an edge of Z and $\tau_Z(G)$ a vertex of t_Z or inversely, $\zeta_Z(G)$ is a vertex of Z and $\tau_Z(G)$ an edge of t_Z. In the former case, $\tau_Z(F)$ is equal to $\tau_Z(G)$ and in the latter $\zeta_Z(F)$ coincides with $\zeta_Z(G)$, which completes the proof. □

We can now prove the following statement similar to that of Lemma 6.4.2.

Lemma 6.4.3. *Consider two* 3*-dimensional zonotopes Z and Z'. If*

$$\psi : \mathcal{F}(Z + t_Z) \to \mathcal{F}(Z' + t_{Z'}) \tag{6.4.6}$$

is an isomorphism, then there exists an isomorphism

$$\theta : \mathcal{F}(Z) \to \mathcal{F}(Z') \tag{6.4.7}$$

such that $\theta \circ \zeta_Z$ is equal to $\zeta_{Z'} \circ \psi$.

Proof. Consider a proper face F of Z and let us show that all the faces of $Z + t_Z$ contained in $\zeta_Z^{-1}(\{F\})$ have the same image by $\zeta_{Z'} \circ \psi$. Assume for contradiction that this is not the case and recall that by Proposition 7.12 from [155], the normal cone $N_Z(F)$ is decomposed into a polyhedral complex by the normal cones of $Z + t_Z$ it contains. Since $N_Z(F)$ is convex and

$$N_{Z'+t_{Z'}} : \mathcal{F}(Z' + t_{Z'}) \to \mathcal{N}(Z' + t_{Z'}) \tag{6.4.8}$$

is an isomorphism, $\zeta_Z^{-1}(\{F\})$ must contain two faces P and Q whose images by $\zeta_{Z'} \circ \psi$ differ and such that the normal cone of $Z + t_Z$ at Q is a facet of the normal cone of $Z + t_Z$ at P. By the correspondence between the face lattice of a polytope and its normal fan, P is a facet of Q. Now recall that

$$P = F + \tau_Z(P) \tag{6.4.9}$$

and

$$Q = F + \tau_Z(Q). \tag{6.4.10}$$

It follows that $\tau_Z(P)$ must differ from $\tau_Z(Q)$ as P and Q would otherwise be equal. Hence, by Lemma 6.4.2, $\psi(P)$ and $\psi(Q)$ have different images by $\tau_{Z'}$. As they also have different images by $\zeta_{Z'}$ and as $\psi(P)$ is a facet of $\psi(Q)$, this contradicts Proposition 6.4.1. As a consequence, all the faces of $Z + t_Z$ contained in $\zeta_Z^{-1}(\{F\})$ have the same image by $\psi \circ \zeta_{Z'}$. In other words, there exists a face $\theta(F)$ of Z' such that ψ sends $\zeta_Z^{-1}(\{F\})$ to a subset of $\zeta_{Z'}^{-1}(\{\theta(F)\})$.

However, observe that

$$\left\{\zeta_Z^{-1}(\{F\}) : F \in \mathcal{F}(Z)\right\} \tag{6.4.11}$$

is a partition of $\mathcal{F}(Z + t_Z)$. Similarly, the sets $\zeta_{Z'}^{-1}(\{F\})$ where F ranges within $\mathcal{F}(Z')$ form a partition of $\mathcal{F}(Z' + t_{Z'})$. As ψ is a bijection, it must then send $\zeta_Z^{-1}(\{F\})$ precisely to $\zeta_{Z'}^{-1}(\{\theta(F)\})$. It follows that θ is a bijection from $\mathcal{F}(Z)$ to $\mathcal{F}(Z')$ such that $\theta \circ \zeta_Z$ is equal to $\zeta_{Z'} \circ \psi$, as desired.

Finally, if G is a face of Z and F a face of G, then observe that some face of $Z + t_Z$ from $\zeta_Z^{-1}(\{F\})$ must be contained in a face from $\zeta_Z^{-1}(\{G\})$. As $\zeta_{Z'} \circ \psi$ is a morphism, this shows that $\theta(F)$ is a face of $\theta(G)$. Hence θ is an isomorphism from the face lattice of Z to that of Z'. □

Just like with Lemma 6.4.2 the statement of Lemma 6.4.3 can be rephrased using a commutative diagram: it tells that, if there exists an isomorphism ψ from $\mathcal{F}(Z + t_Z)$ to $\mathcal{F}(Z' + t_{Z'})$, then there is a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(Z + t_Z) & \xrightarrow{\psi} & \mathcal{F}(Z' + t_{Z'}) \\ \zeta_Z \big\downarrow & & \zeta_{Z'} \big\downarrow \\ \mathcal{F}(Z) & \xrightarrow{\quad\theta\quad} & \mathcal{F}(Z') \end{array}$$

where θ is an isomorphism.

The following is an immediate consequence of Lemma 6.4.3.

Lemma 6.4.4. *Consider two 3-dimensional zonotope Z and Z'. If the Minkowski sums $Z + t_Z$ and $Z' + t_{Z'}$ have the same combinatorial type, then the zonotopes Z and Z' have the same combinatorial type.*

Consider an odd integer k greater than or equal to 9 (under this assumption, there exist zonotopes with $(k-3)/2$ generators). It follows from Lemmas 6.4.1 and 6.4.4 that the number of different combinatorial types of k-equiprojective polytopes is at least the number of zonotopes with $(k-3)/2$ generators. Hence, we can finally state the following

Theorem 6.0.1 in the case of the k-equiprojective polytopes such that k is odd follows from Theorem 6.1.2 and from the observations that

$$\frac{k-3}{\log \frac{k-3}{2}} = O\left(\frac{k}{\log k}\right) \tag{6.4.12}$$

and

$$\left(\frac{k-3}{2}\right)^{\frac{3}{2}(k-3)} = k^{k\left(\frac{3}{2}+O\left(\frac{1}{\log k}\right)\right)} \tag{6.4.13}$$

as k goes to infinity.

6.5 Equiprojective polytopes and decomposability

Our results show that Minkowski sums allow for a sequential construction of equiprojective polytopes, thus providing a partial answer to Shephard's original question. Two of the possible kinds of summands in these Minkowski sums, line segments and polygons, are well understood. However, equiprojective polytopes can also appear as a summand, which leads to asking about the primitive building blocks of these sequential Minkowski sum constructions. More precisely, recall that a polytope P is *decomposable* when it can be written as a Minkowski sum of two polytopes, none of whose is homothetic to P [54, 97, 114, 121, 125, 136, 141]. We ask the following question, in the spirit of Shephard's.

Question 6.5.1. *Are there indecomposable equiprojective polytopes?*

It is a consequence of Lemma 6.2.3 that any 3-dimensional polytope whose aggregated cones at all edge directions are planes or half-planes is necessarily equiprojective. These equiprojective polytopes form a natural superset of the zonotopes since the aggregated cones of 3-dimensional zonotopes at their edge directions are planes. It should be noted that this superset of the zonotopes does not contain all equiprojective polytopes. For instance, the equitruncated tetrahedron described in [85] has an aggregated cone at an edge direction that is neither a plane or a half-plane. Now observe that, if the aggregated cone of a 3-dimensional polytope at some of its edge directions is a plane, then that polytope is decomposable (see for example [54]). However, the above question on polytope decomposability is open for the 3-dimensional polytopes whose aggregated cones at all edge directions are half-planes.

Question 6.5.2. *Does there exist an indecomposable* 3*-dimensional polytope whose aggregated cones at all edge directions are half-planes?*

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