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Cohomologie des variétés de Deligne-Lusztig associées aux espaces de Rapoport-Zink PEL de signature (1,n-1)

Cohomology of Deligne-Lusztig varieties associated to PEL Rapoport-Zink spaces with signature (1,n-1)

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Cohomology of Deligne-Lusztig varieties associated to PEL Rapoport-Zink spaces with signature $(1, n-1)$

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Abstract : *Rapoport-Zink spaces are moduli spaces classifying the deformations of some p-divisible group equipped with additional structures. Its cohomology is expected to play a role in the local Langlands correspondences. In general, it is difficult to compute it especially outside of the supercuspidal part, which is described by the Kottwitz conjecture and which is known in a variety of cases. However, a certain small family of Rapoport-Zink spaces admit a Bruhat-Tits stratification on their special fiber, such that the strata are Deligne-Lusztig varieties of Coxeter type. It is the case in particular of the unitary PEL Rapoport-Zink spaces of signature* $(1, n-1)$ *with* p *inert or ramified. The closure of a Bruhat-Tits stratum is a generalized Deligne-Lusztig variety associated to a finite unitary or symplectic group. In the inert case, we compute the cohomology of an individual stratum entirely, and in the ramified case we describe a substantial part of it. Hyperspecial level in the inert case guarantees the triviality of the nearby cycles, allowing us to carry our computations to the analytical tubes of the closed Bruhat-Tits strata. These tubes form an open cover of the generic fiber of the Rapoport-Zink space, inducing a Čech spectral sequence which computes its cohomology. Exploiting this sequence, we prove that the cohomology of this Rapoport-Zink space in the inert case fails to be admissible in general. Eventually, via p-adic uniformization the cohomology of the Rapoport-Zink space is related to the cohomology of the supersingular locus of the associated PEL Shimura variety at hyperspecial level. For low values of* n*, we compute the cohomology of the supersingular locus through this sequence.*

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Contents

1 Introduction

1.1 Bruhat-Tits stratification in Rapoport-Zink spaces

The Langlands program is a significant collection of conjectures, expected to unveil deep and unifying connections between different fields of mathematics. In particular, it predicts the existence of a correspondence between Galois representations and automorphic or smooth representations of reductive groups over a global or a local field. The first formulations of this research project date back to Langlands in 1967, giving a far-reaching generalization of Harish-Chandra and Gelfand's philosophy of cusp forms. It offered exciting new perspectives to the field of number theory, and ever since several dozen researchers contributed to expanding, deepening and maturing Langlands' intuition.

One approach to tackle Langlands conjectures is based on geometry. It is commonly expected that the correspondence one seeks to establish should be encrypted in the properties of geometric objects, known as Shimura varieties (global case) and Rapoport-Zink spaces (local case). In the PEL case, these are moduli spaces respectively for abelian varieties or for p-divisible groups with extra structures. These symmetric spaces are equipped with actions of reductive groups over a global or a local field, so that their cohomology is expected to give a geometric incarnation of the conjectural Langlands correspondences.

Different techniques have been used by various researchers to access the cohomology in some specific cases. In particular for the local case, the Kottwitz conjecture describes the supercuspidal part of the cohomology of Rapoport-Zink spaces. It has been proved in the Lubin-Tate case by Boyer [Boy99] and by Harris and Taylor [HT01]. It was established for all unramified Rapoport-Zink spaces of EL type by Fargues [Far04] and Shin [Shi12]. Eventually, it was proved for the unramified unitary Rapoport-Zink spaces of PEL type in an odd number of variables by Nguyen and Bertoloni-Meli [Ngu19] and [BMN21].

The non-supercuspidal part is more difficult to grasp, and there is no conjecture to describe it. So far, it has only been computed in the Lubin-Tate case by Boyer [Boy09], and the case of the Drinfeld space then followed by duality (see for instance [FGL08]). The specific geometry of the Lubin-Tate case allowed for explicit computations, however the same approach does not apply to more general cases, so that the non-supercuspidal part seems to be currently out of reach.

There exists however a certain small family of Rapoport-Zink spaces whose special fiber exhibits some very nice geometric properties. Such spaces are said to be "fully Hodge-Newton decomposable" and they have been fully classified by Görtz, He and Nie in [GHN19] using a group theoretic approach. The special fiber of a fully Hodge-Newton decomposable Rapoport-Zink space admits a stratification by Deligne-Lusztig varieties, and the incidence relations of the stratification is closely related to the combinatorics of the Bruhat-Tits building of an underlying p-adic group. Consequently, this stratification is known as the Bruhat-Tits stratification. The Rapoport-Zink space is said to be "of Coxeter type" if it is fully Hodge-Newton decomposable, and if the Deligne-Lusztig varieties occuring in the Bruhat-Tits stratification are of Coxeter

type. This subfamily of Rapoport-Zink space has also been entirely classified by Görtz, He and Nie in their subsequent work [GHN22].
To our knowledge, the first time that Deligne-Lusztig varieties were explicitely mentioned in the context of the Langlands program was in [Yos10], dealing with the Lubin-Tate tower. However, it is the pioneering work of Vollaard and Vollaard-Wedhorn in [Vol10] and [VW11] which coined the notion of Bruhat-Tits stratification. The authors used an approach based on Dieudonné theory and the combinatorics of vertex lattices in a hermitian space. The corresponding space was the $\mathrm{GU}(1, n-1)$ PEL Rapoport-Zink space at inert p and hyperspecial level. Mimicking their approach, Rapoport, Terstiege and Wilson dealt with the case of $\mathrm{GU}(1, n-1)$ at a ramified p and level given by a selfdual lattice in [RTW14]. This paved the way to the study of the geometry of the special fiber on a case-by-case basis by several authors, using either a similar Dieudonné theoretic approach or a group theoretic approach:

- the case of $\mathrm{GU}(2,2)$ at inert p and hyperspecial level by Howard and Pappas in [HP14], and by Wang in [Wan21] using another method which also covers the case of split p,
- the case of $\mathrm{GU}(1, n-1)$ at ramified p and parahoric level of exotic good reduction by Wu in [Wu16],
- the case of spinor groups $\mathrm{GSpin}(n, 2)$ at hyperspecial level by Howard and Pappas in [HP17],
- the case of $\mathrm{G}(\mathrm{U}(1, n-1) \times \mathrm{U}(1, n-1))$ at unramified p and hyperspecial level by Helm, Tian and Xiao in [HTX17],
- the case of $\mathrm{GU}(1, n-1)$ at inert p and arbitrary maximal parahoric level by Cho in [Cho18],
- the case of spinor groups $\mathrm{GSpin}(n, 2)$ at certain non-hyperspecial level by Oki in [Oki20],
- the case of a quaternionic unitary space at parahoric level by Wang in [Wan20] and [Wan22], and independently at maximal special parahoric level by Oki in [Oki22],
- the case of $\mathrm{GU}(2,2)$ at ramified p and at special maximal parahoric level by Oki in [Oki21],
- the case of $\mathrm{GU}(2, n-2)$ at inert p and hyperspecial level by Fox and Imai in [FI21],
- the case of $\mathrm{GL}(4)$ as well as $\mathrm{GU}(2,2)$ at split p and hyperspecial level by Fox in [Fox22].

Aside from the cases studied in [Cho18] and in [FI21], all the Rapoport-Zink spaces cited above are of Coxeter type. The spaces of [Cho18] are fully Hodge-Newton decomposable but not of Coxeter type when the parahoric level is not special, and the space of [FI21] is not fully Hodge-Newton decomposable when $n \geqslant 5$. In particular, the strata which are built in loc. cit. are not necessarily Deligne-Lusztig varieties.

Deligne-Lusztig varieties naturally arise in Deligne-Lusztig theory, a field of mathematics whose aim is the classification of all irreducible complex representations of finite groups of Lie type, ie. reductive groups over finite fields. Let $\mathbf{G}$ be a connected reductive group over an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. Let q be a power of p and assume that $\mathbf{G}$ has an $\mathbb{F}_q$-structure, induced by a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$. Let $G := \mathbf{G}(\mathbb{F}_q) \simeq \mathbf{G}^F$ be the associated finite group of Lie type. A Levi complement $L \subset G$ is the group of $\mathbb{F}_q$-points of some rational Levi complement $\mathbf{L}$ of $\mathbf{G}$. Such a Levi complement L is said to be split if $\mathbf{L}$ is the Levi complement of a rational parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. One way of building irreducible representations of G is to decompose representations parabolically induced from proper split Levi complements L of G. However, this process fails to recover the cuspidal representations. To remedy this issue, Deligne and Lusztig defined in their innovative work [DL76] new induction functors from any (not necessarily split) Levi L of G, generalizing the usual parabolic induction. They did so by associating a certain variety $Y_{\mathbf{L}\subset\mathbf{P}}$ to any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with *rational* Levi complement $\mathbf{L}$, which is naturally equipped with commuting actions of G and of $L = \mathbf{L}^F$. The alternate sum of the cohomology of $Y_{\mathbf{L}\subset\mathbf{P}}$ provides a virtual $G-$bimodule$-L$, which is used to define the Deligne-Lusztig induction functor R_L^G between the categories of representations of L and of G. Reducing to the case where $L = T$ is a maximal torus in G and computing explicitely the decompositions of the induced representations $\mathrm{R}_T^G\theta$ for all characters θ of T, Lusztig managed in [Lus84] to give a complete classification of all irreducible representations of all simple finite groups of Lie type.

To sum up, the geometry of certain Rapoport-Zink spaces can be described in terms of Deligne-Lusztig varieties, and cohomology plays a crucial role in both the Langlands program and Deligne-Lusztig theory. This observation is the starting point of this PhD thesis, whose aim is to derive the consequences of the geometric connections established by the authors cited above for the cohomology. We brought our attention to the two cases that have been chronologically first considered, that is the unitary PEL Rapoport-Zink space of signature $(1, n-1)$ over a prime p which is inert, see [Vol10] and [VW11], or ramified, see [RTW14].
In the following, we first detail the general approach before stating the results reached in the inert case. Eventually, we explain how we plan to adapt the method to the ramified case.

1.2 The case of inert or ramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

If E is a p-adic field where $p > 2$, let $\mathcal{O}_E$ denote its ring of integers,let π denote a uniformizer and let $\kappa(E)$ be the residue field. Let Nilp_E be the category of $\mathcal{O}_E$-schemes where π is locally nilpotent. Assume now that $E/\mathbb{Q}_p$ is quadratic and denote by $\bar{\cdot}$ the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q}_p)$. If $E/\mathbb{Q}_p$ is ramified, we may chose π so that $\overline{\pi} = -\pi$. If $E/\mathbb{Q}_p$ is unramified, then $E \simeq \mathbb{Q}_{p^2} := W(\mathbb{F}_{p^2})_{\mathbb{Q}}$, and $\mathcal{O}_E \simeq \mathbb{Z}_{p^2} := W(\mathbb{F}_{p^2})$, where $W(\cdot)$ denotes the ring of Witt vectors. Let E'/E be an unramified extension. For $S \in \mathrm{Nilp}_{E'}$, a unitary p-divisible group of signature $(1, n-1)$ over S is a triple (X, ι_X, λ_X) such that

- X is a p-divisible group over S,
- $\iota_X : \mathcal{O}_E \to \mathrm{End}(X)$ is a $\mathcal{O}_E$-action on X such that the induced action on its Lie algebra satisfies the Kottwitz signature $(1, n-1)$ condition:
$$\forall a \in \mathcal{O}_E, \qquad \mathrm{char}(\iota(a)\,|\,\mathrm{Lie}(X)) = (T-a)^1(T-\overline{a})^{n-1},$$
- if $E/\mathbb{Q}_p$ is ramified, then the induced action of ι_X on $\mathrm{Lie}(X)$ also satisfies the Pappas condition:
$$\bigwedge^2(\iota(\pi) + \pi\,|\,\mathrm{Lie}(X)) = 0 \text{ and if } n \geqslant 3,\ \bigwedge^n(\iota(\pi) - \pi\,|\,\mathrm{Lie}(X)) = 0,$$

– $\lambda_X : X \xrightarrow{\sim} {}^tX$ is a principal polarization, where ${}^tX$ denotes the Serre dual of X. We assume that the associated Rosati involution induces $\overline{\cdot}$ on $\mathcal{O}_E$.

Note that $\mathrm{char}(\iota(a) \,|\, \mathrm{Lie}(X))$ is a polynomial with coefficients in $\mathcal{O}_S$. The Kottwitz condition compares it with a polynomial with coefficients in $\mathcal{O}_E \subset \mathcal{O}_{E'}$ via the structure morphism $S \to \mathcal{O}_{E'}$.
Let us fix such a unitary p-divisible group $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of signature $(1, n-1)$ over $\kappa(E')$, such that $\mathbb{X}$ is superspecial. We call the triple $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ the framing object of the Rapoport-Zink space. If $E/\mathbb{Q}_p$ is unramified or if $E \simeq \mathbb{Q}_p[\sqrt{-p}]$ then one may take $E' = E$. However if $E \simeq \mathbb{Q}_p[\sqrt{\epsilon p}]$ where $\epsilon \in \mathbb{Z}_p^\times$ is such that $-\epsilon$ is not a square in $\mathbb{Z}_p$, then one must take $E' = E \otimes_{\mathbb{Q}_p} W(\mathbb{F}_{p^2})_{\mathbb{Q}}$ in order to define the framing object. The Rapoport-Zink space is the moduli space $\mathcal{M}$ classifying the deformations of the framing object by quasi-isogenies. More precisely, for $S \in \mathrm{Nilp}_{E'}$, $\mathcal{M}(S)$ is the set of isomorphism classes of tuples $(X, \iota_X, \lambda_X, \rho_X)$ where (X, ι_X, λ_X) is a unitary p-divisible group of signature $(1, n-1)$ over S, and where $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\kappa(E')} \overline{S}$ is an $\mathcal{O}_E$-linear quasi-isogeny such that ${}^t\rho_X \circ \lambda_{\mathbb{X}} \circ \rho_X = c\lambda_X$ for some $c \in \mathbb{Q}_p^\times$. Here $\overline{S}$ is the special fiber of S and ${}^t\rho_X$ is the dual quasi-isogeny. By the work of Rapoport and Zink in [RZ96] and of Pappas in the ramified case, the functor $\mathcal{M}$ is a formal scheme over $\mathrm{Spf}(\mathcal{O}_{E'})$ which is formally of finite type, formally smooth in the inert case, and flat in the ramified case.

Remark. In the inert case and in the ramified case with n odd, any choice of the framing object $\mathbb{X}$ gives the same Rapoport-Zink space. In the ramified case with n even however, there are essentially two choices of framing objects, giving rise to two different spaces. These two cases are refered to as the split and non-split cases, see [RTW14] Remark 4.2.

Let $\mathcal{M}_{\mathrm{red}}$ denote the special fiber of $\mathcal{M}$. The Bruhat-Tits stratification, which is built in [VW11] for the inert case and in [RTW14] for the ramified case, can be written as

$$\mathcal{M}_{\mathrm{red}} = \bigsqcup_{\Lambda \in \mathcal{L}} \mathcal{M}_\Lambda^\circ,$$

where each stratum $\mathcal{M}_\Lambda^\circ$ is a locally closed subvariety which is defined over $\kappa(E)$, and Λ runs over the set $\mathcal{L}$ of so-called vertex lattices. More precisely, the indices Λ are almost self-dual $\mathcal{O}_E$-lattices in a certain $E/\mathbb{Q}_p$-hermitian space of dimension n (denoted by $\mathbf{V}$ in the inert case, and by C in the ramified case). Let J denote the group of unitary similitudes of this hermitian space, so that J acts on the set of vertex lattices $\mathcal{L}$. The group J can also be identified with the group $\mathrm{Aut}(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ of automorphisms of the framing object. In particular, J also acts on $\mathcal{M}$ by $g \cdot (X, \iota_X, \lambda_X, \rho_X) := (X, \iota_X, \lambda_X, g \circ \rho_X)$ for all $g \in J$.
For all $\Lambda \in \mathcal{L}$ let $\mathcal{M}_\Lambda$ denote the closure of the stratum $\mathcal{M}_\Lambda^\circ$. The J-action on the special fiber $\mathcal{M}_{\mathrm{red}}$ is compatible with the Bruhat-Tits stratification, in the sense that any $g \in J$ induces an isomorphism

$$g : \mathcal{M}_\Lambda^\circ \xrightarrow{\sim} \mathcal{M}_{g(\Lambda)}^\circ,$$

and thus an isomorphism between the closed strata $\mathcal{M}_\Lambda \xrightarrow{\sim} \mathcal{M}_{g(\Lambda)}$ as well. Let $J_\Lambda := \mathrm{Fix}_J(\Lambda)$ be the fixator in J of $\Lambda \in \mathcal{L}$. Then J_Λ is a maximal compact subgroup of J, and it admits a finite quotient which is isomorphic to a finite group of unitary similitudes $\mathrm{GU}_{t(\Lambda)}(\mathbb{F}_p)$ in the inert case, and to a finite group of symplectic similitudes $\mathrm{GSp}_{t(\lambda)}(\mathbb{F}_p)$ is the ramified case. Here

$0 \leqslant t(\Lambda) \leqslant n$ is a certain integer called the orbit type of $\Lambda \in \mathcal{L}$, and which is odd in the inert case and even in the ramified case. It turns out that the induced action of J_Λ on $\mathcal{M}_\Lambda$ factors through an action of this finite quotient.
The Bruhat-Tits stratification is very well behaved for the two following reasons.

(1) The set $\mathcal{L}$ of vertex lattices can be given the structure of a polysimplicial complex, whose combinatorics describes the incidence relations between the closed Bruhat-Tits strata.
(2) Each closed Bruhat-Tits stratum, equipped with its action of the finite quotient of the maximal compact subgroup J_Λ, is naturally isomorphic to a generalized Deligne-Lusztig variety for $\mathrm{GU}_{t(\Lambda)}(\mathbb{F}_p)$ in the inert case, and for $\mathrm{GSp}_{t(\Lambda)}(\mathbb{F}_p)$ in the ramified case.

The polysimplicial complex $\mathcal{L}$ of (1) is closely related to the Bruhat-Tits building BT of J over $\mathbb{Q}_p$. In fact, both polysimplicial complexes are equal except in the case of split ramified p with n even. There, $\mathcal{L}$ is a slight modification of BT, see [RTW14] Proposition 3.4. The isomorphism of (2) also induces an isomorphism between $\mathcal{M}_\Lambda^\circ$ and the Coxeter variety for $\mathrm{GU}_{t(\Lambda)}(\mathbb{F}_p)$ or for $\mathrm{GSp}_{t(\Lambda)}(\mathbb{F}_p)$. This is in accordance with the fact that the Rapoport-Zink space $\mathcal{M}$ is of Coxeter type in both the inert and the ramified cases, by [GHN22].
Let $\mathcal{M}^{\mathrm{an}}$ denote the generic fiber of the formal scheme $\mathcal{M}$ in the sense of Berkovich. Thus, $\mathcal{M}^{\mathrm{an}}$ is a smooth analytical space of dimension $n-1$ over E'. Let $\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ denote the reduction map. It is anticontinuous, ie. the preimage of a closed (resp. open) subset is open (resp. closed). In order to derive the consequences of such a stratification for the cohomology, we establish the following strategy.

(A) Understand the cohomology $\mathrm{H}_c^\bullet(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ of an individual closed Bruhat-Tits stratum by using Deligne-Lusztig theory via (2).

(B) Introduce the analytical tubes $U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda)$, and study the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ via the Čech spectral sequence associated to the open cover $\{U_\Lambda\}_\Lambda$, whose combinatorics is described by (1).

Remark. By general theory, there is a connected reductive group G over $\mathbb{Q}_p$, a parahoric subgroup $K_0 \subset G(\mathbb{Q}_p)$, and a finite étale cover $\mathcal{M}_K \to \mathcal{M}^{\mathrm{an}}$ for every open compact subgroup $K \subset K_0$. Here, G is the group of unitary similitudes of some n-dimensional $E/\mathbb{Q}_p$-hermitian space, and J is an inner form of G. Moreover the group K_0 is maximal special, and in the inert case it is hyperspecial. For $K \subset K'$, there are transition maps $\Pi_{K,K'} : \mathcal{M}_{K'} \to \mathcal{M}_K$ so that the spaces $\mathcal{M}_K$ fit together in a projective system $\mathcal{M}_\infty := (\mathcal{M}_K)_K$, called the Rapoport-Zink tower. The action of J on $\mathcal{M}$ can be extended to an action on each $\mathcal{M}_K$ which is compatible with the transition maps. Therefore $\mathcal{M}_\infty$ is equipped with an action of $G(\mathbb{Q}_p) \times J$, where $G(\mathbb{Q}_p)$ acts on the structure level by Hecke correspondences. One may define the cohomology of $\mathcal{M}_\infty$ via the formula

$$\mathrm{H}_c^\bullet(\mathcal{M}_\infty, \overline{\mathbb{Q}_\ell}) := \varinjlim_{K} \varinjlim_{U_K} \varprojlim_{k} \mathrm{H}_c^\bullet(U_K \widehat{\otimes}\, \mathbb{C}_p, \mathbb{Z}/\ell^k\mathbb{Z}) \otimes \overline{\mathbb{Q}_\ell},$$

where U_K runs over all the relatively compact open subsets of $\mathcal{M}_K$. These cohomology groups are representations of $G(\mathbb{Q}_p) \times J \times W$, where $W = W_E$ denotes the absolute Weil group of

E. We note that the action of W_E is induced by Rapoport and Zink's (non effective) Weil descent datum on $\mathcal{M}$, as defined in [RZ96] 3.48. These cohomology groups are the main object of interest in the context of the Langlands program.
In this thesis, our results only deal with the cohomology of the generic fiber $\mathcal{M}^{\mathrm{an}} = \mathcal{M}_{K_0}$. Thus, we focus on the cohomology groups

$$\mathrm{H}^\bullet_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = \mathrm{H}^\bullet_c(\mathcal{M}_\infty, \overline{\mathbb{Q}_\ell})^{K_0},$$

as representations of $J \times W$. The reason is that our approach is fruitful only in situations where one may relate the cohomology of a closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$ in the special fiber, to the cohomology of its analytical tube $U_{\Lambda,K} := \Pi_{K,K_0}^{-1}\mathrm{red}^{-1}(\mathcal{M}_\Lambda)$ in $\mathcal{M}_K$. In the inert case, this can be achieved trivially when $K = K_0$ since the formal smoothness of $\mathcal{M}$ insures the triviality of the nearby cycles. In future works, we hope to generalize our approach to more general parahoric level structures, in which case the semi-stable reduction should allow explicit computations of the nearby cycles. Other cases of bad reduction, such as the ramified case, may also be manageable as we discuss in the end of the introduction.

1.3 Step (A): the cohomology of an individual closed Bruhat-Tits stratum

As mentioned in (2), each closed Bruhat-Tits stratum is isomorphic to a *generalized* Deligne-Lusztig variety. Let us explain what we mean by this. In general, let $\mathbf{G}$ be a connected reductive group over $\overline{\mathbb{F}_p}$ equipped with a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$ inducing an $\mathbb{F}_q$-structure. Let $G := \mathbf{G}^F$ be the associated finite group of Lie type. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. The associated generalized Deligne-Lusztig variety is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

It is defined over $\mathbb{F}_{q^\delta}$ where $\delta \geqslant 1$ is the smallest integer such that $F^\delta(\mathbf{P}) = \mathbf{P}$, and it is equipped with an action $G \curvearrowright X_{\mathbf{P}}$ by left translations. We say that a generalized Deligne-Lusztig variety $X_{\mathbf{P}}$ is classical if in addition, there exists a rational Levi complement $\mathbf{L} \subset \mathbf{P}$. When this condition is satisfied, the Deligne-Lusztig variety inherits an action $X_{\mathbf{P}} \curvearrowleft L := \mathbf{L}^F$ by right translations, which commutes with the action of G. In this case, the cohomology of $X_{\mathbf{P}}$ is a G-bimodule-L, and can be used to defined the Deligne-Lusztig induction functor between the categories of representations of L and of G. We note that the varieties denoted above by $Y_{\mathbf{L}\subset\mathbf{P}}$ are in fact some L-torsor of $X_{\mathbf{P}}$.
Thus, in the context of Deligne-Lusztig theory which focuses on the study of the induction functors afforded by the varieties $Y_{\mathbf{L}\subset\mathbf{P}}$, one is only interested in classical Deligne-Lusztig varieties. For this reason, to our knowledge their generalized versions have not been systematically studied in the literature, except in [BR06] where a criterion for the irreducibility of $X_{\mathbf{P}}$ is proved. It turns out that the Deligne-Lusztig variety to which a closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$ is isomorphic, is not classical. Therefore, no result regarding its cohomology can be directly read from the literature.

However, the works of [VW11] in the inert case, and of [RTW14] in the ramified case, give us enough geometric understanding of $\mathcal{M}_\Lambda$ in order to access its cohomology. Let us write $t(\Lambda) = 2\theta + 1$ in the inert case, and $t(\Lambda) = 2\theta$ in the ramified case. There exists a stratification (called the Ekedahl-Oort stratification in the inert case)

$$\mathcal{M}_\Lambda = \bigsqcup_{0 \leqslant \theta' \leqslant \theta} \mathcal{M}_\Lambda(\theta'),$$

where each $\mathcal{M}_\Lambda(\theta')$ is a locally closed subvariety, and the closure of the stratum associated to θ' is the union of the all the strata associated to $t \leqslant \theta'$. The isomorphism of (2) between $\mathcal{M}_\Lambda$ and a generalized Deligne-Lusztig variety, naturally induces an isomorphism between $\mathcal{M}_\Lambda(\theta')$ and a classical Deligne-Lusztig variety which is, in some sense, parabolically induced from the Coxeter variety for the smaller group of unitary similitudes $\mathrm{GU}_{2\theta'+1}(\mathbb{F}_p)$ in the inert case, and for the smaller group of symplectic similitudes $\mathrm{GSp}_{2\theta'}(\mathbb{F}_p)$ in the ramified case.
In [Lus76], Lusztig has computed the cohomology of the Coxeter varieties for all finite classical groups in terms of unipotent representations. The unipotent representations of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ are classified by the integer partitions λ of $2\theta + 1$ and we denote them ρ_Λ, see 2.2.1. The unipotent representations of $\mathrm{GSp}_{2\theta}(\mathbb{F}_p)$ are classified by Lusztig's notion of symbols S of rank θ and of odd defect, and we denote them ρ_S. In 2.4.3 and 4.3.2, we translate Lusztig's results of [Lus76] in terms of the classification by integer partitions or by symbols respectively. From [GM20] and [GP00], we derive the combinatorical rules to compute parabolic induction of unipotent representations. It allows us to entirely determine the cohomology of a stratum $\mathcal{M}_\Lambda(\theta')$. Then, we study the spectral sequence associated to the stratification

$$E_1^{a,b} = \mathrm{H}_c^{a+b}(\mathcal{M}_\Lambda(a) \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}),$$

which degenerates on the second page thanks to the repartition of the Frobenius eigenvalues throughout the sequence.
The variety $\mathcal{M}_\Lambda$ is projective of dimension θ, and it is smooth only in the inert case, and in the ramified case when $\theta \leqslant 1$. Let $\tau \in \mathrm{Gal}(\overline{\mathbb{F}_p}/\kappa(E))$ be the geometric Frobenius relative to $\kappa(E)$. In the inert case, the purity of the Frobenius on the cohomology of $\mathcal{M}_\Lambda$ allows us to compute all the E_2 terms, and it leads to the following statement.

Theorem (2.5.1)**.** *In the inert case, let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$.*

1. *The cohomology group $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ is zero unless $0 \leqslant j \leqslant 2\theta$.*
2. *The Frobenius τ acts like multiplication by $(-p)^j$ on $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$.*
3. *For $0 \leqslant j \leqslant \theta$ we have*

$$\mathrm{H}_c^{2j}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant j \leqslant \theta - 1$ we have

$$\mathrm{H}_c^{2j+1}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\theta-2s,2s+1)}.$$

In particular, all irreducible representations in the cohomology groups of even index belong to the unipotent principal series, whereas all the ones in the groups of odd index have cuspidal support determined by the unique cuspidal unipotent representation of $\mathrm{GU}_3(\mathbb{F}_p)$, which is denoted ρ_{Δ_2} with Δ_2 equal to the partition $(2,1)$ of 3. The cohomology group $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ contains no cuspidal representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ unless $\theta = j = 0$ or $\theta = j = 1$. If $\theta = 0$ then H_c^0 is the trivial representation of $\mathrm{GU}_1(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times$, and if $\theta = 1$ then H_c^1 is the representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$.

In the ramified case, unless when $\theta = 0$ or 1, in which case $\mathcal{M}_\Lambda$ is respectively isomorphic to a point or to $\mathbb{P}^1$, we do not have a full understanding of the cohomology of $\mathcal{M}_\Lambda$, but we can still get substantial information from the spectral sequence. For $0 \leqslant k \leqslant 2\theta$ the weights of the Frobenius τ on the cohomology group $\mathrm{H}_c^k(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ form a subset of $\{p^i, -p^{j+1}\}$ for $k - \min(k,\theta) \leqslant i \leqslant k - \lceil k/2 \rceil$ and for $k - \min(k,\theta) \leqslant j \leqslant k - \lceil k/2 \rceil - 1$. Among other things, if $i, j > k - \min(k,\theta)$ then we determine the eigenspaces of the Frobenius $\mathrm{H}_c^k(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})_{p^i}$ and $\mathrm{H}_c^k(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})_{-p^{j+1}}$ explicitely up to at most four irreducible representations of $\mathrm{GSp}_{2\theta}(\mathbb{F}_p)$. We refer to 4.4.2 and 4.4.3 for the detailed results, as it would be too long to fit this introduction. In particular, we note that the action of the Frobenius on the cohomology is not pure when $\theta \geqslant 3$ (for $\theta = 2$ the non-purity is undetermined). This is in accordance with $\mathcal{M}_\Lambda$ not being smooth for $\theta \geqslant 2$. All irreducible representations of $\mathrm{GSp}(2\theta, \mathbb{F}_p)$ occuring in an eigenspace of τ for an eigenvalue of the form p^i belong to the unipotent principal series, whereas those corresponding to an eigenvalue of the form $-p^{j+1}$ belong to the cuspidal series determined by the unique cuspidal unipotent representation of $\mathrm{GSp}(4, \mathbb{F}_p)$ which is denoted by ρ_{S_2}, where S_2 is the symbol defined in 4.2.5. We note in particular that $H_c^k(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}_p}, \overline{\mathbb{Q}_\ell})$ contains no cuspidal representation of $\mathrm{GSp}_{2\theta}(\mathbb{F}_p)$, unless $\theta = k = 0$ or $\theta = k = 2$. When $\theta = 0$ then H_c^0 is the trivial representation of $\mathrm{GSp}_0(\mathbb{F}_p) \simeq \{1\}$, and when $\theta = 2$ then the eigenspace of τ in H_c^2 for the eigenvalue $-p$ is ρ_{S_2}.

Remark. We observe that in the inert case, the non-principal cuspidal series determined above contributes to the cohomology of $\mathcal{M}_\Lambda$ for $\theta \geqslant 1$, but in the ramified case it only contributes for $\theta \geqslant 2$. Moreover, in the inert case the two cuspidal series contribute separately to groups of even or odd degrees, but in the ramified case both series contribute to cohomology groups of degrees of any parity.

1.4 Step (B): on the cohomology of the inert Rapoport-Zink space at hyperspecial level

From now on, we consider only the inert case. Let $\mathcal{L}^{\max}$ denote the subset of all vertex lattices $\Lambda \in \mathcal{L}$ having maximal orbit type $t(\Lambda) = t_{\max}$. We have

$$t_{\max} = \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Let us write $t_{\max} = 2\theta_{\max} + 1$. Then $\{U_\Lambda\}_{\Lambda \in \mathcal{L}^{\max}}$ forms an open cover of the generic fiber $\mathcal{M}^{\mathrm{an}}$ to which one can associate the following $J \times W$-equivariant Čech spectral sequence, concentrated in degrees $a \leqslant 0$ and $0 \leqslant b \leqslant 2(n-1)$,

$$E_1^{a,b} : \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

See 3.4.1.4 for some details regarding the definition of the W-action. Here, for $a \leqslant 0$ the set I_{-a+1} is defined by

$$I_{-a+1} := \left\{ \gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \,\middle|\, \forall 1 \leqslant j \leqslant -a+1, \Lambda^j \in \mathcal{L}^{(m)} \text{ and } U(\gamma) := \bigcap_{j=1}^{-a+1} U_{\Lambda^j} \neq \varnothing \right\}.$$

By the properties of the Bruhat-Tits stratification, if $\gamma \in I_{-a+1}$ then there exists a unique vertex lattice $\Lambda(\gamma) \in \mathcal{L}$ such that $U(\gamma) = U_{\Lambda(\gamma)}$. Thus, we must first relate the cohomology of any U_Λ with the cohomology of its special fiber $\mathcal{M}_\Lambda$ that we have investigated in step (A). Each cohomology group $\mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ is naturally a representation of $(J_\Lambda \times I)\tau^{\mathbb{Z}}$ where $I \subset W$ is the inertia subgroup, and $\tau := (p^{-1} \cdot \mathrm{id}, \mathrm{Frob}) \in J \times W$ is called the rational Frobenius element. Here, $\mathrm{Frob} \in W$ is a fixed lift of the geometric Frobenius, and $p^{-1} \cdot \mathrm{id}$ is seen as an element of the center $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}^\times$ (recall that J is a group of unitary similitudes).

Proposition (3.4.1.5)**.** *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. Write $t(\Lambda) = 2\theta + 1$. There is a natural $(J_\Lambda \times I)\tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \overline{\mathbb{F}}_p, \overline{\mathbb{Q}_\ell})(n-1-\theta).$$

On the right-hand side the inertia I acts trivially, the rational Frobenius τ acts like the geometric Frobenius τ defined in step (A), and the J_Λ-action factors through its finite unitary or symplectic similitudes quotient.

This proposition relies on the fact that we consider the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}} = \mathcal{M}_{K_0}$ at hyperspecial level, insuring the triviality of the nearby cycles between U_Λ and $\mathcal{M}_\Lambda$.
It follows that τ acts like multiplication by the scalar $(-p)^b$ on any term $E_1^{a,b}$. Thus, the spectral sequence degenerates on the second page and the filtration on the abutment splits, ie. the k-th cohomology group of $\mathcal{M}^{\mathrm{an}}$ is the direct sum of the $E_2^{a,b}$ terms on the diagonal $a + b = k$, see 3.4.1.7.
In order to study the J-action, we rewrite the terms $E_1^{a,b}$ in terms of compact inductions. Let $\{\Lambda_0, \ldots, \Lambda_{\theta_{\max}}\}$ be a maximal simplex in $\mathcal{L}$ such that for all θ, $t(\Lambda_\theta) = 2\theta + 1$. We write $J_\theta := J_{\Lambda_\theta}$. We also define

$$K_{-a+1}^{(\theta)} := \{\gamma \in I_{-a+1}^{(\theta)} \,|\, \Lambda(\gamma) = \Lambda_\theta\},$$

which is a finite subset of I_{-a+1} equipped with an action of J_θ.

Proposition (3.4.1.10)**.** *We have an equality*

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}] \right)$$

where $\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]$ denotes the permutation representation associated to $J_\theta \curvearrowright K_{-a+1}^{(\theta)}$.

By exploiting this spectral sequence, we are able to compute the cohomology groups of $\mathcal{M}^{\mathrm{an}}$ of highest degree $2(n-1)$. We denote by J° the subgroup of J generated by all the compact subgroups. It corresponds to all the unitary similitudes in J whose multipliers are a unit. We note that J° is normal in J with quotient $J/J^{\circ} \simeq \mathbb{Z}$.

Proposition (3.4.1.12)**.** *There is an isomorphism*

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J^{\circ}}^{J} \mathbf{1},$$

and the rational Frobenius τ acts via multiplication by $p^{2(n-1)}$.

When $\theta_{\max} = 1$ (ie. $n = 3$ or 4), the Bruhat-Tits building of J is essentially a tree. Exploiting its combinatorics and the spectral sequence, we are also able to compute the group of degree $2(n-1)-1$. Recall the representation ρ_{Δ_2} which we introduced in the previous section.

Theorem (3.4.3.4)**.** *Assume that $\theta_{\max} = 1$. We have*

$$\mathrm{H}_c^{2(n-1)-1}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2},$$

with the rational Frobenius τ acting via multiplication by $-p^{2(n-1)-1}$.

In general, the terms $E_2^{a,b}$ in the second page may be difficult to compute. However, the terms corresponding to $a = 0$ and $b \in \{2(n-1-\theta_{\max}), 2(n-1-\theta_{\max})+1\}$ are not touched by any non-zero differential in the alternating version of the Čech spectral sequence, making their computations accessible.

Proposition (3.4.1.11)**.** *We have an isomorphism of J-representations*

$$E_2^{0,2(n-1-\theta_{\max})} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \mathbf{1}.$$

If $n \geqslant 3$ then we also have an isomorphism

$$E_2^{0,2(n-1-\theta_{\max})+1} \simeq \mathrm{c}-\mathrm{Ind}_{J_{\theta_{\max}}}^{J} \rho_{(2\theta_{\max},1)}.$$

Here $\mathbf{1}$ denotes the trivial representation, and $\rho_{(2\theta_{\max},1)}$ denotes (the inflation to $J_{\theta_{\max}}$ of) the unipotent representation of $\mathrm{GU}_{2\theta_{\max}}(\mathbb{F}_p)$ associated to the partition $(2\theta_{\max}, 1)$.
The previous statement has important consequences for the cohomology of $\mathcal{M}^{\mathrm{an}}$. To explain it, let us recall a certain property of compactly induced representations.
Let χ be a continuous character of the center $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}^{\times}$ and let V be a smooth representation of J. Let V_χ be the maximal quotient of V on which $\mathrm{Z}(J)$ acts through χ. Let K be an open compact subgroup of J and let ρ be an irreducible smooth representation of K. Assume that χ agrees with the central character of ρ on $\mathrm{Z}(J) \cap K$. Then

$$(\mathrm{c}-\mathrm{Ind}_K^J \rho)_\chi \simeq \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)K}^{J} \chi \otimes \rho = V_{\rho,\chi,0} \oplus V_{\rho,\chi,\infty},$$

see 3.4.2.2 and 3.4.2.3. The decomposition on the right-hand side follows from a general theorem in [Bus90]. The J-representation $V_{\rho,\chi,0}$ is the sum of all supercuspidal subrepresentations

of $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)K}^{J}\chi\otimes\rho$. This is a finite sum. The space $V_{\rho,\chi,\infty}$ contains no non-zero admissible subrepresentation, in particular it contains no irreducible subrepresentation but it may admit many irreducible quotients and subquotients, none of which is supercuspidal. We note that $V_{\rho,\chi,0}$ or $V_{\rho,\chi,\infty}$ may be zero.
Therefore, the behaviour of a compactly induced representation as above depends greatly on whether there exists some irreducible supercuspidal subquotient in $\mathrm{c}-\mathrm{Ind}_{K}^{J}\rho$. The existence of such subquotients may be elucidated by type theory, especially in the case where ρ is inflated from a finite quotient of K. Combining with the two previous propositions, we deduce the following statements. Note that we consider unramified characters of $\mathrm{Z}(J)$ because any unipotent representation has trivial central character.

Proposition (3.4.2.12)**.** *Let χ be any unramified character of $\mathrm{Z}(J)\simeq\mathbb{Q}_{p^2}^{\times}$.*

- *assume that $n\geqslant 3$. The representation $(E_2^{0,2(n-1-\theta_{\max})})_\chi$ contains no non-zero admissible subrepresentation, and it is not J-semisimple. If $n\geqslant 5$, then the same statement holds for $(E_2^{0,2(n-1-\theta_{\max})+1})_\chi$.*
- *for $n=1$ (resp. $n=3,4$), let $b=0$ (resp. $b=3,5$). Then $(E_2^{0,b})_\chi$ is an irreducible supercuspidal representation of J. If $n=2$, then $(E_2^{0,2})_\chi$ is the sum of two non-isomorphic supercuspidal representations of J.*

In particular, we obtain the following corollary.

Corollary. *Let χ be any unramified character of $\mathrm{Z}(J)$. If $n\geqslant 3$ then $\mathrm{H}_c^{2(n-1-\theta_{\max})}(\mathcal{M}^{\mathrm{an}},\overline{\mathbb{Q}_\ell})_\chi$ is not J-admissible. If $n\geqslant 5$ then $\mathrm{H}_c^{2(n-1-\theta_{\max})+1}(\mathcal{M}^{\mathrm{an}},\overline{\mathbb{Q}_\ell})_\chi$ is not J-admissible.*

This non-admissibility result shows a different behaviour from the cases of the Lubin-Tate tower or of the Drinfeld space.

1.5 The cohomology of the supersingular locus of the associated Shimura variety at an inert prime for $n=3,4$

The Rapoport-Zink space $\mathcal{M}$ is related to the supersingular locus of a certain PEL Shimura variety via the p-adic uniformization theorem, and a certain spectral sequence relates the cohomology of both spaces. In particular, for small values of n, our results so far allow us to compute the cohomology of the supersingular locus both in the inert case. Let us give some more details.
Let $\mathbb{E}$ be an imaginary quadratic field, and let $\mathbb{V}$ be an n-dimensional non-degenerate $\mathbb{E}/\mathbb{Q}$-hermitian space of signature $(1, n-1)$ at infinity, and such that $\mathbb{V}\otimes\mathbb{Q}_p$ is isomorphic to the hermitian space defining the group of unitary similitudes G. In particular $\mathbb{E}_p\simeq\mathbb{Q}_{p^2}$, so that p is inert in $\mathbb{E}$. Let $\mathbb{G}$ be the group of unitary similitudes of $\mathbb{V}$, seen as a reductive group over $\mathbb{Q}$. Then $\mathbb{G}_{\mathbb{Q}_p}=G$ and $\mathbb{G}_{\mathbb{R}}=\mathrm{GU}(1,n-1)$. Assume that there exists a self-dual $\mathcal{O}_{\mathbb{E}}$-lattice Γ in $\mathbb{V}$, and let $\mathrm{Stab}(\Gamma)$ denote the compact subgroup of $\mathbb{G}(\mathbb{A}_f)$ of elements g such that $g(\Gamma\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})=\Gamma\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$. Here $\mathbb{A}_f$ denotes the ring of finite adèles. For any open compact subgroup

$K^p \subset \mathrm{Stab}(\Gamma) \cap \mathbb{G}(\mathbb{A}_f^p)$ which is small enough, there is an integral model S_{K^p} of the associated PEL Shimura variety which is defined over $\mathcal{O}_E$. Since we have hyperspecial level structure at p, the integral model S_{K^p} is smooth and quasi-projective. Let $\overline{S}_{K^p}$ denote the special fiber of S_{K^p}, and let $\overline{S}_{K^p}^{\mathrm{ss}}$ denote the supersingular locus. Let I be the inner form of $\mathbb{G}$ such that $I(\mathbb{Q}_p) = J$, $I_{\mathbb{A}_f^p} = \mathbb{G}_{\mathbb{A}_f^p}$ and $I_{\mathbb{R}} = \mathrm{GU}(0,n)$. The p-adic uniformization theorem of [RZ96] gives natural isomorphisms of analytic spaces over E'

$$I(\mathbb{Q})\backslash(\mathcal{M}^{\mathrm{an}} \times \mathbb{G}(\mathbb{A}_f^p)/K^p) \xrightarrow{\sim} \widehat{S}_{K^p}^{\mathrm{ss,an}} \otimes_E E',$$

which are compatible as the level K^p varies. Here $\widehat{S}_{K^p}^{\mathrm{ss,an}}$ denotes the analytical tube of the supersingular locus inside the analytification of the generic fiber of S_{K^p}. Associated to this geometric identity, Fargues has built in [Far04] a spectral sequence computing the cohomology of $\widehat{S}_{K^p}^{\mathrm{ss,an}}$. Since S_{K^p} is smooth, it amounts to the cohomology of the supersingular locus $\overline{S}_{K^p}^{\mathrm{ss}}$ itself. The $(\mathbb{G}(\mathbb{A}_f^p) \times W)$-equivariant spectral sequence takes the following shape

$$F_2^{a,b} = \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \implies \mathrm{H}_c^{a+b}(\overline{S}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \mathcal{L}_\xi),$$

where ξ is a finite dimensional irreducible algebraic $\overline{\mathbb{Q}_\ell}$-representation of $\mathbb{G}$ of weight $w(\xi) \in \mathbb{Z}$, $\mathcal{L}_\xi$ is the associated local system on the Shimura variety S_{K^p}, $\mathcal{A}_\xi(I)$ is the space of all automorphic representations of $I(\mathbb{A})$ of type $\check{\xi}$ at infinity, and $\mathrm{H}_c^\bullet(\overline{S}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \mathcal{L}_\xi) := \varinjlim_{K^p} \mathrm{H}_c^\bullet(\overline{S}_{K^p}^{\mathrm{ss}} \otimes \overline{\mathbb{F}_p}, \mathcal{L}_\xi)$. By [Far04] Lemme 4.4.12, we have $F_2^{a,b} = 0$ as soon as a is strictly bigger than the semisimple rank of J, which is equal to $\theta_{\max}$. In particular, if $\theta_{\max} \leqslant 1$ then all the differentials are zero and the spectral sequence is already degenerated, allowing us to compute the abutment entirely. Since the case $\theta_{\max} = 0$ is kind of trivial, we now assume $\theta_{\max} = 1$ (ie. $n = 3$ or 4). In particular, the supersingular locus $\overline{S}_{K^p}^{\mathrm{ss}}$ has dimension $\theta_{\max} = 1$. Let $X^{\mathrm{un}}(J)$ denote the set of unramified characters of J. Let St_J denote the Steinberg representation of J. If $x \in \overline{\mathbb{Q}_\ell}^\times$, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of the Weil group W where the inertia acts trivially and Frob acts like multiplication by the scalar x.
Let $\tau_1 := \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^J \widetilde{\rho_{\Delta_2}}$ where $\mathrm{N}_J(J_1)$ is the normalizer of J_1, and $\widetilde{\rho_{\Delta_2}}$ is an extension to $\mathrm{N}_J(J_1)$ of the cuspidal representation ρ_{Δ_2} of J_1. Then τ_1 is an irreducible supercuspidal representation of J. If $\Pi \in \mathcal{A}_\xi(I)$, we define $\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1}\cdot \mathrm{id})p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times$ where ω_{Π_p} is the central character of Π_p, and $p^{-1}\cdot\mathrm{id}$ lies in the center of J. For any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$ we have $|\iota(\delta_{\Pi_p})| = 1$.

Theorem (3.5.2.3)**.** *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi\cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi\cdot\tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

1.6 Adapting the approach to the ramified case

As one of the rapporteur wittingly pointed out to me, an unfortunate typo can be found in [RTW14], where the authors wrote in the course of a paragraph that Pappas' integral model of the unitary PEL $\mathrm{GU}(1, n-1)$ Shimura variety, at a ramified prime and at parahoric level given by the stabilizer of a self-dual lattice, is smooth. This is however not the case, and it is consistent with such a parahoric subgroup being special but not hyperspecial, nor does it correspond to a case of exotic good reduction. We note that this typo has absolutely no impact on the contents of [RTW14], as smoothness is not needed there anyway.

In an earlier version of this thesis, I intended to apply the same approach as in the inert case to the ramified case, and with the smoothness hypothesis it seemed like all steps described above would work the same way. But, this hypothesis being actually wrong, the situation of the ramified case is more complex. Therefore, regarding the ramified case, this thesis only contains the part exploiting Deligne-Lusztig theory in order to get information on the cohomology of a closed Bruhat-Tits stratum.

Let us explain what obstacles we face and propose a slightly unformal strategy to overcome them.

The main issue concerns 3.4.1.5 Proposition, where one identifies the cohomology of an analytical tube U_Λ with the cohomology of its special fiber $\mathcal{M}_\Lambda$. In the proof, we push the Bruhat-Tits stratum into the associated Shimura variety via p-adic uniformization, and we apply a result of Berkovich in [Ber96]. Since the integral model of the Shimura variety is smooth (in the inert case), nearby cycles are trivial and we obtain an isomorphism $\mathrm{H}^\bullet(U_\Lambda, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^\bullet(\mathcal{M}_\Lambda, \overline{\mathbb{Q}_\ell})$. In general however, we only get an isomorphism with the cohomology of $\mathcal{M}_\Lambda$ with coefficients in the nearby cycles sheaf $\mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell}$. In situations where these nearby cycles can not be computed, it seems hopeless to try using the approach of this thesis.

Thus, in the ramified case, the non-smoothness of the integral model means that one must first understand the nearby cycles on the Shimura variety. The integral model S_{K^p} over $\mathrm{Spec}(\mathcal{O}_E)$ (with $E/\mathbb{Q}_p$ quadratic ramified) has been built by Pappas in [Pap00] as a moduli space classifying abelian schemes with usual additional structures, and satisfying a certain "Pappas condition" similar to the one in the definition of the Rapoport-Zink space in the ramified case. Theorem 4.5 of [Pap00] states that S_{K^p} is normal, Cohen-Macaulay and flat over $\mathrm{Spec}(\mathcal{O}_E)$. Moreover its special fiber $\overline{\mathrm{S}}_{K^p}$ is smooth outside of a finite number of singular points, and if $n \geqslant 3$ the blow-up $\mathrm{BL}(\mathrm{S}_{K^p}) \to \mathrm{S}_{K^p}$ at the reduced singular locus has semistable reduction, ie. it is regular and its special fiber is a divisor with normal crossings. If $n = 2$ then S_{K^p} already has semistable reduction.

In this context, the nearby cycles $\mathrm{R}^q\Psi_\eta\overline{\mathbb{Q}_\ell}$ for $q > 0$ are skyscraper and concentrated on the singular points of the special fiber $\overline{\mathrm{S}}_{K^p}$. For any point $x \in \overline{\mathrm{S}}_{K^p}$, there exists a point $y \in \overline{\mathrm{M}}^{\mathrm{loc}}$ (the local model associated to the PEL datum) such that x and y have some isomorphic etale neighborhoods. The stalk $(\mathrm{R}^q\Psi_\eta\overline{\mathbb{Q}_\ell})_x$ may therefore be computed on the local model, which has a much simpler linear algebraic description. In fact, from general theory the integral model

S_{K^p} of the Shimura variety and the local model $\mathrm{M}^{\mathrm{loc}}$ share the same geometric properties, and it is mainly the local model which is studied in [Pap00] ; it is also described in [Krä03]. Since the blow-up $\mathrm{Bl}(\mathrm{M}^{\mathrm{loc}}) \to \mathrm{M}^{\mathrm{loc}}$ at the singular points has semistable reduction, we understand the nearby cycles on the blow-up (for instance via [Ill94]), and via proper base change we could compute the cycles on the local model itself. Then, if one may understand the distribution of singular points of the special fiber $\overline{\mathrm{S}}_{K^p}$ with respect to the Bruhat-Tits stratification on the supersingular locus, it seems reasonable to think that the cohomology groups $\mathrm{H}^\bullet(\mathcal{M}_\Lambda, \mathrm{R}\Psi_\eta\overline{\mathbb{Q}_\ell})$ could be understood, at least sufficiently enough in order to apply the approach described in the inert case.

Organization of the thesis

The body of the thesis consists of the two papers written during the PhD, along with the first part of the 3rd paper. Section 2 is [Mul22b], section 3 is [Mul22a] and Section 4 is the first part of [Mul22c]. Section 2 and 3 deal with the inert case, the former consists of step (A) and the latter of step (B) as explained in the introduction. Section 4 deals with step (A) in the ramified case. Each section may be read independently, however some parts of them may make reference to previous sections as the papers have been written in this chronological order.
We warn the reader that the notations may vary slightly from one section to the other, as well as they may vary from the introduction.

2 Cohomology of the Bruhat-Tits strata in the unramified unitary Rapoport-Zink space of signature $(1, n-1)$

Notations

Throughout this section paper, we fix q a power of an odd prime number p. If k is a perfect field extension of $\mathbb{F}_q$, we denote by $\sigma : x \mapsto x^q$ the q-th power Frobenius of $\mathrm{Gal}(k/\mathbb{F}_q)$. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Unless specified otherwise, $\mathbf{G}$ will denote a connected reductive group over $\mathbb{F}$ equipped with an $\mathbb{F}_q$-structure, induced by a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$. If $\mathbf{H}$ is an F-stable subgroup of $\mathbf{G}$, we denote by $H := \mathbf{H}^F \simeq \mathbf{H}(\mathbb{F}_q)$ its group of $\mathbb{F}_q$-rational points. We fix a pair $(\mathbf{T}, \mathbf{B})$ consisting of a maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$, both of them being F-stable. Such a pair always exists up to $G = \mathbf{G}^F$-conjugation. We obtain a Coxeter system $(\mathbf{W}, \mathbf{S})$ on which F acts, where $\mathbf{W} = \mathbf{W}(\mathbf{T})$ is the Weyl group attached to $\mathbf{T}$ and $\mathbf{S}$ is the set of simple reflexions. It can be identified with the Weyl group of $\mathbf{G}$ as defined in [DL76]. Let ℓ denote the length function on $\mathbf{W}$ relative to $\mathbf{S}$. For $I \subset \mathbf{S}$, we write $\mathbf{P}_I, \mathbf{U}_I, \mathbf{L}_I$ respectively for the standard parabolic subgroup of type I, for its unipotent radical and for its unique Levi complement containing $\mathbf{T}$. We also write $\mathbf{W}_I$ for the parabolic subgroup of $\mathbf{W}$ generated by the simple reflexions in I. Recall that an element $w \in \mathbf{W}$ is said to be I-reduced (resp. reduced-I) if for every $v \in \mathbf{W}_I$, we have $\ell(vw) = \ell(v) + \ell(w)$ (resp. $\ell(wv) = \ell(w) + \ell(v)$). The set of I-reduced (resp. reduced-I) elements is denoted by ${}^I\mathbf{W}$ (resp. $\mathbf{W}^I$). If $I, I' \subset \mathbf{S}$, an element is said to be I-reduced-I' if it belongs to ${}^I\mathbf{W}^{I'} := {}^I\mathbf{W} \cap \mathbf{W}^{I'}$.

2.1 The generalized Deligne-Lusztig variety $X_I(\mathrm{id})$

2.1.1 Let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$. Let F be a Frobenius morphism defining an $\mathbb{F}_q$-structure on it. If $\mathbf{H}$ is an F-stable subgroup of $\mathbf{G}$, we denote by $H := \mathbf{H}^F \simeq \mathbf{H}(\mathbb{F}_q)$ its group of $\mathbb{F}_q$-rational points. We fix a pair $(\mathbf{T}, \mathbf{B})$ consisting of a maximal torus $\mathbf{T}$ contained in a Borel subgroup $\mathbf{B}$, both of them being F-stable. Such a pair always exists up to $G = \mathbf{G}^F$-conjugation. We obtain a Coxeter system $(\mathbf{W}, \mathbf{S})$ on which F acts, where $\mathbf{W} = \mathbf{W}(\mathbf{T})$ is the Weyl group attached to $\mathbf{T}$ and $\mathbf{S}$ is the set of simple reflexions. It can be identified with the Weyl group of $\mathbf{G}$ as defined in [DL76]. Let ℓ denote the length function on $\mathbf{W}$ relative to $\mathbf{S}$. For $I \subset \mathbf{S}$, we write $\mathbf{P}_I, \mathbf{U}_I, \mathbf{L}_I$ respectively for the standard parabolic subgroup of type I, for its unipotent radical and for its unique Levi complement containing $\mathbf{T}$. We also write $\mathbf{W}_I$ for the parabolic subgroup of $\mathbf{W}$ generated by the simple reflexions in I. Recall that an element $w \in \mathbf{W}$ is said to be I-reduced (resp. reduced-I) if for every $v \in \mathbf{W}_I$, we have $\ell(vw) = \ell(v) + \ell(w)$ (resp. $\ell(wv) = \ell(w) + \ell(v)$). The set of I-reduced (resp. reduced-I) elements is denoted by ${}^I\mathbf{W}$ (resp. $\mathbf{W}^I$). If $I, I' \subset \mathbf{S}$, an element is said to be I-reduced-I' if it belongs to ${}^I\mathbf{W}^{I'} := {}^I\mathbf{W} \cap \mathbf{W}^{I'}$.

2.1.2 We recall the definition of Deligne-Lusztig varieties from [BR06]. If $\mathbf{P}$ is any parabolic subgroup of $\mathbf{G}$, the associated generalized parabolic Deligne-Lusztig variety is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

When these varieties were first introduced in [DL76] only the case of Borel subgroups was considered, hence the adjective "parabolic". Moreover, parabolic Deligne-Lusztig varieties have mostly been studied with the additional assumption that $\mathbf{P}$ contains an F-stable Levi complement, see for instance [DM14]. This is not required by the definition above, hence the adjective "generalized".
Using the Coxeter system as above, one may give an equivalent description of these varieties. For $I, I' \subset \mathbf{S}$ the generalized Bruhat decomposition is an isomorphism

$$\mathbf{P}_I\backslash\mathbf{G}/\mathbf{P}_{I'} = \bigsqcup_{w\in {}^I\mathbf{W}^{I'}} \mathbf{P}_I\backslash\mathbf{P}_I w\mathbf{P}_{I'}/\mathbf{P}_{I'} \simeq \mathbf{W}_I\backslash\mathbf{W}/\mathbf{W}_{I'}.$$

For $w \in {}^I\mathbf{W}^{F(I)}$, the generalized parabolic Deligne-Lusztig varieties is defined by

$$X_I(w) = \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \,|\, g^{-1}F(g) \in \mathbf{P}_I w F(\mathbf{P}_I)\}.$$

The families of varieties $X_{\mathbf{P}}$ and $X_I(w)$ are the same and [BR06] explains how to go from one description to the other. The case $I = \varnothing$ corresponds to usual Deligne-Lusztig varieties in $\mathbf{G}/\mathbf{B}$. Moreover, the additional assumption regarding the existence of a rational Levi complement translates into the equation

$$w^{-1}Iw = F(I), \tag{$*$}$$

which is a compatibility condition between the parameters w and I. The variety $X_I(w)$ is defined over $\mathbb{F}_{q^\iota}$, where ι is the least integer such that $F^\iota(I) = I$ and $F^\iota(w) = w$.

2.1.3 In this paragraph, we compute the dimension of a generalized Deligne-Lusztig variety $X_I(w)$. For any $w \in \mathbf{W}$, let $\ell(w)$ denote the length of w with respect to $\mathbf{S}$.

Proposition. *For $I \subset \mathbf{S}$ and $w \in {}^I\mathbf{W}^{F(I)}$, we have*

$$\dim X_I(w) = \ell(w) + \dim \mathbf{G}/\mathbf{P}_{I\cap wF(I)w^{-1}} - \dim \mathbf{G}/\mathbf{P}_I.$$

Let us introduce a few more notations. If $I, I' \subset \mathbf{S}$, the generalized Bruhat decomposition implies that the $\mathbf{G}$-orbits for the diagonal action on $\mathbf{G}/\mathbf{P}_I \times \mathbf{G}/\mathbf{P}_{I'}$ are given by

$$\mathcal{O}_{I,I'}(w) := \{(g\mathbf{P}_I, h\mathbf{P}_{I'}) \,|\, g^{-1}h \in \mathbf{P}_I w\mathbf{P}_{I'}\}$$

for $w \in {}^I\mathbf{W}^{I'}$. The Deligne-Lusztig variety $X_I(w)$ can be seen as the intersection of $\mathcal{O}_{I,F(I)}(w)$ with the graph of the Frobenius $F : \mathbf{G}/\mathbf{P}_I \to \mathbf{G}/\mathbf{P}_{F(I)}$. This intersection is transverse, see [DL76] 9.11 (in loc. cit. the proof deals with the case $I = \varnothing$, but it generalizes to any I). Thus, the proposition follows from the following lemma and the fact that $\dim \mathbf{P}_I = \dim \mathbf{P}_{F(I)}$.

Lemma. *For $I, I' \subset \mathbf{S}$ and $w \in {}^I\mathbf{W}^{I'}$, we have*

$$\dim \mathcal{O}_{I,I'}(w) = \ell(w) + \dim \mathbf{G}/\mathbf{P}_{I\cap wI'w^{-1}}.$$

Proof. Recall that for $I \subset \mathbf{S}$, the standard parabolic subgroup of type I decomposes as a union of Bruhat cells $\mathbf{P}_I = \mathbf{B}\mathbf{W}_I\mathbf{B}$, and any Bruhat cell $\mathbf{B}w\mathbf{B}$ has dimension $\dim \mathbf{B} + \ell(w)$. Therefore

$\dim \mathbf{P}_I = \dim \mathbf{B} + \ell(I)$ where $\ell(I)$ denotes the maximal length of elements of $\mathbf{W}_I$.
Let I, I' and w be as in the lemma. Consider the first projection $\mathcal{O}_{I,I'}(w) \to \mathbf{G}/\mathbf{P}_I$ which is a surjective morphism with fibers isomorphic to $\mathbf{P}_I w \mathbf{P}_{I'}/\mathbf{P}_{I'}$. It is flat since $\mathbf{G} \to \mathbf{G}/\mathbf{P}_I$ is faithfully flat, and the pullback $\mathcal{O}_{I,I'}(w) \times_{\mathbf{G}/\mathbf{P}_I} \mathbf{G}$ is isomorphic to $\mathbf{G} \times \mathbf{P}_I w \mathbf{P}_{I'}/\mathbf{P}_{I'}$. We have

$$\mathbf{P}_I w \mathbf{P}_{I'} = \mathbf{B}\mathbf{W}_I \mathbf{B} w \mathbf{B} \mathbf{W}_{I'} \mathbf{B} = \mathbf{B}\mathbf{W}_I w \mathbf{W}_{I'} \mathbf{B},$$

therefore the dimension of a fiber is given by

$$\dim \mathbf{P}_I w \mathbf{P}_{I'}/\mathbf{P}_{I'} = \dim \mathbf{P}_I w \mathbf{P}_{I'} - \dim \mathbf{P}_{I'} = \max_{v \in \mathbf{W}_I w \mathbf{W}_{I'}} \ell(v) - \ell(I').$$

Since w is I-reduced-I', according to [DM20] Lemma 3.2.2, any element $v \in W_I w \mathbf{W}_{I'}$ can uniquely be written as $v = xwy$ such that $x \in \mathbf{W}_I, y \in \mathbf{W}_{I'}$ and xw is reduced-I'. In particular $\ell(v) = \ell(x) + \ell(w) + \ell(y)$. It follows that

$$\max_{v \in \mathbf{W}_I w \mathbf{W}_{I'}} \ell(v) = \ell(w) + \max_{x \in \mathbf{W}_I \cap \mathbf{W}^{I'} w^{-1}} \ell(x) + \ell(I').$$

We prove that $\mathbf{W}_I \cap \mathbf{W}^{I'} w^{-1} = \mathbf{W}_I \cap \mathbf{W}^{I \cap w I' w^{-1}}$.

Let $x \in \mathbf{W}_I \cap \mathbf{W}^{I'} w^{-1}$, we show that x is reduced-$I \cap wI'w^{-1}$. Let $s \in I \cap wI'w^{-1}$, so that we can write $s = wtw^{-1}$ for some $t \in I'$. Then $xsw = xwt$. Since $xs \in \mathbf{W}_I$ and w is I-reduced, the left hand side has length $\ell(xs) + \ell(w)$. On the other hand, since $t \in I'$ and xw is reduced-I', the right hand side has length $\ell(xw) + 1 = \ell(x) + \ell(w) + 1$. Therefore $\ell(xs) = \ell(x) + 1$ as expected.
For the other inclusion, let $y \in \mathbf{W}_I \cap \mathbf{W}^{I \cap wI'w^{-1}}$. We show that yw is reduced-I'. Towards a contradiction, assume that $\ell(ywt) < \ell(yw)$ for some $t \in I'$. Let $y = s_1 \ldots s_r$ and $w = u_1 \ldots u_{r'}$ be reduced expressions respectively of y and of w, with the s_i in I and the u_j in $\mathbf{S}$. Since w is I-reduced, the concatenation of both reduced expressions give a reduced expression of yw. By the exchange condition (see [DM20] 2.1.2), we have

$$ywt = s_1 \ldots \widehat{s_i} \ldots s_r w \text{ or } y u_1 \ldots \widehat{u_j} \ldots u_{r'}$$

for some $1 \leqslant i \leqslant r$ or $1 \leqslant j \leqslant r'$, where $\widehat{\cdot}$ denotes the product with one omitted term. The second case is impossible, since after simplifying y it would contradict the fact that w is reduced-I'.
Let us write $s := y^{-1} s_1 \ldots \widehat{s_i} \ldots s_r \in \mathbf{W}_I$, so that we have

$$wt = sw.$$

The left hand side has length $\ell(w) + 1$, and the right hand side has length $\ell(s) + \ell(w)$. It follows that $s \in I$ has length 1. Therefore $s = wtw^{-1} \in I \cap wI'w^{-1}$. Eventually, we have $\ell(ys) = \ell(y) + 1$ since y is reduced-$(I \cap wI'w^{-1})$. This is absurd, because $ys = s_1 \ldots \widehat{s_i} \ldots s_r$ has length $r - 1 = \ell(y) - 1$.

To conclude the proof, we recall the following general fact. If $(\mathbf{W}, \mathbf{S})$ is a Coxeter system and $K \subset \mathbf{S}$, then the product map $\mathbf{W}^K \times \mathbf{W}_K \xrightarrow{\sim} \mathbf{W}$ mapping (w^K, w_K) to $w^K w_K$ is a bijection. In particular we have

$$\max_{w \in \mathbf{W}} \ell(w) = \max_{w^K \in \mathbf{W}^K} \ell(w^K) + \max_{w_K \in \mathbf{W}_K} \ell(w_K).$$

We apply this to the Coxeter system $(\mathbf{W}_I, I)$ and $K = I \cap wI'w^{-1}$. It follows that

$$\max_{x \in \mathbf{W}_I \cap \mathbf{W}^{I'}w^{-1}} \ell(x) = \max_{x \in \mathbf{W}_I \cap \mathbf{W}^{I \cap wI'w^{-1}}} \ell(x) = \ell(I) - \ell(I \cap wI'w^{-1}).$$

Putting things together, have proved that

$$\begin{aligned}
\dim \mathcal{O}_{I,I'}(w) &= \dim \mathbf{G}/\mathbf{P}_I + \dim \mathbf{P}_I w \mathbf{P}_{I'}/\mathbf{P}_{I'} \\
&= \dim \mathbf{G} - \dim \mathbf{B} - \ell(I) + \max_{v \in \mathbf{W}_I w \mathbf{W}_{I'}} \ell(v) - \ell(I') \\
&= \dim \mathbf{G} - \dim \mathbf{B} - \ell(I) + \ell(w) + \max_{x \in \mathbf{W}_I \cap \mathbf{W}^{I'}w^{-1}} \ell(x) \\
&= \dim \mathbf{G} - \dim \mathbf{B} - \ell(I \cap wI'w^{-1}) + \ell(w) \\
&= \dim \mathbf{G}/\mathbf{P}_{I \cap wI'w^{-1}} + \ell(w).
\end{aligned}$$

□

Remark. In [VW11] 4.4, the formula given by the authors for the dimension of $\mathcal{O}_{I,I'}(w)$, and as a consequence for the Deligne-Lusztig variety $X_I(w)$ as well, contained a mistake.

2.1.4 Let d be a nonnegative integer and let V be a $(2d+1)$-dimensional $\mathbb{F}_{q^2}$-vector space. Let $(\cdot,\cdot) : V \times V \to \mathbb{F}_{q^2}$ be a non-degenerate hermitian form on V. This hermitian structure on V is unique up to isomorphism. In particular, we may once and for all a basis $\mathcal{B}$ of V in which $(\cdot,\cdot)$ is described by the square matrix $\dot{w}_0$ of size $2d+1$, having 1 on the anti-diagonal and 0 everywhere else. If k is a perfect field extension of $\mathbb{F}_{q^2}$, we may extend the pairing to $V_k := V \otimes_{\mathbb{F}_{q^2}} k$ by setting

$$(v \otimes x, w \otimes y) := xy^\sigma (v, w) \in k$$

for all $v, w \in V$ and $x, y \in k$. If U is a subspace of V_k we denote by $U^\perp$ its orthogonal, that is the subspace of all vectors $x \in V_k$ such that $(x, U) = 0$.

Let J denote the finite group of Lie type $\mathrm{U}(V, (\cdot,\cdot))$. It is defined as the group of F-fixed points of $\mathbf{J} := \mathrm{GL}(V)_{\mathbb{F}}$ with F a non-split Frobenius morphism. Using the basis $\mathcal{B}$, the group $\mathbf{J}$ is identified with GL_{2d+1} with $\mathbb{F}_q$-structure induced by the Frobenius morphism $F(M) := \dot{w}_0 (M^{(q)})^{-t} \dot{w}_0$. Here, $M^{(q)}$ denotes the matrix M having all coefficients raised to the power q. We may then identify J with the usual finite unitary group $\mathrm{U}_{2d+1}(q)$.

The pair $(\mathbf{T}, \mathbf{B})$ consisting of the maximal torus of diagonal matrices and the Borel subgroup of upper-triangular matrices is F-stable. The Weyl system of $(\mathbf{T}, \mathbf{B})$ may be identified with $(\mathfrak{S}_{2d+1}, \mathbf{S})$ in the usual manner, where $\mathbf{S}$ is the set of simple transpositions $s_i := (i \quad i+1)$ for $1 \leqslant i \leqslant 2d$. Under this identification, the Frobenius acts on $\mathbf{W}$ as the conjugation by the element w_0, characterized for having the maximal length. It satisfies $w_0(i) = 2d+2-i$, and a natural representative of w_0 in the normalizer of $\mathbf{T}$ is no other than $\dot{w}_0$. Since w_0 has order 2, the action of the Frobenius on $\mathbf{W}$ is involutive. It also preserves the simple reflexions with the formula $F(s_i) = s_{2d+1-i}$.

2.1.5 We define the following subset of $\mathbf{S}$

$$I := \{s_1, \dots, s_d, s_{d+2}, \dots, s_{2d}\} = \mathbf{S} \backslash \{s_{d+1}\}.$$

We have $F(I) = \mathbf{S}\backslash\{s_d\} \neq I$. We consider the generalized Deligne-Lusztig variety $X_I(\mathrm{id})$. It corresponds to the variety denoted Y_Λ in [VW11] 4.5. It has dimension d and it does not satisfy the compatibility condition ($*$).

Proposition ([VW11] 4.4)**.** *The variety $X_I(\mathrm{id})$ is defined over $\mathbb{F}_{q^2}$ and it is projective, smooth, geometrically irreducible of dimension d.*

Although the proposition in loc. cit. is only stated in the case $q = p$, the arguments carry over to general q. The geometric irreducibility is a consequence of the criterion proved in [BR06].

Remark. Even though the dimension formula for generalized Deligne-Lusztig varieties in [VW11] is wrong, it does give the correct result in the case of $X_I(\mathrm{id})$. It is because for $w = \mathrm{id}$, we have $I \cap wF(I)w^{-1} = I \cap F(I)$. Therefore, that mistake does not change anything regarding the validity of the authors' work.
For example, we may consider the Deligne-Lusztig variety $X_I(s_2s_1)$ for $\mathrm{U}_3(\mathbb{F}_q)$ with $I = \{s_1\}$. It is classical so that $\dim X_I(s_2s_1) = \ell(s_2s_1) = 2$. However, we have $\mathbf{P}_{I\cap F(I)} = \mathbf{B}$ and $\dim \mathbf{G}/\mathbf{B} = 3$ whereas $\dim \mathbf{G}/\mathbf{P}_I = 2$, so that the formula of loc. cit. says that $X_I(s_2s_1)$ would be of dimension $2 + 3 - 2 = 3$.

2.1.6 Rational points of Deligne-Lusztig varieties associated to a unitary group U over $\mathbb{F}_q$ can be described in terms of vectorial flags, in a certain relative position with respect to their image by the Frobenius. Let k be a perfect field extension of $\mathbb{F}_{q^2}$. According to [Vol10] 2.12, the Frobenius acts on a flag $\mathcal{F}$ in V_k by sending it to its orthogonal flag $\mathcal{F}^\perp$. Explicitely, we have

$$\mathcal{F} \;:\; \{0\} \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_r \subset V_k,$$

$$\mathcal{F}^\perp \;:\; \{0\} \subset \mathcal{F}_r^\perp \subset \dots \subset \mathcal{F}_1^\perp \subset V_k.$$

Here, given our choice of I, a k-rational point of $X_I(\mathrm{id})$ corresponds to a flag of the type

$$\mathcal{F} : \{0\} \subset U \subset V_k$$

with U having dimension $d+1$, and which is of relative position id with respect to $\mathcal{F}^\perp$. This precisely means that U must contain $U^\perp$.

Proposition. *The k-rational points of $X_I(\mathrm{id})$ are given by*

$$X_I(\mathrm{id})(k) \simeq \{U \subset V_k \mid \dim U = d+1 \text{ and } U^\perp \subset U\}.$$

2.1.7 In [VW11] 5.3, the authors defined the **Ekedahl-Oort stratification** on the Deligne-Lusztig variety $X_I(\mathrm{id})$. By loc. cit. Corollary 5.12, it turns out that each stratum is itself isomorphic to a parabolic Deligne-Lusztig variety which is not generalized. They are defined

as follows.
For $0 \leqslant t \leqslant d$, we define the subset

$$I_t := \{s_1, \ldots, s_{d-t-1}, s_{d+t+2}, \ldots, s_{2d}\} \subset \mathbf{S}.$$

The subset I_t consists of all $2d$ simple reflexions in $\mathbf{S}$, except that we removed the $2t+2$ ones in the middle. Thus, it has cardinality $2(d-t-1)$. In particular, it is empty for $t = d$ or $d-1$. We also define the cycle $w_t := (d+t+1 \quad d+t \ldots d+1)$. Its decomposition into simple reflexions is $w_t = s_{d+1} \ldots s_{d+t}$. When $t = 0$, it is the identity. We note that even though $I_d = I_{d-1} = \varnothing$, we still have $w_d \neq w_{d-1}$.
One may check that $F(I_t) = I_t$ and that w_t belongs to ${}^{I_t}\mathbf{W}^{I_t}$. Moreover, the compatibility condition $(*)$ is satisfied for the pair (I_t, w_t). Indeed, the reduced decomposition for w_t does not use any simple reflexion that is adjacent to those in I_t.

Proposition ([VW11] 3.3 and 5.3)**.** *The Deligne-Lusztig variety $X_{I_t}(w_t)$ is defined over $\mathbb{F}_{q^2}$ and has dimension t. There is a natural immersion $X_{I_t}(w_t) \hookrightarrow X_I(\mathrm{id})$ inducing a stratification*

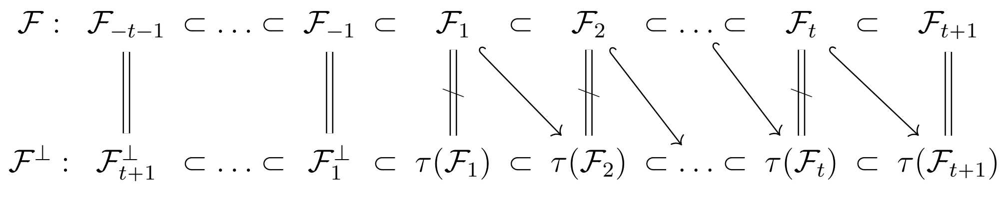
$$X_I(\mathrm{id}) = \bigsqcup_{0 \leqslant t \leqslant d} X_{I_t}(w_t).$$

The closure of the stratum $X_{I_t}(w_t)$ is the union of all the strata $X_{I_s}(w_s)$ for $s \leqslant t$.

2.1.8 Following the proof of Theorem 2.15 of [Vol10], we can describe the stratification at the level of rational points. Let k be a perfect field extension of $\mathbb{F}_{q^2}$. Because of the choice of I_t, a k-point of $X_{I_t}(w_t)$ is a flag

$$\mathcal{F} : \quad \{0\} \subset \mathcal{F}_{-t-1} \subset \ldots \subset \mathcal{F}_{-1} \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{t+1} \subset V_k$$

with $\dim(\mathcal{F}_{-i}) = d + 1 - i$ and $\dim(\mathcal{F}_i) = d + i$ for $1 \leqslant i \leqslant t+1$, and which is in relative position w_t with respect to $\mathcal{F}^\perp$. It means that we have a diagram of the following type.



Here, $\tau := \sigma^2 \cdot \mathrm{id}$ is an $\mathbb{F}_{q^2}$-linear automorphism of V_k, and it satisfies $\tau(U) = (U^\perp)^\perp$ for every subspace $U \subset (V_\Lambda)_k$. This diagram implies that $\tau(\mathcal{F}_i) = \mathcal{F}_{i-1} + \tau(\mathcal{F}_{i-1})$ for all $2 \leqslant i \leqslant t+1$. This rewrites as $\mathcal{F}_i = \mathcal{F}_{i-1} + \tau^{-1}(\mathcal{F}_{i-1})$. We deduce that

$$\mathcal{F}_i = \sum_{l=0}^{i-1} \tau^{-l}(\mathcal{F}_1)$$

for all $1 \leqslant i \leqslant t+1$. Thus, the whole flag is determined by the subspace $\mathcal{F}_1$, which has dimension $d+1$ and contains its orthogonal. The immersion $X_{I_t}(w_t) \hookrightarrow X_I(\mathrm{id})$ maps the flag $\mathcal{F}$ to $\mathcal{F}_1$.

Conversely, a k-point of $X_I(\mathrm{id})$ is given by a subspace $U \subset V_k$ of dimension $d+1$ containing its orthogonal. For $i \geqslant 1$ we define

$$\mathcal{F}_i := \sum_{l=0}^{i-1} \tau^{-1}(U) \subset V_k.$$

Then $(\mathcal{F}_i)_{i \geqslant 1}$ is a nondecreasing sequence of subspaces of V_k. Let t be the smallest integer such that $\mathcal{F}_{t+1} = \mathcal{F}_{t+2}$. It follows that $0 \leqslant t \leqslant d$ and that t is also the smallest integer such that $\mathcal{F}_{t+1} = \tau(\mathcal{F}_{t+1})$. Moreover the orthogonal $U^\perp$ has dimension d and we have $U^\perp \subset U$, so that $U^\perp \subset (U^\perp)^\perp = \tau(U)$. In particular, if $t > 0$ then $U \cap \tau(U) = U^\perp$. Thus, we have $\dim(\mathcal{F}_2) = d+2$. Similarly, we have $\dim(\mathcal{F}_i) = d+i$ for all $1 \leqslant i \leqslant t+1$. By setting $\mathcal{F}_{-i} := \mathcal{F}_i^\perp$, we obtain a flag $\mathcal{F}$ that is the k-rational point of $X_{I_t}(w_t)$ associated to U.

2.1.9 The Deligne-Lusztig varieties $X_{I_t}(w_t)$ are related to Coxeter varieties for smaller unitary groups as we now explain. We define

$$K_t := \{s_1, \ldots s_{d-t-1}, s_{d-t+1}, \ldots, s_{d+t}, s_{d+t+2}, \ldots, s_{2d}\} = \mathbf{S} \backslash \{s_{d-t}, s_{d+t+1}\}.$$

The set K_t is obtained from I_t by adding the $2t$ simple reflexions in the middle. It has cardinality $2d-2$ and satisfies $F(K_t) = K_t$. We have $I_t \subset K_t$ with equality if and only if $t = 0$.

Proposition. *There is a* $\mathrm{U}_{2d+1}(q)$*-equivariant isomorphism*

$$X_{I_t}(w_t) \simeq \mathrm{U}_{2d+1}(q)/U_{K_t} \times_{L_{K_t}} X_{I_t}^{\mathbf{L}_{K_t}}(w_t),$$

where $X_{I_t}^{\mathbf{L}_{K_t}}(w_t)$ *is a Deligne-Lusztig variety for* $\mathbf{L}_{K_t}$*. The zero-dimensional variety* $\mathrm{U}_{2d+1}(q)/U_{K_t}$ *has a left action of* $\mathrm{U}_{2d+1}(q)$ *and a right action of* L_{K_t}*.*

Proof. This is an application of [DM14] Proposition 7.19 which is the geometric identity behind the transitivity of the Deligne-Lusztig functors. It applies to the varieties $X_{I_t}(w_t)$ because they satisfy the compatibility condition ($*$), and satisfies the following conditions: K_t contains I_t, it is stable by the Frobenius and w_t belongs to the parabolic subgroup $\mathbf{W}_{K_t} \simeq \mathfrak{S}_{d-t} \times \mathfrak{S}_{2t+1} \times \mathfrak{S}_{d-t} \subset \mathfrak{S}_{2d+1}$. □

2.1.10 The Levi complement $\mathbf{L}_{K_t}$ is isomorphic to the product $\mathrm{GL}_{d-t} \times \mathrm{GL}_{2t+1} \times \mathrm{GL}_{d-t}$ as a reductive group over $\mathbb{F}$. Given a matrix $M = \mathrm{diag}(A, C, B) \in \mathbf{L}_{K_t}$, we have $F(M) = \mathrm{diag}(F(B), F(C), F(A))$, where we still denote by F the Frobenius morphism for smaller linear groups. Writing $\mathbf{H}$ for the product of the two GL_{d-t} factors, we have $\mathbf{L}_{K_t} \simeq \mathbf{H} \times \mathrm{GL}_{2t+1}$ and both factors inherit an $\mathbb{F}_q$-structure by means of F. We have $L_{K_t} \simeq \mathrm{GL}_{d-t}(q^2) \times \mathrm{U}_{2t+1}(q)$, the first factor corresponding to H.
The Weyl group of $\mathbf{L}_{K_t}$ is isomorphic to $\mathbf{W}_{\mathbf{H}} \times \mathfrak{S}_{2t+1}$ where $\mathbf{W}_{\mathbf{H}} \simeq \mathfrak{S}_{d-t} \times \mathfrak{S}_{d-t}$ is the Weyl group of $\mathbf{H}$. Via this decomposition, the permutation w_t corresponds to $\mathrm{id} \times \widetilde{w}_t$, where $\widetilde{w}_t$ is the restriction of w_t to $\{d-t+1, \ldots, d+t+1\}$. Similarly, the set of simple reflexions $\mathbf{S}$ decomposes as $\mathbf{S}_{\mathbf{H}} \sqcup \widetilde{\mathbf{S}}$, the second term corresponding to the simple reflexions in $\mathfrak{S}_{2t+1}$. Then, we have

$I_t = \mathbf{S_H} \sqcup \varnothing$.
The Deligne-Lusztig variety for $\mathbf{L}_{K_t}$ decompose accordingly as the following product

$$X_{I_t}^{\mathbf{L}_{K_t}}(w_t) = X_{\mathbf{S_H}}^{\mathbf{H}}(\mathrm{id}) \times X_{\varnothing}^{\mathrm{U}_{2t+1}(q)}(\widetilde{w}_t).$$

The variety $X_{\mathbf{S_H}}^{\mathbf{H}}(\mathrm{id})$ is just a point, whereas $X_{\varnothing}^{\mathrm{U}_{2t+1}(q)}(\widetilde{w}_t)$ is a Deligne-Lusztig variety for the unitary group of size $2t+1$. We observe that the permutation $\widetilde{w}_t$ is a **Coxeter element** in $\mathfrak{S}_{2t+1}$, ie. the product of exactly one simple reflexion for each orbit of the Frobenius. Deligne-Lusztig varieties attached to Coxeter elements are called **Coxeter varieties**, and their cohomology with coefficients in $\overline{\mathbb{Q}_\ell}$ where ℓ is a prime number different from p are well understood thanks to the work of Lusztig in [Lus76]. Before stating the results of loc. cit. we recall parts of the representation theory of finite unitary groups.

2.2 Irreducible unipotent representations of the finite unitary group

2.2.1 In this section, we recall the classification of the irreducible unipotent representations of the finite unitary group and we explain the underlying combinatorics.
We use the notations from 2.1.1. For $w \in \mathbf{W}$, let $\dot{w}$ be a representative of w in the normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$. By the Lang-Steinberg theorem, one can find $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. Then ${}^g\mathbf{T} := g\mathbf{T}g^{-1}$ is another F-stable maximal torus, and $w \in \mathbf{W}$ is said to be the **type** of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$. Every F-stable maximal torus arises in this manner. According to [DL76] Corollary 1.14, the G-conjugacy class of ${}^g\mathbf{T}$ only depends on the F-conjugacy class of the image w of the element $g^{-1}F(g) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T})$ in the Weyl group $\mathbf{W}$. Here, two elements w and w' in $\mathbf{W}$ are said to be F-conjugates if there exists some element $u \in \mathbf{W}$ such that $w = uw'F(u)^{-1}$.
For every $w \in \mathbf{W}$, we fix $\mathbf{T}_w$ an F-stable maximal torus of type w with respect to $\mathbf{T}$. The Deligne-Lusztig induction of the trivial representation of $\mathbf{T}_w$ is the virtual representation of G defined by the formula

$$R_w := \sum_{i \geqslant 0} (-1)^i \mathrm{H}_c^i(X_\varnothing(w))$$

where $X_\varnothing(w)$ is a Deligne-Lusztig variety for $\mathbf{G}$ as defined in 2.1.2. According to [DL76] Theorem 1.6, the virtual representation R_w only depends on the F-conjugacy class of w in $\mathbf{W}$. An irreducible representation of G is said to be **unipotent** if it occurs in R_w for some $w \in \mathbf{W}$. The set of isomorphism classes of unipotent representations of G is usually denoted $\mathcal{E}(G, 1)$ following Lusztig's notations.

2.2.2 Assume that the Coxeter graph of the reductive group $\mathbf{G}$ is a union of subgraphs of type A_m (for various m). Let $\breve{\mathbf{W}}$ be the set of isomorphism classes of irreducible representations of its Weyl group $\mathbf{W}$. The action of the Frobenius F on $\mathbf{W}$ induces an action on $\breve{\mathbf{W}}$, and we consider the fixed point set $\breve{\mathbf{W}}^F$. Then, the following classification theorem is well known.

Theorem ([LS77] Theorem 2.2)**.** *There is a bijection between $\breve{\mathbf{W}}^F$ and the set of isomorphism classes of irreducible unipotent representations of $G = \mathbf{G}^F$.*

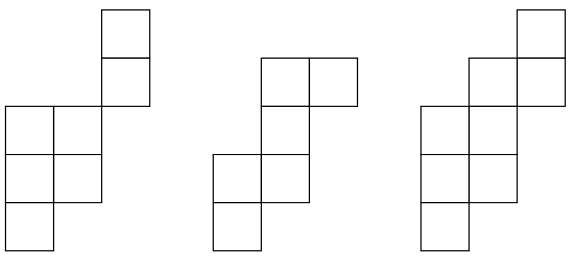
We recall how the bijection is constructed. If $V \in \breve{\mathbf{W}}^F$ is an irreducible F-stable representation of $\mathbf{W}$, according to loc. cit. there is a unique automorphism $\widetilde{F}$ of V of finite order such that

$$R(V) := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \mathrm{Trace}(w \circ \widetilde{F} \,|\, V) R_w$$

is an irreducible representation of G. Then the map $V \mapsto R(V)$ is the desired bijection.
In the case $\mathbf{G} = \mathrm{GL}_n$ with the Frobenius morphism F being either standard or twisted (ie. $G = \mathrm{GL}_n(q)$ or $\mathrm{U}_n(q)$), we have an equality $\breve{\mathbf{W}}^F = \breve{\mathbf{W}}$. Moreover, the automorphism $\widetilde{F}$ is the identity in the former case and multiplication by w_0 on the latter, where w_0 is the element of maximal length in $\mathbf{W}$. Thus, in both cases the irreducible unipotent representations of G are classified by the irreducible representations of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_n$, which in turn are classified by partitions of n or equivalently by Young diagrams. We now recall the underlying combinatorics behind the representation theory of the symmetric group. A general reference is [Jam84].

2.2.3 A partition of n is a tuple $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with $r \geqslant 1$ and the λ_i's are positive integers such that $\lambda_1 + \ldots + \lambda_r = n$. The integer n is called the length of the partition and it is also denoted by $|\lambda|$. If a partition has a series of repeating integers, it is common to write it shortly with an exponent. For instance, the partition $(3, 3, 2, 2, 1)$ of 11 will be denoted $(3^2, 2^2, 1)$. Partitions of n are naturally identified with Young diagrams of size n. The diagram attached to λ has r rows consisting successively of $\lambda_1, \ldots, \lambda_r$ boxes.
To any partition λ of n, one can naturally associate an irreducible representation χ_λ of the symmetric group $\mathfrak{S}_n$. An explicit construction is given, for instance, by the notion of Specht modules as explained in [Jam84] 7.1. In particular, the character $\chi_{(n)}$ is trivial while the character $\chi_{(1^n)}$ is the signature.

2.2.4 We recall the Murnaghan-Nakayama rule which gives a recursive formula to evaluate the characters χ_λ. We first need to introduce skew Young diagrams. Consider a pair λ and μ of two partitions respectively of integers $n+k$ and k. Assume that the Young diagram of μ is contained in the Young diagram of λ. By removing the boxes corresponding to μ from the diagram of λ, one finds a shape consisting of n boxes denoted by $\lambda \backslash \mu$. Any such shape is called a **skew Young diagram** of size n. It is said to be connected if one can go from a given box to any other by moving in a succession of adjacent boxes.
For example, consider the partition $\lambda = (3^2, 2^2, 1)$ and let us define the partitions $\mu_1 = (2^2)$, $\mu_2 = (3, 1^2)$ and $\mu_3 = (2, 1)$. The diagrams below correspond, from left to right, to the skew Young diagrams $\lambda \backslash \mu_i$ for $i = 1, 2, 3$.



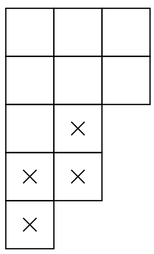
The skew Young diagram $\lambda\backslash\mu_1$ is not connected, whereas the others are connected. A skew Young diagram is said to be a **border strip** if it is connected and if it does not contain any 2×2 square. The **height** of a border strip is defined as its number of rows minus 1. For instance, among the three skew Young diagrams above only $\lambda\backslash\mu_2$ is a border strip. Its size is 6 and its height is 3.
The characters χ_λ are class functions, so we only need to specify their values on conjugacy classes of the symmetric group $\mathfrak{S}_n$. These conjugacy classes are also naturally labelled by partitions of n. Indeed, up to ordering any permutation $\sigma \in \mathfrak{S}_n$ can be uniquely decomposed as a product of $r \geqslant 1$ cycles $c_1, \ldots, c_r$ with disjoint supports. We denote by ν_i the cycle length of c_i and we order them so that $\nu_1 \geqslant \ldots \geqslant \nu_r$. We allow cycles to have length 1, so that the union of the supports of all the c_i's is $\{1, \ldots, n\}$. Thus, we obtain a partition $\nu = (\nu_1, \ldots, \nu_r)$ of n which is called the **cycle type** of the permutation σ. Two permutations are conjugates in $\mathfrak{S}_n$ if and only if they share the same cycle type. We denote by $\chi_\lambda(\nu)$ the value of the character χ_λ on the conjugacy class labelled by ν.

Theorem (Murnaghan-Nakayama rule)**.** *Let λ and ν be two partitions of n. We have*

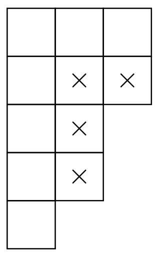
$$\chi_\lambda(\nu) = \sum_S (-1)^{\mathrm{ht}(S)} \chi_{\lambda\backslash S}(\nu\backslash\nu_1),$$

where S runs over the set of all border strips of size ν_1 in the Young diagram of λ, such that removing S from λ gives again a Young diagram. Here, the integer $\mathrm{ht}(S) \in \mathbb{Z}_{\geqslant 0}$ is the height of the border stip S, the Young diagram $\lambda\backslash S$ is the one obtained by removing S from λ, and $\nu\backslash\nu_1$ is the partition of $n - \nu_1$ obtained by removing ν_1 from ν.

Applying the Murnaghan-Nakayama rule in successions results in the value of $\chi_\lambda(\nu)$. We see in particular that $\chi_{(n)}$ is the trivial character whereas $\chi_{(1^n)}$ is the signature. We illustrate the computations with $\lambda = (3^2, 2^2, 1)$ and $\nu = (4^2, 3)$. There are only two elligible border strips of size 4 in the diagram of λ, as marked below.



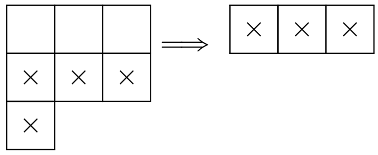
and

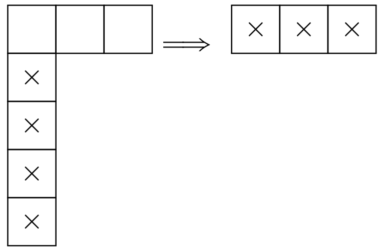


Both border strips have height 2. Thus, the formula gives

$$\chi_{(3^2,2^2,1)}(4^2, 3) = \chi_{(3^2,1)}(4, 3) + \chi_{(3,1^4)}(4, 3).$$

In each of the two Young diagrams obtained after removal of the border strips, there is only one elligible strip of size 4, and eventually the three last remaining boxes form the final border strip of size 3.





Taking the heights of the border strips into account, we find

$$\chi_{(3^2,1)}(4,3) = -\chi_{(3)}(3) = -\chi_{\varnothing} = -1, \qquad \chi_{(3,1^4)}(4,3) = -\chi_{(3)}(3) = -\chi_{\varnothing} = -1.$$

Here, $\varnothing$ denotes the empty partition. The computation finally gives $\chi_{(3^2,2^2,1)}(4^2,3) = -2$.

2.2.5 The irreducible unipotent representation of $\mathrm{U}_n(q)$ (resp. $\mathrm{GL}_n(q)$) associated to χ_λ by the bijection of 2.2.1 Theorem is denoted by $\rho_\lambda^{\mathrm{U}}$ (resp. $\rho_\lambda^{\mathrm{GL}}$). The partition (n) corresponds to the trivial representation and (1^n) to the Steinberg representation in both cases. We will omit the superscript when the group we are talking about is clear from the context.
The degrees of the representations $\rho_\lambda^{\mathrm{GL}}$ and $\rho_\lambda^{\mathrm{U}}$ are given by expressions known as **hook formula**. Given a box $\square$ in the Young diagram of λ, its **hook length** $h(\square)$ is 1 plus the number of boxes lying below it or on its right. For instance, in the following figure the hook length of every box of the Young diagram of $\lambda = (3^2, 2^2, 1)$ has been written inside it.

7	5	2
6	4	1
4	2	
3	1	
1		

Proposition ([GP00] Propositions 4.3.1 and 4.3.5)**.** *Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ be a partition of n. The degrees of the irreducible unipotent representations $\rho_\lambda^{\mathrm{GL}}$ and $\rho_\lambda^{\mathrm{U}}$, respectively of $\mathrm{GL}_n(q)$ and $\mathrm{U}_n(q)$, are given by the following formulas*

$$\deg(\rho_\lambda^{\mathrm{GL}}) = q^{a(\lambda)} \frac{\prod_{i=1}^n q^i - 1}{\prod_{\square \in \lambda} q^{h(\square)} - 1}, \qquad \deg(\rho_\lambda^{\mathrm{U}}) = q^{a(\lambda)} \frac{\prod_{i=1}^n q^i - (-1)^i}{\prod_{\square \in \lambda} q^{h(\square)} - (-1)^{h(\square)}},$$

where $a(\lambda) = \sum_{i=1}^r (i-1)\lambda_i$.

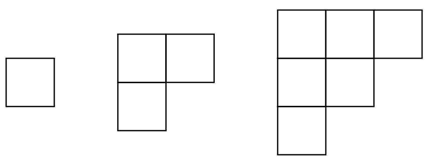
2.2.6 We recall from [GM20] 3.1 and 3.2 some definitions on classical Harish-Chandra theory. A parabolic subgroup of G is a subgroup $P \subset G$ such that there exists an F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ with $P = \mathbf{P}^F$. A Levi complement of G is a subgroup $L \subset G$ such that there exists an F-stable Levi complement $\mathbf{L}$ of $\mathbf{G}$, contained inside some F-stable parabolic subgroup, such that $L = \mathbf{L}^F$. Any parabolic subgroup P of G has a Levi complement L.
Let $L = \mathbf{L}^F$ be a Levi complement of G inside a parabolic subgroup $P = \mathbf{P}^F$. Let $U = \mathbf{U}^F$ be

the F-fixed points of the unipotent radical $\mathbf{U}$ of $\mathbf{P}$. The **Harish-Chandra induction and restriction functors** are defined by the following formulas.

$$\begin{aligned} \mathrm{R}^G_{L\subset P} : \mathrm{Rep}(L) &\to \mathrm{Rep}(G) & \quad {}^*\mathrm{R}^G_{L\subset P} : \mathrm{Rep}(G) &\to \mathrm{Rep}(L) \\ \sigma &\mapsto \mathbb{C}[G/U] \otimes_{\mathbb{C}[L]} \sigma & \rho &\mapsto \mathrm{Hom}_G(\mathbb{C}[G/U], \rho) \end{aligned}$$

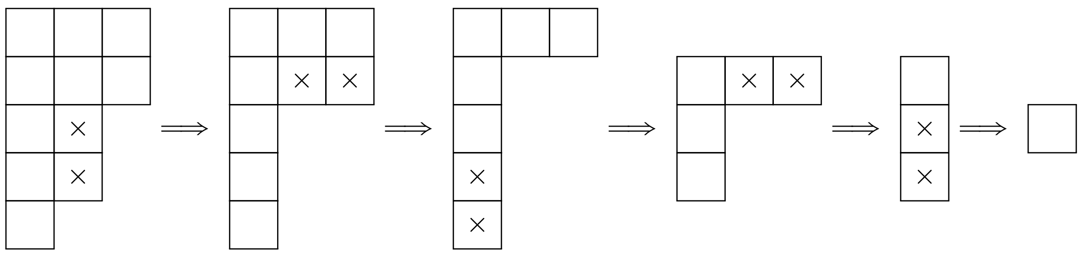
Here, $\mathrm{Rep}(G)$ is the category of complex representations of G, and similarily for $\mathrm{Rep}(L)$. These two functors are adjoint, and up to isomorphism they do not depend on the choice of the parabolic subgroup P containing the Levi complement L. For this reason, we will denote the functors R^G_L and ${}^*\mathrm{R}^G_L$ instead.
An irreducible representation of G is called **cuspidal** if its Harish-Chandra restriction to any proper Levi complement is zero. We consider pairs (L, X) where L is a Levi complement of G and X is an irreducible representation of L. We define an order on the set of such pairs by setting $(L, X) \leqslant (M, Y)$ if $L \subset M$ and if X occurs in the Harish-Chandra restriction of Y to L. A pair is said to be **cuspidal** if it is minimal with respect to this order, in which case X is a cuspidal representation of L. If (L, X) is a cuspidal pair, we will denote by $[L, X]$ its conjugacy class under G.
Given a cuspidal pair (L, X) of G, its associated **Harish-Chandra series** $\mathcal{E}(G, (L, X))$ is defined as the set of isomorphism classes of irreducible constituents in the induction of X to G. Each series is non empty. Two of them are either disjoint or equal, the latter occuring if and only if the two cuspidal pairs are conjugates in G. Thus, the series are indexed by the conjugacy classes of cuspidal pairs $[L, X]$. Moreover, the isomorphism class of any irreducible representation of G belongs to some Harish-Chandra series. Thus, Harish-Chandra series form a partition of the set of isomorphism classes of irreducible representations of G. If ρ is an irreducible representation of G, the conjugacy class $[L, X]$ corresponding to the series to which ρ belongs is called the **cuspidal support** of ρ. If T denotes a maximal torus in G, then the series $\mathcal{E}(G, (T, 1))$ is called the **unipotent principal series** of G.

2.2.7 For the general linear group $\mathrm{GL}_n(q)$, there is no unipotent cuspidal representation unless $n = 1$, in which case the trivial representation is cuspidal. Moreover, the unipotent representations all belong to the principal series. The situation for the unitary group is very different. First, by [Lus77] 9.2 and 9.4 there exists an irreducible unipotent cuspidal representation of $\mathrm{U}_n(q)$ if and only if n is an integer of the form $n = \frac{x(x+1)}{2}$ for some $x \geqslant 0$, and when that is the case it is the one associated to the partition $\Delta_x := (x, x-1, \ldots, 1)$, whose Young diagram has the distinctive shape of a reversed staircase. Here, as a convention $\mathrm{U}_0(q)$ denotes the trivial group.
For example, here are the Young diagrams of Δ_1, Δ_2 and Δ_3. Of course, the one of Δ_0 the empty diagram.



Furthermore the unipotent representations decompose non trivially into various Harish-Chandra series, as we recall from [GM20] 4.3.
We consider an integer $x \geqslant 0$ such that n decomposes as $n = 2a + \frac{x(x+1)}{2}$ for some $a \geqslant 0$. We also consider the standard Levi complement $L_x \simeq \mathrm{GL}_1(q^2)^a \times \mathrm{U}_{\frac{x(x+1)}{2}}(q)$ which corresponds to the choice of simple reflexions $s_{a+1}, \ldots, s_{n-a-1}$. We write ρ_x for the inflation of $\rho^{\mathrm{U}}_{\Delta_x}$ to an irreducible representation of L_x. Then $\mathcal{E}(\mathrm{U}_n(q), 1)$ decomposes as the disjoint union of all the Harish-Chandra series $\mathcal{E}(\mathrm{U}_n(q), (L_x, \rho_x))$ for all possible choices of x. With these notations, the principal unipotent series corresponds to $x = 0$ if n is even and to $x = 1$ if n is odd.

2.2.8 Given an irreducible unipotent representation ρ_λ of $\mathrm{U}_n(q)$, there is a combinatorical way of determining the Harish-Chandra series to which it belongs. We consider the Young diagram of λ. We call **domino** any pair of adjacent boxes in the diagram. It may be either vertical or horizontal. We remove dominoes from the rim of the diagram of λ so that the resulting shape is again a Young diagram, until one can not proceed further. This process results in the Young diagram of the partition Δ_x for some $x \geqslant 0$, and it is called the **2-core** of λ. It does not depend on the successive choices for the dominoes. Then, the representation ρ_λ belongs to the series $\mathcal{E}(\mathrm{U}_n(q), (L_x, \rho_x))$ if and only if λ has 2-core Δ_x.
For instance, the diagram $\lambda = (3^2, 2^2, 1)$ has 2-core Δ_1, as it can be determined by the following steps. We put crosses inside the successive dominoes that we remove from the diagram. Thus, the unipotent representation ρ_λ of $\mathrm{U}_{11}(q)$ belongs to the unipotent principal series $\mathcal{E}(\mathrm{U}_{11}(q), (L_1, \rho_1))$.



2.3 Computing Harish-Chandra induction of unipotent representations in the finite unitary group

2.3.1 In this paragraph, we recover the notations from 2.1.1. We recall from [GM20] 3.2 how Harish-Chandra induction of unipotent representations can be explicitely computed. Let $W = \mathbf{W}^F$ be the Weyl group of G. It is still a Coxeter group, whose set of simple reflexions S is identified with the set of F-orbits on $\mathbf{S}$. Let (L, X) be a cuspidal pair of G. The **relative Weyl group of** L is given by $W_G(L) := N_{\mathbf{G}}(\mathbf{L})^F/L \subset W$. The relative Weyl group of the pair (L, X), also called **the ramification group of** X in [HL83], is the subgroup $W_G(L, X)$ of $W_G(L)$ consisting of elements w such that $wX \simeq X$, where wX denotes the representation $wX(g) := X(wgw^{-1})$ of L. It is yet again a Coxeter group if $\mathbf{G}$ has a connected center or if X is unipotent.
Theorem 3.2.5 of [GM20] establishes an isomorphism between the endomorphism algebra of the induced representation $\mathrm{R}_L^G(X)$ and the complex group ring of the ramification group $W_G(L, X)$.

In particular, this gives an bijection between the Harish-Chandra series $\mathcal{E}(G,(L,X))$ and the set $\mathrm{Irr}(W_G(L,X))$ of isomorphism classes of irreducible complex characters of $W_G(L,X)$. These bijections for G and for various Levi complements in G can be chosen to be compatible with Harish-Chandra induction. This is known as Howlett and Lehrer's comparison theorem which was proved in [HL83].

Theorem ([GM20] Comparison Theorem 3.2.7)**.** *Let (L,X) be a cuspidal pair for the finite group of Lie type G. For every Levi complement M in G containing L, the bijection between* $\mathrm{Irr}(W_M(L,X))$ *and* $\mathcal{E}(M,(L,X))$ *can be taken so that the diagrams*

$$\begin{array}{ccc} \mathbb{Z}\mathcal{E}(G,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_G(L,X)) \\ {\scriptstyle \mathrm{R}_M^G}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{Ind}} \\ \mathbb{Z}\mathcal{E}(M,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_M(L,X)) \end{array} \qquad \begin{array}{ccc} \mathbb{Z}\mathcal{E}(G,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_G(L,X)) \\ {\scriptstyle {}^*\mathrm{R}_M^G}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}} \\ \mathbb{Z}\mathcal{E}(M,(L,X)) & \xrightarrow{\sim} & \mathbb{Z}\mathrm{Irr}(W_M(L,X)) \end{array}$$

are commutative. Here, Ind *and* Res *on the right-hand side of the diagrams are the classical induction and restriction functors for representations of finite groups.*

In other words, computing Harish-Chandra induction and restrictions of representations in G can be entirely done at the level of the associated Coxeter groups. In order to use this statement for unitary groups, we need to make the horizontal arrows explicit and to understand the combinatorics behind induction and restriction of the irreducible representations of the relevant Coxeter groups. This has been explained consistently in [FS90] for classical groups.

2.3.2 We focus on the case of the unitary group. Let $x \geqslant 0$ such that $n = 2a + \frac{x(x+1)}{2}$ for some $a \geqslant 0$. We consider the cuspidal pair (L_x, ρ_x) as in 2.2.7, with $L_x = \mathrm{GL}_1(q^2)^a \times \mathrm{U}_{\frac{x(x+1)}{2}}(q)$. The relative Weyl group $W_{\mathrm{U}_n(q)}(L_x)$ is isomorphic to the Coxeter group of type B_a, which is usually denoted by W_a. Indeed, the Weyl group $W_{\mathrm{U}_n(q)}(L_x)$ admits a presentation by elements $\sigma_1, \ldots, \sigma_{a-1}$ and θ of order 2 satisfying the relations

$$\begin{aligned} \theta\sigma_1\theta\sigma_1 &= \sigma_1\theta\sigma_1\theta, & \theta\sigma_i &= \sigma_i\theta, & &\forall\, 2 \leqslant i \leqslant m-1. \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, & \sigma_i\sigma_j &= \sigma_j\sigma_i, & &\forall\, |i-j| \geqslant 2. \end{aligned}$$

Explicitely, the element σ_i is represented by the permutation matrix of the double transposition $(i \quad i+1)(n-i \quad n-i+1)$ and the element θ by the matrix of the transposition $(1 \quad n)$, all of which belong to $N_{\mathrm{U}_n(q)}(L_x)$. This presentation coincide with the Coxeter group W_a of type B_a, see in [GP00] 1.4.1. Moreover, the ramification group $W_{\mathrm{U}_n(q)}(L_x, \rho_x)$ is equal to the whole of $W_{\mathrm{U}_n(q)}(L_x) \simeq W_a$. The identification between the ramification group and the Coxeter group W_a is naturally induced by the isomorphism between the absolute Weyl group $\mathbf{W}$ and the symmetric group $\mathfrak{S}_n$. In order to proceed further, we need to explain the representation theory of the group W_a.

2.3.3 Let W_a be a Coxeter group of type B_a given with a presentation by elements $\sigma_1, \ldots, \sigma_{a-1}$ and θ satisfying equations as in 2.3.2. For $1 \leqslant i \leqslant a-1$, we define $\theta_i = \sigma_i \ldots \sigma_1\theta\sigma_1 \ldots \sigma_i$. In particular $\theta_0 = \theta$. Following [GP00] 3.4.2, we define **signed blocks** to be elements of the following form. Given $k \geqslant 0$ and $e \geqslant 1$ such that $k + e \leqslant a$, the positive (resp. negative) block of length e starting at k is

$$b^+_{k,e} := \sigma_{k+1}\sigma_{k+2}\dots\sigma_{k+e-1}, \qquad b^-_{k,e} := \theta_k\sigma_{k+1}\sigma_{k+2}\dots\sigma_{k+e-1}.$$

A **bipartition** of a is an ordered pair (α, β) where α is a partition of some integer $0 \leqslant j \leqslant a$ and β is a partition of $a - j$. Given a bipartition (α, β) of a and writing $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_s)$, we define the element

$$w_{\alpha,\beta} := b^-_{k_1,\beta_1}\dots b^-_{k_s,\beta_s} b^+_{k_{s+1},\alpha_1}\dots b^+_{k_{s+r},\alpha_r}$$

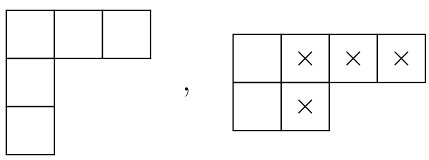
where $k_1 = 0$, $k_{i+1} = k_i + \beta_i$ if $1 \leqslant i \leqslant s$ and $k_{i+1} = k_i + \alpha_{i-s}$ if $s + 1 \leqslant i \leqslant s + r - 1$. In particular, we have $k_{r+s} + \alpha_r = a$. According to [GP00] Proposition 3.4.7, the conjugacy classes in W_a are labelled by bipartitions of a, and a representative of minimal length of the conjugacy class corresponding to the bipartition (α, β) is given by $w_{\alpha,\beta}$. Thus, the irreducible representations of W_a can be labelled by bipartitions of a as well. An explicit construction of these irreducible representations is given in [GP00] 5.5. We will not recall it, however we may again give a method to compute the character values, similar to the Murnaghan-Nakayama formula. The character of the irreducible representation of W_a associated in loc. cit. to the bipartition (α, β) of a will be denoted $\chi_{\alpha,\beta}$. If (γ, δ) is another bipartition of a, we denote by $\chi_{\alpha,\beta}(\gamma, \delta)$ the value of the character $\chi_{\alpha,\beta}$ on the conjugacy class of W_a labelled by (γ, δ).
One can think of a bipartition (α, β) of a as an ordered pair of two Young diagrams of combined size a. A **border strip** of a bipartition (α, β) is a border strip either of the partition α or of β. The height of a border strip is defined in the same way.

Theorem ([GP00] Theorem 10.3.1)**.** *Let (α, β) and (γ, δ) be two bipartitions of a. If $\gamma \neq \varnothing$, let $\epsilon = 1$ and let x be the last integer in the partition γ. If $\gamma = \varnothing$, let $\epsilon = -1$ and let x be the last integer of the partition δ. We have*

$$\chi_{\alpha,\beta}(\gamma, \delta) = \sum_S (-1)^{\mathrm{ht}(S)} \epsilon^{f_S} \chi_{(\alpha,\beta)\backslash S}((\gamma, \delta)\backslash x),$$

where S runs over the set of all border strips of size x in the bipartition (α, β), such that removing S from (α, β) gives again a pair of Young diagrams. Here, the pair of Young diagrams $(\alpha, \beta)\backslash S$ is the one obtained after removing S, and $(\gamma, \delta)\backslash x$ is the bipartition obtained by removing x from (γ, δ). Eventually, the integer f_S is 0 if S is a border strip of α, and it is 1 if S is a border strip of β.

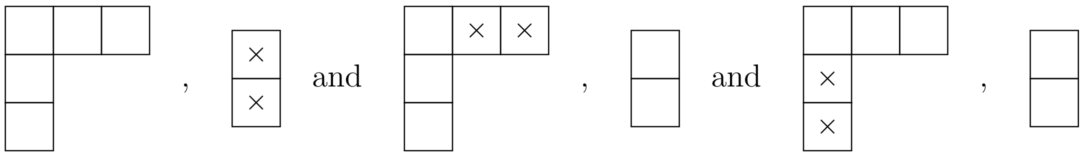
Applying this formula in successions results in the value of $\chi_{(\alpha,\beta)}(\gamma, \delta)$. In particular, one sees that $\chi_{(a),\varnothing}$ is the trivial character and $\chi_{\varnothing,(1^a)}$ is the signature character of W_a. We illustrate the computations with $(\alpha, \beta) = ((3, 1^2), (4, 2))$ and $(\gamma, d) = ((4), (5, 2))$. There is only elligible border strip of size 4 in the pair of diagrams (α, β), as marked below.



This border strip S has height 1. It was taken in the diagram of β so $f_S = 1$. Since $\gamma \neq \varnothing$ we have $\epsilon = 1$. Applying the formula, we obtain

$$\chi_{(3,1^2),(4,2)}((4),(5,2)) = -\chi_{(3,1^2),(1^2)}(\varnothing,(5,2)).$$

We are now looking for border strips of size 2 in the pair of diagrams of the bipartition $(3,1^2),(1^2)$. Three of them are eligible, as marked below.



These three border strips have respective heights $1, 0$ and 1. The corresponding values of f_S are respectively 1, 0 and 0. Moreover, the partition γ is now empty so $\epsilon = -1$. The formula gives

$$\chi_{(3,1^2),(1^2)}(\varnothing,(5,2)) = \chi_{(3,1^2),\varnothing}(\varnothing,(5)) + \chi_{(1^3),(1^2)}(\varnothing,(5)) - \chi_{(3),(1^2)}(\varnothing,(5)).$$

In the bipartitions $((1^3),(1^2))$ and $((3),(1^2))$ there is no border strip of size 5 at all. Thus, the formula tells us that the corresponding character values are 0. On the other hand, the bipartition $((3,1^2),\varnothing)$ consists of a single border strip of size 5 and height 2. The formula gives

$$\chi_{(3,1^2),\varnothing}(\varnothing,(5)) = \chi_{\varnothing} = 1.$$

Putting things together, we deduce that $\chi_{(3,1^2),(4,2)}((4),(5,2)) = -1$.

2.3.4 We may now describe the horizontal arrows in 2.3.1 Theorem for the unitary group. To do this, we need an alternate labelling of the irreducible unipotent representations of the unitary group. We refer to [FS90] for the details.
The new labelling of the irreducible unipotent representations of $\mathrm{U}_n(q)$ involves triples of the form $(\Delta_x, \alpha, \beta)$ where x is a nonnegative integer such that $n = 2a + \frac{x(x+1)}{2}$ for some integer $a \geqslant 0$, and where (α, β) is a bipartition of a. The corresponding representation will be denoted $\rho_{\Delta_x,\alpha,\beta}$. With this labelling, the unipotent Harish-Chandra series $\mathcal{E}(\mathrm{U}_n(q),(L_x,\rho_x))$ consists precisely of all the representations $\rho_{\Delta_x,\alpha,\beta}$ with (α,β) varying over all bipartitions of a. The bijection $\mathbb{Z}\mathcal{E}(\mathrm{U}_n(q),(L_x,\rho_x)) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(W_{\mathrm{U}_n(q)}(L_x,\rho_x))$ involved in the Comparison theorem simply sends $\rho_{\Delta_x,\alpha,\beta}$ to $\chi_{\alpha,\beta}$. Here, we made use of the identification $W_{\mathrm{U}_n(q)}(L_x,\rho_x) \simeq W_a$ as in **??**.
More generally, if M is a standard Levi complement in $\mathrm{U}_n(q)$ containing L_x, we may write $M \simeq \mathrm{U}_b(q) \times \mathrm{GL}_{a_1}(q^2) \times \ldots \times \mathrm{GL}_{a_r}(q^2)$ where $n = 2(a_1 + \ldots + a_r) + b$ and $b \geqslant \frac{x(x+1)}{2}$. The irreducible unipotent representations of M in the Harish-Chandra series $\mathcal{E}(M,(L_x,\rho_x))$ are those of the form $\rho_{\Delta_x,\alpha,\beta} \boxtimes \rho^{\mathrm{GL}}_{\lambda_1} \boxtimes \ldots \boxtimes \rho^{\mathrm{GL}}_{\lambda_r}$ where λ_i is a partition of a_i for $1 \leqslant i \leqslant r$ and (α,β) is a bipartition of the integer $c := \frac{1}{2}\left(b - \frac{x(x+1)}{2}\right)$. On the other hand, the relative Weyl group $W_M(L_x,\rho_x)$ can be identified with the subgroup of $W_{\mathrm{U}_n(q)}(L_x,\rho_x) \simeq W_a$ isomorphic to the product $W_c \times \mathfrak{S}_{a_1} \times \ldots \times \mathfrak{S}_{a_r}$ (note that $c + a_1 + \ldots + a_r = a$). With the notations of 2.3.2, the W_c-component is generated by the elements $\theta, \sigma_1, \ldots \sigma_{c-1}$, the $\mathfrak{S}_{a_1}$-component by the elements $\sigma_{c+1} \ldots, \sigma_{c+a_1-1}$, and so on. Irreducible characters of $W_M(L_x,\rho_x)$ have the shape

$\chi_{\alpha,\beta} \boxtimes \chi_{\lambda_1} \boxtimes \ldots \boxtimes \chi_{\lambda_r}$ where (α, β) is a bipartition of c and λ_i is a partition of a_i for $1 \leqslant i \leqslant r$. Then, according to [FS90] (4.2), the bijection $\mathbb{Z}\mathcal{E}(\mathrm{M}, (L_x, \rho_x)) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(W_M(L_x, \rho_x))$ involved in the Comparison theorem in 2.3.1 sends $\rho_{\Delta_x,\alpha,\beta} \boxtimes \rho^{\mathrm{GL}}_{\lambda_1} \boxtimes \ldots \boxtimes \rho^{\mathrm{GL}}_{\lambda_r}$ to $\chi_{\alpha,\beta} \boxtimes \chi_{\lambda_1} \boxtimes \ldots \boxtimes \chi_{\lambda_r}$.

2.3.5 We explain how the two different labellings of the irreducible unipotent representations of $\mathrm{U}_n(q)$ are related. To do this, one needs the notion of 2-quotient. For the following definitions, we allow partitions to have 0 terms at the end. Thus, let us write $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with $\lambda_r \geqslant 0$. The **β-set** of λ is the sequence of decreasing nonnegative integers $\beta_i := \lambda_i + r - i$ for $1 \leqslant i \leqslant r$. Mapping a partition λ to its β-set gives a bijection between the set of partitions having r terms and the set of decreasing sequences of nonnegative integers of length r. The inverse mapping sends a sequence $(\beta_1 > \ldots > \beta_r \geqslant 0)$ to the partition λ given by $\lambda_i = \beta_i + i - r$. Let λ be a partition of n as above, and let β be its β-set. We let β_{even} (resp. β_{odd}) be the subsequence consisting of all even (resp. odd) integers of β. Then, we define the following sequences.

$$\beta^0 := \left(\frac{\beta_i}{2} \,\middle|\, \beta_i \in \beta_{\mathrm{even}} \right) \qquad\qquad \beta^1 := \left(\frac{\beta_i - 1}{2} \,\middle|\, \beta_i \in \beta_{\mathrm{odd}} \right)$$

The sequences β^0 and β^1 are the β-sets of two partitions, which we call μ^0 and μ^1 respectively. Then, the **2-quotient** of λ is the bipartition (μ^0, μ^1) if r is odd, and (μ^1, μ^0) if r is even. We note that the ordering of μ^0 and μ^1 in the 2-quotient may vary in the literature. Here, we followed the conventions of [FS90] section 1. A different ordering is used in [Jam84] 2.7.29. In loc. cit. Theorem 2.7.37, another construction of the 2-quotient using Young diagrams is proposed.

Let λ' be another partition which differs from λ only by 0 terms at the end. While the β-sets of λ and λ' are not the same, the resulting 2-quotients are equal up to 0 terms at the end of the partitions. Thus, from now on we identify all partitions differing only from 0 terms by removing all of them. The 2-quotient of a partition is then well-defined.

Theorem ([Jam84] Theorem 2.7.30)**.** *A partition λ is uniquely characterized by the data of its 2-core Δ_x and its 2-quotient (λ^0, λ^1). Moreover, the lengths of these partitions are related by the equation*

$$|\lambda| = |\Delta_x| + 2(|\lambda^0| + |\lambda^1|)$$

and $|\Delta_x| = \frac{x(x+1)}{2}$.

For instance, the 2-quotient of the partition $\lambda = (3^2, 2^2, 1)$ is $(2^2, 1)$. Recall that the 2-core of λ is Δ_1. Thus, the equation on the lengths of the partitions is satisfied, as we have $11 = 1 + 2(4+1)$. We may now relate the two labellings $\{\rho^{\mathrm{U}}_\lambda\}$ and $\{\rho_{\Delta_x,\alpha,\beta}\}$ of the irreducible unipotent representations of $\mathrm{U}_n(q)$ together.

Proposition ([FS90] Appendix)**.** *Let λ be a partition of n. Denote by Δ_y its 2-core and by (λ^0, λ^1) its 2-quotient. On the other hand, let $x \geqslant 0$ be such that $n = 2a + \frac{x(x+1)}{2}$ for some $a \geqslant 0$ and let (α, β) be a bipartition of a. Then the irreducible representations ρ^U_λ and $\rho_{\Delta_x,\alpha,\beta}$ are equivalent if and only if $x = y$ and $(\lambda^0, \lambda^1) = (\alpha, \beta)$ if x is even or $(\lambda^0, \lambda^1) = (\beta, \alpha)$ if x is odd.*

For instance, for $\lambda = (3^2, 2^2, 1)$ the representation $\rho^{\mathrm{U}}_{\lambda}$ is equivalent to $\rho_{\Delta_1,(1),(2^2)}$.

2.3.6 In order to apply the comparison theorem 2.3.1 for unitary groups, it remains to understand how to compute inductions in Coxeter groups of type B. Such computations are carried out in [GP00] Section 6.1. It turns out that we will only need one specific case of such inductions, and the corresponding method is known as the Pieri rule for groups of type B.

Proposition ([GP00] 6.1.9)**.** *Let $a \geqslant 1$ and consider $r, s \geqslant 0$ such that $r + s = a$. We think of the group $W_r \times \mathfrak{S}_s$ as a subgroup of W_a as in 2.3.4.*

- *Let (α, β) be a bipartition of r. Then the induced character*

$$\mathrm{Ind}_{W_r \times \mathfrak{S}_s}^{W_a} \left(\chi_{(\alpha,\beta)} \boxtimes \chi_{(s)}\right)$$

is the multiplicity-free sum of all the characters $\chi_{\gamma,\delta}$ such that for some $0 \leqslant k \leqslant s$, the Young diagram of γ (resp. δ) can be obtained from that of α (resp. β) by adding k boxes (resp. $s-k$ boxes) so that no two of them lie in the same column.

- *Let (γ, δ) be a bipartition of a. The restricted character*

$$\mathrm{Res}_{W_r}^{W_a} \left(\chi_{\gamma,\delta}\right)$$

is the multiplicity-free sum of all the characters $\chi_{(\alpha,\beta)}$ such that for some $0 \leqslant k \leqslant s$, the Young diagram of α (resp. β) can be obtained from that of γ (resp. δ) by deleting k boxes (resp. $s-k$ boxes) so that no two of them lie in the same column.

We will use this rule on concrete examples in the sections that follow.

2.4 The cohomology of the Coxeter variety for the unitary group

2.4.1 In this section, we describe the cohomology of the Coxeter varieties for the unitary groups in odd dimension in terms of the classification of unipotent representations that we recalled in the previous section. The cohomology groups are entirely understood by the work of Lusztig in [Lus76].
Let $t \geqslant 0$. The **Coxeter variety** for $\mathrm{U}_{2t+1}(q)$ is the Deligne-Lusztig variety $X_{\varnothing}(\mathrm{cox})$, where cox is any Coxeter element of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_{2t+1}$. Recall that a Coxeter element is a permutation which can be written as the product, in any order, of exactly one simple reflexion for each F-orbit on $\mathbf{S}$. The variety $X_{\varnothing}(\mathrm{cox})$ does not depend on the choice of the Coxeter element. It is defined over $\mathbb{F}_{q^2}$ and is equipped with commuting actions of both $\mathrm{U}_{2t+1}(q)$ and F^2.

Notation. We write $X^t = X_{\varnothing}(\mathrm{cox})$ for the Coxeter variety attached to the unitary group $\mathrm{U}_{2t+1}(q)$. We also write $\mathrm{H}^{\bullet}_c(X^t)$ instead of $\mathrm{H}^{\bullet}_c(X_{\varnothing}(\mathrm{cox}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$, where $\ell \neq p$.

We first recall known facts on the cohomology of X^t from Lusztig's work.

Theorem ([Lus76])**.** *The following statements hold.*

(1) *The variety X^t has dimension t and is affine. The cohomology group $\mathrm{H}_c^{t+i}(X^t)$ is zero unless $0 \leqslant i \leqslant t$.*

(2) *The Frobenius F^2 acts in a semisimple manner on the cohomology of X^t.*

(3) *The group $\mathrm{H}_c^{2t}(X^t)$ is 1-dimensional, the unitary group $\mathrm{U}_{2t+1}(q)$ acts trivially whereas F^2 has a single eigenvalue q^{2t}.*

(4) *The group $\mathrm{H}_c^{t+i}(X^t)$ for $0 \leqslant i < t$ is the direct sum of two eigenspaces of F^2, for the eigenvalues q^{2i} and $-q^{2i+1}$. Each eigenspace is an irreducible unipotent representation of $\mathrm{U}_{2t+1}(q)$.*

(5) *If $0 \leqslant a \leqslant 2t$, the dimension of the eigenspace of $(-q)^a$ inside the sum $\sum_{i \geqslant 0} \mathrm{H}_c^{t+i}(X^t)$ is given by the formula*

$$q^{\frac{(2t-a)(2t+1-a)}{2}} \prod_{j=1}^{2t-a} \frac{q^{a+j} - (-1)^{a+j}}{q^j - (-1)^j}.$$

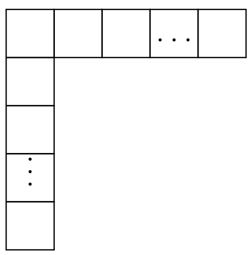
(6) *The sum $\sum_{i \geqslant 0} \mathrm{H}_c^{t+i}(X^t)$ is multiplicity-free as a representation of $\mathrm{U}_{2t+1}(q)$.*

2.4.2 We wish to identify these unipotent representations of $\mathrm{U}_{2t+1}(q)$ occuring in the cohomology of X^t. To this purpose, we start by defining the following partitions. If $0 \leqslant a \leqslant 2t$, we put $\lambda_a^t := (1+a, 1^{2t-a})$. Note that $\lambda_0^t = (1^{2t+1})$ and $\lambda_{2t}^t = (2t+1)$.

Lemma. *For $0 \leqslant i \leqslant t$, the 2-core of λ_{2i}^t is Δ_1 and its 2-quotient is $((1^{t-i}), (i))$.*
For $0 \leqslant i < t$, then the 2-core of λ_{2i+1}^t is Δ_2 and its 2-quotient is $((i), (1^{t-i-1}))$.

In particular, according to 2.3.5 the irreducible unipotent representation $\rho_{\lambda_{2i}^t}$ of $\mathrm{U}_{2t+1}(q)$ is equivalent to the representation $\rho_{\Delta_1,(i),(1^{t-i})}$, and $\rho_{\lambda_{2i+1}^t}$ to $\rho_{\Delta_2,(i),(1^{t-i-1})}$.

Proof. The Young diagram of the partition λ_a^t has the following shape.



The first row has an odd number of boxes when a is even, and an even number of boxes when a is odd. To compute the 2-core, one removes horizontal dominoes from the first row, right to left, and vertical dominoes from the first column, bottom to top. The process results in Δ_1 when a is even and Δ_2 when a is odd.
The partition λ_a^t has $2t+1-a$ non zero terms. Its β-set is given by the sequence

$$\beta = (2t+1, 2t-a, 2t-a-1, \ldots, 1).$$

Assume that $a = 2i$ is even. Then the sequences β^0 and β^1 are given by

$$\beta^0 = (t-i, t-i-1, \dots 1), \qquad \beta^1 = (t, t-i-1, t-i-2, \dots, 0).$$

The sequence β^0 has length $t-i$ while β^1 has length $t-i+1$. The associated permutations are then respectively $\mu_0 = (1^{t-i})$ and $\mu_1 = (i)$. Since $2t+1-a$ is odd, the 2-quotient is given by (μ_0, μ_1) as claimed.
Assume now that $a = 2i+1$ is odd. Then the sequences β^0 and β^1 are given by

$$\beta^0 = (t-i-1, t-i-2, \dots 1), \qquad \beta^1 = (t, t-i-1, t-i-2, \dots, 0).$$

The sequence β^0 has length $t-i-1$ while β^1 has length $t-i+1$. The associated permutations are then respectively $\mu_0 = (1^{t-i-1})$ and $\mu_1 = (i)$. Since $2t+1-a$ is even, the 2-quotient is given by (μ_1, μ_0) as claimed. □

2.4.3 We may now identify the irreducible unipotent representations occuring in the cohomology of the Coxeter variety X^k.

Proposition. *For $0 \leqslant i < t$, the cohomology group of the Coxeter variety for the finite unitary group $\mathrm{U}_{2t+1}(q)$ is given by*

$$\mathrm{H}_c^{t+i}(X^t) = \rho_{\lambda_{2i}^t} \oplus \rho_{\lambda_{2i+1}^t}$$

with the first summand corresponding to the eigenvalue q^{2i} of F^2 and the second to $-q^{2i+1}$. Moreover, $\mathrm{H}_c^{2t}(X^t) = \rho_{\lambda_{2t}^t}$ with eigenvalue q^{2t}.

Before going to the proof, one may notice that the statement is consistent with the dimensions. Indeed, the formula given in 2.4.1 Theorem (5) coincides with the hook formula for the degree of the representation $\rho^{\mathrm{U}}_{\lambda_a^t}$ given in 2.2.5 Proposition.

Proof. First, the statement on the highest cohomology group $\mathrm{H}_c^{2t}(X^t)$ follows from 2.4.1 Theorem (3). It is the only cohomology group in the case $t = 0$. We will prove the formula by induction on t. Let us now assume $t \geqslant 1$ and that the proposition is known for $t-1$. If $0 \leqslant i \leqslant t-1$, we know that $\mathrm{H}_c^{t+i}(X^t)$ is the sum of two irreducible unipotent representations. So let us write

$$\mathrm{H}_c^{t+i}(X^t) = \rho_{\mu_i} \oplus \rho_{\nu_i}$$

where μ_i and ν_i are two partitions of $2t+1$, and so that ρ_{μ_i} corresponds to the eigenvalue q^{2i} of F^2 whereas ρ_{ν_i} corresponds to $-q^{2i+1}$.
We consider the standard Levi complement $L \simeq \mathrm{GL}_1(q^2) \times \mathrm{U}_{2t-1}(q) \subset \mathrm{U}_{2t+1}(q)$. Let V denote the unipotent radical of the standard parabolic subgroup containing L. According to [Lus76] Corollary 2.10, one can build a geometric isomorphism between the quotient variety X^t/V and the product of the Coxeter variety for L and of a copy of $\mathbb{G}_m$. Even though this geometric isomorphism is not L-equivariant, Lusztig proves that the induced map on cohomology is L-equivariant. By a discussion similar to that in 2.1.10, the Coxeter variety for L is isomorphic to the Coxeter variety X^{t-1} for $\mathrm{U}_{2t-1}(q)$. We write ${}^*\mathrm{R}^t_{t-1}$ for the composition of the Harish-Chandra restriction from $\mathrm{U}_{2t+1}(q)$ to L, with the usual restriction from L to the subgroup

$\mathrm{U}_{2t-1}(q)$. For any nonnegative integer i, the $\mathrm{U}_{2t-1}(q), F^2$-equivariant induced map on the cohomology is an isomorphism

$$^*\mathrm{R}^t_{t-1}\left(\mathrm{H}^{t+i}_c(X^t)\right) \simeq \mathrm{H}^{t-1+i}_c(X^{t-1}) \oplus \mathrm{H}^{t-1+(i-1)}_c(X^{t-1})(1). \qquad (**)$$

Here, (1) denotes the Tate twist (the action of F^2 on a twist $M(n)$ is obtained from the action on the space M by multiplication with q^{2n}). The right-hand side of this identity is given by the induction hypothesis. Let us look at the left-hand side.
We fix $0 \leqslant i \leqslant t-1$ and we denote by $(\Delta_x, \alpha, \beta)$ and by $(\Delta_y, \gamma, \delta)$ the alternative labelling of the representations ρ_{μ_i} and ρ_{ν_i} respectively as introduced in 2.3.4 and 2.3.5. By the Howlett-Lehrer comparison theorem for restriction in 2.3.1 and by the Pieri rule in 2.3.6, we know that the restriction $^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_x,\alpha,\beta}\right)$ is the multiplicity-free sum of all the representations $\rho_{\Delta_x,\alpha',\beta'}$ where the bipartition (α', β') can be obtained from (α, β) by removing exactly one box, of either α or β. The similar description also holds for $^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right)$.
By using 2.4.2 Lemma and the induction hypothesis, we may write down the identity $(**)$ explicitely. Moreover, as it is F^2-equivariant we can identify the components corresponding to the same eigenvalues on both sides. We distinguish 4 different cases depending on the values of t and i.

- **Case t = 1**. We only need to consider $i = 0$. On the right-hand side of $(**)$, the second term is 0 because $t-1+(i-1) = -1 < 0$. On the other hand, the first term is $\rho_{\lambda^0_0} \simeq \rho_{\Delta_1,\varnothing,\varnothing}$ and it corresponds to the eigenvalue $(-q)^0 = 1$. By identifying the eigenspaces, we have $^*\mathrm{R}^1_0\left(\rho_{\Delta_x,\alpha,\beta}\right) \simeq \rho_{\Delta_1,\varnothing,\varnothing}$ and $^*\mathrm{R}^1_0\left(\rho_{\Delta_y,\gamma,\delta}\right) = 0$. The second equation implies that there is no box to remove from γ nor from δ. Thus, $\gamma = \delta = \varnothing$. The value of y is given by the relation $2t+1 = 3 = 2(0+0) + \frac{y(y+1)}{2}$, that is $y = 2$. This corresponds to the partition $\nu_0 = \lambda^1_1$. We notice in passing that the representation ρ_{ν_0} is the unique unipotent cuspidal representation of $\mathrm{U}_3(q)$.
  As for μ_0, the equation $^*\mathrm{R}^1_0\left(\rho_{\Delta_x,\alpha,\beta}\right) \simeq \rho_{\Delta_1,\varnothing,\varnothing}$ tells us that there is only one removable box from (α, β). After removal of this box, both partitions are empty. Thus, we deduce that $x = 1$ and $(\alpha, \beta) = (1, \varnothing)$ or $(\varnothing, 1)$. This corresponds respectively to $\mu_0 = \lambda^1_2$ or $\mu_0 = \lambda^1_0$. That is, ρ_{μ_0} is either the trivial or the Steinberg representation of $\mathrm{U}_3(q)$. We can deduce which one it is by comparing the degree of the representations with the formula of 2.4.1 Theorem (5). According to this formula, the dimension of the eigenspace for $(-q)^0$ is q^3. This is precisely the degree of the Steinberg representation $\rho_{\lambda^1_0}$ as given by the hook formula in 2.2.5 Proposition, and it excludes the possibility of ρ_{μ_0} being trivial. Thus, we have $\mu_0 = \lambda^1_0$ as claimed.

From now, we assume $t \geqslant 2$.

- **Case i = 0**. On the right-hand side of $(**)$, the second term is 0 because $t-1+(i-1) = t-2 < t-1$. The first term is $\rho_{\lambda^{t-1}_0} \oplus \rho_{\lambda^{t-1}_1} \simeq \rho_{\Delta_1,\varnothing,(1^{t-1})} \oplus \rho_{\Delta_2,\varnothing,(1^{t-2})}$. Identifying the eigenspaces, we have $^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_x,\alpha,\beta}\right) \simeq \rho_{\Delta_1,\varnothing,(1^{t-1})}$ and $^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right) \simeq \rho_{\Delta_2,\varnothing,(1^{t-2})}$. We deduce that $x = 1$ and $y = 2$. Moreover, it also follows that there is only one removable box in (α, β) and in (γ, δ). After removal, we should obtain respectively $(\varnothing, (1^{t-1}))$ and

$(\varnothing, (1^{t-2}))$. The only possibility is that $(\alpha,\beta) = (\varnothing, (1^t))$ and $(\gamma,\delta) = (\varnothing, (1^{t-1}))$. This corresponds to $\mu_0 = \lambda_0^t$ and $\nu_0 = \lambda_1^t$ as claimed.

– **Case $\mathbf{i = t-1}$**. On the right-hand side of (∗∗), the first term is $\rho_{\lambda^{t-1}_{2(t-1)}} \simeq \rho_{\Delta_1,(t-1),\varnothing}$ and the second term is $\rho_{\lambda^{t-1}_{2(t-2)}} \oplus \rho_{\lambda^{t-1}_{2(t-2)+1}} \simeq \rho_{\Delta_1,(t-2),(1)} \oplus \rho_{\Delta_2,(t-2),\varnothing}$. Identifying the eigenspaces while taking the Tate twist into account, we have ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_x,\alpha,\beta}\right) \simeq \rho_{\Delta_1,(t-1),\varnothing} \oplus \rho_{\Delta_1,(t-2),(1)}$ and ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right) \simeq \rho_{\Delta_2,(t-2),\varnothing}$. We deduce that $x = 1$ and $y = 2$. Moreover, there are two removable boxes in (α,β) and only one removable box in (γ,δ). After removal of one of the two boxes in (α,β), we can get either $((t-1),\varnothing)$ or $((t-2),(1))$; and after removal of the box in (γ,δ) we obtain $((t-2),\varnothing)$. The only possibility is that $(\alpha,\beta) = ((t-1),(1))$ and $(\gamma,\delta) = ((t-1),\varnothing)$. This corresponds to $\mu_{t-1} = \lambda^t_{2(t-1)}$ and $\nu_{t-1} = \lambda^t_{2(t-1)+1}$ as claimed.

– **Case $\mathbf{1 \leqslant i \leqslant t-2}$**. On the right-hand side of (∗∗), the first term is $\rho_{\lambda^{t-1}_{2i}} \oplus \rho_{\lambda^{t-1}_{2i+1}} \simeq \rho_{\Delta_1,(i),(1^{t-1-i})} \oplus \rho_{\Delta_2,(i),(1^{t-2-i})}$. The second term is $\rho_{\lambda^{t-1}_{2(i-1)}} \oplus \rho_{\lambda^{t-1}_{2(i-1)+1}} \simeq \rho_{\Delta_1,(i-1),(1^{t-i})} \oplus \rho_{\Delta_2,(i-1),(1^{t-1-i})}$. Identifying the eigenspaces while taking the Tate twist into account, we have ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_x,\alpha,\beta}\right) \simeq \rho_{\Delta_1,(i),(1^{t-1-i})} \oplus \rho_{\Delta_1,(i-1),(1^{t-i})}$ and ${}^*\mathrm{R}^t_{t-1}\left(\rho_{\Delta_y,\gamma,\delta}\right) \simeq \rho_{\Delta_2,(i),(1^{t-2-i})} \oplus \rho_{\Delta_2,(i-1),(1^{t-1-i})}$. We deduce that $x = 1$ and $y = 2$. Moreover, there are exactly two removable boxes from (α,β) and from (γ,δ). After removal of one of the two boxes in (α,β), we can get either $((i),(1^{t-1-i}))$ or $((i-1),(1^{t-i}))$; and after removal of one of the two boxes in (γ,δ), we can get either $((i),(1^{t-2-i}))$ or $((i-1),(1^{t-1-i}))$. The only possibility is that $(\alpha,\beta) = ((i),(1^{t-i}))$ and $(\gamma,\delta) = ((i),(1^{t-1-i}))$. This corresponds to $\mu_i = \lambda^t_{2i}$ and $\nu_i = \lambda^t_{2i+1}$ as claimed.

□

2.5 The cohomology of the variety $X_I(\mathrm{id})$

2.5.1 We go on with the computation of the cohomology of the variety $X_I(\mathrm{id})$. We use the same notations as in section 1. We first compute the cohomology of each Ekedahl-Oort stratum $X_{I_t}(w_t)$, before using the spectral sequence associated to the stratification to conclude.
Recall that $X_I(\mathrm{id})$ has dimension d, is defined over $\mathbb{F}_{q^2}$ and is equipped with an action of $J \simeq \mathrm{U}_{2d+1}(q)$. As before, we will write $\mathrm{H}^\bullet_c(X_I(\mathrm{id}))$ as a shortcut for $\mathrm{H}^\bullet_c(X_I(\mathrm{id}) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$.

Theorem. *The following statements hold.*

(1) *The cohomology group $\mathrm{H}^i_c(X_I(\mathrm{id}))$ is zero unless $0 \leqslant i \leqslant 2d$. There is an isomorphism $\mathrm{H}^i_c(X_I(\mathrm{id})) \simeq \mathrm{H}^{2d-i}_c(X_I(\mathrm{id}))^\vee(d)$ which is equivariant for the actions of F^2 and of $\mathrm{U}_{2d+1}(q)$.*

(2) *The Frobenius F^2 acts like multiplication by $(-q)^i$ on $\mathrm{H}^i_c(X_I(\mathrm{id}))$.*

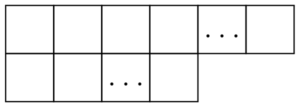
(3) *For $0 \leqslant i \leqslant d$ we have*

$$\mathrm{H}^{2i}_c(X_I(\mathrm{id})) = \bigoplus_{s=0}^{\min(i,d-i)} \rho_{(2d+1-2s,2s)}.$$

For $0 \leqslant i \leqslant d-1$ *we have*

$$\mathrm{H}_c^{2i+1}(X_I(\mathrm{id})) = \bigoplus_{s=0}^{\min(i,d-1-i)} \rho_{(2d-2s,2s+1)}.$$

Thus, in the cohomology of $X_I(\mathrm{id})$ all the representations associated to a Young diagram with at most 2 rows occur, and there is no other. Such a diagram has the following general shape.



We may rephrase the result by using the alternative labelling of the irreducible unipotent representations as in 2.3.5. The partition $(2d+1-2s, 2s)$ has 2-core Δ_1 and 2-quotient $(\varnothing, (d-s, s))$; whereas the partition $(2d-2s, 2s+1)$ has 2-core Δ_2 and 2-quotient $((d-1-s, s), \varnothing)$. Thus, according to 2.3.5 Proposition, we have

$$\rho_{(2d+1-2s,2s)} \simeq \rho_{\Delta_1,(d-s,s),\varnothing}, \qquad \rho_{(2d-2s,2s+1)} \simeq \rho_{\Delta_2,(d-1-s,s),\varnothing}.$$

In particular, all irreducible representations in the cohomology groups of even index belong to the unipotent principal series $\mathcal{E}(\mathrm{U}_{2d+1}(q), (L_1, \rho_1))$, whereas all the ones in the groups of odd index belong to the Harish-Chandra series $\mathcal{E}(\mathrm{U}_{2d+1}(q), (L_2, \rho_2))$.

Proof. Point (1) of the statement follows from a general property of the cohomology groups, namely Poincaré duality. It is due to the fact that $X_I(\mathrm{id})$ is projective and smooth. It also implies the purity of the Frobenius F^2 on the cohomology : we know at this stage that all eigenvalues of F^2 on $\mathrm{H}_c^i(X_I(\mathrm{id}))$ have complex modulus q^i under any choice of an isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$.
We prove the points (2) and (3) by explicit computations. As in 2.4.2, we denote by λ_a^t the partition $(1+a, 1^{2t-a})$ of $2t+1$. Let $0 \leqslant t \leqslant d$. For $0 \leqslant a \leqslant 2t$ we will write

$$\mathrm{R}_a^t := \mathrm{R}_{L_{K_t}}^{\mathrm{U}_{2d+1}(q)} \left(\rho_{(d-t)}^{\mathrm{GL}} \boxtimes \rho_{\lambda_a^t}^{\mathrm{U}}\right).$$

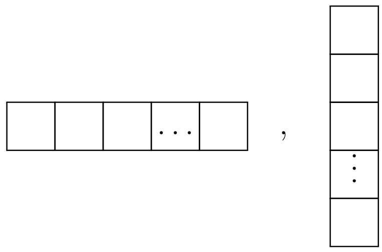
Recall that 2.1.9 Proposition gives an isomorphism between the Ekedahl-Oort stratum $X_{I_t}(w_t)$ and the variety $\mathrm{U}_{2d+1}(q)/U_{K_t} \times_{L_{K_t}} X_{I_t}^{\mathbf{L}_{K_t}}(w_t)$. It implies that the cohomology of the Ekedahl-Oort stratum is the Harish-Chandra induction of the cohomology of the Deligne-Lusztig variety $X_{I_t}^{\mathbf{L}_{K_t}}(w_t)$. According to 2.1.10, this cohomology is related to that of the Coxeter variety for $\mathrm{U}_{2t+1}(q)$. Combining with the formula of 2.4.3 Proposition, for $0 \leqslant i \leqslant t-1$ it follows that

$$\mathrm{H}_c^{t+i}(X_{I_t}(w_t)) = \mathrm{R}_{2i}^t \oplus \mathrm{R}_{2i+1}^t, \qquad \mathrm{H}_c^{2t}(X_{I_t}(w_t)) = \mathrm{R}_{2t}^t.$$

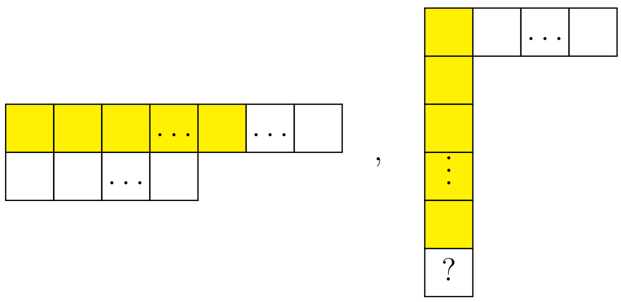
The representation R_a^t in this formula is associated to the eigenvalue $(-q)^a$ of F^2.

We first compute R_a^t explicitely. By the combination of the Howhlett-Lehrer comparison theorem in 2.3.1 and the Pieri rule for groups of type B as in 2.3.6, one can compute the Harish-Chandra induction R_a^t by adding $d-t$ boxes to the bipartition corresponding to the representation $\rho_{\lambda_a^t}^{\mathrm{U}}$ with no two added boxes in the same column. Recall from 2.4.2 Lemma that the

representation $\rho_{\lambda^t_{2i}}$ of $\mathrm{U}_{2t+1}(q)$ is equivalent to the representation $\rho_{\Delta_1,(i),(1^{t-i})}$, and that $\rho_{\lambda^t_{2i+1}}$ is equivalent to $\rho_{\Delta_2,(i),(1^{t-1-i})}$.
In order to illustrate the argument, let us say that we want to add N boxes to a bipartition of the shape as in the figure below, so that no two added boxes lie in the same column.



We will add N_1 boxes to the first diagram and N_2 to the second, where $N = N_1 + N_2$. In the first diagram, the only places where we can add boxes are in the second row from left to right, and at the end of the first row. Because no two added boxes must be in the same column, the number of boxes we add on the second row must be at most the number of boxes already lying in the first row. Of course, it must also be at most N_1.
In the second diagram, the only places where we can add boxes are at the bottom of the first column and at the end of the first row. Because no two added boxes must be in the same column, we can only put up to one box at the bottom of the first column and all the remaining ones will align at the end of the first row.
At the end of the process, we will obtain a bipartition of the following general shape.



We colored in yellow the boxes that were already there before we added new ones. The box with a question mark may or may not be placed there.
We now make the result more precise, and write down exactly what the irreducible components of R^t_a are depending on the parity of a.

- For $0 \leqslant i \leqslant t$, the representation R^t_{2i} is the multiplicity-free sum of all the representations $\rho_{\Delta_1,\alpha,\beta}$ where the bipartition (α, β) satisfies, for some $0 \leqslant x \leqslant d-t$,

$$\begin{cases} \alpha = (i + x - s, s) \text{ for some } 0 \leqslant s \leqslant \min(x, i), \\ \beta = (d - t - x, 1^{t-i}) \text{ or } (d - t - x + 1, 1^{t-i-1}). \end{cases}$$

- For $0 \leqslant i \leqslant t-1$, the representation R^t_{2i+1} is the multiplicity-free sum of all representations

$\rho_{\Delta_2,\alpha,\beta}$ where the bipartition (α, β) satisfies, for some $0 \leqslant x \leqslant d-t$,

$$\begin{cases} \alpha = (i+x-s, s) \text{ for some } 0 \leqslant s \leqslant \min(x, i), \\ \beta = (d-t+1-x, 1^{t-1-i}) \text{ or } (d-t+2-x, 1^{t-2-i}). \end{cases}$$

In our notations, we used the convention that the partitions (0) and (1^0) are the empty partition $\varnothing$. The integer x corresponds to the number of boxes we add to the first partition. We notice that if i takes the maximal value, there is only one possibility for β that is respectively $(d-t-x)$ in the first case and $(d-t+1-x)$ in the second case.
Recall from 2.1.7 that the variety $X_I(\mathrm{id})$ is the union of the Ekedahl-Oort strata $X_{I_t}(w_t)$ for $0 \leqslant t \leqslant d$ and the closure of the stratum for t is the union of all strata $X_{I_s}(w_s)$ for $s \leqslant t$. At the level of cohomology, it translates into the following $F^2, \mathrm{U}_{2d+1}(q)$-equivariant spectral sequence

$$\mathrm{E}_1^{t,i} : \mathrm{H}_c^{t+i}(X_{I_t}(w_t)) \implies \mathrm{H}_c^{t+i}(X_I(\mathrm{id})).$$

The first page of the sequence is drawn in the Figure 1, it has a triangular shape.

$$\begin{array}{ccccccccc}
 & & & & & & & & \mathrm{R}_{2d}^{d} \\
 & & & & & & \mathrm{R}_{2d-2}^{d-1} & \longrightarrow & \mathrm{R}_{2d-2}^{d} \oplus \mathrm{R}_{2d-1}^{d} \\
 & & & & \iddots & & & & \vdots \\
 & & \mathrm{R}_4^2 & \longrightarrow & \cdots & \longrightarrow & \mathrm{R}_4^{d-1} \oplus \mathrm{R}_5^{d-1} & \longrightarrow & \mathrm{R}_4^{d} \oplus \mathrm{R}_5^{d} \\
 & \mathrm{R}_2^1 \longrightarrow & \mathrm{R}_2^2 \oplus \mathrm{R}_3^2 & \longrightarrow & \cdots & \longrightarrow & \mathrm{R}_2^{d-1} \oplus \mathrm{R}_3^{d-1} & \longrightarrow & \mathrm{R}_2^{d} \oplus \mathrm{R}_3^{d} \\
\mathrm{R}_0^0 \longrightarrow & \mathrm{R}_0^1 \oplus \mathrm{R}_1^1 \longrightarrow & \mathrm{R}_0^2 \oplus \mathrm{R}_1^2 & \longrightarrow & \cdots & \longrightarrow & \mathrm{R}_0^{d-1} \oplus \mathrm{R}_1^{d-1} & \longrightarrow & \mathrm{R}_0^{d} \oplus \mathrm{R}_1^{d}
\end{array}$$

Figure 1: The first page of the spectral sequence.

The representation R_a^t corresponds to the eigenvalue $(-q)^a$ of F^2 as before. The only eigenvalues of F^2 on the i-th row of the spectral sequence are q^{2i} and $-q^{2i+1}$. In particular, the eigenvalues on two distinct rows are different. Since the differentials in deeper pages of the sequence map terms from different rows, their F^2-equivariance implies that they vanish. Therefore, the sequence degenerates on the second page.
Moreover, by the machinery of spectral sequences, for $0 \leqslant k \leqslant 2d$ there exists a filtration by $\mathrm{U}_{2d+1}(q) \times \langle F^2 \rangle$-modules on $\mathrm{H}_c^k(X_I(\mathrm{id}))$ whose graded components are the terms of the second page lying on the anti-diagonal $t + i = k$. Since the group algebra $\overline{\mathbb{Q}_\ell}[\mathrm{U}_{2d+1}(q)]$ is semi-simple, the filtration splits, meaning that $\mathrm{H}_c^k(X_I(\mathrm{id}))$ is actually the direct sum of the graded components. The purity of $\mathrm{H}_c^k(X_I(\mathrm{id}))$ then implies that all the terms of the second page lying on the

anti-diagonal $t+i=k$, which are associated to an eigenvalue whose modulus is not equal to q^k, must be zero. Therefore, the second page has the shape described in Figure 2. The Frobenius F^2 acts via q^{2i} on the term $\mathrm{E}_2^{i,i}$, and via $-q^{2i+1}$ on the term $\mathrm{E}_2^{i+1,i}$. Point (2) of the Theorem readily follows.

$$
\begin{array}{cccccc}
 & & & & & \mathrm{E}_2^{d,d} \\
 & & & & \mathrm{E}_2^{d-1,d-1} & \mathrm{E}_2^{d,d-1} \\
 & & & \iddots & & \vdots \\
 & \mathrm{E}_2^{1,1} & \mathrm{E}_2^{2,1} & 0 & \cdots & 0 \\
\mathrm{E}_2^{0,0} & \mathrm{E}_2^{1,0} & 0 & 0 & \cdots & 0
\end{array}
$$

Figure 2: The second page of the spectral sequence.

By the previous computations, we understand precisely all the terms in the first page of the spectral sequence. The key observation to compute the second page is that two terms on the first page which lie on the same row, but are separated by at least 2 arrows, do not have any irreducible component in common. We make the argument more precise in the following two paragraphs, distinguishing the cohomology groups of even and odd index.

We first compute the cohomology group $\mathrm{H}_c^{2t}(X_I(\mathrm{id}))$ for $0 \leqslant t \leqslant d$. We look at the following portion of the first page

$$\mathrm{R}_{2t}^{t} \longrightarrow \mathrm{R}_{2t}^{t+1} \oplus \mathrm{R}_{2t+1}^{t+1} \longrightarrow \mathrm{R}_{2t}^{t+2} \oplus \mathrm{R}_{2t+1}^{t+2} .$$

By extracting the eigenspaces corresponding to q^{2t}, we actually have the following sequence

$$\mathrm{R}_{2t}^{t} \xrightarrow{u} \mathrm{R}_{2t}^{t+1} \xrightarrow{v} \mathrm{R}_{2t}^{t+2} .$$

The representation R_{2t}^{t} is the sum of all the representations $\rho_{\Delta_1,\alpha,\beta}$ where for some $0 \leqslant x \leqslant d-t$ and for some $0 \leqslant s \leqslant \min(x,t)$, we have $\alpha = (t+x-s, s)$ and $\beta = (d-t-x)$.
The representation R_{2t}^{t+1} is the sum of all the representations $\rho_{\Delta_1,\alpha',\beta'}$ where for some $0 \leqslant x' \leqslant d-t-1$ and for some $0 \leqslant s \leqslant \min(x',t)$, we have $\alpha' = (t+x'-s, s)$ and $\beta' = (d-t-x')$ or $(d-t-x'-1, 1)$.
The quotient space $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is isomorphic to the eigenspace of q^{2t} in $E_2^{t+1,t}$, which is zero. Besides, in the representation R_{2t}^{t+2} all the irreducible components have the shape $\rho_{\Delta_1,\alpha'',\beta''}$ with β'' a partition of length 2 or 3. In particular, all the representations $\rho_{\Delta_1,\alpha',\beta'}$ of R_{2t}^{t+1} with β' a

partition of length 1 automatically lie inside $\mathrm{Ker}(v) = \mathrm{Im}(u)$. Such representations correspond to all the irreducible components $\rho_{\Delta_1,\alpha,\beta}$ of R^t_{2t} having $x \neq d-t$. Thus, none of them lies in $\mathrm{Ker}(u) \simeq E_2^{t,t}$.
The remaining components of R^t_{2t} are those having $x = d-t$, and they do not occur in the codomain of u so that they lie in $\mathrm{Ker}(u)$. By the previous argument, they must form the whole of $\mathrm{Ker}(u)$.
Thus, we have proved that

$$E_2^{t,t} \simeq \mathrm{H}^{2t}_c(X_I(\mathrm{id})) \simeq \mathrm{Ker}(u) = \bigoplus_{s=0}^{\min(t,d-t)} \rho_{\Delta_1,(t-s,s),\varnothing}$$

and it coincides with the formula of point (3).

We now compute the cohomology group $\mathrm{H}^{2t+1}_c(X_I(\mathrm{id}))$ for $0 \leqslant t \leqslant d-1$. We look at the following portion of the first page

$$\mathrm{R}^t_{2t} \longrightarrow \mathrm{R}^{t+1}_{2t} \oplus \mathrm{R}^{t+1}_{2t+1} \longrightarrow \mathrm{R}^{t+2}_{2t} \oplus \mathrm{R}^{t+2}_{2t+1} \longrightarrow \mathrm{R}^{t+3}_{2t} \oplus \mathrm{R}^{t+3}_{2t+1} \ .$$

By extracting the eigenspaces corresponding to $-q^{2t+1}$, we actually have the following sequence

$$0 \longrightarrow \mathrm{R}^{t+1}_{2t+1} \xrightarrow{\ u\ } \mathrm{R}^{t+2}_{2t+1} \xrightarrow{\ v\ } \mathrm{R}^{t+3}_{2t+1} \ .$$

The representation R^{t+1}_{2t+1} is the sum of all the representations $\rho_{\Delta_2,\alpha,\beta}$ where for some $0 \leqslant x \leqslant d-t-1$ and for some $0 \leqslant s \leqslant \min(x,t)$, we have $\alpha = (t+x-s, s)$ and $\beta = (d-t-x)$.
The representation R^{t+2}_{2t+1} is the sum of all the representations $\rho_{\Delta_2,\alpha',\beta'}$ where for some $0 \leqslant x' \leqslant d-t-2$ and for some $0 \leqslant s \leqslant \min(x',t)$, we have $\alpha' = (t+x'-s, s)$ and $\beta' = (d-t-1-x', 1)$ or $(d-t-x')$.
The quotient space $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is isomorphic to the eigenspace of $-q^{2t+1}$ in $E_2^{t+2,t}$, which is zero. Besides, in the representation R^{t+3}_{2t+1} all the irreducible components have the shape $\rho_{\Delta_2,\alpha'',\beta''}$ with β'' a partition of length 2 or 3. In particular, all the representations $\rho_{\Delta_2,\alpha',\beta'}$ of R^{t+2}_{2t+1} with β' a partition of length 1 automatically lie inside $\mathrm{Ker}(v) \simeq \mathrm{Im}(u)$. Such representations correspond to all the irreducible components $\rho_{\Delta_2,\alpha,\beta}$ of R^{t+1}_{2t+1} having $x \neq d-t-1$. Thus, none of them lies in $\mathrm{Ker}(u) \simeq E_2^{t+1,t}$.
The remaining components of R^{t+1}_{2t+1} are those having $x = d-t-1$, and they do not occur in the codomain of u so that they lie in $\mathrm{Ker}(u)$. By the argument above, they must form the whole of $\mathrm{Ker}(u)$.
Thus, we have proved that

$$E_2^{t+1,t} \simeq \mathrm{H}^{2t+1}_c(X_I(\mathrm{id})) \simeq \mathrm{Ker}(u) = \bigoplus_{s=0}^{\min(d-t-1,t)} \rho_{\Delta_2,(t-1-s,s),\varnothing}$$

and one may check that it coincides with the formula of point (3). $\square$

3 On the cohomology of the basic unramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

Notations

Throughout this section, we fix an integer $n \geqslant 1$ and we write $m := \lfloor \frac{n-1}{2} \rfloor$ so that $n = 2m+1$ or $2(m+1)$ according to whether n is odd or even. We also fix an odd prime number p. If k is a perfect field of characteristic p, we denote by $W(k)$ the ring of Witt vectors and by $W(k)_{\mathbb{Q}}$ its fraction field, which is an unramified extension of $\mathbb{Q}_p$. We denote by $\sigma_k : x \mapsto x^p$ the Frobenius of $\mathrm{Gal}(k/\mathbb{F}_p)$, and we use the same notation for its (unique) lift to $\mathrm{Gal}(W(k)_{\mathbb{Q}}/\mathbb{Q}_p)$. If k'/k is a perfect field extension then $(\sigma_{k'})_{|k} = \sigma_k$, so we can remove the subscript and write σ unambiguously instead. If $q = p^e$ is a power of p, we write $\mathbb{F}_q$ for the field with q elements. In the special case where $q = p^2$, we also use the alternative notation $\mathbb{Z}_{p^2} = W(\mathbb{F}_{p^2})$ and $\mathbb{Q}_{p^2} = W(\mathbb{F}_{p^2})_{\mathbb{Q}}$. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.

3.1 The Bruhat-Tits stratification on the PEL unitary Rapoport-Zink space of signature $(1, n-1)$

3.1.1 The PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$

3.1.1.1 In [VW11], the authors introduce the PEL unitary Rapoport-Zink space $\mathcal{M}$ of signature $(1, n-1)$ as a moduli space, classifying the deformations of a given p-divisible group equipped with additional structures. We briefly recall the construction. Let Nilp denote the category of schemes over $\mathbb{Z}_{p^2}$ where p is locally nilpotent. For $S \in \mathrm{Nilp}$, a **unitary p-divisible group of signature $(1, n-1)$** over S is a triple (X, ι_X, λ_X) where

- X is a p-divisible group over S.
- $\iota_X : \mathbb{Z}_{p^2} \to \mathrm{End}(X)$ is a $\mathbb{Z}_{p^2}$-action on X such that the induced action on its Lie algebra satisfies the **signature $(1, n-1)$ condition**: for every $a \in \mathbb{Z}_{p^2}$, the characteristic polynomial of $\iota_X(a)$ acting on $\mathrm{Lie}(X)$ is given by

$$(T-a)^1(T-\sigma(a))^{n-1} \in \mathbb{Z}_{p^2}[T] \subset \mathcal{O}_S[T].$$

- $\lambda_X : X \xrightarrow{\sim} {}^tX$ is a $\mathbb{Z}_{p^2}$-linear polarization where ${}^tX$ denotes the Serre dual of X.

The $\mathbb{Z}_{p^2}$-linearity of λ_X is with respect to the $\mathbb{Z}_{p^2}$-actions ι_X and the induced action $\iota_{{}^tX}$ on the dual. A specific example of unitary p-divisible group over $\mathbb{F}_{p^2}$ is given in [VW11] 2.4 by means of covariant Dieudonné theory. We denote it by $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ and call it the **standard unitary p-divisible group**. The p-divisible group $\mathbb{X}$ is superspecial. The following set-valued functor $\mathcal{M}$ defines a moduli problem classifying deformations of $\mathbb{X}$ by quasi-isogenies. More precisely, for $S \in \mathrm{Nilp}$ the set $\mathcal{M}(S)$ consists of all isomorphism classes of tuples $(X, \iota_X, \lambda_X, \rho_X)$ such that

– (X, λ_X, ρ_X) is a unitary p-divisible group of signature $(1, n-1)$ over S.

– $\rho_X : X \times_S \overline{S} \to \mathbb{X} \times_{\mathbb{F}_{p^2}} \overline{S}$ is a $\mathbb{Z}_{p^2}$-linear quasi-isogeny compatible with the polarizations, in the sense that ${}^t\rho_X \circ \lambda_{\mathbb{X}} \circ \rho_X$ is a $\mathbb{Q}_p^\times$-multiple of λ_X.

In the second condition, $\overline{S}$ denotes the special fiber of S. By [RZ96] Corollary 3.40, this moduli problem is represented by a separated formal scheme $\mathcal{M}$ over $\mathrm{Spf}(\mathbb{Z}_{p^2})$, called a **Rapoport-Zink space**. It is formally locally of finite type, and because the associated PEL datum is unramified it is also formally smooth over $\mathbb{Z}_{p^2}$. The **reduced special fiber** of $\mathcal{M}$ is the reduced $\mathbb{F}_{p^2}$-scheme $\mathcal{M}_{\mathrm{red}}$ defined by the maximal ideal of definition. By loc. cit. Proposition 2.32, each irreducible component of $\mathcal{M}_{\mathrm{red}}$ is projective. The geometry of the special fiber has been thoroughly described in [Vol10] and [VW11], and we recall some of their constructions.

3.1.1.2 Rational points of $\mathcal{M}$ over a perfect field extension k of $\mathbb{F}_{p^2}$ can be understood in terms of semi-linear algebra by means of Dieudonné theory. We denote by $M(\mathbb{X})$ the Dieudonné module of $\mathbb{X}$, this is a free $\mathbb{Z}_{p^2}$-module of rank $2n$. We denote by $N(\mathbb{X}) := M(\mathbb{X}) \otimes \mathbb{Q}_{p^2}$ its isocrystal. By construction, the Frobenius and the Verschiebung agree on $N(\mathbb{X})$. In particular, we have $\mathbf{F}^2 = p \cdot \mathrm{id}$ on the isocrystal. The $\mathbb{Z}_{p^2}$-action $\iota_{\mathbb{X}}$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading $M(\mathbb{X}) = M(\mathbb{X})_0 \oplus M(\mathbb{X})_1$ as a sum of two free $\mathbb{Z}_{p^2}$-modules of rank n. The same goes for the isocrystal $N(\mathbb{X}) = N(\mathbb{X})_0 \oplus N(\mathbb{X})_1$ where $N(\mathbb{X})_i = M(\mathbb{X})_i \otimes \mathbb{Q}_{p^2}$ for $i = 0, 1$. The polarization $\lambda_{\mathbb{X}}$ induces a perfect σ-symplectic form on $N(\mathbb{X})$ which stabilizes the lattice $M(\mathbb{X})$ and for which $\mathbf{F}$ is self-adjoint. Compatibility with $\iota_{\mathbb{X}}$ implies that the pieces $N(\mathbb{X})_i$ are totally isotropic for $i = 0, 1$ and dual of each other. Moreover, the Frobenius $\mathbf{F}$ is then 1-homogeneous with respect to this grading. As in [VW11] 2.6, it is possible to modify the symplectic pairing so that it restricts to a non-degenerate $\mathbb{Q}_{p^2}$-valued σ-hermitian form $\{\cdot,\cdot\}$ on $N(\mathbb{X})_0$.

Notation. From now on, we will write $\mathbf{V} := N(\mathbb{X})_0$ and $\mathbf{M} := M(\mathbb{X})_0$.

Then $\mathbf{V}$ is a $\mathbb{Q}_{p^2}$-hermitian space of dimension n, and $\mathbf{M}$ is a given $\mathbb{Z}_{p^2}$-lattice, ie. a $\mathbb{Z}_{p^2}$-submodule containing a basis of $\mathbf{V}$. Given two lattices M_1 and M_2, the notation $M_1 \overset{d}{\subset} M_2$ means that $M_1 \subset M_2$ and the quotient module M_2/M_1 has length d. The integer d is called the **index** of M_1 in M_2, and is denoted $d = [M_2 : M_1]$. We have $0 \leqslant d \leqslant n$. Given a lattice $M \subset \mathbf{V}$, the dual lattice is denoted $M^\vee$. It consists of all the vectors $v \in \mathbf{V}$ such that $\{v, M\} \subset \mathbb{Z}_{p^2}$. Then, by construction the lattice $\mathbf{M}$ satisfies

$$p\mathbf{M}^\vee \overset{1}{\subset} \mathbf{M} \overset{n-1}{\subset} \mathbf{M}^\vee.$$

The existence of such a lattice $\mathbf{M}$ in $\mathbf{V}$ implies that the σ-hermitian structure on $\mathbf{V}$ is isomorphic to any one described by the following two matrices

$$T_{\mathrm{odd}} := A_{2m+1}, \qquad T_{\mathrm{even}} := \begin{pmatrix} & & & A_m \\ & 1 & 0 & \\ & 0 & p & \\ A_m & & & \end{pmatrix}.$$

Here, A_k denotes the $k \times k$ matrix with 1's in the antidiagonal and 0 everywhere else.

Proposition ([Vol10] 1.15). *There exists a basis of* $\mathbf{V}$ *such that* $\{\cdot,\cdot\}$ *is represented by the matrix* T_{odd} *is* n *is odd and by* T_{even} *if* n *is even.*

3.1.1.3 A **Witt decomposition** on $\mathbf{V}$ is a set $\{L_i\}_{i\in I}$ of isotropic lines in $\mathbf{V}$ such that the following conditions are satisfied:

- For every $i \in I$, there is a unique $i' \in I$ such that $\{L_i, L_{i'}\} \neq 0$.
- The sum of the L_i's is direct.
- The orthogonal in $\mathbf{V}$ of the direct sum of the L_i's is an anisotropic subspace of $\mathbf{V}$.

Because each line L_i is isotropic, in the first condition one necessarily has $(i')' = i$ and $i \neq i'$. As a consequence, the cardinality of the index set I is an even number $\#I = 2w(\mathbf{V})$. The integer $w = w(\mathbf{V})$ is called the **Witt index** of $\mathbf{V}$ and it does not depend on the choice of a Witt decomposition. We write L^{an} for the orthogonal of the direct sum of the L_i's. The dimension of L^{an} is $n^{\mathrm{an}} := n - 2w$, therefore it is also independent on the choice of the Witt decomposition.
Given any Witt decomposition, one may always find vectors $e_i \in L_i$ such that $\{e_i, e_j\} = \delta_{j,i'}$. Together with a choice of an orthogonal basis for L^{an}, these vectors define a basis of $\mathbf{V}$ which is said to be **adapted to the Witt decomposition**. For any $i \in I$, the direct sum $L_i \oplus L_{i'}$ is isometric to the hyperbolic plane $\mathbf{H}$. Therefore, we obtain a decomposition

$$\mathbf{V} = w\mathbf{H} \oplus L^{\mathrm{an}}.$$

We may always rearrange the index set so that $I = \{-w, \dots, -1, 1, \dots, w\}$ and for every $i \in I$, we have $\{L_i, L_{-i}\} \neq 0$. Thus, the i' associated to i by the first condition is $-i$. Of course, this process is not unique as it relies on a choice of an ordering for the lines $\{L_i\}_{i\in I}$. In this context, we write L_0 instead of L^{an}.

3.1.1.4 We fix once and for all a basis e of $\mathbf{V}$ in which the hermitian form is represented by the matrix T_{odd} or T_{even}. In the case $n = 2m+1$ is odd, we will denote it

$$e = (e_{-m}, \dots, e_{-1}, e_0^{\mathrm{an}}, e_1, \dots, e_m),$$

and in the case $n = 2(m+1)$ is even we will denote it

$$e = (e_{-m}, \dots, e_{-1}, e_0^{\mathrm{an}}, e_1^{\mathrm{an}}, e_1, \dots, e_m).$$

In this way, for every $1 \leqslant s \leqslant m$ the subspace generated by e_{-s} and e_s is isomorphic to the hyperbolic plane $\mathbf{H}$. Moreover, the vectors with a superscript $\cdot^{\mathrm{an}}$ generate an anisotropic subspace $\mathbf{V}^{\mathrm{an}}$ of $\mathbf{V}$. The choice of such a basis gives a Witt decomposition

$$\mathbf{V} = m\mathbf{H} \oplus \mathbf{V}^{\mathrm{an}}$$

consisting of an orthogonal sum of m copies of $\mathbf{H}$ and of the anisotropic subspace $\mathbf{V}^{\mathrm{an}}$. In particular, the Witt index of $\mathbf{V}$ is m and we have $n^{\mathrm{an}} = 1$ or 2 depending on whether n is odd or even respectively.

3.1.1.5 Given a perfect field extension k of $\mathbb{F}_{p^2}$, we denote by $\mathbf{V}_k$ the base change $\mathbf{V} \otimes_{\mathbb{Q}_{p^2}} W(k)_{\mathbb{Q}}$. The form may be extended to $\mathbf{V}_k$ by the formula

$$\{v \otimes x, w \otimes y\} := xy^{\sigma}\{v, w\} \in W(k)_{\mathbb{Q}}$$

for all $v, w \in \mathbf{V}$ and $x, y \in W(k)_{\mathbb{Q}}$. The notions of index and duality for $W(k)$-lattices can be extended as well. We have the following description of the rational points of the Rapoport-Zink space.

Proposition ([Vol10] 1.10)**.** *Let k be a perfect field extension of $\mathbb{F}_{p^2}$. There is a natural bijection between $\mathcal{M}(k) = \mathcal{M}_{\mathrm{red}}(k)$ and the set of lattices M in $\mathbf{V}_k$ such that for some integer $i \in \mathbb{Z}$, we have*

$$p^{i+1}M^{\vee} \overset{1}{\subset} M \overset{n-1}{\subset} p^i M^{\vee}.$$

3.1.1.6 There is a decomposition $\mathcal{M} = \bigsqcup_{i\in\mathbb{Z}} \mathcal{M}_i$ into formal connected subschemes which are open and closed. The rational points of $\mathcal{M}_i$ are those lattices M satisfying the relation above with the given integer i. Similarly, we have a decomposition into open and closed connected subschemes $\mathcal{M}_{\mathrm{red}} = \bigsqcup_{i\in\mathbb{Z}} \mathcal{M}_{i,\mathrm{red}}$. In particular, the lattice $\mathbf{M}$ defined in the previous paragraph is an element of $\mathcal{M}_0(\mathbb{F}_{p^2})$. Not all integers i can occur though, as a parity condition must be satisfied by the following lemma.

Lemma ([Vol10] 1.7)**.** *The formal scheme $\mathcal{M}_i$ is empty if ni is odd.*

3.1.1.7 Let $J = \mathrm{GU}(\mathbf{V})$ be the group of unitary similitudes attached to $\mathbf{V}$. It consists of all linear transformations g which preserve the hermitian form up to a unit $c(g) \in \mathbb{Q}_p^{\times}$, called the **multiplier**. One may think of J as the group of $\mathbb{Q}_p$-rational point of a reductive algebraic group. The space $\mathcal{M}$ is endowed with a natural action of J. At the level of points, the element g acts by sending a lattice M to $g(M)$.
By [Vol10] 1.16, the action of $g \in J$ induces, for every integer i, an isomorphism $\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_{i+\alpha(g)}$ where $\alpha(g)$ is the p-adic valuation of the multiplier $c(g)$. This defines a continous homomorphism

$$\alpha : J \to \mathbb{Z}$$

where $\mathbb{Z}$ is given the discrete topology. According to 1.17 in loc. cit. the image of α is $\mathbb{Z}$ if n is even, and it is $2\mathbb{Z}$ if n is odd. The center $\mathrm{Z}(J)$ of J consists of all the multiple of the identity. Therefore it can be identified with $\mathbb{Q}_{p^2}^{\times}$. If $\lambda \in \mathbb{Q}_{p^2}^{\times}$, then $c(\lambda\cdot\mathrm{id}) = \lambda\sigma(\lambda) = \mathrm{Norm}(\lambda) \in \mathbb{Q}_p^{\times}$, where Norm is the norm map relative to the quadratic extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$. In particular, $\alpha(\mathrm{Z}(J)) = 2\mathbb{Z}$. Thus, the restriction of α to the center of J is surjective onto the image of α only when n is odd. When n is even, we define the following element

$$g_0 := \begin{pmatrix} & & & I_m \\ & 0 & p & \\ & 1 & 0 & \\ pI_m & & & \end{pmatrix}$$

where I_m denotes the $m \times m$ identity matrix. Then $g_0 \in J$ and $c(g_0) = p$ so that $\alpha(g_0) = 1$. Moreover $g_0^2 = p \cdot \mathrm{id}$ belongs to $\mathrm{Z}(J)$.
Let i and i' be two integers such that ni and ni' are even. Following [Vol10] Proposition 1.18, we define a morphism $\psi_{i,i'} : \mathcal{M}_i \to \mathcal{M}_{i'}$ by sending, for any perfect field extension $k/\mathbb{F}_{p^2}$, a point $M \in \mathcal{M}_i$ to

$$\psi_{i,i'}(M) = \begin{cases} p^{\frac{i'-i}{2}} \cdot M & \text{if } i \equiv i' \mod 2. \\ p^{\frac{i'-i-1}{2}} g_0 \cdot M & \text{if } i \not\equiv i' \mod 2. \end{cases}$$

This is well defined as the second case may only happen when n is even. We obtain the following proposition.

Proposition ([Vol10] 1.18)**.** *The map $\psi_{i,i'}$ is an isomorphism between $\mathcal{M}_i$ and $\mathcal{M}_{i'}$. Moreover they are compatible with each other in the sense that if i, i' and i'' are three integers such that ni, ni' and ni'' are even, then we have $\psi_{i',i''} \circ \psi_{i,i'} = \psi_{i,i''}$.*

The same statement also holds for the special fiber $\mathcal{M}_{\mathrm{red}}$. In particular, we have $\mathcal{M}_i \neq \varnothing$ if and only if ni is even.

3.1.2 The Bruhat-Tits stratification of the special fiber $\mathcal{M}_{\mathrm{red}}$

3.1.2.1 We now recall the construction of the Bruhat-Tits stratification on $\mathcal{M}_{\mathrm{red}}$ as in [VW11]. Let i be an integer such that ni is even. We define

$$\mathcal{L}_i := \{\Lambda \subset \mathbf{V} \text{ a lattice} \,|\, p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i\Lambda^\vee\}.$$

If $\Lambda \in \mathcal{L}_i$, we define its **orbit type** $t(\Lambda) := [\Lambda : p^{i+1}\Lambda^\vee]$. We also call it the type of Λ. In particular, the lattices in $\mathcal{L}_i$ of type 1 are precisely the $\mathbb{F}_{p^2}$-rational points of $\mathcal{M}_{i,\mathrm{red}}$. By sending Λ to $g(\Lambda)$, an element $g \in J$ defines a map $\mathcal{L}_i \to \mathcal{L}_{i+\alpha(g)}$.

Proposition ([Vol10] Remark 2.3 and [VW11] Remark 4.1)**.** *Let i be an integer such that ni is even and let $\Lambda \in \mathcal{L}_i$.*

- *The map $\mathcal{L}_i \to \mathcal{L}_{i+\alpha(g)}$ induced by an element $g \in J$ is an inclusion preserving, type preserving bijection.*
- *We have $1 \leqslant t(\Lambda) \leqslant n$. Furthermore $t(\Lambda)$ is odd.*
- *The sets $\mathcal{L}_i$'s for various i's are pairwise disjoint.*

Moreover, two lattices $\Lambda, \Lambda' \in \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{L}_i$ are in the same orbit under the action of J if and only if $t(\Lambda) = t(\Lambda')$.

Proof. The first three points are proved in [Vol10]. Thus, we only explain the last statement. If Λ and Λ' are in the same J-orbit, because the action of J preserves the type we have $t(\Lambda) = t(\Lambda')$.
For the converse, assume that Λ and Λ' have the same type. Let i and i' be the integers such that $\Lambda \in \mathcal{L}_i$ and $\Lambda' \in \mathcal{L}_{i'}$. According to 3.1.1.7, we can always find $g \in J$ such that $\alpha(g) = i - i'$. Hence, replacing Λ' by $g \cdot \Lambda'$ we may assume that $i = i'$. Then the statement follows from [VW11] Remark 4.1. □

We write $\mathcal{L} := \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{L}_i$. For any integer i such that ni is even and any odd number t between 1 and n, there exists a lattice $\Lambda \in \mathcal{L}_i$ of orbit type t. Indeed, by fixing a bijection $\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_0$ it is enough to find such a lattice for $i = 0$. Then, examples of lattices in $\mathcal{L}_0$ of any type are given in 3.1.2.6 below.

3.1.2.2 Write $t_{\max} := 2m+1$, so that the orbit type t of any lattice in $\mathcal{L}$ satisfies $1 \leqslant t \leqslant t_{\max}$. The following lemma will be useful later.

Lemma. *Let $i \in \mathbb{Z}$ such that ni is even, and let $\Lambda \in \mathcal{L}$. We have $\Lambda^\vee \in \mathcal{L}$ if and only if either n is even, either n is odd and $t(\Lambda) = t_{\max}$.*
If this condition is satisfied and n is even, then $\Lambda^\vee \in \mathcal{L}_{-i-1}$ and $t(\Lambda^\vee) = n - t(\Lambda)$. If on the contrary n is odd, then $\Lambda^\vee \in \mathcal{L}_{-i}$ and $t(\Lambda^\vee) = t(\Lambda)$.

Proof. First we prove the converse. We have the following chain of inclusions

$$p^{-i}\Lambda \overset{n-t(\Lambda)}{\subset} \Lambda^\vee \overset{t(\Lambda)}{\subset} p^{-i-1}\Lambda.$$

If n is even, then $-n(i+1)$ is also even and $n - t(\Lambda) \neq 0$. Since $(\Lambda^\vee)^\vee = \Lambda$, we deduce that $\Lambda^\vee \in \mathcal{L}_{-i-1}$ with orbit type $n - t(\Lambda)$. Assume now that n is odd and that $t(\Lambda) = t_{\max} = n$. Then $\Lambda^\vee = p^{-i}\Lambda \in \mathcal{L}_{-i}$.
Let us now assume that $\Lambda^\vee \in \mathcal{L}$ and that n is odd. Let $i' \in 2\mathbb{Z}$ such that $\Lambda^\vee \in \mathcal{L}_{i'}$. We have

$$\Lambda^\vee \overset{n-t(\Lambda^\vee)}{\subset} p^{i'}\Lambda \overset{n-t(\Lambda)}{\subset} p^{i'+i}\Lambda^\vee, \qquad \Lambda^\vee \overset{t(\Lambda)}{\subset} p^{-i-1}\Lambda \overset{t(\Lambda^\vee)}{\subset} p^{-i-i'-2}\Lambda^\vee,$$

therefore $-2 \leqslant i+i' \leqslant 0$. Since $i+i'$ is even it is either -2 or 0. If it were -2, then we would have $t(\Lambda) = t(\Lambda^\vee) = 0$ which is absurd. Therefore $i+i' = 0$, and we have $n-t(\Lambda) = n-t(\Lambda^\vee) = 0$. ☐

3.1.2.3 With the help of $\mathcal{L}_i$, one may construct an abstract simplicial complex $\mathcal{B}_i$. For $s \geqslant 0$, an s-simplex of $\mathcal{B}_i$ is a subset $S \subset \mathcal{L}_i$ of cardinality $s+1$ such that for some ordering $\Lambda_0, \ldots, \Lambda_s$ of its elements, we have a chain of inclusions $p^{i+1}\Lambda_s^\vee \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \ldots \subsetneq \Lambda_s$. We must have $0 \leqslant s \leqslant m$ for such a simplex to exist.
We introduce $\tilde{J} = \mathrm{SU}(\mathbf{V})$, the derived group of J. We consider the abstract simplicial complex $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ of the Bruhat-Tits building of $\tilde{J}$ over $\mathbb{Q}_p$. A concrete description of this complex is given in [Vol10], while proving the following theorem.

Theorem ([Vol10] 3.5)**.** *The abstract simplicial complex $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ of the Bruhat-Tits building of $\tilde{J}$ is naturally identified with $\mathcal{B}_i$ for any fixed integer i such that ni is even. There is in particular an identification of $\mathcal{L}_i$ with the set of vertices of $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$. The identification is $\tilde{J}$-equivariant.*

Apartments in the Bruhat-Tits building $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ are in 1 to 1 correspondence with Witt decompositions of $\mathbf{V}$. Let $L = \{L_j\}_{i \in I}$ be a Witt decomposition of $\mathbf{V}$ and let $f = (f_i)_{i \in I} \sqcup B^{\mathrm{an}}$ be a basis of $\mathbf{V}$ adapted to the decomposition, where B^{an} is an orthogonal basis of L^{an}. Under the identification of $\mathrm{BT}(\tilde{J}, \mathbb{Q}_p)$ with $\mathcal{B}_i$, the vertices inside the apartment associated to L

correspond to the lattices $\Lambda \in \mathcal{L}_i$ which are equal to the direct sum of $\Lambda \cap L^{\mathrm{an}}$ and of the modules $p^{r_i}\mathbb{Z}_{p^2} f_i$ for some integers $(r_i)_{i \in I}$. The subset of $\mathcal{L}_i$ consisting of all such lattices will be denoted $\mathcal{A}_i^L$ or, with an abuse of notations, $\mathcal{A}_i^f$. We call such a set $\mathcal{A}_i^L$ the **apartment associated to L in $\mathcal{L}_i$.**

Remark. The set of vertices of the Bruhat-Tits building of $J = \mathrm{GU}(\mathbf{V})$ may then be identified with the disjoint union $\mathcal{L}$ of the $\mathcal{L}_i$'s. The subsets of lattices in a common apartment correspond to the sets $\mathcal{A}^L := \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{A}_i^L$ where L is some Witt decomposition of $\mathbf{V}$. The set $\mathcal{A}^L$ will be called the apartment associated to L.

We recall a general result regarding Bruhat-Tits buildings.

Proposition. *Let i be an integer such that ni is even. Any two lattices Λ and Λ' in $\mathcal{L}_i$ (resp. $\mathcal{L}$) lie inside a common apartment $\mathcal{A}_i^L$ (resp. $\mathcal{A}^L$) for some Witt decomposition L.*
Moreover, the action of the group $\tilde{J}$ sends apartments to apartments. It acts transitively on the set $\{\mathcal{A}_i^L\}_L$. The same is true for J acting on the set $\{\mathcal{A}^L\}_L$.

3.1.2.4 Recall the basis e of $\mathbf{V}$ that we fixed in **1.4**. We will denote by

$$\Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$$

the $\mathbb{Z}_{p^2}$-lattice generated by the vectors $p^{r_j} e_j$ for all $j = \pm 1, \ldots, \pm m$, by $p^{s_0} e_0^{\mathrm{an}}$ and if n is even, by $p^{s_1} e_1^{\mathrm{an}}$ too. Here, the r_j's are integers and s denotes either the integer s_0 if n is odd or the pair of integers (s_0, s_1) if n is even.

Proposition. *Let i be an integer such that ni is even. Let (r_j, s) be a family of integers as above. The corresponding lattice $\Lambda = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$ belongs to $\mathcal{L}_i$ if and only if the following conditions are satisfied*

- *for all $1 \leqslant j \leqslant m$, we have $r_{-j} + r_j \in \{i, i+1\}$,*
- $s_0 = \lfloor \frac{i+1}{2} \rfloor$,
- *if n is even, then $s_1 = \lfloor \frac{i}{2} \rfloor$.*

Moreover, when that is the case the type of Λ is given by

$$t(\Lambda) = 1 + 2\#\{1 \leqslant j \leqslant m \,|\, r_{-j} + r_j = i\}.$$

Proof. The lattice Λ belongs to $\mathcal{L}_i$ if and only if the following chain of inclusions holds:

$$p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i \Lambda^\vee.$$

The dual lattice $\Lambda^\vee$ is equal to the lattice $\Lambda(-r_m, \ldots, -r_1, s', -r_{-1}, \ldots, -r_{-m})$, where $s' = -s_0$ when n is odd, and $s' = (-s_0, -s_1 - 1)$ when n is even. Thus, the inclusions above are equivalent to the following inequalities:

$$i - r_{-j} \leqslant r_j \leqslant i + 1 - r_{-j}, \qquad i - s_0 \leqslant s_0 \leqslant i + 1 - s_0,$$
$$i - 1 - s_1 \leqslant s_1 \leqslant i - s_1 \text{ (if } n \text{ is even).}$$

This proves the desired condition on the integers r_j's and on s.
Let us now assume that $\Lambda \in \mathcal{L}_i$. Its orbit type is equal to the index $[\Lambda, p^{i+1}\Lambda^\vee]$. This corresponds to the number of times equality occurs with the left-hand side in all the inequalities above. Of course, if the equality $i - r_{-j} = r_j$ occurs for some j, then it occurs also for $-j$. Moreover, if i is even then the equality $i - s_0 = s_0$ occurs whereas $i - 1 - s_1 \neq s_1$. On the contrary if i is odd, then the equality $i - 1 - s_1 = s_1$ occurs whereas $i - s_0 \neq s_0$. Thus in all cases, only one of s_0 and s_1 contributes to the index. Putting things together, we deduce the desired formula. □

3.1.2.5 We deduce the following corollary.

Corollary. *The apartment A_i^e (resp. A^e) consists of all the lattices of the form*

$$\Lambda = \Lambda(r_{-m}, \dots, r_{-1}, s, r_1, \dots, r_m)$$

which belong to $\mathcal{L}_i$ (resp. to $\mathcal{L}$).

Proof. According to the previous proposition, it is clear that all lattices which belong to $\mathcal{L}_i$ and are of the form $\Lambda(r_{-m}, \dots, r_{-1}, s, r_1, \dots, r_m)$ are elements of $\mathcal{A}_i^e$. We shall prove the converse. Let $\Lambda \in \mathcal{A}_i^e$. By definition, there exists integers (r_j) such that

$$\Lambda = \Lambda \cap \mathbf{V}^{\mathrm{an}} \oplus \bigoplus_{1 \leqslant j \leqslant m} (p^{r_{-j}}\mathbb{Z}_{p^2}e_{-j} \oplus p^{r_j}\mathbb{Z}_{p^2}e_j)\,.$$

Write $\Lambda' = \Lambda \cap \mathbf{V}^{\mathrm{an}}$. This is a lattice in $\mathbf{V}^{\mathrm{an}}$ which satisfies the chain of inclusions

$$p^{i+1}\Lambda'^{\,\vee} \subset \Lambda' \subset p^i\Lambda'^{\,\vee},$$

where the duals are taken with respect to the restriction of $\{\cdot,\cdot\}$ to $\mathbf{V}^{\mathrm{an}}$. Since $\mathbf{V}^{\mathrm{an}}$ is anisotropic, there is only a single lattice satisfying the chain of inclusions above. If we write $a := \lfloor \frac{i+1}{2} \rfloor$ and $b := \lfloor \frac{i}{2} \rfloor$, it is given by $p^a\mathbb{Z}_{p^2}e_0^{\mathrm{an}}$ if n is odd, and by $p^a\mathbb{Z}_{p^2}e_0^{\mathrm{an}} \oplus p^b\mathbb{Z}_{p^2}e_1^{\mathrm{an}}$ if n is even. Thus, it must be equal to Λ' and it concludes the proof. □

3.1.2.6 We fix a maximal simplex in $\mathcal{L}_0$ lying inside the apartment $\mathcal{A}_0^e$. For $0 \leqslant \theta \leqslant m$ we define

$$\Lambda_\theta := \Lambda(\underbrace{0, \dots, 0}_{m}, 0, \underbrace{0, \dots, 0}_{\theta}, \underbrace{1, \dots, 1}_{m-\theta}).$$

Here, the 0 in the middle stands for $(0,0)$ in case n is even. The lattice Λ_θ belongs to $\mathcal{L}_0$, its orbit type is $2\theta + 1$ and together they fit inside the following chain of inclusions

$$p\Lambda_0^\vee \subsetneq \Lambda_0 \subset \dots \subset \Lambda_m.$$

Thus, they form an m-simplex in $\mathcal{L}_0$.

3.1.2.7 Given a lattice $\Lambda \in \mathcal{L}_i$, the authors of [VW11] define a subfunctor $\mathcal{M}_\Lambda$ of $\mathcal{M}_{i,\mathrm{red}}$ classifying those p-divisible groups for which a certain quasi-isogeny, depending on Λ, is in fact an actual isogeny. In Lemma 4.2, they prove that it is representable by a projective scheme over $\mathbb{F}_{p^2}$, and that the natural morphism $\mathcal{M}_\Lambda \hookrightarrow \mathcal{M}_{i,\mathrm{red}}$ is a closed immersion. The schemes $\mathcal{M}_\Lambda$ are called the **closed Bruhat-Tits strata of $\mathcal{M}$**. Their rational points are described as follows.

Proposition ([VW11] Lemma 4.3)**.** *Let k be a perfect field extension of $\mathbb{F}_{p^2}$, and let $M \in \mathcal{M}_{i,\mathrm{red}}(k)$. Then we have the equivalence*

$$M \in \mathcal{M}_\Lambda(k) \iff M \subset \Lambda_k := \Lambda \otimes_{\mathbb{Z}_{p^2}} W(k).$$

The set of lattices satisfying the condition above was conjectured in [Vol10] to be the set of points of a subscheme of $\mathcal{M}_{i,\mathrm{red}}$, and it was proved in the special cases $n = 2, 3$. In [VW11], the general argument is given by the construction of $\mathcal{M}_\Lambda$. The action of an element $g \in J$ on $\mathcal{M}_{\mathrm{red}}$ induces an isomorphism $\mathcal{M}_\Lambda \xrightarrow{\sim} \mathcal{M}_{g\cdot\Lambda}$.

3.1.2.8 Let $\Lambda \in \mathcal{L}$, we denote by J_Λ the fixator of Λ under the action of J. If $\Lambda = \Lambda_\theta$ for some $0 \leqslant \theta \leqslant m$, we will write J_θ instead. These are **maximal parahoric subgroups** of J. In unramified unitary similitude groups, maximal parahoric subgroups and maximal compact subgroups are the same. A general **parahoric subgroup** is an intersection $J_{\Lambda_1} \cap \ldots \cap J_{\Lambda_s}$ where $\{\Lambda_1, \ldots, \Lambda_s\}$ is an s-simplex in $\mathcal{L}_i$ for some i. Any parahoric subgroup is compact and open in J.
Let i be the integer such that $\Lambda \in \mathcal{L}_i$. We define $V_\Lambda^0 := \Lambda/p^{i+1}\Lambda^\vee$ and $V_\Lambda^1 := p^i\Lambda^\vee/\Lambda$. Since $p\Lambda \subset p\cdot p^i\Lambda^\vee$ and $p \cdot p^i\Lambda^\vee \subset \Lambda$, these are both $\mathbb{F}_{p^2}$-vector space of dimensions respectively $t(\Lambda)$ and $n - t(\Lambda)$. Both spaces come together with a non-degenerate σ-hermitian form $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ with values in $\mathbb{F}_{p^2}$, respectively induced by $p^{-i}\{\cdot,\cdot\}$ and by $p^{-i+1}\{\cdot,\cdot\}$. If k is a perfect field extension of $\mathbb{F}_{p^2}$ and if $\epsilon \in \{0,1\}$, we may extend the pairings to $(V_\Lambda^\epsilon)_k = V_\Lambda^\epsilon \otimes_{\mathbb{F}_{p^2}} k$ by setting

$$(v \otimes x, w \otimes y)_\epsilon := xy^\sigma(v,w)_\epsilon \in k$$

for all $v, w \in V_\Lambda^\epsilon$ and $x, y \in k$. If U is a subspace of $(V_\Lambda^\epsilon)_k$ we denote by $U^\perp$ its orthogonal, that is the subspace of all vectors $x \in (V_\Lambda^\epsilon)_k$ such that $(x, U)_\epsilon = 0$.
Denote by J_Λ^+ the pro-unipotent radical of J_Λ and write $\mathcal{J}_\Lambda := J_\Lambda/J_\Lambda^+$. This is a finite group of Lie type, called the **maximal reductive quotient** of J_Λ. We have an identification $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$, that is the group of pairs (g_0, g_1) where for $\epsilon \in \{0,1\}$ we have $g_\epsilon \in \mathrm{GU}(V_\Lambda^\epsilon)$ and $c(g_0) = c(g_1)$. Here, $c(g_\epsilon) \in \mathbb{F}_p^\times$ denotes the multiplier of g_ϵ.
For $0 \leqslant \theta \leqslant m$ and $\epsilon \in \{0,1\}$, we will write V_θ^ϵ and $\mathcal{J}_\theta$ instead of $V_{\Lambda_\theta}^\epsilon$ and $\mathcal{J}_{\Lambda_\theta}$. A basis of V_θ^0 is given by the images of the $2\theta+1$ vectors $e_{-\theta} \ldots, e_{-1}, e_0^{\mathrm{an}}, e_1, \ldots, e_\theta$. As for V_θ^1, a basis is given by the images of the $n - 2\theta - 1$ vectors $p^{-1}e_{-m}, \ldots, p^{-1}e_{-\theta-1}, e_{\theta+1}, \ldots, e_m$ when n is odd, and in case n is even one must add the image of $p^{-1}e_1^{\mathrm{an}}$ to the basis.

3.1.2.9 Let $\Lambda \in \mathcal{L}_i$ where ni is even. We write $t(\Lambda) = 2\theta + 1$. Let k be a perfect field extension of $\mathbb{F}_{p^2}$. Let T be any $W(k)$-lattice in $\mathbf{V}_k$ such that

$$p^{i+1}T^\vee \overset{2\theta'+1}{\subset} T \subset \Lambda_k$$

where $0 \leqslant \theta' \leqslant \theta$. Then T must contain $p^{i+1}\Lambda_k^\vee$ and $[\Lambda_k : T] = \theta - \theta'$. We may consider $\overline{T} := T/p^{i+1}\Lambda_k^\vee$ the image of T in $V_\Lambda^{(0)}$. Then $\overline{T}$ is an $\mathbb{F}_{p^2}$-subspace of dimension $\theta + \theta' + 1$. Moreover, one may check that $\overline{p^{i+1}T^\vee} = \overline{T}^\perp$, therefore the subspace $\overline{T}$ contains its orthogonal. These observations lead to the following proposition.

Proposition ([Vol10] 2.7)**.** *The mapping $T \mapsto \overline{T}$ defines a bijection between the set of $W(k)$-lattices T in $\mathbf{V}_k$ such that $p^{i+1}T^\vee \overset{2\theta'+1}{\subset} T \subset \Lambda_k$ and the set*

$$\{U \subset (V_\Lambda^0)_k \mid \dim U = \theta + \theta' + 1 \text{ and } U^\perp \subset U\}.$$

In particular taking $\theta' = 0$, this set is in bijection with $\mathcal{M}_\Lambda(k)$.

Remark. Similarly, the set of $W(k)$-lattices T such that $\Lambda_k \subset T \overset{n-2\theta'-1}{\subset} p^iT^\vee$ for some $\theta \leqslant \theta' \leqslant m$ is in bijection with

$$\{U \subset (V_\Lambda^1)_k \mid \dim U = n - \theta' - \theta - 1 \text{ and } U^\perp \subset U\}.$$

The bijection is given by $T \mapsto \overline{T}^\perp$ where $\overline{T} := T/\Lambda_k \subset V_k^{(1)}$. These sets can be seen as the k-rational points of some flag variety for $\mathrm{GU}(V_\Lambda^{(0)})$ and $\mathrm{GU}(V_\Lambda^{(1)})$, which are special instances of Deligne-Lusztig varieties. This is accounted for in the next paragraph.

3.1.2.10 Let $\Lambda \in \mathcal{L}$. The action of J on the Rapoport-Zink space $\mathcal{M}$ restricts to an action of the parahoric subgroup J_Λ on the closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. This action factors through the maximal reductive quotient $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$. This action is trivial on the normal subgroup $\{\mathrm{id}\} \times \mathrm{U}(V_\Lambda^1) \subset \mathcal{J}_\Lambda$, thus it factors again through the quotient which is isomorphic to $\mathrm{GU}(V_\Lambda^0)$.

Theorem ([VW11] Theorem 4.8)**.** *There is an isomorphism between $\mathcal{M}_\Lambda$ and a certain "generalized" parabolic Deligne-Lusztig variety for the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$, compatible with the actions. In particular, if $t(\Lambda) = 2\theta + 1$ then the scheme $\mathcal{M}_\Lambda$ is projective, smooth, geometrically irreducible of dimension θ.*

We refer to [Mul22a] Section 1 for the definition of Deligne-Lusztig varieties. In particular, the adjective "generalized" is understood according to loc. cit. The Deligne-Lusztig variety isomorphic to $\mathcal{M}_\Lambda$ is introduced in [VW11] 4.5, and it is denoted by Y_Λ there.

3.1.2.11 We now explain how the different closed Bruhat-Tits strata behave together.

Theorem ([VW11] Theorem 5.1)**.** *Let $i \in \mathbb{Z}$ such that ni is even. Consider Λ and Λ' two lattices in $\mathcal{L}_i$. The following statements hold.*

(1) *The inclusion* $\Lambda \subset \Lambda'$ *is equivalent to the scheme-theoretic inclusion* $\mathcal{M}_\Lambda \subset \mathcal{M}_{\Lambda'}$. *It also implies* $t(\Lambda) \leqslant t(\Lambda')$ *and there is equality if and only if* $\Lambda = \Lambda'$.

(2) *The three following assertions are equivalent.*

(i) $\Lambda \cap \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda \cap \Lambda'$ *contains a lattice of* $\mathcal{L}_i$. (iii) $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} \neq \varnothing$.

If these conditions are satisfied, then $\mathcal{M}_\Lambda \cap \mathcal{M}_{\Lambda'} = \mathcal{M}_{\Lambda \cap \Lambda'}$, *where we understand the left hand side as the scheme theoretic intersection inside* $\mathcal{M}_{i,\mathrm{red}}$.

(3) *The three following assertions are equivalent*

(i) $\Lambda + \Lambda' \in \mathcal{L}_i$. (ii) $\Lambda + \Lambda'$ *is contained in a lattice of* $\mathcal{L}_i$.
(iii) $\mathcal{M}_\Lambda, \mathcal{M}_{\Lambda'} \subset \mathcal{M}_{\widetilde{\Lambda}}$ *for some* $\widetilde{\Lambda}$ *in* $\mathcal{L}_i$.

If these conditions are satisfied, then $\mathcal{M}_{\Lambda+\Lambda'}$ *is the smallest subscheme of the form* $\mathcal{M}_{\widetilde{\Lambda}}$ *containing both* $\mathcal{M}_\Lambda$ *and* $\mathcal{M}_{\Lambda'}$.

(4) *If* k *is a perfect field field extension of* $\mathbb{F}_{p^2}$ *then* $\mathcal{M}_i(k) = \bigcup_{\Lambda \in \mathcal{L}_i} \mathcal{M}_\Lambda(k)$.

In essence, the previous statements explain how the stratification given by the $\mathcal{M}_\Lambda$ mimics the combinatorics of the Bruhat-Tits building of $\tilde{J}$, hence the name.

3.1.3 On the maximal parahoric subgroups of J

3.1.3.1 In this section we give a few results that will be useful later regarding the maximal parahoric subgroups J_Λ. First, we study their conjugacy classes. It starts with the following lemma.

Lemma. *Let* $\Lambda, \Lambda' \in \mathcal{L}$.

(i) The parahoric subgroup J_Λ *acts transitively on the set of apartments containing* Λ.
(ii) We have $J_\Lambda = J_{\Lambda'}$ *if and only if there exists* $k \in \mathbb{Z}$ *such that* $\Lambda = p^k\Lambda'$ *or* $\Lambda = p^k\Lambda'^{\vee}$.

Proof. The first point is a general fact from the theory of Bruhat-Tits buildings.
For the second point, the converse is clear. Indeed, if $x \in \mathbb{Q}_{p^2}^\times$ then $J_{x\Lambda} = J_\Lambda$, and an element $g \in J$ fixes a lattice Λ if and only if it fixes its dual $\Lambda^\vee$.
Now, let $\Lambda, \Lambda' \in \mathcal{L}$ such that $J_\Lambda = J_{\Lambda'}$. Up to replacing Λ' by an appropriate lattice $g \cdot \Lambda'$, it is enough to treat the case $\Lambda' = \Lambda_\theta$ for some $0 \leqslant \theta \leqslant m$. By 3.1.2.3 Proposition, we can find an apartment $\mathcal{A}^L$ containing both Λ_θ and Λ. By the first point, we can find $g \in J_\theta = J_\Lambda$ which sends $\mathcal{A}^L$ to $\mathcal{A}^e$. Therefore $g \cdot \Lambda = \Lambda$ belongs to $\mathcal{A}^e$. According to 3.1.2.5, we may write

$$\Lambda = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m)$$

for some integers (r_j, s). Let i be the integer such that $\Lambda \in \mathcal{L}_i$. Then according to 3.1.2.4 we have

- $\forall 1 \leqslant j \leqslant m, r_{-j} + r_j \in \{i, i+1\}$.
- $s_0 = \lfloor \frac{i+1}{2} \rfloor$.

– if n is even then $s_1 = \lfloor \frac{i}{2} \rfloor$.

For $1 \leqslant j \leqslant \theta$, let g_j be the automorphism of $\mathbf{V}$ which exchanges e_{-j} and e_j while fixing all the other vectors in the basis e. Then, from the definition of Λ_θ we have $g_j \in J_\theta$. Therefore g_j must fix Λ too, which implies that $r_{-j} = r_j$. And for $\theta + 1 \leqslant j \leqslant m$, let g_j be the automorphism sending e_j to $p^{-1}e_{-j}$ and e_{-j} to pe_j while fixing all the other vectors in the basis e. Then again we have $g_j \in J_\theta = J_\Lambda$ which implies that $r_{-j} = r_j - 1$.
Assume first that $i = 2i'$ is even. Combining the previous observations, we have $r_j = i'$ for all $1 \leqslant j \leqslant \theta$ and $r_j = i' + 1$ for all $\theta + 1 \leqslant j \leqslant m$. Moreover we have $s_0 = i'$ and if n is even, we have $s_1 = i'$. In other words, we have $\Lambda = p^{i'}\Lambda_\theta$.
Assume now that $i = 2i' + 1$ is odd. This implies that n is even. Combining the previous observations, we have $r_j = i' + 1$ for all $1 \leqslant j \leqslant m$. Moreover we have $s_0 = i' + 1$ and if n is even, we have $s_1 = i'$. In other words, we have $\Lambda = p^{i'+1}\Lambda_\theta^\vee$. □

3.1.3.2 We may now describe the conjugacy classes of these maximal parahoric subgroups.

Corollary. *Let* $\Lambda, \Lambda' \in \mathcal{L}$.

(i) If n is odd, then $t(\Lambda) = t(\Lambda')$ if and only if the associated maximal parahoric subgroups J_Λ and $J_{\Lambda'}$ are conjugate in J. Each such subgroup is conjugate to J_θ for a unique $0 \leqslant \theta \leqslant m$.
(ii) If n is even, then $t(\Lambda) \in \{t(\Lambda'), n - t(\Lambda')\}$ if and only if the associated maximal parahoric subgroups J_Λ and $J_{\Lambda'}$ are conjugate in J. Each such subgroup is conjugate to J_θ for a unique $0 \leqslant \theta \leqslant \lfloor \frac{m}{2} \rfloor$.

Thus, there are $m + 1$ conjugacy classes of maximal parahoric subgroups when n is odd, and only $\lfloor \frac{m}{2} \rfloor + 1$ when n is even. If n is odd the subgroups J_θ are pairwise non conjugate, whereas J_θ is conjugate to $J_{m-\theta}$ when n is even.

Remark. The special maximal compact subgroups are the conjugates of J_0 and of J_m. When n is odd, the conjugates of J_m are hyperspecial.

Proof. For the first point, assume that $t(\Lambda) = t(\Lambda')$. By 3.1.2.1 Proposition, we can find $g \in J$ such that $g \cdot \Lambda = \Lambda'$. Therefore $J_{\Lambda'} = J_{g\cdot\Lambda} = {}^gJ_\Lambda$, the two parahoric subgroups are conjugate. For the converse, assume that $J_{\Lambda'} = {}^gJ_\Lambda$ for some $g \in J$. Then $J_{\Lambda'} = J_{g\cdot\Lambda}$. By 3.1.3.1 there is some $k \in \mathbb{Z}$ such that $\Lambda' = p^k g \cdot \Lambda$ or $(\Lambda')^\vee = p^k g \cdot \Lambda$. This implies that $t(\Lambda) = t(\Lambda')$. Indeed, it is clear in the first case, and in the second case we have in particular $(\Lambda')^\vee \in \mathcal{L}$. Since n is odd, by 3.1.2.2 we have $t(\Lambda') = t((\Lambda')^\vee)$, so that we are done.
For the second point, if $t(\Lambda') = t(\Lambda)$ then we reason the same way as above. If $t(\Lambda') = n - t(\Lambda)$ then Λ' and $\Lambda^\vee$ have the same type. By the first case, we know that $J_{\Lambda'}$ and $J_{\Lambda^\vee} = J_\Lambda$ are conjugate. The converse goes the same way as above, except that the case $(\Lambda')^\vee = p^k g \cdot \Lambda$ now implies that $t(\Lambda') = n - t(\Lambda)$ therefore we are done. □

3.1.3.3 As another corollary of 3.1.3.1 we may also describe the normalizers of the maximal parahoric subgroups.

Corollary. *Let $\Lambda \in \mathcal{L}$. If $t(\Lambda) \neq n - t(\Lambda)$ then the normalizer of J_Λ in J is $\mathrm{N}_J(J_\Lambda) = \mathrm{Z}(J)J_\Lambda$. Otherwise, n is even and there exists an element $h_0 \in J$ such that $h_0^2 = p \cdot \mathrm{id}$ and $N_J(J_\lambda)$ is the subgroup generated by J_Λ and h_0. In particular, $\mathrm{Z}(J)J_\Lambda$ is a subgroup of index 2 in $N_J(J_\Lambda)$.*

Remark. The condition $t(\Lambda) \neq n - t(\Lambda)$ is automatically satisfied if n is odd. If n is even, it is satisfied when $t(\Lambda) \neq m + 1$, this is the case in particular when m is odd.

Proof. It is clear that $\mathrm{Z}(J)J_\Lambda \subset \mathrm{N}_J(J_\Lambda)$. Conversely, let $g \in N_J(J_\Lambda)$, so that we have $J_\Lambda = {}^gJ_\Lambda = J_{g\cdot\Lambda}$. We apply 3.1.3.1 to deduce the existence of $k \in \mathbb{Z}$ such that $g \cdot \Lambda = p^k\Lambda$ (case 1) or $g \cdot \Lambda = p^k\Lambda^\vee$ (case 2). If we are in case 1, then $g \in p^kJ_\Lambda \subset \mathrm{Z}(J)J_\Lambda$ and we are done. If n is even, the assumption that $t(\Lambda) \neq n - t(\Lambda)$ makes the case 2 impossible. If n is odd and we are in case 2, then in particular $\Lambda^\vee \in \mathcal{L}$. By 3.1.2.2, we must have $\Lambda = p^i\Lambda^\vee$ for some even $i \in \mathbb{Z}$. In particular, we are also in case 1. Therefore, no matter the parity of n, we are always in case 1.
Assume now that $t(\Lambda) = n - t(\Lambda)$, in particular n and m are both even. We write $m = 2m'$ so that $t(\Lambda) = 2m' + 1$ and we solve the case $\Lambda = \Lambda_{m'}$ first. Recall the element g_0 that was defined in 3.1.1.7. By direct computation, we see that $g_0 \cdot \Lambda_{m'} = p\Lambda_{m'}^\vee$. Therefore ${}^{g_0}J_{m'} = J_{p\Lambda_{m'}^\vee} = J_{m'}$ so that $g_0 \in \mathrm{N}_J(J_{m'})$. Now let g be any element normalizing J_m, so that $J_{m'} = {}^gJ_{m'} = J_{g\cdot\Lambda_{m'}}$. According to 3.1.3.1 there exists $k \in \mathbb{Z}$ such that $g\cdot\Lambda_{m'} = p^k\Lambda_{m'}$ or $g\cdot\Lambda_{m'} = p^k\Lambda_{m'}^\vee = p^{k-1}g_0\cdot\Lambda_{m'}$. In the first case we have $g \in p^kJ_{m'}$ and in the second case we have $g \in p^{k-1}g_0J_{m'}$. Because $g_0^2 = p \cdot \mathrm{id}$, the claim is proved with $h_0 = g_0$.
In the general case, we have $t(\Lambda) = 2m' + 1 = t(\Lambda_{m'})$. By 3.1.2.1 there exists some $g \in J$ such that $\Lambda = g \cdot \Lambda_{m'}$. Then $\mathrm{N}_J(\Lambda) = {}^g\mathrm{N}_J(\Lambda_{m'})$ so that the claim follows with $h_0 := gg_0g^{-1}$. □

3.1.3.4 Let J° be the kernel of $\alpha : J \to \mathbb{Z}$. In other words, J° is the subgroup of J consisting of all $g \in J$ whose multiplier $c(g)$ is a unit in $\mathbb{Z}_p^\times$. We have an isomorphism $J/J^\circ \simeq \mathbb{Z}$ induced by α when n is even, and by $\frac{1}{2}\alpha$ when n is odd. Note that J° contains all the compact subgroups of J, in particular $J_\Lambda \subset J^\circ$ for every $\Lambda \in \mathcal{L}$. Let K be the subgroup generated by all the J_Λ for $\Lambda \in \mathcal{L}$ having maximal orbit type $t(\Lambda) = 2m + 1$. We will prove the following result.

Proposition. *We have $K = J^\circ$.*

The proof requires the following lemma.

Lemma. *Let $i \in \mathbb{Z}$ such that ni is even and let $\Lambda \in \mathcal{L}_i$ be a lattice of maximal orbit type. Let $\Lambda', \Lambda'' \in \mathcal{L}_i$ such that $\Lambda' \cap \Lambda$ and $\Lambda'' \cap \Lambda$ belong to $\mathcal{L}_i$. There exists $g \in J_\Lambda$ such that $g \cdot \Lambda' = \Lambda''$ if and only if $t(\Lambda') = t(\Lambda'')$ and $t(\Lambda' \cap \Lambda) = t(\Lambda'' \cap \Lambda)$.*

Proof. The forward direction is clear because the action of J preserves the types of the lattices. We prove the converse. Since J acts transitively on $\mathcal{L}$ while preserving types and inclusions, it is enough to look at the case $i = 0$ and $\Lambda = \Lambda_m = \Lambda(0, \dots, 0)$. Let $0 \leqslant \theta_- \leqslant \theta_+ \leqslant m$. We fix a certain $\Lambda' \in \mathcal{L}_0$ such that $t(\Lambda') = 2\theta_+ + 1$ and $t(\Lambda' \cap \Lambda) = 2\theta_- + 1$, and we prove that any $\Lambda'' \in \mathcal{L}_0$ satisfying the hypotheses of the lemma is in the J_m-orbit of Λ'. We define

$$\Lambda' = \Lambda(0^{\theta_-}, 1^{\theta_+-\theta_-}, 1^{m-\theta_+}, 0, 0^{m-\theta_+}, -1^{\theta_+-\theta_-}, 0^{\theta_-})$$

where the 0 in the middle stands for 0 when n is odd and the pair $(0,0)$ when n is even. Then, we have

$$\Lambda' \cap \Lambda = \Lambda(0^{\theta_-}, 1^{m-\theta_-}, 0, 0^{m-\theta_-}, 0^{\theta_-})$$

so that Λ' satisfies the required conditions. Let Λ'' be as in the lemma. Let L be a Witt decomposition of $\mathbf{V}$ such that the corresponding apartment $\mathcal{A}^L$ contains both Λ and Λ''. Since J_m acts transitively on the set of apartments containing Λ_m, we can find some $g \in J_m$ such that $g \cdot \mathcal{A}^L = \mathcal{A}^e$. Up to replacing Λ'' by $g \cdot \Lambda''$, we may then assume that $\Lambda'' \in \mathcal{A}^e$. Therefore, there exists integers $r_{-m}, \ldots, r_m, s$ such that

$$\Lambda'' = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m).$$

Since $\Lambda'' \in \mathcal{L}_0$, by 3.1.2.4 we have $s = 0$ and $r_j + r_{-j} \in \{0,1\}$ for all $1 \leqslant j \leqslant m$. Let us write $r_{-j} = r_j + \epsilon_j$ where $\epsilon_j \in \{0,1\}$. Since $t(\Lambda'') = 2\theta_+ + 1$, there are θ_+ indices $1 \leqslant j_1 \leqslant \ldots \leqslant j_{\theta_+} \leqslant m$ such that $\epsilon_j = 0$ if and only if j is one of the j_k's. Moreover, we have

$$\Lambda'' \cap \Lambda = \Lambda\left(\max(-r_m + \epsilon_m, 0), \ldots, \max(-r_1 + \epsilon_1, 0), 0, \max(r_1, 0), \ldots, \max(r_m, 0)\right).$$

This lattice is in $\mathcal{L}_0$, thus for every $1 \leqslant j \leqslant m$ we have $0 \leqslant \max(-r_j + \epsilon_j, 0) + \max(r_j, 0) \leqslant 1$. Hence, if $j = j_k$ for some k then $\epsilon_j = 0$ and

$$\max(-r_j + \epsilon_j, 0) + \max(r_j, 0) = \max(-r_j, 0) + \max(r_j, 0) = |r_j|.$$

Thus, $|r_j| = 0$ or 1. If $j \neq j_k$ for all k, then $\epsilon_j = 1$ and

$$\max(-r_j + \epsilon_j, 0) + \max(r_j, 0) = \max(-r_j + 1, 0) + \max(r_j, 0) = \frac{1}{2} + \frac{|r_j| + |r_j - 1|}{2}.$$

This sum is a positive integer between 0 and 1, therefore it is always 1. It means that $|r_j| + |r_j - 1| = 1$ and as a consequence, $r_j = 0$ or 1.
Lastly, we have $t(\Lambda'' \cap \Lambda) = 2\theta_- + 1$ so there are exactly θ_- indices j for which the sum $\max(-r_j + \epsilon_j, 0) + \max(r_j, 0)$ is zero. As we have just seen, this may only happen when j is one of the j_k's. Thus, among the indices $j = j_1, \ldots, j_{\theta_+}$, there are exactly θ_- of them for which $(r_{-j}, r_j) = (0,0)$, and for the others we have $(r_{-j}, r_j) = (1,-1)$ or $(-1,1)$. If j is not one of the j_k's, we have $(r_{-j}, r_j) = (0,1)$ or $(1,0)$. In other words, the pairs of indices (r_{-j}, r_j) are, up to shifts and ordering, the same as the corresponding pairs of indices defining Λ'. By considering appropriate permutation matrices, we may change a pair (r_{-j}, r_j) into (r_j, r_{-j}) and we may change the order so that Λ'' is sent to Λ'. This transformation defines an element of J which stabilizes $\Lambda = \Lambda(0, \ldots, 0)$. □

3.1.3.5 We may now prove the proposition.

Proof. It is clear that $K \subset \mathrm{Ker}(\alpha)$, so we prove the reverse inclusion. Let $g^0 \in J^\circ$. We will write g^0 as a product of elements in J, each of which fixes some lattice of maximal orbit type in the Bruhat-Tits building. We write $\Lambda := \Lambda_m = \Lambda(0, \ldots, 0)$ and $\Lambda^0 := g^0 \cdot \Lambda$. Since $g^0 \in J^\circ$, we have $\Lambda^0 \in \mathcal{L}_0$. We would like to send Λ^0 back to Λ by using elements of K only. Let L be some

Witt decomposition of $\mathbf{V}$ such that the corresponding apartment $\mathcal{A}^L$ contains both Λ and Λ^0. We can find some $g_1 \in J_\Lambda$ which sends $\mathcal{A}^L$ to $\mathcal{A}^e$. We define $g^1 := g_1 g^0$ and $\Lambda^1 := g^1 \cdot \Lambda$. Then $\Lambda^1 \in \mathcal{L}_0$ and it belongs to the apartment $\mathcal{A}^e$. Therefore, there exists integers $r_{-m}, \ldots, r_m, s$ such that

$$\Lambda^1 = \Lambda(r_{-m}, \ldots, r_{-1}, s, r_1, \ldots, r_m).$$

Since $\Lambda^1 \in \mathcal{L}_0$ and its orbit type is maximal, we have $s = 0$ and $r_{-j} = -r_j$ for all $1 \leqslant j \leqslant m$. Let $1 \leqslant j_1 < \ldots < j_a \leqslant m$ be the indices j for which r_j is odd. We have $0 \leqslant a \leqslant m$. For $1 \leqslant j \leqslant m$ we write $r_j = 2r'_j + 1$ if j is some of the $j'_k s$ and $r_j = 2r'_j$ otherwise. We also write $r'_{-j} = -r'_j$, so that we have $r_{-j} = 2r'_{-j} - 1$ if j is some of the j_k's and $r_{-j} = 2r'_{-j}$ otherwise. We define g_2 the endomorphism of $\mathbf{V}$ sending e_{-j} to $p^{2r'_j} e_j$ for $-m \leqslant j \leqslant m$ and $j \neq 0$, and which acts like identity on $\mathbf{V}^{\mathrm{an}}$. Then g_2 is an element of J with multiplier equal to 1. Moreover, g_2 stabilizes the lattice $\Lambda(r'_{-m}, \ldots, r'_{-1}, 0, r'_1, \ldots, r'_m) \in \mathcal{L}_0$ whose orbit type is maximal, therefore $g_2 \in K$. We define $g^2 := g_2 g^1$ and $\Lambda^2 := g^2 \cdot \Lambda \in \mathcal{L}_0$. Concretely, the lattice Λ^2 still lies in the apartment $\mathcal{A}^e$ and its coefficients are obtained from those of Λ^1 by replacing each pair (r_{-j_k}, r_{j_k}) by $(1, -1)$ and the other pairs (r_{-j}, r_j) by $(0, 0)$. Let us note that if $a = 0$ then we already have $\Lambda^2 = \Lambda$.

Let us now assume that $a > 0$. The intersection of the lattices Λ^2 and Λ has the following shape.

$$\Lambda^2 \cap \Lambda = \Lambda(\underbrace{0 \text{ or } 1, \ldots, 0 \text{ or } 1}_{a \text{ times } 1 \text{ and } m-a \text{ times } 0}, 0, 0^m).$$

The coefficient takes the value 1 if and only if its index is one of the $-j_k$'s. This is a lattice in $\mathcal{L}_0$ of orbit type $2(m-a)+1$. We will use 3.1.3.4 Lemma in order to send Λ^2 to Λ while fixing some lattice of maximal orbit type. In order to find this lattice, we need to leave the apartment $\mathcal{A}^e$. Let $\delta \in \mathbb{Z}_{p^2}^\times$ such that $\sigma(\delta) = -\delta$. We define the following vectors

$$f_j = \begin{cases} e_j & \text{if } j \text{ is not one of the } \pm j_k\text{'s.} \\ pe_{-j_k} & \text{if } j = -j_k. \\ p^{-1}e_{j_k} + \delta e_{-j_k} & \text{if } j = j_k. \end{cases}$$

We also define $f_i^{\mathrm{an}} = e_i^{\mathrm{an}}$ for $i \in \{0, 1\}$ (the case $i = 1$ only occurs if n is even). All together, these vectors form a basis f of $\mathbf{V}$. We write Λ_f for the $\mathbb{Z}_{p^2}$-lattice generated by the basis f. One may check that $\langle f_j, f_{j'} \rangle = \delta_{j',-j}$ for every j and j'. It follows that $\Lambda_f \in \mathcal{L}_0$ and it has maximal orbit type. It turns out that both intersections $\Lambda^2 \cap \Lambda_f$ and $\Lambda \cap \Lambda_f$ are equal to $\Lambda^2 \cap \Lambda$, as we prove in the following two points.

- $\underline{\Lambda^2 \cap \Lambda_f}$: The lattice $\Lambda^2 \cap \Lambda_f$ contains all the vectors e_j where j is not of the $\pm j_k$'s. It also contains the vectors pe_{-j_k} and $p \cdot (p^{-1}e_{j_k} + \delta e_{-j_k}) = e_{j_k} + \delta p e_{-j_k}$ for all $1 \leqslant k \leqslant a$. Therefore, it must contain the vectors e_{j_k}'s as well. This gives the inclusion $\Lambda^2 \cap \Lambda \subset \Lambda^2 \cap \Lambda_f$. For the converse, if $x \in \Lambda_f$ then we may write

$$\begin{aligned} x &= \sum_{j \neq \pm j_k} \mu_j e_j + \sum_{k=1}^{s} \lambda_k p e_{-j_k} + \lambda'_k (p^{-1} e_{j_k} + \delta e_{-j_k}) \\ &= \sum_{j \neq \pm j_k} \mu_j e_j + \sum_{k=1}^{s} (\lambda_k p + \lambda'_k \delta) e_{-j_k} + \lambda'_k p^{-1} e_{j_k} \end{aligned}$$

with the scalars μ_j, λ_k and λ'_k in $\mathbb{Z}_{p^2}$. If moreover $x \in \Lambda^2$ then in the last formula, we must have $\lambda_k p + \lambda'_k \delta \in p\mathbb{Z}_{p^2}$. It follows that the scalars λ'_k belong to $p\mathbb{Z}_{p^2}$ and thus $x \in \Lambda^2 \cap \Lambda$.

– $\underline{\Lambda \cap \Lambda_f}$: By the same arguments as above, we prove that $\Lambda^2 \cap \Lambda \subset \Lambda \cap \Lambda_f$. For the converse, let $x \in \Lambda_f$ as above. If moreover $x \in \Lambda$ then the scalars λ'_k are elements of $p\mathbb{Z}_{p^2}$. It implies that $\lambda_k p + \lambda'_k \delta \in p\mathbb{Z}_{p^2}$, whence $x \in \Lambda^2 \cap \Lambda$.

Eventually we may apply 3.1.3.4 Lemma to the lattices Λ_f, Λ^2 and Λ. It gives the existence of an element $g_3 \in J$ which stabilizes Λ_f and sends Λ^2 to Λ. We write $g^3 := g_3 g^2$. It follows that $g^3 \cdot \Lambda = \Lambda$, therefore $g^3 \in J_\Lambda \subset K$. But $g^3 = g_3 g_2 g_1 g^0$ and each of the elements g_1, g_2 and g_3 also lies in K. Therefore $g^0 \in K$ as well. □

3.1.4 Counting the closed Bruhat-Tits strata

3.1.4.1 In this section we count the number of closed Bruhat-Tits strata which contain or which are contained in another given one. Let $d \geqslant 0$ and consider V a d-dimensional $\mathbb{F}_{p^2}$-vector space equipped with a non degenerate hermitian form. This structure is uniquely determined up to isomorphism as we are working over a finite field. As in [VW11], for $\lceil \frac{d}{2} \rceil \leqslant r \leqslant d$, we define

$$\begin{aligned} N(r, V) &:= \{U \mid U \text{ is an } r\text{-dimensional subspace of } V \text{ such that } U^\perp \subset U\}, \\ \nu(r, d) &:= \#N(r, V), \end{aligned}$$

where $U^\perp$ denotes the orthogonal of U with respect to the hermitian form on V. As remarked in [VW11], the set $N(r, V)$ can be seen as the set of rational points of a certain flag variety for the unitary group of V.

Proposition ([VW11] Corollary 5.7)**.** *Let $\Lambda \in \mathcal{L}$. Write $t(\Lambda) = 2\theta + 1$ for some $0 \leqslant \theta \leqslant m$.*

- *Let θ' be an integer such that $0 \leqslant \theta' \leqslant \theta$. The number of closed Bruhat-Tits strata of dimension θ' contained in $\mathcal{M}_\Lambda$ is $\nu(\theta + \theta' + 1, 2\theta + 1)$.*
- *Let θ' be an integer such that $\theta \leqslant \theta' \leqslant m$. The number of closed Bruhat-Tits strata of dimension θ' containing $\mathcal{M}_\Lambda$ is $\nu(n - \theta - \theta' - 1, n - 2\theta - 1)$.*

These follows from 3.1.2.9 Proposition and Remark. Another way to formulate the proposition is to say that $\nu(\theta + \theta' + 1, 2\theta + 1)$ (resp. $\nu(n - \theta - \theta' - 1, n - 2\theta - 1)$) is the number of vertices of type $2\theta' + 1$ in the Bruhat-Tits building of $\tilde{J}$ which are neighbors of a given vertex of type $2\theta + 1$ for $\theta' \leqslant \theta$ (resp. $\theta' \geqslant \theta$).

3.1.4.2 In [VW11], an explicit formula is given for $\nu(d-1,d)$. The next proposition gives a formula to compute $\nu(r,d)$ for general r and d.

Proposition. *Let $d \geqslant 0$ and let $\left\lceil \frac{d}{2} \right\rceil \leqslant r \leqslant d$. We have*

$$\nu(r,d) = \frac{\prod_{j=1}^{2(d-r)} \left(p^{2r-d+j} - (-1)^{2r-d+j}\right)}{\prod_{j=1}^{d-r} (p^{2j}-1)}$$

Proof. Recall that for any integer k, we denote by A_k the $k \times k$ matrix having 1 in the antidiagonal and 0 everywhere else. We fix a basis $(e_1, \ldots, e_d)$ of V in which the hermitian form is represented by the matrix A_d. We denote by U_0 the subspace generated by the vectors $e_1, \ldots, e_r$. Then the orthogonal of U_0 is generated by $e_1, \ldots, e_{d-r}$. Since r is an integer between $\left\lceil \frac{d}{2} \right\rceil$ and d, we have $0 \leqslant d-r \leqslant r$ and therefore U_0 contains its orthogonal. Thus, U_0 defines an element of $N(r,V)$. The unitary group $\mathrm{U}(V) \simeq \mathrm{U}_d(\mathbb{F}_p)$ acts on the set $N(r,V)$: an element $g \in \mathrm{U}(V)$ sends the subspace U to $g(U)$. This action is transitive. Indeed, any $U \in N(r,V)$ can be sent to U_0 by using an equivalent of the Gram-Schmidt orthogonalization process over $\mathbb{F}_{p^2}$ (note that $p \neq 2$). The stabilizer of U_0 in $\mathrm{U}_d(\mathbb{F}_p)$ is the standard parabolic subgroup

$$P_0 := \left\{ \begin{pmatrix} B & * & * \\ 0 & M & * \\ 0 & 0 & F(B) \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; B \in \mathrm{GL}_{d-r}(\mathbb{F}_{p^2}), M \in \mathrm{U}_{2r-d}(\mathbb{F}_p) \right\}.$$

Here, $F(B) = A_{d-r}(B^{(p)})^{-T} A_{d-r}$ where $B^{(p)}$ is the matrix B with all coefficients raised to the power p. Therefore, the set $N(r,V)$ is in bijection with the quotient $\mathrm{U}_d(\mathbb{F}_p)/P_0$. The order of $\mathrm{U}_d(\mathbb{F}_p)$ is well known and given by the formula

$$\#\mathrm{U}_d(\mathbb{F}_p) = p^{\frac{d(d-1)}{2}} \prod_{j=1}^{d} \left(p^j - (-1)^j\right).$$

It remains to compute the order of P_0. We have a Levi decomposition $P_0 = L_0 N_0$ with $L_0 \cap N_0 = \{1\}$ where

$$L_0 := \left\{ \begin{pmatrix} B & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & F(B) \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; B \in \mathrm{GL}_{d-r}(\mathbb{F}_{p^2}), M \in \mathrm{U}_{2r-d}(\mathbb{F}_p) \right\},$$

$$N_0 := \left\{ \begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}_d(\mathbb{F}_p) \;\middle|\; X \in \mathrm{M}_{d-r,2r-d}(\mathbb{F}_{p^2}), Y \in \mathrm{M}_{2r-d,d-r}(\mathbb{F}_{p^2}), Z \in \mathrm{M}_{d-r}(\mathbb{F}_{p^2}) \right\}.$$

The order of L_0 is given by

$$\#L_0 = \#\mathrm{GL}_{d-r}(\mathbb{F}_{p^2})\#\mathrm{U}_{2r-d}(\mathbb{F}_p) = p^{(d-r)(d-r-1)+\frac{(2r-d)(2r-d-1)}{2}} \prod_{j=1}^{d-r} \left(p^{2j}-1\right) \prod_{j=1}^{2r-d} \left(p^j - (-1)^j\right).$$

As for N_0, we need some more conditions on the matrices X, Y and Z. By direct computations, one checks that such a matrix belongs to $\mathrm{U}_d(\mathbb{F}_p)$ if and only if

$$Y = -A_{2r-d}(X^{(p)})^T A_{d-r}, \qquad Z + A_{d-r}(Z^{(p)})^T A_{d-r} = XY \in \mathrm{M}_{d-r}(\mathbb{F}_{p^2}).$$

Thus, X is any matrix of size $(d-r) \times (2d-r)$ and Y is determined by X. Let us look at the second equation. The matrix $A_{d-r}(Z^{(p)})^T A_{d-r}$ is the reflexion of $Z^{(p)}$ with respect to the antidiagonal. The equation implies that the coefficients below the antidiagonal of Z determine those above the antidiagonal. Furthermore, if z is a coefficient in the antidiagonal then the equation determines the value of $\mathrm{Tr}(z) = z + z^p$, where $\mathrm{Tr} : \mathbb{F}_{p^2} \to \mathbb{F}_p$ is the trace relative to the extension $\mathbb{F}_{p^2}/\mathbb{F}_p$. The trace is surjective and its kernel has order p. Thus, there are only p possibilities for each antidiagonal coefficient. Putting things together, the order of N_0 is given by

$$\#N_0 = p^{2(d-r)(2r-d)} \cdot p^{2\frac{(d-r)(d-r-1)}{2}} \cdot p^{d-r} = p^{(d-r)(3r-d)}$$

where the three terms take account respectively of the choice of X, the choice of the coefficients below the antidiagonal of Z and the choice of the coefficients in the antidiagonal of Z.
Hence the order of P_0 is given by

$$\#P_0 = \#L_0 \#N_0 = p^{\frac{d(d-1)}{2}} \prod_{j=1}^{d-r} (p^{2j} - 1) \prod_{j=1}^{2r-d} \left(p^j - (-1)^j\right).$$

Upon taking the quotient $\nu(r, d) = \#\mathrm{U}_d(\mathbb{F}_p)/\#P_0$, the result follows. □

In particular with $r = d - 1$, we obtain

$$\nu(d-1, d) = \frac{(p^{d-1} - (-1)^{d-1})(p^d - (-1)^d)}{p^2 - 1}.$$

If $d = 2\delta$ is even, it is equal to $(p^{d-1} + 1) \sum_{j=0}^{\delta-1} p^{2j}$, and if $d = 2\delta + 1$ is odd, it is equal to $(p^d + 1) \sum_{j=0}^{\delta-1} p^{2j}$. This coincides with the formula given in [VW11] Example 5.6.

3.2 The cohomology of a closed Bruhat-Tits stratum

3.2.1 In [Mul22a], we computed the cohomology groups $\mathrm{H}^\bullet_c(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ of the closed Bruhat-Tits strata (recall that $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_p$). The computation relies on the Ekedahl-Oort stratification on $\mathcal{M}_\Lambda$ which, in the language of Deligne-Lusztig varieties, translates into a stratification by Coxeter varieties for unitary groups of smaller sizes. The cohomology of Coxeter varieties is well known thanks to the work of Lusztig in [Lus76]. In order to state our results, we recall the classification of unipotent representations of the finite unitary group over $\overline{\mathbb{Q}_\ell}$.

3.2.2 Let q be a power of prime number p, and let $\mathbf{G}$ be a reductive connected group over an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Assume that $\mathbf{G}$ is equipped with an $\mathbb{F}_q$-structure induced by a Frobenius morphism F. Let $G = \mathbf{G}^F$ be the associated finite group of Lie type. Let $(\mathbf{T}, \mathbf{B})$ be a pair consisting of an F-stable maximal torus $\mathbf{T}$ and an F-stable Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\mathbf{W} = \mathbf{W}(\mathbf{T})$ denote the Weyl group of $\mathbf{G}$. The Frobenius F induces an action on $\mathbf{W}$. For $w \in \mathbf{W}$, let $\dot{w}$ be a representative of w in the normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of $\mathbf{T}$. By the Lang-Steinberg theorem, one can find $g \in \mathbf{G}$ such that $\dot{w} = g^{-1}F(g)$. Then ${}^g\mathbf{T} := g\mathbf{T}g^{-1}$ is another F-stable maximal torus, and $w \in \mathbf{W}$ is said to be the **type** of ${}^g\mathbf{T}$ with respect to $\mathbf{T}$. Every F-stable maximal torus arises in this manner. According to [DL76] Corollary 1.14, the G-conjugacy class of ${}^g\mathbf{T}$ only depends on the F-conjugacy class of w in the Weyl group $\mathbf{W}$. Here, two elements w and w' in $\mathbf{W}$ are said to be F-conjugates if there exists some element $\tau \in \mathbf{W}$ such that $w = \tau w' F(\tau)^{-1}$. For every $w \in \mathbf{W}$, we fix $\mathbf{T}_w$ an F-stable maximal torus of type w with respect to $\mathbf{T}$. The Deligne-Lusztig induction of the trivial representation of $\mathbf{T}_w$ is the virtual representation of G defined by the formula

$$R_w := \sum_{i \geqslant 0} (-1)^i \mathrm{H}^i_c(X_\varnothing(w) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}),$$

where $X_\varnothing(w)$ is the Deligne-Lusztig variety for $\mathbf{G}$ given by

$$X_\varnothing(w) := \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}w\mathbf{B}\}.$$

According to [DL76] Theorem 1.6, the virtual representation R_w only depends on the F-conjugacy class of w in $\mathbf{W}$. An irreducible representation of G is said to be **unipotent** if it occurs in R_w for some $w \in \mathbf{W}$. The set of isomorphism classes of unipotent representations of G is usually denoted $\mathcal{E}(G, 1)$ following Lusztig's notations.

Remark. Since the center $\mathrm{Z}(G)$ acts trivially on the variety $X_\varnothing(w)$, any irreducible unipotent representation of G has trivial central character.

3.2.3 Let $\mathbf{G}$ and $\mathbf{G}'$ be two reductive connected group over $\mathbb{F}$ both equipped with an $\mathbb{F}_q$-structure. We denote by F and F' the respective Frobenius morphisms. Let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy, that is a homomorphism defined over $\mathbb{F}_q$ whose kernel is contained in the center of $\mathbf{G}$ and whose image contains the derived subgroup of $\mathbf{G}'$. Then, according to [DM14] Proposition 11.3.8, we have an equality

$$\mathcal{E}(G, 1) = \{\rho \circ f \mid \rho \in \mathcal{E}(G', 1)\}.$$

Thus, the irreducible unipotent representations of G and of G' can be identified. We will use this observation in the case $G = \mathrm{U}_k(\mathbb{F}_q)$ and $G' = \mathrm{GU}_k(\mathbb{F}_q)$. The corresponding reductive groups are $\mathbf{G} = \mathrm{GL}_k$ and $\mathbf{G}' = \mathrm{GL}_k \times \mathrm{GL}_1$. The Frobenius morphisms can be defined as

$$F(M) = \dot{w}_0 (M^{(q)})^{-T} \dot{w}_0, \qquad F'(M, c) = (c^q \dot{w}_0 (M^{(q)})^{-T} \dot{w}_0, c^q).$$

Here, $\dot{w}_0$ is the $k \times k$ matrix with only 1's in the antidiagonal and $M^{(q)}$ is the matrix M whose entries are all raised to the power q. The isotypy $f : \mathbf{G} \to \mathbf{G}'$ is defined by $f(M) = (M, 1)$. It satisfies $F' \circ f = f \circ F$, it is injective and its image contains the derived subgroup $\mathrm{SL}_n \times \{1\} \subset \mathbf{G}'$. Hence, we obtain the following result.

Proposition. *The irreducible unipotent representations of the finite groups of Lie type* $\mathrm{U}_k(\mathbb{F}_q)$ *and* $\mathrm{GU}_k(\mathbb{F}_q)$ *can be naturally identified.*

3.2.4 Assume that the Coxeter graph of the reductive group $\mathbf{G}$ is a union of subgraphs of type A_m (for various m). Let $\widecheck{\mathbf{W}}$ be the set of isomorphism classes of irreducible representations of its Weyl group $\mathbf{W}$. The action of the Frobenius F on $\mathbf{W}$ induces an action on $\breve{\mathbf{W}}$, and we consider the fixed point set $\breve{\mathbf{W}}^F$. The following theorem classifies the irreducible unipotent representations of G.

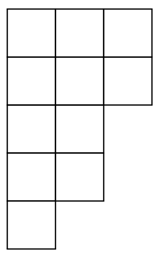
Theorem ([LS77] Theorem 2.2)**.** *There is a bijection between $\breve{\mathbf{W}}^F$ and the set of isomorphism classes of irreducible unipotent representations of G.*

We recall how the bijection is constructed. According to loc. cit. if $V \in \breve{\mathbf{W}}^F$ there is a unique automorphism $\widetilde{F}$ of V of finite order such that

$$R(V) := \frac{1}{|\mathbf{W}|} \sum_{w \in \mathbf{W}} \mathrm{Trace}(w \circ \widetilde{F} \,|\, V) R_w$$

is an irreducible representation of G. Then the map $V \mapsto R(V)$ is the desired bijection. In the case of $\mathrm{U}_k(\mathbb{F}_q)$ or $\mathrm{GU}_k(\mathbb{F}_q)$, the Weyl group $\mathbf{W}$ is identified with the symmetric group $\mathfrak{S}_k$ and we have an equality $\breve{\mathbf{W}}^F = \breve{\mathbf{W}}$. Moreover, the automorphism $\widetilde{F}$ is the multiplication by w_0, where w_0 is the element of maximal length in $\mathbf{W}$. Thus, in both cases the irreducible unipotent representations of G are classified by the irreducible representations of the Weyl group $\mathbf{W} \simeq \mathfrak{S}_k$, which in turn are classified by partitions of k or equivalently by Young diagrams, as we briefly recall in the next paragraph.

3.2.5 A partition of k is a tuple $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ with $r \geqslant 1$ and each λ_i is a positive integer, such that $\lambda_1 + \ldots + \lambda_r = k$. The integer k is called the length of the partition, and it is denoted by $|\lambda|$. A Young diagram of size k is a top left justified collection of k boxes, arranged in rows and columns. There is a correspondance between Young diagrams of size k and partitions of k, by associating to a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ the Young diagram having r rows consisting successively of $\lambda_1, \ldots, \lambda_r$ boxes. We will often identify a partition with its Young diagram, and conversely. For example, the Young diagram associated to $\lambda = (3^2, 2^2, 1)$ is the following one.



To any partition λ of k, one can naturally associate an irreducible character χ_λ of the symmetric group $\mathfrak{S}_k$. An explicit construction is given, for instance, by the notion of Specht modules as explained in [Jam84] 7.1. We will not recall their definition.

3.2.6 The irreducible unipotent representation of $\mathrm{U}_k(\mathbb{F}_q)$ (resp. $\mathrm{GU}_k(\mathbb{F}_q)$) associated to χ_λ by the bijection of 3.2.4 is denoted by $\rho_\lambda^{\mathrm{U}}$ (resp. $\rho_\lambda^{\mathrm{GU}}$). In virtue of 3.2.3, for every λ we have $\rho_\lambda^{\mathrm{U}} = \rho_\lambda^{\mathrm{GU}} \circ f$, where $f : \mathrm{U}_k(\mathbb{F}_q) \to \mathrm{GU}_k(\mathbb{F}_q)$ is the inclusion. Thus, it is harmless to identify $\rho_\lambda^{\mathrm{U}}$ and $\rho_\lambda^{\mathrm{GU}}$ so that from now on, we will omit the superscript. The partition (k)

corresponds to the trivial representation and (1^k) to the Steinberg representation. The degree of the representations ρ_λ is given by expressions known as **hook formula**. Given a box $\square$ in the Young diagram of λ, its **hook length** $h(\square)$ is 1 plus the number of boxes lying below it or on its right. For instance, in the following figure the hook length of every box of the Young diagram of $\lambda = (3^2, 2^2, 1)$ has been written inside it.

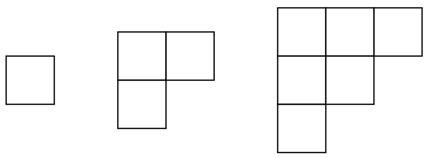
7	5	2
6	4	1
4	2	
3	1	
1		

Proposition ([GP00] Propositions 4.3.5)**.** *Let $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_r)$ be a partition of n. The degree of the irreducible unipotent representation ρ_λ is given by the following formula*

$$\deg(\rho_\lambda) = q^{a(\lambda)} \frac{\prod_{i=1}^{k} q^i - (-1)^i}{\prod_{\square \in \lambda} q^{h(\square)} - (-1)^{h(\square)}}$$

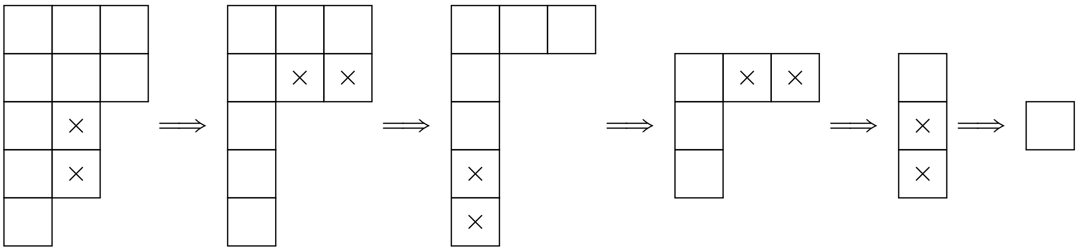
where $a(\lambda) = \sum_{i=1}^{r} (i-1)\lambda_i$.

3.2.7 We may describe the cuspidal support of the unipotent representations ρ_λ. According to [Lus77] 9.2 and 9.4 there exists an irreducible unipotent cuspidal character of $\mathrm{U}_k(\mathbb{F}_q)$ (or $\mathrm{GU}_k(\mathbb{F}_q)$) if and only if k is an integer of the form $k = \frac{t(t+1)}{2}$ for some $t \geqslant 0$, and when that is the case it is the one associated to the partition $\Delta_t := (t, t-1, \ldots, 1)$, whose Young diagram has the distinctive shape of a reversed staircase. Here, as a convention $U_0(\mathbb{F}_q) = \mathrm{GU}_0(\mathbb{F}_q)$ denotes the trivial group. For example, here are the Young diagrams of Δ_1, Δ_2 and Δ_3. Of course, the one of Δ_0 the empty diagram.



We consider an integer $t \geqslant 0$ such that k decomposes as $k = 2e + \frac{t(t+1)}{2}$ for some $e \geqslant 0$. Let G denote $\mathrm{U}_k(\mathbb{F}_q)$ or $\mathrm{GU}_k(\mathbb{F}_q)$, and consider L_t the subgroup consisting of block-diagonal matrices having one middle block of size $\frac{t(t+1)}{2}$ and all other blocks of size 1. This is a standard Levi subgroup of G. For $\mathrm{U}_k(\mathbb{F}_q)$, it is isomorphic to $\mathrm{GL}_1(\mathbb{F}_{q^2})^e \times \mathrm{U}_{\frac{t(t+1)}{2}}(\mathbb{F}_q)$ whereas in the case of $\mathrm{GU}_k(\mathbb{F}_q)$ it is isomorphic to $\mathrm{G}\left(\mathrm{U}_1(\mathbb{F}_q)^e \times \mathrm{U}_{\frac{t(t+1)}{2}}(\mathbb{F}_q)\right)$. In both cases, L_t admits a quotient which is isomorphic to a group of the same type as G but of size $\frac{t(t+1)}{2}$. We write ρ_t for the inflation to L_t of the unipotent cuspidal representation ρ_{Δ_t} of this quotient. If λ is a partition of k, the cuspidal support of the representation ρ_λ is given by exactly one of the pair (L_t, ρ_t) up to conjugacy, where $t \geqslant 0$ is an integer such that for some $e \geqslant 0$ we have $k = 2e + \frac{t(t+1)}{2}$. Note that in particular k and $\frac{t(t+1)}{2}$ must have the same parity. With these notations, the irreducible unipotent representations belonging to the principal series are those with cuspidal support (L_0, ρ_0) if k is even and (L_1, ρ_1) is k is odd.

3.2.8 Given an irreducible unipotent representation ρ_λ, there is a combinatorical way to determine the Harish-Chandra series to which it belongs, as we recalled in [Mul22a] Section 2. We consider the Young diagram of λ. We call **domino** any pair of adjacent boxes in the diagram. It may be either vertical or horizontal. We remove dominoes from the diagram of λ so that the resulting shape is again a Young diagram, until one can not proceed further. This process results in the Young diagram of the partition Δ_t for some $t \geqslant 0$, and it is called the 2-**core** of λ. It does not depend on the successive choices for the dominoes. Then, the representation ρ_λ has cuspidal support (L_t, ρ_t) if and only if λ has 2-core Δ_t. For instance, the diagram $\lambda = (3^2, 2^2, 1)$ given in 3.2.5 has 2-core Δ_1, as it can be determined by the following steps. We put crosses inside the successive dominoes that we remove from the diagram. Thus, the unipotent representation ρ_λ of $\mathrm{U}_{11}(\mathbb{F}_q)$ or $\mathrm{GU}_{11}(\mathbb{F}_q)$ has cuspidal support (L_1, ρ_1), so in particular it is a principal series representation.



3.2.9 From now on, we take $q = p$. We consider the ℓ-adic cohomology with compact support of a closed Bruhat-Tits stratum $\mathcal{M}_\Lambda \otimes \mathbb{F}$, where ℓ is a prime number different from p and $\Lambda \in \mathcal{L}$ has orbit type $t(\Lambda) = 2\theta + 1$, $0 \leqslant \theta \leqslant m$. Recall from 3.1.2.10 that the stratum $\mathcal{M}_\Lambda$ is equipped with an action of the finite group of Lie type $\mathrm{GU}(V_\Lambda^0) \simeq \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$, and as such it is isomorphic to a Deligne-Lusztig variety. Let F be the Frobenius morphism of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ as defined in 3.2.3. Then F^2 induces a geometric Frobenius morphism $\mathcal{M}_\Lambda \otimes \mathbb{F} \to \mathcal{M}_\Lambda \otimes \mathbb{F}$ relative to the $\mathbb{F}_{p^2}$-structure of $\mathcal{M}_\Lambda$. Because it is a finite morphism, it induces a linear endomorphism on the cohomology groups, and it is in fact an automorphism. In [Mul22a], we computed these cohomology groups in terms of a $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p) \times \langle F^2 \rangle$-representation.

Theorem. *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$ for some $0 \leqslant \theta \leqslant m$.*

(1) *The cohomology group $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ is zero unless $0 \leqslant j \leqslant 2\theta$. There is an isomorphism*

$$\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}_c^{2\theta-j}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})^\vee(\theta)$$

which is equivariant for the action of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p) \times \langle F^2 \rangle$.

(2) *The Frobenius F^2 acts like multiplication by $(-p)^j$ on $\mathrm{H}_c^j(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$.*

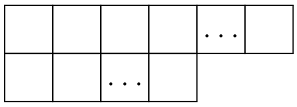
(3) *For $0 \leqslant j \leqslant \theta$ we have*

$$\mathrm{H}_c^{2j}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant j \leqslant \theta - 1$ we have

$$\mathrm{H}_c^{2j+1}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\theta-2s,2s+1)}.$$

Thus, in the cohomology of $\mathcal{M}_\Lambda$ all the representations associated to a Young diagram with at most 2 rows occur, and there is no other. Such a diagram has the following general shape.



Remarks. Let us make a few comments.

- Part (1) of the theorem follows from general theory of etale cohomology given that the variety $\mathcal{M}_\Lambda$ is smooth and projective over $\mathbb{F}_{p^2}$. The identity is a consequence of Poincaré duality. The notation (θ) is a Tate twist, it modifies the action of F^2 by multiplying it with $p^{2\theta}$.
- The cohomology groups of index 0 and 2θ are the trivial representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$.
- All irreducible representations in the cohomology groups of even index belong to the unipotent principal series, whereas all the ones in the groups of odd index have cuspidal support (L_2, ρ_2).
- The cohomology group $\mathrm{H}^j_c(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ contains no cuspidal representation of $\mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ unless $\theta = j = 0$ or $\theta = j = 1$. If $\theta = 0$ then H^0_c is the trivial representation of $\mathrm{GU}_1(\mathbb{F}_p) = \mathbb{F}_{p^2}^\times$, and if $\theta = 1$ then H^1_c is the representation ρ_{Δ_2} of $\mathrm{GU}_3(\mathbb{F}_p)$. Both of them are cuspidal.

3.3 Shimura variety and p-adic uniformization of the basic stratum

3.3.1 In this section, we introduce the PEL unitary Shimura variety with signature $(1, n-1)$ as in [VW11] 6.1 and 6.2, and we recall the p-adic uniformization theorem of its basic (or supersingular) locus. The Shimura variety can be defined as a moduli problem classifying abelian varieties with additional structures, as follows. Let E be a quadratic imaginary extension of $\mathbb{Q}$ in which p **is inert**. Let B/F be a simple central algebra of degree $d \geqslant 1$ which splits over p and at infinity. Let $*$ be a positive involution of the second kind on B, and let $\mathbb{V}$ be a non-zero finitely generated left B-module equipped with a non-degenerate $*$-alternating form $\langle \cdot, \cdot \rangle$ taking values in $\mathbb{Q}$. Assume also that $\dim_E(\mathbb{V}) = nd$. Let G be the connected reductive group over $\mathbb{Q}$ whose points over a $\mathbb{Q}$-algebra R are given by

$$G(R) := \{g \in \mathrm{GL}_{E\otimes R}(\mathbb{V} \otimes R) \,|\, \exists c \in R^\times \text{ such that for all } v, w \in \mathbb{V} \otimes R, \langle gv, gw \rangle = c\langle v, w \rangle\}.$$

We denote by $c : G \to \mathbb{G}_m$ the **multiplier** character. The base change $G_\mathbb{R}$ is isomorphic to a group of unitary similitudes $\mathrm{GU}(r, s)$ of a hermitian space with signature (r, s) where $r + s = n$. We assume that $r = 1$ and $s = n - 1$. We consider a Shimura datum of the form (G, X), where X denotes the unique $G(\mathbb{R})$-conjugacy class of homorphisms $h : \mathbb{C}^\times \to G_\mathbb{R}$ such that for all $z \in \mathbb{C}^\times$ we have $\langle h(z)\cdot, \cdot \rangle = \langle \cdot, h(\overline{z})\cdot \rangle$, and such that the $\mathbb{R}$-pairing $\langle \cdot, h(i)\cdot \rangle$ is positive definite. Such a homomorphism h induces a decomposition $\mathbb{V} \otimes \mathbb{C} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Concretely, $\mathbb{V}_1$ (resp. $\mathbb{V}_2$) is the subspace where $h(z)$ acts like z (resp. like $\overline{z}$). The reflex field associated to this PEL data, that is the field of definition of $\mathbb{V}_1$ as a complex representation of B, is E unless $n = 2$ in which case it is $\mathbb{Q}$. Nonetheless, for simplicity we will consider the associated Shimura varieties over E even in the case $n = 2$.

Remark. As remarked in [Vol10] Section 6, the group G satisfies the Hasse principle, ie. $\ker^1(\mathbb{Q}, G)$ is a singleton. Therefore, the Shimura variety associated to the Shimura datum (G, X) coincides with the moduli space of abelian varieties that we are going to define.

3.3.2 Let $\mathbb{A}_f$ denote the ring of finite adèles over $\mathbb{Q}$ and let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup. We define a functor Sh_K by associating to an E-scheme S the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta})$ where

- A is an abelian scheme over S.
- $\lambda : A \to \widehat{A}$ is a polarization.
- $\iota : B \to \mathrm{End}(A) \otimes \mathbb{Q}$ is a morphism of algebras such that $\iota(b^*) = \iota(b)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ, and such that the Kottwitz determinant condition is satisfied:
$$\forall b \in B,\ \det(\iota(b)) = \det(b \,|\, \mathbb{V}_1).$$
- $\overline{\eta}$ is a K-level structure, that is a K-orbit of isomorphisms of $B \otimes \mathbb{A}_f$-modules $\mathrm{H}_1(A, \mathbb{A}_f) \xrightarrow{\sim} \mathbb{V} \otimes \mathbb{A}_f$ that is compatible with the other data.

The Kottwitz condition in the third point is independent on the choice of $h \in X$. If K is sufficiently small, this moduli problem is represented by a smooth quasi-projective scheme Sh_K over E. When the level K varies, the Shimura varieties form a projective system $(\mathrm{Sh}_K)_K$ equipped with an action of $G(\mathbb{A}_f)$ by Hecke correspondences.

3.3.3 We assume the existence of a $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ in B, stable under the involution $*$, such that its p-adic completion is a maximal order in $B_{\mathbb{Q}_p}$. We also assume that there is a $\mathbb{Z}_p$-lattice Γ in $\mathbb{V} \otimes \mathbb{Q}_p$, invariant under $\mathcal{O}_B$ and self-dual for $\langle\cdot,\cdot\rangle$. We may fix isomorphisms $E_p \simeq \mathbb{Q}_{p^2}$ and $B_{\mathbb{Q}_p} \simeq \mathrm{M}_d(\mathbb{Q}_{p^2})$ such that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is identified with $\mathrm{M}_d(\mathbb{Z}_{p^2})$.
As a consequence of the existence of Γ, the group $G_{\mathbb{Q}_p}$ is unramified. Let $K_0 := \mathrm{Fix}(\Gamma)$ be the subgroup of $G(\mathbb{Q}_p)$ consisting of all g such that $g \cdot \Gamma = \Gamma$. It is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We will consider levels of the form $K = K_0K^p$ where K^p is an open compact subgroup of $G(\mathbb{A}_f^p)$. Note that K is sufficiently small as soon as K^p is sufficiently small. By the work of Kottwitz in [**kottwitzpoints**], the Shimura varieties $\mathrm{Sh}_{K_0K^p}$ admit integral models over $\mathcal{O}_{E,(p)}$ which have the following moduli interpretation. We define a functor S_{K^p} by associating to an $\mathcal{O}_{E,(p)}$-scheme S the set of isomorphism classes of tuples $(A, \lambda, \iota, \overline{\eta}^p)$ where

- A is an abelian scheme over S.
- $\lambda : A \to \widehat{A}$ is a polarization whose order is prime to p.
- $\iota : \mathcal{O}_B \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is a morphism of algebras such that $\iota(b^*) = \iota(b)^\dagger$ where $\cdot^\dagger$ denotes the Rosati involution associated to λ, and such that the Kottwitz determinant condition is satisfied:
$$\forall b \in \mathcal{O}_B,\ \det(\iota(b)) = \det(b \,|\, \mathbb{V}_1).$$

– $\overline{\eta}^p$ is a K^p-level structure, that is a K^p-orbit of isomorphisms of $B\otimes\mathbb{A}_f^p$-modules $\mathrm{H}_1(A,\mathbb{A}_f^p) \xrightarrow{\sim} \mathbb{V}\otimes\mathbb{A}_f^p$ that is compatible with the other data.

If K^p is sufficiently small, this moduli problem is also representable by a smooth quasi-projective scheme over $\mathcal{O}_{E,(p)}$. When the level K^p varies, these integral Shimura varieties form a projective system $(\mathrm{S}_{K^p})_{K^p}$ equipped with an action of $G(\mathbb{A}_f^p)$ by Hecke correspondences. We have a family of isomorphisms

$$\mathrm{Sh}_{K_0K^p} \simeq \mathrm{S}_{K^p} \otimes_{\mathcal{O}_{E,(p)}} E$$

which are compatible as the level K^p varies.

Notation. Unless explicitly mentioned, from now on the notation S_{K^p} will refer to the smooth quasi-projective $\mathbb{Z}_{p^2}$-scheme $\mathrm{S}_{K^p} \otimes_{\mathcal{O}_{E,(p)}} \mathbb{Z}_{p^2}$. Here, we implicitly use the identification of E_p with $\mathbb{Q}_{p^2}$.

Therefore, with this convention we have isomorphisms $\mathrm{Sh}_{K_0K^p}\otimes_E\mathbb{Q}_{p^2} \simeq \mathrm{S}_{\mathrm{K}^\mathrm{p}}\otimes_{\mathbb{Z}_{p^2}}\mathbb{Q}_{p^2}$ compatible as the level K^p varies.

3.3.4 Let $\overline{\mathrm{S}}_{K^p} := \mathrm{S}_{K^p} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ denote the special fiber of the Shimura variety. It is a smooth quasi-projective variety over $\mathbb{F}_{p^2}$. Its geometry can be described in terms of the Newton stratification as follows. Recall the Shimura datum introduced in 3.3.1. To any homomorphism $h \in X$, we can associate the cocharacter

$$\mu_h : \mathbb{C}^\times \to G_{\mathbb{C}} = \bigsqcup_{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} G_{\mathbb{R}}$$

which is given by $h : \mathbb{C}^\times \to G_{\mathbb{R}}$ into the summand corresponding to the identity in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The conjugacy class μ of μ_h is well-determined by X. The field of definition of μ is by definition the reflex field of the Shimura datum, that is E when $n \neq 2$ and $\mathbb{Q}$ otherwise. We fix an algebraic closure $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}_p}$) containing E (resp. $\mathbb{Q}_{p^2}$). We also fix an embedding $\nu : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ compatible with the identification $E_p \simeq \mathbb{Q}_{p^2}$. We may then consider the local datum $(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$ where $\mu_{\overline{\mathbb{Q}_p}}$ is the conjugacy class of cocharacters $\overline{\mathbb{Q}_p}^\times \to G_{\overline{\mathbb{Q}_p}}$ induced by μ and ν. Let $B(G_{\mathbb{Q}_p})$ denote the set of σ-conjugacy classes in $G(\breve{\mathbb{Q}}_p)$ where $\breve{\mathbb{Q}}_p := \widehat{W(\mathbb{F})}_{\mathbb{Q}}$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. As in [**kottwitziso**], we may associate the Kottwitz set $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}}) \subset B(G_{\mathbb{Q}_p})$. It is a finite set equipped with a partial order. An element $b \in B(G_{\mathbb{Q}_p})$ is said to be $\mu_{\overline{\mathbb{Q}_p}}$**–admissible** when it belongs to $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$. The set $B(G_{\mathbb{Q}_p})$ (resp. $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$) canonically classifies the isomorphism classes of isocrystals with a $G_{\mathbb{Q}_p}$-structure (resp. compatible $\mu_{\overline{\mathbb{Q}_p}}, G_{\mathbb{Q}_p}$-structures).
Let $\mathcal{A}_{K^p}$ denote the universal abelian scheme over S_{K^p}, and let $\overline{\mathcal{A}}_{K^p}$ denote its reduction modulo p. The associated p-divisible group $\mathcal{A}_{K^p}[p^\infty]$ is denoted by X_{K^p}. For any geometric point $x \in \overline{\mathrm{S}}_{K^p}$, the p-divisible group $(X_{K^p})_x$ is equipped with compatible $\mu_{\overline{\mathbb{Q}_p}}, G_{\mathbb{Q}_p}$-structures therefore it determines an element $b_x \in B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$. For $b \in B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$, the set

$$\overline{\mathrm{S}}_{K^p}(b) := \{x \in \overline{\mathrm{S}}_{K^p} \mid b_x = b\}$$

is locally closed in $\overline{\mathrm{S}}_{K^p}$. It is the underlying topological space of a reduced subscheme which we still denote by $\overline{\mathrm{S}}_{K^p}(b)$. They are called the **Newton strata** of the special fiber of the Shimura variety. For a fixed b, as the level K^p varies the strata form a projective tower $(\overline{\mathrm{S}}_{K^p}(b))_{K^p}$ equipped with an action of $G(\mathbb{A}_f^p)$ by Hecke correspondences.

3.3.5 In [BW05], the combinatorics of the Newton stratification is described in the case of a PEL unitary Shimura variety of signature $(1, n-1)$. The set $B(G_{\mathbb{Q}_p}, \mu_{\overline{\mathbb{Q}_p}})$ contains $\lfloor \frac{n}{2} \rfloor + 1$ elements $b_0 < b_1 < \ldots < b_{\lfloor \frac{n}{2} \rfloor}$ and we have

$$\overline{\mathrm{S}}_{K^p} = \bigsqcup_{i=0}^{\lfloor \frac{n}{2} \rfloor} \overline{\mathrm{S}}_{K^p}(b_i).$$

The stratification is linear, that is the closure of a stratum $\overline{\mathrm{S}}_{K^p}(b_i)$ is the union of all the strata $\overline{\mathrm{S}}_{K^p}(b_j)$ for $j \leqslant i$. The stratum corresponding to b_i has dimension $m + i$. The element $b_{\lfloor \frac{n}{2} \rfloor}$ is μ-ordinary, and the corresponding stratum $\overline{\mathrm{S}}_{K^p}(b_{\lfloor \frac{n}{2} \rfloor})$ is called the **μ-ordinary locus**. It is open and dense in $\overline{\mathrm{S}}_{K^p}$. The unique basic element is b_0, and the corresponding stratum $\overline{\mathrm{S}}_{K^p}(b_0)$ is called the **basic stratum**. It coincides with the **supersingular locus**. It is a closed subscheme of $\overline{\mathrm{S}}_{K^p}$.

3.3.6 The geometry of the basic stratum can be described using the Rapoport-Zink space $\mathcal{M}$ in a process called p-adic uniformization, see [RZ96] and [Far04]. Let x be a geometric point of $\overline{\mathrm{S}}_{K^p}(b_0)$. Since G satisfies the Hasse principle, according to [Far04] Proposition 3.1.8 the isogeny class of the triple $(\mathcal{A}_x, \lambda, \iota)$, consisting of the abelian variety $\mathcal{A}_x$ together with its additional structures, does not depend on the choice of x. We define

$$I := \mathrm{Aut}(\mathcal{A}_x, \lambda, \iota).$$

It is a reductive group over $\mathbb{Q}$. In fact, since we are considering the basic stratum, according to loc. cit. the group I is the inner form of G such that $I(\mathbb{A}_f) = J \times G(\mathbb{A}_f^p)$ and $I(\mathbb{R}) \simeq \mathrm{GU}(0, n)$, that is the unique inner form of $G(\mathbb{R})$ which is compact modulo center. In particular, one can think of $I(\mathbb{Q})$ as a subgroup both of J and of $G(\mathbb{A}_f^p)$. Let $(\widehat{\mathrm{S}}_{K^p})_{|b_0}$ denote the formal completion of S_{K^p} along the basic stratum.

Theorem ([RZ96] Theorem 6.24)**.** *There is an isomorphism of formal schemes over* $\mathrm{Spf}(\mathbb{Z}_{p^2})$

$$\Theta_{K^p} : I(\mathbb{Q}) \backslash \big(\mathcal{M} \times G(\mathbb{A}_f^p)/K^p\big) \xrightarrow{\sim} (\widehat{\mathrm{S}}_{K^p})_{|b_0}$$

which is compatible with the $G(\mathbb{A}_f^p)$-action by Hecke correspondences as the level K^p varies.

This isomorphism is known as the **p-adic uniformization** of the basic stratum. The induced map on the special fiber is an isomorphism

$$(\Theta_{K^p})_s : I(\mathbb{Q}) \backslash \big(\mathcal{M}_{\mathrm{red}} \times G(\mathbb{A}_f^p)/K^p\big) \xrightarrow{\sim} \overline{\mathrm{S}}_{K^p}(b_0)$$

of schemes over $\mathrm{Spec}(\mathbb{F}_{p^2})$. We denote by $\mathcal{M}^{\mathrm{an}}$ (resp. $(\widehat{\mathrm{S}}_{K^p})_{|b_0}^{\mathrm{an}}$) the smooth analytic space over $\mathbb{Q}_{p^2}$ associated to the formal scheme $\mathcal{M}$ (resp. $(\widehat{\mathrm{S}}_{K^p})_{|b_0}$) by the Berkovich functor as defined in [Ber96]. Note that both formal schemes are special in the sense of loc. cit. so that we may use Berkovich's constructions. These analytic spaces play the role of the generic fibers of the formal schemes over $\mathrm{Spf}(\mathbb{Z}_{p^2})$. By [Far04] Théorème 3.2.6, p-adic uniformization induces an isomorphism

$$\Theta^{\mathrm{an}}_{K^p} : I(\mathbb{Q})\backslash \left(\mathcal{M}^{\mathrm{an}} \times G(\mathbb{A}_f^p)/K^p\right) \xrightarrow{\sim} (\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}$$

of analytic spaces over $\mathbb{Q}_{p^2}$. We denote by red the reduction map from the generic fiber to the special fiber. It is an anticontinuous map of topological spaces, which means that the preimage of an open subset is closed and the preimage of a closed subet is open. Then, the uniformization on the generic and special fibers are compatible in the sense that the diagram

$$\begin{array}{ccc} I(\mathbb{Q})\backslash \left(\mathcal{M}^{\mathrm{an}} \times G(\mathbb{A}_f^p)/K^p\right) & \xrightarrow{\Theta^{\mathrm{an}}_{K^p}} & (\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0} \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ I(\mathbb{Q})\backslash \left(\mathcal{M}_{\mathrm{red}} \times G(\mathbb{A}_f^p)/K^p\right) & \xrightarrow{(\Theta_{K^p})_s} & \overline{\mathrm{S}}_{K^p}(b_0) \end{array}$$

is commutative.

3.3.7 The double coset space $I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)/K^p$ is finite, so that we may fix a system of representatives $g_1, \ldots, g_s \in G(\mathbb{A}_f^p)$. For every $1 \leqslant k \leqslant s$, we define $\Gamma_k := I(\mathbb{Q}) \cap g_k K^p g_k^{-1}$, which we see as a discrete subgroup of J that is cocompact modulo the center. The left hand side of the p-adic uniformization theorem is isomorphic to the disjoint union of the various quotients of $\mathcal{M}$ (or $\mathcal{M}_{\mathrm{red}}$ or $\mathcal{M}^{\mathrm{an}}$) by the subgroups $\Gamma_k \subset J$. In particular for the special fiber, it is an isomorphism

$$(\Theta_{K^p})_s : \bigsqcup_{k=1}^{s} \Gamma_k\backslash\mathcal{M}_{\mathrm{red}} \xrightarrow{\sim} \overline{\mathrm{S}}_{K^p}(b_0).$$

Let $\Phi^k_{K^p}$ be the composition $\mathcal{M}_{\mathrm{red}} \to \Gamma_k\backslash\mathcal{M}_{\mathrm{red}} \to \overline{\mathrm{Sh}}^{\mathrm{ss}}_{C^p}$ and let Φ_{K^p} be the disjoint union of the $\Phi^k_{K^p}$. The map Φ_{K^p} is surjective onto $\overline{\mathrm{S}}_{K^p}(b_0)$. According to [VW11] Section 6.4, it is a local isomorphism which can be used in order to transport the Bruhat-Tits stratification from $\mathcal{M}_{\mathrm{red}}$ to $\overline{\mathrm{S}}_{K^p}(b_0)$. Recall the notations of 3.1.2.3.

Proposition ([VW11] Proof of Proposition 6.5)**.** *Let $\Lambda \in \mathcal{L}$. For any $1 \leqslant k \leqslant s$, the restriction of $\Phi^k_{K^p}$ to $\mathcal{M}_\Lambda$ is an isomorphism onto its image.*

We will denote by $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ the scheme theoretic image of $\mathcal{M}_\Lambda$ through Φ^k. A subscheme of the form $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is called a **closed Bruhat-Tits stratum** of the Shimura variety. Together, they form the Bruhat-Tits stratification of the basic stratum, whose combinatorics is described by the union of the complexes $\Gamma_k\backslash\mathcal{L}$.

3.4 The cohomology of the Rapoport-Zink space at maximal level

3.4.1 The spectral sequence associated to an open cover of $\mathcal{M}^{\mathrm{an}}$

3.4.1.1 As in 3.3.6, we consider the generic fiber $\mathcal{M}^{\mathrm{an}}$ of the Rapoport-Zink space as a smooth Berkovich analytic space over $\mathbb{Q}_{p^2}$. Let $\mathrm{red} : \mathcal{M}^{\mathrm{an}} \to \mathcal{M}_{\mathrm{red}}$ be the reduction map. If Z

is a locally closed subset of $\mathcal{M}_{\mathrm{red}}$, then the preimage $\mathrm{red}^{-1}(Z)$ is called the **analytical tube over** Z. It is an analytic domain in $\mathcal{M}^{\mathrm{an}}$ and it coincides with the generic fiber of the formal completion of $\mathcal{M}_{\mathrm{red}}$ along Z. If $i \in \mathbb{Z}$ such that ni is even, then the tube $\mathrm{red}^{-1}(\mathcal{M}_i) = \mathcal{M}_i^{\mathrm{an}}$ is open and closed in $\mathcal{M}^{\mathrm{an}}$ and we have

$$\mathcal{M}^{\mathrm{an}} = \bigsqcup_{ni \in 2\mathbb{Z}} \mathcal{M}_i^{\mathrm{an}}.$$

If $\Lambda \in \mathcal{L}$, we define

$$U_\Lambda := \mathrm{red}^{-1}(\mathcal{M}_\Lambda)$$

the tube over $\mathcal{M}_\Lambda$. The action of J on $\mathcal{M}$ induces an action on the generic fiber $\mathcal{M}^{\mathrm{an}}$ such that red is J-equivariant. By restriction it induces an action of J_Λ on U_Λ. The analytic space $\mathcal{M}^{\mathrm{an}}$ and each of the open subspaces U_Λ have dimension $n-1$.

3.4.1.2 We fix a prime number $\ell \neq p$. In [Ber93], Berkovich developped a theory of étale cohomology for his analytic spaces. Using it we may define the cohomology of the Rapoport-Zink space $\mathcal{M}^{\mathrm{an}}$ by the formula

$$\begin{aligned}\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) &:= \varinjlim_U \mathrm{H}_c^\bullet(U \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \\ &= \varinjlim_U \varprojlim_n \mathrm{H}_c^\bullet(U \widehat{\otimes}\, \mathbb{C}_p, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes \overline{\mathbb{Q}_\ell}\end{aligned}$$

where U goes over all relatively compact open of $\mathcal{M}^{\mathrm{an}}$. These cohomology groups are equipped with commuting actions of J and of W, the Weyl group of $\mathbb{Q}_{p^2}$. The J-action causes no problem of interpretation, but the W-action needs explanations. Let $\tau := \sigma^2$ be the Frobenius relative to $\mathbb{F}_{p^2}$. We fix a lift $\mathrm{Frob} \in W$ of the geometric Frobenius $\tau^{-1} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_{p^2})$. The inertia subgroup $I \subset W$ acts on $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}} \widehat{\otimes}\, \mathbb{C}_p, \overline{\mathbb{Q}_\ell})$ via the coefficients $\mathbb{C}_p$, whereas Frob acts via the **Weil descent datum** defined by Rapoport and Zink in [RZ96] 3.48, as we explain now.
Recall the standard unitary p-divisible group $\mathbb{X}$ introduced in 3.1.1.1. Let

$$\mathcal{F}_{\mathbb{X}} : \mathbb{X} \otimes \mathbb{F} \to \tau^*(\mathbb{X} \otimes \mathbb{F})$$

denote the Frobenius morphism relative to $\mathbb{F}_{p^2}$. Let $(\mathcal{M} \widehat{\otimes}\, \mathcal{O}_{\breve{\mathbb{Q}}_p})^\tau$ be the functor defined by

$$(\mathcal{M} \widehat{\otimes}\, \mathcal{O}_{\breve{\mathbb{Q}}_p})^\tau(S) := \mathcal{M}(S_\tau)$$

for all $\mathcal{O}_{\breve{\mathbb{Q}}_p}$-scheme S where p is locally nilpotent. Here, S_τ denotes the scheme S but with structure morphism the composition $S \to \mathrm{Spec}(\mathcal{O}_{\breve{\mathbb{Q}}_p}) \xrightarrow{\tau} \mathrm{Spec}(\mathcal{O}_{\breve{\mathbb{Q}}_p})$. The Weil descent datum is the isomorphism $\alpha_{\mathrm{RZ}} : \mathcal{M} \widehat{\otimes}\, \mathcal{O}_{\breve{\mathbb{Q}}_p} \xrightarrow{\sim} (\mathcal{M} \widehat{\otimes}\, \mathcal{O}_{\breve{\mathbb{Q}}_p})^\tau$ given by $(X, \iota, \lambda, \rho) \in \mathcal{M}(S) \mapsto (X, \iota, \lambda, \mathcal{F}_{\mathbb{X}} \circ \rho)$.
We may describe this in terms of k-rational points, where k is a perfect field extension of $\mathbb{F}$. Since we use covariant Dieudonné theory, the relative Frobenius $\mathcal{F}_{\mathbb{X}}$ corresponds to the Verschiebung $\mathbf{V}^2$ in the Dieudonné module. By construction of $\mathbb{X}$, we have $\mathbf{V}^2 = p\tau^{-1}$. Therefore, if $S = \mathrm{Spec}(k)$ with $k/\mathbb{F}_{p^2}$ perfect, then α_{RZ} sends a Dieudonné module $M \in \mathcal{M}(k)$ to $p\tau^{-1}(M)$.
Since $\mathrm{Frob} \in W$ is a *geometric* Frobenius element, its action on the cohomology of $\mathcal{M}^{\mathrm{an}}$ is induced by the inverse $\alpha_{\mathrm{RZ}}^{-1}$.

Remark. The Rapoport-Zink space is defined over $\mathbb{Z}_{p^2}$ and this rational structure is induced by the effective descent datum $p\alpha_{\mathrm{RZ}}^{-1}$, with $p = p \cdot \mathrm{id}$ seen as an element of the center of J. It sends a point M to $\tau(M)$. Consequently, in the following we will write $\tau := (p^{-1}\cdot \mathrm{id}, \mathrm{Frob}) \in J \times W$, and we refer to it as the *rational Frobenius*. We note that $p^{-1}\cdot \mathrm{id}$ comes from contravariance of cohomology with compact support: the action of $g \in J$ on the cohomology of $\mathcal{M}^{\mathrm{an}}$ is induced by the action of g^{-1} on the space $\mathcal{M}^{\mathrm{an}}$.

Notation. In order to shorten the notations, we will omit the coefficients $\mathbb{C}_p$. Thefore we write $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ and similarly for subspaces of $\mathcal{M}^{\mathrm{an}}$.

3.4.1.3 The cohomology groups $\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ are concentrated in degrees 0 to $2\dim(\mathcal{M}^{\mathrm{an}}) = 2(n-1)$. According to [Far04] Corollaire 4.4.7, these groups are smooth for the J-action and continous for the I-action. In a similar way as for $\mathcal{M}^{\mathrm{an}}$, we can also define the cohomology groups $\mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ for every $i \in \mathbb{Z}$ such that ni is even. The action of an element $g \in J$ induces an isomorphism

$$g : \mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^\bullet(\mathcal{M}_{i+\alpha(g)}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

In particular, the action of Frob gives an isomorphism from the cohomology of $\mathcal{M}_i^{\mathrm{an}}$ to that of $\mathcal{M}_{i+2}^{\mathrm{an}}$. Let $(J \times W)^\circ$ be the subgroup of $J \times W$ consisting of all elements of the form $(g, u\mathrm{Frob}^j)$ with $u \in I$ and $\alpha(g) = -2j$. In fact, we have $(J \times W)^\circ = (J^\circ \times I)\tau^{\mathbb{Z}}$ where $J^\circ \subset J$ is the subgroup introduced in 3.1.3.4. Each group $\mathrm{H}_c^\bullet(\mathcal{M}_i^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is a $(J \times W)^\circ$-representation, and we have an isomorphism

$$\mathrm{H}_c^\bullet(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{(J\times W)^\circ}^{J\times W} \mathrm{H}_c^\bullet(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

In particular, when $\mathrm{H}_c^k(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is non-zero it is infinite dimensional. However, by loc. cit. Proposition 4.4.13, these cohomology groups are always of finite type as J-modules.

3.4.1.4 In order to obtain information on the cohomology of $\mathcal{M}^{\mathrm{an}}$, we study the spectral sequence associated to the covering by the open subspaces U_Λ for $\Lambda \in \mathcal{L}$. The spaces U_Λ satisfy the same incidence relations as the $\mathcal{M}_\Lambda$, as described in 3.1.2.11 Theorem (1), (2) and (3). As a consequence, the open covering of $\mathcal{M}^{\mathrm{an}}$ by the $\{U_\Lambda\}$ is locally finite. For $i \in \mathbb{Z}$ such that ni is even and for $0 \leqslant \theta \leqslant m$, we denote by $\mathcal{L}_i^{(\theta)}$ the subset of $\mathcal{L}_i$ whose elements are those lattices of orbit type $2\theta+1$. We also write $\mathcal{L}^{(\theta)}$ for the union of the $\mathcal{L}_i^{(\theta)}$. Then $\{U_\Lambda\}_{\Lambda \in \mathcal{L}^{(m)}}$ is an open cover of $\mathcal{M}^{\mathrm{an}}$. We may apply [Far04] Proposition 4.2.2 to deduce the existence of the following Čech spectral sequence computing the cohomology of the Rapoport-Zink space, concentrated in degrees $a \leqslant 0$ and $0 \leqslant b \leqslant 2(n-1)$,

$$E_1^{a,b} : \bigoplus_{\gamma \in I_{-a+1}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

Here, for $s \geqslant 1$ the set I_s is defined by

$$I_s := \left\{ \gamma = (\Lambda^1, \dots, \Lambda^s) \,\middle|\, \forall 1 \leqslant j \leqslant s, \Lambda^j \in \mathcal{L}^{(m)} \text{ and } U(\gamma) := \bigcap_{j=1}^s U_{\Lambda^j} \neq \varnothing \right\}.$$

Necessarily, if $\gamma = (\Lambda^1, \ldots, \Lambda^s) \in I_s$ then there exists a unique i such that ni is even and $\Lambda^j \in \mathcal{L}_i^{(m)}$ for all j. We then define

$$\Lambda(\gamma) := \bigcap_{j=1}^{s} \Lambda^j \in \mathcal{L}_i,$$

so that $U(\gamma) = U_{\Lambda(\gamma)}$. In particular, the open subspace $U(\gamma)$ depends only on the intersection $\Lambda(\gamma)$ of the elements in the s-tuple γ.
For $s \geqslant 2$ and $\gamma = (\Lambda^1, \ldots, \Lambda^s) \in I_s$, define $\gamma_j := (\Lambda^1, \ldots, \widehat{\Lambda^j}, \ldots, \Lambda^s) \in I_{s-1}$ for the $(s-1)$-tuple obtained from γ by removing the j-th term. Besides, for $\Lambda, \Lambda' \in \mathcal{L}_i$ with $\Lambda' \subset \Lambda$, we write $f^b_{\Lambda',\Lambda}$ for the natural map $\mathrm{H}^b_c(U_{\Lambda'}, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell})$ induced by the inclusion $U_{\Lambda'} \subset U_\Lambda$.
For $a \leqslant -1$, the differential $E_1^{a,b} \to E_1^{a+1,b}$ is denoted by φ^b_{-a}. It is the direct sum over all $\gamma \in I_{-a+1}$ of the maps

$$\mathrm{H}^b_c(U(\gamma), \overline{\mathbb{Q}_\ell}) \to \bigoplus_{\delta \in \{\gamma_1, \ldots \gamma_{-a+1}\}} \mathrm{H}^b_c(U(\delta), \overline{\mathbb{Q}_\ell})$$

$$v \mapsto \sum_{j=1}^{-a+1} \gamma_j \cdot (-1)^{j+1} f^b_{\Lambda(\gamma),\Lambda(\gamma_j)}(v).$$

Here, the notation $\gamma_j \cdot (-1)^{j+1} f^b_{\Lambda(\gamma),\Lambda(\gamma_j)}(v)$ means the vector $(-1)^{j+1} f^b_{\Lambda(\gamma),\Lambda(\gamma_j)}(v)$ considered inside the summand $\mathrm{H}^b_c(U(\delta), \overline{\mathbb{Q}_\ell})$ corresponding to $\delta = \gamma_j$. We observe that we may have $\Lambda(\gamma_j) = \Lambda(\gamma_{j'})$ even though $\gamma_j \neq \gamma_{j'}$. In such a case, the vectors $f^b_{\Lambda(\gamma),\Lambda(\gamma_j)}(v)$ and $f^b_{\Lambda(\gamma),\Lambda(\gamma_{j'})}(v)$ are equal in $\mathrm{H}^b_c(U(\gamma_j), \overline{\mathbb{Q}_\ell}) = \mathrm{H}^b_c(U(\gamma_{j'}), \overline{\mathbb{Q}_\ell})$, but they contribute to two distinct summands in the codomain, namely associated to $\delta = \gamma_j$ and $\delta = \gamma_{j'}$.

An element $g \in J$ acts on the set I_s by sending γ to $g \cdot \gamma := (g\Lambda^1, \ldots, g\Lambda^s)$. The action of g^{-1} induces an isomorphism

$$\mathrm{H}^b_c(U(\gamma), \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^b_c(U(g \cdot \gamma), \overline{\mathbb{Q}_\ell}).$$

This defines a natural J-action on the terms $E_1^{a,b}$, with respect to which the spectral sequence is equivariant.

Remark. The map $p\alpha_{\mathrm{RZ}}^{-1}$ defines a Weil descent datum on $\mathcal{M}_\Lambda \otimes \mathbb{F}$ which is effective, and coincides with the natural $\mathbb{F}_{p^2}$-structure. Hence, the same holds for the analytical tube $U_\Lambda \widehat{\otimes} \mathbb{C}_p$. The descent datum $p\alpha_{\mathrm{RZ}}^{-1}$ induces the action of τ on the cohomology of U_Λ. If $\gamma \in I_{-a+1}$ then $p \cdot \gamma \in I_{-a+1}$. It follows that each term $E_1^{a,b}$ is equipped with an action of W. The spectral sequence E is in fact $J \times W$-equivariant.

3.4.1.5 First we relate the cohomology of a tube U_Λ to the cohomology of the corresponding closed Bruhat-Tits stratum $\mathcal{M}_\Lambda$. We observe that $\mathrm{H}^\bullet_c(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is naturally a representation of the subgroup $(J_\Lambda \times I)\tau^{\mathbb{Z}} \subset J \times W$.

Proposition. *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. There is a $(J_\Lambda \times I)\tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}^b(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$$

where, on the left-hand side, the inertia I acts trivially and τ acts like the geometric Frobenius F^2.

In particular, the inertia acts trivially on the cohomology of U_Λ.

Proof. Recall the notations of 3.3.7 regarding the Bruhat-Tits stratification on the Shimura variety $\overline{\mathrm{S}}_{K^p}$, where K^p is any open compact subgroup of $G(\mathbb{A}_f^p)$ that is small enough. Fix an integer $1 \leqslant k \leqslant s$ and consider the closed Bruhat-Tits stratum $\overline{\mathrm{S}}_{K^p,\Lambda,k}$, that is the isomorphic image of $\mathcal{M}_\Lambda$ through $\Phi^k_{K^p}$. Let $\mathrm{Sh}_{K^p,\Lambda,k}$ be the analytic tube of $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ inside $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}$. By compatibility of the p-adic uniformization, the tube $\mathrm{Sh}_{K^p,\Lambda,k}$ is the isomorphic image of U_Λ through $(\Phi^k_{K^p})^{\mathrm{an}}$, which is the composition $\mathcal{M}^{\mathrm{an}} \to \Gamma_k\backslash\mathcal{M}^{\mathrm{an}} \to (\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}$. Thus, the following diagram is commutative.

$$\begin{array}{ccc} U_\Lambda & \xrightarrow{\sim} & \mathrm{Sh}_{K^p,\Lambda,k} \\ {\scriptstyle \mathrm{red}}\downarrow & & \downarrow{\scriptstyle \mathrm{red}} \\ \mathcal{M}_\Lambda & \xrightarrow{\sim} & \overline{\mathrm{S}}_{K^p,\Lambda,k} \end{array}$$

Berkovich's comparison theorem gives the desired isomorphism. More precisely, let $\widehat{\mathrm{S}}_{K^p}$ denote the formal completion of the Shimura variety S_{K^p} along its special fiber. Since it is a smooth formal scheme over $\mathrm{Spf}(\mathbb{Z}_{p^2})$, we may apply [Ber96] Corollary 3.7 to deduce the existence of a natural isomorphism

$$\mathrm{H}^b(\overline{\mathrm{S}}_{K^p,\Lambda,k} \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}^b(\mathrm{Sh}_{K^p,\Lambda,k}, \overline{\mathbb{Q}_\ell}).$$

This isomorphism is equivariant for the action of $(J_\Lambda \times I)\tau^{\mathbb{Z}}$, with the rational Frobenius τ on the right-hand side corresponding to F^2 on the left-hand side. □

Remark. It is a priori not possible to use Berkovich's result directly on the Rapoport-Zink space because $\mathcal{M}$ is not a smooth formal scheme over $\mathrm{Spf}(\mathbb{Z}_p^2)$. In fact, it is not adic unless $n = 1$ or 2, see [Far04] Remarque 2.3.5. It is the reason why we have to introduce the Shimura variety in the proof.

Corollary. *Let $\Lambda \in \mathcal{L}$ and let $0 \leqslant b \leqslant 2(n-1)$. There is a $(J_\Lambda \times I)\tau^{\mathbb{Z}}$-equivariant isomorphism*

$$\mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})(n-1-\theta)$$

where $t(\Lambda) = 2\theta + 1$.

Proof. This is a consequence of algebraic and analytic Poincaré duality, respectively for U_Λ and for $\mathcal{M}_\Lambda$. Indeed, we have

$$\begin{aligned} \mathrm{H}^b_c(U_\Lambda, \overline{\mathbb{Q}_\ell}) &\simeq \mathrm{H}^{2(n-1)-b}(U_\Lambda, \overline{\mathbb{Q}_\ell})^\vee(n-1) \\ &\simeq \mathrm{H}^{2(n-1)-b}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})^\vee(n-1) \\ &\simeq \mathrm{H}_c^{b-2(n-1-\theta)}(\mathcal{M}_\Lambda \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})(n-1-\theta). \end{aligned}$$

□

3.4.1.6 Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$. If λ is a partition of $2\theta + 1$, recall the unipotent irreducible representation ρ_λ of $\mathrm{GU}(V_\Lambda^0) \simeq \mathrm{GU}_{2\theta+1}(\mathbb{F}_p)$ that we introduced in 3.2.6. It can be inflated to the maximal reductive quotient $\mathcal{J}_\Lambda \simeq \mathrm{G}(\mathrm{U}(V_\Lambda^0) \times \mathrm{U}(V_\Lambda^1))$, and then to the maximal parahoric subgroup J_Λ. With an abuse of notation, we still denote this inflated representation by ρ_λ. In virtue of 3.2.9, the isomorphism in the last paragraph translates into the following result.

Proposition. *Let $\Lambda \in \mathcal{L}$ and write $t(\Lambda) = 2\theta + 1$. The following statements hold.*

(1) *The cohomology group $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$ is zero unless $2(n-1-\theta) \leqslant b \leqslant 2(n-1)$.*
(2) *The action of J_Λ on the cohomology factors through an action of the finite group of Lie type $\mathrm{GU}(V_\Lambda^0)$. The rational Frobenius τ acts like multiplication by $(-p)^b$ on $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell})$.*
(3) *For $0 \leqslant b \leqslant \theta$ we have*

$$\mathrm{H}_c^{2b+2(n-1-\theta)}(U_\Lambda, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-j)} \rho_{(2\theta+1-2s,2s)}.$$

For $0 \leqslant b \leqslant \theta - 1$ we have

$$\mathrm{H}_c^{2b+1+2(n-1-\theta)}(U_\Lambda, \overline{\mathbb{Q}_\ell}) = \bigoplus_{s=0}^{\min(j,\theta-1-j)} \rho_{(2\delta-2s,2s+1)}.$$

3.4.1.7 The description of the rational Frobenius action yields the following result.

Corollary. *The spectral sequence degenerates on the second page E_2. For $0 \leqslant b \leqslant 2(n-1)$, the induced filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ splits, ie. we have an isomorphism*

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b \leqslant b' \leqslant 2(n-1)} E_2^{b-b',b'}.$$

The action of W on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is trivial on the inertia subgroup and the action of the rational Frobenius element τ is semisimple. The subspace $E_2^{b-b',b'}$ is identified with the eigenspace of τ associated to the eigenvalue $(-p)^{b'}$.

Remark. In the previous statement, the terms $E_2^{b-b',b'}$ may be zero.

Proof. The (a, b)-term in the first page of the spectral sequence is the direct sum of the cohomology groups $H_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ for all $\gamma \in I_{-a+1}$. On each of these cohomology groups, the rational Frobenius τ acts like multiplication by $(-p)^b$. This action is in particular independant of γ and of a. Thus, on the b-th row of the first page of the sequence, the Frobenius acts everywhere as multiplication by $(-p)^b$. Starting from the second page, the differentials in the sequence connect two terms lying in different rows. Since the differentials are equivariant for the τ-action, they must all be zero. Thus, the sequence degenerates on the second page. By the machinery of spectral sequences, there is a filtration on $\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ whose graded factors are given by the terms $E_2^{b-b',b'}$ of the second page. Only a finite number of these terms are non-zero, and since they all lie on different rows, the Frobenius τ acts like multiplication by a different scalar on each graded factor of the filtration. It follows that the filtration splits, ie. the abutment is the direct sum of the graded pieces of the filtration, as they correspond to the eigenspaces of τ. Consequently, its action is semisimple. ∎

3.4.1.8 The spectral sequence $E_1^{a,b}$ has non-zero terms extending indefinitely in the range $a \leqslant 0$. For instance, if $\Lambda \in \mathcal{L}^{(m)}$ then $(\Lambda, \ldots, \Lambda) \in I_{-a+1}$ so that $E_1^{a,b} \neq 0$ for all $a \leqslant 0$ and $2(n-1-m) \leqslant b \leqslant 2(n-1)$. To rectify this, we introduce the alternating Čech spectral sequence. If $v \in E_1^{a,b}$ and $\gamma \in I_{-a+1}$, we denote by $v_\gamma \in \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell})$ the component of v in the summand of $E_1^{a,b}$ indexed by γ. Besides, if $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$ and if $\sigma \in \mathfrak{S}_{-a+1}$ then we write $\sigma(\gamma) := (\Lambda^{\sigma(1)}, \ldots, \Lambda^{\sigma(-a+1)}) \in I_{-a+1}$. For all a, b we define

$$E_{1,\mathrm{alt}}^{a,b} := \{v \in E_1^{a,b} \,|\, \forall \gamma \in I_{-a+1}, \forall \sigma \in \mathfrak{S}_{-a+1}, v_{\sigma(\gamma)} = \mathrm{sgn}(\sigma) v_\gamma\}.$$

In particular, if $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1})$ with $\Lambda^j = \Lambda^{j'}$ for some $j \neq j'$ then $v \in E_{1,\mathrm{alt}}^{a,b} \implies v_\gamma = 0$. The subspace $E_{1,\mathrm{alt}}^{a,b} \subset E_1^{a,b}$ is stable under the action of $J \times W$, and the differential $\varphi_{-a}^b : E_1^{a,b} \to E_1^{a+1,b}$ sends $E_{1,\mathrm{alt}}^{a,b}$ to $E_{1,\mathrm{alt}}^{a+1,b}$. Thus, for all b we have a chain complex $E_{1,\mathrm{alt}}^{\bullet,b}$ and the following proposition is well-known.

Proposition ([Sta23] Lemma 01FM)**.** *The inclusion map $E_{1,\mathrm{alt}}^{\bullet,b} \hookrightarrow E_1^{\bullet,b}$ is a homotopy equivalence. In particular we have canonical isomorphisms $E_{2,\mathrm{alt}}^{a,b} \simeq E_2^{a,b}$ for all a, b.*

The advantage of the alternating Čech spectral sequence is that it is concentrated in a finite strip. Indeed, if $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$, let $i \in \mathbb{Z}$ such that $\Lambda(\gamma) \in \mathcal{L}_i$. Then all the Λ^j's belong to the set of lattices in $\mathcal{L}_i^{(m)}$ containing $\Lambda(\gamma)$. This set is finite of cardinality $\nu(n-\theta-m-1, n-2\theta-1)$ where $t(\Lambda(\gamma)) = 2\theta+1$ according to 3.1.4.1. Thus, if $-a+1$ is big enough then all the γ's in I_{-a+1} will have some repetition, so that $E_{1,\mathrm{alt}}^{a,b} = 0$.

Remark. The Lemma 01FM of [Sta23] is stated in the context of Čech cohomology of an abelian presheaf $\mathcal{F}$ on a topological space X. However, the proof may be adapted to Čech homology of precosheaves such as $U \mapsto \mathrm{H}_c^b(U, \overline{\mathbb{Q}_\ell})$.

3.4.1.9 For $a = 0$, we have $E_{1,\mathrm{alt}}^{0,b} = E_1^{0,b}$ by definition. Let us consider the cases $b = 2(n-1-m)$ and $b = 2(n-1-m)+1$. For such b, it follows from 3.4.1.6 that $\mathrm{H}_c^b(U_\Lambda, \overline{\mathbb{Q}_\ell}) = 0$ if $t(\Lambda) < t_{\max}$. If $a \leqslant -1$, we have $-a+1 \geqslant 2$ so that for all $\gamma = (\Lambda^1, \ldots, \Lambda^{-a+1}) \in I_{-a+1}$, if there exists $j \neq j'$ such that $\Lambda^j \neq \Lambda^{j'}$, then $t(\Lambda(\gamma)) < t_{\max}$ and $\mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) = 0$. It follows that $E_{1,\mathrm{alt}}^{a,b} = 0$ for all $a \leqslant -1$ and b as above. This observation, along with the previous paragraph, yields the following proposition.

Proposition. *We have $E_2^{0,2(n-1-m)} \simeq E_1^{0,2(n-1-m)}$. If moreover $m \geqslant 1$ (ie. $n \geqslant 3$), then we have $E_2^{0,2(n-1-m)+1} \simeq E_1^{0,2(n-1-m)+1}$ as well.*

3.4.1.10 In order to study the action of J, we may rewrite $E_1^{a,b}$ conveniently in terms of compactly induced representations. To do this, let us introduce a few more notations. For $0 \leqslant \theta \leqslant m$ and $s \geqslant 1$, we define

$$I_s^{(\theta)} := \{\gamma \in I_s \,|\, t(\Lambda(\gamma)) = 2\theta + 1\}.$$

The subset $I_s^{(\theta)} \subset I_s$ is stable under the action of J. We denote by $N(\Lambda_\theta)$ the finite set $N(n-\theta-m-1, V_\theta^1)$ as defined in paragraph 3.1.4.1. It corresponds to the set of lattices $\Lambda \in \mathcal{L}_0$ of maximal orbit type $t(\Lambda) = 2m+1$ containing Λ_θ. For $s \geqslant 1$ we define

$$K_s^{(\theta)} := \{\delta = (\Lambda^1, \ldots, \Lambda^s) \,|\, \forall 1 \leqslant j \leqslant s, \Lambda^j \in N(\Lambda_\theta) \text{ and } \Lambda(\delta) = \Lambda_\theta\}.$$

Then $K_s^{(\theta)}$ is a finite subset of $I_s^{(\theta)}$ and it is stable under the action of J_θ. If $\gamma \in I_s^{(\theta)}$, there exists some $g \in J$ such that $g \cdot \Lambda(\gamma) = \Lambda_\theta$ because both lattices share the same orbit type. Moreover, the coset $J_\theta \cdot g$ is uniquely determined, and $g \cdot \gamma$ is an element of $K_s^{(\theta)}$. This mapping results in a natural bijection between the orbit sets

$$J \backslash I_s^{(\theta)} \xrightarrow{\sim} J_\theta \backslash K_s^{(\theta)}.$$

The bijection sends the orbit $J \cdot \alpha$ to the orbit $J_\theta \cdot (g \cdot \alpha)$ where g is chosen as above. The inverse sends an orbit $J_\theta \cdot \beta$ to $J \cdot \beta$. We note that both orbit sets are finite.
We may now rearrange the terms in the spectral sequence.

Proposition. *We have an isomorphism*

$$\begin{aligned} E_1^{a,b} &\simeq \bigoplus_{\theta=0}^{m} \bigoplus_{[\delta] \in J_\theta \backslash K_{-a+1}^{(\theta)}} \mathrm{c}-\mathrm{Ind}_{\mathrm{Fix}(\delta)}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)} \\ &\simeq \bigoplus_{\theta=0}^{m} \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}] \right), \end{aligned}$$

where $\overline{\mathbb{Q}_\ell}[K_{-a+1}^{(\theta)}]$ is the permutation representation associated to the action of J_θ on the finite set $K_{-a+1}^{(\theta)}$.

Remark. For $\delta \in K_s^{(\theta)}$, the group $\mathrm{Fix}(\delta)$ consists of the elements $g \in J$ such that $g \cdot \delta = \delta$. Any such g satisfies $g\Lambda(\delta) = \Lambda(\delta)$, and since $\Lambda(\delta) = \Lambda_\theta$ we have $\mathrm{Fix}(\delta) \subset J_\theta$. If $\delta = (\Lambda^1, \ldots, \Lambda^s)$ then $\mathrm{Fix}(\delta)$ is the intersection of the maximal parahoric subgroups $J_{\Lambda^1}, \ldots, J_{\Lambda^s}$. We note that in general, $\mathrm{Fix}(\delta)$ is itself not a parahoric subgroup of J since the lattices $\Lambda^1, \ldots, \Lambda^s$ need not form a simplex in $\mathcal{L}$, as they all share the same orbit type. If however $\Lambda^1 = \ldots = \Lambda^s$ then $\mathrm{Fix}(\delta) = J_{\Lambda^1}$ is a conjugate of the maximal parahoric subgroup J_m.

Proof. First, by decomposing I_{-a+1} as the disjoint union of the $I_{-a+1}^{(\theta)}$ for $0 \leqslant \theta \leqslant m$, we may write

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \bigoplus_{\gamma \in I_{-a+1}^{(\theta)}} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}).$$

For each orbit $X \in J \backslash I_{-a+1}^{(\theta)}$, we fix a representative δ_X which lies in $K_{-a+1}^{(\theta)}$. We may write

$$E_1^{a,b} = \bigoplus_{\theta=0}^{m} \bigoplus_{X \in J \backslash I_{-a+1}^{(\theta)}} \bigoplus_{\gamma \in X} \mathrm{H}_c^b(U(\gamma), \overline{\mathbb{Q}_\ell}) = \bigoplus_{\theta=0}^{m} \bigoplus_{X \in J \backslash I_{-a+1}^{(\theta)}} \bigoplus_{g \in J/\mathrm{Fix}(\delta_X)} g \cdot \mathrm{H}_c^b(U(\delta_X), \overline{\mathbb{Q}_\ell}).$$

The rightmost sum can be identified with a compact induction from $\mathrm{Fix}(\delta_X)$ to J. Identifying the orbit sets $J \backslash I_{-a+1}^{(\theta)} \xrightarrow{\sim} J_\theta \backslash K_{-a+1}^{(\theta)}$, we have

$$E_1^{a,b} \simeq \bigoplus_{\theta=0}^{m} \bigoplus_{[\delta] \in J_\theta \backslash K_{-a+1}^{(\theta)}} \mathrm{c}-\mathrm{Ind}_{\mathrm{Fix}(\delta)}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)}.$$

By transitivity of compact induction, we have

$$\mathrm{c}-\mathrm{Ind}_{\mathrm{Fix}(\delta)}^{J} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)} = \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \mathrm{c}-\mathrm{Ind}_{\mathrm{Fix}(\delta)}^{J_\theta} \mathrm{H}_c^b(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)}.$$

Since $H^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell})_{|\mathrm{Fix}(\delta)}$ is the restriction of a representation of J_θ to $\mathrm{Fix}(\delta)$, applying compact induction from $\mathrm{Fix}(\delta)$ to J_θ results in tensoring with the permutation representation of $J_\theta/\mathrm{Fix}(\delta)$. Thus

$$E_1^{a,b} \simeq \bigoplus_{\theta=0}^{m} \bigoplus_{[\delta]\in J_\theta\backslash K^{(\theta)}_{-a+1}} \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \overline{\mathbb{Q}_\ell}[J_\theta/\mathrm{Fix}(\delta)]\right)$$

$$\simeq \bigoplus_{\theta=0}^{m} \mathrm{c}-\mathrm{Ind}_{J_\theta}^{J} \left(\mathrm{H}^b_c(U_{\Lambda_\theta}, \overline{\mathbb{Q}_\ell}) \otimes \bigoplus_{[\delta]\in J_\theta\backslash K^{(\theta)}_{-a+1}} \overline{\mathbb{Q}_\ell}[J_\theta/\mathrm{Fix}(\delta)]\right),$$

where on the second line we used additivity of compact induction. Now, $J_\theta/\mathrm{Fix}(\delta)$ is identified with the J_θ-orbit $J_\theta\cdot\delta$ of δ in $K^{(\theta)}_{-a+1}$, so that

$$\bigoplus_{[\delta]\in J_\theta\backslash K^{(\theta)}_{-a+1}} \overline{\mathbb{Q}_\ell}[J_\theta/\mathrm{Fix}(\delta)] \simeq \overline{\mathbb{Q}_\ell}[\bigsqcup_{[\delta]\in J_\theta\backslash K^{(\theta)}_{-a+1}} J_\theta\cdot\delta] \simeq \overline{\mathbb{Q}_\ell}[K^{(\theta)}_{-a+1}],$$

which concludes the proof. □

3.4.1.11 By 3.1.2.9, we may identify $N(\Lambda_\theta)$ with the set

$$\overline{N}(\Lambda_\theta) := \{U \subset V^1_\theta \mid \dim U = m-\theta \text{ and } U \subset U^\perp\}.$$

Thus, for $s \geqslant 1$, $K^{(\theta)}_s$ is naturally identified with

$$\overline{K}^{(\theta)}_s \simeq \left\{\overline{\delta} = (U^1,\ldots,U^s) \,\middle|\, \forall 1\leqslant j\leqslant s, U^j \in \overline{N}(\Lambda_\theta) \text{ and } \bigcap_{j=1}^{s} U^j = \{0\}\right\}.$$

The action of J_θ on $K^{(\theta)}_s$ corresponds to the natural action of $\mathrm{GU}(V^1_\theta)$ on $\overline{K}^{(\theta)}_s$, which factors through an action of the finite projective unitary group $\mathrm{PU}(V^1_\theta) := \mathrm{U}(V^1_\theta)/\mathrm{Z}(\mathrm{U}(V^1_\theta)) \simeq \mathrm{GU}(V^1_\theta)/\mathrm{Z}(\mathrm{GU}(V^1_\theta))$. Thus, the representation $\overline{\mathbb{Q}_\ell}[K^{(\theta)}_{-a+1}]$ of J_θ is the inflation, via the maximal reductive quotient as in 3.1.2.8, of the representation $\overline{\mathbb{Q}_\ell}[\overline{K}^{(\theta)}_{-a+1}]$ of the finite projective unitary group $\mathrm{PU}(V^1_\theta)$.
When $\theta = m$ or when $s = 1$, we trivially have the following proposition.

Proposition. *For $s \geqslant 1$, we have $\overline{\mathbb{Q}_\ell}[K^{(m)}_s] = \mathbf{1}$.*
For $0 \leqslant \theta \leqslant m-1$, we have $\overline{\mathbb{Q}_\ell}[K^{(\theta)}_1] = 0$.

Proof. If $\delta = (\Lambda^1,\ldots,\Lambda^s) \in K^{(m)}_s$ then $\Lambda(\delta) = \Lambda_m$ has maximal orbit type $t_{\max} = 2m+1$. For any $1 \leqslant j \leqslant s$ we have $\Lambda_m \subset \Lambda^j$, therefore $\Lambda^1 = \ldots = \Lambda^s = \Lambda_m$. Thus $K^{(m)}_s$ is a singleton and so $\overline{\mathbb{Q}_\ell}[K^{(m)}_s]$ is trivial. Besides, if $\theta < m$ then $K^{(\theta)}_s$ is clearly empty. □

Recall 3.4.1.9 Proposition. We obtain the following corollary.

Corollary. *We have*

$$E_1^{0,b} \simeq \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \mathrm{H}^b_c(U_{\Lambda_m}, \overline{\mathbb{Q}_\ell}).$$

In particular, we have

$$E_2^{0,b} \simeq \begin{cases} \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \rho_{(2m+1)} & \text{if } b = 2(n-1-m), \\ \mathrm{c}-\mathrm{Ind}_{J_m}^{J} \rho_{(2m,1)} & \text{if } m \geqslant 1 \text{ and } b = 2(n-1-m)+1. \end{cases}$$

Remark. The representation $\rho_{(2m+1)} = \mathbf{1}$ is the trivial representation of J_m.

3.4.1.12 Let us now consider the top row of the spectral sequence, corresponding to $b = 2(n-1)$. For $\Lambda' \subset \Lambda$, recall the map $f^{2(n-1)}_{\Lambda',\Lambda} : \mathrm{H}^{2(n-1)}_c(U_{\Lambda'}, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}^{2(n-1)}_c(U_\Lambda, \overline{\mathbb{Q}_\ell})$. By Poincaré duality, it is the dual map of the restriction morphism $\mathrm{H}^0(U_\Lambda, \overline{\mathbb{Q}_\ell}) \to \mathrm{H}^0(U_{\Lambda'}, \overline{\mathbb{Q}_\ell})$. Since U_Λ is connected for every $\Lambda \in \mathcal{L}$, we have $\mathrm{H}^0(U_\Lambda, \overline{\mathbb{Q}_\ell}) \simeq \overline{\mathbb{Q}_\ell}$ and the restriction maps for $\Lambda' \subset \Lambda$ are all identity. Thus, $E_1^{a,2(n-1)}$ is the $\overline{\mathbb{Q}_\ell}$-vector space generated by I_{-a+1}, and the differential $\varphi^{2(n-1)}_{-a}$ is given by

$$\gamma \in I_{-a+1} \mapsto \sum_{j=1}^{-a+1} (-1)^{j+1} \gamma_j.$$

Using this description, we may compute the highest cohomology group $\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ explicitely.

Proposition. *There is an isomorphism*

$$\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c} - \mathrm{Ind}_{J^\circ}^{J} \mathbf{1},$$

and the rational Frobenius τ acts via multiplication by $p^{2(n-1)}$.

Proof. The statement on the Frobenius action is already known by 3.4.1.7 Corollary. Besides, we have $\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq E_2^{0,2(n-1)} = \mathrm{Coker}(\varphi_1^{2(n-1)})$. The differential $\varphi_1^{2(n-1)}$ is described by

$$\begin{aligned} (\Lambda, \Lambda) &\mapsto 0, && \forall \Lambda \in \mathcal{L}^{(m)}, \\ (\Lambda, \Lambda') &\mapsto (\Lambda') - (\Lambda), && \forall \Lambda, \Lambda' \in \mathcal{L}^{(m)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \neq \varnothing. \end{aligned}$$

Let $i \in \mathbb{Z}$ such that ni is even, and let $\Lambda, \Lambda' \in \mathcal{L}_i^{(m)}$. Since the Bruhat-Tits building $\mathrm{BT}(\widetilde{J}, \mathbb{Q}_p) \simeq \mathcal{L}_i$ is connected, there exists a sequence $\Lambda = \Lambda^0, \ldots, \Lambda^d = \Lambda'$ of lattices in $\mathcal{L}_i$ such that for all $0 \leqslant j \leqslant d-1$, $\{\Lambda^j, \Lambda^{j+1}\}$ is an edge in $\mathcal{L}_i$. Assume that $d \geqslant 0$ is minimal satisfying this property. Since $t(\Lambda) = t(\Lambda') = t_{\max}$, the integer d is even and we may assume that $t(\Lambda^j)$ is equal to $t_{\max}$ when j is even, and equal to 1 when j is odd. In particular, for all $0 \leqslant j \leqslant \frac{d}{2} - 1$ we have $\Lambda^{2j}, \Lambda^{2j+2} \in \mathcal{L}_i^{(m)}$ and $U_{\Lambda^{2j}} \cap U_{\Lambda^{2j+2}} \neq \varnothing$. Consider the vector

$$w := \sum_{j=0}^{\frac{d}{2}-1} (\Lambda^{2j}, \Lambda^{2j+2}) \in E_1^{-1,2(n-1)}.$$

Then we compute $\varphi_1^{2(n-1)}(w) = (\Lambda') - (\Lambda)$. Thus, $\mathrm{Coker}(\varphi_1^{2(n-1)})$ consists of one copy of $\overline{\mathbb{Q}_\ell}$ for each $i \in \mathbb{Z}$ such that ni is even. Considering the action of J as well, it readily follows that $\mathrm{Coker}(\varphi_1^{2(n-1)}) \simeq \mathrm{c} - \mathrm{Ind}_{J^\circ}^{J} \mathbf{1}$. □

Remark. The cohomology group $\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ can also be computed in another way which does not require the spectral sequence. Indeed, we have an isomorphism

$$\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c} - \mathrm{Ind}_{J^\circ}^{J} \mathrm{H}^{2(n-1)}_c(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}).$$

By definition, we have

$$\mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) = \varinjlim_{U} \mathrm{H}_c^{2(n-1)}(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}),$$

where U runs over the relatively compact open subspaces of $\mathcal{M}_0^{\mathrm{an}}$. Since U is smooth, Poincaré duality gives

$$\mathrm{H}_c^{2(n-1)}(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{H}^0(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell})^\vee.$$

And since $\mathcal{M}_0^{\mathrm{an}}$ is connected, we can insure that all the U's involved are connected as well. Therefore $\mathrm{H}^0(U \widehat{\otimes} \mathbb{C}_p, \overline{\mathbb{Q}_\ell}) \simeq \overline{\mathbb{Q}_\ell}$, and all the transition maps in the direct limit are identity. It follows that $\mathrm{H}_c^{2(n-1)}(\mathcal{M}_0^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})$ is trivial.

3.4.2 Compactly induced representations and type theory

3.4.2.1 Let $\mathrm{Rep}(J)$ denote the category of smooth $\overline{\mathbb{Q}_\ell}$-representations of G. Let χ be a continuous character of the center $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}^\times$ and let $V \in \mathrm{Rep}(J)$. We define **the maximal quotient of V on which the center acts like** χ as follows. Let us consider the set

$$\Omega := \{W \,|\, W \text{ is a subrepresentation of } V \text{ and } \mathrm{Z}(J) \text{ acts like } \chi \text{ on } V/W\}.$$

The set Ω is stable under arbitrary intersection, so that $W_\circ := \bigcap_{W \in \Omega} W \in \Omega$. The maximal quotient is defined by

$$V_\chi := V/W_\circ.$$

It satisfies the following universal property.

Proposition. *Let χ be a continuous character of $\mathrm{Z}(J)$ and let $V, V' \in \mathrm{Rep}(J)$. Assume that $\mathrm{Z}(J)$ acts like χ on V'. Then any morphism $V \to V'$ factors through V_χ.*

Proof. Let $f : V \to V'$ be a morphism of J-representations. Since $V/\mathrm{Ker}(f) \simeq \mathrm{Im}(f) \subset V'$, the center $\mathrm{Z}(J)$ acts like χ on the quotient $V/\mathrm{Ker}(f)$. Therefore $\mathrm{Ker}(f) \in \Omega$. It follows that $\mathrm{Ker}(f)$ contains $W_\circ$ and as a consequence, f factors through V_χ. □

3.4.2.2 As representations of J, the terms $E_1^{a,b}$ of the spectral sequence 3.4.1.4 consist of representations of the form

$$\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \rho,$$

where ρ is the inflation to J_θ of a representation of the finite group of Lie type $\mathcal{J}_\theta$. We note that such a compactly induced representation does not contain any smooth irreducible subrepresentation of J. Indeed, the center $\mathrm{Z}(J) \simeq \mathbb{Q}_{p^2}^\times$ does not fix any finite dimensional subspace. In order to rectify this, it is customary to fix a continuous character χ of $\mathrm{Z}(J)$ which agrees with the central character of ρ on $\mathrm{Z}(J) \cap J_\theta \simeq \mathbb{Z}_{p^2}^\times$, and to describe the space $(\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \rho)_\chi$ instead.

Lemma. *We have $(\mathrm{c} - \mathrm{Ind}_{J_\theta}^{J} \rho)_\chi \simeq \mathrm{c} - \mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J} \chi \otimes \rho$.*

Proof. By Frobenius reciprocity, the identity map on $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho$ gives a morphism $\chi\otimes\rho \to \left(\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho\right)_{|\mathrm{Z}(J)J_\theta}$ of $\mathrm{Z}(J)J_\theta$-representations. Restricting further to J_θ, we obtain a morphism $\rho \to \left(\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho\right)_{|J_\theta}$. By Frobenius reciprocity, this corresponds to a morphism $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho \to \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho$ of J-representations. Because $\mathrm{Z}(J)$ acts via the character χ on the target space, this morphism factors through a map $(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi \to \mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho$. In order to prove that this is an isomorphism, we build its inverse. The quotient morphism $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi$ corresponds, via Frobenius reciprocity, to a morphism $\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_{\chi\,|J_\theta}$ of J_θ-representations. Because $\mathrm{Z}(J)$ acts via the character χ on the target space, this arrow may be extended to a morphism $\chi\otimes\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_{\chi\,|\mathrm{Z}(J)J_\theta}$ of $\mathrm{Z}(J)J_\theta$-representations. By Frobenius reciprocity, this corresponds to a morphism $\mathrm{c}-\mathrm{Ind}_{\mathrm{Z}(J)J_\theta}^{J}\chi\otimes\rho \to (\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi$, and this is our desired inverse. □

3.4.2.3 We recall a general theorem from [Bus90] describing certain compactly induced representations. In this paragraph only, let G be any p-adic group, and let L be an open subgroup of G which contains the center $\mathrm{Z}(G)$ and which is compact modulo $\mathrm{Z}(G)$.

Theorem ([Bus90] Theorem 2 (supp)). *Let (σ, V) be an irreducible smooth representation of L. There is a canonical decomposition*

$$\mathrm{c}-\mathrm{Ind}_{L}^{G}\sigma \simeq V_0 \oplus V_\infty,$$

where V_0 is the sum of all supercuspidal subrepresentations of $\mathrm{c}-\mathrm{Ind}_{L}^{G}\sigma$, and where V_∞ contains no non-zero admissible subrepresentation. Moreover, V_0 is a finite sum of irreducible supercuspidal subrepresentations of G.

The spaces V_0 or V_∞ could be zero. Note also that since G is p-adic, any irreducible representation is admissible. So in particular, V_∞ does not contain any irreducible subrepresentation. However, it may have many irreducible quotients and subquotients. Thus, the space V_∞ is in general not G-semisimple. Hence, the structure of the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{L}^{G}\sigma$ heavily depends on the supercuspidal supports of its irreducible subquotients.
We go back to our previous notations. Let $0 \leqslant \theta \leqslant m$, let ρ be a smooth irreducible representation of J_θ and let χ be a character of $\mathrm{Z}(J)$ agreeing with the central character of ρ on $\mathrm{Z}(J)\cap J_\theta$. Since the group $\mathrm{Z}(J)J_\theta$ contains the center and is compact modulo the center, we have a canonical decomposition

$$(\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho)_\chi \simeq V_{\rho,\chi,0} \oplus V_{\rho,\chi,\infty}.$$

In order to describe the spaces $V_{\rho,\chi,0}$ and $V_{\rho,\chi,\infty}$, we determine the supercuspidal supports of the irreducible subquotients of $\mathrm{c}-\mathrm{Ind}_{J_\theta}^{J}\rho$ through type theory, with the assumption that ρ is inflated from $\mathcal{J}_\theta$. For our purpose, it will be enough to analyze only the case $\theta = m$. In this case, $\dim V_m^1$ is equal to 0 or 1 so that $\mathrm{GU}(V_m^1) = \{1\}$ or $\mathbb{F}_{p^2}^\times$ has no proper parabolic subgroup. In particular, if ρ is a cuspidal representation of $\mathrm{GU}(V_m^0)$, then its inflation to the reductive quotient

$$\mathcal{J}_m \simeq \mathrm{G}(\mathrm{U}(V_m^0)\times\mathrm{U}(V_m^1))$$

is also cuspidal.

3.4.2.4 In the following paragraphs, we recall a few general facts from type theory. For more details, we refer to [BK98] and [Mor99]. Let G be the group of F-rational points of a reductive connected group $\mathbf{G}$ over a p-adic field F. A parabolic subgroup P (resp. Levi complement L) of G is defined as the group of F-rational points of an F-rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ (resp. an F-rational Levi complement $\mathbf{L} \subset \mathbf{G}$). Every parabolic subgroup P admits a Levi decomposition $P = LU$ where U is the unipotent radical of P. We denote by $X_F(G)$ the set of F-rational $\overline{\mathbb{Q}_\ell}$-characters of $\mathbf{G}$, and by $X^{\mathrm{un}}(G)$ the set of **unramified characters** of G, ie. the continuous characters of G which are trivial on all compact subgroups. We consider pairs (L, τ) where L is a Levi complement of G and τ is a supercuspidal representation of L. Two pairs (L, τ) and (L', τ') are said to be **inertially equivalent** if for some $g \in G$ and $\chi \in X^{\mathrm{un}}(G)$ we have $L' = L^g$ and $\tau' \simeq \tau^g \otimes \chi$ where τ^g is the representation of L^g defined by $\tau^g(l) := \tau(g^{-1}lg)$. This is an equivalence relation, and we denote by $[L, \tau]_G$ or $[L, \tau]$ the inertial equivalence class of (L, τ) in G. The set of all inertial equivalence classes is denoted $\mathrm{IC}(G)$. If P is a parabolic subgroup of G, we write ι_P^G for the normalised parabolic induction functor. Any smooth irreducible representation π of G is isomorphic to a subquotient of some parabolically induced representation $\iota_P^G(\tau)$ where $P = LU$ for some Levi complement L and τ is a supercuspidal representation of L. We denote by $\ell(\pi) \in \mathrm{IC}(G)$ the inertial equivalence class $[L, \tau]$. This is uniquely determined by π and it is called the **inertial support** of π.

3.4.2.5 Let $\mathfrak{s} \in \mathrm{IC}(G)$. We denote by $\mathrm{Rep}^{\mathfrak{s}}(G)$ the full subcategory of $\mathrm{Rep}(G)$ whose objects are the smooth representations of G all of whose irreducible subquotients have inertial support $\mathfrak{s}$. This definition corresponds to the one given in [BD84] 2.8. If $\mathfrak{S} \subset \mathrm{IC}(G)$, we write $\mathrm{Rep}^{\mathfrak{S}}(G)$ for the direct product of the categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ where $\mathfrak{s}$ runs over $\mathfrak{S}$. We recall the main results from loc. cit.

Theorem ([BD84] 2.8 and 2.10)**.** *The category* $\mathrm{Rep}(G)$ *decomposes as the direct product of the subcategories* $\mathrm{Rep}^{\mathfrak{s}}(G)$ *where* $\mathfrak{s}$ *runs over* $\mathrm{IC}(G)$*. Moreover, if* $\mathfrak{S} \subset \mathrm{IC}(G)$ *then the category* $\mathrm{Rep}^{\mathfrak{S}}(G)$ *is stable under direct sums and subquotients.*

Type theory was then introduced in [BK98] in order to describe the categories $\mathrm{Rep}^{\mathfrak{s}}(G)$ which are called the **Bernstein blocks**.

3.4.2.6 Let $\mathfrak{S}$ be a subset of $\mathrm{IC}(G)$. A **$\mathfrak{S}$-type** in G is a pair (K, ρ) where K is an open compact subgroup of G and ρ is a smooth irreducible representation of K, such that for every smooth irreducible representation π of G we have

$$\pi_{|K} \text{ contains } \rho \iff \ell(\pi) \in \mathfrak{S}.$$

When $\mathfrak{S}$ is a singleton $\{\mathfrak{s}\}$, we call it an $\mathfrak{s}$-type instead.

Remark. By Frobenius reciprocity, the condition that $\pi_{|K}$ contains ρ is equivalent to π being isomorphic to an irreducible quotient of $\mathrm{c}-\mathrm{Ind}_K^G \rho$. In fact, we can say a little bit more. Let K be an open compact subgroup of G and let ρ be an irreducible smooth representation of K. Let $\mathrm{Rep}_\rho(G)$ denote the full subcategory of $\mathrm{Rep}(G)$ whose objects are those representations which are generated by their ρ-isotypic component. If (K,ρ) is an $\mathfrak{S}$-type, then [BK98] Theorem 4.3 establishes the equality of categories $\mathrm{Rep}_\rho(G) = \mathrm{Rep}^{\mathfrak{S}}(G)$. By definition of compact induction, the representation $\mathrm{c}-\mathrm{Ind}_K^G\,\rho$ is generated by its ρ-isotypic vectors. Therefore any irreducible subquotient of $\mathrm{c}-\mathrm{Ind}_K^G\,\rho$ has inertial support in $\mathfrak{S}$.

3.4.2.7 An important class of types are those of depth zero, and they are the only ones we shall encounter. First, we recall the following result. If K is a parahoric subgroup of G, we denote by $\mathcal{K}$ its maximal reductive quotient. It is a finite group of Lie type over the residue field of F.

Proposition ([Mor99] 4.1)**.** *Let K be a maximal parahoric subgroup of G and let ρ be an irreducible cuspidal representation of $\mathcal{K}$. We see ρ as a representation of K by inflation. Let π be an irreducible smooth representation of G and assume that $\pi_{|K}$ contains ρ. Then π is supercuspidal and there exists an irreducible smooth representation $\tilde{\rho}$ of the normalizer $\mathrm{N}_G(K)$ such that $\tilde{\rho}_{|K}$ contains ρ and $\pi \simeq \mathrm{c}-\mathrm{Ind}_{N_G(K)}^G\tilde{\rho}$.*

Such representations π are called **depth-0 supercupidal representations** of G. More generally, a smooth irreducible representation π of G is said to be of **depth-0** if it contains a non-zero vector that is fixed by the pro-unipotent radical of some parahoric subgroup of G. A **depth-0 type** in G is a pair (K,ρ) where K is a parahoric subgroup of G and ρ is an irreducible cuspidal representation of $\mathcal{K}$, inflated to K. The name is justified by the following theorem.

Theorem ([Mor99] 4.8)**.** *Let (K,ρ) be a depth-0 type. Then there exists a (unique) finite set $\mathfrak{S} \subset \mathrm{IC}(G)$ such that (K,ρ) is an $\mathfrak{S}$-type of G.*

In loc. cit. it is also proved that any depth-0 supercuspidal representation of G contains a unique conjugacy class of depth-0 types. Let K be a parahoric subgroup of G. Using the Bruhat-Tits building of G, one may canonically associate a Levi complement L of G such that $K_L := L \cap K$ is a maximal parahoric subgroup of L, whose maximal reductive quotient $\mathcal{K}_L$ is naturally identified with $\mathcal{K}$. This is precisely described in [Mor99] 2.1. Moreover, we have $L = G$ if and only if K is a maximal parahoric subgroup of G. Now, let (K,ρ) be a depth-0 type of G and denote by $\mathfrak{S}$ the finite subset of $\mathrm{IC}(G)$ such that it is an $\mathfrak{S}$-type of G. Since ρ is a cuspidal representation of $\mathcal{K} \simeq \mathcal{K}_L$, we may inflate it to K_L. Then, the pair (K_L,ρ) is a depth-0 type of L. We say that (K,ρ) is a **G-cover** of (K_L,ρ). By the previous theorem, there is a finite set $\mathfrak{S}_L \subset \mathrm{IC}(L)$ such that (K_L,ρ) is an $\mathfrak{S}_L$-type of L. Then the proof of Theorem 4.8 in [Mor99] shows that we have the relation

$$\mathfrak{S} = \left\{[M,\tau]_G \,\middle|\, [M,\tau]_L \in \mathfrak{S}_L\right\}.$$

In this set, M is some Levi complement of L, therefore it may also be seen as a Levi complement in G. Thus, an inertial equivalence class $[M,\tau]_L$ in L gives rise to a class $[M,\tau]_G$ in G. Since K_L is maximal in L, in virtue of the proposition above any element of $\mathfrak{S}_L$ has the form $[L,\pi]_L$ for some supercuspidal representation π of L. In particular, every smooth irreducible representation of G containing the type (K,ρ) has a conjugate of L as cuspidal support. We deduce the following corollary.

Corollary. *Let (K, ρ) be a depth-0 type in G and assume that K is not a maximal parahoric subgroup. Then no smooth irreducible representation π of G containing the type (K, ρ) is supercuspidal.*

3.4.2.8 Thus, up to replacing G with a Levi complement, the study of any depth-0 type (K, ρ) can be reduced to the case where K is a maximal parahoric subgroup. Let us assume that it is the case, and let $\mathfrak{S}$ be the associated finite subset of $\mathrm{IC}(G)$. While $\mathfrak{S}$ is in general not a singleton, it becomes one once we modify the pair (K, ρ) a little bit. Let $\widehat{K}$ be the maximal open compact subgroup of $\mathrm{N}_G(K)$. We have $K \subset \widehat{K}$ but in general this inclusion may be strict. Let $\tilde{\rho}$ be a smooth irreducible representation of $\mathrm{N}_G(K)$ such that $\tilde{\rho}_{|K}$ contains ρ. Let $\widehat{\rho}$ be any irreducible component of the restriction $\tilde{\rho}_{|\widehat{K}}$. Eventually, let $\pi := \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_G(K)}^G \tilde{\rho}$ be the associated depth-0 supercuspidal representation of G.

Theorem ([Mor99] Variant 4.7)**.** *The pair $(\widehat{K}, \widehat{\rho})$ is a $[G, \pi]$-type.*

The conclusion does not depend on the choice of $\widehat{\rho}$ as an irreducible component of $\tilde{\rho}_{|\widehat{K}}$. Any one of them affords a type for the same singleton $\mathfrak{s} = [G, \pi]$.

3.4.2.9 Let us now consider a parahoric subgroup K along with an irreducible representation ρ of its maximal reductive quotient $\mathcal{K} = K/K^+$, where K^+ is the pro-unipotent radical of K. Assume that ρ is not cuspidal. Thus, there exists a proper parabolic subgroup $\mathcal{P} \subset \mathcal{K}$ with Levi complement $\mathcal{L}$, and a cuspidal irreducible representation τ of $\mathcal{L}$, such that ρ is an irreducible component of the Harish-Chandra induction $\iota_{\mathcal{P}}^{\mathcal{K}} \tau$. The preimage of $\mathcal{P}$ via the quotient map $K \twoheadrightarrow \mathcal{K}$ is a parahoric subgroup $K' \subsetneq K$, whose maximal reductive quotient $\mathcal{K}' := K'/K'^+$ is naturally identified with $\mathcal{L}$. We have $K^+ \subset K'^+ \subset K'$ and the intermediate quotient K'^+/K^+ is identified with the unipotent radical $\mathcal{N}$ of $\mathcal{P} \simeq K'/K^+$. Consider ρ as an irreducible representation of K inflated from $\mathcal{K}$. The invariants $\rho^{K'^+}$ form a representation of K' which coincides with the inflation of the Harish-Chandra restriction of ρ (as a representation of $\mathcal{K}$) to $\mathcal{L}$. Thus, $\rho^{K'^+}$ contains the inflation of τ to a representation of K'. In other words, we have a K'-equivariant map

$$\tau \rightarrow \rho_{|K'}.$$

By Frobenius reciprocity, it gives a map

$$\mathrm{c}-\mathrm{Ind}_{K'}^{K} \tau \rightarrow \rho,$$

which is surjective by irreducibility of ρ. Applying the functor $\mathrm{c}-\mathrm{Ind}_K^G : \mathrm{Rep}(K) \rightarrow \mathrm{Rep}(G)$, which is exact, and using transitivity of compact induction, we deduce the existence of a natural surjection

$$\mathrm{c}-\mathrm{Ind}_{K'}^{G} \tau \twoheadrightarrow \mathrm{c}-\mathrm{Ind}_K^G \rho.$$

Now, (K', τ) is a depth-0 type in G. Let $\mathfrak{S} \subset \mathrm{IC}(G)$ be the subset such that (K', τ) is an $\mathfrak{S}$-type, and let L be the (proper) Levi complement of G associated to K' as in the previous paragraph. By 3.4.2.6 Remark, it follows that any irreducible subquotient of $\mathrm{c}-\mathrm{Ind}_K^G \rho$ has inertial support in $\mathfrak{S}$. Since all elements of $\mathfrak{S}$ are of the form $[L, \pi]$ for some supercuspidal representation π of L, we reach the following conclusion.

Proposition. *Let K be a parahoric subgroup of G and let ρ be a non cuspidal irreducible representation of its maximal reductive quotient $\mathcal{K}$. Then no irreducible subquotient of* $\mathrm{c}-\mathrm{Ind}_K^G\,\rho$ *is supercuspidal.*

3.4.2.10 We go back to the context of the unitary similitude group J. We may now determine the inertial support of any irreducible subquotient of a representation of the form $\mathrm{c}-\mathrm{Ind}_{J_m}^J\,\rho$ with ρ inflated from a unipotent representation of $\mathrm{GU}(V_m^0)$. In particular, all the terms $E_1^{0,b}$ are of this form according to 3.4.1.11 Corollary. More precisely, let λ be a partition of $2m+1$ and let Δ_t be its 2-core (see 3.2.8). Thus $2m+1 = \frac{t(t+1)}{2} + 2e$ for some $e \geqslant 0$. The integer $\frac{t(t+1)}{2}$ is odd, so it can be written as $2f+1$ for some $f \geqslant 0$, and we have $m = f+e$. Using the basis of V_m^0 fixed in 3.1.2.8, we identify $\mathrm{GU}(V_m^0)$ with the matrix group $\mathrm{GU}_{2m+1}(\mathbb{F}_p)$. The cuspidal support of ρ_λ is (L_t, ρ_t) according to 3.2.8. Let P_t be the standard parabolic subgroup with Levi complement L_t. By direct computation, one may check that the preimage of P_t in J_m is the parahoric subgroup $J_{f,\ldots,m} := J_f \cap J_{f+1} \cap \ldots \cap J_m$. Let L_f be the Levi complement of J that is associated to the parahoric subgroup $J_{f,\ldots,m}$. Using the basis of $\mathbf{V}$ fixed in 3.1.1.4, let $\mathbf{V}^f$ be the subspace of $\mathbf{V}$ generated by $\mathbf{V}^{\mathrm{an}}$ and by the vectors $e_{\pm 1}, \ldots, e_{\pm f}$. It is equipped with the restriction of the hermitian form of $\mathbf{V}$. Then $L_f \simeq \mathrm{G}(\mathrm{U}(\mathbf{V}^f) \times \mathrm{U}_1(\mathbb{Q}_p)^e)$.
The group $L_f \cap J_{f,\ldots,m}$ is a maximal parahoric subgroup of L_f, and ρ_t can be inflated to it. In particular, the pair $(L_f \cap J_{f,\ldots,m}, \rho_t)$ is a level-0 type in L_f. Since we work with unitary groups over an unramified quadratic extension, $L_f \cap J_{f,\ldots,m}$ is also a maximal compact subgroup of L_f. In particular, $(L_f \cap J_{f,\ldots,m}, \rho_t)$ is a type for a singleton of the form $[L_f, \tau_f]_{L_f}$. Then τ_f has the form

$$\tau_f = \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_{L_f}(L_f \cap J_{f,\ldots,m})}^{L_f}\,\widetilde{\rho}_t,$$

where $\widetilde{\rho}_t$ is some smooth irreducible representation of $\mathrm{N}_{L_f}(L_f \cap J_{f,\ldots,m})$ containing ρ_t upon restriction. It follows that if we inflate ρ_t to $J_{f,\ldots,m}$ then $(J_{f,\ldots,m}, \rho_t)$ is a $[L_f, \tau_f]$-type in J. Moreover the compactly induced representation $\mathrm{c}-\mathrm{Ind}_{J_m}^J\,\rho_\lambda$ is a quotient of $\mathrm{c}-\mathrm{Ind}_{J_{f,\ldots,m}}^J\,\rho_t$. In particular, we reach the following conclusion.

Proposition. *Let λ be a partition of $2m+1$ with 2-core Δ_t. Write $\frac{t(t+1)}{2} = 2f+1$ for some $f \geqslant 0$. Any irreducible subquotient of* $\mathrm{c}-\mathrm{Ind}_{J_m}^J\,\rho_\lambda$ *has inertial support* $[L_f, \tau_f]$.

In particular, if $f < m$ then none of these irreducible subquotients are supercuspidal.

3.4.2.11 Let us keep the notations of the previous paragraph. Since unipotent representations of finite groups of Lie type have trivial central characters, if χ is an unramified character of $\mathrm{Z}(J)$ then $\chi_{\mathrm{Z}(J)\cap J_m}$ coincides with the central character of ρ_λ inflated to J_m. As in 3.4.2.3, we have

$$\left(\mathrm{c}-\mathrm{Ind}_{J_m}^J\,\rho_\lambda\right)_\chi \simeq V_{\rho_\lambda,\chi,0} \oplus V_{\rho_\lambda,\chi,\infty}.$$

If $f < m$, then no irreducible supercuspidal representation can occur. Thus $V_{\rho_\lambda,\chi,0} = 0$.
On the other hand, assume now that $f = m$ so that $L_f = J$ and ρ_λ is equal to the cuspidal representation ρ_{Δ_m}. As seen in 3.1.3.3, we have $\mathrm{N}_J(J_m) = \mathrm{Z}(J)J_m$ unless $n = 2$ (thus $m = 0$) in

which case $J_0 = J^\circ$ and $\mathrm{Z}(J)J_0$ is of index 2 in $\mathrm{N}_J(J_0) = J$. A representative of the non-trivial coset is given by g_0 as defined in 3.1.1.7. If $n \neq 2$, define

$$\tau_{m,\chi} := \mathrm{c} - \mathrm{Ind}_{\mathrm{Z}(J)J_m}^{J} \chi \otimes \rho_\lambda.$$

Then $\tau_{m,\chi}$ is an irreducible supercuspidal representation of J, and we have

$$\left(\mathrm{c} - \mathrm{Ind}_{J_m}^{J} \rho_\lambda\right)_\chi \simeq \mathrm{c} - \mathrm{Ind}_{\mathrm{Z}(J)J_m}^{J} \chi \otimes \rho_\lambda = \tau_{m,\chi}.$$

Thus $V_{\rho_\lambda,\chi,\infty} = 0$ and $V_{\rho_\lambda,\chi\infty} = \tau_{m,\chi}$ in this case.
When $n = 2$, $\rho_\lambda = \rho_{\Delta_0} = \mathbf{1}$ is the trivial representation of $J_0 = J^\circ$. Let $\chi_0 : J \to \overline{\mathbb{Q}_\ell}^\times$ be the unique non-trivial character of J which is trivial on $\mathrm{Z}(J)J_0$. Then $\left(\mathrm{c} - \mathrm{Ind}_{J_0}^{J} \mathbf{1}\right)_\chi$ is the sum of an unramified character $\tau_{0,\chi}$ of J whose central character is χ, and of the character $\chi_0\tau_{0,\chi}$. Both characters are supercuspidal, and they are the only unramified characters of J with central character χ.

3.4.2.12 According to 3.4.1.6 and 3.4.1.11, the terms $E_1^{0,b}$ are a sum of representations of the form

$$\mathrm{c} - \mathrm{Ind}_{J_m}^{J} \rho_\lambda,$$

with λ a partition of $2m+1$ having 2-core Δ_0 if b is even, and Δ_1 if b is odd. Moreover, by 3.4.1.11 we have

$$E_2^{0,2(n-1-m)} \simeq \mathrm{c} - \mathrm{Ind}_{J_m}^{J} \mathbf{1}, \qquad E_2^{0,2(n-1-m)+1} \simeq \mathrm{c} - \mathrm{Ind}_{J_m}^{J} \rho_{(2m,1)}.$$

In particular, summing up the discussion of the previous paragraph, we have reached the following statement.

Proposition. *Let χ be an unramified character of $\mathrm{Z}(J)$.*

- *Assume that $n \geqslant 3$. The representation $(E_2^{0,2(n-1-m)})_\chi$ contains no non-zero admissible subrepresentation, and it is not J-semisimple. Moreover, any irreducible subquotient has inertial support $[L_0, \tau_0]$. If $n \geqslant 5$, then the same statement holds for $(E_2^{0,2(n-1-m)+1})_\chi$ with the inertial support being $[L_1, \tau_1]$.*
- *For $n = 1, 2, 3, 4$, let $b = 0, 2, 3, 5$ respectively. Then $m = 0$ when $1, 2$ and $m = 1$ when $n = 3, 4$. Let χ be an unramified character of $\mathrm{Z}(J)$. The representation $\tau_{m,\chi}$ is irreducible supercuspidal, and we have*

$$(E_2^{0,b})_\chi \simeq \begin{cases} \tau_{m,\chi} & \text{if } n = 1, 3, 4, \\ \tau_{m,\chi} \oplus \chi_0\tau_{m,\chi} & \text{if } n = 2. \end{cases}$$

In particular, we deduce the following important corollary.

Corollary. *Let χ be an unramified character of $\mathrm{Z}(J)$. If $n \geqslant 3$ then $\mathrm{H}_c^{2(n-1-m)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})_\chi$ is not J-admissible. If $n \geqslant 5$ then the same holds for $\mathrm{H}_c^{2(n-1-m)+1}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})_\chi$.*

3.4.3 The case $n = 3, 4$

3.4.3.1 Let us focus on the case $m = 1$, that is $n = 3$ or 4. Recall that $N(\Lambda_0)$ denotes the set of lattices $\Lambda \in \mathcal{L}_0$ with type $t(\Lambda) = t_{\max} = 3$ containing Λ_0. It has cardinality $\nu(1,2) = p+1$ when $n = 3$ and $\nu(2,3) = p^3+1$ when $n = 4$. In particular, we may locate the non zero terms $E_{1,\mathrm{alt}}^{a,b}$ of the alternating Čech spectral sequence as follows.

$$E_{1,\mathrm{alt}}^{a,b} \neq 0 \iff \begin{cases} (a,b) \in \{(0,2); (0,3); (-k,4) \mid 0 \leqslant k \leqslant p\} & \text{if } n = 3, \\ (a,b) \in \{(0,4); (0,5); (-k,6) \mid 0 \leqslant k \leqslant p^3\} & \text{if } n = 4. \end{cases}$$

In Figure 3 below, we draw the shape of the first page $E_{1,\mathrm{alt}}$ for $n = 3$. The case of $n = 4$ is similar, except that two more 0 rows should be added at the bottom. To alleviate the notations, we write φ_{-a} for the differential $\varphi_{-a}^{2(n-1)}$.

$$\ldots \xrightarrow{\varphi_4} E_{1,\mathrm{alt}}^{-3,4} \xrightarrow{\varphi_3} E_{1,\mathrm{alt}}^{-2,4} \xrightarrow{\varphi_2} E_{1,\mathrm{alt}}^{-1,4} \xrightarrow{\varphi_1} \mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}$$

$$\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\rho_{\Delta_2}$$

$$\mathrm{c}-\mathrm{Ind}_{J_1}^{J}\mathbf{1}$$

$$0$$

$$0$$

Figure 3: The first page $E_{1,\mathrm{alt}}$ of the alternating Čech spectral sequence when $n = 3$.

3.4.3.2 Let $i \in \mathbb{Z}$ such that ni is even. For $\Lambda, \Lambda' \in \mathcal{L}_i$, recall that the distance $d(\Lambda, \Lambda')$ is the smallest integer $d \geqslant 0$ such that there exists a sequence $\Lambda = \Lambda^0, \ldots, \Lambda^d = \Lambda'$ of lattices of $\mathcal{L}_i$ with $\{\Lambda^j, \Lambda^{j+1}\}$ being an edge for all $0 \leqslant j \leqslant d-1$. When $m = 1$, any lattice $\Lambda \in \mathcal{L}_i$ has type 1 or 3, and two lattices forming an edge can not have the same type. Therefore, the value of $t(\Lambda^j)$ alternates between 1 and 3. In particular, if $t(\Lambda) = t(\Lambda')$ then $d(\Lambda, \Lambda')$ is even. According to [Vol10] Proposition 3.7, the simplicial complex $\mathcal{L}_i$ is in fact a tree. We will use this to prove the following proposition.

Proposition. *Let $b = 4$ when $n = 3$, and $b = 6$ when $n = 4$. We have $E_2^{-1,b} = 0$.*

By 3.4.1.8 Proposition, we may use the alternating Čech spectral sequence to show that $E_2^{-1,b} = \mathrm{Ker}(\varphi_1)/\mathrm{Im}(\varphi_2)$ vanishes. As we have observed in 3.4.1.12, the term $E_1^{a,b}$ is the $\overline{\mathbb{Q}_\ell}$-vector space generated by the set I_{-a+1}, and $E_{1,\mathrm{alt}}^{a,b}$ is the subspace consisting of all the vectors $v = \sum_{\gamma \in I_{-a+1}} \lambda_\gamma \gamma$ such that for all $\sigma \in \mathfrak{S}_{-a+1}$ we have $\lambda_{\sigma(\gamma)} = \mathrm{sgn}(\sigma)\lambda_\gamma$. Here the λ_γ's are scalars which are almost all zero. To prove the proposition, let us look at the differential φ_2. It acts on the basis vectors in the following way.

$$\left.\begin{array}{l}(\Lambda,\Lambda,\Lambda)\\(\Lambda,\Lambda,\Lambda')\\(\Lambda',\Lambda,\Lambda)\end{array}\right\}\mapsto(\Lambda,\Lambda), \qquad \forall \Lambda,\Lambda' \in \mathcal{L}^{(1)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \neq \varnothing,$$

$$(\Lambda,\Lambda',\Lambda)\mapsto(\Lambda',\Lambda)+(\Lambda,\Lambda')-(\Lambda,\Lambda), \qquad \forall \Lambda,\Lambda' \in \mathcal{L}^{(1)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \neq \varnothing,$$

$$(\Lambda,\Lambda',\Lambda'')\mapsto(\Lambda,\Lambda')+(\Lambda',\Lambda'')-(\Lambda,\Lambda''), \quad \forall \Lambda,\Lambda',\Lambda'' \in \mathcal{L}^{(1)} \text{ such that } U_\Lambda \cap U_{\Lambda'} \cap U_{\Lambda''} \neq \varnothing.$$

We note that for a collection of lattices $\Lambda^1,\ldots,\Lambda^s \in \mathcal{L}_i^{(1)}$, the condition $U_{\Lambda^1}\cap\ldots\cap U_{\Lambda^s} \neq \varnothing$ is equivalent to $d(\Lambda^j,\Lambda^{j'}) = 2$ for all $1 \leqslant j \neq j' \leqslant s$.
Towards a contradiction, we assume that $\mathrm{Im}(\varphi_2) \subsetneq \mathrm{Ker}(\varphi_1)$. Let $v \in \mathrm{Ker}(\varphi_1)\backslash\mathrm{Im}(\varphi_2)$. Since $v \in E_{1,\mathrm{alt}}^{-1,b}$, it decomposes under the form

$$v = \sum_{j=1}^{r} \lambda_j(\gamma_j - \tau(\gamma_j)),$$

where $r \geqslant 1$, the γ_j's are of the form (Λ,Λ') with $\Lambda \neq \Lambda'$ and $U_\Lambda \cap U_{\Lambda'} \neq \varnothing$, the scalars λ_j's are non zero and $\tau \in \mathfrak{S}_2$ is the transposition. We may assume that r is minimal among all the vectors in the complement $\mathrm{Ker}(\varphi_1)\backslash\mathrm{Im}(\varphi_2)$. In particular, there exists a single $i \in \mathbb{Z}$ such that ni is even, and for all j the lattices in γ_j belong to $\mathcal{L}_i^{(1)}$. We may further assume $i = 0$ without loss of generality.
We say that an element $\gamma \in I_2$ occurs in v if $\gamma = \gamma_j$ or $\tau(\gamma_j)$ for some j. Similarly, we say that a lattice $\Lambda \in \mathcal{L}_0^{(1)}$ occurs in v if it is a constituent of some γ_j.

Lemma. *Let $\gamma = (\Lambda',\Lambda) \in I_2$ be an element occuring in v. Then there exists $\Lambda'' \in \mathcal{L}_0^{(1)}$ such that $(\Lambda'',\Lambda) \in I_2$ occurs in v and $d(\Lambda',\Lambda'') = 4$.*

Proof. Let us write $(\Lambda^j,\Lambda) \in I_2, 1 \leqslant j \leqslant s$ for the various elements occuring in v whose first component is Λ. Up to reordering the γ_j's and swapping them with $\tau(\gamma_j)$ if necessary, we may assume that $(\Lambda^j,\Lambda) = \gamma_j$ for all $1 \leqslant j \leqslant s$, and that $\Lambda^1 = \Lambda'$. The coordinate of $\varphi_1(v)$ along the basis vector (Λ) is equal to $2\sum_{j=1}^{s}\lambda_j$. Since $\varphi_1(v) = 0$, the sum of the λ_j's from 1 to s is zero. In particular, we have $s \geqslant 2$.
For all $2 \leqslant j \leqslant s$, we have $2 \leqslant d(\Lambda',\Lambda^j) \leqslant 4$ by the triangular inequality. Towards a contradiction, assume that $d(\Lambda',\Lambda^j) = 2$ for all $2 \leqslant j \leqslant s$. In particular, $\delta_j := (\Lambda^j,\Lambda',\Lambda) \in I_3$ for all $2 \leqslant j \leqslant s$. Consider the vector

$$w := \frac{1}{3}\sum_{j=2}^{s}\sum_{\sigma\in\mathfrak{S}_6}\mathrm{sgn}(\sigma)\lambda_j\sigma(\delta_j) \in E_{1,\mathrm{alt}}^{-2,b}.$$

Then we compute

$$\varphi_2(w) = -\lambda_1((\Lambda',\Lambda)-(\Lambda,\Lambda')) - \sum_{j=2}^{s}\lambda_j((\Lambda^j,\Lambda)-(\Lambda,\Lambda^j)) + \sum_{j=2}^{s}\lambda_j((\Lambda^j,\Lambda')-(\Lambda',\Lambda^j)).$$

In particular, we get

$$v + \varphi_2(w) = \sum_{j=s+1}^{r} \lambda_j(\gamma_j - \tau(\gamma_j)) + \sum_{j=2}^{s} \lambda_j((\Lambda^j, \Lambda') - (\Lambda', \Lambda^j)) \in \mathrm{Ker}(\varphi_1)\backslash\mathrm{Im}(\varphi_2),$$

which contradicts the minimality of r. □

3.4.3.3 To conclude the proof of the proposition, let us pick $\Lambda = \Lambda^0 \in \mathcal{L}_0^{(1)}$ which occurs in v, say in a pair $(\Lambda', \Lambda) \in I_2$. Write $\Lambda^1 := \Lambda'$. By induction, we build a sequence $(\Lambda^k)_{k \geqslant 0}$ of lattices in $\mathcal{L}_0^{(1)}$ such that for all k, the pair $(\Lambda^{k+1}, \Lambda^k)$ occurs in v and we have $d(\Lambda^0, \Lambda^k) = 2k$. It follows that the Λ^k's are pairwise distinct, and it leads to a contradiction since only a finite number of such lattices can occur in v.
Let us assume that $\Lambda^0, \ldots, \Lambda^k$ are already built for some $k \geqslant 1$. By the Lemma applied to Λ^k, there exists $\Lambda^{k+1} \in \mathcal{L}_0^{(1)}$ such that the pair $(\Lambda^{k+1}, \Lambda^k)$ occurs in v and $d(\Lambda^{k-1}, \Lambda^{k+1}) = 4$. By the triangular inequality, we have

$$d(\Lambda^0, \Lambda^{k+1}) \geqslant |d(\Lambda^0, \Lambda^k) - d(\Lambda^k, \Lambda^{k+1})| = 2k - 2 = 2(k-1).$$

Thus $d(\Lambda^0, \Lambda^{k+1}) = 2(k-1), 2k$ or $2(k+1)$. We prove that it must be equal to the latter.

Assume $d(\Lambda^0, \Lambda^{k+1}) = 2(k-1)$. There exists a path $\Lambda^0 = L^0, \ldots, L^{2(k-1)} = \Lambda^{k+1}$. We obtain a cycle

$$\begin{array}{ccccc} & \Lambda^0 \cap \Lambda^1 - \Lambda^1 - \ldots - \Lambda^{k-1} - \Lambda^{k-1} \cap \Lambda^k & \\ \Lambda^0 & & \Lambda^k \\ & L^1 - L^2 - \ldots - L^{2(k-1)} = \Lambda^{k+1} - \Lambda^k \cap \Lambda^{k+1} & \end{array}$$

Since $\mathcal{L}_0$ is a tree, this cycle must be trivial, ie. the lower and upper paths, which are of the same length, are the same. In particular, we have $\Lambda^{k-1} = \Lambda^{k+1}$, which is absurd since $d(\Lambda^{k-1}, \Lambda^{k+1}) = 4$.

Assume $d(\Lambda^0, \Lambda^{k+1}) = 2k$. There exists a path $\Lambda^0 = L_0, \ldots, L^{2k} = \Lambda^{k+1}$. We obtain a cycle

$$\begin{array}{ccccc} & \Lambda^0 \cap \Lambda^1 - \Lambda^1 - \ldots - \Lambda^{k-1} \cap \Lambda^k - \Lambda^k & \\ \Lambda^0 & & \Lambda^k \cap \Lambda^{k+1} \\ & L^1 - L^2 - \ldots - L^{2k-1} - L^{2k} = \Lambda^{k+1} & \end{array}$$

Since $\mathcal{L}_0$ is a tree, this cycle must be trivial, ie. the lower and upper paths, which are of the same length, are the same. In particular, we have $\Lambda^k = \Lambda^{k+1}$, which is absurd since $d(\Lambda^k, \Lambda^{k+1}) = 2$.

Thus, we have $d(\Lambda^0, \Lambda^{k+1}) = 2(k+1)$ so that Λ^{k+1} meets all the requirements. It concludes the proof.

3.4.3.4 In particular, we obtain the following statement.

Theorem. *Assume that $n = 3$ or 4. Let $b = 3$ if $n = 3$, and let $b = 5$ if $n = 4$. We have*

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^J \rho_{\Delta_2},$$

with the rational Frobenius τ acting like multiplication by $-p^b$.

3.5 The cohomology of the basic stratum of the Shimura variety for $n = 3, 4$

3.5.1 The Hochschild-Serre spectral sequence induced by p-adic uniformization

3.5.1.1 In this section, we still assume that n is any integer $\geqslant 1$. We recover the notations of Part 3.3 regarding Shimura varieties. As we have seen in 3.3.6, p-adic uniformization is a geometric identity relating the Rapoport-Zink space $\mathcal{M}$ with the basic stratum $\overline{\mathrm{S}}_{K^p}(b_0)$. In [Far04], Fargues constructed a Hochschild-Serre spectral sequence using the uniformization theorem on the generic fibers, which we introduce in the following paragraphs.
Recall the PEL datum introduced in 3.3.1. Let $\xi : G \to W_\xi$ be a finite-dimensional irreducible algebraic $\overline{\mathbb{Q}_\ell}$-representation of G. Such representations have been classified in [HT01] III.2. We look at $\mathbb{V}_{\overline{\mathbb{Q}_\ell}} := \mathbb{V} \otimes \overline{\mathbb{Q}_\ell}$ as a representation of G, whose dual is denoted by $\mathbb{V}_0$. Using the alternating form $\langle\cdot,\cdot\rangle$, we have an isomorphism $\mathbb{V}_0 \simeq \mathbb{V}_{\overline{\mathbb{Q}_\ell}} \otimes c^{-1}$, where c is the multiplier character of G.

Proposition ([HT01] III.2)**.** *There exists unique integers $t(\xi), m(\xi) \geqslant 0$ and an idempotent $\epsilon(\xi) \in \mathrm{End}(\mathbb{V}_0^{\otimes m(\xi)})$ such that*

$$W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)}).$$

The weight $w(\xi)$ is defined by

$$w(\xi) := m(\xi) - 2t(\xi).$$

To any ξ as above, we can associate a local system $\mathcal{L}_\xi$ which is defined on the tower $(\mathrm{S}_{K^p})_{K^p}$ of Shimura varieties. We still write $\mathcal{L}_\xi$ for its restriction to the generic fiber $\mathrm{Sh}_{K_0K^p} \otimes_E \mathbb{Z}_{p^2}$, and we denote by $\overline{\mathcal{L}_\xi}$ its restriction to the special fiber $\overline{\mathrm{S}}_{K^p}$. Let $\mathcal{A}_{K^p}$ be the universal abelian scheme over S_{K^p}. We write $\pi_{K^p}^m : \mathcal{A}_{K^p}^m \to \mathrm{S}_{K^p}$ for the structure morphism of the m-fold product of $\mathcal{A}_{K^p}$ with itself over S_{K^p}. If $m = 0$ it is just the identity on S_{K^p}. According to [HT01] III.2, we have an isomorphism

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)}\left(\mathrm{R}^{m(\xi)}(\pi_{K^p}^{m(\xi)})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right),$$

where $\epsilon_{m(\xi)}$ is some idempotent. In particular, if ξ is the trivial representation of G then $\mathcal{L}_\xi = \overline{\mathbb{Q}_\ell}$.

3.5.1.2 We fix an irreducible algebraic representation $\xi : G \to W_\xi$ as above. We associate the space $\mathcal{A}_\xi$ of **automorphic forms of I of type ξ at infinity**. Explicitly, it is given by

$$\mathcal{A}_\xi = \{f : I(\mathbb{A}_f) \to W_\xi \,|\, f \text{ is } I(\mathbb{A}_f)\text{-smooth by right translations and } \forall \gamma \in I(\mathbb{Q}), f(\gamma\,\cdot) = \xi(\gamma)f(\cdot)\}\,.$$

We denote by $\mathcal{L}_\xi^{\mathrm{an}}$ the analytification of $\mathcal{L}_\xi$ to $\mathrm{Sh}_{K_0K^p}^{\mathrm{an}}$, as well as for its restriction to any open subspace.

Notation. We write $\mathrm{H}^\bullet((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}})$ for the cohomology of $(\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0} \widehat{\otimes}\, \mathbb{C}_p$ with coefficients in $\mathcal{L}_\xi^{\mathrm{an}}$.

Theorem ([Far04] 4.5.12)**.** *There is a W-equivariant spectral sequence*

$$F_2^{a,b}(K^p) : \mathrm{Ext}^a_J\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \mathcal{A}_\xi^{K^p}\right) \implies \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}}).$$

These spectral sequences are compatible as the open compact subgroup K^p varies in $G(\mathbb{A}_f^p)$.

The W-action on $F_2^{a,b}(K^p)$ is inherited from the cohomology group $\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n)$. By the compatibility with K^p, we may take the limit $\varinjlim_{K^p}$ for all terms and obtain a $G(\mathbb{A}_f^p)\times W$-equivariant spectral sequence. Since m is the semisimple rank of J, the terms $F_2^{a,b}(K^p)$ are zero for $a > m$ according to [Far04] Lemme 4.4.12. Therefore, the non-zero terms $F_2^{a,b}$ are located in the finite strip delimited by $0 \leqslant a \leqslant m$ and $0 \leqslant b \leqslant 2(n-1)$.
Let us look at the abutment of the sequence. Since the formal completion $\widehat{\mathrm{S}}_{K^p}$ of S_{K^p} along its special fiber is a smooth formal scheme, Berkovich's comparison theorem ([Ber96] Corollary 3.7) gives an isomorphism

$$\mathrm{H}_c^{a+b}(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) = \mathrm{H}^{a+b}(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \xrightarrow{\sim} \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}}).$$

The first equality follows from $\overline{\mathrm{S}}_{K^p}(b_0)$ being a proper variety. Since this variety has dimension m, the cohomology $\mathrm{H}^\bullet((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}})$ is concentrated in degrees 0 to $2m$.

3.5.1.3 Let $\mathcal{A}(I)$ denote the set of all automorphic representations of I counted with multiplicities. We write $\check{\xi}$ for the dual of ξ. We also define

$$\mathcal{A}_\xi(I) := \{\Pi \in \mathcal{A}(I) \,|\, \Pi_\infty = \check{\xi}\}.$$

According to [Far04] 4.6, we have an identification

$$\mathcal{A}_\xi^{K_p} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \Pi_p \otimes (\Pi^p)^{K_p}.$$

It yields, for every a and b, an isomorphism

$$F_2^{a,b}(K^p) \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^a_J\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes (\Pi^p)^{K_p}.$$

Taking the limit over K^p, we deduce that

$$F_2^{a,b} := \varinjlim_{K^p} F_2^{a,b}(K^p) \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^a_J\left(\mathrm{H}_c^{2(n-1)-b}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p.$$

The spectral sequence defined by the terms $F_2^{a,b}$ computes $\mathrm{H}^{a+b}(\widehat{\mathrm{S}}^{\,\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}}) := \varinjlim_{K^p} \mathrm{H}^{a+b}((\widehat{\mathrm{S}}_{K^p})^{\mathrm{an}}_{|b_0}, \mathcal{L}_\xi^{\mathrm{an}})$. It is isomorphic to $\mathrm{H}_c^{a+b}(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) := \varinjlim_{K^p} \mathrm{H}_c^{a+b}(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$.

3.5.1.4 Recall from 3.4.1.7 that we have a decomposition

$$\mathrm{H}_c^b(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{b \leqslant b' \leqslant 2(n-1)} E_2^{b-b',b'},$$

and $E_2^{b-b',b'}$ corresponds to the eigenspace of τ associated to the eigenvalue $(-p)^b$. Accordingly, we have a decomposition

$$F_2^{a,b} \simeq \bigoplus_{\substack{2(n-1)-b \leqslant \\ b' \leqslant 2(n-1)}} \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^a\left(E_2^{2(n-1)-b-b',b'}(1-n), \Pi_p\right) \otimes \Pi^p.$$

For $\Pi \in \mathcal{A}_\xi(I)$, we denote by ω_Π the central character. We define

$$\delta_{\Pi_p} := \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id}) p^{-w(\xi)} \in \overline{\mathbb{Q}_\ell}^\times.$$

Let ι be any isomorphism $\overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, and write $|\cdot|_\iota := |\iota(\cdot)|$. Since I is a group of unitary similitudes of an $E/\mathbb{Q}$-hermitian space, its center is $E^\times \cdot \mathrm{id}$. The element $p^{-1} \cdot \mathrm{id} \in \mathrm{Z}(J)$ can be seen as the image of $p^{-1} \cdot \mathrm{id} \in \mathrm{Z}(I(\mathbb{Q}))$. We have $\omega_\Pi(p^{-1} \cdot \mathrm{id}) = 1$. Moreover, for any finite place $q \neq p$, the element $p^{-1} \cdot \mathrm{id}$ lies inside the maximal compact subgroup of $\mathrm{Z}(I(\mathbb{Q}_q))$, so $|\omega_{\Pi_q}(p^{-1}\mathrm{id})|_\iota = 1$. Besides $\Pi_\infty = \check{\xi}$, so we have

$$|\omega_{\Pi_p}(p^{-1} \cdot \mathrm{id})|_\iota = |\omega_{\check{\xi}}(p^{-1} \cdot \mathrm{id})|_\iota^{-1} = |\omega_\xi(p^{-1} \cdot \mathrm{id})|_\iota = |p^{w(\xi)}|_\iota = p^{w(\xi)}.$$

The last equality comes from the isomorphism $W_\xi \simeq c^{t(\xi)} \otimes \epsilon(\xi)(\mathbb{V}_0^{\otimes m(\xi)})$, see 3.5.1.1. In particular $|\delta_{\Pi_p}|_\iota = 1$ for any isomorphism ι.

Proposition. *The W-action on $\mathrm{Ext}_J^a(E_2^{2(n-1)-b-b',b'}(1-n), \Pi_p)$ is trivial on the inertia I, and the Frobenius element* Frob *acts like multiplication by $(-1)^{-b'} \delta_{\Pi_p} p^{-b'+2(n-1)+w(\xi)}$.*

Proof. Let us write $X := E_2^{2(n-1)-b-b',b'}(1-n)$. By convention, the action of Frob on a space $\mathrm{Ext}_J^a(X, \Pi_p)$ is induced by functoriality of Ext applied to $\mathrm{Frob}^{-1} : X \to X$. Let us consider a projective resolution of X in the category of smooth representations of J

$$\ldots \xrightarrow{u_3} P_2 \xrightarrow{u_2} P_1 \xrightarrow{u_1} P_0 \xrightarrow{u_0} X \longrightarrow 0.$$

Since Frob^{-1} commutes with the action of J, we can choose a lift $\mathcal{F} = (\mathcal{F}_i)_{i \geqslant 0}$ of Frob^{-1} to a morphism of chain complexes.

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{u_3} & P_2 & \xrightarrow{u_2} & P_1 & \xrightarrow{u_1} & P_0 & \xrightarrow{u_0} & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathcal{F}_2} & & \downarrow{\scriptstyle \mathcal{F}_1} & & \downarrow{\scriptstyle \mathcal{F}_0} & & \downarrow{\scriptstyle \mathrm{Frob}^{-1}} & & \\
\ldots & \xrightarrow{u_3} & P_2 & \xrightarrow{u_2} & P_1 & \xrightarrow{u_1} & P_0 & \xrightarrow{u_0} & X & \longrightarrow & 0
\end{array}$$

After applying $\mathrm{Hom}_J(\cdot, \Pi_p)$ and forgetting about the first term, we obtain a morphism $\mathcal{F}^*$ of chain complexes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_J(P_0, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_1, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_2, \Pi_p) & \longrightarrow & \ldots \\
& & \downarrow{\scriptstyle \mathcal{F}_0^*} & & \downarrow{\scriptstyle \mathcal{F}_1^*} & & \downarrow{\scriptstyle \mathcal{F}_2^*} & & \\
0 & \longrightarrow & \mathrm{Hom}_J(P_0, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_1, \Pi_p) & \longrightarrow & \mathrm{Hom}_J(P_2, \Pi_p) & \longrightarrow & \ldots
\end{array}$$

Here $\mathcal{F}_i^* f(v) := f(\mathcal{F}_i(v))$. It induces morphisms on the cohomology

$$\mathcal{F}_i^* : \mathrm{Ext}^i_J(X, \Pi_p) \to \mathrm{Ext}^i_J(X, \Pi_p),$$

which do not depend on the choice of the lift $\mathcal{F}$. Recall that Frob is the composition of τ and $p \cdot \mathrm{id} \in J$. Since τ is multiplication by the scalar $(-1)^{b'} p^{b'-2(n-1)}$ on X, we may choose the lift $\mathcal{F}_i := (-1)^{-b'} p^{-b'+2(n-1)} (p^{-1} \cdot \mathrm{id})$ for all i.
Consider an element of $\mathrm{Ext}^i_J(X, \Pi_p)$ represented by a morphism $f : P_i \to \Pi_p$. For any $v \in P_i$ we have

$$\mathcal{F}_i^* f(v) = f(\mathcal{F}_i(v)) = (-1)^{-b'} p^{-b'+2(n-1)} f((p^{-1} \cdot \mathrm{id}) \cdot v) = (-1)^{-b'} p^{-b'+2(n-1)} \omega_{\Pi_p}(p^{-1} \cdot \mathrm{id}) f(v).$$

It follows that Frob acts on $\mathrm{Ext}^i_J(X, \Pi_p)$ via multiplication by the scalar $(-1)^{-b'} \delta_{\Pi_p} p^{-b'+2(n-1)+w(\xi)}$. □

3.5.1.5 In general, the Hochschild-Serre spectral sequence has many differentials between non-zero terms. However, focusing on the diagonal defined by $a + b = 0$, it is possible to compute $\mathrm{H}^0_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. Recall that $X^{\mathrm{un}}(J)$ denotes the set of unramified characters of J. If $x \in \overline{\mathbb{Q}_\ell}^\times$ is any non-zero scalar, we denote by $\overline{\mathbb{Q}_\ell}[x]$ the 1-dimensional representation of W where the inertia I acts trivially and the geometric Frobenius Frob acts like $x \cdot \mathrm{id}$.

Proposition. *We have an isomorphism of $G(\mathbb{A}_f^p) \times W$-representations*

$$\mathrm{H}^0_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}].$$

Proof. The only non-zero term $F_2^{a,b}$ on the diagonal defined by $a + b = 0$ is $F_2^{0,0}$. Since there is no non-zero arrow pointing at nor coming from this term, it is untouched in all the successive pages of the sequence. Therefore we have an isomorphism

$$F_2^{0,0} \simeq \mathrm{H}^0_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

Using 3.4.1.12, we also have isomorphisms

$$\begin{aligned} F_2^{0,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J \left(\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J \left((\mathrm{c} - \mathrm{Ind}_{J^\circ}^J \mathbf{1})(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J^\circ} \left(\mathbf{1}(1-n), \Pi_{p|J^\circ}\right) \otimes \Pi^p. \end{aligned}$$

Thus, only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ with $\Pi_p^{J^\circ} \neq 0$ contribute to the sum. Consider such a Π. The irreducible representation Π_p is generated by a J°-invariant vector. Since J° is normal in J, the whole representation Π_p is trivial on J°. Thus, it is an irreducible representation of $J/J^\circ \simeq \mathbb{Z}$. Therefore, it is one-dimensional. Since J° is generated by all compact subgroups of J, it follows that $\Pi_p^{J^\circ} \neq 0 \iff \Pi_p \in X^{\mathrm{un}}(J)$. When it is satisfied, the W-representation $V_\Pi^0 := \mathrm{Hom}_{J^\circ}(\mathbf{1}(1-n), \Pi_p)$ has dimension one and the Frobenius action was described in 3.5.1.4. □

3.5.2 The case $n = 3, 4$

3.5.2.1 In this section, we assume that $m = 1$, ie. $n = 3$ or 4. We recover the notations of 3.4.3.1. We use our knowledge so far on the cohomology of the Rapoport-Zink space to entirely compute the cohomology of the basic locus of the Shimura variety via p-adic uniformization.
Let ξ be an irreducible finite dimensional algebraic representation of G as in 3.5.1.1. When $n = 3$ or 4, the semisimple rank of J is $m = 1$, therefore the terms $F_2^{a,b}$ are zero for $a > 1$. In particular, the spectral sequence degenerates on the second page. Since it computes the cohomology of the basic locus $\overline{\mathrm{S}}(b_0)$ which is 1-dimensional, we also have $F_2^{0,b} = 0$ for $b \geqslant 3$, and $F_2^{1,b} = 0$ for $b \geqslant 2$. In Figure 4, we draw the second page F_2 and we write between brackets the *complex modulus* of the possible eigenvalues of Frob on each term under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$, as computed in 3.5.1.4.

Remark. The fact that no eigenvalue of complex modulus $p^{w(\xi)}$ appears in $F_2^{0,1}$ nor in $F_2^{1,1}$ follows from 3.4.3.2 Proposition, where we proved that $E_2^{-1,b} = 0$ for $b = 4$ (resp. 6) when $n = 3$ (resp. 4).

$$\begin{array}{cc} F_2^{0,2}[p^{w(\xi)+2}, p^{w(\xi)}] & 0 \\ F_2^{0,1}[p^{w(\xi)+1}] & F_2^{1,1}[p^{w(\xi)+1}] \\ F_2^{0,0}[p^{w(\xi)}] & F_2^{1,0}[p^{w(\xi)}] \end{array}$$

Figure 4: The second page F_2 with the complex modulus of possible eigenvalues of Frob on each term.

Proposition. *We have $F_2^{1,1} = 0$ and the eigenspaces of* Frob *on $F_2^{0,2}$ attached to any eigenvalue of complex modulus $p^{w(\xi)}$ are zero.*

Proof. By the machinery of spectral sequences, there is a $G(\mathbb{A}_f^p) \times W$-subspace of $\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ isomorphic to $F_2^{1,1}$, and the quotient by this subspace is isomorphic to $F_2^{0,2}$. We prove that all eigenvalues of Frob on $\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ have complex modulus $p^{w(\xi)+2}$. The proposition then readily follows.
We need the Ekedahl-Oort stratification on the basic stratum of the Shimura variety. Let $K^p \subset G(\mathbb{A}_f^p)$ be small enough. In [VW11] 3.3 and 6.3, the authors define the Ekedahl-Oort stratification on $\mathcal{M}_{\mathrm{red}}$ and on $\overline{\mathrm{S}}_{K^p}(b_0)$ respectively, and they are compatible via the p-adic uniformization isomorphism. For $n = 3$ or 4, the stratification on the basic stratum take the following form

$$\overline{\mathrm{S}}_{K^p}(b_0) = \overline{\mathrm{S}}_{K^p}[1] \sqcup \overline{\mathrm{S}}_{K^p}[3].$$

The stratum $\overline{\mathrm{S}}_{K^p}[1]$ is closed and 0-dimensional, whereas the other stratum $\overline{\mathrm{S}}_{K^p}[3]$ is open, dense and 1-dimensional. In particular, we have a Frobenius equivariant isomorphism between the cohomology groups of highest degree

$$\mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

According the [VW11] 5.3, the closed Bruhat-Tits strata $\mathcal{M}_\Lambda$ and $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ also admit an Ekedahl-Oort stratification of a similar form, and we have a decomposition

$$\overline{\mathrm{S}}_{K^p}[3] = \bigsqcup_{\Lambda,k} \overline{\mathrm{S}}_{K^p,\Lambda,k}[3]$$

into a finite disjoint union of open and closed subvarieties. As a consequence, we have the following Frobenius equivariant isomorphisms

$$\mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\Lambda,k} \mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p,\Lambda,k}[3] \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\Lambda,k} \mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p,\Lambda,k} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$$

where the last isomorphism between cohomology groups of highest degree follows from the stratification on the closed Bruhat-Tits strata $\overline{\mathrm{S}}_{K^p,\Lambda,k}$. Now, recall from 3.5.1.1 that the local system $\mathcal{L}_\xi$ is given by

$$\mathcal{L}_\xi \simeq \epsilon(\xi)\epsilon_{m(\xi)}\left(\mathrm{R}^{m(\xi)}(\pi^{m(\xi)}_{K^p})_*\overline{\mathbb{Q}_\ell}(t(\xi))\right).$$

It implies that $\overline{\mathcal{L}_\xi}$ is pure of weight $w(\xi)$. Since the variety $\overline{\mathrm{S}}_{K^p,\Lambda,k}$ is smooth and projective, it follows that all eigenvalues of Frob on the cohomology group $\mathrm{H}^2_c(\overline{\mathrm{S}}_{K^p,\Lambda,k} \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ must have complex modulus $p^{w(\xi)+2}$ under any isomorphism $\iota : \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}$. The result follows by taking the limit over K^p. □

3.5.2.2 In this paragraph, let us compute the term

$$\begin{aligned} F_2^{1,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^1_J\left(\mathrm{H}^{2(n-1)}_c(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}^1_J\left(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1}(1-n), \Pi_p\right) \otimes \Pi^p. \end{aligned}$$

Let St_J denote the Steinberg representation of J, and recall that $X^{\mathrm{un}}(J)$ denotes the set of unramified characters of J.

Proposition. *Let π be an irreducible smooth representation of J. Then*

$$\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1}, \pi) = \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } \exists \chi \in X^{\mathrm{un}}(J), \pi \simeq \chi \cdot \mathrm{St}_J, \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove this proposition, we need a few general facts about restriction of smooth representations to normal subgroups. Let G be a locally profinite group and let H be a closed normal subgroup. If (σ, W) is a representation of H, for $g \in G$ we define the representation (σ^g, W) by $\sigma^g : h \mapsto \sigma(g^{-1}hg)$. The representation σ is irreducible if and only if σ^g is for any (or for all) $g \in G$.

Lemma. *Assume that $\mathrm{Z}(G)H$ has finite index in G.*

(1) *Let π be a smooth irreducible admissible representation of G. There exists a smooth irreducible representation σ of H, an integer $r \geqslant 1$ and $g_1, \ldots, g_r \in G$ such that*

$$\pi_{|H} \simeq \sigma^{g_1} \oplus \ldots \oplus \sigma^{g_r}.$$

Moreover $r \leqslant [\mathrm{Z}(G)H : G]$, and for any $g \in G$ there exists some $1 \leqslant i \leqslant r$ such that $\sigma^g \simeq \sigma^{g_i}$.

(2) *Assume furthermore that G/H is abelian. Let π_1 and π_2 be two smooth admissible irreducible representations of G. The three following statements are equivalent.*
   - $(\pi_1)_{|H} \simeq (\pi_2)_{|H}$.
   - *There exists a smooth character χ of G which is trivial on H such that $\pi_2 \simeq \chi \cdot \pi_1$.*
   - $\mathrm{Hom}_H(\pi_1, \pi_2) \neq 0$.

(3) *Assume that G/H is abelian and that $[\mathrm{Z}(G)H : G] = 2$. Let $g_0 \in G \backslash \mathrm{Z}(G)H$ and let π be a smooth admissible irreducible representation of G. If there exists an irreducible representation σ of H such that $\pi_{|H} \simeq \sigma \oplus \sigma^{g_0}$, then $\sigma \not\simeq \sigma^{g_0}$.*

Proof. For (1) and (2), we refer to [Ren09] VI.3.2 Proposition. The result there is stated in the context of a p-adic group G with normal subgroup $H = {}^0G$ such that $G/{}^0G \simeq \mathbb{Z}^d$ for some $d \geqslant 0$, but the same arguments work as verbatim in the generality of the lemma. Admissibility of the representations involved is assumed only in order to apply Schur's lemma, insuring for instance the existence of central characters of smooth irreducible representations. In particular, if G/K is at most countable for any open compact subgroup K of G, then it is not necessary to assume admissibility.

Let us prove (3). Assume towards a contradiction that $\pi_{|H} \simeq \sigma \oplus \sigma^{g_0}$ and that $\sigma \simeq \sigma^{g_0}$. We build a smooth admissible irreducible representation Π of G such that $\Pi_{|H} = \sigma$, which results in a contradiction in regards to (2) since $\mathrm{Hom}_H(\Pi, \pi) \neq 0$ but $\Pi_{|H} \not\simeq \pi_{|H}$. Let χ be the central character of π. Then $\chi_{|\mathrm{Z}(G)\cap H}$ coincides with the central character of σ.

Let W denote the underlying vector space of σ. By hypothesis, there exists a linear automorphism $f : W \to W$ such that for every $h \in H$ and $w \in W$,

$$f(\sigma(g_0^{-1} h g_0) \cdot w) = \sigma(h) \circ f(w).$$

Let us write $g_0^2 = z_0 h_0$ for some $z_0 \in \mathrm{Z}(G)$ and $h_0 \in H$. We define $\varphi := f^2 \circ \sigma(h_0)^{-1}$. Then for all $h \in H$ and $w \in W$, we have

$$\begin{aligned} \varphi(\sigma(h) \cdot w) = f^2(\sigma(h_0^{-1} h) \cdot w) &= f^2(\sigma(h_0^{-1} h h_0)\sigma(h_0^{-1}) \cdot w) \\ &= f^2(\sigma(g_0^{-2} h g_0^2)\sigma(h_0^{-1}) \cdot w) \\ &= \sigma(h) \circ f^2(\sigma(h_0)^{-1} \cdot w) \\ &= \sigma(h) \circ \varphi(w). \end{aligned}$$

Thus $\varphi : \sigma \xrightarrow{\sim} \sigma$. By Schur's lemma we have $\varphi = \lambda \cdot \mathrm{id}$ for some $\lambda \in \overline{\mathbb{Q}_\ell}$. Up to replacing f by $(\chi(z_0)\lambda^{-1})^{1/2} f$, we may assume that $\varphi = \chi(z_0) \cdot \mathrm{id}$, ie. $f^2 = \chi(z_0)\sigma(h_0)$.

We build a G-representation Π on W which extends σ. Let $g \in G$ and define

$$\Pi(g) = \begin{cases} \chi(z)\sigma(h) & \text{if } g = zh \in Z(G)H, \\ \chi(z) f \circ \sigma(h) & \text{if } g = g_0 z h \in g_0 Z(G)H. \end{cases}$$

Then one may check that Π is a well defined group morphism $G \to \mathrm{GL}(W)$. The fact that it is smooth irreducible and admissible follows from $\Pi_{|H} \simeq \sigma$ by construction, and it concludes the proof. □

Remark. Under the hypotheses of (3), as long as σ is a smooth irreducible admissible representation of H such that $\sigma^{g_0} \simeq \sigma$ and whose central character $\chi_{|\mathrm{Z}(G)\cap H}$ can be extended to a character of $\mathrm{Z}(G)$, then one may build Π as in the proof of the lemma.

We may now move on to the proof of the proposition.

Proof. By Frobenius reciprocity we have

$$\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J\,\mathbf{1},\pi) \simeq \mathrm{Ext}^1_{J^\circ}(\mathbf{1},\pi_{|J^\circ}).$$

By functoriality of Ext, we have $\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\pi_{|J^\circ}) = 0$ if the central character of π is not unramified. Thus, let us now assume that it is unramified. According to 3.1.3.4, we have $J/J^\circ \simeq \mathbb{Z}$, and $\mathrm{Z}(J)J^\circ = J$ when n is odd, and is of index 2 in J when n is even. Thus, $\pi_{|J^\circ}$ is irreducible when n is odd, and can either be irreducible, either decompose as $\sigma \oplus \sigma^{g_0}$ for some irreducible representation σ of J° such that $\sigma^{g_0} \not\simeq \sigma$ when n is even. Here, g_0 may be defined as in 3.1.1.7. Thus, we are reduced to computing $\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\sigma)$ for any irreducible representation σ of J° with trivial central character. Let $J^1 = \mathrm{U}(\mathbf{V})$ denote the unitary group of $\mathbf{V}$ (recall that $J = \mathrm{GU}(\mathbf{V})$ is the group of unitary similitudes). Then J^1 is a normal subgroup both of J° and of J. Moreover, J°/J^1 is isomorphic to the image of the multiplier $c_{|J^\circ}: J^\circ \to \mathbb{Z}_p^\times$, in particular it is compact. Thus, we have

$$\mathrm{Ext}^1_{J^\circ}(\mathbf{1},\sigma) \simeq \mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1})^{J^\circ/J^1}.$$

Since σ has trivial central character, the J°-action on $\mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1})$ is actually trivial on $\mathrm{Z}(J^\circ)J^1$. But this group is equal to the whole of J°. Indeed, let $g \in J^\circ$. Since $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ is unramified, there exists some $\lambda \in \mathbb{Z}_{p^2}^\times$ such that $\mathrm{Norm}(\lambda) = c(g)$. Thus $c(\lambda^{-1}g) = 1$ so that g is the product of $\lambda\cdot\mathrm{id} \in \mathrm{Z}(J^\circ)$ and of an element of J^1. Hence, J° acts trivially on $\mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1})$. Since J^1 is an algebraic group, we may use Theorem 2 of [NP20], a generalization of a duality theorem of Schneider and Stühler, to finish the computation. Namely, we have

$$\mathrm{Ext}^1_{J^1}(\mathbf{1},\sigma_{|J^1}) \simeq \mathrm{Hom}_{J^1}(\sigma_{|J^1}, D(\mathbf{1}))^\vee,$$

where D denotes the Aubert-Zelevinsky involution in J^1. We note that $D(\mathbf{1}) = \mathrm{St}_{J^1}$ is the Steinberg representation of J^1.

Let us justify that the restriction of St_J to J^1 is equal to St_{J^1}. The Steinberg representation St_J (resp. St_{J^1}) can be characterized as the unique irreducible representation ρ of J (resp. of J^1) such that $\mathrm{Ext}^2_J(\mathbf{1},\rho) \neq 0$ (resp. $\mathrm{Ext}^1_{J^1}(\mathbf{1},\rho) \neq 0$). The gap between the degrees of the Ext groups for J and for J^1 is explained by the non-compactness of the center of J. Since St_J has trivial central character, by [NP20] Proposition 3.4 we have

$$\mathrm{Ext}^2_J(\mathbf{1},\mathrm{St}_J) \simeq \mathrm{Ext}^1_{J,\mathbf{1}}(\mathbf{1},\mathrm{St}_J) \oplus \mathrm{Ext}^2_{J,\mathbf{1}}(\mathbf{1},\mathrm{St}_J),$$

where the Ext groups on the right-hand side are taken in the category of smooth representations of J on which the center acts trivially. Equivalently, this is the category of smooth representations of $J/\mathrm{Z}(J)$. Consider the normal subgroup $\mathrm{Z}(J)J^1/\mathrm{Z}(J) \simeq J^1/\mathrm{Z}(J)\cap J^1 = J^1/\mathrm{Z}(J^1)$, with quotient isomorphic to $J/\mathrm{Z}(J)J^1$, which is trivial if n is odd and $\mathbb{Z}/2\mathbb{Z}$ is n is even. Thus, we have

$$\begin{aligned}\mathrm{Ext}^{\bullet}_{J,\mathbf{1}}(\mathbf{1}, \mathrm{St}_J) &\simeq \mathrm{Ext}^{\bullet}_{J/\mathrm{Z}(J)}(\mathbf{1}, \mathrm{St}_J)\\ &\simeq \mathrm{Ext}^{\bullet}_{J^1/\mathrm{Z}(J^1)}(\mathbf{1}, (\mathrm{St}_J)_{|J^1})^{J/\mathrm{Z}(J)J^1}\\ &\simeq \mathrm{Ext}^{\bullet}_{J^1,\mathbf{1}}(\mathbf{1}, (\mathrm{St}_J)_{|J^1})^{J/\mathrm{Z}(J)J^1}\\ &\simeq \mathrm{Ext}^{\bullet}_{J^1}(\mathbf{1}, (\mathrm{St}_J)_{|J^1})^{J/\mathrm{Z}(J)J^1},\end{aligned}$$

the last line following from the same Proposition 3.4 as above, but applied to J^1. In [Far04] Lemme 4.4.12, it is explained that $\mathrm{Ext}^i_{J^1}(\pi_1, \pi_2)$ vanishes for any smooth representations π_1, π_2 of J^1 as soon as i is greater than the semisimple rank of J, that is 1 in our case. Hence, $\mathrm{Ext}^2_{J,\mathbf{1}}(\mathbf{1}, \mathrm{St}_J) = 0$ and we have

$$\mathrm{Ext}^2_J(\mathbf{1}, \mathrm{St}_J) \simeq \mathrm{Ext}^1_{J,\mathbf{1}}(\mathbf{1}, \mathrm{St}_J) \simeq \mathrm{Ext}^1_{J^1}(\mathbf{1}, (\mathrm{St}_J)_{|J^1})^{J/\mathrm{Z}(J)J^1}.$$

In particular, the right-hand side is non zero, which proves that $(\mathrm{St}_J)_{|J^1}$ contains St_{J^1}. If n is odd so that $\mathrm{Z}(J)J^1 = J$, it follows that $(\mathrm{St}_J)_{|J^1} = \mathrm{St}_{J^1}$. If n is even, in virtue of point (3) of the lemma, it remains to justify that for any $g \in J$ we have $\mathrm{St}^g_{J^1} \simeq \mathrm{St}_{J^1}$. This follows from the following computation

$$\mathrm{Ext}^1_{J^1}(\mathbf{1}, \mathrm{St}^g_{J^1}) = \mathrm{Ext}^1_{J^1}(\mathbf{1}^{g^{-1}}, \mathrm{St}_{J^1}) = \mathrm{Ext}^1_{J^1}(\mathbf{1}, \mathrm{St}_{J^1}) \neq 0.$$

Let us go back to the irreducible representation π of J with unramified central character. Summing up the previous paragraphs, we have that $\pi_{|J^1}$ contains St_{J^1} if and only if $\pi \simeq \chi \cdot \mathrm{St}_J$ for some character χ of J that is trivial on J^1 (and thus trivial on $\mathrm{Z}(J^\circ)J^1 = J^\circ$ by the unramifiedness of the central character), and

$$\mathrm{Ext}^1_J(\mathrm{c}-\mathrm{Ind}_{J^\circ}^J \mathbf{1}, \pi) \simeq \mathrm{Hom}_{J^1}(\sigma_{|J^1}, \mathrm{St}_{J^1})^\vee \simeq \begin{cases} \overline{\mathbb{Q}_\ell} & \text{if } \pi_{|J^1} \simeq \mathrm{St}_{J^1},\\ 0 & \text{otherwise.}\end{cases}$$

□

3.5.2.3 We may now compute the cohomology of the basic stratum. Recall the supercuspidal representation τ_1 of the Levi complement $M_1 \subset J$ that we defined in **??**. When $n = 3$ or 4, we actually have $M_1 = J$ and

$$\tau_1 = \mathrm{c}-\mathrm{Ind}^J_{\mathrm{N}_J(J_1)} \widetilde{\rho_{\Delta_2}}$$

is a supercuspidal representation of J, where $\mathrm{N}_J(J_1) = \mathrm{Z}(J)J_1$ (see 3.1.3.3) and $\widetilde{\rho_{\Delta_2}}$ is the inflation of ρ_{Δ_2} to $\mathrm{N}_J(J_1) = \mathrm{Z}(J)J_1$ (see 3.1.3.3) obtained by letting the center act trivially. We use the same notations as in 3.5.1.5.

Theorem. *There are $G(\mathbb{A}_f^p) \times W$-equivariant isomorphisms*

$$\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}],$$

$$\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}],$$

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

Proof. The statement regarding $\mathrm{H}_c^0(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ was already proved in 3.5.1.5.
Let us prove the statement regarding $\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$ first. By 3.5.2.1, we have

$$\mathrm{H}_c^2(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq F_2^{0,2} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J \left(E_2^{0,b}(1-n), \Pi_p \right) \otimes \Pi^p,$$

where $b = 2$ if $n = 3$ and $b = 4$ if $n = 4$. The term $E_2^{0,b}$ is isomorphic to $\mathrm{c} - \mathrm{Ind}_{J_1}^J \mathbf{1}$. Therefore, by Frobenius reciprocity we have

$$\mathrm{Hom}_J \left(E_2^{0,b}(1-n), \Pi_p \right) \simeq \mathrm{Hom}_{J_1} \left(\mathbf{1}(1-n), \Pi_p \right).$$

Hence, only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ with $\Pi_p^{J_1} \neq 0$ contribute to $F_2^{0,2}$. Such a representation Π_p is said to be J_1**-spherical**. Since J_1 is a special maximal compact subgroup of J, according to [**minguez**] 2.1, we have $\dim(\pi^{J_1}) = 1$ for every smooth irreducible J_1-spherical representation π of J. The result follows using 3.5.1.4 to describe the eigenvalues of Frob.

We now prove the statement regarding $\mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. By the Hochschild-Serre spectral sequence, there exists a $G(\mathbb{A}_f^p) \times W$-subspace V' of this cohomology group such that

$$V' \simeq F_2^{1,0} \quad \text{and} \quad \mathrm{H}_c^1(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})/V' \simeq F_2^{0,1}.$$

We have

$$\begin{aligned} F_2^{1,0} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^1 \left(\mathrm{H}_c^{2(n-1)}(\mathcal{M}^{\mathrm{an}}, \overline{\mathbb{Q}_\ell})(1-n), \Pi_p \right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Ext}_J^1 \left(\mathrm{c} - \mathrm{Ind}_{J^\circ}^J \mathbf{1}(1-n), \Pi_p \right) \otimes \Pi^p \\ &\simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}], \end{aligned}$$

according to 3.5.2.2, and with the eigenvalues of Frob being given by 3.5.1.4.
On the other hand, we have

$$F_2^{0,1} \simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J \left(E_2^{0,2(n-1)-1}(1-n), \Pi_p \right) \otimes \Pi^p.$$

By 3.5.1.4, Frob acts on a summand of $F_2^{0,1}$ by the scalar $-\delta_{\Pi_p} p^{w(\xi)+1}$. Since $\mathrm{Frob}_{|V'}$ has no eigenvalue of complex modulus $p^{w(\xi)+1}$, the quotient actually splits so that $F_2^{0,1}$ is naturally a subspace of $\mathrm{H}^1_c(\overline{\mathrm{S}}(b_0) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi})$. It remains to compute it.
We have

$$E_2^{0,2(n-1)-1} \simeq \mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2},$$

with τ acting like multiplication by $-p^3$ when $n = 3$ and by $-p^5$ when $n = 4$, and $\Delta_2 = (2, 1)$ is the partition of $2m + 1 = 3$ defined in 3.2.7. Hence, we have an isomorphism

$$\begin{aligned} F_2^{0,1} &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_J\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2}(1-n), \Pi_p\right) \otimes \Pi^p \\ &\simeq \bigoplus_{\Pi \in \mathcal{A}_\xi(I)} \mathrm{Hom}_{J_1}\left(\rho_{\Delta_2}(1-n), \Pi_{p|J_1}\right) \otimes \Pi^p. \end{aligned}$$

It follows that only the automorphic representations $\Pi \in \mathcal{A}_\xi(I)$ whose p-component Π_p contains the supercuspidal representation ρ_{Δ_2} when restricted to J_1, contribute to the sum. According to 3.4.2.7, such Π_p are precisely those of the form $\chi \cdot \tau_1$ for some $\chi \in X^{\mathrm{un}}(J)$. By the Mackey formula we have

$$\begin{aligned} \mathrm{Hom}_J\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2}, \chi \cdot \tau_1\right) &\simeq \mathrm{Hom}_{J_1}\left(\rho_{\Delta_2}, \tau_{1|J_1}\right) \\ &\simeq \mathrm{Hom}_{J_1}\left(\rho_{\Delta_2}, (\mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J} \widetilde{\rho_{\Delta_2}})_{|J_1}\right) \\ &\simeq \bigoplus_{h \in J_1 \backslash J / \mathrm{N}_J(J_1)} \mathrm{Hom}_{J_1 \cap {}^h\mathrm{N}_J(J_1)}(\rho_{\Delta_2}, {}^h\widetilde{\rho_{\Delta_2}}), \end{aligned}$$

where in the last formula we omitted to write the restrictions to $J_1 \cap {}^h\mathrm{N}_J(J_1)$. We used the fact that $\chi_{|J_1}$ is trivial. Since $\widetilde{\rho_{\Delta_2}}$ is just the inflation of ρ_{Δ_2} from J_1 to $\mathrm{N}_J(J_1) = \mathrm{Z}(J)J_1$ obtained by letting $\mathrm{Z}(J)$ act trivially, we have a bijection

$$\mathrm{Hom}_{J_1 \cap {}^h\mathrm{N}_J(J_1)}(\rho_{\Delta_2}, {}^h\widetilde{\rho_{\Delta_2}}) \simeq \mathrm{Hom}_{\mathrm{N}_J(J_1) \cap {}^h\mathrm{N}_J(J_1)}(\widetilde{\rho_{\Delta_2}}, {}^h\widetilde{\rho_{\Delta_2}}).$$

Now, $\mathrm{N}_J(J_1)$ contains the center, is compact modulo the center, and $\tau_1 = \mathrm{c}-\mathrm{Ind}_{\mathrm{N}_J(J_1)}^{J} \widetilde{\rho_{\Delta_2}}$ is supercuspidal. It follows that an element $h \in J$ intertwines $\widetilde{\rho_{\Delta_2}}$ if and only if $h \in \mathrm{N}_J(J_1)$ (see for instance [**bushnellbook**] 11.4 Theorem along with Remarks 1 and 2). Therefore, only the trivial double coset contributes to the sum and we have

$$\mathrm{Hom}_J\left(\mathrm{c}-\mathrm{Ind}_{J_1}^{J} \rho_{\Delta_2}, \chi \cdot \tau_1\right) \simeq \mathrm{Hom}_{J_1}(\rho_{\Delta_2}, \rho_{\Delta_2}) \simeq \overline{\mathbb{Q}_\ell}.$$

To sum up, we have

$$F_2^{0,1} \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}].$$

It concludes the proof. □

3.5.3 On the cohomology of the ordinary locus when $n = 3$

3.5.3.1 In this section, we assume that the Shimura variety is of Kottwitz-Harris-Taylor type. According to [HT01] I.7, it amounts to assuming that the algebra B from 3.3.1 is a division

algebra satisfying a few additional conditions. In particular, B_v is either split either a division algebra for every place v of $\mathbb{Q}$, and there must be at least one prime number p' (different from p) which splits in E and such that B splits over p'. In this situation, the Shimura variety is compact.
According to 3.3.5, when $n = 3$ there is a single Newton stratum other than the basic one. It is the μ-ordinary locus $\overline{\mathrm{S}}_{K^p}(b_1)$, and it is an open dense subscheme of the special fiber of the Shimura variety. Moreover, since the Shimura variety is compact, the ordinary locus is also an affine scheme according to [**goldringnicole**] and [**koskivirtawedhorn**]. By using the spectral sequence associated to the stratification

$$\overline{\mathrm{S}}_{K^p} = \overline{\mathrm{S}}_{K^p}(b_0) \sqcup \overline{\mathrm{S}}_{K^p}(b_1),$$

we may deduce information on the cohomology of the ordinary locus. The spectral sequence is given by

$$G_1^{a,b} : \mathrm{H}_c^b(\overline{\mathrm{S}}_{K^p}(b_a) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \implies \mathrm{H}_c^{a+b}(\mathrm{S}_{K^p} \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}).$$

In figure 5, we draw the first page of this sequence.

$$\begin{array}{ccc} & & \mathrm{H}_c^4(\overline{\mathrm{S}}_{K^p}(b_1) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \\ \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) & \xrightarrow{\phi} & \mathrm{H}_c^3(\overline{\mathrm{S}}_{K^p}(b_1) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \\ \mathrm{H}_c^1(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) & \xrightarrow{\psi} & \mathrm{H}_c^2(\overline{\mathrm{S}}_{K^p}(b_1) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) \\ \mathrm{H}_c^0(\overline{\mathrm{S}}_{K^p}(b_0) \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell}) & & \end{array}$$

Figure 5: The first page G_1.

3.5.3.2 Let v be a place of E above p'. The cohomology of the Shimura variety $\mathrm{Sh}_{C_0K^p} \otimes_E E_v$ has been entirely computed in [Boy10]. Note that as $G(\mathbb{A}_f^p)$-representations, the cohomology of $\mathrm{Sh}_{C_0K^p} \otimes_E E_v$ is isomorphic to the cohomology of $\mathrm{Sh}_{C_0K^p} \otimes_E \mathbb{Q}_{p^2}$, which in turn is isomorphic to the cohomology of the special fiber $\overline{\mathrm{S}}_{K^p}$ using nearby cycles. In particular, we understand perfectly the abutment of the spectral sequence $G_1^{a,b}$. Since $\overline{\mathrm{S}}_{K^p}$ is smooth and projective, its cohomology admits a symmetry with respect to the middle degree 2. Moreover, by the results of loc. cit. the groups of degree 1 and 3 are zero. It follows that ϕ is surjective and ψ is injective. Combining with our computations, we deduce the following proposition.

Proposition. *There is a $G(\mathbb{A}_f^p) \times W$-equivariant isomorphism*

$$\mathrm{H}_c^4(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \simeq \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p \in X^{\mathrm{un}}(J)}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+4}].$$

There is a $G(\mathbb{A}_f^p) \times W$*-equivariant monomorphism*

$$\mathrm{H}_c^3(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}) \hookrightarrow \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \Pi_p^{J_1} \neq 0}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)+2}].$$

There is a $G(\mathbb{A}_f^p) \times W$*-equivariant monomorphism*

$$\bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \mathrm{St}_J}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[\delta_{\Pi_p} p^{w(\xi)}] \oplus \bigoplus_{\substack{\Pi \in \mathcal{A}_\xi(I) \\ \exists \chi \in X^{\mathrm{un}}(J), \\ \Pi_p = \chi \cdot \tau_1}} \Pi^p \otimes \overline{\mathbb{Q}_\ell}[-\delta_{\Pi_p} p^{w(\xi)+1}] \hookrightarrow \mathrm{H}_c^2(\overline{\mathrm{S}}(b_1) \otimes \mathbb{F}, \overline{\mathcal{L}_\xi}).$$

4 On the cohomology of a closed Bruhat-Tits stratum in the ramified PEL unitary Rapoport-Zink space of signature $(1, n-1)$

Notations

Throughout the chapter, we fix an integer $n \geqslant 1$ and an odd prime number p. If k is a perfect field of characteristic p, we denote by $\sigma : x \mapsto x^p$ the Frobenius of $\mathrm{Aut}(k/\mathbb{F}_p)$. If $q = p^e$ is a power of p, we write $\mathbb{F}_q$ for the field with q elements. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$.

4.1 The closed Deligne-Lusztig variety isomorphic to a closed Bruhat-Tits stratum

4.1.1 Let q be a power of p and let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$, together with a split $\mathbb{F}_q$-structure given by a geometric Frobenius morphism F. For $\mathbf{H}$ any F-stable subgroup of $\mathbf{G}$, we write $H := \mathbf{H}^F$ for its group of $\mathbb{F}_q$-rational points. Let $(\mathbf{T}, \mathbf{B})$ be a pair consisting of a maximal F-stable torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$. Let $(\mathbf{W}, \mathbf{S})$ be the associated Coxeter system, where $\mathbf{W} = \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. Since the $\mathbb{F}_q$-structure on $\mathbf{G}$ is split, the Frobenius F acts trivially on $\mathbf{W}$. For $I \subset \mathbf{S}$, let $\mathbf{P}_I, \mathbf{U}_I, \mathbf{L}_I$ be respectively the standard parabolic subgroup of type I, its unipotent radical and its unique Levi complement containing $\mathbf{T}$. Let $\mathbf{W}_I$ be the subgroup of $\mathbf{W}$ generated by I.
For $\mathbf{P}$ any parabolic subgroup of $\mathbf{G}$, the associated **generalized parabolic Deligne-Lusztig variety** is

$$X_{\mathbf{P}} := \{g\mathbf{P} \in \mathbf{G}/\mathbf{P} \mid g^{-1}F(g) \in \mathbf{P}F(\mathbf{P})\}.$$

We say that the variety is **classical** (as opposed to generalized) when in addition the parabolic subgroup $\mathbf{P}$ contains an F-stable Levi complement. Note that $\mathbf{P}$ itself needs not be F-stable.
We may give an equivalent definition using the Coxeter system $(\mathbf{W}, \mathbf{S})$. For $I \subset \mathbf{S}$, let ${}^I\mathbf{W}^I$ be the set of elements $w \in \mathbf{W}$ which are I-reduced-I. For $w \in {}^I\mathbf{W}^I$, the associated generalized parabolic Deligne-Lusztig variety is

$$X_I(w) := \{g\mathbf{P}_I \in \mathbf{G}/\mathbf{P}_I \mid g^{-1}F(g) \in \mathbf{P}_I w F(\mathbf{P}_I)\}.$$

The variety $X_I(w)$ is classical when $w^{-1}Iw = I$, and it is defined over $\mathbb{F}_q$. The dimension is given by $\dim X_I(w) = l(w)$ where $l(w)$ denotes the length of w with respect to $\mathbf{S}$.

4.1.2 Let $\mathbf{G}$ and $\mathbf{G}'$ be two reductive connected group over $\mathbb{F}$ both equipped with an $\mathbb{F}_q$-structure. We denote by F and F' the respective Frobenius morphisms. Let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy, that is a homomorphism defined over $\mathbb{F}_q$ whose kernel is contained in the center of $\mathbf{G}$ and whose image contains the derived subgroup of $\mathbf{G}'$. Then, according to [DM14] proof of Proposition 11.3.8, we have $\mathbf{G}' = f(\mathbf{G})\mathrm{Z}(\mathbf{G}')^0$, where $\mathrm{Z}(\mathbf{G}')^0$ is the connected component of unity of the center of $\mathbf{G}'$. Thus intersecting with $f(\mathbf{G})$ defines a bijection between parabolic subgroups of $\mathbf{G}'$ and those of $f(\mathbf{G})$. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{P}' = f(\mathbf{P})\mathrm{Z}(\mathbf{G}')^0$ be the corresponding parabolic of $\mathbf{G}'$. Then the map $g\mathbf{P} \mapsto f(g\mathbf{P})$ induces an isomorphism $f : X_{\mathbf{P}} \xrightarrow{\sim} X_{\mathbf{P}'}$ which is compatible with the actions of G and G' via f. Therefore $\mathbf{G}$ and $\mathbf{G}'$ generate the same Deligne-Lusztig varieties.

4.1.3 Let $\theta \geqslant 0$ and let V be a 2θ-dimensional $\mathbb{F}_q$-vector space equipped with a non-degenerate symplectic form $(\cdot,\cdot) : V \times V \to \mathbb{F}_q$. Fix a basis $(e_1, \ldots, e_{2\theta})$ in which $(\cdot,\cdot)$ is described by the matrix

$$\begin{pmatrix} 0 & A_\theta \\ -A_\theta & 0 \end{pmatrix},$$

where A_θ denotes the matrix having 1 on the anti-diagonal and 0 everywhere else. If k is a perfect field extension of $\mathbb{F}_q$, let $V_k := V \otimes_{\mathbb{F}_q} k$ denote the scalar extension to k equipped with its induced k-symplectic form $(\cdot,\cdot)$. Let $\tau : V_k \xrightarrow{\sim} V_k$ denote the map $\mathrm{id} \otimes \sigma$. If $U \subset V_k$, let $U^\perp$ denote its orthogonal.
We consider the finite symplectic group $\mathrm{Sp}(V, (\cdot,\cdot)) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)$. It can be identified with $G = \mathbf{G}^F$ where $\mathbf{G}$ is the symplectic group $\mathrm{Sp}(V_{\mathbb{F}}, (\cdot,\cdot)) \simeq \mathrm{Sp}(2\theta, \mathbb{F})$ and F is the Frobenius raising the entries of a matrix to their q-th power. Let $\mathbf{T} \subset \mathbf{G}$ be the maximal torus of diagonal symplectic matrices and let $\mathbf{B} \subset \mathbf{G}$ be the Borel subgroup of upper-triangular symplectic matrices. The Weyl system of $(\mathbf{T}, \mathbf{B})$ is identified with $(W_\theta, \mathbf{S})$ where W_θ is the finite Coxeter group of type B_θ and $\mathbf{S} = \{s_1, \ldots, s_\theta\}$ is the set of simple reflexions. They satisfy the following relations

$$s_\theta s_{\theta-1} s_\theta s_{\theta-1} = s_{\theta-1} s_\theta s_{\theta-1} s_\theta, \qquad s_i s_{i-1} s_i = s_{i-1} s_i s_{i-1}, \qquad \forall\, 2 \leqslant i \leqslant \theta - 1,$$
$$s_i s_j = s_j s_i, \qquad \forall\, |i-j| \geqslant 2.$$

Concretely, the simple reflexion s_i acts on V by exchanging e_i and e_{i+1} as well as $e_{2\theta-i}$ and $e_{2\theta-i+1}$ for $1 \leqslant i \leqslant \theta - 1$, whereas s_θ exchanges e_θ and $e_{\theta+1}$. The Frobenius F acts trivially on W_θ.

4.1.4 We define the following subset of $\mathbf{S}$

$$I := \{s_1, \ldots, s_{\theta-1}\} = \mathbf{S} \backslash \{s_\theta\}.$$

We consider the generalized Deligne-Lusztig variety $X_I(s_\theta)$. Since $s_\theta s_{\theta-1} s_\theta \notin I$, it is not a classical Deligne-Lusztig variety. Let $S_\theta := \overline{X_I(s_\theta)}$ be its closure in $\mathbf{G}/\mathbf{P}_I$. This normal projective variety occurs as a closed Bruhat-Tits stratum in the special fiber of the ramified unitary PEL Rapoport-Zink space of signature $(1, n-1)$, as established in [RTW14]. In loc. cit. the authors describe the geometry of S_θ. We summarize their analysis.

Proposition ([RTW14] 5.3, 5.4)**.** *Let k be a perfect field extension of $\mathbb{F}_q$. The k-rational points of S_θ are given by*

$$S_\theta(k) \simeq \{U \subset V_k \,|\, U^\perp = U \text{ and } U \overset{\leqslant 1}{\subset} U + \tau(U)\},$$

where $\overset{\leqslant 1}{\subset}$ denotes an inclusion of subspaces with index at most 1. There is a decomposition

$$S_\theta = X_I(\mathrm{id}) \sqcup X_I(s_\theta),$$

where $X_I(\mathrm{id})$ *is closed and of dimension* 0*, and* $X_I(s_\theta)$ *is open, dense of dimension* θ*. They correspond respectively to points* U *having* $U = \tau(U)$ *and* $U \subsetneq U + \tau(U)$*.*
If $\theta \geqslant 2$ *then* S_θ *is singular at the points of* $X_I(\mathrm{id})$*. When* $\theta = 1$*, we have* $S_1 \simeq \mathbb{P}^1$*.*

4.1.5 For $0 \leqslant \theta' \leqslant \theta$, define

$$I_{\theta'} := \{s_1, \ldots, s_{\theta-\theta'-1}\},$$

and $w_{\theta'} := s_{\theta+1-\theta'} \ldots s_\theta$. In particular $I_0 = I$, $I_{\theta-1} = I_\theta = \varnothing$, $w_0 = \mathrm{id}$ and $w_1 = s_\theta$.

Proposition ([RTW14] 5.5)**.** *There is a stratification into locally closed subvarieties*

$$S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'}).$$

The stratum $X_{I_{\theta'}}(w_{\theta'})$ *corresponds to points* U *such that* $\dim(U + \tau(U) + \ldots + \tau^{\theta'+1}(U)) = \theta + \theta'$*. The closure in* S_θ *of a stratum* $X_{I_{\theta'}}(w_{\theta'})$ *is the union of all the strata* $X_{I_t}(w_t)$ *for* $t \leqslant \theta'$*. The stratum* $X_{I_{\theta'}}(w_{\theta'})$ *is of dimension* θ'*, and* $X_{I_\theta}(w_\theta)$ *is open, dense and irreducible. In particular* S_θ *is irreducible.*

Remark. This stratification plays the role of the Ekedahl-Oort stratification $\mathcal{M}_\Lambda = \bigsqcup_t \mathcal{M}_\Lambda(t)$ of the closed Bruhat-Tits strata in the unramified case, see [VW11].

4.1.6 It turns out that the strata $X_{I_{\theta'}}(w_{\theta'})$ are related to Coxeter varieties for symplectic groups of smaller sizes. For $0 \leqslant \theta' \leqslant \theta$, define

$$K_{\theta'} := \{s_1, \ldots s_{\theta-\theta'-1}, s_{\theta-\theta'+1}, \ldots, s_\theta\} = \mathbf{S} \backslash \{s_{\theta-\theta'}\}.$$

Note that $K_0 = I_0 = I$ and $K_\theta = \mathbf{S}$. We have $I_{\theta'} \subset K_{\theta'}$ with equality if and only if $\theta' = 0$.

Proposition. *There is an* $\mathrm{Sp}(2\theta, \mathbb{F}_p)$*-equivariant isomorphism*

$$X_{I_{\theta'}}(w_{\theta'}) \simeq \mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}} \times_{L_{K_{\theta'}}} X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'}),$$

where $X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'})$ *is a Deligne-Lusztig variety for* $\mathbf{L}_{K_{\theta'}}$*. The zero-dimensional variety* $\mathrm{Sp}(2\theta, \mathbb{F}_q)/U_{K_{\theta'}}$ *has a left action of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ *and a right action of* $L_{K_{\theta'}}$*.*

Proof. It is similar to [Mul22b] Proposition 8. □

4.1.7 The Levi complement $\mathbf{L}_{K_{\theta'}}$ is isomorphic to $\mathrm{GL}(\theta - \theta') \times \mathrm{Sp}(2\theta')$, and its Weyl group is isomorphic to $\mathfrak{S}_{\theta-\theta'} \times W_{\theta'}$. Via this decomposition, the permutation $w_{\theta'}$ corresponds to $\mathrm{id} \times w_{\theta'}$. The Deligne-Lusztig variety $X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'})$ decomposes as a product

$$X_{I_{\theta'}}^{\mathbf{L}_{K_{\theta'}}}(w_{\theta'}) = X_{\mathbf{I}_{\theta'}}^{\mathrm{GL}(\theta-\theta')}(\mathrm{id}) \times X_{\varnothing}^{\mathrm{Sp}(2\theta')}(w_{\theta'}).$$

The variety $X_{\mathbf{I}_{\theta'}}^{\mathrm{GL}(\theta-\theta')}(\mathrm{id})$ is just a single point, but $X_{\varnothing}^{\mathrm{Sp}(2\theta')}(w_{\theta'})$ is the Coxeter variety for the symplectic group of size $2\theta'$. Indeed, $w_{\theta'}$ is a Coxeter element, ie. the product of all the simple reflexions of the Weyl group of $\mathrm{Sp}(2\theta')$.

4.2 Unipotent representations of the finite symplectic group

4.2.1 Recall that a (complex) irreducible representation of a finite group of Lie type $G = \mathbf{G}^F$ is said to be **unipotent**, if it occurs in the Deligne-Lusztig induction of the trivial representation of some maximal rational torus. Equivalently, it is unipotent if it occurs in the cohomology (with coefficient in $\overline{\mathbb{Q}_\ell}$ with $\ell \neq p$) of some Deligne-Lusztig variety of the form $X_{\mathbf{B}}$, with $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing a maximal rational torus.
Let $\mathbf{G}, \mathbf{G}'$ and let $f : \mathbf{G} \to \mathbf{G}'$ be an $\mathbb{F}_q$-isotypy as in 4.1.2. If $\mathbf{B}$ is such a Borel in $\mathbf{G}$, then $\mathbf{B}' := f(\mathbf{B})\mathrm{Z}(\mathbf{G}')^0$ is such a Borel in $\mathbf{B}'$, and f induces an isomorphism $X_{\mathbf{B}} \xrightarrow{\sim} X_{\mathbf{B}'}$ compatible with the actions. As a consequence, the map

$$\rho \mapsto f \circ \rho$$

defines a bijection between the sets of equivalence classes of unipotent representations of G' and of G. We will use this observation later in the case $\mathbf{G} = \mathrm{Sp}(2\theta)$ and $\mathbf{G}' = \mathrm{GSp}(2\theta)$, the symplectic group and the group of symplectic similitudes, the morphism f being the inclusion.

4.2.2 In this section, we recall the classification of the unipotent representations of the finite symplectic groups. The underlying combinatorics is described by Lusztig's notion of symbols. Our reference is [GM20] Section 4.4.

Definition. Let $\theta \geqslant 1$ and let d be an odd positive integer. The set of **symbols of rank** θ **and defect** d is

$$\mathcal{Y}^1_{d,\theta} := \left\{ S = (X,Y) \;\middle|\; \begin{matrix} X = (x_1, \ldots, x_{r+d}) \\ Y = (y_1, \ldots, y_r) \end{matrix} \text{ with } x_i, y_j \in \mathbb{Z}_{\geqslant 0}, \begin{matrix} x_{i+1} - x_i \geqslant 1, \\ y_{j+1} - y_j \geqslant 1, \end{matrix} \mathrm{rk}(S) = \theta \right\} \Big/ (\text{shift}),$$

where the shift operation is defined by $\mathrm{shift}(X,Y) := (\{0\} \sqcup (X+1), \{0\} \sqcup (Y+1))$, and where the rank of S is given by

$$\mathrm{rk}(S) := \sum_{s \in S} s - \left\lfloor \frac{(\#S-1)^2}{4} \right\rfloor.$$

Note that the formula defining the rank is invariant under the shift operation, therefore it is well defined. By [Lus77], we have $\mathrm{rk}(S) \geqslant \left\lfloor \frac{d^2}{4} \right\rfloor$ so in particular $\mathcal{Y}^1_{d,\theta}$ is empty for d big enough. We write $\mathcal{Y}^1_\theta$ for the union of the $\mathcal{Y}^1_{d,\theta}$ with d odd, this is a finite set.

Example. In general, a symbol $S = (X,Y)$ will be written

$$S = \begin{pmatrix} x_1 & \ldots & x_r & \ldots & x_{r+d} \\ y_1 & \ldots & y_r & & \end{pmatrix}.$$

We refer to X and Y as the first and second rows of S. The 6 elements of $\mathcal{Y}^1_2$ are given by

$$\begin{pmatrix} 2 \\ \; \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 \\ & & \end{pmatrix}.$$

The last symbol has defect 3 whereas all the other symbols have defect 1.

4.2.3 The symbols can be used to classify the unipotent representations of the finite symplectic group.

Theorem ([Lus77] Theorem 8.2). *There is a natural bijection between $\mathcal{Y}_\theta^1$ and the set of equivalence classes of unipotent representations of* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

If $S \in \mathcal{Y}_\theta^1$ we write ρ_S for the associated unipotent representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$. The classification is done so that the symbols

$$\begin{pmatrix} \theta \\ \end{pmatrix}, \qquad \begin{pmatrix} 0 & \dots & \theta-1 & \theta \\ 1 & \dots & \theta & \end{pmatrix},$$

correspond respectively to the trivial and the Steinberg representations.

4.2.4 Let $S = (X, Y)$ be a symbol and let $k \geqslant 1$. A k**-hook** h in S is an integer $z \geqslant k$ such that $z \in X, z-k \notin X$ or $z \in Y, z-k \notin Y$. A k**-cohook** c in S is an integer $z \geqslant k$ such that $z \in X, z-k \notin Y$ or $z \in Y, z-k \notin X$. The integer k is referred to as the **length** of the hook h or the cohook c, it is denoted $\ell(h)$ or $\ell(c)$. The **hook formula** gives an expression of $\dim(\rho_S)$ in terms of hooks and cohooks.

Proposition ([GM20] Proposition 4.4.17). *We have*

$$\dim(\rho_S) = q^{a(S)} \frac{\prod_{i=1}^{\theta} (q^{2i}-1)}{2^{b'(S)} \prod_h (q^{\ell(h)}-1) \prod_c (q^{\ell(c)}+1)},$$

where the products in the denominator run over all the hooks h and all the cohooks c in S, and the numbers $a(S)$ and $b'(S)$ are given by

$$a(S) = \sum_{\{s,t\} \subset S} \min(s,t) - \sum_{i \geqslant 1} \binom{\#S - 2i}{2}, \qquad b'(S) = \left\lfloor \frac{\#S-1}{2} \right\rfloor - \#\left(X \cap Y\right).$$

4.2.5 For $\delta \geqslant 0$, we define the symbol

$$S_\delta := \begin{pmatrix} 0 & \dots & 2\delta \\ & & \end{pmatrix} \in \mathcal{Y}^1_{2\delta+1, \delta(\delta+1)}.$$

Definition. The **core** of a symbol $S \in \mathcal{Y}^1_{d,\theta}$ is defined by $\mathrm{core}(S) := S_\delta$ where $d = 2\delta + 1$. We say that S is **cuspidal** if $S = \mathrm{core}(S)$.

Remark. In general, we have $\mathrm{rk}(\mathrm{core}(S)) \leqslant \mathrm{rk}(S)$ with equality if and only if S is cuspidal.

The next theorem states that cuspidal unipotent representations correspond to cuspidal symbols.

Theorem ([GM20] Theorem 4.4.28). *The group* $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ *admits a cuspidal unipotent representation if and only if $\theta = \delta(\delta+1)$ for some $\delta \geqslant 0$. When this is the case, the cuspidal unipotent representation is unique and given by ρ_{S_δ}.*

4.2.6 The determination of the cuspidal unipotent representations leads to a description of the unipotent Harish-Chandra series.

Definition. Let $\delta \geqslant 0$ such that $\theta = \delta(\delta+1) + a$ for some $a \geqslant 0$. We write

$$L_\delta \simeq \mathrm{GL}(1, \mathbb{F}_q)^a \times \mathrm{Sp}(2\delta(\delta+1), \mathbb{F}_q)$$

for the block-diagonal Levi complement in $\mathrm{Sp}(2\theta, \mathbb{F}_q)$, with one middle block of size $2\delta(\delta+1)$ and other blocks of size 1. We write $\rho_\delta := (\mathbf{1})^a \boxtimes \rho_{S_\delta}$, which is a cuspidal representation of L_δ.

Proposition ([GM20] Proposition 4.4.29)**.** *Let $S \in \mathcal{Y}^1_{\theta,d}$. The cuspidal support of ρ_S is (L_δ, ρ_δ) where $d = 2\delta + 1$.*

In particular, the defect of the symbol S of rank θ classifies the unipotent Harish-Chandra series of $\mathrm{Sp}(2\theta, \mathbb{F}_p)$.

4.2.7 As it will be needed later, we explain how to compute a Harish-Chandra induction of the form

$$\mathrm{R}_L^G \, \mathbf{1} \boxtimes \rho_{S'},$$

where $G = \mathrm{Sp}(2\theta, \mathbb{F}_q)$, L is a block-diagonal Levi complement of the form $L \simeq \mathrm{GL}(a, \mathbb{F}_q) \times \mathrm{Sp}(2\theta', \mathbb{F}_q)$ and $S' \in \mathcal{Y}^1_{d,\theta'}$ is a symbol.

Definition. Let $S = (X, Y) \in \mathcal{Y}^1_{d,\theta}$ and let h be a k-hook of S given by some integer z. Assume that $z \in X$ and $z - k \notin X$ (resp. $z \in Y$ and $z - k \notin Y$). The **leg length** of h is given by the number of integers $s \in X$ (resp. Y) such that $z - k < s < z$.
Consider the symbol $S' = (X', Y')$ obtained by deleting z and replacing it with $z - k$ in the same row. We say that S' is obtained from S by **removing a k-hook**, or equivalently that S is obtained from S' by **adding a k-hook**.

Theorem ([FS90] Statement 4.B')**.** *Let $S' = (X', Y') \in \mathcal{Y}^1_{d,\theta'}$. We have*

$$\mathrm{R}_L^G \, \mathbf{1} \boxtimes \rho_{S'} = \sum_S \rho_S$$

where S runs over all the symbols in $\mathcal{Y}^1_{d,\theta}$ such that, for some $a_1, a_2 \geqslant 0$ with $a = a_1 + a_2$, S is obtained from S' by adding an a_1-hook of leg length 0 to its first row and an a_2-hook of leg length 0 to its second row.

This computation is a consequence of the Howlett-Lehrer comparison theorem [HL83] as well as the Pieri rule for Coxeter groups of type B, see [GP00] 6.1.9. We will use it in concrete examples in the following sections.

4.2.8 There is a similar rule to compute Harish-Chandra restrictions. Let $0 \leqslant \theta' \leqslant \theta$ and consider the embedding $G' \hookrightarrow L \hookrightarrow G$ where $G' = \mathrm{Sp}(2\theta', \mathbb{F}_q), G = \mathrm{Sp}(2\theta, \mathbb{F}_q)$ and L is the block diagonal Levi complement $\mathrm{GL}(a, \mathbb{F}_q) \times \mathrm{Sp}(2\theta', \mathbb{F}_q)$ where $a = \theta - \theta'$. We write ${}^*\mathrm{R}_{G'}^G$ for the composition of the Harish-Chandra restriction functor ${}^*\mathrm{R}_L^G$ with the usual restriction from L to G'.

Theorem. *Let $S = (X, Y) \in \mathcal{Y}^1_{d,\theta}$. We have*

$$^*\mathrm{R}^G_{G'}\,\rho_S = \sum_{S'} \rho_{S'}$$

where S' runs over all the symbols in $\mathcal{Y}^1_{d,\theta'}$ such that, for some $a_1, a_2 \geqslant 0$ with $a = a_1 + a_2$, S' is obtained from S by removing an a_1-hook of leg length 0 to its first row and an a_2-hook of leg length 0 to its second row.

4.3 The cohomology of the Coxeter variety for the symplectic group

4.3.1 In this section we compute the cohomology of Coxeter varieties of finite symplectic groups, in terms of the classification of the unipotent characters that we recalled in 4.2.3.

Notation. We write $X^k := X_{\varnothing}(\mathrm{cox})$ for the Coxeter variety attached to the symplectic group $\mathrm{Sp}(2k, \mathbb{F}_q)$, and $\mathrm{H}^\bullet_c(X^k)$ instead of $\mathrm{H}^\bullet_c(X^k \otimes \mathbb{F}, \overline{\mathbb{Q}_\ell})$ where $\ell \neq p$.

We first recall known facts on the cohomology of X^k from Lusztig's work.

Theorem ([Lus76])**.** *The following statements hold.*

1. *The variety X^k has dimension k and is affine. The cohomology group $\mathrm{H}^i_c(X^k)$ is zero unless $k \leqslant i \leqslant 2k$.*
2. *The Frobenius F acts in a semisimple manner on the cohomology of X^k.*
3. *The groups $\mathrm{H}^{2k-1}_c(X^k)$ and $\mathrm{H}^{2k}_c(X^k)$ are irreducible as $\mathrm{Sp}(2k, \mathbb{F}_q)$-representations, and the latter is the trivial representation. The Frobenius F acts with eigenvalues respectively q^{k-1} and q^k.*
4. *The group $\mathrm{H}^{k+i}_c(X^k)$ for $0 \leqslant i \leqslant k-2$ is the direct sum of two eigenspaces of F, for the eigenvalues q^i and $-q^{i+1}$. Each eigenspace is an irreducible unipotent representation of $\mathrm{Sp}(2k, \mathbb{F}_q)$.*
5. *The sum $\bigoplus_{i \geqslant 0} \mathrm{H}^i_c(X^k)$ is multiplicity-free as a representation of $\mathrm{Sp}(2k, \mathbb{F}_q)$.*

In other words, there exists a uniquely determined family of pairwise distinct symbols $S^k_0, \ldots, S^k_k$ and $T^k_0, \ldots, T^k_{k-2}$ in $\mathcal{Y}^1_k$ such that

$$\forall 0 \leqslant i \leqslant k-2, \qquad \mathrm{H}^{k+i}_c(X^k) \simeq \rho_{S^k_i} \oplus \rho_{T^k_i},$$
$$\forall k-1 \leqslant i \leqslant k, \qquad \mathrm{H}^{k+1}_c(X^k) \simeq \rho_{S^k_i}.$$

The representation $\rho_{S^k_i}$ (resp. $\rho_{T^k_i}$) corresponds to the eigenspace of the Frobenius F on $\bigoplus_{i \geqslant 0} \mathrm{H}^i_c(X^k)$ attached to p^i (resp. to $-p^{i+1}$). Moreover, we know that $\rho_{S^k_k}$ is the trivial representation, therefore

$$S^k_k = \begin{pmatrix} k \\ \end{pmatrix}.$$

Lusztig also gives a formula computing the dimension of the eigenspaces. Specializing to the case of the symplectic group, it reduces to the following statement.

Proposition ([Lus76]). *For $0 \leqslant i \leqslant k$ we have*

$$\deg(\rho_{S_i^k}) = q^{(k-i)^2} \prod_{s=1}^{k-i} \frac{q^{s+i}-1}{q^s-1} \prod_{s=0}^{k-i-1} \frac{q^{s+i}+1}{q^s+1}.$$

For $0 \leqslant j \leqslant k-2$ we have

$$\deg(\rho_{T_j^k}) = q^{(k-j-1)^2} \frac{(q^{k-1}-1)(q^k-1)}{2(q+1)} \prod_{s=1}^{k-j-2} \frac{q^{s+j}-1}{q^s-1} \prod_{s=2}^{k-j-1} \frac{q^{s+j}+1}{q^s+1}.$$

4.3.2 Our goal in this section is to determine the symbols S_i^k and T_j^k explicitly. This is done in the following proposition.

Proposition. *For $0 \leqslant i \leqslant k$ and $0 \leqslant j \leqslant k-2$, we have*

$$S_i^k = \begin{pmatrix} 0 \dots k-i-1 & k \\ 1 \dots \quad k-i & \end{pmatrix}, \qquad T_j^k = \begin{pmatrix} 0 & \dots & k-j-3 & k-j-2 & k-j-1 & k \\ 1 & \dots & k-j-2 & & & \end{pmatrix}.$$

We note that the statement is coherent with the two dimension formulae that we provided earlier. That is, the degree of $\rho_{S_i^k}$ (resp. of $\rho_{T_j^k}$) computed with the hook formula 4.2.4, agrees with the dimension of the eigenspace of p^i (resp. of $-p^{j+1}$) in the cohomology of X^k as given in the previous paragraph.

Proof. We use induction on $k \geqslant 0$. Since we already know that S_k^k is the symbol corresponding to the trivial representation, the proposition is proved for $k = 0$. Thus we may assume $k \geqslant 1$. We consider the block diagonal Levi complement $L \simeq \mathrm{GL}(1, \mathbb{F}_q) \times \mathrm{Sp}(2(k-1), \mathbb{F}_q)$, and we write ${}^*\mathrm{R}_{k-1}^k$ for the composition of the Harish-Chandra restriction from $\mathrm{Sp}(2k, \mathbb{F}_q)$ to L, with the usual restriction from L to $\mathrm{Sp}(2(k-1), \mathbb{F}_q)$. As in the proof of [Mul22b] Proposition 19, for all $0 \leqslant i \leqslant k$ we have an $\mathrm{Sp}(2(k-1), \mathbb{F}_q) \times \langle F \rangle$-equivariant isomorphism

$${}^*\mathrm{R}_{k-1}^k \left(\mathrm{H}_c^{k+i}(X^k)\right) \simeq \mathrm{H}_c^{k-1+i}(X^{k-1}) \oplus \mathrm{H}_c^{k-1+(i-1)}(X^{k-1})(1). \qquad (*)$$

Here, (1) denotes the Tate twist. This recursive formula is established by Lusztig in [Lus76] Corollary 2.10. The right-hand side is known by induction hypothesis whereas the left-hand side can be computed using 4.2.8 Theorem. We establish the proposition by comparing the different eigenspaces of F on both sides.
If $S \in \mathcal{Y}_{d,k}^1$ is any symbol, the restriction ${}^*\mathrm{R}_{k-1}^k \, \rho_S$ is the sum of all the representations $\rho_{S'}$ where S' is obtained from S by removing a 1-hook from any of its rows.
We distinguish different cases depending on the values of k and i.

- **Case k = 1**. We only need to determine S_0^1. For $i = 0$, the right-hand side of $(*)$ is $\rho_{S_0^0}$ with eigenvalue 1. Thus, the symbol $S_0^1 \in \mathcal{Y}_1^1$ has defect 1 and admits only one 1-hook. If we remove this hook we obtain S_0^0. Therefore, S_0^1 must be one of the two following symbols

$$\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ \ \end{pmatrix}.$$

  By 4.3.1, we know that $\rho_{S_0^1}$ has degree q, thus S_0^1 must be equal to the former symbol.

From now, we assume $k \geqslant 2$ and we determine S_i^k for $0 \leqslant i < k$.

– **Case k = 2 and i = 0**. The eigenspace attached to 1 on the right-hand side of $(*)$ is $\rho_{S_0^1}$. Thus, the symbol $S_0^2 \in \mathcal{Y}_k^1$ has defect 1 and admits only one 1-hook. If we remove this hook we obtain S_0^1. Therefore, S_0^2 must be one of the two following symbols

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 2 & \end{pmatrix}.$$

By 4.3.1, we know that $\rho_{S_0^2}$ has degree q^4, thus S_0^2 must be equal to the former symbol.

– **Case k > 2 and i = 0**. The eigenspace attached to 1 on the right-hand side of $(*)$ is $\rho_{S_0^{k-1}}$. Thus, the symbol $S_0^k \in \mathcal{Y}_k^1$ has defect 1 and admits only one 1-hook. If we remove this hook we obtain S_0^{k-1}. The only such symbol is

$$S_0^k = \begin{pmatrix} 0 & \ldots & k-1 & k \\ 1 & \ldots & k & \end{pmatrix}.$$

– **Case 1 ⩽ i ⩽ k − 1**. The eigenspace attached to p^i on the right-hand side of $(*)$ is $\rho_{S_i^{k-1}} \oplus \rho_{S_{i-1}^{k-1}}$. Thus, the symbol $S_i^k \in \mathcal{Y}_k^1$ has defect 1 and admits only two 1-hooks. If we remove one of these hooks we obtain either S_i^{k-1} or S_{i-1}^{k-1}. The only such symbol is

$$S_i^k = \begin{pmatrix} 0 & \ldots & k-i-1 & k \\ 1 & \ldots & k-i & \end{pmatrix}.$$

It remains to determine T_j^k for $0 \leqslant j \leqslant k-2$.

– **Case k = 2**. The eigenspace attached to $-p$ on the right-hand side of $(*)$ is 0. Thus, the symbol $T_0^2 \in \mathcal{Y}_2^1$ has no hook at all, implying that it is cuspidal in the sense of 4.2.5. Since $\mathrm{Sp}(4, \mathbb{F}_q)$ admits only 1 unipotent cuspidal representation, we deduce that

$$T_0^2 = \begin{pmatrix} 0 & 1 & 2 \\ & & \end{pmatrix}.$$

– **Case k = 3**. First when $j = 0$, the eigenspace attached to $-p$ on the right-hand side of $(*)$ is $\rho_{T_0^2}$. Thus, the symbol $T_0^3 \in \mathcal{Y}_3^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^2. Therefore, T_0^3 must be one of the two following symbols

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & & & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 3 \\ & & \end{pmatrix}.$$

By 4.3.1, we know that $\rho_{T_0^3}$ has degree $q^4 \frac{(q^2-1)(q^3-1)}{2(q+1)}$, thus T_0^3 must be equal to the former symbol.

Then when $j = 1$, the eigenspace attached to $-p^2$ on the right-hand side of $(*)$ is $\rho_{T_0^2}$. Thus, the symbol $T_1^3 \in \mathcal{Y}_3^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^2. Thus T_1^3 is also one of the two symbols above. We can deduce that it is equal to the latter by comparing the dimensions or by using the fact that the symbols T_j^k are pairwise distinct.

From now, we assume $k \geqslant 4$ and we determine T_j^k for $0 \leqslant j \leqslant k-2$.

– **Case $\mathbf{k = 4}$ and $\mathbf{j = 0}$.** The eigenspace attached to $-p$ on the right-hand side of ($*$) is $\rho_{T_0^3}$. Thus, the symbol $T_0^4 \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^3. Therefore, T_0^4 must be one of the two following symbols

$$\begin{pmatrix} 0\ 1\ 2\ 3\ 4 \\ 1\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & & & \end{pmatrix}.$$

By 4.3.1, we know that $\rho_{T_0^4}$ has degree $q^9\frac{(q^3-1)(q^4-1)}{2(q+1)}$, thus T_0^4 must be equal to the former symbol.

– **Case $\mathbf{k > 4}$ and $\mathbf{j = 0}$.** The eigenspace attached to $-p$ on the right-hand side of ($*$) is $\rho_{T_0^{k-1}}$. Thus, the symbol $T_0^k \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_0^{k-1}. The only such symbol is

$$T_0^k = \begin{pmatrix} 0 \ldots k-3\ k-2\ k-1\ k \\ 1 \ldots k-2 \end{pmatrix}.$$

– **Case $\mathbf{k = 4}$ and $\mathbf{j = k-2}$.** The eigenspace attached to $-p^3$ on the right-hand side of ($*$) is $\rho_{T_1^3}$. Thus, the symbol $T_2^4 \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_1^3. Therefore, T_2^4 must be one of the two following symbols

$$\begin{pmatrix} 0\ 1\ 4 \\ \ \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 & 3 \\ & & \end{pmatrix}.$$

By 4.3.1, we know that $\rho_{T_2^4}$ has degree $q\frac{(q^3-1)(q^4-1)}{2(q+1)}$, thus T_2^4 must be equal to the former symbol.

– **Case $\mathbf{k > 4}$ and $\mathbf{j = k-2}$.** The eigenspace attached to $-p^{k-1}$ on the right-hand side of ($*$) is $\rho_{T_{k-3}^{k-1}}$. Thus, the symbol $T_{k-2}^k \in \mathcal{Y}_k^1$ has defect 3 and admits only one 1-hook. If we remove this hook we obtain T_{k-3}^{k-1}. The only such symbol is

$$T_{k-2}^k = \begin{pmatrix} 0\ 1\ k \\ \ \end{pmatrix}.$$

– **Case $\mathbf{1 \leqslant j \leqslant k-3}$.** The eigenspace attached to $-p^{j+1}$ on the right-hand side of ($*$) is $\rho_{T_j^{k-1}} \oplus \rho_{T_{j-1}^{k-1}}$. Thus, the symbol $T_j^k \in \mathcal{Y}_k^1$ has defect 3 and admits only two 1-hooks. If we remove one of these hooks we obtain either T_j^{k-1} or T_{j-1}^{k-1}. The only such symbol is

$$T_j^k = \begin{pmatrix} 0 \ldots k-j-3 & k-j-2 & k-j-1 & k \\ 1 \ldots k-j-2 & & & \end{pmatrix}.$$

□

4.4 On the cohomology of a closed Bruhat-Tits stratum

4.4.1 Recall from 4.1.4 the θ-dimensional normal projective variety $S_\theta := \overline{X_I(s_\theta)}$ defined over $\mathbb{F}_q$. It is equipped with an action of the finite symplectic group $\mathrm{Sp}(2\theta, \mathbb{F}_q)$. We use the stratification of 4.1.5 Proposition to study its cohomology over $\overline{\mathbb{Q}_\ell}$. If λ is a scalar, we write $\mathrm{H}^\bullet_c(S_\theta)_\lambda$ to denote the eigenspace of the Frobenius F associated to λ (we do not in principle assume the eigenspace to be non zero). We give a series of statements before proving all of them at once in the remaining of this section.

Proposition. *The Frobenius F acts semi-simply on $\mathrm{H}^\bullet_c(S_\theta)$. Its eigenvalues form a subset of*

$$\{q^i \,|\, 0 \leqslant i \leqslant \theta\} \cup \{-q^{j+1} \,|\, 0 \leqslant j \leqslant \theta - 2\}.$$

4.4.2 In a first statement, we give our results regarding the eigenspaces attached to a scalar of the form q^i for some i. Recall from 4.2.6 the cuspidal supports (L_δ, ρ_δ) for the finite symplectic group $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

Theorem. *Let $0 \leqslant i \leqslant \theta$ and $\theta' \in \mathbb{Z}$.*

(1) *The eigenspace $\mathrm{H}^{\theta'+i}_c(S_\theta)_{q^i}$ is zero when $\theta' < i$ or $\theta' > \theta$.*

We now assume that $0 \leqslant i \leqslant \theta' \leqslant \theta$.

(2) *All the irreducible representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ in the eigenspace $\mathrm{H}^{\theta'+i}_c(S_\theta)_{q^i}$ belong to the unipotent principal series, ie. they have cuspidal support (L_0, ρ_0).*

(3) *We have*

$$\mathrm{H}^0_c(S_\theta) = \mathrm{H}^0_c(S_\theta)_1 \simeq \rho_{\begin{pmatrix}\theta\\ \end{pmatrix}}, \qquad \mathrm{H}^{2\theta}_c(S_\theta) = \mathrm{H}^{2\theta}_c(S_\theta)_{q^\theta} \simeq \rho_{\begin{pmatrix}\theta\\ \end{pmatrix}}.$$

(4) *If $i + 2 \leqslant \theta'$ then*

$$\bigoplus_{0 \leqslant d \leqslant \theta-\theta'-1} \rho_{\begin{pmatrix}0 & \dots & \theta'-i-2 & \theta'-i-1 & \theta'+d\\ 1 & \dots & \theta'-i-1 & \theta-i-d & \end{pmatrix}} \oplus \bigoplus_{\substack{1 \leqslant d \leqslant \\ \min(i, \theta-\theta'-1)}} \rho_{\begin{pmatrix}0 & \dots & \theta'-i-2 & \theta'-i-1+d & \theta'\\ 1 & \dots & \theta'-i-1 & \theta-i-d & \end{pmatrix}} \hookrightarrow \mathrm{H}^{\theta'+i}_c(S_\theta)_{q^i}.$$

The cokernel of this map consists of at most 4 irreducible representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

(5) *When $i = \theta' \neq \theta$, we have*

$$\rho_{\begin{pmatrix}\theta\\ \end{pmatrix}} \hookrightarrow \mathrm{H}^{2i}_c(S_\theta)_{q^i} \text{ if } 2i < \theta, \qquad \rho_{\begin{pmatrix}\theta\\ \end{pmatrix}} \oplus \rho_{\begin{pmatrix}\theta-i & i+1\\ 0 & \end{pmatrix}} \hookrightarrow \mathrm{H}^{2i}_c(S_\theta)_{q^i} \text{ if } 2i \geqslant \theta.$$

(6) *When $\theta' = \theta$ we have*

$$\mathrm{H}^{\theta+i}_c(S_\theta)_{q^i} \simeq 0 \text{ or } \rho_{\begin{pmatrix}0 & \dots & \theta-i-1 & \theta\\ 1 & \dots & \theta-i & \end{pmatrix}}.$$

(7) *When $\theta' = 1$ and $i = 0$, we have*

$$\mathrm{H}^1_c(S_1) = 0, \qquad \mathrm{H}^1_c(S_\theta) = \mathrm{H}^1_c(S_\theta)_1 \simeq 0 \text{ or } \rho_{\begin{pmatrix}0 & 1 & \theta\\ 1 & 2 & \end{pmatrix}} \text{ when } \theta \geqslant 2.$$

We note that when $\theta' = \theta$, the formula of (4) does not say anything about the eigenspace $\mathrm{H}^{\theta+i}_c(S_\theta)_{q^i}$ since the sums are empty. However, by (6) we understand that this eigenspace is either 0 either irreducible.

We note also that the theorem does not give any information in the case $i + 1 = \theta'$, except when $\theta' = 1$ and $i = 0$ which corresponds to (7).

4.4.3 In a second statement, we give our results regarding the eigenspaces attached to a scalar of the form $-q^{j+1}$ for some j.

Theorem. *Let $0 \leqslant j \leqslant \theta - 2$ and $\theta' \in \mathbb{Z}$.*

(1) *The eigenspace $\mathrm{H}_c^{\theta'+j}(S_\theta)_{-q^{j+1}}$ is zero when $\theta' < j+2$ or $\theta' > \theta$.*

We now assume that $2 \leqslant j+2 \leqslant \theta' \leqslant \theta$.

(2) *All the irreducible representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$ in the eigenspace $\mathrm{H}_c^{\theta'+j}(S_\theta)_{-q^{j+1}}$ are unipotent with cuspidal support (L_1, ρ_1).*

(3) *We have*

$$\mathrm{H}_c^{2\theta-2}(S_\theta)_{-q^{\theta-1}} \simeq \rho_{\begin{pmatrix} 0 & 1 & \theta \end{pmatrix}}.$$

(4) *If $j+4 \leqslant \theta' \leqslant \theta$ then*

$$\bigoplus_{0 \leqslant d \leqslant \theta-\theta'-1} \rho_{\begin{pmatrix} 0 & \dots & \theta'-i-4 & \theta'-i-3 & \theta'-j-2 & \theta'-j-1 & \theta'+d \\ 1 & \dots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix}} \bigoplus$$
$$\bigoplus_{\substack{1 \leqslant d \leqslant \\ \min(i, \theta-\theta'-1)}} \rho_{\begin{pmatrix} 0 & \dots & \theta'-i-4 & \theta'-i-3 & \theta'-j-2 & \theta'-j-1+d & \theta' \\ 1 & \dots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{\theta'+j}(S_\theta)_{-q^{j+1}}.$$

The cokernel of this map consists of at most 4 irreducible representations of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

(5) *When $j+2 = \theta' \neq \theta$, we have*

$$\rho_{\begin{pmatrix} 0 & 1 & \theta \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{2(j+1)}(S_\theta)_{-q^{j+1}} \qquad \text{if } 2(j+1) < \theta,$$

$$\rho_{\begin{pmatrix} 0 & 1 & \theta \end{pmatrix}} \oplus \rho_{\begin{pmatrix} 0 & \theta-i-1 & i+2 \end{pmatrix}} \hookrightarrow \mathrm{H}_c^{2(j+1)}(S_\theta)_{-q^{j+1}} \qquad \text{if } 2(j+1) \geqslant \theta.$$

(6) *When $\theta' = \theta$ we have*

$$\mathrm{H}_c^{\theta+j}(S_\theta)_{-q^{j+1}} \simeq 0 \text{ or } \rho_{\begin{pmatrix} 0 & \dots & \theta-j-3 & \theta-j-2 & \theta-j-1 & \theta \\ 1 & \dots & \theta-j-2 & & & \end{pmatrix}}.$$

We note that when $\theta' = \theta$, the formula of (4) does not say anything about the eigenspace $\mathrm{H}_c^{\theta+j}(S_\theta)_{-q^{j+1}}$ since the sums are empty. However, by (6) we understand that this eigenspace is either 0 either irreducible.

We note also that the theorem does not give any information in the case $j+3 = \theta'$.

Remark. A cuspidal representation occurs in the cohomology of S_θ only in the cases $\theta = 0$ and $\theta = 2$. When $\theta = 0$ it corresponds to $\mathrm{H}_c^0(S_0)$ which is trivial. When $\theta = 2$ it corresponds to $\mathrm{H}_c^2(S_2)_{-q}$ as described by (3) in the theorem above.

4.4.4 The remaining of this section is dedicated to proving the theorems stated above. Recall from 4.1.5 that we have a stratification $S_\theta = \bigsqcup_{\theta'=0}^{\theta} X_{I_{\theta'}}(w_{\theta'})$. It induces a spectral sequence on the cohomology whose first page is given by

$$E_1^{a,b} = \mathrm{H}_c^{a+b}(X_{I_a}(w_a)) \implies \mathrm{H}_c^{a+b}(S_\theta). \tag{E}$$

Now, recall that the strata $X_{I_{\theta'}}(w_{\theta'})$ are related to Coxeter varieties for the finite symplectic group $\mathrm{Sp}(2\theta', \mathbb{F}_q)$. Using 4.1.7, the geometric isomorphism given in 4.1.6 Proposition induces an isomorphism on the cohomology

$$\mathrm{H}_c^\bullet(X_{I_{\theta'}}(w_{\theta'})) \simeq \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \mathrm{H}_c^\bullet(X^{\mathrm{Sp}(2\theta')}(w_{\theta'})), \tag{$**$}$$

where $L_{K_{\theta'}}$ denotes the block-diagonal Levi complement isomorphic to $\mathrm{GL}(\theta-\theta', \mathbb{F}_q)\times\mathrm{Sp}(2\theta', \mathbb{F}_q)$. The variety $X^{\mathrm{Sp}(2\theta')}(w_{\theta'})$ is nothing but the Coxeter variety that we denoted by $X^{k'}$ in 4.3.1, and whose cohomology we have described. For $0 \leqslant i \leqslant \theta'$ and $0 \leqslant j \leqslant \theta'-2$, recall from 4.3.2 the symbols $S_i^{\theta'}$ and $T_j^{\theta'}$. We define

$$\mathrm{R}_{i,\theta'}^S := \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{S_i^{\theta'}}, \qquad \mathrm{R}_{j,\theta'}^T := \mathrm{R}_{L_{K_{\theta'}}}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)} \mathbf{1} \boxtimes \rho_{T_j^{\theta'}}.$$

Then by ($**$), we have

$$\begin{aligned} \mathrm{H}_c^{\theta'+i}(X_{I_{\theta'}}(w_{\theta'})) &\simeq \mathrm{R}_{i,\theta'}^S \oplus \mathrm{R}_{i,\theta'}^T & \forall 0 \leqslant i \leqslant \theta'-2, \\ \mathrm{H}_c^{\theta'+i}(X_{I_{\theta'}}(w_{\theta'})) &\simeq \mathrm{R}_{i,\theta'}^S & \forall \theta'-1 \leqslant i \leqslant \theta'. \end{aligned}$$

The cohomology groups of other degrees vanish. The representation $\mathrm{R}_{i,\theta'}^S$ corresponds to the eigenvalue q^i of F, whereas $\mathrm{R}_{j,\theta'}^T$ corresponds to $-q^{j+1}$.

Lemma. *Let $0 \leqslant \theta' \leqslant \theta$, $0 \leqslant i \leqslant \theta'$ and $0 \leqslant j \leqslant \theta'-2$.*

- *If $i < \theta'$, the representation $\mathrm{R}_{i,\theta'}^S$ is the multiplicity-free sum of the unipotent representations ρ_S where $S \in \mathcal{Y}_{1,\theta}^1$ runs over the following 4 distinct families of symbols*

(S1) $$\begin{pmatrix} 0 & \dots & \theta'-i-2 & \theta'-i-1 & \theta'+d \\ 1 & \dots & \theta'-i-1 & \theta-i-d & \end{pmatrix} \qquad \forall 0 \leqslant d \leqslant \theta-\theta',$$

(S2) $$\begin{pmatrix} 0 & \dots & \theta'-i-2 & \theta'-i-1+d & \theta' \\ 1 & \dots & \theta'-i-1 & \theta-i-d & \end{pmatrix} \qquad \forall 1 \leqslant d \leqslant \min(i, \theta-\theta'),$$

(S Exc 1) $$\begin{pmatrix} 0 & \dots & \theta'-i-1 & \theta'-i & \theta \\ 1 & \dots & \theta'-i & \theta'-i+1 & \end{pmatrix} \qquad \text{if } \theta' \neq \theta,$$

(S Exc 2) $$\begin{pmatrix} 0 & \dots & \theta'-i-1 & \theta-i-1 & \theta'+1 \\ 1 & \dots & \theta'-i & \theta'-i+1 & \end{pmatrix} \qquad \text{if } \theta' \neq \theta, \theta-1 \text{ and } \theta \leqslant \theta'+i+1.$$

- *The representation $\mathrm{R}_{\theta',\theta'}^S$ is the multiplicity-free sum of the unipotent representations ρ_S where $S \in \mathcal{Y}_{1,\theta}^1$ runs over the following 2 distinct families of symbols*

(S1') $$\begin{pmatrix} 0 & \theta'+1+d \\ \theta-\theta'-d & \end{pmatrix} \qquad \forall 0 \leqslant d \leqslant \theta-\theta',$$

(S2') $$\begin{pmatrix} d & \theta'+1 \\ \theta-\theta'-d & \end{pmatrix} \qquad \forall 1 \leqslant d \leqslant \min(\theta', \theta-\theta').$$

– *If $j+2 < \theta'$, the representation $\mathrm{R}^T_{j,\theta'}$ is the multiplicity-free sum of the unipotent representations ρ_T where $T \in \mathcal{Y}^1_{3,\theta}$ runs over the following 4 distinct families of symbols*

(T1) $$\begin{pmatrix} 0 & \dots & \theta'-j-4 & \theta'-j-3 & \theta'-j-2 & \theta'-j-1 & \theta'+d \\ 1 & \dots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix} \quad \forall 0 \leqslant d \leqslant \theta-\theta',$$

(T2) $$\begin{pmatrix} 0 & \dots & \theta'-j-4 & \theta'-j-3 & \theta'-j-2 & \theta'-j-1+d & \theta' \\ 1 & \dots & \theta'-j-3 & \theta-j-2-d & & & \end{pmatrix} \quad \forall 1 \leqslant d \leqslant \min(j, \theta-\theta'),$$

(T Exc 1) $$\begin{pmatrix} 0 & \dots & \theta'-j-2 & \theta'-j-1 & \theta'-j & \theta \\ 1 & \dots & \theta'-j-1 & & & \end{pmatrix} \quad \textit{if } \theta' \neq \theta,$$

(T Exc 2) $$\begin{pmatrix} 0 & \dots & \theta'-j-2 & \theta'-j-1 & \theta-j-1 & \theta'+1 \\ 1 & \dots & \theta'-j-1 & & & \end{pmatrix} \quad \textit{if } \theta' \neq \theta, \theta-1 \textit{ and } \theta \leqslant \theta'+j+1.$$

– *The representation $\mathrm{R}^T_{\theta'-2,\theta'}$ is the multiplicity-free sum of the unipotent representations ρ_T where $T \in \mathcal{Y}^1_{3,\theta}$ runs over the following 2 distinct families of symbols*

(T1') $$\begin{pmatrix} 0 & 1 & 2 & \theta'+1+d \\ \theta-\theta'-d & & & \end{pmatrix} \quad \forall 0 \leqslant d \leqslant \theta-\theta',$$

(T2') $$\begin{pmatrix} 0 & 1 & 2+d & \theta'+1 \\ \theta-\theta'-d & & & \end{pmatrix} \quad \forall 1 \leqslant d \leqslant \min(\theta'-2, \theta-\theta').$$

This lemma results directly from the computational rule explained in 4.2.7. In concrete terms, an induction of the form
$$\mathrm{R}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}_{L_{K_{\theta'}}} \mathbf{1} \boxtimes \rho_{S'}$$
is the sum of all the representations ρ_S where S is obtained from S' by adding a hook of leg length 0 to both rows, whose lengths sum to $\theta-\theta'$. We illustrate the arguments by looking at a concrete example.

With $\theta = 6, \theta' = 3$ and $i = 2$ let us explain the computation of
$$\mathrm{R}^S_{2,3} = \mathrm{R}^{\mathrm{Sp}(12,\mathbb{F}_q)}_{L_{K_3}} \mathbf{1} \boxtimes \rho_{S^3_2}.$$
Recall that
$$S^3_2 = \begin{pmatrix} 0 & 3 \\ 1 & \end{pmatrix}.$$
For $0 \leqslant d \leqslant \theta-\theta' = 3$, we add a d-hook of leg length 0 to the first row of S^3_2, and a $(3-d)$-hook of leg length 0 to its second row.
We may always add the hooks to the last entries of each row. By doing so we obtain the representations corresponding to the family of symbols (S1):
$$\begin{pmatrix} 0 & 3 \\ 4 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 4 \\ 3 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 5 \\ 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 6 \\ 1 & \end{pmatrix}.$$

When $d \leqslant \min(\theta - \theta', i) = \min(3,2) = 2$, we may also add the first hook to the penultimate entry of the first row. Note that since $i < \theta'$, the first row of $S_i^{\theta'}$ has at least 2 entries. By doing so, we obtain the representations corresponding to the family of symbols (S2):

$$\begin{pmatrix} 1\ 3 \\ 3 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 3 \\ 2 & \end{pmatrix}.$$

Now, recall that symbols are equal up to shifts. Therefore, one may rewrite S_2^3 as

$$S_2^3 = \text{shift}(S_2^3) = \begin{pmatrix} 0\ 1\ 4 \\ 0\ 2 \end{pmatrix}.$$

Written this way, we notice that a 1-hook can be added to the first entry of the second row, which is a 0. Then one must add to the first row a hook of length $d = \theta - \theta' - 1 = 2$. One may always add it to the last entry, which results in the first "exceptional" representation (S Exc 1). Moreover if $d \leqslant i$, which is the case here, one may also add this hook to the penultimate entry of the first row, which leads to the second "exceptional" representation (S Exc 2):

$$\begin{pmatrix} 0\ 1\ 6 \\ 1\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 2 & \end{pmatrix}.$$

The sum of the representations attached to all the 8 symbols written above is isomorphic to $\mathrm{R}^S_{2,3}$.

We also explain in detail the special case $i = \theta$. Thus we compute

$$\mathrm{R}^S_{\theta,\theta} = \mathrm{R}^{\mathrm{Sp}(2\theta,\mathbb{F}_q)}_{L_{K_{\theta'}}} \mathbf{1} \boxtimes \rho_{S^{\theta'}_{\theta'}}.$$

Recall that

$$S^{\theta'}_{\theta'} = \begin{pmatrix} \theta' \\ \ \end{pmatrix}$$

corresponds to the trivial representation of $\mathrm{Sp}(2\theta', \mathbb{F}_q)$. In order to compute this induction, we shift the symbol $S^{\theta'}_{\theta'}$ first:

$$S^{\theta'}_{\theta'} = \begin{pmatrix} 0\ \theta' + 1 \\ 0 \end{pmatrix}.$$

For $0 \leqslant d \leqslant \theta - \theta'$, we add a d-hook of leg length 0 to the first row and a $(\theta - \theta' - d)$-hook of leg length 0 to the second row. We may always add the hooks to the last entries of each row. By doing so, we obtain the representations corresponding to the family of symbols $(S1')$. Moreover when $d \leqslant \min(\theta', \theta - \theta')$, we may also add the first hook to the 0 in the first row. It leads to the representations corresponding to the family of symbols $(S2')$.
In particular, we notice that the symbol of $(S1')$ with $d = \theta - \theta'$ corresponds to the trivial representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & & & & & & & \mathrm{R}^S_{\theta,\theta} \\
 & & & & & & & & & \mathrm{R}^S_{\theta-1,\theta-1} & \longrightarrow & & \mathrm{R}^S_{\theta-1,\theta} \\
 & & & & & & & \mathrm{R}^S_{\theta-2,\theta-2} & \longrightarrow & \mathrm{R}^S_{\theta-2,\theta-1} & \longrightarrow & & \mathrm{R}^S_{\theta-2,\theta} \oplus \mathrm{R}^T_{\theta-2,\theta} \\
 & & & & & \iddots & & & & & & & \vdots \\
 & & & \mathrm{R}^S_{2,2} & \longrightarrow & \dots & \longrightarrow & \mathrm{R}^S_{2,\theta-2} \oplus \mathrm{R}^T_{2,\theta-2} & \longrightarrow & \mathrm{R}^S_{2,\theta-1} \oplus \mathrm{R}^T_{2,\theta-1} & \longrightarrow & & \mathrm{R}^S_{2,\theta} \oplus \mathrm{R}^T_{2,\theta} \\
 & & \mathrm{R}^S_{1,1} & \longrightarrow \mathrm{R}^S_{1,2} & \longrightarrow & \dots & \longrightarrow & \mathrm{R}^S_{1,\theta-2} \oplus \mathrm{R}^T_{1,\theta-2} & \longrightarrow & \mathrm{R}^S_{1,\theta-1} \oplus \mathrm{R}^T_{1,\theta-1} & \longrightarrow & & \mathrm{R}^S_{1,\theta} \oplus \mathrm{R}^T_{1,\theta} \\
\mathrm{R}^S_{0,0} & \longrightarrow & \mathrm{R}^S_{0,1} & \longrightarrow \mathrm{R}^S_{0,2} \oplus \mathrm{R}^T_{0,2} & \longrightarrow & \dots & \longrightarrow & \mathrm{R}^S_{0,\theta-2} \oplus \mathrm{R}^T_{0,\theta-2} & \longrightarrow & \mathrm{R}^S_{0,\theta-1} \oplus \mathrm{R}^T_{0,\theta-1} & \longrightarrow & & \mathrm{R}^S_{0,\theta} \oplus \mathrm{R}^T_{0,\theta}
\end{array}
$$

Figure 6: The first page of the spectral sequence.

4.4.5 Now, we have an explicit description of the terms $E_1^{a,b}$ in the first page of the spectral sequence (E). In the Figure 6, we draw the shape of the first page.

First, since the Frobenius F acts with the eigenvalue q^i (resp. $-q^{j+1}$) on the representations $\mathrm{R}^S_{i,\theta'}$ (resp. $\mathrm{R}^T_{j,\theta'}$), 4.4.1 Proposition as well as point (1) of 4.4.2 and 4.4.3 Theorems follow from the triangular shape of the spectral sequence. Point (2) also follows from 4.4.4 Lemma.
Next, we notice that on the b-th row of the first page E_1, the eigenvalues of F which occur are q^b and $-q^{b+1}$. In particular, the eigenvalues on different rows are all distinct. It follows that all the arrows in the deeper pages of the sequence are zero, therefore it degenerates on the second page. Moreover, the filtration induced by the spectral sequence on the abutment splits, so that $\mathrm{H}^k_c(S_\theta)$ is isomorphic to the direct sum of the terms $E_2^{k-b,b}$ on the k-th diagonal of the second page.

We prove point (3) of 4.4.2 and 4.4.3 Theorems. By the shape of the spectral sequence, we see that

$$
\mathrm{H}^{2\theta}_c(S_\theta) = \mathrm{H}^{2\theta}_c(S_\theta)_{q^\theta} \simeq \mathrm{R}^S_{\theta,\theta} \simeq \rho_{\begin{pmatrix}\theta \\ \ \end{pmatrix}}, \qquad \mathrm{H}^{2\theta-2}_c(S_\theta)_{-q^{\theta-1}} \simeq \mathrm{R}^T_{\theta-2,\theta} \simeq \rho_{\begin{pmatrix}0 & 1 & \theta \\ & & \end{pmatrix}}.
$$

Moreover, by the spectral sequence we know that $\mathrm{H}^0_c(S_\theta)$ is a subspace of $\mathrm{R}^S_{0,0}$, thus the Frobenius F acts like the identity. Since S_θ is projective and irreducible, the cohomology group $\mathrm{H}^0_c(S_\theta) = \mathrm{H}^0(S_\theta)$ is trivial.

We now prove point (4) of 4.4.2 and 4.4.3 Theorems. Let $2 \leqslant i+2 \leqslant \theta' \leqslant \theta - 1$. By extracting the eigenvalue q^i in the spectral sequence, we have a chain

$$\dots \longrightarrow \mathrm{R}^S_{i,\theta'-1} \xrightarrow{u} \mathrm{R}^S_{i,\theta'} \xrightarrow{v} \mathrm{R}^S_{i,\theta'+1} \longrightarrow \dots$$

The quotient $\mathrm{Ker}(v)/\mathrm{Im}(u)$ is isomorphic to the eigenspace $\mathrm{H}_c^{\theta'+i}(S_\theta)_{q^i}$.
The middle term $\mathrm{R}^S_{i,\theta'}$ is the sum of the representations ρ_S where S runs over the families of symbols (S1), (S2), (S Exc 1) and (S Exc 2) as in 4.4.4 Lemma. All these symbols are written in their "reduced" form, meaning that they can not be written as the shift of another symbol. Let us look at the length of the second row of these symbols. If S belongs to (S1) or (S2), then the second row has length $\theta' - i$. If S belongs to (S Exc 1) or (S Exc 2), then the second row has length $\theta' - i + 1$.
We may do a similar analysis for the left term (resp. the right term) by replacing θ' with $\theta' - 1$ (resp. $\theta' + 1$). In the left term $\mathrm{R}^S_{i,\theta'-1}$, all the representations corresponding to the families (S1) and (S2) have second row of length $\theta' - i - 1$. No such representation occurs in the middle term, therefore they all automatically lie in the $\mathrm{Ker}(u)$. Then, in the left term the representation corresponding to (S Exc 1) occurs since $\theta' - 1 \neq \theta$. We observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'}$ with S in the family (S1) and $d = \theta - \theta'$. Further, assume that $\theta \leqslant \theta' + i$ so that the representation corresponding to (S Exc 2) occurs in $\mathrm{R}^S_{i,\theta'-1}$. Then we observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'}$ with S in the family (S2) and $d = \theta - \theta' = \min(i, \theta - \theta')$. Hence, it follows that $\mathrm{Im}(u)$ consists of at most 2 irreducible subrepresentations of $\mathrm{R}^S_{i,\theta'}$, and they correspond to the symbols of (S1) and (S2) with $d = \theta - \theta'$. Next, all the subrepresentations ρ_S of $\mathrm{R}^S_{i,\theta'}$ with S in (S1) or (S2) belong to $\mathrm{Ker}(v)$, since no component of $\mathrm{R}^S_{i,\theta'+1}$ correspond to a symbol whose second row has length $\theta' - i$. Since $\theta' \neq \theta$, the represensation corresponding to (S Exc 1) occurs in $\mathrm{R}^S_{i,\theta'}$. We observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'+1}$ with S in the family (S1) and $d = \theta - \theta' - 1$. Assume that $\theta' \leqslant \theta - 2$ and $\theta \leqslant \theta' + i + 1$, so that the representation corresponding to (S Exc 2) occurs in $\mathrm{R}^S_{i,\theta'}$. Then we observe that it is equivalent to the representation ρ_S occuring in $\mathrm{R}^S_{i,\theta'+1}$ with S in the family (S2) and $d = \theta - \theta' - 1 = \min(i, \theta - \theta' - 1)$. Therefore, it is not possible to tell whether the components of $\mathrm{R}^S_{i,\theta'}$ corresponding to (S Exc 1) and (S Exc 2) are in $\mathrm{Ker}(v)$ or not. In all cases, we conclude that $\mathrm{Ker}(v)/\mathrm{Im}(u)$ contains at least all the representations corresponding to the symbols S in (S1) and (S2) with $d < \theta - \theta'$. With this description we miss up to four irreducible representations, which correspond to (S1) and (S2) with $d = \theta - \theta'$, (S Exc 1) and (S Exc 2). This proves point (4) of 4.4.2 Theorem.
The point (4) of 4.4.3 Theorem is proved by identical arguments.

We now prove point (5) of 4.4.2 and 4.4.3 Theorems. We consider $i = \theta' \neq \theta$. By extracting the eigenvalue q^i in the spectral sequence, we have a chain

$$\mathrm{R}^S_{i,i} \xrightarrow{u} \mathrm{R}^S_{i,i+1} \longrightarrow \dots$$

The kernel $\mathrm{Ker}(u)$ is isomorphic to the eigenspace $\mathrm{H}_c^{2i}(S_\theta)_{q^i}$. The left term $\mathrm{R}^S_{i,i}$ is the sum of the representations $\rho_{S'}$ where S' runs over the families of symbols (S1') and (S2'). We observe that the representation $\rho_{S'}$ with S' in (S1') corresponding to some $0 \leqslant d' \leqslant \theta - i - 1$ is equivalent to the component ρ_S of $\mathrm{R}^S_{i,i+1}$ with S in (S1) corresponding to $d = d'$. Similarly, we observe

that the representation $\rho_{S'}$ with S' in (S2') corresponding to some $1 \leqslant d' \leqslant \min(i, \theta - i - 1)$ is equivalent to the component ρ_S of $\mathrm{R}^S_{i,i+1}$ with S in (S2) corresponding to $d = d'$.
Therefore, the representation ρ_S corresponding to S in (S1') with $d' = \theta - i$ belongs to $\mathrm{Ker}(u)$. This is no other than the trivial representation. Moreover, if $\min(i, \theta - i - 1) \neq \min(i, \theta - i)$, ie. if $2i \geqslant \theta$, then the representation ρ_S corresponding to S in (S2') with $d' = \theta - i$ also belongs to $\mathrm{Ker}(u)$. This proves point (5) of 4.4.2 Theorem.
The point (5) of 4.4.3 Theorem is proved by identical arguments.

Points (6) of 4.4.2 and 4.4.3 Theorems follows easily from the shape of the spectral sequence. Indeed, it suffices to notice that all the terms $\mathrm{R}^S_{i,\theta}$ and $\mathrm{R}^T_{j,\theta}$ in the rightmost column of the sequence are irreducible. Thus, they may either vanish, either remain the same in the second page.

Lastly we prove point (7) of 4.4.2. Assume first that $\theta = 1$. The 0-th row of the spectral sequence is given by

$$\rho_{\begin{pmatrix} 1 \\ \end{pmatrix}} \oplus \rho_{\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}} \xrightarrow{u} \rho_{\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}}$$

We have $\mathrm{H}^1_c(S_1) \simeq \mathrm{Coker}(u)$. Since we already know that $\mathrm{H}^0_c(S_1) \simeq \mathrm{Ker}(u)$ is the trivial representation of $\mathrm{Sp}(2, \mathbb{F}_q)$, we see that u must be surjective. Therefore $\mathrm{H}^1_c(S_1) = 0$.

Remark. The vanishing of $\mathrm{H}^1_c(S_1)$ also follows directly from the fact that $S_1 \simeq \mathbb{P}^1$.

Let us now assume $\theta \geqslant 2$. The first terms of the 0-th row of the spectral sequence are

$$\mathrm{R}^S_{0,0} \xrightarrow{u} \mathrm{R}^S_{0,1} \xrightarrow{v} \mathrm{R}^S_{0,2} \longrightarrow \dots$$

We have $\mathrm{H}^1_c(S_\theta) = \mathrm{H}^1_c(S_\theta)_1 \simeq \mathrm{Ker}(v)/\mathrm{Im}(u)$. The middle term $\mathrm{R}^S_{0,1}$ is the sum of all the representations corresponding to the following symbols

$$\begin{pmatrix} 0 & 1 & \theta \\ 1 & 2 & \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1+d \\ \theta - d & \end{pmatrix}, \qquad \forall 0 \leqslant d \leqslant \theta - 1.$$

On the other hand, the left term $\mathrm{R}^S_{0,0}$ is the sum of all the representations corresponding to the following symbols

$$\begin{pmatrix} 0 & 1+d \\ \theta - d & \end{pmatrix}, \qquad \forall 0 \leqslant d \leqslant \theta.$$

Since we already know that $\mathrm{H}^0_c(S_\theta) \simeq \mathrm{Ker}(u)$ is the trivial representation of $\mathrm{Sp}(2\theta, \mathbb{F}_q)$, we see that $\mathrm{Im}(u)$ contains all the components of $\mathrm{R}^S_{0,1}$ associated to a symbol whose second row has length 1. Therefore, $\mathrm{H}^1_c(S_\theta)$ is either 0 either irreducible, depending on whether the remaining component

$$\begin{pmatrix} 0 & 1 & \theta \\ 1 & 2 & \end{pmatrix}$$

is in $\mathrm{Ker}(v)$ or not. This proves point (7) and concludes the proof of 4.4.2 and 4.4.3 Theorems.

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