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Contrôle d'équations des ondes linéaires et non-linéaires

THÈSE DE DOCTORAT

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Résumé

Le sujet de cette thèse est l'étude de différents problèmes de contrôle pour des équations des ondes linéaires et non-linéaires. Le chapitre 1, en français, est une introduction générale aux propriétés de contrôle et aux équations des ondes étudiées, et le chapitre 2, également en français, résume les résultats obtenus.

On s'intéresse au chapitre 3 à l'observabilité au bord (ou à la contrôlabilité au bord) de systèmes d'équations des ondes linéaires. On construit en détails les solutions pour des données au bord de type Dirichlet homogène et inhomogène, à tout niveau de régularité. On démontre ensuite avec un argument microlocal que l'observabilité au bord (et donc la contrôlabilité au bord) à un niveau de régularité est équivalente à l'observabilité au bord à tout autre niveau de régularité.

Le chapitre 4 se concentre sur les solutions globales d'équations des ondes non-linéaires focalisantes sous-critiques, avec un terme d'amortissement, sur un domaine borné. On démontre des estimations uniformes en temps, le résultat principal étant qu'une solution globale est bornée dans l'espace d'énergie, pour certaines non-linéarités.

On étudie ensuite au chapitre 5 le cas de l'équation des ondes cubique sur un domaine borné de dimension 3, avec un terme d'amortissement. On démontre que sous l'énergie de l'état fondamental, la dichotomie classique entre existence globale et explosion reste valide. En particulier, les solutions explosives ne sont pas stabilisées. À l'inverse, en supposant la condition de contrôle géométrique vérifiée, on établit la stabilisation des solutions globales sous l'énergie de l'état fondamental.

Enfin, au chapitre 6, on considère une équation des ondes non-linéaire sous-critique, avec des hypothèses assez générales, et on établit la contrôlabilité locale au voisinage d'une trajectoire régulière de l'équation. Dans le cas d'un domaine non-borné, on montre également la contrôlabilité à zéro en temps long des solutions scattering. En corollaire, on obtient la contrôlabilité à zéro en temps long de solutions initialement proches de l'état fondamental pour une équation focalisante, ce qui implique en particulier la contrôlabilité de solutions explosives, et la contrôlabilité exacte en temps long de certaines équations défocalisantes.

Abstract

The subject of this thesis is the study of various control problems for linear and nonlinear wave equations. Chapter 1, in French, is a general introduction to the control properties and the wave equations under consideration, and Chapter 2, also in French, summarizes the obtained results.

In Chapter 3, we focus on boundary observability (or boundary controllability) of systems of linear wave equations. We construct solutions for both homogeneous and inhomogeneous Dirichlet boundary conditions at every regularity level. Using microlocal techniques, we prove that boundary observability (and thus boundary exact controllability) at one regularity level is equivalent to boundary observability at any other level.

Chapter 4 presents uniform-in-time estimates for global solutions of subcritical focusing nonlinear wave equations, with the main result being that a global solution is bounded in the energy space for some nonlinearities.

In Chapter 5, we delve into the study of the focusing cubic wave equation on a bounded domain of dimension 3, with a damping term. We demonstrate that under the energy of the ground state, the classic dichotomy between global existence and blow-up remains valid. In particular, explosive solutions are not stabilized. Conversely, assuming that the geometric control condition is satisfied, we establish the stabilization of global solutions under the energy of the ground state.

Finally, in Chapter 6, we consider a subcritical nonlinear wave equation, with fairly general assumptions, and we establish local controllability around a regular trajectory of the equation. In the case of an unbounded domain, we also demonstrate the null-controllability of scattering solutions in a long time. As a corollary, we obtain the null-controllability of solutions initially close to the ground state in a long time for a focusing equation (which implies, in particular, the controllability of some explosive solutions), and the exact controllability in a long time of some defocusing equations.

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Chapitre 1

Introduction générale

1.1 Contrôlabilité

Exemple introductif. On considère l'équation des ondes

$$\begin{cases} \partial_t^2 u - \Delta u &= f & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & x \in \mathbb{R}^3, \end{cases}$$

pour $(u^0, u^1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ et $f \in L^2(\mathbb{R} \times \mathbb{R}^3)$. Si $f = 0$, alors u décrit le comportement d'une onde qui évolue librement dans $\mathbb{R}^3$: la théorie du contrôle s'intéresse à l'influence que peut avoir le terme source f, appelé *contrôle*, sur la solution. Par exemple, pour un temps $T > 0$, une question naturelle est la suivante : pour toute donnée initiale $(u^0, u^1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, existe-t-il un contrôle $f \in L^2(\mathbb{R} \times \mathbb{R}^3)$ telle que la solution u associée vérifie $(u(T), \partial_t u(T)) = 0$? Sur cet exemple très simple, on peut facilement donner une réponse positive à cette question, pour tout $T > 0$. Dans cette thèse, on s'intéressera à différents types de contrôles, pour plusieurs équations des ondes, ainsi qu'à des questions reliées, comme l'observabilité et la stabilisation.

1.1.1 Contrôlabilité exacte, locale, interne, au bord

On introduit ici les notions liées à la contrôlabilité que l'on rencontrera. Le cadre des définitions est approximatif, pour donner une idée générale : les propriétés étudiées dans cette thèse seront ensuite définies rigoureusement dans le chapitre 2, qui résume l'ensemble des résultats obtenus.

Contrôlabilité d'un ensemble vers un autre. On prend un espace de Banach X, et on considère une équation d'évolution sous la forme générale

$$\begin{cases} \partial_t u + P(u) &= f & (t,x) \in]0,T[\times \Omega, \\ u(0) &= u^0 & x \in \Omega, \\ Bu &= 0 & (t,x) \in]0,T[\times \partial\Omega, \end{cases} \tag{1.1.1}$$

pour $T > 0$, Ω un domaine spatial, P un opérateur (qui n'est pas supposé linéaire), B un opérateur de bord, $f :]0,T[\times \Omega \to X$ un terme source, et $u^0 : \Omega \to X$ une donnée initiale. On prend $\mathcal{F}_\Omega$ un espace de fonctions de Ω dans X, et on suppose que pour toutes les données $u^0 \in \mathcal{F}_\Omega$ et f considérées, il existe une unique solution $u : [0,T] \to \mathcal{F}_\Omega$. On rencontrera souvent des équations aux dérivées partielles qui sont réversibles en temps, ce qui signifie ici que si u est une solution de (1.1.1), alors $t \mapsto u(T-t)$ l'est aussi.

Définition 1.1.1 (Contrôlabilité d'un ensemble vers un autre en temps T)**.** Soient A et B des sous-ensembles de $\mathcal{F}_\Omega$. On dit qu'il y a *contrôlabilité de A vers B en temps T* si pour tout $(u^0, v^0) \in A \times B$, il existe un contrôle f tel que la solution u de (1.1.1) associée à u^0 et f vérifie $u(T) = v^0$.

Dans le cas d'une équation réversible en temps, la contrôlabilité de A vers B est équivalente à la contrôlabilité de B vers A.

Définition 1.1.2 (Contrôlabilité exacte en temps T)**.** On dit qu'il y a *contrôlabilité exacte en temps T* si il y a contrôlabilité de $\mathcal{F}_\Omega$ vers $\mathcal{F}_\Omega$ en temps T.

En une phrase, la contrôlabilité exacte correspond au fait de pouvoir emmener la solution de n'importe quel état à n'importe quel état, en choisissant le contrôle f.

Définition 1.1.3 (Contrôlabilité locale en temps T)**.** Soit v une solution de (1.1.1) pour $f = 0$. On dit qu'il y a *contrôlabilité locale au voisinage de v en temps T* s'il existe un voisinage $\mathcal{O}$ de $v(0)$ dans $\mathcal{F}_\Omega$ tel qu'il y a contrôlabilité de $\mathcal{O}$ vers $\{v(T)\}$ en temps T.

Pour le contrôle local, on a l'intuition que le contrôle f doit être petit : on part très proche de la solution v, on modifie légèrement l'équation en ajoutant le terme source f, et on arrive précisément à la solution v au temps T. Une illustration du contrôle local est donnée à la Figure 1.1.

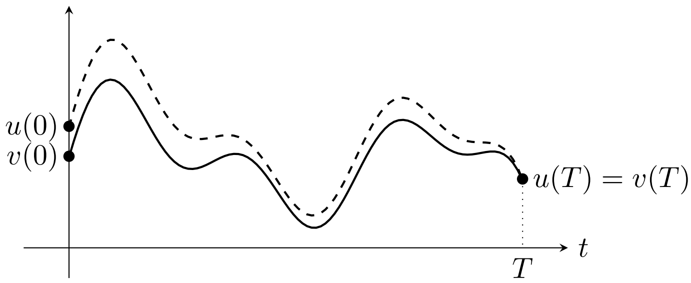


FIGURE 1.1 : Contrôle local au voisinage d'une solution.

Définition 1.1.4 (Contrôlabilité en temps long)**.** Soient A et B des sous-ensembles de $\mathcal{F}_\Omega$. On dit qu'il y a *contrôlabilité de A vers B en temps long* si pour tout $(u^0, v^0) \in A \times B$, il existe un temps $T > 0$ et un contrôle f tel que la solution u de (1.1.1) associée à u^0 et f vérifie $u(T) = v^0$.

Dans la définition précédente, le temps de contrôle T dépend de la donnée initiale u^0 et de l'état final v^0. On se retrouve confronté à ce type de résultat par exemple quand une démonstration utilise plusieurs phénomènes, dont un qui comporte une quantité qui tend vers 0 au cours du temps : on attend alors que cette quantité soit suffisamment petite pour invoquer un deuxième argument. La question de l'optimalité du temps T n'est alors pas traitée et reste ouverte.

Régularité du contrôle. Une question qui nous intéressera dans cette thèse concerne la régularité du contrôle. Si on suppose qu'il y a contrôlabilité entre deux ensembles A et B, en choisissant nos contrôles dans un certain espace C, peut-on en déduire qu'il y a contrôlabilité entre deux autres ensembles A_0 et B_0, avec $A_0 \subset A$, $B_0 \subset B$, en choisissant nos contrôles dans un ensemble $C_0 \subset C$? En d'autres termes, la régularité de l'état initial et de la cible permettent-elles d'utiliser un contrôle plus régulier ?

Contrôle interne. Pour simplifier les définitions précédentes, on a omis d'indiquer la région de l'espace où agissait le contrôle. On rencontrera deux cas de figure : le cas d'un contrôle interne, et le cas d'un contrôle au bord. On reprend les notations de l'équation (1.1.1).

Définition 1.1.5 (Contrôle interne)**.** Soit $\omega \subset \Omega$ un ouvert. Soient A et B des sous-ensembles de $\mathcal{F}_\Omega$. On dit qu'il y a *contrôlabilité de A vers B en temps T par la région* ω si pour toute fonction $a \in \mathscr{C}^\infty(\Omega, \mathbb{R})$ vérifiant $a \geq C > 0$ dans ω, il y a contrôlabilité de A vers B en temps T pour l'équation

$$\begin{cases} \partial_t u + P(u) &= af & (t,x) \in]0,T[\times\Omega, \\ u(0) &= u^0 & x \in \Omega, \\ Bu &= 0 & (t,x) \in]0,T[\times\partial\Omega. \end{cases}$$

De façon plus synthétique, le contrôle interne correspond au fait de placer le terme de contrôle en second membre de l'équation, et de le choisir supporté dans une région de l'espace donnée.

Remarque 1.1.6*.* Dans la pratique, on fixe parfois la fonction a sans spécifier l'ouvert dans lequel on contrôle. On pourrait être tenté de simplifier la définition précédente dans deux directions. La première consisterait à considérer une fonction a non-lisse, comme par exemple $a = \mathbb{1}_\omega$. Cette approche peut introduire des irrégularités, dans le cas où les contrôles sont choisis dans des espaces de fonctions assez régulières, et on l'évitera systématiquement dans cette thèse. La deuxième direction serait de remplacer l'hypothèse $a \geq C > 0$ sur ω, par $a > 0$ sur $\overline{\omega}$, ou encore $a > 0$ sur ω. On évite cette dernière condition qui peut créer des difficultés dans le cas où a s'annule au bord de ω. Dans le cas où $\overline{\omega}$ est borné, les conditions $a \geq C > 0$ sur ω et $a > 0$ sur $\overline{\omega}$ sont équivalentes, mais des difficultés peuvent apparaître dans le cas d'un domaine non-borné.

Contrôle au bord. Dans le cas d'un contrôle au bord, le contrôle est placé dans la condition de bord, et est supporté dans une région de la frontière donnée.

Définition 1.1.7 (Contrôle au bord)**.** Soit $\Gamma \subset \partial\Omega$ un ouvert. Soient A et B des sous-ensembles de $\mathcal{F}_\Omega$. On dit qu'il y a *contrôlabilité de A vers B en temps T par la région* Γ si pour toute fonction $a \in \mathscr{C}^\infty(\partial\Omega, \mathbb{R})$ vérifiant $a \geq C > 0$ dans Γ, il y a contrôlabilité de A vers B en temps T pour l'équation

$$\begin{cases} \partial_t u + P(u) &= 0 & (t,x) \in]0,T[\times\Omega, \\ u(0) &= u^0 & x \in \Omega, \\ Bu &= af & (t,x) \in]0,T[\times\partial\Omega. \end{cases}$$

Par exemple, l'opérateur B peut être la trace de Dirichlet $Bu = u_{|\partial\Omega}$, ou la trace de Neumann $Bu = \partial_\nu u$.

1.1.2 Observabilité, stabilisation

On étudiera deux autres propriétés reliées à la contrôlabilité.

Observabilité. L'observabilité correspond à une inégalité duale de la contrôlabilité. Une propriété de contrôle peut être reformulée comme la surjectivité d'un opérateur T entre deux espaces de Banach $\mathcal{X}$ et $\mathcal{Y}$. Si cet opérateur est linéaire, on note T^* son adjoint, et sa surjectivité est alors équivalente à l'inégalité

$$\|y\|_{\mathcal{Y}'} \lesssim \|\mathsf{T}^* y\|_{\mathcal{X}'}, \quad y \in \mathcal{Y}',$$

appelée inégalité d'observabilité. Un énoncé précis est donné au Théorème 3.1.15. On présente ici des inégalités d'observabilité sous une forme générale, l'équivalence entre contrôlabilité et observabilité sera donnée au cas par cas. Dans le cas où P est linéaire, le lien entre observabilité et contrôlabilité a été mis en évidence par différents auteurs (voir notamment [DR77], et le livre de Lions [Lio88]). On conserve les notations de l'équation (1.1.1).

Définition 1.1.8 (Observabilité interne)**.** Soit $\omega \subset \Omega$ un ouvert. On dit qu'il y a *observabilité en temps T depuis ω* si pour toute fonction $a \in \mathscr{C}^\infty(\Omega, \mathbb{R})$ vérifiant $a \geq C > 0$ dans ω, il existe $C' > 0$ telle que $\|u^0\|_{\mathcal{F}_\Omega} \leq C' \|au\|_{L^2(]0,T[,\mathcal{F}_\Omega)}$ pour tout $u^0 \in \mathcal{F}_\Omega$, où u est la solution de

$$\left\{ \begin{array}{rcll} \partial_t u + P(u) & = & 0 & (t,x) \in]0,T[\times\Omega, \\ u(0) & = & u^0 & x \in \Omega, \\ Bu & = & 0 & (t,x) \in]0,T[\times\partial\Omega. \end{array} \right. \tag{1.1.2}$$

En particulier, l'observabilité implique qu'une solution nulle sur le support de a est nulle, et l'inégalité est une version quantifiée de cette propriété. Le terme *observabilité* traduit le fait qu'une information sur la taille de la solution sur un ouvert ω donne une information sur la taille de la solution sur l'ensemble du domaine Ω.

Définition 1.1.9 (Observabilité au bord)**.** Soit $\Gamma \subset \partial\Omega$ un ouvert, B_0 un opérateur du bord, et $\mathcal{F}_{\partial\Omega}$ un espace de fonctions de $\partial\Omega$ dans X. On dit qu'il y a *observabilité en temps T par Γ* si pour toute fonction $a \in \mathscr{C}^\infty(\partial\Omega, \mathbb{R})$ vérifiant $a \geq C > 0$ dans Γ, il existe $C' > 0$ telle que $\|u^0\|_{\mathcal{F}_\Omega} \leq C' \|aB_0 u\|_{L^2(]0,T[,\mathcal{F}_{\partial\Omega})}$ pour tout $u^0 \in \mathcal{F}_\Omega$, où u est la solution de (1.1.2).

Dans notre cas, B_0 sera la dérivée normale au bord d'une solution de l'équation des ondes.

Stabilisation. On conserve les notations de l'équation (1.1.1), et on prend un opérateur A, que l'on appelle *amortissement*. On considère l'équation

$$\left\{ \begin{array}{rcll} \partial_t u + P(u) + A(u) & = & 0 & (t,x) \in]0,T[\times\Omega, \\ u(0) & = & u^0 & x \in \Omega, \\ Bu & = & 0 & (t,x) \in]0,T[\times\partial\Omega. \end{array} \right. \tag{1.1.3}$$

On se place ici dans un cadre où les solutions de (1.1.3) sont globales : on prend $T = +\infty$ et on suppose que pour toutes les données initiales $u^0 \in \mathcal{F}_\Omega$ considérées, il existe une unique solution $u : [0, +\infty[\to \mathcal{F}_\Omega$ de (1.1.3).

Définition 1.1.10 (Stabilisation)**.** On dit qu'il y a *stabilisation pour l'amortissement A* si pour tout $u^0 \in \mathcal{F}_\Omega$, il existe des constantes $C, \lambda > 0$ telle que la solution u de (1.1.3) vérifie $\|u(t)\|_{\mathcal{F}_\Omega} \leq Ce^{-\lambda t}$, pour tout $t \geq 0$.

Pour l'équation des ondes, on prendra un amortissement de la forme $A(u) = \gamma\partial_t u$, où γ est une fonction positive bornée. Cette notion est parfois appelée *stabilisation exponentielle*. Des formes plus faibles sont possibles, mais elles ne seront pas étudiées ici.

1.2 Équations des ondes linéaires

1.2.1 Définitions, existence des solutions

Soient Ω l'intérieur et $\partial\Omega$ la frontière d'une variété riemannienne à bord, de dimension $d \geq 1$, de métrique g. On supposera toujours que g et $\partial\Omega$ sont lisses, et que Ω est connexe. On note

$\Delta = \Delta_g$ l'opérateur de Laplace-Beltrami. L'équation des ondes linéaire associée est

$$\begin{cases} \partial_t^2 u - \Delta u = F & (t,x) \in \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) = (u^0, u^1) & x \in \Omega, \\ u = f & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{1.2.1}$$

avec, par exemple, une donnée initiale $(u^0, u^1) \in H^1(\Omega) \times L^2(\Omega)$, un second membre $F \in L^1_{\text{loc}}(\mathbb{R}, L^2(\Omega))$, et une donnée au bord de type Dirichlet $f \in L^2_{\text{loc}}(\mathbb{R}, L^2(\Omega))$. On utilisera parfois la notation $\square = \partial_t^2 - \Delta$. On construit généralement la solution de (1.2.1) en prenant le point de vue des semi-groupes. On note

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & 0 \end{pmatrix} : D(A) \subset \mathcal{H} \to \mathcal{H}$$

avec $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$, et $D(A) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$. On peut montrer que l'opérateur non-borné A vérifie les hypothèses du théorème de Hille-Yosida (voir par exemple [Vra03], Theorem 3.1.1), qui indique que A est le générateur infinitésimal d'un semi-groupe fortement continu $S(t)$. On a alors $S(t)(u^0, u^1) = (u(t), \partial_t u(t))$, pour $t \in \mathbb{R}$, où u est la solution de (1.2.1) avec $f = F = 0$. La solution avec $f = 0$ et $F \neq 0$ est ensuite donnée par la formule de Duhamel

$$(u(t), \partial_t u(t)) = S(t)\left(u^0, u^1\right) + \int_0^t S(t-s)\,(0, F(s))\,\mathrm{d}s, \quad t \in \mathbb{R}. \tag{1.2.2}$$

On construit la solution dans le cas $f \neq 0$ par une méthode de dualité (par exemple, voir le théorème 1.3.2).

Équation de Klein-Gordon. Pour $\beta > 0$, l'équation

$$\begin{cases} \partial_t^2 u - \Delta u + \beta u = F & (t,x) \in \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) = (u^0, u^1) & x \in \Omega, \\ u = f & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases}$$

s'appelle l'équation de Klein-Gordon. Le terme βu est appelé *terme de masse*. On essayera dans cette thèse de traiter à la fois le cas de l'équation des ondes, et le cas de l'équation de Klein-Gordon, quand c'est possible : pour cela, on prendra généralement $\beta \in \mathbb{R}$, on notera

$$\|u\|^2_{H^1_0} = \int_\Omega \left(|\nabla u|^2 + \beta |u|^2\right) \mathrm{d}x, \quad u \in H^1_0(\Omega),$$

et on supposera que l'inégalité de Poincaré $\|u\|_{H^1_0} \gtrsim \|u\|_{L^2}$ est vérifiée, pour $u \in H^1_0(\Omega)$. Si le domaine Ω est tel que cette inégalité est vraie pour $\beta = 0$ (comme par exemple dans le cas où Ω est un ouvert borné), alors un résultat qui prend cette inégalité comme hypothèse sera valide pour l'équation des ondes et l'équation de Klein-Gordon, et à l'inverse, si Ω est tel que cette inégalité est vraie seulement pour $\beta > 0$ (comme par exemple dans le cas où $\Omega = \mathbb{R}^d$), alors un résultat qui prend cette inégalité comme hypothèse sera valide seulement pour l'équation de Klein-Gordon.

Remarque 1.2.1. D'autres types de données au bord sont possibles pour l'équation (1.2.1), comme par exemple des données au bord de type Neumann. Dans cette thèse, on ne considérera que le cas des données au bord de type Dirichlet, homogène ($u_{|\partial\Omega} = 0$) ou inhomogène ($u_{|\partial\Omega} = f \neq 0$).

1.2.2 Bicaractéristiques

On introduit ici des courbes qui jouent un rôle particulier dans l'étude des équations des ondes. Ces courbes sont définies dans l'espace cotangeant $T^*(\mathbb{R}\times\Omega)$, dont on notera (t,x,τ,ξ) les variables. Le symbole principal de l'opérateur des ondes est $p(t,x,\tau,\xi) = |\xi|_x^2 - \tau^2$. On introduit le *champ hamiltonien de p*, défini par

$$H_p = \left(\frac{\partial p}{\partial \tau}, \frac{\partial p}{\partial \xi_1}, \cdots, \frac{\partial p}{\partial \xi_d}, -\frac{\partial p}{\partial t}, -\frac{\partial p}{\partial x_1}, \cdots, -\frac{\partial p}{\partial x_d}\right).$$

Les courbes intégrales associées à H_p sont appelées *courbes hamiltoniennes de p*. Par définition, une courbe hamiltonienne et donc une solution maximale $s \mapsto (t(s), x(s), \tau(s), \xi(s))$ de l'équation différentielle ordinaire

$$\frac{\mathrm{d}}{\mathrm{d}s}(t,x,\tau,\xi)(s) = H_p\left(t(s), x(s), \tau(s), \xi(s)\right), \quad s \in I,$$

sur un intervalle $I \subset \mathbb{R}$. Comme $H_p p = 0$, la fonction $s \mapsto p\left(t(s), x(s), \tau(s), \xi(s)\right)$ est constante ; on appelle *bicaractéristique de p* une courbe hamiltonienne de p sur laquelle $p = 0$. Les singularités des solutions d'équations des ondes sont transportées le long des courbes bicaractéristiques.

Dans le cas $\Omega = \mathbb{R}^d$, avec la métrique euclidienne, les bicaractéristiques de p sont les courbes de la forme

$$(t(s), x(s), \tau(s), \xi(s)) = \left(-2\tau_0 s + t_0, 2\xi_0 s + x_0, \tau_0, \xi_0\right), \quad s \in \mathbb{R},$$

pour $(t_0, x_0, \tau_0, \xi_0)$ vérifiant $\tau_0^2 = |\xi_0|^2$. Dans le cas d'une variété lisse sans bord Ω, on peut montrer que les bicaractéristiques sont définies sur $\mathbb{R}$, et que la fonction $s \mapsto x(s)$ est une géodésique, à une reparamétrisation près.

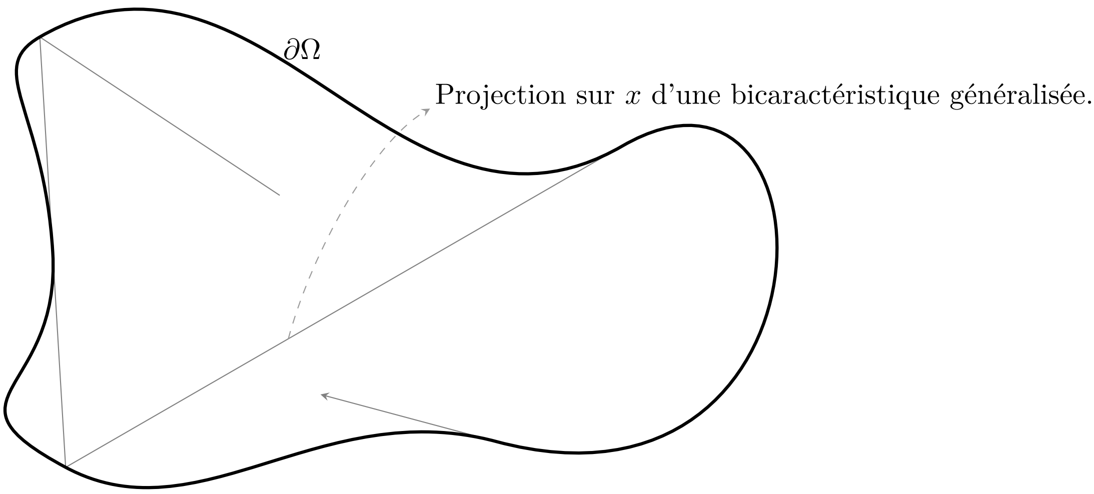


FIGURE 1.2 : Les bicaractéristiques généralisées correspondent aux rayons de l'optique géométrique.

La compréhension de la propagation des singularités d'une solution de l'équation des ondes dans le cas d'une variété à bord est due au travail fondateur de Melrose et Sjöstrand

([MS78], [MS82]), qui ont introduit les *bicaractéristiques généralisées*. Ces courbes coïncident avec les bicaractéristiques en dehors de la frontière. On se contente ici d'une description approximative (Figure 1.2) de leur comportement au bord. Si une bicaractéristique rencontre la frontière de manière transverse, elle rebondit comme le ferait un rayon lumineux, selon les lois de l'optique géométrique (on parle de point *hyperbolique*). Si une bicaractéristique rencontre la frontière de façon tangente, on distingue deux cas : si c'est possible, on prolonge la bicaractéristique de façon à ce que le seul point d'intersection avec le bord soit le point de contact (on parle d'un point *diffractif*), et sinon, la bicaractéristique est prolongée par une trajectoire à l'intérieur du bord. On peut rendre cette définition rigoureuse, montrer que les bicaractéristiques existent globalement, et sont uniques si aucun contact d'ordre infini n'a lieu entre la frontière et une bicaractéristique. L'unicité permet de définir un flot. Les bicaractéristiques sont de plus continues, hormis au niveau des points hyperboliques.

Toujours de façon approximative, on dit qu'un point de contact $(t(s), x(s), \tau(s), \xi(s))$ entre une bicaractéristique et la frontière est *non-diffractif* si dans un voisinage $]s-\varepsilon, s+\varepsilon[$, la bicaractéristique généralisée ne coïncide pas avec une bicaractéristique classique. En de tels points, le contact avec le bord affecte la dynamique. Enfin, on appelle *géodésiques généralisées* les projections sur x des bicaractéristiques généralisées.

Remarque 1.2.2. Dans toute cette thèse, on supposera qu'il n'y a pas de contact d'ordre infini entre le bord et les bicaractéristiques.

1.3 Systèmes d'équations des ondes linéaires à différents niveaux de régularité

Nous commençons par l'étude d'équations des ondes linéaires avant de nous consacrer à des questions concernant des équations non-linéaires. Un travail effectué sur des problèmes inverses pour des systèmes d'équations des ondes, avec Lauri Oksanen, l'année précédant cette thèse, a naturellement mené à une question de contrôle. On présente ici ces systèmes d'équations des ondes linéaires couplées, qui s'écrivent de façon équivalente sous la forme d'équations des ondes à valeurs vectorielles.

1.3.1 Définitions, existence des solutions

On considère Ω l'intérieur et $\partial\Omega$ la frontière d'une variété riemannienne compacte à bord, de dimension $d \geq 1$, de métrique g, et on fixe un entier $N \geq 1$. On note $\overline{\Omega} = \Omega \cup \partial\Omega$. Soit $X \in \mathscr{C}^\infty(\overline{\Omega}, T\overline{\Omega} \otimes \mathbb{C}^{N\times N})$ un opérateur différentiel d'ordre 1, donné dans une carte (U, x) par

$$X = X^j \frac{\partial}{\partial x^j}, \qquad \text{avec } X^j \in \mathscr{C}^\infty(U, \mathbb{C}^{N\times N}) \text{ pour } j \in [\![1, d]\!].$$

On prend également $q \in \mathscr{C}^\infty(\overline{\Omega}, \mathbb{C}^{N\times N})$. On note $\mathsf{P} = \Delta - X - q$, où $\Delta = \Delta \mathrm{Id}_{\mathbb{C}^N}$ est la version vectorielle de l'opérateur de Laplace-Beltrami, définie composante par composante. L'adjoint de P, noté P^*, est de la même forme que P.

Conditions de compatibilité. Si u est une fonction régulière qui vérifie $u(0) = u^0$ et $u_{|\partial\Omega} = f$, alors $(u^0)_{|\partial\Omega} = f(0)$. Si cette condition n'est pas vérifiée, alors une solution de (1.2.1) aura nécessairement une régularité assez faible. Pour construire des solutions à différents niveaux de régularité, il faut donc introduire des espaces de fonctions vérifiant certaines conditions, appelées *conditions de compatibilité*. Notons $\mathsf{P}_{\mathscr{D}'} : \mathscr{D}'(\Omega, \mathbb{C}^N) \longrightarrow \mathscr{D}'(\Omega, \mathbb{C}^N)$ l'action de P au sens des distributions. On pose $\mathcal{K}^0 = L^2(\Omega, \mathbb{C}^N)$, et pour $m \in \mathbb{N}^*$,

$$\mathcal{K}^m = \left\{ u \in H^m(\Omega, \mathbb{C}^N), \mathsf{P}_{\mathscr{D}'}^k u \in H_0^1(\Omega, \mathbb{C}^N) \text{ pour } k \in \left[\!\left[0, \left\lfloor \frac{m-1}{2} \right\rfloor \right]\!\right] \right\},$$

où $\lfloor \cdot \rfloor$ est la fonction partie entière. On a par exemple $\mathcal{K}^1 = H_0^1(\Omega, \mathbb{C}^N)$ et $\mathcal{K}^2 = H^2(\Omega, \mathbb{C}^N) \cap H_0^1(\Omega, \mathbb{C}^N)$. Pour $m \in \mathbb{N}$, $\mathcal{K}^m$ est un espace de Hilbert pour le produit scalaire de $H^m(\Omega, \mathbb{C}^N)$. On définit ensuite $\mathcal{K}^s$ pour $s \geq 0$ par interpolation. On note $\mathcal{K}_*^s$, pour $s \geq 0$, l'espace obtenu en remplaçant P par P^* dans la définition précédente, et on définit, pour $s < 0$, $\mathcal{K}^s$ comme le dual de $\mathcal{K}_*^{-s}$, et $\mathcal{K}_*^s$ comme le dual de $\mathcal{K}^{-s}$. On dispose ainsi d'une échelle de régularité compète, similaire à une régularité de type Sobolev, qui généralise les domaines itérés du laplacien de Dirichlet. Pour $s_1 > s_2$, on peut définir une injection naturelle de $\mathcal{K}^{s_1}$ dans $\mathcal{K}^{s_2}$, que l'on notera $\iota_{\mathcal{K}^{s_1} \to \mathcal{K}^{s_2}}$, et pour $s \in \mathbb{R}$, on peut donner un sens à l'action de P en tant qu'opérateur de $\mathcal{K}^{s+1}$ dans $\mathcal{K}^{s-1}$, que l'on notera P_s. On peut démontrer que toutes les formules naturelles faisant intervenir P_s et les injections sont vraies (voir Proposition 3.1.8). Ces notations, claires, mais un peu lourdes, sont nécessaires pour définir rigoureusement des solutions d'équations des ondes de faible régularité.

Systèmes d'équations à différents niveaux de régularité. Soient $T > 0$, et $\Theta = (\Theta_1, \cdots, \Theta_N) \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$. On note

$$\operatorname{diag}(\Theta) = \begin{pmatrix} \Theta_1 & 0 & \cdots & 0 \\ 0 & \Theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Theta_N \end{pmatrix}.$$

Pour $s \in \mathbb{R}$, on considérera deux types de systèmes d'équations des ondes : les équations avec donnée au bord homogène,

$$\left\{ \begin{array}{rcll} \partial_t^2 u - \mathsf{P}u & = & 0 & (t,x) \in]0,T[\times\Omega, \\ (u(0,\cdot), \partial_t u(0,\cdot)) & = & (u^0, u^1) & x \in \Omega, \\ u & = & 0 & (t,x) \in]0,T[\times\partial\Omega, \end{array} \right. \tag{1.3.1}$$

et les équations avec donnée au bord inhomogène,

$$\left\{ \begin{array}{rcll} \partial_t^2 v - \mathsf{P}^* v & = & 0 & (t,x) \in]0,T[\times\Omega, \\ (v(0,\cdot), \partial_t v(0,\cdot)) & = & 0 & x \in \Omega, \\ v & = & \operatorname{diag}(\Theta) f & (t,x) \in]0,T[\times\partial\Omega, \end{array} \right. \tag{1.3.2}$$

avec $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$, et $f \in H^s((0,T) \times \partial\Omega, \mathbb{C}^N)$. On dispose d'un résultat d'existence et d'unicité pour ces deux équations.

Théorème 1.3.1 (Solution d'un système d'équations des ondes avec donnée au bord homogène)**.** *Soient $T > 0$, $s \in \mathbb{R}$ et $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. Il existe un unique*

$$u \in \mathscr{C}^0([0,T], \mathcal{K}^{s+1}) \cap \mathscr{C}^1([0,T], \mathcal{K}^s) \cap \mathscr{C}^2([0,T], \mathcal{K}^{s-1})$$

tel que $(u(0), \partial_t u(0)) = (u^0, u^1)$ et $\partial_t^2 u(t) = \mathsf{P}_s u(t)$, pour tout $t \in [0,T]$, que l'on appelle solution de l'équation (1.3.1) avec donnée initiale (u^0, u^1)*. On a également les résultats suivants.*

(i) Pour tout $\delta > 0$*, si* $\tilde{u}$ *est la solution de l'équation* (1.3.1) *avec donnée initiale*

$$\left(\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{s+1-\delta}}u^0, \iota_{\mathcal{K}^{s}\to\mathcal{K}^{s-\delta}}u^1\right),$$

alors pour tout $t \in [0,T]$*, on a* $\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{s+1-\delta}}u(t) = \tilde{u}(t)$*.*

(ii) On peut définir une dérivée normale au bord de u*, notée* $\partial_\nu u$*, qui est un élément de* $H^s((0,T)\times\partial\Omega,\mathbb{C}^N)$*, et qui coïncide avec la dérivée normale usuelle si* u *est suffisamment régulière.*

(iii) On suppose ici que $s \geq 0$*. Pour* $F \in L^1((0,T),H_0^s(\Omega,\mathbb{C}^N))$*, on définit la solution de*

$$\begin{cases} \partial_t^2 u - \mathsf{P}u &= F & (t,x) \in]0,T[\times\Omega, \\ (u(0,\cdot),\partial_t u(0,\cdot)) &= (u^0,u^1) & x\in\Omega, \\ u &= 0 & (t,x)\in]0,T[\times\partial\Omega, \end{cases} \tag{1.3.3}$$

avec la formule de Duhamel (1.2.2)*. Cette solution vérifie* $u \in \mathscr{C}^0((0,T),\mathcal{K}^{s+1}) \cap \mathscr{C}^1((0,T),\mathcal{K}^s)$*, et on peut également donner un sens* $H^s((0,T)\times\partial\Omega,\mathbb{C}^N)$ *à sa dérivée normale.*

Théorème 1.3.2 (Solution d'un système d'équations des ondes avec donnée au bord inhomogène)**.** *Soient* $T > 0$*,* $s \in \mathbb{R}$*,* $\Theta \in \mathscr{C}_c^\infty((0,T)\times\partial\Omega,\mathbb{C}^N)$*, et* $f \in H^s((0,T)\times\partial\Omega,\mathbb{C}^N)$*. Pour* $s \leq 0$*, on définit la* solution de l'équation (1.3.2) avec donnée au bord f *par dualité avec les solutions de* (1.3.3) *au niveau de régularité* $|s|$*. Si* $s > 0$*, on définit la solution comme pour* $s = 0$*. Dans tous les cas, on a*

$$v \in \mathscr{C}^0((0,T),H^s(\Omega,\mathbb{C}^N)) \cap \mathscr{C}^1((0,T),H^{s-1}(\Omega,\mathbb{C}^N)) \cap \mathscr{C}^2((0,T),H^{s-2}(\Omega,\mathbb{C}^N)),$$

et $\partial_t^2 v = \mathsf{P}^*_{\mathscr{D}'}v$ *au sens* $\mathscr{D}'((0,T)\times\Omega,\mathbb{C}^N)$*. De plus, si* $s \geq 1$*, alors* $v(t)_{|\partial\Omega} = (\mathrm{diag}(\Theta)f)_{|\{t\}\times\partial\Omega}$ *dans* $H^{s-\frac{1}{2}}(\partial\Omega,\mathbb{C}^N)$*, au sens des opérateurs de trace usuels sur les espaces de Sobolev.*

Une construction rigoureuse des solutions d'une équation des ondes avec donnée au bord de type Dirichlet inhomogène, à différents niveaux de régularité, se trouve dans l'article de Lasiecka, Lions et Triggiani [LLT86]. Par rapport aux résultats de [LLT86], la principale nouveauté des théorèmes 1.3.1 et 1.3.2 réside dans le fait de considérer des systèmes. Une version détaillée, avec une démonstration complète, de ces deux théorèmes, est donnée au chapitre 3.

1.3.2 Schéma de démonstration des théorèmes 1.3.1 et 1.3.2

Expliquons, de façon approximative et simplifiée, les étapes de la constructions des solutions.

La première étape, très classique, consiste à prendre un point de vue semi-groupe, et à construire la solution de (1.3.1) pour $s \geq 0$ en utilisant le théorème de Hille-Yosida. La seule difficulté consiste à vérifier que les espaces contenant les conditions de compatibilité correspondent bien aux domaines itérés du générateur infinitésimal considéré.

L'idée de la deuxième étape est de construire la solution de (1.3.1) pour $s < 0$, en utilisant la construction pour $s \geq 0$, et le fait que si u est une solution, on souhaite faire en sorte que $\mathsf{P}^k u$ soit également une solution (pour une régularité plus faible), pour $k \in \mathbb{N}$. Plus précisément, on démontre qu'il existe $z \in \mathbb{C}$ tel que l'opérateur $\mathsf{P} + z\mathrm{Id}$ est un isomorphisme entre deux niveaux de régularité, qui préserve les conditions de compatibilité. Étant donnée une donnée initiale (u^0,u^1) de régularité $s < 0$, on considère alors la solution $\tilde{u}$ de donnée initiale $\left((\mathsf{P}+z\mathrm{Id})^{-k}u^0,(\mathsf{P}+z\mathrm{Id})^{-k}u^1\right)$, avec k assez grand, puis on définit la solution de donnée initiale (u^0,u^1) par $u=(\mathsf{P}+z\mathrm{Id})^k\tilde{u}$.

La troisième étape consiste à terminer la démonstration du théorème 1.3.1, en montrant notamment les résultats concernant la régularité de la dérivée normale, et ceux sur les solutions de l'équation avec terme source. Pour la dérivée normale dans le cas $s\geq 0$, la démonstration est assez classique et suit essentiellement [LLT86]. Pour une solution de régularité $s<0$, on emploie la stratégie suivante, en supposant ici pour simplifier que l'on peut prendre $z=0$ dans la deuxième étape : par définition, une solution peu régulière u est de la forme $\mathsf{P}^k\tilde{u}$, avec $\tilde{u}$ une solution assez régulière, et qui a donc une dérivée normale bien définie. Poser $\partial_\nu u=\mathsf{P}^k\partial_\nu\tilde{u}$ n'ayant pas de sens, on écrit $\mathsf{P}^k\tilde{u}=\partial_t^{2k}\tilde{u}$, puis on prend comme définition $\partial_\nu u=\partial_t^{2k}\partial_\nu\tilde{u}$. Cette formule est un peu compliquée, mais on montre qu'elle implique toutes les propriétés naturelles attendues. Pour les solutions avec terme source, on les définit avec la formule de Duhamel, et on montre les résultats de régularité annoncés.

Dans une quatrième étape, on peut alors démontrer le théorème 1.3.2 pour $s\leq 0$, par dualité avec les résultats du théorème 1.3.1 pour $s\geq 0$. Pour montrer les propriétés de régularité annoncées, si f est une donnée au bord régulière, l'idée consiste à considérer une fonction $\tilde{f}$ régulière, définie sur tout le domaine, et coïncidant avec f au bord. On montre alors que si v est la solution de (1.3.2) avec donnée au bord f, alors $v-\tilde{f}$ est une solution de l'équation homogène (1.3.1).

Enfin, dans une cinquième et dernière étape, on démontre le théorème 1.3.2 pour $s>0$. La solution pour $s>0$ est définie comme la solution pour $s=0$, et on démontre par récurrence sur $s\in\mathbb{N}^*$ qu'elle vérifie les résultats de régularité attendus. L'idée de considérer $v-\tilde{f}$ fonctionne à nouveau pour le cas $s=1$, mais pas pour les cas suivants, à cause des conditions de compatibilité. Pour l'hérédité de la récurrence, on gagne un cran en régularité en considérant la dérivée en temps d'une solution.

1.4 Équations des ondes non-linéaires

Toutes les équations non-linéaires considérées sont à valeurs dans $\mathbb{R}$.

1.4.1 Équation des ondes cubique

Soient Ω l'intérieur et $\partial\Omega$ la frontière d'une variété riemannienne à bord de dimension 3, supposée lisse et connexe. Soit $\beta\in\mathbb{R}$ tel que l'inégalité de Poincaré

$$\|u\|_{H_0^1}^2=\int_\Omega\left(|\nabla u|^2+\beta|u|^2\right)\mathrm{d}x\gtrsim\|u\|_{L^2}^2\,,\quad u\in H_0^1(\Omega),\tag{1.4.1}$$

est vérifiée. Pour $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$, on considère l'équation des ondes (ou de Klein-Gordon) cubique donnée par

$$\left\{\begin{array}{rcll}\partial_t^2u-\Delta u+\beta u&=&\pm u^3&(t,x)\in\mathbb{R}\times\Omega,\\(u(0),\partial_tu(0))&=&(u^0,u^1)&x\in\Omega,\\u&=&0&(t,x)\in\mathbb{R}\times\partial\Omega.\end{array}\right.\tag{1.4.2}$$

On dit que cette équation est défocalisante quand la non-linéarité est $-u^3$, et focalisante quand la non-linéarité est $+u^3$. Ces deux types d'équations ont des propriétés très différentes.

Existence locale, explosion. Dans les deux cas, on peut démontrer l'*existence locale* et l'*unicité* des solutions, par une méthode de point fixe. L'idée, très classique, consiste à considérer l'équation

$$\begin{cases} \partial_t^2 u - \Delta u + \beta u &= \pm U^3 & (t,x) \in]-\varepsilon,\varepsilon[\times\Omega, \\ (u(0),\partial_t u(0)) &= (u^0,u^1) & x\in\Omega, \\ u &= 0 & (t,x)\in]-\varepsilon,\varepsilon[\times\partial\Omega, \end{cases}$$

avec $\varepsilon > 0$, et $U \in X_\varepsilon = \mathscr{C}^0(]-\varepsilon,\varepsilon[,H_0^1(\Omega)) \cap \mathscr{C}^1(]-\varepsilon,\varepsilon[,L^2(\Omega))$. Une estimation d'énergie pour l'équation des ondes linéaire avec terme source donne alors

$$\|u\|_{X_\varepsilon} \lesssim \left\| \left(u^0,u^1\right) \right\|_{H_0^1\times L^2} + \left\| U^3 \right\|_{L^1(]-\varepsilon,\varepsilon[,L^2(\Omega))} \lesssim \left\| \left(u^0,u^1\right) \right\|_{H_0^1\times L^2} + \varepsilon \|U\|_{X_\varepsilon}^3 , \qquad (1.4.3)$$

d'après l'injection de Sobolev $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. Si ε est suffisamment petit, on peut également montrer que l'opérateur $U \mapsto u$ est contractant. Le théorème de point fixe de Banach-Picard permet alors de construire la solution u sur $]-\varepsilon,\varepsilon[$. On obtient aussi un *critère d'explosion* : si la solution de (1.4.2) n'est pas définie sur $\mathbb{R}_+$, alors il existe un temps $T > 0$ tel que

$$\|(u(t),\partial_t u(t))\|_{H_0^1\times L^2} \longrightarrow +\infty, \text{ quand } t \longrightarrow T^-.$$

On définit l'*énergie d'une solution* au temps t par

$$E\left(u(t),\partial_t u(t)\right) = \frac{1}{2}\|u(t)\|_{H_0^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 \mp \frac{1}{4}\|u(t)\|_{L^4}^4 .$$

Cette énergie est constante au cours du temps. Dans le cas d'une non-linéarité défocalisante, on a donc

$$\frac{1}{2}\|u(t)\|_{H_0^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 \leq E\left(u(t),\partial_t u(t)\right) = E\left(u^0,u^1\right),$$

impliquant que toutes les solutions sont globales. Ce raisonnement ne fonctionne pas pour une non-linéarité focalisante : le fait que l'énergie soit constante ne donne pas de contrôle sur la taille de la solution. On peut montrer qu'il existe des solutions qui explosent en temps fini (voir théorème 2.3.1). Un exemple explicite, dans le cas d'une variété compacte sans bord, avec $\beta = 1$, est donné par la fonction

$$u(t,x) = \frac{\sqrt{2}}{\sin\left(\frac{\pi}{2}-t\right)}, \quad (t,x) \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\times \Omega.$$

Solutions stationnaires. On appelle *solution stationnaire* une solution de (1.4.2) indépendante du temps. Si $u(t) = u^0$ est une telle solution, on voit en intégrant par parties que

$$\left\|u^0\right\|_{H_0^1}^2 = \pm \left\|u^0\right\|_{L^4}^4 .$$

En particulier, la seule solution stationnaire de l'équation cubique défocalisante est zéro. À l'inverse, on peut montrer que des solutions stationnaires non-nulles existent dans le cas focalisant. On a notamment le résultat suivant, démontré en 1975 par Payne et Sattinger [PS75].

Théorème 1.4.1. *On suppose que Ω est un ouvert borné de $\mathbb{R}^3$. Il existe une unique solution stationnaire positive de l'équation cubique focalisante, appelée* état fondamental*, et notée Q, d'énergie $E(Q,0) > 0$ minimale.*

On verra que l'énergie de l'état fondamental joue un rôle particulier dans l'étude des solutions de l'équation focalisante. Supposer que Ω est un ouvert borné de $\mathbb{R}^3$ n'est pas nécessaire pour construire un état fondamental : on fait cette hypothèse pour assurer son unicité. Dans le cas $\Omega = \mathbb{R}^d$, on peut montrer que l'état fondamental est unique aux translations près, et est radial (voir [Cof72]).

1.4.2 Équation non-linéaire plus générale

Soient maintenant Ω l'intérieur et $\partial\Omega$ la frontière d'une variété riemannienne à bord de dimension $d \geq 2$, lisse et connexe. Soit $\beta \in \mathbb{R}$ tel que l'inégalité de Poincaré (1.4.1) est vérifiée. On prend une non-linéarité $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ vérifiant $f(0) = 0$, et

$$|f(s_1) - f(s_2)| \leq C|s_1 - s_2| \left(1 + |s_1|^{p_0-1} + |s_2|^{p_0-1}\right), \quad s_1, s_2 \in \mathbb{R},$$

avec $C > 0$ et $1 < p_0 < +\infty$. On considère ensuite l'équation

$$\left\{\begin{array}{rcll} \partial_t^2 u - \Delta u + \beta u & = & f(u) & (t,x) \in \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) & = & (u^0, u^1) & x \in \Omega, \\ u & = & 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{array}\right. \tag{1.4.4}$$

pour $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. Pour montrer l'existence locale de solutions, en regardant (1.4.3), on voit qu'il est naturel de chercher des estimations sur $\|f(u)\|_{L^1(]-\varepsilon,\varepsilon[,L^2(\Omega))}$, pour $\varepsilon > 0$. Si $d > 2$ et $p_0 \leq \frac{d}{d-2}$, alors l'injection de Sobolev $H_0^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$ donne

$$\|u^{p_0}\|_{L^1(]-\varepsilon,\varepsilon[,L^2(\Omega))} \lesssim \varepsilon \|u\|_{L^\infty(]-\varepsilon,\varepsilon[,H_0^1(\Omega))},$$

ce qui permet de construire localement la solution. Ce point de vue n'est pas optimal : on peut en réalité construire la solution localement pour certains exposants $p_0 > 1$ ne vérifiant pas la condition $p_0 \leq \frac{d}{d-2}$, en utilisant des estimations appelées *inégalités de Strichartz.*

Théorème 1.4.2 (Inégalités de Strichartz (locales en temps))**.** *Il existe une famille de couples d'exposants $\Lambda_\Omega \subset \left\{(p,q), 1 \leq p < +\infty, 2 \leq q < \frac{d+2}{d-2}\right\}$, vérifiant la propriété suivante. Pour tous $(p,q) \in \Lambda_\Omega$ et $T > 0$, il existe une constante $C > 0$ telle que*

$$\|u\|_{L^p([0,T],L^q(\Omega))} \leq C\left(\left\|\left(u^0, u^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|g\|_{L^1([0,T],L^2(\Omega))}\right),$$

pour tous $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $g \in L^1([0,T], L^2(\Omega))$, où u est la solution de

$$\left\{\begin{array}{rcll} \partial_t^2 u - \Delta u + \beta u & = & g & (t,x) \in \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) & = & (u^0, u^1) & x \in \Omega, \\ u & = & 0 & (t,x) \in \mathbb{R} \times \partial\Omega. \end{array}\right.$$

L'ensemble Λ_Ω dépend de la géométrie de Ω. Si $3 \leq d \leq 5$, et si Ω est une variété sans bord, on peut montrer que

$$\left\{(p,q),\ 1 \leq p \leq \infty,\ 2 \leq q \leq \frac{2d}{d-3},\ (q,d) \neq (+\infty, 3),\ \frac{1}{p} + \frac{d}{q} \geq \frac{d}{2} - 1\right\} \subset \Lambda_\Omega.$$

En particulier, dans ce cas, on trouve $(p_0, 2p_0) \in \Lambda_\Omega$ pour $1 \leq p_0 \leq \frac{d+2}{d-2}$, avec $\frac{1}{p_0} + \frac{d}{q_0} > \frac{d}{2} - 1$ si $p_0 < \frac{d+2}{d-2}$, ce qui permet de construire localement la solution. On dit que $\frac{d+2}{d-2}$ est l'*exposant critique*, et on parle de non-linéarité sous-critique, critique ou sur-critique, selon la position de p_0 par rapport à cet exposant.

Si Ω est une variété à bord, déterminer l'ensemble Λ_Ω n'est pas simple : des contre-exemples de Ivanovici [Iva12] montrent que les inégalités de Strichartz ne sont pas vérifiées pour tous les exposants du cas sans bord. Dans le cas où il existe $q_0 > p_0$ tel que $(q_0, 2p_0) \in \Lambda_\Omega$, la méthode précédente permet de construire localement la solution. Cependant, comme dans le cas cubique, sans hypothèse supplémentaire sur f, les solutions ne sont pas toutes globales.

On dispose également d'une fonctionnelle d'*énergie*, qui généralise celle du cas cubique : on note $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$, et on introduit

$$E\left(u(t), \partial_t u(t)\right) = \frac{1}{2}\left\|u(t)\right\|_{H_0^1}^2 + \frac{1}{2}\left\|\partial_t u(t)\right\|_{L^2}^2 - \int_\Omega F\left(u(t)\right)\mathrm{d}x, \quad t \in \mathbb{R}.$$

En multipliant l'équation (1.4.4) par $\partial_t u$ et en intégrant par parties, on voit (au moins formellement) que l'énergie est une quantité conservée au cours du temps. Dans certains cas, comme par exemple si $F(s) \leq 0$ pour tout $s \in \mathbb{R}$, cela implique en particulier que toutes les solutions sont globales. Comme dans le cas cubique, on peut démontrer pour certaines non-linéarités que des solutions stationnaires ou explosives existent. On retrouve la valeur de l'exposant critique comme étant celle qui permet d'assurer que l'énergie est bien définie grâce à l'injection de Sobolev $H_0^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$: en effet, en prenant par exemple $f(s) = \pm s\left|s\right|^{p_0-1}$, on trouve

$$\left|\int_\Omega F\left(u(t)\right)\mathrm{d}x\right| \lesssim \left\|u(t)\right\|_{H_0^1}^{p_0+1}, \quad t \in \mathbb{R},$$

si $p_0 + 1 \leq \frac{2d}{d-2}$, c'est-à-dire $p_0 \leq \frac{d+2}{d-2}$.

1.4.3 Comportement en temps long

L'étude du comportement en temps long des solutions d'équations des ondes non-linéaires est un domaine de recherche très actif. De nombreux cas ont été considérés : par exemple, selon le domaine ($\Omega = \mathbb{R}^d$, ou bien $\Omega = \mathbb{R}^d \backslash K$, avec K compact), selon la non-linéarité (notamment le cas où f est une puissance, focalisante ou défocalisante, sous-critique, critique ou sur-critique), selon l'énergie des données initiales (par rapport à l'énergie de l'état fondamental dans le cas focalisant), ou encore selon la régularité des solutions (solutions dans $H^s \times H^{s-1}$, pour différentes valeurs de s, ou bien solutions dans les espaces de Sobolev homogènes $\dot{H}^s \times \dot{H}^{s-1}$). On ne présentera bien sûr pas tous ces résultats : on se contentera ici du cas de l'équation de Klein-Gordon cubique, $f(s) = \pm s^3$, sur $\Omega = \mathbb{R}^3$. On commence par une définition.

Définition 1.4.3. Soient u une solution de (1.4.4), et φ une solution stationnaire. On dit que u est une *solution scattering vers φ en $+\infty$* si u existe sur $\mathbb{R}_+$, et s'il existe $\left(v^0, v^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ telle que la solution v de l'équation linéaire

$$\left\{\begin{array}{rcll} \partial_t^2 v - \Delta v + \beta v &=& 0 & (t,x) \in \mathbb{R} \times \Omega, \\ (v(0), \partial_t v(0)) &=& \left(v^0, v^1\right) & x \in \Omega, \\ v &=& 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{array}\right.$$

vérifie $\left(u - v, \partial_t u - \partial_t v\right)(t) \longrightarrow (\varphi, 0)$ dans $H_0^1(\Omega) \times L^2(\Omega)$, quand $t \longrightarrow +\infty$.

On dit qu'une solution u est scattering vers φ en $-\infty$ si la solution $t \mapsto u(-t)$ est scattering vers φ en $+\infty$. On dit simplement qu'une solution est scattering si elle est scattering vers 0. Par définition, une solution scattering est une solution asymptotiquement proche d'une solution de l'équation linéaire. Ce comportement est un phénomène dispersif, et ne sera étudiée que lorsque Ω est non-borné, comme par exemple dans le cas $\Omega = \mathbb{R}^d$. On peut montrer dans certains cas qu'une solution est scattering si et seulement si elle a une *norme de Strichartz globale en temps* finie. On sera donc amené à démontrer des résultats analogues au théorème 1.4.2, dans le cas $T = +\infty$.

Décrivons maintenant les résultats connus concernant le comportement en temps long des solutions dans le cas $f(s) = \pm s^3$, $\beta = 1$, et $\Omega = \mathbb{R}^3$. Le cas défocalisant $f(s) = -s^3$ a été considéré par de nombreux auteurs, et dans ce cas, toutes les solutions de (1.4.4) sont scattering (vers 0) en $+\infty$ et en $-\infty$ (voir par exemple l'article de Brenner [Bre84]). Le cas focalisant $f(s) = s^3$ présente des phénomènes plus variés. En 1975, Payne et Sattinger [PS75] ont établi une partition explicite des données initiales d'énergie $E\left(u^0, u^1\right) < E\left(Q, 0\right)$ en deux sous-ensembles $\mathcal{K}^+$ et $\mathcal{K}^-$, $\mathcal{K}^-$ contenant les données initiales des solutions qui explosent en temps fini positif et négatif, et $\mathcal{K}^+$ contenant les données initiales des solutions définies sur $\mathbb{R}$ (voir théorème 2.3.1 ci-dessous). En 2011, Ibrahim, Masmoudi et Nakanishi [IMN11] ont démontré que les solutions avec données initiales dans $\mathcal{K}^+$ sont toutes scattering (avec une méthode introduite par Kenig et Merle [KM08]). Le comportement des solutions d'énergie initiale $E\left(u^0, u^1\right) < E\left(Q, 0\right) + \varepsilon$, avec $\varepsilon > 0$ petit, a ensuite été classifié par Nakanishi et Schlag dans [NS11a] (cas radial) et dans [NS12] (cas non-radial). En temps positifs ou négatifs, une solution peut exploser, être scattering vers 0, ou être scattering vers Q ; cela donne donc neuf types de comportements possibles pour une solution, et chaque combinaison possible en $-\infty$ et $+\infty$ peut se produire. Dans le cas de l'équation des ondes critique, l'existence de solutions dites hétéroclines avait été démontrée par Duyckaerts et Merle [DM08]. Dans le cas d'une donnée initiale d'énergie plus grande, des expérimentations numériques ont été effectuées (voir par exemple [DS11]). On pourra consulter le livre [NS11b] pour une présentation pédagogique de ces résultats.

Chapitre 2

Résumé des travaux

Chacune des sections suivantes correspond à un chapitre de cette thèse.

2.1 Changement de régularité pour la contrôlabilité de systèmes d'équations des ondes

2.1.1 Résultats principaux et contexte bibliographique

On se place ici dans le cadre des systèmes d'équations des ondes couplées, présenté dans la section 1.3. On fixe $s \in \mathbb{R}$, $T > 0$ et $\Theta \in \mathscr{C}_{\mathrm{c}}^{\infty}(]0,T[\times\partial\Omega,\mathbb{C}^N)$. On s'intéresse aux deux propriétés suivantes.

Définition 2.1.1 (Observabilité H^s pour Θ)**.** On dit qu'il y a *observabilité H^s pour* Θ s'il existe $C > 0$ tel que pour tout $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$, la solution u de (1.3.1) vérifie

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s} \leq C \left\|\operatorname{diag}(\Theta)\partial_\nu u\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)}.$$

Définition 2.1.2 (Contrôlabilité exacte H^s pour Θ)**.** On dit qu'il y a *contrôlabilité exacte H^s pour* Θ si pour tout $(\varphi^0, \varphi^1) \in \mathcal{K}_*^s \times \mathcal{K}_*^{s-1}$, il existe un contrôle $f \in H^s((0,T)\times\partial\Omega,\mathbb{C}^N)$ tel que la solution v de (1.3.2) vérifie $(v(T), \partial_t v(T)) = (\varphi^0, \varphi^1)$.

Le lien entre contrôlabilité et observabilité est donné par le résultat suivant.

Lemme 2.1.3. *Pour tout $s \in \mathbb{R}$, la contrôlabilité H^s pour Θ et l'observabilité H^{-s} pour Θ sont équivalentes.*

La condition suivante, fondamentale, a été introduite par Bardos, Lebeau et Rauch [BLR92]. Dans le cas d'un domaine sans bord, l'utilisation des bicaractéristiques dans des problèmes de contrôle a été initiée par Rauch et Taylor [RT74].

Définition 2.1.4 (Condition de contrôle géométrique GCC)**.** Soient $\Gamma \subset \partial\Omega$ un ouvert, et $T > 0$. On dit que (Γ, T) *contrôle géométriquement* Ω, si pour toute bicaractéristique généralisée $s \mapsto (t(s), x(s), \tau(s), \xi(s))$ (associée à l'opérateur des ondes scalaire $\partial_t^2 - \Delta_g$), il existe un point non-diffractif $(t(s), x(s), \tau(s), \xi(s))$ tel que $t(s) \in]0,T[$ et $x(s) \in \Gamma$. On dit que Γ vérifie GCC si un tel temps T existe. On note $T_{\mathrm{GCC}}(\Gamma)$ l'infimum des T tels que (Γ, T) contrôle géométriquement Ω.

Dans le cas scalaire, Bardos, Lebeau et Rauch ont montré le résultat suivant.

Théorème 2.1.5. *On suppose ici que $N = 1$. Soient $\Gamma \subset \partial\Omega$ un ouvert qui vérifie GCC et tel que $T_{\mathrm{GCC}}(\Gamma) < T$. Soit $\Theta \in \mathscr{C}_c^\infty(]0,T[\times\partial\Omega,\mathbb{C})$ vérifiant $\Theta(t,x) \neq 0$ pour tout $(t,x) \in [t_0,t_1] \times \overline{\Gamma}$, avec $0 < t_0 < t_1 < T$ et $t_1 - t_0 > T_{\mathrm{GCC}}(\Gamma)$. Alors pour tout $s \in \mathbb{R}$, il y a contrôlabilité H^s pour Θ.*

Dans le cas particulier $\Theta = (\theta, \cdots, \theta)$, avec $\theta \in \mathscr{C}_c^\infty(]0,T[\times\partial\Omega,\mathbb{C})$, le terme de contrôle agit de la même façon sur toutes les composantes de la solution, et on peut raisonnablement conjecturer que le théorème 2.1.5 est vrai. Dans le cas général, on peut conjecturer qu'une condition géométrique plus compliquée est équivalente à la contrôlabilité H^s. Dans le cas d'un contrôle interne, ce travail a été effectué par Cui, Laurent et Wang [CLW20] (au niveau de régularité $s = 0$).

Dans le cas scalaire ($N = 1$), Burq et Gérard [BG97] ont montré que la condition de contrôle géométrique est essentiellement équivalente à la contrôlabilité L^2 (et donc à la contrôlabilité H^s, d'après le théorème 2.1.5), toujours dans le cas $N = 1$. Une conséquence de ces résultats est que la contrôlabilité H^s est en réalité indépendante de $s \in \mathbb{R}$. Cette remarque soulève la question naturelle suivante : peut-on déduire la contrôlabilité H^{s_1}, pour un certain $s_1 \in \mathbb{R}$, de la contrôlabilité H^{s_2}, pour un certain $s_2 \neq s_1$, sans passer par la condition géométrique de contrôle ? On donne une réponse positive à cette question, pour tout $N \geq 1$, avec le théorème suivant, qui est notre résultat principal concernant les systèmes d'équations des ondes linéaires.

Théorème 2.1.6. *Soient $s_1, s_2 \in \mathbb{R}$. Si $s_1 > s_2$, alors pour tout $\Theta \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$, la contrôlabilité H^{s_1} pour Θ implique la contrôlabilité H^{s_2} pour Θ. Si $s_1 < s_2$, alors pour tout $\Theta^1 = (\Theta^1_1, \cdots, \Theta^1_N) \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$ et $\Theta^2 = (\Theta^2_1, \cdots, \Theta^2_N) \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$, vérifiant $\Theta^2_k \neq 0$ sur $\operatorname{supp} \Theta^1_k$ pour tout $k \in [\![1, N]\!]$, la contrôlabilité H^{s_1} pour Θ^1 implique la contrôlabilité H^{s_2} pour Θ^2.*

De façon équivalente, avec le lemme 2.1.3, on peut changer le niveau de régularité d'une inégalité d'observabilité.

Théorème 2.1.7. *Soient $s_1, s_2 \in \mathbb{R}$. Si $s_1 < s_2$, alors pour tout $\Theta \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$, l'observabilité H^{s_1} pour Θ implique l'observabilité H^{s_2} pour Θ. Si $s_1 > s_2$, alors pour tout $\Theta^1 = (\Theta^1_1, \cdots, \Theta^1_N) \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$ et $\Theta^2 = (\Theta^2_1, \cdots, \Theta^2_N) \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$, vérifiant $\Theta^2_k \neq 0$ sur $\operatorname{supp} \Theta^1_k$ pour tout $k \in [\![1, N]\!]$, l'observabilité H^{s_1} pour Θ^1 implique l'observabilité H^{s_2} pour Θ^2.*

Remarque 2.1.8. Le fait d'utiliser deux fonctions différentes, Θ_1 et Θ_2, peut être évité en prenant une définition de l'observabilité qui fait intervenir des ouverts. Considérons $s \in \mathbb{R}$, $T' \in]0,T[$, et $\Gamma_1, \cdots, \Gamma_N$ des ouverts de $\partial\Omega$. On dit qu'il y a observabilité H^s pour $(T', \Gamma_1, \cdots, \Gamma_N)$ si pour tout $\Theta = (\Theta_1, \cdots, \Theta_N) \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$ tel que pour tout $k \in [\![1, N]\!]$, $\Theta_k \neq 0$ sur $\left[\frac{T-T'}{2}, \frac{T+T'}{2}\right] \times \overline{\Gamma_k}$, l'observabilité H^s pour Θ est vraie (au sens de la définition 2.1.1). Avec ce point de vue, le théorème 2.1.7 donne l'équivalence entre l'observabilité H^{s_1} pour $(T', \Gamma_1, \cdots, \Gamma_N)$ et l'observabilité H^{s_2} pour $(T', \Gamma_1, \cdots, \Gamma_N)$, pour tout $s_1, s_2 \in \mathbb{R}$.

Les solutions à un très faible niveau de régularité ne sont pas simples à définir et à manipuler, et grâce au théorème 2.1.6, pour démontrer qu'une certaine condition géométrique implique la contrôlabilité H^s pour tout $s \in \mathbb{R}$, il suffit de montrer par exemple que cette condition implique la contrôlabilité L^2. Au niveau de régularité L^2, une stratégie pourrait consister à utiliser des mesures de défaut, dans l'esprit de l'article [Bur97a] (voir Burq et Lebeau [BL01] pour le cas des systèmes). Un résultat analogue à celui de Cui, Laurent et

Wang [CLW20] pour le cas du contrôle au bord au niveau de régularité L^2, avec le théorème 2.1.6, donnerait immédiatement une caractérisation du contrôle au bord au niveau de régularité H^s, pour $s \in \mathbb{R}$. De plus, la stratégie employée pour démontrer le théorème 2.1.6 pourrait s'appliquer dans d'autres contextes, où une condition géométrique équivalente à la contrôlabilité n'est pas connue.

Un résultat de changement de régularité analogue est vrai pour la contrôlabilité (ou l'observabilité) interne, avec une démonstration plus simple. En utilisant les propriétés de contrôlabilité interne démontrées par Cui, Laurent et Wang [CLW20], on obtient une caractérisation de la contrôlabilité interne de systèmes, à tous les niveaux de régularité $s \in \mathbb{R}$.

Un projet en collaboration avec Lauri Oksanen, commencé avant le début de cette thèse, s'intéresse à des problèmes inverses pour l'équation (1.3.2) dans le cas $\Theta = (\theta, \cdots, \theta)$, avec $\theta \in \mathscr{C}_c^\infty(]0, T[\times \partial\Omega, \mathbb{C})$. La méthode employée utilise des solutions concentrées dans une certaine région de l'espace donnée, à la fois au niveau de régularité $s = 0$ et à un niveau de régularité élevé $s \gg 1$. Actuellement, notre résultat principal concernant les problèmes inverses prend la contrôlabilité H^s, pour tout $s \geq 0$, comme une hypothèse. Un résultat analogue à Cui, Laurent et Wang [CLW20], dans le cas du contrôle au bord, mais avec l'hypothèse restrictive $\Theta = (\theta, \cdots, \theta)$, permettrait d'énoncer notre résultat principal en prenant seulement comme hypothèse la condition de contrôle géométrique.

Les propriétés connues concernant la régularité du contrôle sont de natures très différentes du théorème 2.1.6. Des auteurs se sont intéressés à la question suivante : si on suppose que la contrôlabilité L^2 est vraie, on peut alors considérer un opérateur qui, à un état final à atteindre donné, associe un contrôle emmenant la solution de zéro à cet état final. Un tel opérateur n'est pas unique, mais la méthode HUM (Lions [Lio88]) en fournit un naturellement, qui donne est le contrôle de norme L^2 minimale. Il est naturel de se demander si cet opérateur préserve la régularité. Explicitement, si on considère par exemple l'opérateur HUM $\mathsf{g} : H_0^1(\Omega) \times L^2(\Omega) \to L^2((0,T) \times \partial\Omega)$, alors étant donné $(u^0, u^1) \in H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$, a-t-on $\mathsf{g}\,(u^0, u^1) \in H^1((0,T) \times \partial\Omega)$? Dehman et Lebeau [DL09] ont initié l'étude de la régularité de l'opérateur HUM pour un contrôle interne, et Ervedoza et Zuazua [EZ10] ont ensuite considéré le cas d'une équation réversible assez générale. Dans le cas de la contrôlabilité au bord pour l'équation des ondes, pour une donnée au niveau de régularité $s \geq 0$, le théorème 5.4 de [EZ10] donne un contrôle dans l'espace

$$H^s((0,T), L^2(\partial\Omega)) \cap \bigcap_{k=0}^{\lfloor s \rfloor} \mathscr{C}^k([0,T], H^{s-k-\frac{1}{2}}(\partial\Omega)),$$

alors que le théorème 2.1.6 montre qu'il existe un contrôle dans $H^s((0,T) \times \partial\Omega)$. Le désavantage de ce contrôle, par rapport à celui donné par [EZ10], est qu'il n'est pas construit de la même façon à tous les niveaux de régularité : on sait seulement que si un résultat de contrôle est vrai à un niveau de régularité, alors il est vrai aux autres niveaux de régularité, sans faire le lien entre les contrôles eux-mêmes.

2.1.2 Schémas de démonstration

Schéma de démonstration des théorèmes 2.1.6 et 2.1.7. D'après le lemme 2.1.3, il suffit de démontrer le théorème 2.1.7. L'idée de la démonstration est très naturelle, et consiste essentiellement à se demander si une inégalité d'observabilité, appliquée à $\Delta^{\pm r} u$, avec $r \in \mathbb{N}^*$, permet d'obtenir une autre inégalité d'observabilité. Pour simplifier, on omet les indices sur l'opérateur P, et les injections entre les différents espaces de régularité. Par interpolation, il suffit de montrer que l'observabilité H^s implique l'observabilité $H^{s\pm 2r}$, pour $s \in \mathbb{R}$ et $r \in \mathbb{N}^*$.

Commençons par regarder le cas le plus simple, qui est celui où l'on souhaite augmenter le niveau de régularité d'une inégalité d'observabilité. Supposons l'observabilité H^s pour Θ vraie, pour un certain $s \in \mathbb{R}$, et expliquons comment montrer l'observabilité H^{s+2} pour Θ. Soit u une solution de (1.3.1) de donnée initiale $\left(u^0, u^1\right) \in \mathcal{K}^{s+3} \times \mathcal{K}^{s+2}$. On peut démontrer que $\partial_t^2 u = \mathsf{P}u = \tilde{u}$, où $\tilde{u}$ est la solution de (1.3.1) de donnée initiale $\left(\mathsf{P}u^0, \mathsf{P}u^1\right) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. L'observabilité H^s pour Θ donne donc

$$\left\|\left(\mathsf{P}u^0, \mathsf{P}u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s} \lesssim \left\|\mathrm{diag}(\Theta)\partial_\nu \partial_t^2 u\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)} .$$

On utilise une inégalité elliptique à gauche, et la continuité de la dérivée en temps (entre H^{s+2} et H^s), pour obtenir l'*inégalité d'observabilité faible*

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+3}\times\mathcal{K}^{s+2}} \lesssim \left\|\mathrm{diag}(\Theta)\partial_\nu u\right\|_{H^{s+2}((0,T)\times\partial\Omega,\mathbb{C}^N)} + \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+2}\times\mathcal{K}^{s+1}} .$$

Il reste à montrer que cette inégalité d'observabilité faible implique l'observabilité H^{s+2} pour Θ, c'est-à-dire que le terme de reste à droite peut être retiré. Par compacité de l'injection $\mathcal{K}^{s+3}\times\mathcal{K}^{s+2} \hookrightarrow \mathcal{K}^{s+2}\times\mathcal{K}^{s+1}$, il suffit de démontrer que l'opérateur $\left(u^0, u^1\right) \in \mathcal{K}^{s+3}\times\mathcal{K}^{s+2} \mapsto \mathrm{diag}(\Theta)\partial_\nu u$ est injectif. La stratégie habituelle, à ce stade, consiste à utiliser un résultat de prolongement unique. On s'en passe ici, en utilisant à nouveau l'observabilité H^s pour Θ. En notant ι l'injection de $\mathcal{K}^{s+3}$ dans $\mathcal{K}^{s+1}$, si u est une solution avec donnée initiale dans l'espace $\mathcal{K}^{s+3} \times \mathcal{K}^{s+2}$ qui vérifie $\mathrm{diag}(\Theta)\partial_\nu u = 0$ dans H^{s+2}, alors ιu est une solution avec donnée initiale dans l'espace $\mathcal{K}^{s+1} \times \mathcal{K}^s$, qui vérifie $\mathrm{diag}(\Theta)\partial_\nu u = 0$ dans H^s. L'observabilité H^s pour Θ permet donc de conclure.

Regardons maintenant le cas où l'on souhaite diminuer le niveau de régularité d'une inégalité d'observabilité. On considère $s \in \mathbb{R}$ et $\Theta^1 = \left(\Theta_1^1, \cdots, \Theta_N^1\right)$ tels que l'observabilité H^s pour Θ^1 est vraie. D'après le cas précédent, on peut en fait supposer $s > 0$: cela sera utile dans la démonstration, mais nécessite de démontrer ensuite l'observabilité H^{s-2r}, pour tout $r \in \mathbb{N}^*$, et pas seulement l'observabilité H^{s-2}. Comme dans la démonstration des théorèmes 1.3.1 et 1.3.2, on considère $z \in \mathbb{C}$ tel que l'opérateur $\mathsf{P} + z\mathrm{Id}$ est un isomorphisme entre deux niveaux de régularité, qui préserve les conditions de compatibilité. Soient $r \in \mathbb{N}^*$, $\left(u^0, u^1\right) \in \mathcal{K}^{s+1-2r} \times \mathcal{K}^{s-2r}$, et u la solution de (1.3.1) de donnée initiale $\left(u^0, u^1\right)$. L'observabilité H^s pour Θ^1, appliquée à la solution $\tilde{u} = (\mathsf{P} + z\mathrm{Id})^{-r} u$, donne

$$\left\|\left((\mathsf{P} + z\mathrm{Id})^{-r} u^0, (\mathsf{P} + z\mathrm{Id})^{-r} u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s} \lesssim \left\|\mathrm{diag}(\Theta^1)\partial_\nu \tilde{u}\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)} . \tag{2.1.1}$$

À gauche, pour obtenir le terme $\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1-2r}\times\mathcal{K}^{s-2r}}$, à un terme de reste près, on utilise simplement la continuité de l'opérateur $(\mathsf{P} + z\mathrm{Id})^r$. À droite, on rencontre une difficulté. Dans le cas $z = 0$, on a $u = \mathsf{P}^r \tilde{u} = \partial_t^{2r} \tilde{u}$, on voit donc que l'inégalité dont on a besoin, à un terme de reste près, est

$$\left\|\mathrm{diag}(\Theta^1)\partial_\nu \tilde{u}\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)} \lesssim \left\|\mathrm{diag}(\Theta^1)\partial_\nu \partial_t^{2r} \tilde{u}\right\|_{H^{s-2r}((0,T)\times\partial\Omega,\mathbb{C}^N)} .$$

Dans le cas $z \neq 0$, une estimation de ce type peut également être utilisée, quitte à modifier le terme de reste. Cette inégalité, non-triviale, exprime l'ellipticité de la dérivée en temps sur les dérivées normales de solutions. Plus précisément, on démontre le résultat suivant.

Théorème 2.1.9 (Ellipticité de la dérivée en temps sur les traces Neumann de solutions)**.** *Pour $\Theta \in \mathscr{C}_c^\infty((0,T) \times \partial\Omega, \mathbb{C}^N)$, $s > -1$, et $r \in \mathbb{N}^*$, il existe $C > 0$ tel que pour tout $\left(u^0, u^1\right) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$, on a*

$$\begin{aligned} &\left\|\mathrm{diag}(\Theta)\partial_\nu u\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)} \\ &\leq C\left(\left\|\mathrm{diag}(\Theta)\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial\Omega,\mathbb{C}^N)} + \left\|u^0\right\|_{\mathcal{K}^{s+\frac{1}{2}}} + \left\|u^1\right\|_{\mathcal{K}^{s-\frac{1}{2}}}\right), \end{aligned}$$

où u est la solution de (1.3.1) *de donnée initiale* $\left(u^0, u^1\right)$.

Expliquons comment terminer la démonstration du théorème 2.1.7. En utilisant le théorème 2.1.9 et l'estimation (2.1.1), on obtient l'inégalité d'observabilité faible

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1-2r}\times\mathcal{K}^{s-2r}} \lesssim \left\|\operatorname{diag}(\Theta^1)\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial\Omega,\mathbb{C}^N)} + \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+\frac{1}{2}-2r}\times\mathcal{K}^{s-\frac{1}{2}-2r}}.$$

On souhaite ensuite retirer le terme de reste, en utilisant uniquement l'observabilité H^s pour Θ^1. On ne peut malheureusement pas faire cela en toute généralité : on considère donc ici $\Theta^2 = (\Theta^2_1, \cdots, \Theta^2_N) \in \mathscr{C}^\infty_c((0,T)\times\partial\Omega,\mathbb{C}^N)$, vérifiant $\Theta^2_k \neq 0$ sur $\operatorname{supp}\Theta^1_k$ pour tout $k \in [\![1, N]\!]$. Par compacité de l'injection $\mathcal{K}^{s+1-2r}\times\mathcal{K}^{s-2r} \hookrightarrow \mathcal{K}^{s+\frac{1}{2}-2r}\times\mathcal{K}^{s-\frac{1}{2}-2r}$, il suffit de montrer que l'opérateur $\left(u^0, u^1\right) \mapsto \operatorname{diag}(\Theta^2)\partial_\nu u$ est injectif. Un argument classique permet de montrer que son noyau est de dimension finie, et est stable par l'opérateur $\left(u^0, u^1\right) \mapsto \left(\partial_t u(0), \partial_t^2 u(0)\right)$, ce qui permet ensuite de conclure.

Schéma de démonstration du théorème 2.1.9. Pour comprendre l'ellipticité de la dérivée en temps sur la dérivée normale de solutions, il faut avoir une approche microlocale. On commence par se ramener au cas d'un demi-espace, et on note (t, x, τ, ξ) un point de l'espace cotangent. On écrit $\operatorname{Op}(p)$ pour l'opérateur pseudo-différentiel de symbole p, et $\operatorname{Op}_{\mathsf{T}}(p)$ pour l'opérateur pseudo-différentiel tangentiel de symbole p. L'idée est ensuite de décomposer $\partial_\nu u$ en deux termes, un sur lequel ∂_t est elliptique, qui sera le terme principal, et un autre sur lequel l'opérateur des ondes est elliptique, qui sera un terme de reste. Plus précisément, on écrit

$$\begin{aligned}\|\operatorname{diag}(\Theta)\partial_\nu u\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)} \leq{}& \left\|\operatorname{diag}(\Theta)\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)} \\ &+ \left\|\operatorname{diag}(\Theta)\operatorname{Op}_{\mathsf{T}}(1-\chi_{|x^n=0})\partial_\nu u\right\|_{H^s((0,T)\times\partial\Omega,\mathbb{C}^N)}, \qquad (2.1.2)\end{aligned}$$

avec χ une fonction lisse à déterminer. Dans les grandes lignes, on construit χ en assurant les propriétés suivantes : il existe des constantes $c_0, c_1, c_2 > 0$ telles que si $\tau^2 + |\xi|^2 \geq c_0$, alors

$$\begin{cases} \chi_{|x^n=0}(x,\tau,\xi',0) = 1, & \text{si } \tau^2 \geq c_1\left(1+\tau^2+|\xi'|^2\right), \\ \chi_{|x^n=0}(x,\tau,\xi',0) = 0, & \text{si } \tau^2 \leq \frac{c_1}{2}\left(1+\tau^2+|\xi'|^2\right), \\ \chi(x,\tau,\xi) = 0, & \text{si } |\xi|^2-\tau^2 \geq c_3\left(\tau^2+|\xi|^2\right), \\ \chi(x,\tau,\xi) = 1, & \text{si } |\xi|^2-\tau^2 \leq \frac{c_3}{2}\left(\tau^2+|\xi|^2\right). \end{cases}$$

Le symbole $\chi_{|x^n=0}(x,\tau,\xi',0)\tau^{-2r}$ est alors bien défini, et le premier terme à droite de (2.1.2) donne le premier terme à droite de l'estimation du théorème 2.1.9. On montre ensuite que le deuxième terme à droite de (2.1.2) est un reste, en utilisant le fait que u est une solution de l'équation des ondes, ainsi qu'une parametrix de $\operatorname{Op}(1-\chi)\mathsf{P}$.

2.1.3 Perspectives

Pour terminer le projet en cours avec Lauri Oksanen, il faut montrer l'observabilité L^2 pour $\Theta = (\theta, \cdots, \theta)$, avec $\theta \in \mathscr{C}^\infty_c(]0,T[\times\partial\Omega,\mathbb{C})$ satisfaisant la condition géométrique de contrôle, ce qui peut-être fait en utilisant des mesures de défaut, ou en adaptant la démonstration originale de Bardos, Lebeau et Rauch [BLR92]. Il serait intéressant d'essayer de démontrer des résultats analogues au théorème 2.1.7 pour d'autres équations au dérivées partielles, pour lesquelles une condition géométrique caractérisant l'observabilité n'est pas connue.

2.2 Estimations uniformes de solutions globales d'équations des ondes non-linéaires focalisantes

2.2.1 Résultats principaux et contexte bibliographique

On s'intéresse ici à la question suivante : une solution globale en temps positifs, donc qui existe pour tout $t \in \mathbb{R}_+$, est-elle nécessairement bornée dans l'espace d'énergie $H_0^1(\Omega) \times L^2(\Omega)$? Une réponse positive à ce genre de question est d'une grande aide pour classifier le comportement des solutions d'une équation, puisqu'elle donne accès à des résultats dits de *compacité asymptotique* (voir théorème 2.3.4). L'équation que l'on a en tête est l'équation des ondes cubique amortie sur un domaine borné, mais on considère ici une non-linéarité un peu plus générale. L'idée est de généraliser les résultats de Cazenave [Caz85] à une équation amortie.

Soient Ω l'intérieur et $\partial\Omega$ le bord d'une variété compacte de dimension $d \geq 3$. Comme précédemment, on fixe $\beta \in \mathbb{R}$ tel que l'inégalité de Poincaré

$$\left\|u^0\right\|_{H_0^1}^2 = \int_\Omega \left(|\nabla u|^2 + \beta|u|^2\right) \mathrm{d}x \gtrsim \int_\Omega |u|^2 \mathrm{d}x, \quad u \in H_0^1(\Omega), \tag{2.2.1}$$

est vérifiée. On considère ensuite l'équation des ondes (ou de Klein-Gordon)

$$\begin{cases} \Box u + \gamma\partial_t u + \beta u &= f(u) & (t,x) \in \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & x \in \Omega, \\ u &= 0 & (t,x) \in \mathbb{R} \times \partial\Omega, \end{cases} \tag{2.2.2}$$

pour une donnée initiale $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ à valeurs réelles, un amortissement $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, et une non-linéarité $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$. On note $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$, et on fait les hypothèses suivantes. On suppose que $f(0) = 0$, et qu'il existe $1 < p < +\infty$, $1 < p_0 < +\infty$, $C > 0$, $\varepsilon > 0$ tels que

$$|f(s_1) - f(s_2)| \leq C|s_1 - s_2| \left(1 + |s_1|^{p-1} + |s_2|^{p-1}\right), \quad s_1, s_2 \in \mathbb{R}, \tag{2.2.3}$$

$$sf(s) \geq (2+\varepsilon)F(s), \quad s \in \mathbb{R}, \tag{2.2.4}$$

et

$$\|u\|_{L^2}^{p_0+1} \leq C \int_\Omega F(u)\mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{2.2.5}$$

On supposera toujours que Ω et f sont tels que l'on dispose d'une théorie de Cauchy locale : plus précisément, on suppose que pour toute donnée initiale (u^0, u^1), il existe un temps maximal d'existence $T \in]0, +\infty]$ et une solution $u \in \mathscr{C}^0([0,T), H_0^1(\Omega)) \cap \mathscr{C}^1([0,T), L^2(\Omega))$ de (2.2.2).

Pour $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, on introduit la fonctionnelle d'énergie

$$E\left(u^0, u^1\right) = \frac{1}{2}\left\|u^0\right\|_{H_0^1}^2 + \frac{1}{2}\left\|u^1\right\|_{L^2}^2 - \int_\Omega F(u^0)\mathrm{d}x.$$

Si la solution u de (2.2.2) existe sur un intervalle $[0,T]$, alors on a l'égalité d'énergie

$$E\left(u(t_1), \partial_t u(t_1)\right) = E\left(u(t_0), \partial_t u(t_0)\right) - \int_{t_0}^{t_1} \int_\Omega \gamma \left|\partial_t u\right|^2 \mathrm{d}t\mathrm{d}x, \quad 0 \leq t_0 \leq t_1 \leq T.$$

Notre résultat principal est le suivant.

Théorème 2.2.1. *On considère f vérifiant* (2.2.3) *pour un certain $p < \frac{d+2}{d-2}$,* (2.2.4)*, et* (2.2.5)*.*

(i) Il existe $C = C(f,\gamma) > 0$ tel que pour tout $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, si la solution u de (2.2.2) *existe sur $\mathbb{R}_+$, alors*

$$E\left(u^0, u^1\right) \geq E\left(u(t), \partial_t u(t)\right) \geq -C, \quad t \geq 0.$$

(ii) Il existe $c_0 = c_0(f,\gamma) > 0$, $c_1 = c_1(f,\gamma) > 0$, et $c_2 = c_2(f) > 0$ tels que pour tout $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, si la solution u de (2.2.2) *existe sur $\mathbb{R}_+$, alors*

$$\|u(t)\|_{L^2}^2 \leq \left\|u^0\right\|_{L^2}^2 e^{-c_2 t} + \left(c_0 + c_1 \left|E\left(u^0, u^1\right)\right|\right)\left(1 - e^{-c_2 t}\right), \quad t \geq 0.$$

(iii) On suppose de plus que f vérifie (2.2.3) *pour un certain $p \leq \frac{d}{d-2}$. Alors il existe $c = c(f,\gamma) > 0$ tel que la fonction $\alpha : s \mapsto c\exp(cs)$ vérifie la propriété suivante. Pour tout $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, si la solution u de* (2.2.2) *existe sur $\mathbb{R}_+$, alors*

$$\|(u(t), \partial_t u(t))\|_{H_0^1 \times L^2} \leq \alpha\left(\left\|\left(u^0, u^1\right)\right\|_{H_0^1 \times L^2}^2\right), \quad t \geq 0.$$

De plus, il existe $T \geq 0$, qui dépend de f, γ et $\|(u^0, u^1)\|_{H_0^1 \times L^2}$, tel que

$$\|(u(t), \partial_t u(t))\|_{H_0^1 \times L^2} \leq \alpha\left(\left|E\left(u^0, u^1\right)\right|\right), \quad t \geq T.$$

Par exemple, si f est donnée par

$$f(s) = \lambda_1 s|s|^{\alpha_1 - 1} + \cdots + \lambda_n s|s|^{\alpha_n - 1}, \quad s \in \mathbb{R},$$

pour $n \in \mathbb{N}^*$, $\lambda_1, \cdots, \lambda_n > 0$, et $1 < p_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq p$, alors f vérifie (2.2.3), (2.2.4), et (2.2.5). Notons que le fait que Ω est borné est utilisé pour obtenir (2.2.5). Les points *(i)* et *(ii)* du théorème 2.2.1 peuvent être appliqués à f si $p < \frac{d+2}{d-2}$, et le point *(iii)* peut aussi être appliqué si $p \leq \frac{d}{d-2}$. En particulier, le théorème 2.2.1 implique que toute solution globale (en temps positifs) de l'équation cubique sur une variété compacte de dimension 3 est bornée dans l'espace d'énergie.

Commentons les hypothèses du théorème 2.2.1, et comparons-les à celles faites par Cazenave dans [Caz85]. Notons déjà que les cas $d = 1, 2$ peuvent être traités en adaptant les méthodes de [Caz85] et les idées de la démonstration du théorème 2.2.1 : ce n'est pas fait ici seulement par souci de simplification. Pour avoir une théorie de Cauchy locale, il est naturelle de supposer (2.2.3) avec $p < \frac{d+2}{d-2}$, ce qui revient à dire que f se comporte comme une puissance, et est sous-critique. Dans le cas d'une variété à bord, les estimations de Strichartz ne sont pas valides avec tous les couples d'exposants du cas sans bord, d'où notre hypothèse supplémentaire sur la théorie de Cauchy. L'inégalité (2.2.4) traduit le caractère focalisant de la non-linéarité, et est exactement l'hypothèse faite par Cazenave. Enfin, (2.2.5) est également une hypothèse focalisante, et ne figure pas dans l'article de Cazenave : c'est une hypothèse supplémentaire, qui nous permet de traiter le terme d'amortissement. Notons que (2.2.5) est une hypothèse "près de 0", et peut être remplacée par $F(s) \gtrsim s^{2+\varepsilon}$, pour s proche de 0. La question de savoir si le théorème 2.2.1 est vrai sans l'hypothèse (2.2.5) (ou une hypothèse analogue) est ouverte, à notre connaissance. Le fait de supposer Ω borné sert uniquement à assurer l'existence de non-linéarité vérifiant toutes ces hypothèses. Notons également que Cazenave suppose que $p \leq \frac{d}{d-2}$ dans tout son article, mais une lecture attentive montre que cette hypothèse n'est utilisée que pour la démonstration de la borne H^1.

Des résultats analogues à ceux de [Caz85] ont été démontrés par d'autres auteurs. Feireisl [Fei98] a considéré le cas $\Omega = \mathbb{R}^d$, avec un amortissement constant en espace et en temps, et une non-linéarité vérifiant la condition restrictive $1 < p < \min\left(\frac{d}{d-2}, \frac{d+4}{d}\right)$. Le Lemme 4.2 de [Fei98] indique qu'une solution avec une énergie bornée est globale (en temps positifs), et est bornée dans $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Cependant, [Fei98] ne semble pas contenir de démonstration du fait qu'une solution globale a une énergie bornée.

Des questions reliées ont été étudiées par Burq, Raugel et Schlag. Dans [BRS17], ces auteurs ont démontré qu'une solution radiale globale est bornée dans l'espace d'énergie, dans le cas $\Omega = \mathbb{R}^d$, avec un amortissement égal à une constante, pour une non-linéarité sous-critique générale. La démonstration utilise des arguments de systèmes dynamiques. Dans [BRS18], les mêmes auteurs ont considéré le cas d'un amortissement constant en espace, et qui tend vers zéro quand t tend vers l'infini, toujours avec $\Omega = \mathbb{R}^d$ et pour des solutions radiales. La démonstration du fait que les solutions globales sont bornées utilise la méthode de Cazenave pour la borne L^2, et le fait que l'amortissement tend vers zéro lorsque t tend vers l'infini.

2.2.2 Schéma de démonstration du théorème 2.2.1

Considérons une solution u définie sur $\mathbb{R}_+$, et notons $M(t) = \|u(t)\|_{L^2}^2$, pour $t \geq 0$. L'idée de s'intéresser à cette quantité M n'est pas nouvelle, et sert notamment pour montrer que certaines solutions sont explosives (voir par exemple [PS75], et le théorème 2.3.1 plus bas). On peut montrer, en utilisant la dérivée seconde de M, les hypothèses faites sur f, et l'estimation d'énergie, qu'il existe une constante $C > 0$ telle que si $E\left(u(T), \partial_t u(T)\right) < -C$ pour un certain $T > 0$, alors

$$M''(t) \gtrsim 1, \quad t \geq T. \tag{2.2.6}$$

et

$$\left(M'(t)\right)^2 \leq \delta M(t) M''(t), \quad t \geq T, \tag{2.2.7}$$

pour un certain $\delta \in]0, 1[$. L'inégalité (2.2.6) implique que $M(t)$ tend vers l'infini quand t tend vers l'infini, et (2.2.7) implique alors l'explosion en temps fini de u, ce qui contredit notre hypothèse de départ. Cela démontre *(i)*. Notons que dans le cas d'une équation sans amortissement, on retrouve, avec l'expression explicite de la constante C, le fait qu'une solution globale a une énergie positive, ce qui avait déjà été démontré par Cazenave dans [Caz85].

Pour démontrer *(ii)*, on utilise le fait que l'énergie est bornée, l'expression de M'', et l'inégalité d'énergie, pour obtenir une estimation de la forme

$$M''(t) \geq (4 + \varepsilon') \|\partial_t u(t)\|_{L^2}^2 + C_1 M(t) - C_2, \quad t \geq 0, \tag{2.2.8}$$

avec $\varepsilon', C_1, C_2 > 0$. En particulier, la fonction $M_0 : t \mapsto C_1 M(t) - C_2$ vérifie $M_0''(t) \geq C_1 M_0(t)$, pour tout $t \geq 0$. Une telle inégalité laisse deux possibilités : ou bien M_0 est bornée, ou bien $M_0(t)$ tend vers l'infini quand t tend vers l'infini. Dans ce dernier cas, on obtient une contradiction en utilisant (2.2.8), comme dans la démonstration de *(i)*. Donc M_0 est bornée, ce qui permet de conclure. L'inégalité (2.2.8) implique également une estimation de la forme

$$|M'(t)| \lesssim M''(t) + 1, \quad t \geq 0.$$

En intégrant cette inégalité, et en utilisant le fait que M est bornée, on en déduit que M' est bornée.

Enfin, expliquons comment démontrer *(iii)*. L'hypothèse restrictive $p \leq \frac{d}{d-2}$ permet de considérer la dérivée de la norme H^1. En effet, en définissant

$$E_{\mathscr{L}}(t) = \frac{1}{2} \|u(t)\|_{H_0^1}^2 + \frac{1}{2} \|\partial_t u(t)\|_{L^2}^2 = E\left(u(t), \partial_t u(t)\right) + \int_\Omega F(u(t)) \mathrm{d}x, \quad t \geq 0,$$

on trouve

$$E'_{\mathscr{L}}(t) = -\int_\Omega \gamma\,|\partial_t u|^2\,\mathrm{d}x + \int_\Omega \partial_t u(t) f(u(t)), \quad t \geq 0.$$

Pour étudier $E_{\mathscr{L}}$, il est donc naturel de supposer $f(u(t)) \in L^2(\Omega)$, pour tout $t \geq 0$, conséquence de l'injection de Sobolev $H^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$ et de l'hypothèse $p \leq \frac{d}{d-2}$. On en déduit

$$E'_{\mathscr{L}}(t) \lesssim \|\partial_t u(t)\|_{L^2} \left(\|u(t)\|_{H_0^1} + \|u(t)\|^p_{H_0^1}\right) \lesssim E_{\mathscr{L}}(t)\,(1 + E_{\mathscr{L}}(t)), \quad t \geq 0,$$

en utilisant de plus le fait que $p \leq \frac{d}{d-2}$ (et $d \geq 3$) implique $p \leq 3$. On complète la démonstration en intégrant cette inégalité, et en s'appuyant sur (2.2.8) et sur l'estimation obtenue sur M'.

2.2.3 Perspectives

Chronologiquement, on s'est intéressés à ces questions à la fin de notre étude de l'équation cubique sous l'énergie de l'état fondamental. Le théorème 2.2.1 permet de rendre plus général le résultat de compacité asymptotique qui sera présenté plus loin (théorème 2.3.4), qui pourrait être utilisé pour étudier l'équation cubique amortie un peu au-dessus de l'énergie de l'état fondamental. Il serait intéressant d'essayer de généraliser le théorème 2.2.1 dans deux directions : ou bien en affaiblissant l'hypothèse (2.2.5), qui sert à contrôler l'amortissement, ou bien en affaiblissant l'hypothèse $p \leq \frac{d}{d-2}$ (dans le cas de l'équation non-amortie pour commencer). Ces deux questions semblent complètement ouvertes.

2.3 Stabilisation de l'équation des ondes cubique focalisante sur un domaine borné

2.3.1 Résultats principaux et contexte bibliographique

Soient Ω l'intérieur et $\partial\Omega$ la frontière d'une variété riemannienne compacte à bord de dimension 3, et soit $\beta \in \mathbb{R}$ tel que l'inégalité de Poincaré (2.2.1) est vérifiée. Pour $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, on considère l'équation des ondes (ou de Klein-Gordon) cubique focalisante amortie

$$\left\{\begin{array}{rcll} \partial_t^2 u - \Delta u + \gamma\partial_t u + \beta u & = & u^3 & (t,x) \in \mathbb{R}_+ \times \Omega, \\ (u(0), \partial_t u(0)) & = & (u^0, u^1) & x \in \Omega, \\ u & = & 0 & (t,x) \in \mathbb{R}_+ \times \partial\Omega, \end{array}\right. \tag{2.3.1}$$

avec donnée initiale réelle $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. On note

$$\left\|u^0\right\|^2_{H_0^1} = \int_\Omega \left(\left|\nabla u^0\right|^2 + \beta|u^0|^2\right)\mathrm{d}x,$$

et on introduit deux fonctionnelles, l'*énergie statique* et l'*énergie*, définies par

$$J(u^0) = \frac{1}{2}\left\|u^0\right\|^2_{H_0^1} - \frac{1}{4}\left\|u^0\right\|^4_{L^4} \quad \text{et} \quad E\left(u^0, u^1\right) = J(u^0) + \frac{1}{2}\left\|u^1\right\|^2_{L^2}.$$

Comme précédemment, si u est une solution de (2.3.1) qui existe sur un intervalle $[0,T]$, on a l'égalité d'énergie

$$E\,(u(t_1), \partial_t u(t_1)) = E\,(u(t_0), \partial_t u(t_0)) - \int_{t_0}^{t_1}\int_\Omega \gamma\,|\partial_t u|^2\,\mathrm{d}t\mathrm{d}x, \quad 0 \leq t_0 \leq t_1 \leq T.$$

La construction de Payne et Sattinger [PS75] d'un état fondamental (voir théorème 1.4.1) est valide dans le cas d'une variété Ω, mais l'unicité ne semble pas être démontrée en toute généralité dans la littérature. Si $u \in H_0^1(\Omega)$ est une solution stationnaire, alors en notant $K(u) = \|u\|_{H_0^1}^2 - \|u\|_{L^4}^4$, on vérifie que $K(u) = 0$. On peut montrer que

$$m_0 = \inf \left\{ J(u), u \in H_0^1(\Omega), u \neq 0, K(u) = 0 \right\} > 0,$$

et on verra que ce niveau d'énergie m_0 permet de faire fonctionner des arguments de type *puits de potentiel*. Dans ce qui suit, on note Q un état fondamental, c'est-à-dire une solution stationnaire telle que $J(Q) = m_0$: on sait que Q est unique (au signe près) si Ω est un ouvert de $\mathbb{R}^3$, mais on ne s'en servira pas. On introduit ensuite une partition des données initiales d'énergie inférieure à m_0, on notant

$$\begin{cases} \mathscr{K}^+ = \left\{ (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega), E\left(u^0, u^1\right) < m_0, K(u^0) \geq 0 \right\}, \\ \mathscr{K}^- = \left\{ (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega), E\left(u^0, u^1\right) < m_0, K(u^0) < 0 \right\}. \end{cases}$$

Notre premier résultat sur l'équation (2.3.1) est le suivant.

Théorème 2.3.1. *Soit $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$. Les espaces $\mathscr{K}^+$ et $\mathscr{K}^-$ sont invariants par le flot de l'équation* (2.3.1)*. De plus, une solution ayant une donnée initiale dans $\mathscr{K}^+$ est définie sur $\mathbb{R}_+$, et une solution ayant une donnée initiale dans $\mathscr{K}^-$ explose en temps fini.*

Ce résultat a été montré par Payne et Sattinger [PS75] dans le cas $\gamma = 0$. L'équation (2.3.1) avec $\gamma = 0$ étant réversible, le résultat original de Payne et Sattinger est valide en temps positifs comme en temps négatifs, ce qui n'est pas le cas du théorème 2.3.1. On peut voir l'explosion d'une solution initiée dans $\mathscr{K}^-$ comme un résultat de *non-stabilisation* : aucun amortissement γ n'empêche l'explosion, donc en particulier, aucun amortissement ne permet de stabiliser la solution. Pour une solution initiée dans $\mathscr{K}^+$, on a un résultat de stabilisation. Commençons par rappeler la condition de contrôle géométrique (déjà introduite dans le cas de la contrôlabilité au bord à la définition 2.1.4).

Définition 2.3.2. Soit $\omega \subset \overline{\Omega}$ un ouvert, et $T > 0$. On dit que (ω, T) vérifie la *Condition de Contrôle Géométrique* (abrégée en GCC), si pour toute bicaractéristique généralisée $s \mapsto (t(s), x(s), \tau(s), \xi(s))$, il existe un point $(t(s), x(s), \tau(s), \xi(s))$ tel que $t(s) \in]0, T[$ et $x(s) \in \omega$. On dit simplement que ω vérifie la GCC s'il existe $T > 0$ tel que (ω, T) vérifie la GCC.

On a le résultat de stabilisation suivant.

Théorème 2.3.3. *Supposons que $\gamma(x) \geq \alpha$ pour presque tout $x \in \omega$, avec $\alpha > 0$ une constante et $\omega \subset \overline{\Omega}$ un ouvert vérifiant la condition de contrôle géométrique. Alors pour tout $E_0 \in [0, m_0[$, il existe $C > 0$ et $\lambda > 0$ tels que pour tout $\left(u^0, u^1\right) \in \mathscr{K}^+$ tel que $E\left(u^0, u^1\right) \leq E_0$, la solution u de* (2.3.1) *vérifie*

$$\|u(t)\|_{H_0^1} + \|\partial_t u(t)\|_{L^2} \leq Ce^{-\lambda t}, \quad t \geq 0.$$

Le théorème 2.3.3 peut être vu comme l'extension au cas focalisant (et sous l'énergie de l'état fondamental) d'un résultat de Joly et Laurent [JL13], qui démontre une propriété de stabilisation analogue pour toutes les solutions de la version défocalisante de (2.3.1). L'article de Joly et Laurent couvre le cas d'un domaine Ω de dimension 3 qui peut être non-borné, et une non-linéarité de type puissance sous-critique ; les méthodes utilisées permettent probablement de démontrer le théorème 2.3.3 dans ce cadre, mais nous ne l'avons pas fait pour simplifier.

La démonstration du théorème 2.3.3 est fondée sur un résultat appelé *compacité asymptotique*, qui implique le théorème suivant.

Théorème 2.3.4. *Soient $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$ vérifiant les hypothèses du théorème 2.3.3, et $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ tel que la solution associée u de* (2.3.1) *existe sur $\mathbb{R}_+$.*

(i) Pour toute suite $(T_n)_{n\in\mathbb{N}}$ telle que $T_n \to +\infty$, il existe une extraction $\phi : \mathbb{N} \to \mathbb{N}$ et une solution stationnaire w de (2.3.1) *telle que*

$$\Big(u(T_{\phi(n)} + \cdot), \partial_t u(T_{\phi(n)} + \cdot)\Big) \xrightarrow{n\to\infty} (w, 0)$$

dans $L^\infty_{\text{loc}}(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$.

(ii) Supposons qu'il existe un ensemble au plus dénombrable de solutions stationnaires w de (2.3.1) *telles que*

$$J(w) = \liminf_{t\to+\infty} E\left(u(t), \partial_t u(t)\right).$$

Alors il existe une solution stationnaire w de (2.3.1) *telle que*

$$(u(t + \cdot), \partial_t u(t + \cdot)) \xrightarrow{t\to+\infty} (w, 0)$$

dans $L^\infty_{\text{loc}}(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$.

Par exemple, si $\liminf_{t\to+\infty} E\left(u(t), \partial_t u(t)\right) < m_0$, on a nécessairement $w = 0$ dans *(ii)*, et si Ω est un ouvert de $\mathbb{R}^3$, alors $\liminf_{t\to+\infty} E\left(u(t), \partial_t u(t)\right) = m_0$ implique $w = \pm Q$. Comme mentionné précédemment, la démonstration de ce théorème utilise le fait qu'une solution globale est bornée (théorème 2.2.1).

Comparons nos résultats à ceux existants dans la littérature. Dans le cas défocalisant, l'énergie d'une solution est positive et correspond essentiellement à sa taille. En plus de l'article [JL13] déjà mentionné, Joly et Laurent [JL14] ont considéré le cas d'une non-linéarité qui est seulement asymptotiquement défocalisante, et ont établi un résultat de contrôlabilité dit semi-global : pour tout sous-ensemble borné de $H_0^1(\Omega) \times L^2(\Omega)$, il existe un temps T tel que la contrôlabilité de cet ensemble dans lui-même est vraie en temps T. Pour le type de non-linéarité considéré, des solutions stationnaires existent, mais il n'y a pas de phénomène d'explosion en temps fini. Dans le cas focalisant, pour un domaine égal à l'espace entier ou bien à l'extérieur d'un obstacle étoilé, Alaoui, Ibrahim, et Nakanishi [AIN10] ont démontré la stabilisation des solutions initiée dans $\mathcal{K}^+$ (avec l'hypothèse que l'amortissement est plus grand qu'une constante strictement positive en dehors d'une boule). Le fait de supposer que l'obstacle est étoilé est plus fort que la condition de contrôle géométrique. Dans le cas de l'équation de Klein-Gordon radiale cubique focalisante sur une boule de $\mathbb{R}^3$, Krieger et Xiang [KX20] ont étudié la stabilisation et la stabilité au voisinage de la solution stationnaire $u = 1$, avec des conditions de dissipations aux bord.

Les articles de Burq, Raugel et Schlag mentionnés plus haut contiennent des résultats à mettre en parallèle avec le théorème 2.3.4. Dans [BRS17], ces auteurs ont montré une propriété de dichotomie pour l'équation de Klein-Gordon radiale focalisante amortie sur l'espace entier, avec un amortissement constant en temps et en espace : une solution peut exploser en temps fini, ou bien être globale (en temps positifs), bornée, et converger vers un état stationnaire. Notons que supposer que l'amortissement est constant est une hypothèse plus forte que la condition de contrôle géométrique. Une propriété similaire est démontrée dans [BRS18] pour un amortissement constant en espace et qui tend vers 0 quand t tend vers $+\infty$.

2.3.2 Schémas de démonstration

Schéma de démonstration du théorème 2.3.1. La démonstration du fait que $\mathcal{K}^+$ et $\mathcal{K}^-$ sont invariants par le flot de l'équation (2.3.1) est similaire au cas de l'équation non-amortie, car l'énergie est décroissante. C'est dans cette démonstration que l'on voit le rôle de puits de potentiel joué par m_0 : intuitivement, on ne peut passer de $\mathcal{K}^+$ à $\mathcal{K}^-$ qu'en passant au-dessus de l'énergie de Q. Une représentation schématique de $\mathcal{K}^+$ et $\mathcal{K}^-$ est donnée à la figure 2.1.

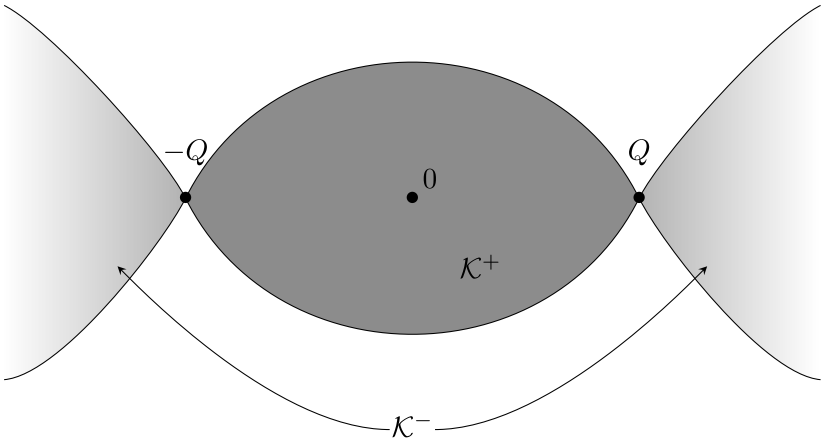


FIGURE 2.1 : Les ensembles $\mathcal{K}^+$ (en gris foncé) et $\mathcal{K}^-$ (en dégradé).

Donnons un peu plus de détails, en expliquant comment démontrer que pour tout $\delta \in]0, m_0[$, si $u \in H_0^1(\Omega)$ vérifie $J(u) \leq m_0 - \delta$ et $K(u) < 0$, alors il existe $c = c(\delta) > 0$ tel que $K(u) \leq -c$. On commence par montrer l'inégalité de Poincaré explicite

$$\|u\|_{L^4} \|Q\|_{L^4} \leq \|u\|_{H_0^1}, \quad u \in H_0^1(\Omega),$$

et on en déduit que si $u \in H_0^1(\Omega)$ vérifie $J(u) \leq m_0 - \delta$, alors $\alpha\left(\|u\|_{H_0^1}\right) \leq J(u) \leq m_0 - \delta$, avec

$$\alpha : x \in \mathbb{R} \longmapsto \frac{x^2}{2} - \frac{x^4}{4\|Q\|_{L^4}^4}.$$

Le graphe de α (figure 2.2) fait clairement apparaître le puits de potentiel. On voit qu'il intersecte la droite $y = m_0 - \delta$ en exactement deux points, dont on note $x^\pm = x^\pm(\delta)$ les abscisses. La condition $K(u) < 0$ implique alors $\|u\|_{H_0^1} \geq x^+$, ce qui permet de conclure.

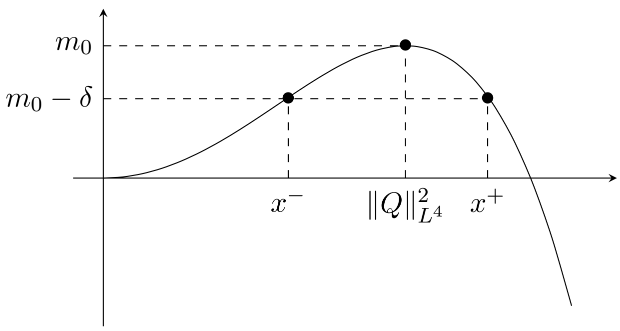


FIGURE 2.2 : Le graphe de α.

Une fois l'invariance de $\mathcal{K}^+$ et $\mathcal{K}^-$ établie, le fait qu'une solution initiée dans $\mathcal{K}^+$ est globale suit immédiatement, en écrivant, pour $(u^0, u^1) \in \mathcal{K}^+$ et $t \geq 0$,

$$E\left(u(t), \partial_t u(t)\right) = \frac{1}{4} K(u(t)) + \frac{1}{4}\|u(t)\|_{H_0^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 \geq \frac{1}{4}\|u(t)\|_{H_0^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2. \quad (2.3.2)$$

Pour montrer qu'une solution initiée dans $\mathcal{K}^-$ explose en temps fini, on utilise des arguments analogues à ceux présentés dans le schéma de démonstration du théorème 2.2.1, en faisant attention au fait que quand t tend vers l'infini, l'énergie peut ou bien tendre vers $-\infty$, ou bien converger vers une limite finie.

Schéma de démonstration du théorème 2.3.3. La démonstration est fondée sur le résultat principal de la démonstration du théorème 2.3.4, que l'on n'énonce pas pour simplifier, qui est essentiellement une version du théorème 2.3.4 qui s'applique à une suite uniformément bornée de solutions globales, le long d'une suite de temps.

L'équation (2.3.2) montre que dans $\mathcal{K}^+$, l'énergie d'une solution contrôle sa taille, ce qui permet d'adapter les méthodes utilisées par Joly et Laurent [JL13] dans le cas défocalisant. En utilisant (2.3.2) et l'estimation d'énergie, on ramène l'étude de la stabilisation à la démonstration d'une inégalité d'observabilité : il suffit de montrer qu'il existe $T > 0$ et $C > 0$ tels que

$$E\left(u^0, u^1\right) \leq C \int_0^T \int_\Omega \gamma(x) \left|\partial_t u(t,x)\right|^2 \mathrm{d}x\mathrm{d}t,$$

pour tout $\left(u^0, u^1\right) \in \mathscr{K}^+$ vérifiant $E\left(u^0, u^1\right) \leq E_0$. On raisonne par l'absurde et on applique le résultat principal de la démonstration du théorème 2.3.4, ce qui fournit une suite de temps $T_n \to +\infty$ et une suite de données initiales $E\left(u_n^0, u_n^1\right) \to 0$, telles que

$$E\left(u_n(T_n), \partial_t u_n(T_n)\right) \gtrsim E\left(u_n^0, u_n^1\right), \quad n \in \mathbb{N}. \tag{2.3.3}$$

On considère la suite $v_n = E\left(u_n^0, u_n^1\right)^{-\frac{1}{2}} u_n$. En utilisant (2.3.3), la formule de Duhamel, et la stabilisation du semi-groupe linéaire, on obtient une contradiction. Si cette démonstration est assez courte, c'est parce que toute la difficulté est dans la démonstration du théorème 2.3.4.

Schéma de démonstration du théorème 2.3.4. On utilise ici encore une méthode introduite par Joly et Laurent dans le cas défocalisant. On donne seulement les idées de la démonstration, de façon un peu approximative. La première étape consiste à appliquer un résultat de Dehman, Lebeau et Zuazua [DLZ03], qui indique qu'une non-linéarité sous-critique donne un gain de régularité. Plus précisément, on montre que si u est une solution de (2.3.1) telle que

$$u \in \mathscr{C}^0([0,T], H^{1+s}(\Omega) \cap H_0^1(\Omega)) \cap \mathscr{C}^0([0,T], H_0^s(\Omega)),$$

pour un $s \in [0,1[$, alors on a $u^3 \in L^1([0,T], H_0^{s+\varepsilon}(\Omega))$, pour un certain $\varepsilon > 0$ que l'on peut expliciter. Notons S le semi-groupe linéaire associé à l'équation (2.3.1) sans le terme u^3. En appliquant la formule de Duhamel, et en utilisant la stabilisation du semi-groupe linéaire et le résultat précédent sur des blocs de taille 1, on montre que la suite

$$\left((u(T_n), \partial_t u(T_n)) - S\left(u^0, u^1\right)(T_n)\right)_{n \in \mathbb{N}}$$

est bornée dans $H^{1+\varepsilon}(\Omega) \times H^\varepsilon(\Omega)$. Le théorème de Rellich entraîne donc la convergence forte de la suite dans $H_0^1(\Omega) \times L^2(\Omega)$, et on démontre en itérant le raisonnement précédent que la limite u_∞ est une solution globale de (2.3.1) vérifiant $(u_\infty(0), \partial_t u_\infty(0)) \in H^2 \cap H_0^1(\Omega) \times H_0^1(\Omega)$ et $\partial_t u_\infty = 0$ sur le support de γ. On applique ensuite un résultat de Hale et Raugel [HR03], qui montre que u_∞ est analytique en temps. On en déduit par régularité elliptique que u_∞ est lisse par rapport à la variable d'espace. Un résultat de prolongement unique, dû à Robianno et Zuilly [RZ98], qui concerne les équations à coefficients partiellement analytiques, permet de conclure que u_∞ est une solution stationnaire de (2.3.1).

2.3.3 Perspectives

La question naturelle suivante est d'étudier (2.3.1) un peu au-dessus de l'énergie de l'état fondamental. On pourrait par exemple essayer de montrer, avec l'hypothèse du théorème 2.3.3 sur γ, qu'il existe $\varepsilon > 0$ tel que pour toute donnée initiale $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ vérifiant $E(u^0, u^1) \leq m_0 + \varepsilon$, si u est la solution associée, alors ou bien u explose en temps fini, ou bien u converge à vitesse exponentielle vers une solution stationnaire v, qui peut être la solution nulle, ou un état fondamental.

Une autre question intéressante serait d'étudier plus en détails les solutions explosives de l'équation amortie. On sait que si $(u^0, u^1) \in \mathcal{K}^-$, alors pour tout amortissement γ, la solution u de (2.3.1) explose en temps fini, mais on peut se demander si γ a une influence sur la durée maximale d'existence de la solution. Plus précisément, si on note $T_{\exp}(\gamma)$ le temps $T > 0$ tel que u existe sur $[0, T[$, et

$$\|u(t)\|_{H_0^1} + \|\partial_t u(t)\|_{L^2} \xrightarrow{t \to T^-} +\infty,$$

alors peut-on établir une majoration ou une minoration de $T_{\exp}(\gamma)$ en fonction de γ ?

2.4 Contrôlabilité locale et contrôlabilité à zéro des solutions scattering

2.4.1 Résultats principaux et contexte bibliographique

On s'intéresse ici à la contrôlabilité locale, avec pour objectif de montrer un résultat assez général, avec un domaine qui peut être non-borné, à bord, une non-linéarité qui n'est pas seulement une puissance, et en se plaçant au voisinage d'une trajectoire qui n'est pas nécessairement la solution nulle. Dans le cas d'un domaine non-borné, avec des hypothèses légèrement plus fortes, on a aussi la contrôlabilité à zéro des solutions scattering. On a regroupé ces deux résultats dans un même chapitre pour deux raisons : premièrement, la contrôlabilité locale (au voisinage de la solution nulle) est utilisée pour démontrer la contrôlabilité à zéro, et deuxièmement, les deux démonstrations nécessitent d'énoncer des inégalités de Strichartz avec un ensemble d'exposants aussi grand que possible.

Soient $d \in [\![2, 5]\!]$, et Ω l'intérieur d'une variété riemannienne lisse de dimension d, avec ou sans bord, qui peut être ou bien compacte, ou bien une perturbation compacte de $\mathbb{R}^d$, c'est-à-dire le complémentaire dans $\mathbb{R}^d$ d'un ouvert borné, avec une métrique euclidienne hors d'une boule. Pour raccourcir les énoncés, on dit que $\partial\Omega = \emptyset$ si Ω est égal à $\mathbb{R}^d$ ou bien à une variété compacte sans bord, et $\partial\Omega \neq \emptyset$ si Ω est ou bien une perturbation compacte de $\mathbb{R}^d$ (avec $\Omega \neq \mathbb{R}^d$) ou bien une variété riemannienne compacte à bord, avec un bord non-vide. Si $\partial\Omega \neq \emptyset$, alors on note $\partial\Omega$ le bord de Ω, et $\overline{\Omega} = \Omega \cup \partial\Omega$, et si $\partial\Omega = \emptyset$, alors on note $\overline{\Omega} = \Omega$. De plus, on dit que Ω est non-borné si Ω est une perturbation compacte de $\mathbb{R}^d$ (ce qui inclut le cas $\Omega = \mathbb{R}^d$). Comme précédemment, on fixe $\beta \in \mathbb{R}$ tel que l'inégalité de Poincaré (2.2.1) est vérifiée, et on note $\|u^0\|_{H_0^1}^2 = \int_\Omega \left(|\nabla u^0|^2 + \beta |u^0|^2\right) \mathrm{d}x$.

On considère une non-linéarité $f \in \mathscr{C}^2(\mathbb{R}, \mathbb{R})$ telle que $f(0) = f'(0) = 0$, et vérifiant l'une des deux hypothèses suivantes. Pour la contrôlabilité locale, on suppose qu'il existe $C_0 > 0$ et α tels que

$$\begin{gathered} |f''(s)| \leq C_0 \left(1 + |s|\right)^{\alpha - 2} \text{ pour tout } s \in \mathbb{R}, \\ 2 \leq \alpha < \tfrac{d+2}{d-2}, \quad \text{et} \quad (\alpha = 2 \text{ si } d = 5 \text{ et } \partial\Omega \neq \emptyset)\,. \end{gathered} \tag{2.4.1}$$

Pour la contrôlabilité à zéro, on suppose que Ω est non-borné, et qu'il existe $C_0 > 0$ et

$\alpha_0 \leq \alpha_1$ tels que

$$\begin{aligned} &|f''(s)| \leq C_0\left(|s|^{\alpha_0-2} + |s|^{\alpha_1-2}\right) \text{ pour tout } s \in \mathbb{R}, \\ &2 < \alpha_0 \leq \alpha_1 < \tfrac{d+2}{d-2}, \quad \text{ et } \quad (d \neq 5 \text{ si } \partial\Omega \neq \emptyset)\,. \end{aligned} \tag{2.4.2}$$

Si $d = 2$, alors α, α_0 et α_1 peuvent être arbitrairement grands. Notons que la condition (2.4.2) implique $f''(0) = 0$. Aucune hypothèse de signe n'est faite sur f, qui peut donc en particulier être focalisante ou défocalisante. On peut par exemple prendre

$$f(s) = \sum_{j=0}^{n} \lambda_j s|s|^{\alpha_j - 1}, \quad s \in \mathbb{R},$$

avec $n \in \mathbb{N}$, $\lambda_0, \cdots, \lambda_n \in \mathbb{R}$, et $2 \leq \alpha_0 \leq \cdots \leq \alpha_n$, en ajustant la taille des exposants pour que f vérifie (2.4.1) ou (2.4.2). Enfin, remarquons que l'hypothèse faite pour la contrôlabilité à zéro est plus forte que l'hypothèse faite pour la contrôlabilité locale : plus précisément, (2.4.2) pour $\alpha_0 \leq \alpha_1$ implique (2.4.1) pour $\alpha = \alpha_1$.

Pour une donnée initiale réelle $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, on considère l'équation des ondes (ou de Klein-Gordon)

$$\left\{ \begin{array}{rcll} \Box u + \beta u &=& f(u) & (t,x) \in \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) &=& (u^0, u^1) & x \in \Omega, \\ u &=& 0 & (t,x) \in \mathbb{R} \times \partial\Omega. \end{array} \right. \tag{2.4.3}$$

Dans le cas $\partial\Omega = \emptyset$, la condition de bord est à retirer.

Introduisons maintenant la contrôlabilité locale au voisinage d'une trajectoire. On considère un temps $T > 0$, une fonction $a \in \mathscr{C}^\infty(\Omega, \mathbb{R})$ servant à délimiter la région où le contrôle agit, et une donnée initiale $(\mathbf{u}^0, \mathbf{u}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ telle que la solution $\mathbf{u}$ de (2.4.3) associée existe sur $[0, T]$.

Définition 2.4.1. On dit qu'il y a *contrôlabilité locale au voisinage de* $\mathbf{u}$ *en temps* T s'il existe $\delta > 0$ tel que pour toute donnée initiale $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ telle que

$$\left\| \left(u^0, u^1\right) - \left(\mathbf{u}^0, \mathbf{u}^1\right) \right\|_{H_0^1 \times L^2} \leq \delta,$$

il existe un contrôle $g \in L^1((0,T), L^2(\Omega))$ tel que la solution u de

$$\left\{ \begin{array}{rcll} \Box u + \beta u &=& f(u) + ag & (t,x) \in]0,T[\times \Omega, \\ (u(0), \partial_t u(0)) &=& (u^0, u^1) & x \in \Omega, \\ u &=& 0 & (t,x) \in]0,T[\times \Omega, \end{array} \right. \tag{2.4.4}$$

vérifie $(u(T), \partial_t u(T)) = (\mathbf{u}(T), \partial_t \mathbf{u}(T))$.

On dit qu'une fonction lisse $F : (0,T) \times \Omega \to \mathbb{R}$ est *analytique en temps* si pour tout $(t_0, x_0) \in (0,T) \times \Omega$, il existe un voisinage $\mathcal{O} \subset (0,T) \times \Omega$ de (t_0, x_0) tel que

$$F(t,x) = \sum_{k=0}^{\infty} \partial_t^k F(t_0, x) \frac{(t - t_0)^k}{k!}, \quad (t,x) \in \mathcal{O}.$$

Notre résultat de contrôlabilité locale est le suivant.

Théorème 2.4.2. *Soient f vérifiant* (2.4.1), *et* $(\mathbf{u}^0, \mathbf{u}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *une donnée initiale telle que la solution* $\mathbf{u}$ *de* (2.4.3) *associée existe sur* $[0, T]$. *On fait les hypothèses suivantes.*

(i) On suppose qu'il existe un ouvert $\omega \subset \overline{\Omega}$ et une constante $c > 0$ tels que $a \geq c$ sur ω, avec (ω, T) vérifiant la condition de contrôle géométrique. Si Ω est non-borné, on suppose de plus qu'il existe $R_0 > 0$ tel que $\mathbb{R}^d \backslash B(0, R_0) \subset \omega$.

(ii) On suppose que $f'(\mathbf{u}) \in L^\infty((0,T) \times \Omega)$, et que $f'(\mathbf{u})$ est lisse et analytique en temps. Si Ω est non-borné, on suppose de plus que pour tout $t \in [0,T]$,

$$\left|\nabla f'(\mathbf{u}(t,x))\right| + \left|f'(\mathbf{u}(t,x))\right| \xrightarrow{|x| \to \infty} 0,$$

où ∇ est le gradient par rapport à la variable d'espace x.

Alors il y a contrôlabilité locale au voisinage de $\mathbf{u}$ en temps T.

Passons maintenant à la contrôlabilité à zéro. On rappelle que si Ω est non-borné, alors $\beta > 0$, et (2.4.4) est l'équation de Klein-Gordon.

Définition 2.4.3. On dit qu'il y a *contrôlabilité à zéro en temps long pour* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ s'il existe $T > 0$ et $g \in L^1((0,T), L^2(\Omega))$ tel que la solution u de (2.4.4) vérifie $(u(T), \partial_t u(T)) = 0$.

On aura besoin de supposer que le domaine Ω considéré est *non-captif*, ce qui correspond au fait que toutes les géodésiques généralisées sortent de tout compact en temps fini. Cette condition implique des inégalités appelées *estimations de résolvante* : pour simplifier, on prend ces estimations comme définition de non-captif.

Définition 2.4.4. On suppose ici que Ω est non-borné. On dit que Ω est *non-captif* si pour tout $\chi \in \mathscr{C}_c^\infty(\Omega)$, il existe une constante $C > 0$ telle que

$$\sqrt{1 + |\lambda|} \left\| \chi \left(-\Delta + \lambda\right)^{-1} \chi u \right\|_{L^2} \leq C \|u\|_{L^2}, \quad u \in L^2(\Omega), \quad \operatorname{Im} \lambda \neq 0.$$

On rappelle que l'expression *solution scattering* désigne une solution scattering vers 0 au sens de la définition 1.4.3. Notre résultat de contrôlabilité à zéro est le suivant.

Théorème 2.4.5. *On suppose que Ω est non-borné et non-captif, et on prend f vérifiant* (2.4.2)*. On considère un ouvert $\omega \subset \overline{\Omega}$ vérifiant la condition de contrôle géométrique, et tel que $\mathbb{R}^d \backslash B(0, R_0) \subset \omega$ pour un certain $R_0 > 0$. On suppose qu'il existe $c > 0$ tel que $a \geq c$ sur ω. Alors pour tout $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, si la solution u de* (2.4.3) *est scattering, alors il y a contrôlabilité à zéro en temps long pour (u^0, u^1).*

De nombreux auteurs ont démontré des propriétés de contrôle pour des équations des ondes non-linéaires, dans des cadres variés. Plus l'hypothèse sur la non-linéarité est forte (par exemple, une non-linéarité asymptotiquement linéaire, ou logarithmique), plus le résultat obtenu est fort (par exemple, contrôlabilité exacte au lieu de contrôlabilité locale). Certains auteurs ont considéré le cas du contrôle au bord, ou le cas de la dimension 1. On ne fera pas ici une bibliographie exhaustive de tous ces résultats. Dans un cadre où la stabilisation et la contrôlabilité locale au voisinage de zéro sont vraies, alors la contrôlabilité exacte en temps long est vraie : cet argument est utilisé par Dehman, Lebeau et Zuazua [DLZ03], puis par Laurent et Joly [JL13] pour une non-linéarité défocalisante sous-critique en dimension 3. Pour des résultats de contrôlabilité au bord, on fait référence à Lasiecka et Triggiani [LT91 ; LT05], Lei et Zhou [ZL07], Ton [Ton19], et Zuazua [Zua90]. Le théorème 2.4.2 semble être le premier résultat de contrôlabilité locale au voisinage d'une trajectoire dépendant du temps pour l'équation des ondes non-linéaire : en ce sens, les résultats les plus proches sont ceux de Rosier et Zhang [RZ09] et Laurent [Lau10] sur l'équation de Schrödinger. Notons que le résultat de contrôlabilité locale de [Lau10] repose sur le prolongement unique pour l'équation linéarisé, qui est pris comme une hypothèse.

2.4.2 Conséquences

On donne ici différents exemples d'applications des résultats de cette section et de la précédente.

Exemples dans des cas focalisants. On considère ici deux exemples explicites. Pour le premier, reprenons l'équation des ondes cubique sur un domaine borné de dimension 3, dont on note Q un état fondamental. On vérifie facilement que si $\varepsilon > 0$ est assez petit, alors $((1-\varepsilon)Q, 0) \in \mathcal{K}^+$ et $((1+\varepsilon)Q, 0) \in \mathcal{K}^-$. Le théorème 2.3.1 implique donc que la solution de (2.3.1) (avec $\gamma = 0$) de donnée initiale $((1+\varepsilon)Q, 0)$ explose en temps fini, et le théorème 2.3.3 implique qu'il existe un amortissement γ, dont le support vérifie la GCC, telle que la solution de (2.3.1) de donnée initiale $((1-\varepsilon)Q, 0)$ et d'amortissement γ, tend vers zéro à vitesse exponentielle. En appliquant la contrôlabilité locale (théorème 2.4.2) plusieurs fois, et en utilisant la réversibilité en temps de l'équation, on obtient donc des résultats de contrôlabilité en temps long permettant de passer d'un voisinage de $(Q, 0)$ à un voisinage de zéro, ou bien d'une donnée initiale explosive de la forme $((1+\varepsilon)Q, 0)$ à zéro, ou encore d'un voisinage de zéro à une donné initiale explosive de cette forme. Notons que la stabilisation des solutions de donnée initiale dans $\mathcal{K}^+$ et la contrôlabilité locale au voisinage de zéro impliquent la contrôlabilité exacte en temps long dans $\mathcal{K}^+$.

Pour notre deuxième exemple, on s'intéresse à l'équation de Klein-Gordon cubique focalisante sur $\Omega = \mathbb{R}^3$, avec $\beta = 1$. La dichotomie du théorème 2.3.1 est toujours valide, et Ibrahim, Masmoudi et Nakanishi [IMN11] ont montré que les solutions avec donnée initiale dans $\mathcal{K}^+$ sont scattering. En utilisant de plus le fait que $((1 \pm \varepsilon)Q, 0) \in \mathcal{K}^\mp$, la contrôlabilité locale (théorème 2.4.2) et la contrôlabilité à zéro des solutions scattering (théorème 2.4.5), on obtient les mêmes résultats de contrôlabilité en temps long, entre un voisinage de $(Q, 0)$, un voisinage de 0, et certaines solutions explosives. On peut également obtenir ces résultats sans utiliser le fait que les solutions avec donnée initiale dans $\mathcal{K}^+$ sont scattering, mais en se basant sur l'existence d'une solution hétérocline : il existe une solution W qui est scattering (en temps positifs) et qui vérifie

$$\|(W(t), \partial_t W(t)) - (Q, 0)\|_{H^1 \times L^2} \xrightarrow{t \to -\infty} 0.$$

L'existence d'une telle solution a été démontrée à l'origine pour l'équation des ondes critique par Duyckaerts et Merle [DM08]. Pour une démonstration dans le cas de l'équation de Klein-Gordon, on pourra consulter le livre [NS11b], ou les articles [NS12 ; NS11a], de Nakanishi et Schlag. Remarquons que dans cet exemple, on a aussi la contrôlabilité exacte en temps long dans $\mathcal{K}^+$, en utilisant le fait que les solutions de donnée initiale dans $\mathcal{K}^+$ sont scattering, la contrôlabilité locale au voisinage de zéro, et la contrôlabilité à zéro des solutions scattering.

Exemples dans des cas défocalisants. Pour certaines non-linéarités défocalisantes, des auteurs ont montré que toutes les solutions sont scattering (voir par exemple Brenner [Bre84], Ginibre et Vélo [GV89], Nakanishi [Nak01]). Dans le cas d'un domaine non-borné et non-captif, si la non-linéarité vérifie de plus (2.4.2), alors la contrôlabilité à zéro des solutions scattering (théorème 2.4.5) implique la contrôlabilité à zéro de toutes les solutions. En particulier, en utilisant comme précédemment la contrôlabilité locale au voisinage de zéro, et la réversibilité en temps de l'équation, on obtient un résultat de contrôlabilité exacte en temps long.

2.4.3 Schémas de démonstration

Schéma de démonstration du théorème 2.4.2. L'idée de la démonstration, assez classique, est de linéariser l'équation au voisinage de la trajectoire considérée, d'établir la contrôlabilité exacte du problème linéarisé, et d'en déduire la contrôlabilité locale de l'équation non-linéaire avec un théorème de point fixe. Donnons un peu plus de détails. Fixons $\mathbf{u}$ une solution de (2.4.3) qui existe sur $[0,T]$, de donnée initiale $(\mathbf{u}^0,\mathbf{u}^1)\in H_0^1(\Omega)\times L^2(\Omega)$. Pour tout $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ proche de $(\mathbf{u}^0,\mathbf{u}^1)$, on cherche un contrôle g tel que la solution u de l'équation avec donnée initiale (u^0,u^1) et contrôle g vérifie $(u(T),\partial_t u(T))=(\mathbf{u}(T),\partial_t\mathbf{u}(T))$. En écrivant $u=\mathbf{u}+h$, on voit que de façon équivalente, pour tout $(h^0,h^1)\in H_0^1(\Omega)\times L^2(\Omega)$ proche de zéro, on cherche g tel que la solution h de

$$\begin{cases} \Box h+\beta h &= f(\mathbf{u}+h)-f(\mathbf{u})+ag & (t,x)\in]0,T[\times\Omega,\\ (h(T),\partial_t h(T)) &= 0 & x\in\Omega,\\ h &= 0 & (t,x)\in]0,T[\times\Omega,\end{cases}$$

vérifie $(h(0),\partial_t h(0))=(h^0,h^1)$. La première étape de la démonstration consiste à vérifier que h (et donc u) existe sur $[0,T]$ si g est assez petit.

On étudie ensuite, dans une deuxième étape, la contrôlabilité exacte pour l'*équation linéarisée au voisinage de* $\mathbf{u}$, qui est

$$\begin{cases} \Box\phi+\beta\phi &= f'(\mathbf{u})\phi+ag & (t,x)\in]0,T[\times\Omega,\\ (\phi(T),\partial_t\phi(T)) &= 0 & x\in\Omega,\\ \phi &= 0 & (t,x)\in]0,T[\times\Omega.\end{cases} \tag{2.4.5}$$

On voit que (2.4.5) est une équation des ondes linéaire avec un potentiel qui dépend du temps, ce qui justifie notre hypothèse forte sur la régularité de $f'(\mathbf{u})$. Notons également que le domaine Ω n'est pas supposé borné, ce qui peut rajouter une difficulté supplémentaire. Pour démontrer la contrôlabilité exacte de (2.4.5), avec l'hypothèse que le support de a vérifie la GCC, on procède comme suit. D'après la méthode HUM, il suffit de démontrer une inégalité d'observabilité. On commence par considérer le cas où $f'(\mathbf{u})=0$. Si Ω est compact, alors la contrôlabilité exacte (ou l'observabilité) de (2.4.5) est connue depuis le travail fondateur de Bardos, Lebeau et Rauch [BLR92]. Si Ω est un domaine extérieur, on utilise le fait que la région où agit le contrôle contient le complémentaire d'une boule. On propose deux arguments : le premier, plus astucieux, consiste à couper la solution en deux, à utiliser l'observabilité sur un domaine compact et l'observabilité sur $\mathbb{R}^d$, et le second, plus classique, repose sur l'utilisation d'une mesure de défaut microlocale. Une fois l'observabilité établie dans le cas $f'(\mathbf{u})=0$, on utilise la décroissance en l'infini du potentiel et un argument de compacité pour se ramener à l'absence de solutions invisibles pour l'équation (2.4.5). On fait alors appel au résultat de prolongement unique de Laurent et Léautaud [LL15] pour des équations à coefficients partiellement analytiques, qui permet de conclure.

La troisième et dernière étape consiste à appliquer un théorème de point fixe. Prenons $(h^0,h^1)\in H_0^1(\Omega)\times L^2(\Omega)$. Pour $H:[0,T]\times\Omega\to\mathbb{R}$, on note $\mathrm{NL}_{\mathbf{u}}(H)=f(\mathbf{u}+H)-f(\mathbf{u})-f'(\mathbf{u})H$, et on utilise la contrôlabilité de l'équation linéarisé pour trouver un contrôle $g(H)$ tel que la solution de

$$\begin{cases} \Box h+\beta h &= f'(\mathbf{u})h+\mathrm{NL}_{\mathbf{u}}(H)+ag(H) & (t,x)\in]0,T[\times\Omega,\\ (h(T),\partial_t h(T)) &= 0 & x\in\Omega,\\ h &= 0 & (t,x)\in]0,T[\times\Omega,\end{cases}$$

vérifie $(h(0),\partial_t h(0))=(h^0,h^1)$. On montre alors que l'opérateur $H\mapsto h$ admet un point fixe si (h^0,h^1) est suffisamment proche de zéro, à l'aide d'estimations de Strichartz.

Schéma de démonstration du théorème 2.4.5. On commence par un exemple introductif très simple, qui est le cas de l'équation linéaire, avec un contrôle agissant sur tout le domaine Ω. Fixons $\phi \in \mathscr{C}^\infty([0,T],\mathbb{R})$ telle que $\phi(t)=1$ pour t proche de 0, et $\phi(t)=0$ pour t proche de T. Si u est une solution de $\Box u+\beta u=0$, alors la fonction $v:(t,x)\mapsto \phi(t)u(t,x)$ est solution de

$$\begin{cases} \Box v+\beta v &= (\Box+\beta)(\phi u) & (t,x)\in]0,T[\times\Omega, \\ (v(0),\partial_t v(0)) &= (u(0),\partial_t u(0)) & x\in\Omega, \\ v &= 0 & (t,x)\in]0,T[\times\Omega, \end{cases}$$

et vérifie $(v(T),\partial_t v(T))=0$. Le terme $(\Box+\beta)(\phi u)$ peut donc être vu comme un contrôle, permettant de passer d'un état $(u(0),\partial_t u(0))$ à 0. On va imiter cette stratégie pour démontrer la contrôlabilité à zéro des solutions scattering. Il y a deux obstacles : la présence de la non-linéarité, et le fait que la région où agit le contrôle n'est pas égale à tout l'espace.

On considère une donnée initiale $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ telle que la solution correspondante u_{NL} de (2.4.3) est asymptotiquement proche d'une solution u_{L} de l'équation linéaire $\Box u_{\mathrm{L}}+\beta u_{\mathrm{L}}=0$. On montre ensuite que si $T>0$ est assez grand, alors on a les trois propriétés suivantes. Premièrement, par hypothèse, si T est assez grand, alors la quantité

$$\|(u_{\mathrm{NL}}(t),\partial_t u_{\mathrm{NL}}(t))-(u_{\mathrm{L}}(t),\partial_t u_{\mathrm{L}}(t))\|_{H_0^1\times L^2}$$

est petite, pour tout $t\geq T$.

Deuxièmement, le domaine considéré est non-captif, ce qui implique la décroissance locale de l'énergie : l'énergie de $u_{\mathrm{L}}(t)$ sur une boule tend vers 0 quand t tend vers l'infini. Pour le montrer, on utilise la stratégie de [Bur03], avec une idée supplémentaire due à [Bur], fondée sur deux arguments TT^*, à deux niveaux de régularité différentes. En notant B une boule dont le complémentaire est inclus dans la région où le contrôle agit, on peut donc supposer que la norme $H_0^1\times L^2$ de $(\mathbb{1}_B u_{\mathrm{L}}(t),\mathbb{1}_B\partial_t u_{\mathrm{L}}(t))$ est petite, pour tout $t\geq T$.

Troisièmement, on dispose d'une inégalité de Strichartz globale en temps. Le fait que u_{NL} est scattering permet de montrer que

$$u_{\mathrm{NL}}\in L^{\alpha_i}(]0,+\infty[,L^{2\alpha_i}(\Omega)),\quad i=0,1, \tag{2.4.6}$$

où α_0 et α_1 sont les exposants apparaissant dans (2.4.2). Pour cela, on commence par montrer que u_{L} vérifie (2.4.6), en adaptant la stratégie de [Bur03] et [BSS09], qui repose sur la décroissance locale de l'énergie. En utilisant le fait que u_{L} et u_{NL} sont proches, et des estimations de Strichartz locales en temps, on obtient (2.4.6). En particulier, on peut choisir T tel que la quantité

$$\|u_{\mathrm{NL}}\|_{L^{\alpha_0}(]T,+\infty[,L^{2\alpha_0})}+\|u_{\mathrm{NL}}\|_{L^{\alpha_1}(]T,+\infty[,L^{2\alpha_1})}$$

est petite.

On choisit maintenant T vérifiant les trois propriétés précédentes. On prend ensuite $\phi\in\mathscr{C}^\infty(\mathbb{R}_+,\mathbb{R})$ telle que $\phi(t)=1$ pour $t\leq T$, et $\phi(t)=0$ pour $t\geq T+1$. On note $v=\varphi u_{\mathrm{NL}}$, et

$$g=\mathbb{1}_{\Omega\setminus B}\left(\Box v+\beta v-f(v)\right).$$

La fonction g peut être utilisée comme un contrôle, permettant de passer de (u^0,u^1) à un état proche de zéro. On conclut avec la contrôlabilité locale au voisinage de zéro.

2.4.4 Perspectives.

Dans nos exemples, on a seulement utilisé la contrôlabilité locale au voisinage d'une solution stationnaire. Il serait intéressant de chercher des trajectoires de l'équation vérifiant les hypothèses du théorème de contrôlabilité locale, mais qui dépendent du temps, et d'en déduire d'autres résultats de contrôlabilité. Une autre question à étudier est celle d'essayer de généraliser les théorèmes 2.4.2 et 2.4.5 à des plus grandes dimensions.

Nous avons mentionné plus haut que certaines solutions explosives peuvent être contrôlées. Cependant, ces résultats concernent des données initiales très proches d'une solution globale, donc des solutions explosives qui ont un temps d'existence assez long. Peut-on montrer des résultats de contrôlabilité plus généraux pour des solutions explosives ? Une première conjecture pourrait être la suivante : pour toute donnée initiale (u^0, u^1) telle que la solution associée a un temps maximal d'existence fini T_1, il existe un contrôle g satisfaisant une hypothèse géométrique naturelle telle que la solution avec donnée initiale (u^0, u^1) et avec contrôle g a un temps maximal d'existence $T_2 > T_1$. Par vitesse finie de propagation, une condition géométrique nécessaire est que tous les points de Ω où la solution explose aient une distance au support du contrôle plus petite que T_1. Il serait intéressant de se demander si cette condition géométrique est également suffisante pour que la conjecture soit vraie.

Chapter 3

Change of regularity in controllability and observability of systems of wave equations

This chapter is based on the article [Per23a], which has been prepublished and submitted in a journal.

Contents

Introduction

Let (M,g) be a n-dimensional compact Riemannian manifold with boundary. We write ∂M for its boundary and $\operatorname{Int} M = M\backslash\partial M$. Let N be a positive integer. Consider a first-order differential operator $X \in \mathscr{C}^\infty(M, TM \otimes \mathbb{C}^{N\times N})$, acting on functions from M to $\mathbb{C}^N$, given in a coordinate chart (U,x) by

$$X = X^j \frac{\partial}{\partial x^j}, \quad \text{with } X^j \in \mathscr{C}^\infty(U, \mathbb{C}^{N\times N}) \text{ for } j \in [\![1,n]\!].$$

Consider also $q \in \mathscr{C}^\infty(M, \mathbb{C}^{N\times N})$, and write P for the operator $\mathsf{P} = \Delta - X - q$, where $\Delta = \Delta \mathrm{Id}_{\mathbb{C}^N}$ is the (vectorial) Laplace-Beltrami operator associated with the metric g. Denote by P^* the adjoint of P. It has the same form as P. We will define a family of spaces of Sobolev type, written $\mathcal{K}^s$ and $\mathcal{K}^s_*$, corresponding to compatibility conditions adapted to P and P^*.

Consider $s \in \mathbb{R}$, $T > 0$, $\Theta = (\Theta_1, \cdots, \Theta_N) \in \mathscr{C}^\infty_{\mathrm{c}}((0,T) \times \partial M, \mathbb{C}^N)$, and set

$$\operatorname{diag}(\Theta) = \begin{pmatrix} \Theta_1 & 0 & \cdots & 0 \\ 0 & \Theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Theta_N \end{pmatrix}.$$

Solutions of the wave equations

$$\begin{cases} \partial_t^2 u - \mathsf{P} u = 0 & \text{in } (0,T) \times M, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (u^0, u^1) & \text{in } M, \\ u = 0 & \text{on } (0,T) \times \partial M, \end{cases} \tag{3.0.1}$$

$$\begin{cases} \partial_t^2 v - \mathsf{P}^* v = 0 & \text{in } (0,T) \times M, \\ (v(0,\cdot), \partial_t v(0,\cdot)) = 0 & \text{in } M, \\ v = \operatorname{diag}(\Theta) f & \text{on } (0,T) \times \partial M, \end{cases} \tag{3.0.2}$$

are given, for $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$ and $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$. We write $\partial_\nu u = (\partial_\nu \mathrm{Id}_{\mathbb{C}^N})\, u$ for the normal derivative of u.

Definition 3.0.1 (H^s-observability for Θ)**.** We say that H^s-observability for Θ holds if there exists $C > 0$ such that for all $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$,

$$\left\| \left(u^0, u^1\right) \right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s} \le C \left\| \operatorname{diag}(\Theta) \partial_\nu u \right\|_{H^s((0,T) \times \partial M, \mathbb{C}^N)}.$$

Definition 3.0.2 (H^s-exact controllability for Θ)**.** We say that H^s-exact controllability for Θ holds if for all $(\varphi^0, \varphi^1) \in \mathcal{K}^s_* \times \mathcal{K}^{s-1}_*$, there exists $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$ such that

$$(v(T), \partial_t v(T)) = (\varphi^0, \varphi^1).$$

A duality property for solutions of (3.0.1) and (3.0.2) implies that the classical controllability - observability equivalence is satisfied.

Lemma 3.0.3. *For $s \in \mathbb{R}$, H^s-exact controllability for Θ and H^{-s}-observability for Θ are equivalent.*

The main result of this chapter is the following.

Theorem 3.0.4. *Consider $s_1, s_2 \in \mathbb{R}$. If $s_1 < s_2$ then for all $\Theta \in \mathscr{C}^\infty_{\mathrm{c}}((0,T) \times \partial M, \mathbb{C}^N)$, H^{s_1}-observability for Θ implies H^{s_2}-observability for Θ. If $s_1 > s_2$ then for all $\Theta^1 = (\Theta^1_1, \cdots, \Theta^1_N) \in \mathscr{C}^\infty_{\mathrm{c}}((0,T) \times \partial M, \mathbb{C}^N)$ and $\Theta^2 = (\Theta^2_1, \cdots, \Theta^2_N) \in \mathscr{C}^\infty_{\mathrm{c}}((0,T) \times \partial M, \mathbb{C}^N)$ such that for all $k \in [\![1, N]\!]$, $\Theta^2_k \neq 0$ on $\operatorname{supp} \Theta^1_k$, H^{s_1}-observability for Θ^1 implies H^{s_2}-observability for Θ^2.*

An analogue of Theorem 3.0.4 for internal controllability holds, with a simpler proof (see Appendix 3.A). We refer to Section 2.1 for a discussion about Theorem 3.0.4 and its connection to the existing literature.

Outline of the chapter. In Section 3.1, we gather some basic results about the spaces $\mathcal{K}^s$ and wave systems, and we prove the controllability / observability equivalence (Lemma 3.0.3). In Section 3.2, we show the ellipticity estimate for ∂_t^2 acting on Neumann traces of solutions. In Section 3.3, we prove Theorem 3.0.4. In Appendix 3.A, we briefly explain how the methods of the proof of Theorem 3.0.4 can be adapted to the case of internal observability. Proofs of the results of Section 1 are provided in Appendix 3.B.

Notation. For $x \in M$ and $U, V \in T_xM$, we write $\langle U, V\rangle_g$ for the inner product of U and V with respect to the metric g. The gradient with respect to g of a function $u : M \to \mathbb{C}$ is denoted by ∇u, and the divergence with respect to g of a vector field X on M is denoted by $\operatorname{div} X$. We write $\mathrm{d}V_g$ for the Riemannian density on M. Finally, $\langle \cdot, \cdot\rangle_{\mathcal{X}',\mathcal{X}}$ denotes the bilinear duality product between a Banach space $\mathcal{X}$ and its dual space $\mathcal{X}'$, and $\langle \cdot, \cdot\rangle_{\mathcal{H}}$ denotes the inner product of a Hilbert space $\mathcal{H}$, which is linear in the first variable and antilinear in the the second. We write $(\pi_1, \cdots, \pi_N)$ for the projections associated with the canonical basis of $\mathbb{C}^N$.

3.1 Controllability - observability equivalence

3.1.1 Adjoint operator

Here, we give the precise expression of the adjoint of P. The operator X is compatible with change of coordinates, meaning that for a second set of coordinates $(\tilde{x}^1, \cdots, \tilde{x}^n)$, with $X = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}$, one has

$$\tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j} X^j, \quad i \in [\![1, n]\!]. \tag{3.1.1}$$

Denote by $(e_1, \cdots, e_N)$ the canonical basis of $\mathbb{C}^N$, and consider $u = (u^1, \cdots, u^N) \in \mathscr{C}^\infty(M, \mathbb{C}^N)$. We use Einstein summation convention in M (for indices between 1 and n) but not on $\mathbb{C}^{\mathbb{N}}$ (for indices between 1 and N). One writes

$$Xu = \sum_{k,\ell=1}^N X^j_{k\ell} \frac{\partial u^\ell}{\partial x^j} e_k,$$

where $X^j_{k\ell}$ is the coefficient (k, ℓ) of the matrix X^j. For $k, \ell \in [\![1, N]\!]$, write $X_{k\ell} = X^j_{k\ell} \frac{\partial}{\partial x^j}$, so that

$$Xu = \sum_{k,\ell=1}^N \left\langle \nabla u^\ell, X_{k\ell} \right\rangle_g e_k.$$

Using formula (3.1.1), one sees that $X_{k\ell}$ is a vector field for all $k, \ell \in [\![1, N]\!]$. We will use the notation

$$\langle X, V\rangle_g \, u = \sum_{k,\ell=1}^N \langle X_{k\ell}, V\rangle_g \, u^\ell e_k \in \mathbb{C}^N,$$

if V is a vector field on M and $u = (u^1, \cdots, u^N)$ is a function with values in $\mathbb{C}^N$. With integration by parts, one derives the following results.

Lemma 3.1.1. *For $u, v \in H^1(M, \mathbb{C}^N)$, one has*

$$\langle Xu, v\rangle_{L^2(M,\mathbb{C}^N)} = \langle u, X^*v\rangle_{L^2(M,\mathbb{C}^N)} + \langle \langle X, \nu\rangle_g u, v\rangle_{L^2(\partial M,\mathbb{C}^N)},$$

where X^* is the first-order differential operator given by

$$(X^*v)^\ell = -\sum_{k=1}^N \left\langle \nabla v^k, \overline{X_{k\ell}} \right\rangle_g - \left(\left(\operatorname{div} \overline{X}\right) v\right)^\ell, \quad \ell \in [\![1, N]\!].$$

Lemma 3.1.2. *For $u, v \in H^2(M, \mathbb{C}^N)$, one has*

$$\langle \mathsf{P}u, v\rangle_{L^2(M,\mathbb{C}^N)} = \langle u, \mathsf{P}^* v\rangle_{L^2(M,\mathbb{C}^N)} + \langle \langle X, \nu\rangle_g u, v\rangle_{L^2(\partial M,\mathbb{C}^N)} + \langle \partial_\nu u, v\rangle_{L^2(\partial M,\mathbb{C}^N)} - \langle u, \partial_\nu v\rangle_{L^2(\partial M,\mathbb{C}^N)},$$

with $\mathsf{P}^* = \Delta - X^* - q^*$, where q^* denotes the adjoint of q.

Remark 3.1.3. In particular, if $u, v \in H^2(M, \mathbb{C}^N) \cap H^1_0(M, \mathbb{C}^N)$, then

$$\langle \mathsf{P}u, v\rangle_{L^2(M,\mathbb{C}^N)} = \langle u, \mathsf{P}^* v\rangle_{L^2(M,\mathbb{C}^N)}.$$

Remark 3.1.4. The operators P and P^* are of the same form. Indeed, set $\tilde{X} = \left(X^j\right)^* \frac{\partial}{\partial x^j}$ and let $\tilde{q} \in \mathscr{C}^\infty(M, \mathbb{C}^{N\times N})$ be given by $\tilde{q}_{k\ell} = -\operatorname{div} \overline{X_{\ell k}} + \overline{q_{\ell k}}$ for $k, \ell \in [\![1, N]\!]$. One has $\mathsf{P}^* = \Delta - \tilde{X} - \tilde{q}$.

3.1.2 A family of spaces of Sobolev type

Denote by $\mathsf{P}_{\mathscr{D}'}$ the action of P on distributions, that is,

$$\mathsf{P}_{\mathscr{D}'} : \mathscr{D}'(M, \mathbb{C}^N) \longrightarrow \mathscr{D}'(M, \mathbb{C}^N),$$

with $\langle \mathsf{P}_{\mathscr{D}'} u, \phi\rangle_{\mathscr{D}',\mathscr{D}} = \langle u, \mathsf{P}^*\phi\rangle_{\mathscr{D}',\mathscr{D}}$ for $u \in \mathscr{D}'(M, \mathbb{C}^N)$ and $\phi \in \mathscr{D}(M, \mathbb{C}^N)$.

We define a Sobolev-like regularity, adapted to the operator P. Write $\mathcal{K}^0 = L^2(M, \mathbb{C}^N)$ and for $m \in \mathbb{N}^*$, set

$$\mathcal{K}^m = \left\{u \in H^m(M, \mathbb{C}^N), \mathsf{P}^k_{\mathscr{D}'} u \in H^1_0(M, \mathbb{C}^N) \text{ for } k \in \left[\!\!\left[0, \left\lfloor \frac{m-1}{2} \right\rfloor\right]\!\!\right]\right\},$$

endowed with the H^m inner product. Here, $\lfloor\cdot\rfloor$ is the floor function. Note that $\mathcal{K}^1 = H^1_0(M, \mathbb{C}^N)$ and $\mathcal{K}^2 = H^2(M, \mathbb{C}^N) \cap H^1_0(M, \mathbb{C}^N)$. For $m \in \mathbb{N}$, one checks that $\mathcal{K}^m$ is a complete subspace of $H^m(M, \mathbb{C}^N)$, and, in particular, is a Hilbert space.

The space $\mathcal{K}^s$ is defined for $s \geq 0$ by interpolation. Since the operators P and P^* are of the same form, one can define the space $\mathcal{K}^s_*$ for $s \geq 0$, by replacing P with P^* in the previous definitions. Then, for $s < 0$, define $\mathcal{K}^s$ as the dual of $\mathcal{K}^{-s}_*$, and $\mathcal{K}^s_*$ as the dual of $\mathcal{K}^{-s}$. Note that for $m \in \mathbb{N}$ sufficiently large, $H^m_0(M, \mathbb{C}^N)$ is not dense in $\mathcal{K}^m$, so that $\mathcal{K}^{-m} \not\subseteq H^{-m}(M, \mathbb{C}^N)$. For $s \geq 0$, set

$$\|u\|_{\mathcal{K}^s} = \|u\|_{H^s(M,\mathbb{C}^N)},$$

and for $s < 0$, set

$$\|u\|_{\mathcal{K}^s} = \sup\left\{\left|\langle u, v\rangle_{\mathcal{K}^s,\mathcal{K}^{-s}_*}\right|, \|v\|_{\mathcal{K}^{-s}_*} \leq 1\right\},$$

the usual norm of a dual space. For $s \in \mathbb{R}$, a norm on $\mathcal{K}^s_*$ is defined similarly.

Next, we define the natural action of P on $\mathcal{K}^s$. With interpolation, the definition of $\mathsf{P}_s : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ is only needed for $s \in \mathbb{Z}$.

Definition 3.1.5 (Definition of P_s).

(i) Suppose $s \in \mathbb{N}^*$. Then the operator $\mathsf{P}_s : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ is the differential operator P on $\mathcal{K}^{s+1}$. It is a bounded operator. The operator $\mathsf{P}^*_s : \mathcal{K}^{s+1}_* \longrightarrow \mathcal{K}^{s-1}_*$ is defined similarly.

(ii) Suppose $s \in \mathbb{Z}$, $s \leq -1$. Define $\mathsf{P}_s : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ as the adjoint of $\mathsf{P}^*_{-s} : \mathcal{K}^{-s+1}_* \longrightarrow \mathcal{K}^{-s-1}_*$, and $\mathsf{P}^*_s : \mathcal{K}^{s+1}_* \longrightarrow \mathcal{K}^{s-1}_*$ as the adjoint of $\mathsf{P}_{-s} : \mathcal{K}^{-s+1} \longrightarrow \mathcal{K}^{-s-1}$.

(iii) If $s = 0$, then $\mathcal{K}^{s+1} = \mathcal{K}_*^{s+1} = H_0^1(M, \mathbb{C}^N)$ and $\mathcal{K}^{s-1} = \mathcal{K}_*^{s-1} = H^{-1}(M, \mathbb{C}^N)$. For $u \in H_0^1(M, \mathbb{C}^N)$, define $\mathsf{P}_0 u \in H^{-1}(M, \mathbb{C}^N)$ by

$$\langle \mathsf{P}_0 u, v\rangle_{H^{-1},H_0^1} = -\sum_{k=1}^{N} \left\langle \nabla u^k, \overline{\nabla v^k}\right\rangle_{L^2(M)} - \langle (X+q)u, \overline{v}\rangle_{L^2(M,\mathbb{C}^N)}, \quad v \in H_0^1(M, \mathbb{C}^N).$$

This gives an operator $\mathsf{P}_0 : \mathcal{K}^1 \longrightarrow \mathcal{K}^{-1}$. The operator $\mathsf{P}_0^* : \mathcal{K}_*^1 \longrightarrow \mathcal{K}_*^{-1}$ is defined similarly.

(iv) For $r \in \mathbb{N}$ and $s \in \mathbb{R}$, also define $\mathsf{P}_s^0 = \mathrm{Id}_{\mathcal{K}^s}$, $\mathsf{P}_s^r : \mathcal{K}^{s+r} \to \mathcal{K}^{s-r}$ by

$$\mathsf{P}_s^r : \mathcal{K}^{s+r} \xrightarrow{\mathsf{P}_{s+r-1}} \mathcal{K}^{s+r-2} \xrightarrow{\mathsf{P}_{s+r-3}} \cdots \xrightarrow{\mathsf{P}_{s-r+3}} \mathcal{K}^{s-r+2} \xrightarrow{\mathsf{P}_{s-r+1}} \mathcal{K}^{s-r},$$

and $\mathsf{P}_s^{*r} : \mathcal{K}_*^{s+r} \to \mathcal{K}_*^{s-r}$ similarly.

Note that for all $s \in \mathbb{R}$, as $\mathcal{K}^{s+1}$ and $\mathcal{K}^{s-1}$ are Hilbert spaces, one has

$$\mathsf{P}_s = \left(\mathsf{P}_{-s}^*\right)^*. \tag{3.1.2}$$

We check that our definitions make sense in the following lemma.

Lemma 3.1.6. *For $s \in \mathbb{R}$ and $r \in \mathbb{N}^*$, the operator $\mathsf{P}_s^r : \mathcal{K}^{s+r} \to \mathcal{K}^{s-r}$ is well-defined and continuous. The same is true for $\mathsf{P}_s^{*r} : \mathcal{K}_*^{s+r} \to \mathcal{K}_*^{s-r}$. If $s \in \mathbb{R}$ and $r \in \mathbb{N}^*$ are such that $s - r \geq -1$, then for $u \in \mathcal{K}^{s+r}$ and $v \in \mathcal{K}^{s-r}$ such that $v = \mathsf{P}_s^r u$, one has $v = \mathsf{P}_{\mathscr{D}'}^r u$ in $\mathscr{D}'(M, \mathbb{C}^N)$.*

Proof. By definition of P_s^r, we can always assume that $r = 1$, and by interpolation, we may assume that $s \in \mathbb{Z}$. The connection between P_s and $\mathsf{P}_{\mathscr{D}'}$ follows from our definition of P_s for $s \geq 0$.

Consider $s \in \mathbb{N}^*$. For $u \in \mathcal{K}^{s+1}$, one has $u \in H^{s+1}(M, \mathbb{C}^N)$ and $\mathsf{P}_s u = \mathsf{P}_{\mathscr{D}'} u$ in $\mathscr{D}'(M, \mathbb{C}^N)$, implying $\mathsf{P}_s u \in H^{s-1}(M, \mathbb{C}^N)$ and

$$\|\mathsf{P}_s u\|_{H^{s-1}} \lesssim \|u\|_{H^{s+1}}.$$

Thus, we only have to check that the boundary conditions of the definition of the spaces $\mathcal{K}^s$ are such that the operators P_s are well-defined. For $s = 0$, the result is true.

Assume that s is even and write $s = 2\sigma$. By definition, $u \in \mathcal{K}^{2\sigma+1}$ gives

$$\mathsf{P}_{\mathscr{D}'}^k u \in H_0^1(M, \mathbb{C}^N) \text{ for } k \in [\![0, \sigma]\!].$$

As $\mathsf{P}_{\mathscr{D}'} u = \mathsf{P}_s u$ in $\mathscr{D}'(M, \mathbb{C}^N)$, one has

$$\mathsf{P}_{\mathscr{D}'}^k (\mathsf{P}_s u) \in H_0^1(M, \mathbb{C}^N) \text{ for } k \in [\![0, \sigma - 1]\!],$$

that is, $\mathsf{P}_s u \in \mathcal{K}^{2\sigma-1}$.

Assume that s is odd and write $s = 2\sigma + 1$. By definition, $u \in \mathcal{K}^{2\sigma+2}$ gives

$$\mathsf{P}_{\mathscr{D}'}^k u \in H_0^1(M, \mathbb{C}^N) \text{ for } k \in [\![0, \sigma]\!].$$

If $\sigma = 0$, one has $\mathsf{P}_s u \in \mathcal{K}^{2\sigma}$. If $\sigma \geq 1$, then as $\mathsf{P}_{\mathscr{D}'} u = \mathsf{P}_s u$ in $\mathscr{D}'(M, \mathbb{C}^N)$, one has

$$\mathsf{P}_{\mathscr{D}'}^k (\mathsf{P}_s u) \in H_0^1(M, \mathbb{C}^N) \text{ for } k \in [\![0, \sigma - 1]\!],$$

that is, $\mathsf{P}_s u \in \mathcal{K}^{2\sigma}$.

Finally, the adjoint of a continuous linear operator between Hilbert spaces is a well-defined continuous operator, so the result is true for $s \in \mathbb{Z}$, $s \leq -1$. ☐

Remark 3.1.7. (i) The fact that $v = \mathsf{P}_s^r u$ implies $v = \mathsf{P}_{\mathscr{D}'}^r u$ does not hold for $s < -1$, because $\mathcal{K}^s$ is not included in $\mathscr{D}'(M, \mathbb{C}^N)$ if $s < -1$.

(ii) The previous definitions are very natural, but note that some non-intuitive phenomena can occur when dealing with the operator $\mathsf{P}_s : \mathcal{K}^{s+1} \to \mathcal{K}^{s-1}$. To illustrate this, take $N = 1$, $\mathsf{P} = \Delta$ and $s = -1$. In that case, one has $\mathsf{P}_s^* = \mathsf{P}_{-s}$ for $s \in \mathbb{R}$. Recall that by definition, $H^{-2}(M)$ is the dual of $H_0^2(M)$. The constant function $u = 1$ belongs to $L^2(M)$, and is sent to zero by the differential operator $\Delta : L^2(M) \to H^{-2}(M)$. However, by definition, the operator $\mathsf{P}_{-1} : L^2(M) \to \mathcal{K}^{-2}$ is the adjoint of the operator

$$\mathsf{P}_1 : \mathcal{K}^2 = H^2(M) \cap H_0^1(M) \longrightarrow L^2(M),$$

implying

$$\langle \mathsf{P}_{-1} u, v \rangle_{\mathcal{K}^{-2}, \mathcal{K}^2} = \left\langle 1, \overline{\mathsf{P}_1 v} \right\rangle_{L^2(M)} = \langle 1, \Delta \overline{v} \rangle_{L^2(M)} = \langle 1, \partial_\nu \overline{v} \rangle_{L^2(\partial M)}, \quad v \in \mathcal{K}^2.$$

Hence, the function $u = 1$ is not sent to zero by the operator $\mathsf{P}_{-1} : L^2(M) \to \mathcal{K}^{-2}$.

In the following proposition, we gather the properties of the spaces $\mathcal{K}^s$ that are needed for what follows.

Proposition 3.1.8. *(i)* Embeddings properties. *For $s \in \mathbb{R}$ and $\delta > 0$, the map*

$$\iota_{\mathcal{K}^{s+\delta} \to \mathcal{K}^s} : \mathcal{K}^{s+\delta} \hookrightarrow \mathcal{K}^s$$

is a well-defined, compact embedding with a dense range. If $s + \delta < 0$, the embedding corresponds to a restriction operator. If $s + \delta \geq 0 > s$, then the embedding is defined by using $L^2(M, \mathbb{C}^N)$ as a pivot space. The operator P *commutes with the embeddings: more precisely, for $r \in \mathbb{N}$, $s \in \mathbb{R}$ and $\delta > 0$, one has*

$$\mathsf{P}_s^r \circ \iota_{\mathcal{K}^{s+r+\delta} \to \mathcal{K}^{s+r}} = \iota_{\mathcal{K}^{s-r+\delta} \to \mathcal{K}^{s-r}} \circ \mathsf{P}_{s+\delta}^r : \mathcal{K}^{s+r+\delta} \to \mathcal{K}^{s-r}. \tag{3.1.3}$$

(ii) Elliptic estimate of P. *Consider $s \in \mathbb{R}$, $r \in \mathbb{N}^*$, and $w \in \mathcal{K}^{s+r-1}$. We already know that $\mathsf{P}_{s-1}^r w \in \mathcal{K}^{s-r-1}$. Assume that there exists $v \in \mathcal{K}^{s-r}$ such that $\mathsf{P}_{s-1}^r w = \iota_{\mathcal{K}^{s-r} \to \mathcal{K}^{s-r-1}}(v)$. Then there exists $u \in \mathcal{K}^{s+r}$ such that*

$$\iota_{\mathcal{K}^{s+r} \to \mathcal{K}^{s+r-1}}(u) = w \quad \text{and} \quad \mathsf{P}_s^r u = v.$$

Moreover, there exists $C > 0$ such that

$$\|u\|_{\mathcal{K}^{s+r}} \leq C \left(\|\mathsf{P}_s^r u\|_{\mathcal{K}^{s-r}} + \|\iota_{\mathcal{K}^{s+r} \to \mathcal{K}^{s+r-1}} u\|_{\mathcal{K}^{s+r-1}} \right), \quad u \in \mathcal{K}^{s+r}. \tag{3.1.4}$$

(iii) The shift operator. *For $s \in \mathbb{R}$ and $r \in \mathbb{N}^*$, there exists a continuous isomorphism $\mathcal{S}_s^r : \mathcal{K}^{s+r} \longrightarrow \mathcal{K}^{s-r}$ such that the following property holds: for $r, r' \in \mathbb{N}$, $s \in \mathbb{R}$ and $\delta > 0$,*

$$\mathcal{S}_s^{r+r'} = \mathcal{S}_{s-r'}^r \circ \mathcal{S}_{s+r}^{r'} : \mathcal{K}^{s+r+r'} \to \mathcal{K}^{s-r-r'}, \tag{3.1.5}$$

$$\mathcal{S}_{s-1}^r \circ \mathsf{P}_{s+r} = \mathsf{P}_{s-r} \circ \mathcal{S}_{s+1}^r : \mathcal{K}^{s+r+1} \to \mathcal{K}^{s-r-1},$$

and

$$\mathcal{S}_s^r \circ \iota_{\mathcal{K}^{s+r+\delta} \to \mathcal{K}^{s+r}} = \iota_{\mathcal{K}^{s-r+\delta} \to \mathcal{K}^{s-r}} \circ \mathcal{S}_{s+\delta}^r : \mathcal{K}^{s+r+\delta} \to \mathcal{K}^{s-r}. \tag{3.1.6}$$

In addition, for $r \in \mathbb{N}$ and $s \in \mathbb{R}$, one has

$$\|(\mathsf{P}_s^r - \mathcal{S}_s^r)\, u\|_{\mathcal{K}^{s-r}} \leq C_{r,s} \|\iota_{\mathcal{K}^{s+r} \to \mathcal{K}^{s+r-1}} u\|_{\mathcal{K}^{s+r-1}}, \quad u \in \mathcal{K}^{s+r}, \tag{3.1.7}$$

for some $C_{r,s} > 0$. The operator $\mathcal{S}_s^1$ will be defined by $\mathsf{P}_s + i\mu\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s-1}}$, for $\mu \in \mathbb{R}$ chosen sufficiently large.

Remark 3.1.9. If we omit the embedding notation, then *(ii)* can be written as

$$u \in \mathcal{K}^{s+r-1} \text{ and } \mathsf{P}^r_{s-1} u \in \mathcal{K}^{s-r} \quad \Longrightarrow \quad u \in \mathcal{K}^{s+r}.$$

Note that we cannot replace $\mathsf{P}^r_{s-1} u \in \mathcal{K}^{s-r}$ by $\mathsf{P}^r_{\mathscr{D}'} u \in \mathcal{K}^{s-r}$. With the same example as in Remark 3.1.7, take $N = 1$, $\mathsf{P} = \Delta$, $s = 0$, $r = 1$, and let u be the constant function $u = 1 \in \mathcal{K}^0 = L^2(M)$. One has $\mathsf{P}_{\mathscr{D}'} u = 0$, implying $\mathsf{P}_{\mathscr{D}'} u \in \mathcal{K}^{-1} = H^{-1}(M)$. However, u does not belong to the space $\mathcal{K}^1 = H^1_0(M)$.

Remark 3.1.10. By definition, for $s \in \mathbb{R}$ and $\delta > 0$, one has

$$\iota_{\mathcal{K}^{s+\delta} \to \mathcal{K}^s} = \left(\iota_{\mathcal{K}_*^{-s} \to \mathcal{K}_*^{-s-\delta}} \right)^*. \tag{3.1.8}$$

The proof of Proposition 3.1.8 is given in appendix. The proof of our main result will use the following interpolation lemma.

Lemma 3.1.11. *For $\eta \in [0,1]$, $s \in \mathbb{R}$, one has $\left[\mathcal{K}^{s+2}, \mathcal{K}^s\right]_\eta = \mathcal{K}^{s+2-2\eta}$, with equivalent norms, where $\left[\mathcal{K}^{s+2}, \mathcal{K}^s\right]_\eta$ denotes the complex interpolation space between $\mathcal{K}^{s+2}$ and $\mathcal{K}^s$.*

Proof. First, note that the result is standard for $s \in [-2, 0]$, as $\mathcal{K}^s = D(\Delta_{\text{dir}}^{\frac{s}{2}})^N$ for $s \in [-2, 2]$. Second, we prove Lemma 3.1.11 for $s > 0$, using the shift operator and the definition of complex interpolation spaces (see, for example, [BL76]). If A_0 and A_1 are subspaces of a Banach space $\mathcal{X}$, we write $\mathscr{F}_{A_0, A_1}$ for the set of continuous functions

$$f : \{z \in \mathbb{C}, 0 \leq \operatorname{Re} z \leq 1\} \to A_0 + A_1$$

satisfying the following two properties: f is analytic on the open strip $\{z \in \mathbb{C}, 0 < \operatorname{Re} z < 1\}$, and for $j = 0, 1$, the function $t \mapsto f(j + it)$ maps continuously $\mathbb{R}$ to A_j, and tends to zero as $|t|$ tends to infinity. To ease notation, we omit embeddings and subscripts of the shift operator, identifying $\mathcal{K}^s$ and $\iota_{\mathcal{K}^s \to \mathcal{K}^{-2}}(\mathcal{K}^s)$, for $s \geq -2$, and writing $\mathcal{S}^k = \mathcal{S}^k_{k-2} : \mathcal{K}^{2k-2} \to \mathcal{K}^{-2}$. Consider $s > 0$, $k \in \mathbb{N}$ such that $s - 2k \in [-2, 0]$, and $\eta \in [0, 1]$. By definition, one has

$$\left[\mathcal{K}^{s+2}, \mathcal{K}^s\right]_\eta = \left\{u \in \mathcal{K}^{s+2} + \mathcal{K}^s, u = f(\eta) \text{ for some } f \in \mathscr{F}_{\mathcal{K}^{s+2}, \mathcal{K}^s}\right\}.$$

As $\left\{\mathcal{S}^k \circ f, f \in \mathscr{F}_{\mathcal{K}^{s+2}, \mathcal{K}^s}\right\} = \mathscr{F}_{\mathcal{K}^{s+2-2k}, \mathcal{K}^{s-2k}}$, one has

$$\begin{aligned} u \in \left[\mathcal{K}^{s+2}, \mathcal{K}^s\right]_\eta &\Longleftrightarrow \mathcal{S}^k u = v \text{ for some } v \in \left[\mathcal{K}^{s+2-2k}, \mathcal{K}^{s-2k}\right]_\eta, \\ &\Longleftrightarrow u = \left(\mathcal{S}^k\right)^{-1} v \text{ for some } v \in \mathcal{K}^{s+2-2k-2\eta}, \\ &\Longleftrightarrow u \in \mathcal{K}^{s+2-2\eta}, \end{aligned}$$

by the case $s \in [-2, 0]$, and Proposition 3.1.8. Third, for $s < -2$, using Corollary 4.5.2 and Theorem 4.2.1 of [BL76], one obtains $\left(\left[\mathcal{K}^{s+2}, \mathcal{K}^s\right]_\eta\right)' = \left[\mathcal{K}_*^{-s}, \mathcal{K}_*^{-s-2}\right]_{1-\eta} = \mathcal{K}_*^{-s-2+2\eta}$, as P and P^* are of the same form. This completes the proof. ☐

3.1.3 Solutions of the wave equations

Most of the ideas used here can be found in [LLT86]. For wave equations with Dirichlet boundary condition, one has the following theorem.

Theorem 3.1.12. *Consider $s \in \mathbb{R}$ and $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. There exists a unique*

$$u \in \mathscr{C}^0(\mathbb{R}, \mathcal{K}^{s+1}) \cap \mathscr{C}^1(\mathbb{R}, \mathcal{K}^s) \cap \mathscr{C}^2(\mathbb{R}, \mathcal{K}^{s-1})$$

such that $(u(0), \partial_t u(0)) = (u^0, u^1)$ and $\partial_t^2 u(t) = \mathsf{P}_s u(t)$ for all $t \in \mathbb{R}$. We will say that u is the solution of the wave equation

$$\begin{cases} \partial_t^2 u - \mathsf{P} u &= 0 & \text{in } \mathbb{R} \times M, \\ (u(0,\cdot), \partial_t u(0,\cdot)) &= (u^0, u^1) & \text{in } M, \\ u &= 0 & \text{on } \mathbb{R} \times \partial M. \end{cases}$$

The following additional results hold.

(i) One has

$$u \in \bigcap_{k \in \mathbb{N}} \mathscr{C}^k(\mathbb{R}, \mathcal{K}^{s+1-k}),$$

and $\partial_t^{2k} u(t) = \mathsf{P}_{s+1-k}^k u(t) \in \mathcal{K}^{s+1-2k}$ for $k \in \mathbb{N}$, and $t \in \mathbb{R}$. For all $k \in \mathbb{N}$ and $T > 0$, there exists $C > 0$ such that

$$\left\| \partial_t^k u \right\|_{L^\infty((0,T), \mathcal{K}^{s+1-k})} \leq C \left\| \left(u^0, u^1\right) \right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s}, \quad \left(u^0, u^1\right) \in \mathcal{K}^{s+1} \times \mathcal{K}^s.$$

In particular, if $s \geq -2$, then $u \in H^{s+1}((0,T) \times M, \mathbb{C}^N)$ for all $T > 0$, with the corresponding inequality.

(ii) For $\delta > 0$, if $\tilde{u}$ is the solution with initial data

$$\left(\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s+1-\delta}} u^0, \iota_{\mathcal{K}^s \to \mathcal{K}^{s-\delta}} u^1\right),$$

then for $t \in \mathbb{R}$, one has $\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s+1-\delta}} u(t) = \tilde{u}(t)$. In particular, a solution can be approximated by solutions with higher regularity.

(iii) Consider $T > 0$. A normal derivative $\partial_\nu u$ at the boundary, that lies in $H^s((0,T) \times \partial M, \mathbb{C}^N)$, can be defined extending the usual normal derivative if u is sufficiently smooth. For $\delta > 0$, one has

$$\partial_\nu \left(\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s+1-\delta}} u\right) = \iota_{H^s \to H^{s-\delta}} \partial_\nu u,$$

where $\iota_{H^s \to H^{s-\delta}}$ denotes the embedding from $H^s((0,T) \times \partial M, \mathbb{C}^N)$ into $H^{s-\delta}((0,T) \times \partial M, \mathbb{C}^N)$. There exists $C > 0$ such that

$$\|\partial_\nu u\|_{H^s((0,T) \times \partial M, \mathbb{C}^N)} \leq C \left\| \left(u^0, u^1\right) \right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s}, \quad \left(u^0, u^1\right) \in \mathcal{K}^{s+1} \times \mathcal{K}^s.$$

For $k \in \mathbb{N}$, $\partial_t^{2k} u$ is the solution associated with $\left(\mathsf{P}_{s+1-k}^k u^0, \mathsf{P}_{s-k}^k u^1\right) \in \mathcal{K}^{s+1-2k} \times \mathcal{K}^{s-2k}$, and one has

$$\partial_\nu \partial_t^{2k} u = \partial_t^{2k} \partial_\nu u \in H^{s-2k}((0,T) \times \partial M, \mathbb{C}^N).$$

(iv) Assume that $s \geq 0$. For $F \in L^1((0,T), H_0^s(M, \mathbb{C}^N))$, we define the solution of

$$\begin{cases} \partial_t^2 u - \mathsf{P} u &= F & \text{in } (0,T) \times M, \\ (u(0,\cdot), \partial_t u(0,\cdot)) &= 0 & \text{in } M, \\ u &= 0 & \text{on } (0,T) \times \partial M, \end{cases}$$

using the Duhamel formula. One has $u \in \mathscr{C}^0((0,T),\mathcal{K}^{s+1}) \cap \mathscr{C}^1((0,T),\mathcal{K}^s)$, $\partial_\nu u \in H^s((0,T)\times\partial M,\mathbb{C}^N)$, *and there exists* $C>0$ *such that*

$$\|(u,\partial_t u)\|_{L^\infty((0,T),\mathcal{K}^{s+1}\times\mathcal{K}^s)} + \|\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \leq C\,\|F\|_{L^1((0,T),H^s)},$$

for all $F\in L^1((0,T),H_0^s(M,\mathbb{C}^N))$. *If in addition* $F\in\mathscr{C}^0((0,T),H^{s-1}(M,\mathbb{C}^N))$, *then* $u\in\mathscr{C}^2((0,T),\mathcal{K}^{s-1})$, *with* $\partial_t^2 u = \mathsf{P}_s u + F$ *and*

$$\left\|\partial_t^2 u\right\|_{L^\infty((0,T),\mathcal{K}^{s-1})} \leq C\left(\|F\|_{L^1((0,T),H_0^s(M,\mathbb{C}^N))} + \|F\|_{L^\infty((0,T),H^{s-1}(M,\mathbb{C}^N))}\right),$$

for some $C>0$ *independent of* F.

Using Theorem 3.1.12-*(i)*, Theorem 3.1.12-*(ii)* and Proposition 3.1.8-*(iii)*, one obtains the following corollary.

Corollary 3.1.13. *Consider* $s\in\mathbb{R}$, $r\in\mathbb{N}$, $(u^0,u^1)\in\mathcal{K}^{s+1}\times\mathcal{K}^s$, *and denote by* u *the solution with initial data* (u^0,u^1), *given by Theorem 3.1.12. Then,* $w=\mathcal{S}^r_{s-r+1}u$ *is the solution of*

$$\begin{cases} \partial_t^2 w - \mathsf{P}w = 0 & \text{in } (0,T)\times M,\\ (w(0,\cdot),\partial_t w(0,\cdot)) = (w^0,w^1) & \text{in } M,\\ w = 0 & \text{on } (0,T)\times\partial M,\end{cases}$$

where $(w^0,w^1) = (\mathcal{S}^r_{s-r+1}u^0, \mathcal{S}^r_{s-r}u^1)\in\mathcal{K}^{s-2r+1}\times\mathcal{K}^{s-2r}$.

For wave equations with inhomogeneous boundary condition, one has the following theorem.

Theorem 3.1.14. *Consider* $T>0$, $\Theta\in\mathscr{C}_c^\infty((0,T)\times\partial M,\mathbb{C}^N)$, $s\in\mathbb{R}$ *and* $f\in H^s((0,T)\times\partial M,\mathbb{C}^N)$. *If* $s\leq 0$, *we define the solution of the wave equation*

$$\begin{cases} \partial_t^2 v - \mathsf{P}^* v = 0 & \text{in } \mathbb{R}\times M,\\ (v(0,\cdot),\partial_t v(0,\cdot)) = 0 & \text{in } M,\\ v = \operatorname{diag}(\Theta)f & \text{on } \mathbb{R}\times\partial M.\end{cases} \tag{3.1.9}$$

by duality with Theorem 3.1.12-(iv): v *is the unique element of* $L^\infty((0,T),H^s(M,\mathbb{C}^N))$ *such that*

$$\langle v,F\rangle_{L^\infty(H^s),L^1(H_0^{-s})} = -\left\langle f,\operatorname{diag}(\Theta)\partial_\nu u\right\rangle_{H^s,H_0^{-s}},$$

for all $F\in L^1((0,T),H_0^{-s}(M,\mathbb{C}^N))$, *where* u *is the solution associated with* F *defined in Theorem 3.1.12-(iv). If* $s>0$, *we define the solution of the previous wave equation as in the case* $s=0$. *In any case, one has*

$$v\in\mathscr{C}^0((0,T),H^s(M,\mathbb{C}^N))\cap\mathscr{C}^1((0,T),H^{s-1}(M,\mathbb{C}^N))\cap\mathscr{C}^2((0,T),H^{s-2}(M,\mathbb{C}^N)),$$

$\partial_t^2 v = \mathsf{P}^*_{\mathscr{D}'}v$ *in* $\mathscr{D}'((0,T)\times M,\mathbb{C}^N)$, *and there exists* $C>0$ *such that*

$$\sum_{j=0}^{2}\left\|\partial_t^j v\right\|_{L^\infty([0,T],H^{s-j})} \leq C\,\|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)},\quad f\in H^s((0,T)\times\partial M,\mathbb{C}^N).$$

If $s\geq 1$, *then* $v(t)_{|\partial M} = (\operatorname{diag}(\Theta)f)_{|\{t\}\times\partial M}$ *in* $H^{s-\frac{1}{2}}(\partial M,\mathbb{C}^N)$, *in the sense of classical Sobolev trace operators. In addition, as* Θ *is compactly supported in* $(0,T)\times\partial M$, *one has*

$(v(T), \partial_t v(T)) \in \mathcal{K}_*^s \times \mathcal{K}_*^{s-1}$, *with the following duality equality: for* $(u^0, u^1) \in \mathcal{K}^{-s+1} \times \mathcal{K}^{-s}$, *if* u *is the solution of*

$$\left\{\begin{array}{rcll} \partial_t^2 u - \mathsf{P}u & = & 0 & \text{in } (0,T) \times M, \\ (u(T,\cdot), \partial_t u(T,\cdot)) & = & (u^0, u^1) & \text{in } M, \\ u & = & 0 & \text{on } (0,T) \times \partial M, \end{array}\right.$$

then

$$\left\langle u^1, v(T) \right\rangle_{\mathcal{K}^{-s+1}, \mathcal{K}_*^{s-1}} - \left\langle u^0, \partial_t v(T) \right\rangle_{\mathcal{K}^{-s}, \mathcal{K}_*^{s}} = \left\{\begin{array}{ll} \langle \partial_\nu u, \operatorname{diag}(\Theta) f \rangle_{H^{-s}, H_0^s} & \text{if } s \geq 0 \\ \langle f, \operatorname{diag}(\Theta) \partial_\nu u \rangle_{H^s, H_0^{-s}} & \text{if } s < 0 \end{array}\right. . \quad (3.1.10)$$

The proof of Theorems 3.1.12 and 3.1.14 is given in appendix.

3.1.4 The duality argument

Here, we prove Lemma 3.0.3. The proof is based on the following classical result. In our applications, $\mathcal{X}$ and $\mathcal{Y}$ will be Hilbert spaces.

Theorem 3.1.15. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be Banach spaces, and* $K : \mathcal{X} \to \mathcal{Y}$ *be a linear continuous operator. Assume that* $\mathcal{X}$ *is reflexive. Then* K *is surjective if and only if there exists* $C > 0$ *such that*

$$\|\ell\|_{\mathcal{Y}'} \leq C \|K^* \ell\|_{\mathcal{X}'}, \quad \ell \in \mathcal{Y}'. \quad (3.1.11)$$

Proof. Assume that (3.1.11) holds, and consider $y \in \mathcal{Y}$. For $x \in \mathcal{X}$, one has $y = Kx$ if and only if

$$\ell y = (K^* \ell)\, x, \quad \ell \in \mathcal{Y}', \quad (3.1.12)$$

by the following lemma, which is a classic consequence of the Hahn-Banach theorem.

Lemma 3.1.16. *Let* $\mathcal{X}$ *be a normed vector space. For* $x \in \mathcal{X}$, $x = 0$ *if and only if* $\ell x = 0$ *for all* $\ell \in \mathcal{X}'$.

By (3.1.11), K^* is one-to-one, implying that the operator

$$\begin{array}{rccc} \psi : & \operatorname{Im} K^* & \longrightarrow & \mathcal{Y}' \\ & K^* \ell & \longmapsto & \ell \end{array}$$

is well-defined. Note that (3.1.11) also implies that ψ is continuous. In addition, (3.1.12) is equivalent with

$$\left(\psi \tilde{\ell}\right) y = \tilde{\ell} x, \quad \tilde{\ell} \in \operatorname{Im} K^*.$$

For $\tilde{\ell} \in \operatorname{Im} K^*$, set $\Psi \tilde{\ell} = \left(\psi \tilde{\ell}\right) y$. Using the Hahn-Banach theorem, one can extend Ψ into a continuous linear form (still denoted Ψ) on $\mathcal{X}'$. As $\mathcal{X}$ is reflexive, there exists $x \in \mathcal{X}$ such that

$$\Psi \tilde{\ell} = \tilde{\ell} x, \quad \tilde{\ell} \in \mathcal{X}',$$

yielding $y = Kx$. Hence, K is surjective.

Conversely, assume that K is surjective. Then K^* is one-to-one, implying that the operator

$$\begin{array}{rccc} A : & \mathcal{Y}' & \longrightarrow & \operatorname{Im} K^* \\ & \tilde{\ell} & \longmapsto & K^* \tilde{\ell} \end{array}$$

is an isomorphism. Note that A is continuous. We claim that $\operatorname{Im} K^*$ is closed. Together with the Banach isomorphism theorem, it implies that the inverse of A is continuous, yielding (3.1.11).

To prove that $\operatorname{Im} K^*$ is closed, consider a sequence $(\ell_p)_{p\in\mathbb{N}}$ of elements of $\mathcal{Y}'$ such that $(K^*\ell_p)_{p\in\mathbb{N}}$ converges in $\mathcal{X}'$ to a limit $\tilde{\ell} \in \mathcal{X}'$. Consider $y \in \mathcal{Y}$. As K is surjective, there exists $x \in \mathcal{X}$ such that $y = Kx$. One has

$$\ell_p y = (K^*\ell_p)\, x \xrightarrow{p\to\infty} \tilde{\ell}x.$$

implying that the sequence $(\ell_p)_{p\in\mathbb{N}}$ converges pointwise. Write ℓ for its limit. Then ℓ is linear, and by the Banach-Steinhaus theorem, ℓ is continuous. In addition, one has

$$\ell\,(Kx) = \lim_{p\to\infty} (K^*\ell_p)\, x = \tilde{\ell}x, \quad x \in \mathcal{X},$$

yielding $K^*\ell = \tilde{\ell}$. This completes the proof. □

For $s \in \mathbb{R}$, if one denotes by $H^s_0((0,T)\times\partial M, \mathbb{C}^N)$ the closure of $\mathscr{C}^\infty_{\mathrm{c}}((0,T)\times\partial M, \mathbb{C}^N)$ in $H^s((0,T)\times\partial M, \mathbb{C}^N)$, then $H^s_0((0,T)\times\partial M, \mathbb{C}^N)$ is the dual of $H^{-s}_0((0,T)\times\partial M, \mathbb{C}^N)$ for all $s \in \mathbb{R}$. Consider $s \in \mathbb{R}$, $T > 0$ and $\Theta \in \mathscr{C}^\infty_{\mathrm{c}}((0,T)\times\partial M, \mathbb{C}^N)$. By Theorem 3.1.14, one can define

$$\begin{array}{rccc} K: & H^s_0((0,T)\times\partial M, \mathbb{C}^N) & \longrightarrow & \mathcal{K}^s \times \mathcal{K}^{s-1} \\ & f & \longmapsto & (v(T), \partial_t v(T)) \end{array}$$

where v is the solution of

$$\left\{\begin{array}{rcll} \partial_t^2 v - \mathsf{P}^* v & = & 0 & \text{in } (0,T)\times M, \\ (v(0,\cdot), \partial_t v(0,\cdot)) & = & 0 & \text{in } M, \\ v & = & \operatorname{diag}(\Theta) f & \text{on } (0,T)\times\partial M. \end{array}\right.$$

Note that in the definition of H^s-exact controllability, one can consider $f \in H^s_0((0,T)\times\partial M, \mathbb{C}^N)$ instead of $H^s(0,T)\times\partial M, \mathbb{C}^N)$. Hence, H^s-exact controllability for Θ holds if and only if the operator K is surjective. By Theorem 3.1.14, the adjoint of K is given by

$$\begin{array}{rccc} K^*: & \mathcal{K}^{-s+1} \times \mathcal{K}^{-s} & \longrightarrow & H^{-s}_0((0,T)\times\partial M, \mathbb{C}^N) \\ & (u^1, u^0) & \longmapsto & \operatorname{diag}(\Theta)\partial_\nu u \end{array}$$

where u is the solution

$$\left\{\begin{array}{rcll} \partial_t^2 u - \mathsf{P} u & = & 0 & \text{in } (0,T)\times M, \\ (u(T,\cdot), \partial_t u(T,\cdot)) & = & (-u^0, u^1) & \text{in } M, \\ u & = & 0 & \text{on } (0,T)\times\partial M. \end{array}\right. \tag{3.1.13}$$

By Theorem 3.1.15, H^s-exact controllability for Θ is equivalent with the inequality

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{-s+1}\times\mathcal{K}^{-s}} \lesssim \|\operatorname{diag}(\Theta)\partial_\nu u\|_{H^{-s}((0,T)\times\partial M, \mathbb{C}^N)}, \quad \left(u^0, u^1\right) \in \mathcal{K}^{-s+1}\times\mathcal{K}^{-s}, \tag{3.1.14}$$

where u is the solution of (3.1.13). One has

$$\|(u(0), \partial_t u(0))\|_{\mathcal{K}^{-s+1}\times\mathcal{K}^{-s}} \lesssim \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{-s+1}\times\mathcal{K}^{-s}} \lesssim \|(u(0), \partial_t u(0))\|_{\mathcal{K}^{-s+1}\times\mathcal{K}^{-s}},$$

for $(u^0, u^1) \in \mathcal{K}^{-s+1} \times \mathcal{K}^{-s}$, where u is the solution of (3.1.13), implying that (3.1.14) and H^{-s}-observability are equivalent. This completes the proof of Lemma 3.0.3.

3.2 Ellipticity of the time-derivative on Neumann traces

3.2.1 Statement of the main estimate and beginning of the proof

The main result of this section is the following theorem.

Theorem 3.2.1 (Ellipticity of the time-derivative on the Neumann trace)**.** *For $s > -1$, $\Theta \in \mathscr{C}_{\mathrm{c}}^{\infty}((0,T) \times \partial M, \mathbb{C}^N)$, and $r \in \mathbb{N}^*$, there exists $C > 0$ such that for all $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$, one has*

$$\begin{aligned} &\|\mathrm{diag}(\Theta)\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \\ &\leq C\left(\left\|\mathrm{diag}(\Theta)\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} + \left\|u^0\right\|_{\mathcal{K}^{s+\frac{1}{2}}} + \left\|u^1\right\|_{\mathcal{K}^{s-\frac{1}{2}}}\right) \end{aligned}$$

where u is the solution of

$$\begin{cases} \partial_t^2 u - \mathsf{P}u = 0 & \text{in } \mathbb{R}\times M, \\ (u(0,\cdot),\partial_t u(0,\cdot)) = (u^0,u^1) & \text{in } M, \\ u = 0 & \text{on } \mathbb{R}\times\partial M. \end{cases}$$

Remark 3.2.2. For clarity, embeddings have been omitted in the statement of Theorem 3.2.1. The notation $\|u^0\|_{\mathcal{K}^{s+\frac{1}{2}}} + \|u^1\|_{\mathcal{K}^{s-\frac{1}{2}}}$ stands for

$$\left\|\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{s+\frac{1}{2}}} u^0\right\|_{\mathcal{K}^{s+\frac{1}{2}}} + \left\|\iota_{\mathcal{K}^{s}\to\mathcal{K}^{s-\frac{1}{2}}} u^1\right\|_{\mathcal{K}^{s-\frac{1}{2}}}.$$

Proof. Let $(O^j)_{j\in J}$ be a finite family of open subsets of M satisfying the following properties:

(i) One has
$$\bigcup_{j\in J}\left(O^j\cap\partial M\right) = \partial M.$$

(ii) For each $j \in J$, there exists a smooth diffeomorphism κ^j such that
$$\kappa^j : \tilde{O}^j \longrightarrow O^j \cap \partial M$$
where $\tilde{O}^j$ is a non-empty subset of $\mathbb{R}^{n-1}$.

(iii) We can use boundary normal coordinates on each O^j: more precisely, we assume that there exists $\delta > 0$ such that for all $j \in J$, the map
$$\begin{aligned} \tilde{\kappa}^j :\ \tilde{O}^j\times[0,\delta) &\longrightarrow O^j \\ (x', x^n) &\longmapsto \gamma(\nu_{\kappa^j(x')}, x^n) \end{aligned}$$
is a smooth diffeomorphism, where for $y \in \partial M$, ν_y is the inward-pointing unit vector normal to the boundary at y, and $\gamma(\nu_{\kappa^j(x')},\cdot)$ is the geodesic starting from $\kappa^j(x')$ and of initial velocity $\nu_{\kappa^j(x')}$.

It is well-know that in the coordinates given by $\tilde{\kappa}^j$, the Laplace-Beltrami operator becomes an elliptic operator $\tilde{P}^j$ on $\mathbb{R}^n_+$ with principal part

$$\tilde{P}^j = \partial_{x^n}^2 + \sum_{1\leq p,q\leq n-1} \alpha_j^{pq}(x)\partial_{x^p}\partial_{x^q}. \tag{3.2.1}$$

The coefficients (α_j^{pq}) can be smoothly extended to $\mathbb{R}^n$ in such a way that $\tilde{P}^j$ is an elliptic operator on $\mathbb{R}^n$.

We take a partition of the unity associated to the sets $(O^j)_{j\in J}$: there exists a family of functions $(\psi^j)_{j\in J}$ such that for each $j\in J$, $\psi^j\in\mathscr{C}_c^\infty(O^j,[0,1])$ and such that

$$\sum_{j\in J}(\psi^j)^2=1$$

in a neighborhood of ∂M in M. Also, take $\psi^0\in\mathscr{C}_c^\infty((0,T),[0,1])$ such that $\psi^0\Theta=\Theta$.

Consider $(u^0,u^1)\in\mathcal{K}^{s+1}\times\mathcal{K}^s$. We start the proof by writing

$$\begin{aligned}\|\mathrm{diag}(\Theta)\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}&=\|\mathrm{diag}(\Theta)\partial_\nu u\|_{H^s(\mathbb{R}\times\partial M,\mathbb{C}^N)}\\&=\left\|\psi^0\,\mathrm{diag}(\Theta)\partial_\nu u\right\|_{H^s(\mathbb{R}\times\partial M,\mathbb{C}^N)}\\&\leq\sum_{j\in J}\left\|(\psi^j)^2\psi^0\,\mathrm{diag}(\Theta)\partial_\nu u\right\|_{H^s(\mathbb{R}\times(O^j\cap\partial M),\mathbb{C}^N)}.\end{aligned}\tag{3.2.2}$$

For $j\in J$, $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$, we define

$$u^j(t,x)=\psi^0(t)\psi^j(\tilde\kappa^j(x))u(t,\tilde\kappa^j(x))\quad\text{ and }\quad\Theta^j(t,x')=\psi^j(\kappa^j(x'))\Theta(t,\kappa^j(x')).$$

Note that those functions are well-defined because ψ^j is compactly supported in O^j. As $u_{|\partial M}=0$, one has

$$\partial_\nu u^j(t,x',0)=\psi^0(t)\psi^j\left(\kappa^j(x')\right)\partial_\nu u\left(t,\kappa^j(x')\right).\tag{3.2.3}$$

By definition of the H^s-norm on a Riemannian manifold, coming back to (3.2.2), we thus have

$$\begin{aligned}\|\mathrm{diag}(\Theta)\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}&\lesssim\sum_{j\in J}\left\|\mathrm{diag}(\Theta^j)\partial_\nu u^j\right\|_{H^s(\mathbb{R}\times\tilde O^j,\mathbb{C}^N)}\\&=\sum_{j\in J}\left\|\mathrm{diag}(\Theta^j)\partial_\nu u^j\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)}.\end{aligned}$$

Recall that $(\pi_1,\cdots,\pi_N)$ denotes the projections associated with the canonical basis of $\mathbb{C}^N$. By definition of the $H^s(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)$-norm, one has

$$\|\mathrm{diag}(\Theta)\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\lesssim\sum_{k=1}^N\sum_{j\in J}\left\|\left(\pi_k\Theta^j\right)\partial_\nu\left(\pi_k u^j\right)\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}.\tag{3.2.4}$$

We see that we are reduced to the study of scalar functions defined on the half-space $\mathbb{R}\times\mathbb{R}^n_+$. We gather the properties satisfied by the functions $\pi_k u^j$. First, as $s\geq-2$, one has $\pi_k u^j\in H^{s+1}(\mathbb{R}\times\mathbb{R}^n_+)$ by Theorem 3.1.12. Second, one has $\pi_k u^j(t,x',0)=0$ for all $(t,x')\in\mathbb{R}\times\mathbb{R}^{n-1}$ by the Dirichlet boundary condition, and

$$\partial_\nu\left(\pi_k u^j\right)\in H^s(\mathbb{R}\times\mathbb{R}^{n-1})$$

by Theorem 3.1.12 and (3.2.3). Third, we know that

$$\partial_t^2u-\Delta u=-Xu-qu$$

where Δ is the Laplace-Beltrami operator, so by the Leibniz formula, there exists a differential operator R^j of order 1, supported in $(0,T)\times O^j$ such that

$$\left(\partial_t^2-\tilde P^j\right)u^j(t,x)=R^ju(t,\tilde\kappa^j(x))$$

where $\tilde P^j$ is defined by (3.2.1). In particular, one has

$$\left(\partial_t^2-\tilde P^j\right)(\pi_k u^j)\in H^s(\mathbb{R}\times\mathbb{R}^n_+).$$

Proposition 3.2.3. *Suppose*

$$P = \partial_t^2 - \partial_{x^n}^2 - \sum_{1\le i,j\le n-1} \alpha^{ij}(x)\partial_{x^i}\partial_{x^j}$$

on $\mathbb{R}_t \times \mathbb{R}^n_x$, where the coefficients (α^{ij}) are such that

$$\xi_n^2 + \sum_{1\le i,j\le n-1} \alpha^{ij}(x)\xi_i\xi_j$$

is uniformly elliptic on $\mathbb{R}^n$. Take $\theta \in \mathscr{C}^\infty_c(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C})$. There exists $C>0$ such that

$$\begin{aligned}\|\theta\partial_\nu u\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} \le C\bigg(&\left\|\theta\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1})} + \|Pu\|_{H^{s-\frac12}(\mathbb{R}\times\mathbb{R}^n_+)}\\ &+ \|\partial_\nu u\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})} + \|u\|_{H^{s+\frac12}(\mathbb{R}\times\mathbb{R}^n_+)}\bigg).\end{aligned}$$

for all $u\in H^{s+1}(\mathbb{R}\times\mathbb{R}^n_+)$ such that $Pu\in H^s(\mathbb{R}\times\mathbb{R}^n_+)$, $u_{|x_n=0}=0$ and $\partial_\nu u\in H^s(\mathbb{R}\times\mathbb{R}^{n-1})$.

A proof is given in section 3.2. This proposition allows us to complete the proof of Theorem 3.2.1. One obtains

$$\begin{aligned}\left\|\left(\pi_k\Theta^j\right)\partial_\nu\pi_k u^j\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} &\lesssim \left\|\left(\pi_k\Theta^j\right)\partial_t^{2r}\partial_\nu\pi_k u^j\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1})}\\ &+\left\|\left(\partial_t^2-\tilde P^j\right)(\pi_k u^j)\right\|_{H^{s-\frac12}(\mathbb{R}\times\mathbb{R}^n_+)} + \left\|\partial_\nu\pi_k u^j\right\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})} + \left\|\pi_k u^j\right\|_{H^{s+\frac12}(\mathbb{R}\times\mathbb{R}^n_+)}.\end{aligned}$$

Using (3.2.4) and the definition of the H^s-norms of vectors, one has

$$\begin{aligned}\|\mathrm{diag}(\Theta)\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} &\lesssim \sum_{j\in J}\left\|\mathrm{diag}(\Theta^j)\partial_t^{2r}\partial_\nu u^j\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)}\\ +\sum_{j\in J}&\left(\left\|\left(\partial_t^2-\tilde P^j\right)u^j\right\|_{H^{s-\frac12}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{C}^N)} + \left\|\partial_\nu u^j\right\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1}),\mathbb{C}^N)} + \left\|u^j\right\|_{H^{s+\frac12}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{C}^N)}\right).\end{aligned}$$

We estimate the terms on the right-hand side one by one.

First term. We prove

$$\begin{aligned}&\sum_{j\in J}\left\|\mathrm{diag}(\Theta^j)\partial_t^{2r}\partial_\nu u^j\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)}\\ &\lesssim \left\|\mathrm{diag}(\Theta)\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} + \|\partial_\nu u\|_{H^{s-1}((0,T)\times\partial M,\mathbb{C}^N)},\end{aligned}\tag{3.2.5}$$

meaning that the first term yields the main term of the estimate up to a remainder term.

Using (3.2.3) and the Leibniz formula, one finds

$$\begin{aligned}&\left\|\mathrm{diag}(\Theta^j)\partial_t^{2r}\partial_\nu u^j\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)}\\ &\lesssim\sum_{k=0}^{2r}\left\|\mathrm{diag}(\Theta^j)\left(\psi^j\circ\kappa^j\right)\partial_t^k\psi^0\partial_t^{2r-k}\partial_\nu u(t,\kappa^j(x'))\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)}.\end{aligned}$$

As ψ^j is supported in a coordinate chart of ∂M, one has

$$\left\|\mathrm{diag}(\Theta^j)\partial_t^{2r}\partial_\nu u^j\right\|_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)} \lesssim \sum_{k=0}^{2r}\left\|\mathrm{diag}(\Theta)\left(\psi^j\right)^2\partial_t^k\psi^0\partial_t^{2r-k}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)}.$$

One has

$$\left\|\operatorname{diag}(\Theta)\left(\psi^j\right)^2\psi^0\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\operatorname{diag}(\Theta)\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)},$$

and for $k \in [\![1,2r]\!]$,

$$\left\|\operatorname{diag}(\Theta)\left(\psi^j\right)^2\partial_t^k\psi^0\partial_t^{2r-k}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \|\partial_\nu u\|_{H^{s-1}((0,T)\times\partial M,\mathbb{C}^N)}.$$

This gives (3.2.5).

Second term. For the second term, one has

$$\sum_{j\in J}\left\|\left(\partial_t^2-\tilde{P}^j\right)u^j\right\|_{H^{s-\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{C}^N)} \lesssim \|u\|_{H^{s+\frac{1}{2}}((0,T)\times M,\mathbb{C}^N)}.$$

This holds since for all j, there exists a differential operator R^j of order 1, supported in $(0,T)\times O^j$ such that

$$\left(\partial_t^2-\tilde{P}^j\right)u^j(t,x) = R^j u(t,\tilde{\kappa}^j(x)).$$

Third and forth term. Arguing as above, one finds

$$\begin{aligned}&\sum_{j\in J}\left(\left\|\partial_\nu u^j\right\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1},\mathbb{C}^N)}+\left\|u^j\right\|_{H^{s+\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{C}^N)}\right)\\ &\lesssim \|\partial_\nu u\|_{H^{s-1}((0,T)\times\partial M,\mathbb{C}^N)}+\|u\|_{H^{s+\frac{1}{2}}((0,T)\times M,\mathbb{C}^N)}.\end{aligned}$$

Conclusion. Gathering all our estimates, one finds

$$\begin{aligned}\|\operatorname{diag}(\Theta)\partial_\nu u\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim{}& \left\|\operatorname{diag}(\Theta)\partial_t^{2r}\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)}\\ &+\|\partial_\nu u\|_{H^{s-1}((0,T)\times\partial M,\mathbb{C}^N)}+\|u\|_{H^{s+\frac{1}{2}}((0,T)\times M,\mathbb{C}^N)}.\end{aligned}$$

By Theorem 3.1.12, one has

$$\|\partial_\nu u\|_{H^{s-1}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \|u^0\|_{\mathcal{K}^s}+\|u^1\|_{\mathcal{K}^{s-1}}$$

and

$$\|u\|_{H^{\left(s-\frac{1}{2}\right)+1}((0,T)\times M,\mathbb{C}^N)} \lesssim \|u^0\|_{\mathcal{K}^{s+\frac{1}{2}}}+\|u^1\|_{\mathcal{K}^{s-\frac{1}{2}}},$$

as $s-\frac{1}{2}\geq -2$. This completes the proof. □

3.2.2 Analysis in a half-space.

Here, we prove Proposition 3.2.3. Write $\mathsf{S}^m(\mathbb{R}_t\times\mathbb{R}^n_x)$ for the set of symbols of order m, $\mathsf{S}^m_{\mathsf{T}}(\mathbb{R}_t\times\mathbb{R}^n_x)$ for the set of tangential symbols of order m, and $\Psi^m(\mathbb{R}_t\times\mathbb{R}^n_x)$ and $\Psi^m_{\mathsf{T}}(\mathbb{R}_t\times\mathbb{R}^n_x)$ for the associated sets of pseudo-differential operators. Let p be the principal symbol of P, that is

$$p(x,\tau,\xi) = -\tau^2+\xi_n^2+\sum_{1\leq i,j\leq n-1}\alpha^{ij}(x)\xi_i\xi_j.$$

Write

$$|\xi'|_x^2 = \sum_{1\leq i,j\leq n-1}\alpha^{ij}(x)\xi_i\xi_j \quad\text{and}\quad \rho(x,\tau,\xi') = -\tau^2+|\xi'|_x^2$$

so that $p(x,\tau,\xi)=\xi_n^2+\rho(x,\tau,\xi')$. On the boundary, we sometimes use the notation $|\xi'|_{x'}=|\xi'|_{(x',0)}$.

Consider $u\in H^{s+1}(\mathbb{R}\times\mathbb{R}^n_+)$ satisfying the assumptions of Proposition 3.2.3. The idea of the proof is to split miscellaneously $\theta\partial_\nu u$ into two terms: one on which ∂_t is elliptic, and one on which the wave operator is elliptic. More precisely, let $\chi_0\in\mathscr{C}^\infty(\mathbb{R}_+,[0,1])$ be such that $\chi_0=1$ on $\left[\frac{1}{4},+\infty\right)$ and $\chi_0=0$ on $\left[0,\frac{1}{5}\right]$. We define

$$\chi(x,\tau,\xi')=\chi_0\left(\frac{\rho(x,\tau,\xi')}{1+\tau^2+|\xi'|_x^2}\right).$$

Then $\chi\in\mathsf{S}^0_{\mathsf{T}}(\mathbb{R}_t\times\mathbb{R}^n_x)$ by Lemma 18.1.10 of [Hör07]. One has $\chi(x,\tau,\xi')=1$ if $1+\tau^2+|\xi'|_x^2\leq 4\rho(x,\tau,\xi')$, and χ is supported in the set

$$\left\{(x,\tau,\xi')\in\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n, 1+\tau^2+|\xi'|_x^2\leq 5\rho(x,\tau,\xi')\right\}.$$

One can write

$$\|\theta\partial_\nu u\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\lesssim\left\|\theta\operatorname{Op}_{\mathsf{T}}(1-\chi_{|x^n=0})(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}+\left\|\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}.$$

We will study the two terms on the right-hand side separately: for the first one, ∂_t turns out to be elliptic, and for the second one, P is elliptic.

Remark 3.2.4. We use the notation $\operatorname{Op}_{\mathsf{T}}$ both for tangential pseudo-differential operators on $\mathbb{R}_t\times\mathbb{R}^n_x$ and for pseudo-differential operators on the boundary $\mathbb{R}_t\times\mathbb{R}^{n-1}_{x'}$. They coincide at $x_n=0$ up to a $\frac{1}{2\pi}$ factor.

Ellipticity of the time-derivative

We prove the following estimate.

Lemma 3.2.5. *There exists $C>0$ such that*

$$\left\|\theta\operatorname{Op}_{\mathsf{T}}(1-\chi_{|x^n=0})(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\leq C\left(\left\|\theta\partial_t^{2r}\partial_\nu u\right\|^2_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1})}+\|\partial_\nu u\|^2_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})}\right).$$

Proof. Let χ_1 be a smooth compactly supported function such that $\chi_1(x',\tau,\xi')=1$ if $\tau^2+|\xi'|^2_{x'}\leq 1$. Write

$$\begin{aligned}\left\|\theta\operatorname{Op}_{\mathsf{T}}(1-\chi_{|x^n=0})(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\leq&\left\|\theta\operatorname{Op}_{\mathsf{T}}\left(\chi_1(1-\chi_{|x^n=0})\right)(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\\&+\left\|\theta\operatorname{Op}_{\mathsf{T}}\left((1-\chi_1)(1-\chi_{|x^n=0})\right)(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}.\end{aligned}$$

As $\chi_1(1-\chi_{|x^n=0})\in\mathsf{S}^{-\infty}(\mathbb{R}_t\times\mathbb{R}^{n-1}_{x'})$, one has

$$\left\|\theta\operatorname{Op}_{\mathsf{T}}\left(\chi_1(1-\chi_{|x^n=0})\right)(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\lesssim\|\partial_\nu u\|^2_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})}.$$

Thus, to complete the proof of the lemma, it suffices to show that

$$\begin{aligned}&\left\|\theta\operatorname{Op}_{\mathsf{T}}\left((1-\chi_1)(1-\chi_{|x^n=0})\right)(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\\\lesssim&\left\|\theta\partial_t^{2r}\partial_\nu u\right\|^2_{H^{s-2r}(\mathbb{R}\times\mathbb{R}^{n-1})}+\|\partial_\nu u\|^2_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})}.\end{aligned}\tag{3.2.6}$$

Set $\chi_2 = (1-\chi_1)(1-\chi_{|x^n=0}) \in \mathsf{S}^0(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$. If $(x', \tau, \xi') \in \operatorname{supp} \chi_2$, then $\tau^2 + |\xi'|^2_{x'} > 1$ and

$$1 + \tau^2 + |\xi'|^2_{x'} > 4\left(|\xi'|^2_{x'} - \tau^2\right).$$

Combining those two inequalities, one finds

$$2\left(\tau^2 + |\xi'|^2_{x'}\right) + 1 + \tau^2 + |\xi'|^2_{x'} > 2 + 4\left(|\xi'|^2_{x'} - \tau^2\right)$$

that is

$$7\tau^2 > 1 + |\xi'|^2_{x'}. \tag{3.2.7}$$

In particular, $\tilde{\chi}_2(x', \tau, \xi') = \tau^{-2r}\chi_2(x', \tau, \xi')$ is well-defined. Using (3.2.7), one finds $\tilde{\chi}_2 \in \mathsf{S}^{-2r}(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$. Since $\operatorname{Op}_\mathsf{T}(\chi_2) = \operatorname{Op}_\mathsf{T}(\tilde{\chi}_2)\partial_t^{2r}$, one has

$$\begin{aligned}\left\|\theta \operatorname{Op}_\mathsf{T}(\chi_2)(\partial_\nu u)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} \leq{}& \left\|\operatorname{Op}_\mathsf{T}(\tilde{\chi}_2)\left(\theta\partial_t^{2r}\partial_\nu u\right)\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} \\ &+ \left\|\left[\theta, \operatorname{Op}_\mathsf{T}(\tilde{\chi}_2)\right]\partial_t^{2r}\partial_\nu u\right\|^2_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\end{aligned}$$

and this gives (3.2.6), as $\left[\theta, \operatorname{Op}_\mathsf{T}(\tilde{\chi}_2)\right]\partial_t^{2r} \in \Psi^{-1}(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$. □

Ellipticity of the wave operator

We denote by $\underline{u}$ and $\underline{f}$ the extensions by 0 of u and f on the whole space. In the sense of distributions on $\mathbb{R}^{n+1}$, since $u_{|x_n=0} = 0$, one has

$$P\underline{u} = \underline{f} + \delta_{x^n=0} \otimes \partial_\nu u,$$

and this holds in fact in $\mathscr{S}'(\mathbb{R}^{n+1})$. As $\operatorname{Op}_\mathsf{T}(\chi)$ sends $\mathscr{S}'(\mathbb{R}^{n+1})$ to $\mathscr{S}'(\mathbb{R}^{n+1})$, one has

$$P\operatorname{Op}_\mathsf{T}(\chi)\underline{u} + \left[\operatorname{Op}_\mathsf{T}(\chi), P\right]\underline{u} = \operatorname{Op}_\mathsf{T}(\chi)\underline{f} + \delta_{x^n=0} \otimes \left(\operatorname{Op}_\mathsf{T}(\chi_{|x^n=0})\partial_\nu u\right) \tag{3.2.8}$$

in $\mathscr{S}'(\mathbb{R}^{n+1})$.

To get an estimate on $\operatorname{Op}_\mathsf{T}(\chi_{|x^n=0})\partial_\nu u$, we apply a parametrix of P. Thus, one has to find a non-tangential symbol $\tilde{\chi}$ of order 0 supported where P is elliptic, and such that $\tilde{\chi}(x, \tau, \xi) = 1$ if $(x, \tau, \xi') \in \operatorname{supp}\chi$. If (x, τ, ξ) is such that $(x, \tau, \xi') \in \operatorname{supp}\chi$, then

$$1 + \tau^2 + |\xi'|^2_x \leq 5\left(|\xi'|^2_x - \tau^2\right)$$

and this implies

$$1 + \tau^2 + \xi_n^2 + |\xi'|^2_x \leq 5\left(|\xi'|^2_x + \xi_n^2 - \tau^2\right) = 5p(x, \tau, \xi).$$

Set

$$\tilde{\chi}(x, \tau, \xi) = \eta\left(\frac{p(x, \tau, \xi)}{1 + \xi_n^2 + |\xi'|^2_x + \tau^2}\right)$$

where $\eta \in \mathscr{C}^\infty(\mathbb{R}_+, [0, 1])$ is such that $\eta(\sigma) = 1$ if $\sigma \geq \frac{1}{5}$, and $\eta(\sigma) = 0$ if $\sigma \leq \frac{1}{10}$. Then $\tilde{\chi}$ is supported where P is elliptic, and $\tilde{\chi}(x, \tau, \xi) = 1$ if $(x, \tau, \xi') \in \operatorname{supp}\chi$. The function $\tilde{\chi}$ is a symbol of order 0 by Lemma 18.1.10 of [Hör07].

Set $Q = \operatorname{Op}(q) \in \Psi^{-2}(\mathbb{R}_t \times \mathbb{R}^n_x)$, with

$$q(x, \tau, \xi) = \frac{\tilde{\chi}(x, \tau, \xi)}{p(x, \tau, \xi)}.$$

Pseudo-differential calculus gives $QP = \mathrm{Op}(\tilde{\chi}) + R_1$, with $R_1 \in \Psi^{-1}(\mathbb{R}_t \times \mathbb{R}^n_x)$. Note that one can construct $\tilde{Q} \in \Psi^{-2}(\mathbb{R}_t \times \mathbb{R}^n_x)$ such that

$$\tilde{Q}P - \mathrm{Op}(\tilde{\chi}) \in \Psi^{-\infty}(\mathbb{R}_t \times \mathbb{R}^n_x)$$

as in Theorem 18.1.9 of [Hör07], but such a refinement is not needed here.

Applying Q to Equation (3.2.8), one finds

$$\begin{aligned} &\mathrm{Op}(\tilde{\chi})\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} + R_1\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} + Q\left[\mathrm{Op}_{\mathsf{T}}(\chi), P\right]\underline{u} \\ &= Q\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{f} + Q\left(\delta_{x^n=0} \otimes \left(\mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right)\right). \end{aligned} \tag{3.2.9}$$

Since $u_{|x_n=0} = 0$, one has

$$\mathrm{Op}(\tilde{\chi})\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x_n=0} = \mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x_n=0} + \mathrm{Op}(\tilde{\chi}-1)\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x_n=0} = \mathrm{Op}(\tilde{\chi}-1)\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x_n=0}.$$

Thus, computing the trace of (3.2.9) at $x^n = 0$ gives

$$Q\left(\delta_{x^n=0} \otimes \left(\mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right)\right)_{|x_n=0} = -Q\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{f}_{|x_n=0} + R_3 u_{|x_n=0} \tag{3.2.10}$$

where the rest $R_3 u$ is

$$R_3 u = \mathrm{Op}(\tilde{\chi}-1)\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} + R_1\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} + Q\left[\mathrm{Op}_{\mathsf{T}}(\chi), P\right]\underline{u}.$$

Lemma 3.2.6. *There exists $C > 0$ such that*

$$\left\| Q\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{f}_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} \le C\left(\|f\|_{H^{s-\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^n_+)} \right) \tag{3.2.11}$$

and

$$\left\| R_3 u_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} \le C\left(\|\partial_\nu u\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})} + \|u\|_{H^{s+\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^n_+)} \right). \tag{3.2.12}$$

Proof. As $s > -1$ and as Q is of order -2, one has

$$\left\| Q\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{f}_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim \left\| Q\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{f} \right\|_{H^{s+\frac{3}{2}}(\mathbb{R}\times\mathbb{R}^n_+)} \lesssim \left\| \mathrm{Op}_{\mathsf{T}}(\chi)\underline{f} \right\|_{H^{s-\frac{1}{2}}(\mathbb{R}\times\mathbb{R}^n_+)}.$$

As $\chi \in \Psi^0_{\mathsf{T}}(\mathbb{R}_t \times \mathbb{R}^n_x)$, one obtains (3.2.11). Next, we prove (3.2.12).

Term 1. For the term $\mathrm{Op}(\tilde{\chi}-1)\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}$, we use Theorem 18.1.35 of [Hör07].

Lemma 3.2.7. *The symbol $1 - \tilde{\chi}$ satisfies the assumption of Theorem 18.1.35 of [Hör07]: there exists $\varepsilon > 0$ such that*

$$1 - \tilde{\chi}(x,\tau,\xi) = 0$$

if $\varepsilon|\xi_n| > 1$ and $|(\tau,\xi')| \le \varepsilon|\xi_n|$.

Proof. There exists $C > 0$ such that

$$|\xi'|^2_x \le c|\xi'|^2$$

for all (x,ξ'). Hence, if $\varepsilon|\xi_n| > 1$ and $|(\tau,\xi')| \le \varepsilon|\xi_n|$, then

$$\begin{aligned} \frac{\xi_n^2 + |\xi'|_x^2 - \tau^2}{1 + \xi_n^2 + |\xi'|_x^2 + \tau^2} &\ge \frac{\xi_n^2 - \tau^2}{1 + \xi_n^2 + |\xi'|_x^2 + \tau^2} \ge \frac{\xi_n^2 - \varepsilon^2\xi_n^2}{1 + \xi_n^2 + c\varepsilon^2\xi_n^2 + \varepsilon^2\xi_n^2} \\ &\ge \frac{\xi_n^2 - \varepsilon^2\xi_n^2}{\varepsilon^2\xi_n^2 + \xi_n^2 + c\varepsilon^2\xi_n^2 + \varepsilon^2\xi_n^2} \ge \frac{1-\varepsilon^2}{1 + c\varepsilon^2 + 2\varepsilon^2}. \end{aligned}$$

Thus, if ε is sufficiently small, one has

$$\frac{\xi_n^2 + |\xi'|_x^2 - \tau^2}{1 + \xi_n^2 + |\xi'|_x^2 + \tau^2} \geq \frac{1}{5}$$

implying $\tilde{\chi}(x, \tau, \xi) = 1$. □

Thus, one has $\mathrm{Op}(\tilde{\chi} - 1)\,\mathrm{Op}_{\mathsf{T}}(\chi) \in \Psi^0(\mathbb{R}_t \times \mathbb{R}_x^n)$, with vanishing symbol. Hence, $\mathrm{Op}(\tilde{\chi} - 1)\,\mathrm{Op}_{\mathsf{T}}(\chi) \in \Psi^{-\infty}(\mathbb{R}_t \times \mathbb{R}_x^n)$, yielding

$$\begin{aligned} \left\| \mathrm{Op}(\tilde{\chi} - 1)\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} &\lesssim \left\| \mathrm{Op}(\tilde{\chi} - 1)\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} \right\|_{H^{s+\frac{3}{2}}(\mathbb{R}\times\mathbb{R}_+^n)} \\ &\lesssim \|u\|_{H^{s+\frac{3}{2}-N}(\mathbb{R}\times\mathbb{R}_+^n)}, \end{aligned}$$

for any $N \geq 0$.

Term 2. For the term $R_1\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}$, as $R_1 \in \Psi^{-1}(\mathbb{R}_t \times \mathbb{R}_x^n)$, one has

$$\left\| R_1\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} \lesssim \left\| R_1\,\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} \right\|_{H^{s+\frac{3}{2}}(\mathbb{R}\times\mathbb{R}_+^n)} \lesssim \|u\|_{H^{s+\frac{1}{2}}(\mathbb{R}\times\mathbb{R}_+^n)}.$$

Term 3. Pseudo-differential calculus gives $Q\left[\mathrm{Op}_{\mathsf{T}}(\chi), P\right] \in \Psi^{-1}(\mathbb{R}_t \times \mathbb{R}_x^n)$, yielding

$$\begin{aligned} \left\| Q\left[\mathrm{Op}_{\mathsf{T}}(\chi), P\right]\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u}_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} &\lesssim \left\| Q\left[\mathrm{Op}_{\mathsf{T}}(\chi), P\right]\mathrm{Op}_{\mathsf{T}}(\chi)\underline{u} \right\|_{H^{s+\frac{3}{2}}(\mathbb{R}\times\mathbb{R}_+^n)} \\ &\lesssim \|u\|_{H^{s+\frac{1}{2}}(\mathbb{R}\times\mathbb{R}_+^n)}. \end{aligned}$$

Gathering all those estimates, one finds (3.2.12). This completes the proof of Lemma 3.2.6. □

We now turn to the study of the left-hand side of (3.2.10).

Lemma 3.2.8. *There exists $C > 0$ such that*

$$\begin{aligned} &\left\| \mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u \right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} \\ &\leq C\left(\left\| Q\left(\delta_{x^n=0} \otimes \left(\mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right)\right)_{|x^n=0} \right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})} + \|\partial_\nu u\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})} \right). \end{aligned}$$

Proof. The idea is to find a pseudo-differential expression of

$$\mathrm{Op}(q)\left(\delta_{x^n=0} \otimes \left(\mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right)\right)_{|x^n=0}.$$

By definition, one has

$$\mathrm{Op}(q)\left(\delta_{x^n=0} \otimes \left(\mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right)\right)_{|x_n=0}(t, x') = \mathrm{Op}_{\mathsf{T}}(q_{\mathsf{T}})\,\mathrm{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u(t, x) \tag{3.2.13}$$

where q_{T} is the symbol

$$q_{\mathsf{T}}(x', \tau, \xi') = \int_{\mathbb{R}} \frac{\tilde{\chi}(x', 0, \tau, \xi)}{\xi_n^2 + \rho(x', 0, \tau, \xi')}\mathrm{d}\xi_n.$$

Lemma 3.2.9. *One has $q_{\mathsf{T}} \in \mathsf{S}^{-1}(\mathbb{R}_t \times \mathbb{R}_{x'}^{n-1})$.*

Proof. Note that an explicit formula for q_{T} is not needed. The idea of the proof is to write $q_{\mathsf{T}} = a \times (F \circ b)$ where a is a symbol of order -1, F is a smooth function, and b is a symbol of order 0, so that the conclusion will be a consequence of Lemma 18.1.10 of [Hör07]. Recall that

$$\tilde{\chi}(x,\tau,\xi) = \eta\left(\frac{p(x,\tau,\xi)}{1+\xi_n^2+|\xi'|_x^2+\tau^2}\right)$$

where η is a real nonnegative smooth function such that $\eta(\sigma) = 1$ if $\sigma \geq \frac{1}{5}$, and $\eta(\sigma) = 0$ if $\sigma \leq \frac{1}{10}$. Write $|\xi'|_{x'}$ instead of $|\xi'|_{(x',0)}$, and set

$$b(x',\tau,\xi') = \frac{|\xi'|_{x'}^2 - \tau^2}{1+|\xi'|_{x'}^2+\tau^2}.$$

Then $b \in \mathsf{S}^0(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$ and a change of variable gives

$$q_{\mathsf{T}}(x',\tau,\xi') = \frac{1}{\sqrt{1+|\xi'|_{x'}^2+\tau^2}} \int_{\mathbb{R}} \frac{1}{\sigma^2 + b(x',\tau,\xi')} \eta\left(\frac{\sigma^2 + b(x',\tau,\xi')}{\sigma^2+1}\right) \mathrm{d}\sigma.$$

Set

$$F(\sigma') = \int_{\mathbb{R}} \frac{1}{\sigma^2+\sigma'} \eta\left(\frac{\sigma^2+\sigma'}{\sigma^2+1}\right) \mathrm{d}\sigma.$$

Lemma 3.2.10. *The function F is smooth.*

Proof. Consider $\sigma' \in \mathbb{R}$ and write

$$g(\sigma) = \frac{\sigma^2+\sigma'}{\sigma^2+1}$$

for $\sigma \in \mathbb{R}$. Note that if $\sigma' \geq \frac{1}{5}$, then $g(\sigma) \geq \frac{1}{5}$ for all $\sigma \in \mathbb{R}$, so that

$$F(\sigma') = \int_{\mathbb{R}} \frac{1}{\sigma^2+\sigma'} \mathrm{d}\sigma.$$

Hence, we may assume that $\sigma' < \frac{1}{5}$. In particular, as $\sigma' < 1$, the function g is a bijection from $\mathbb{R}_+$ to $[\sigma', 1]$. A change of variables gives

$$F(\sigma') = \int_{\sigma'}^{1} \frac{\eta(\sigma)}{\sigma\sqrt{1-\sigma}\sqrt{\sigma-\sigma'}} \mathrm{d}\sigma.$$

As $\eta(\sigma) = 0$ for $\sigma \leq \frac{1}{10}$, one finds that if $\sigma' \leq \frac{1}{10}$, then

$$F(\sigma') = \int_{\frac{1}{10}}^{1} \frac{\eta(\sigma)}{\sigma\sqrt{1-\sigma}\sqrt{\sigma-\sigma'}} \mathrm{d}\sigma,$$

implying that F is smooth on $\left(-\infty, \frac{1}{10}\right]$. Finally, note that

$$F(\sigma') = \int_0^1 \frac{\eta\left(\sigma' + (1-\sigma')\sigma\right)}{(\sigma' + (1-\sigma')\sigma)\sqrt{1-\sigma}\sqrt{\sigma}} \mathrm{d}\sigma,$$

by a last change of variable. As $\sigma' + (1-\sigma')\sigma > \frac{1}{10}$ for $\sigma \in (0,1)$, this proves that F is smooth on $[\frac{1}{10}, \frac{1}{5}]$. □

Lemma 18.1.10 of [Hör07] gives $q_{\mathsf{T}} \in \mathsf{S}^{-1}(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$, completing the proof of Lemma 3.2.9. □

With the same construction as for χ, consider $\chi_3 \in \mathsf{S}^0(\mathbb{R}_t\times\mathbb{R}^{n-1}_{x'})$ such that $\chi_3(x',\tau,\xi') = 1$ if $(x',0,\tau,\xi') \in \operatorname{supp}\chi$ and $1+\tau^2+|\xi'|^2_x \lesssim \rho(x',0,\tau,\xi')$ on $\operatorname{supp}\chi_3$. The function $\tilde\chi_3(x',\tau,\xi') = \sqrt{\rho(x',0,\tau,\xi')}\chi_3(x,\tau,\xi')$ is well-defined, and one has $\tilde\chi_3 \in \mathsf{S}^1(\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'})$. By Lemma 3.2.9, one obtains

$$\operatorname{Op}_{\mathsf{T}}(\tilde\chi_3)\operatorname{Op}_{\mathsf{T}}(q_{\mathsf{T}})\operatorname{Op}_{\mathsf{T}}(\chi_{|x_n=0}) = \operatorname{Op}_{\mathsf{T}}\left(\sqrt{\rho}q_{\mathsf{T}}\chi_{|x_n=0}\right) + R_4$$

where R_4 is a tangential pseudo-differential operator of order -1. As $\tilde\chi(x',\tau,\xi) = 1$ if $(x,\tau,\xi') \in \operatorname{supp}\chi$, one has

$$q_{\mathsf{T}}(x',0,\tau,\xi')\chi(x',0,\tau,\xi') = \chi(x',0,\tau,\xi')\int_{\mathbb{R}}\frac{1}{\xi_n^2+\rho(x',0,\tau,\xi')}\mathrm{d}\xi_n = \frac{\pi\chi(x',0,\tau,\xi')}{\sqrt{\rho(x',0,\tau,\xi')}},$$

and this gives

$$\operatorname{Op}_{\mathsf{T}}(\tilde\chi_3)\operatorname{Op}_{\mathsf{T}}(q_{\mathsf{T}})\operatorname{Op}_{\mathsf{T}}(\chi_{|x_n=0})\partial_\nu u = \pi\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u + R_4\partial_\nu u.$$

This yields

$$\begin{aligned}\left\|\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} &\lesssim \left\|\operatorname{Op}_{\mathsf{T}}(\tilde\chi_3)\operatorname{Op}_{\mathsf{T}}(q_{\mathsf{T}})\operatorname{Op}_{\mathsf{T}}(\chi_{|x_n=0})\partial_\nu u\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}\\ &\quad + \left\|R_4\partial_\nu u\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})}.\end{aligned}$$

As $\tilde\chi_3$ is of order 1 and R_4 of order -1, this gives

$$\begin{aligned}\left\|\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} &\lesssim \left\|\operatorname{Op}_{\mathsf{T}}(q_{\mathsf{T}})\operatorname{Op}_{\mathsf{T}}(\chi_{|x_n=0})\partial_\nu u\right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})}\\ &\quad + \left\|\partial_\nu u\right\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})}.\end{aligned}$$

Using (3.2.13), one obtains

$$\begin{aligned}\left\|\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right\|_{H^s(\mathbb{R}\times\mathbb{R}^{n-1})} &\lesssim \left\|\operatorname{Op}(q)\left(\delta_{x^n=0}\otimes\left(\operatorname{Op}_{\mathsf{T}}(\chi_{|x^n=0})\partial_\nu u\right)\right)_{|x^n=0}\right\|_{H^{s+1}(\mathbb{R}\times\mathbb{R}^{n-1})}\\ &\quad + \left\|\partial_\nu u\right\|_{H^{s-1}(\mathbb{R}\times\mathbb{R}^{n-1})}.\end{aligned}$$

This completes the proof of Lemma 3.2.8. ☐

Using Lemma 3.2.6 and Lemma 3.2.8 in (3.2.10), one finds the estimate of Proposition 3.2.3.

3.3 Change of regularity in observability inequalities

We prove our main result, Theorem 3.0.4, that is, the equivalence between observability at different levels of regularity. First, we show that for all $r \in \mathbb{N}$ and $s \in \mathbb{R}$, H^s-observability for Θ implies H^{s+2r}-observability for Θ. Second, we show that for all $r \in \mathbb{N}$ and $s > -1$, H^s-observability for Θ implies H^{s-2r}-observability for $\tilde\Theta$, for all $\tilde\Theta$ such that $\pi_k\tilde\Theta \neq 0$ on $\operatorname{supp}\pi_k\Theta$, for all $k \in [\![1,N]\!]$. This is enough to give the full conclusion, by Lemma 3.1.11.

3.3.1 Increasing the level of regularity

Consider $\Theta \in \mathscr{C}^\infty_c((0,T)\times\partial M,\mathbb{C}^N)$, $s \in \mathbb{R}$ and assume that H^s-observability for Θ holds. We prove that H^{s+2}-observability for Θ holds, implying by induction that H^{s+2r}-observability for Θ holds for all $r \geq 1$. Consider also $\left(u^0, u^1\right) \in \mathcal{K}^{s+3} \times \mathcal{K}^{s+2}$, and write u for the solution with initial data $\left(u^0, u^1\right)$. We show that

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+3}\times\mathcal{K}^{s+2}} \lesssim \|\operatorname{diag}(\Theta)\partial_\nu u\|_{H^{s+2}((0,T)\times\partial M,\mathbb{C}^N)} .$$

For $t \in (0,T)$, set $\tilde{u}(t) = \mathsf{P}_{s+2}u(t)$. Then, $\tilde{u}$ is the solution of

$$\begin{cases} \partial_t^2\tilde{u} - \mathsf{P}\tilde{u} &= 0 & \text{in } (0,T)\times M, \\ (\tilde{u}(0,\cdot), \partial_t\tilde{u}(0,\cdot)) &= \left(\mathsf{P}_{s+2}u^0, \mathsf{P}_{s+1}u^1\right) & \text{in } M, \\ \tilde{u} &= 0 & \text{on } (0,T)\times\partial M. \end{cases}$$

Since $\left(\mathsf{P}_{s+2}u^0, \mathsf{P}_{s+1}u^1\right) \in \mathcal{K}^{s+1}\times\mathcal{K}^s$, H^s-observability for Θ gives

$$\left\|\mathsf{P}_{s+2}u^0\right\|_{\mathcal{K}^{s+1}} + \left\|\mathsf{P}_{s+1}u^1\right\|_{\mathcal{K}^s} \lesssim \|\operatorname{diag}(\Theta)\partial_\nu\tilde{u}\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} .$$

By Theorem 3.1.12, one has $\tilde{u}(t) = \mathsf{P}_{s+2}u(t) = \partial_t^2 u(t)$ in $\mathcal{K}^{s+1}$ for all $t \in [0,T]$, and $\partial_\nu\tilde{u} = \partial_\nu\partial_t^2 u = \partial_t^2\partial_\nu u$. Hence, the previous estimate reads

$$\left\|\mathsf{P}_{s+2}u^0\right\|_{\mathcal{K}^{s+1}} + \left\|\mathsf{P}_{s+1}u^1\right\|_{\mathcal{K}^s} \lesssim \left\|\operatorname{diag}(\Theta)\partial_t^2\partial_\nu u\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} .$$

Using the ellipticity estimate for P (Proposition 3.1.8-*(ii)*), one finds

$$\begin{aligned}\left\|u^0\right\|_{\mathcal{K}^{s+3}} + \left\|u^1\right\|_{\mathcal{K}^{s+2}} \lesssim{}& \left\|\operatorname{diag}(\Theta)\partial_t^2\partial_\nu u\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \\ &+ \left\|\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+2}}u^0\right\|_{\mathcal{K}^{s+2}} + \left\|\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}}u^1\right\|_{\mathcal{K}^{s+1}} .\end{aligned}$$

To estimate the term with the normal derivative, note that

$$\operatorname{diag}(\Theta)\partial_t^2\partial_\nu u = \partial_t^2\left(\operatorname{diag}(\Theta)\partial_\nu u\right) - 2\partial_t\operatorname{diag}(\Theta)\partial_t\partial_\nu u - \partial_t^2\operatorname{diag}(\Theta)\partial_\nu u,$$

implying

$$\left\|\operatorname{diag}(\Theta)\partial_t^2\partial_\nu u\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \|\operatorname{diag}(\Theta)\partial_\nu u\|_{H^{s+2}((0,T)\times\partial M,\mathbb{C}^N)} + \|\partial_\nu u\|_{H^{s+1}((0,T)\times\partial M,\mathbb{C}^N)} ,$$

where the embedding $\iota_{H^{s+2}\to H^{s+1}}$ has been omitted. By Theorem 3.1.12, if v is the solution of

$$\begin{cases} \partial_t^2 v - \mathsf{P}v &= 0 & \text{in } \mathbb{R}\times M, \\ (v(0,\cdot), \partial_t v(0,\cdot)) &= \left(\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+2}}u^0, \iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}}u^1\right) & \text{in } M, \\ v &= 0 & \text{on } \mathbb{R}\times\partial M, \end{cases}$$

then $\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+2}}u = v$ and $\iota_{H^{s+2}\to H^{s+1}}\partial_\nu u = \partial_\nu v$. Hence, using Theorem 3.1.12 again, one obtains

$$\begin{aligned}\|\partial_\nu u\|_{H^{s+1}((0,T)\times\partial M,\mathbb{C}^N)} &= \|\partial_\nu v\|_{H^{s+1}((0,T)\times\partial M,\mathbb{C}^N)} \\ &\lesssim \|v(0)\|_{\mathcal{K}^{s+2}} + \|\partial_t v(0)\|_{\mathcal{K}^{s+1}} \\ &= \left\|\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+2}}u^0\right\|_{\mathcal{K}^{s+2}} + \left\|\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}}u^1\right\|_{\mathcal{K}^{s+1}} ,\end{aligned}$$

yielding

$$\begin{aligned}\left\|u^0\right\|_{\mathcal{K}^{s+3}} + \left\|u^1\right\|_{\mathcal{K}^{s+2}} \lesssim{}& \|\operatorname{diag}(\Theta)\partial_\nu u\|_{H^{s+2}((0,T)\times\partial M,\mathbb{C}^N)} \\ &+ \left\|\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+2}}u^0\right\|_{\mathcal{K}^{s+2}} + \left\|\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}}u^1\right\|_{\mathcal{K}^{s+1}} .\end{aligned}$$

To complete the proof, we prove that the remainder terms on the right-hand side can be removed. The embedding

$$\begin{array}{rccc} K: & \mathcal{K}^{s+3}\times\mathcal{K}^{s+2} & \longrightarrow & \mathcal{K}^{s+2}\times\mathcal{K}^{s+1} \\ & (u^0,u^1) & \longmapsto & \left(\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+2}}u^0, \iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}}u^1\right) \end{array}$$

is compact, by Proposition 3.1.8-*(i)*. We show that the operator

$$\begin{array}{rccc} A: & \mathcal{K}^{s+3}\times\mathcal{K}^{s+2} & \longrightarrow & H^{s+2}((0,T)\times\partial M,\mathbb{C}^N) \\ & (u^0,u^1) & \longmapsto & \operatorname{diag}(\Theta)\partial_\nu u \end{array}$$

is one-to-one, using the fact that the operator

$$\begin{array}{ccc} \mathcal{K}^{s+1}\times\mathcal{K}^{s} & \longrightarrow & H^{s}((0,T)\times\partial M,\mathbb{C}^N) \\ (u^0,u^1) & \longmapsto & \operatorname{diag}(\Theta)\partial_\nu u \end{array}$$

is one-to-one, by H^s-observability. Consider $\left(u^0,u^1\right)\in\mathcal{K}^{s+3}\times\mathcal{K}^{s+2}$ such that $\operatorname{diag}(\Theta)\partial_\nu u=0$. As above, one has $\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+1}}u=v$ and $\iota_{H^{s+2}\to H^s}\partial_\nu u=\partial_\nu v$, where v is the solution of

$$\left\{\begin{array}{rcll} \partial_t^2 v-\mathsf{P}v & = & 0 & \text{in } \mathbb{R}\times M, \\ (v(0,\cdot),\partial_t v(0,\cdot)) & = & \left(\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+1}}u^0,\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s}}u^1\right) & \text{in } M, \\ v & = & 0 & \text{on } \mathbb{R}\times\partial M. \end{array}\right.$$

Since $\operatorname{diag}(\Theta)\partial_\nu v=0$ in $H^s((0,T)\times\partial M,\mathbb{C}^N)$, H^s-observability gives

$$\left(\iota_{\mathcal{K}^{s+3}\to\mathcal{K}^{s+1}}u^0,\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s}}u^1\right)=0.$$

Thus, one finds $\left(u^0,u^1\right)=0$, and H^{s+2}-observability is a consequence of the following lemma. Here, A is one-to-one: the information from Lemma 3.3.1 about the kernel is used below.

Lemma 3.3.1. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Hilbert spaces. Consider two continuous linear operators $A:\mathcal{X}\to\mathcal{Y}$ and $K:\mathcal{X}\to\mathcal{Z}$. Assume that K is compact and that there exists $C>0$ such that*

$$\|x\|_{\mathcal{X}}\leq\left(\|Ax\|_{\mathcal{Y}}+\|Kx\|_{\mathcal{Z}}\right),\quad x\in\mathcal{X}.$$

Then the kernel of A is finite-dimensional. If moreover A is one-to-one, there exists $C'>0$ such that for all $x\in\mathcal{X}$, one has

$$\|x\|_{\mathcal{X}}\leq C'\|Ax\|_{\mathcal{Y}},\quad x\in\mathcal{X}. \tag{3.3.1}$$

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of elements of $\operatorname{Ker}A$. Up to a subsequence, as K is compact, we may assume that the sequence (Kx_n) converges. Writing

$$\|x_n-x_m\|_{\mathcal{X}}\lesssim\|Kx_n-Kx_m\|_{\mathcal{Z}},$$

one finds that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. This proves that the unit ball of $\operatorname{Ker}A$ is compact, implying that $\operatorname{Ker}A$ is finite-dimensional.

Now, we assume that A is one-to-one. We prove (3.3.1) by contradiction : we assume that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\|x_n\|_{\mathcal{X}}=1$ for all $n\in\mathbb{N}$ and

$$Ax_n\xrightarrow{n\to\infty}0.$$

Up to a subsequence, we can assume that the sequence (Kx_n) converges. One has

$$\|x_n-x_m\|_{\mathcal{X}}\lesssim\|Ax_n\|_{\mathcal{Y}}+\|Ax_m\|_{\mathcal{Y}}+\|Kx_n-Kx_m\|_{\mathcal{Z}},$$

implying that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Write $x\in\mathcal{X}$ its limit. For $n\in\mathbb{N}$, using $\|x_n\|_{\mathcal{X}}=1$, one finds $\|x\|_{\mathcal{X}}=1$. As A is continuous, one also has $Ax=0$, a contradiction. □

3.3.2 Decreasing the level of regularity

Consider $\Theta^1 \in \mathscr{C}_c^\infty((0,T)\times\partial M,\mathbb{C}^N)$ and assume that H^s-observability for Θ^1 holds. As the level of regularity can be increased, by the part of the proof of Section 3.1, we may assume that $s \geq 1$, without loss of generality. We prove that for $r \in \mathbb{N}^*$, H^{s-2r}-observability for Θ^2 holds, for all $\Theta^2 \in \mathscr{C}_c^\infty((0,T)\times\partial M,\mathbb{C}^N)$ such that $\pi_k\Theta^2 \neq 0$ on $\operatorname{supp}\pi_k\Theta^1$, for all $k \in [\![1,N]\!]$. Consider $(u^0,u^1) \in \mathcal{K}^{s-2r+1}\times\mathcal{K}^{s-2r}$, and denote by u the associated solution.

Following the proof of Section 3.1, one might be inclined to define $\tilde{u}(t) = \mathsf{P}^{-r}u(t)$. However, this is not always possible, for example if $\mathsf{P} = \Delta + \lambda$, with λ in the spectrum of the Dirichlet Laplacian. To overcome this difficulty, we use the shift operator of Proposition 3.1.8-*(iii)*. We introduce $(\tilde{u}^0,\tilde{u}^1)$ as the unique element of $\mathcal{K}^{s+1}\times\mathcal{K}^s$ such that $(u^0,u^1) = (\mathcal{S}^r_{s-r+1}\tilde{u}^0, \mathcal{S}^r_{s-r}\tilde{u}^1)$, and set $\tilde{u}$ as the solution associated with $(\tilde{u}^0,\tilde{u}^1)$.

By H^s-observability for Θ^1, one has

$$\left\|\tilde{u}^0\right\|_{\mathcal{K}^{s+1}} + \left\|\tilde{u}^1\right\|_{\mathcal{K}^s} \lesssim \left\|\operatorname{diag}(\Theta^1)\partial_\nu\tilde{u}\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}.$$

Since $\mathcal{S}^r_{s-r+1}$ and $\mathcal{S}^r_{s-r}$ are continuous, one has $\|u^0\|_{\mathcal{K}^{s-2r+1}} + \|u^1\|_{\mathcal{K}^{s-2r}} \lesssim \|\tilde{u}^0\|_{\mathcal{K}^{s+1}} + \|\tilde{u}^1\|_{\mathcal{K}^s}$, implying

$$\left\|u^0\right\|_{\mathcal{K}^{s-2r+1}} + \left\|u^1\right\|_{\mathcal{K}^{s-2r}} \lesssim \left\|\operatorname{diag}(\Theta^1)\partial_\nu\tilde{u}\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}.$$

Now, Theorem 3.2.1 gives

$$\left\|u^0\right\|_{\mathcal{K}^{s-2r+1}} + \left\|u^1\right\|_{\mathcal{K}^{s-2r}} \lesssim \left\|\operatorname{diag}(\Theta^1)\partial_t^{2r}\partial_\nu\tilde{u}\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} + \left\|\tilde{u}^0\right\|_{\mathcal{K}^{s+\frac12}} + \left\|\tilde{u}^1\right\|_{\mathcal{K}^{s-\frac12}}.$$

Note that embeddings in the remainder terms are omitted here, as $s \geq 1$ (see Remark 3.2.2). Using Proposition 3.1.8-*(iii)*, one finds

$$\begin{aligned}\left\|\tilde{u}^0\right\|_{\mathcal{K}^{s+\frac12}} + \left\|\tilde{u}^1\right\|_{\mathcal{K}^{s-\frac12}} = &\left\|\left(\mathcal{S}^r_{s-r+\frac12}\right)^{-1}\circ\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r+\frac12}}u^0\right\|_{\mathcal{K}^{s+\frac12}}\\ &+\left\|\left(\mathcal{S}^r_{s-r-\frac12}\right)^{-1}\circ\iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-\frac12}}u^1\right\|_{\mathcal{K}^{s-\frac12}}.\end{aligned}$$

With the continuity of $\left(\mathcal{S}^r_{s-r+\frac12}\right)^{-1}$ and $\left(\mathcal{S}^r_{s-r-\frac12}\right)^{-1}$, one obtains

$$\left\|\tilde{u}^0\right\|_{\mathcal{K}^{s+\frac12}} + \left\|\tilde{u}^1\right\|_{\mathcal{K}^{s-\frac12}} \lesssim \left\|\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r+\frac12}}u^0\right\|_{\mathcal{K}^{s-2r+\frac12}} + \left\|\iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-\frac12}}u^1\right\|_{\mathcal{K}^{s-2r-\frac12}},$$

implying

$$\begin{aligned}\left\|u^0\right\|_{\mathcal{K}^{s-2r+1}} + \left\|u^1\right\|_{\mathcal{K}^{s-2r}} \lesssim &\left\|\operatorname{diag}(\Theta^1)\partial_t^{2r}\partial_\nu\tilde{u}\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)}\\ &+\left\|\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r+\frac12}}u^0\right\|_{\mathcal{K}^{s-2r+\frac12}} + \left\|\iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-\frac12}}u^1\right\|_{\mathcal{K}^{s-2r-\frac12}}.\end{aligned}$$

Next, we want to replace $\|\operatorname{diag}(\Theta^1)\partial_t^{2r}\partial_\nu\tilde{u}\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)}$ by

$$\left\|\operatorname{diag}(\Theta^1)\partial_\nu u\right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)},$$

up to a remainder term. The idea is the following: if $\mathcal{S}^r_{s-r} = \mathsf{P}^r_{s-r}$, as in the case $\mathsf{P} = \Delta$ for example, then we can prove that $\partial_t^{2r}\tilde{u} = u$. In the general case, one has the following lemma.

Lemma 3.3.2. *For $s \in \mathbb{R}$, $r \in \mathbb{N}^*$, $\Theta \in \mathscr{C}^\infty((0,T) \times \partial M, \mathbb{C}^N)$, $(v^0, v^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$, and v the solution associated with (v^0, v^1), one has*

$$\left\| \operatorname{diag}(\Theta) \partial_t^{2r} \partial_\nu v \right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\| \operatorname{diag}(\Theta) \partial_\nu \left(\mathcal{S}^r_{s-r+1} v \right) \right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} + \left\| \iota_{\mathcal{K}^{s+1}\to\mathcal{K}^s} v^0 \right\|_{\mathcal{K}^s} + \left\| \iota_{\mathcal{K}^{s}\to\mathcal{K}^{s-1}} v^1 \right\|_{\mathcal{K}^{s-1}}.$$

A proof of Lemma 3.3.2 is given below. Arguing as above, one finds

$$\begin{aligned} &\left\| \iota_{\mathcal{K}^{s+1}\to\mathcal{K}^s} \tilde{u}^0 \right\|_{\mathcal{K}^s} + \left\| \iota_{\mathcal{K}^{s}\to\mathcal{K}^{s-1}} \tilde{u}^1 \right\|_{\mathcal{K}^{s-1}} \\ &\lesssim \left\| \iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r}} u^0 \right\|_{\mathcal{K}^{s-2r}} + \left\| \iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-1}} u^1 \right\|_{\mathcal{K}^{s-2r-1}} \\ &\lesssim \left\| \iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r+\frac12}} u^0 \right\|_{\mathcal{K}^{s-2r+\frac12}} + \left\| \iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-\frac12}} u^1 \right\|_{\mathcal{K}^{s-2r-\frac12}}. \end{aligned}$$

As $\mathcal{S}^r_{s-r+1} \tilde{u} = u$, by Corollary 3.1.13, Lemma 3.3.2 gives

$$\begin{aligned} \left\| u^0 \right\|_{\mathcal{K}^{s-2r+1}} + \left\| u^1 \right\|_{\mathcal{K}^{s-2r}} &\lesssim \left\| \operatorname{diag}(\Theta^1) \partial_\nu u \right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} \\ &\quad + \left\| \iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r+\frac12}} u^0 \right\|_{\mathcal{K}^{s-2r+\frac12}} + \left\| \iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-\frac12}} u^1 \right\|_{\mathcal{K}^{s-2r-\frac12}}. \end{aligned} \tag{3.3.2}$$

Note that (3.3.2) holds true if Θ^1 is replaced by some $\Theta^2 \in \mathscr{C}^\infty_c((0,T) \times \partial M, \mathbb{C}^N)$ such that $\pi_k \Theta^2 \neq 0$ on $\operatorname{supp} \pi_k \Theta^1$, for all $k \in [\![1, N]\!]$. To complete the proof, we show that the remainder terms on the right-hand side of (3.3.2) can be removed, when Θ^1 is replaced by such Θ^2. The embedding

$$\begin{array}{rccc} K: & \mathcal{K}^{s-2r+1} \times \mathcal{K}^{s-2r} & \longrightarrow & \mathcal{K}^{s-2r+\frac12} \times \mathcal{K}^{s-2r-\frac12} \\ & (u^0, u^1) & \longmapsto & \left(\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r+\frac12}} u^0, \iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-\frac12}} u^1 \right) \end{array}$$

is compact, by Proposition 3.1.8-*(i)*. For $s' \in \mathbb{R}$ and $\Theta \in \mathscr{C}^\infty_c((0,T) \times \partial M, \mathbb{C}^N)$, introduce

$$\begin{array}{rccc} A_{\Theta,s'}: & \mathcal{K}^{s'+1} \times \mathcal{K}^{s'} & \longrightarrow & H^{s'}((0,T) \times \partial M, \mathbb{C}^N) \\ & (u^0, u^1) & \longmapsto & \operatorname{diag}(\Theta) \partial_\nu u \end{array}.$$

By (3.3.2) and Lemma 3.3.1, the kernel of $A_{\Theta^1,s-2r}$ is finite-dimensional. Note also that H^s-observability for Θ^1 implies that $A_{\Theta^1,s}$ is one-to-one.

Lemma 3.3.3. *Consider $s \in \mathbb{R}$ and $\Theta^1 \in \mathscr{C}^\infty_c((0,T) \times \partial M, \mathbb{C}^N)$ such that $A_{\Theta^1,s}$ is one-to-one, and such that for all $r \in \mathbb{N}^*$, the kernel of $A_{\Theta^1,s-2r}$ is finite-dimensional. Then, for $r \in \mathbb{N}^*$ and $\Theta^2 \in \mathscr{C}^\infty_c((0,T) \times \partial M, \mathbb{C}^N)$ such that $\pi_k \Theta^2 \neq 0$ on $\operatorname{supp} \pi_k \Theta^1$, for all $k \in [\![1, N]\!]$, $A_{\Theta^2,s-2r}$ is one-to-one.*

A proof of Lemma 3.3.3 is given below. Now, using (3.3.2) with Θ^2 instead of Θ^1, and Lemma 3.3.1 again, one concludes that H^{s-2r}-observability for Θ^2 holds. As explained in the beginning of Section 3, this completes the proof of Theorem 3.0.4.

Now, we prove Lemma 3.3.2 and Lemma 3.3.3.

Proof of Lemma 3.3.2. By interpolation, one may assume that $s \in \mathbb{Z}$. By Theorem 3.1.12, one has $\partial_t^{2r} \partial_\nu v = \partial_\nu \partial_t^{2r} v = \partial_\nu \mathsf{P}^r_{s-r+1} v$. The triangular inequality gives

$$\begin{aligned} &\left\| \operatorname{diag}(\Theta) \partial_t^{2r} \partial_\nu v \right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} \\ &\lesssim \left\| \operatorname{diag}(\Theta) \partial_\nu \left(\mathcal{S}^r_{s-r+1} v \right) \right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)} + \left\| \partial_\nu \left(\mathsf{P}^r_{s-r+1} v - \mathcal{S}^r_{s-r+1} v \right) \right\|_{H^{s-2r}((0,T)\times\partial M,\mathbb{C}^N)}. \end{aligned}$$

Set $w = \left(\mathsf{P}^r_{s-r+1} - \mathcal{S}^r_{s-r+1}\right) v$. By Theorem 3.1.12 and Corollary 3.1.13, w is the solution of

$$\begin{cases} \partial_t^2 w - \mathsf{P} w &= 0 & \text{in } (0,T)\times M, \\ (w(0,\cdot), \partial_t w(0,\cdot)) &= (w^0, w^1) & \text{in } M, \\ w &= 0 & \text{on } (0,T)\times \partial M, \end{cases}$$

where $(w^0, w^1) = \left(\left(\mathsf{P}^r_{s-r+1} - \mathcal{S}^r_{s-r+1}\right) v^0, \left(\mathsf{P}^r_{s-r} - \mathcal{S}^r_{s-r}\right) v^1\right)$. Hence, using Theorem 3.1.12 and Proposition 3.1.8-*(iii)*, one obtains

$$\begin{aligned} \|\partial_\nu w\|_{H^{s-2r}((0,T)\times\partial M, \mathbb{C}^N)} &\lesssim \left\|w^0\right\|_{\mathcal{K}^{s-2r+1}} + \left\|w^1\right\|_{\mathcal{K}^{s-2r}} \\ &\lesssim \left\|\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^s} v^0\right\|_{\mathcal{K}^s} + \left\|\iota_{\mathcal{K}^{s}\to\mathcal{K}^{s-1}} v^1\right\|_{\mathcal{K}^{s-1}}, \end{aligned}$$

and this gives the desired result. ☐

Proof of Lemma 3.3.3. For $s' \in \mathbb{R}$ and $\Theta \in \mathscr{C}^\infty_{\mathrm{c}}((0,T)\times\partial M, \mathbb{C}^N)$, denote by $N_{\Theta,s'}$ the kernel of $A_{\Theta,s'}$, that is,

$$N_{\Theta,s'} = \left\{\left(u^0, u^1\right) \in \mathcal{K}^{s'+1}\times\mathcal{K}^{s'}, \operatorname{diag}(\Theta)\partial_\nu u = 0\right\}.$$

Note that by definition, one has

$$N_{\Theta,s'} \subset N_{\tilde{\Theta},s'}, \quad s' \in \mathbb{R}, \tag{3.3.3}$$

if $\pi_k\Theta \neq 0$ on $\operatorname{supp}\pi_k\tilde{\Theta}$, for all $k \in [\![1,N]\!]$, and for $s_1 > s_2$, the map

$$\begin{aligned} \Phi_{s_1,s_2}(\Theta): \quad N_{\Theta,s_1} &\longrightarrow N_{\Theta,s_2} \\ (u^0,u^1) &\longmapsto \left(\iota_{\mathcal{K}^{s_1+1}\to\mathcal{K}^{s_2+1}} u^0, \iota_{\mathcal{K}^{s_1}\to\mathcal{K}^{s_2}} u^1\right) \end{aligned}$$

is well-defined, injective, and compact.

Consider $\Theta^2 \in \mathscr{C}^\infty_{\mathrm{c}}((0,T)\times\partial M, \mathbb{C}^N)$ such that $\pi_k\Theta^2 \neq 0$ on $\operatorname{supp}\pi_k\Theta^1$, for all $k \in [\![1,N]\!]$. We claim that $N_{\Theta^2,s-2r}$ is finite-dimensional for all $r \geq 0$. Indeed, for $r \in \mathbb{N}$, it follows from the assumptions of Lemma 3.3.3 and (3.3.3). For $r \geq 0$, as $\Phi_{s-2r,s-\lfloor 2r\rfloor-1}(\Theta^2)$ is one-to-one, $N_{\Theta^2,s-2r}$ is isomorphic to a subspace of $N_{\Theta^2,s-\lfloor 2r\rfloor-1}$, and hence, is finite-dimensional.

Consider $r \geq 0$. We prove that

$$\Phi_{s-2r,s-2r-1}(\Theta^2) \text{ is an isomorphism.} \tag{3.3.4}$$

It suffices to show that $\Phi_{s-2r,s-2r-1}(\Theta^2)$ is onto. Consider $\left(u^0,u^1\right) \in N_{\Theta^2,s-2r-1}$, and write $U(t) = (u(t), \partial_t u(t))$ for $t \in \mathbb{R}$, where u is the solution associated with $\left(u^0,u^1\right)$. As the distance between $\operatorname{supp}\pi_k\Theta^1$ and $(\operatorname{supp}\pi_k\Theta^2)^\complement$ is positive for all $k \in [\![1,N]\!]$, there exists $\varepsilon > 0$ such that for $t \in [0,\varepsilon)$, $U(t) \in N_{\Theta^1,s-2r-1}$.

For $t \in (0,\varepsilon)$, set $V_t = \frac{1}{t}(U(t) - U(0)) \in N_{\Theta^1,s-2r-1}$. One has

$$\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r}}\left(\frac{1}{t}(u(t)-u^0)\right) \xrightarrow{t\to 0^+} \partial_t u(0) = u^1 \in \mathcal{K}^{s-2r}, \tag{3.3.5}$$

and

$$\iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-1}}\left(\frac{1}{t}(\partial_t u(t)-u^1)\right) \xrightarrow{t\to 0^+} \partial_t^2 u(0) = \mathsf{P}_{s-2r} u^0 \in \mathcal{K}^{s-2r-1}. \tag{3.3.6}$$

As $N_{\Theta^1,s-2r-1}$ is finite-dimensional, the norm of $\mathcal{K}^{s-2r+1}\times\mathcal{K}^{s-2r}$ is equivalent to the norm

$$\mathcal{N}\left(u^0,u^1\right) = \left\|\left(\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r}} u^0, \iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-1}} u^1\right)\right\|_{\mathcal{K}^{s-2r}\times\mathcal{K}^{s-2r-1}}.$$

By (3.3.5) and (3.3.6), $(V_t)_{t>0}$ is a Cauchy sequence for the norm $\mathcal{N}$, and thus, it converges in $N_{\Theta^1,s-2r}$. Write (v^0, v^1) for its limit. Using (3.3.5) and (3.3.6) again, one finds

$$\left(\iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r}} v^0, \iota_{\mathcal{K}^{s-2r}\to\mathcal{K}^{s-2r-1}} v^1\right) = \left(u^1, \mathsf{P}_{s-2r} u^0\right).$$

By Proposition 3.1.8-*(ii)*, there exists $\tilde{u}^0 \in \mathcal{K}^{s-2r+2}$ such that $u^0 = \iota_{\mathcal{K}^{s-2r+2}\to\mathcal{K}^{s-2r+1}} \tilde{u}^0$ and $v^1 = \mathsf{P}_{s-2r+1}\tilde{u}^0$. One has

$$\left(\iota_{\mathcal{K}^{s-2r+2}\to\mathcal{K}^{s-2r+1}} \tilde{u}^0, \iota_{\mathcal{K}^{s-2r+1}\to\mathcal{K}^{s-2r}} v^0\right) = \left(u^0, u^1\right).$$

This gives $\Phi_{s-2r,s-2r-1}(\Theta^2)\left(\tilde{u}^0, v^0\right) = \left(u^0, u^1\right)$, if we show that $\left(\tilde{u}^0, v^0\right) \in N_{\Theta^2,s-2r-1}$. If $\tilde{u}$ is the solution associated with $\left(\tilde{u}^0, v^0\right)$, then by Theorem 3.1.12, one has $\iota_{\mathcal{K}^{s-2r+2}\to\mathcal{K}^{s-2r+1}}\tilde{u} = u$ and

$$\iota_{H^{s-2r+1}\to H^{s-2r}} \partial_\nu \tilde{u} = \partial_\nu u.$$

As $\operatorname{diag}(\Theta^2)\partial_\nu u = 0 \in \mathscr{D}'((0,T)\times\partial M, \mathbb{C}^N)$, this implies $\left(\tilde{u}^0, v^0\right) \in N_{\Theta^2,s-2r-1}$, completing the proof of (3.3.4).

By iteration, one obtains an isomorphism between $N_{\Theta^2,s}$ and $N_{\Theta^2,s-2r}$ for $r \in \mathbb{N}^*$. As $N_{\Theta^1,s} = \{0\}$, (3.3.3) gives $N_{\Theta^2,s} = \{0\}$. This completes the proof of Lemma 3.3.3. □

3.A The case of internal observability

Here, we explain how to adapt the methods of this chapter to the case of internal observability. Consider $\chi \in \mathscr{C}^\infty(M, \mathbb{C}^N)$.

Definition 3.A.1 ($\mathcal{K}^s$-observability for (χ, T))**.** We say that $\mathcal{K}^s$-observability for (χ, T) holds if there exists $C > 0$ such that for all $(u^0, u^1) \in \mathcal{K}^s \times \mathcal{K}^{s-1}$,

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^s\times\mathcal{K}^{s-1}} \leq C \left\|\operatorname{diag}(\chi) u\right\|_{L^2((0,T),\mathcal{K}^s)},$$

where u is the solution of (3.0.1) with initial data (u^0, u^1).

Note that the multiplication operator $u \in \mathcal{K}^s \mapsto \operatorname{diag}(\chi)u \in \mathcal{K}^s$ is well-defined, and commutes with the embeddings of Proposition 3.1.8. As in the boundary case, one can prove that $\mathcal{K}^s$-observability for (χ, T) is equivalent with a controllability property, for the equation (3.0.1) with a source term of the form $\operatorname{diag}(\chi)F$, with $F \in L^2((0,T), \mathcal{K}_*^{-s})$. One can check that the solution of such a system is well-defined, by adapting the proof of Theorem 3.1.12. The analogue of Theorem 3.0.4 is the following result.

Theorem 3.A.2. *Consider $s_1, s_2 \in \mathbb{R}$, and $\chi \in \mathscr{C}^\infty(M, \mathbb{C}^N)$. If $s_1 < s_2$, then for all $T > 0$, $\mathcal{K}^{s_1}$-observability for (χ, T) implies $\mathcal{K}^{s_2}$-observability for (χ, T). If $s_1 > s_2$, then for all $0 < T_1 < T_2$, $\mathcal{K}^{s_1}$-observability for (χ, T_1) implies $\mathcal{K}^{s_2}$-observability for (χ, T_2).*

The proof of Theorem 3.A.2 is simpler than that of Theorem 3.0.4, so we only sketch it. To increase the regularity level, one uses the following lemma.

Lemma 3.A.3. *Consider $s \in \mathbb{R}$ and $\chi \in \mathscr{C}^\infty(M, \mathbb{C}^N)$. There exists $C > 0$ such that*

$$\left\|\left[\operatorname{diag}(\chi), \mathsf{P}_{s+1}\right] u\right\|_{\mathcal{K}^s} \leq C \left\|\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}} u\right\|_{\mathcal{K}^{s+1}}, \quad u \in \mathcal{K}^{s+2}.$$

Sketch of proof of Lemma 3.A.3. By interpolation, it suffices to prove Lemma 3.A.3 for $s \in \mathbb{Z}$. For $s \in \mathbb{N}$ and $u \in \mathcal{K}^{s+2}$, one has

$$\|[\mathrm{diag}(\chi), \mathsf{P}_{s+1}]\, u\|_{\mathcal{K}^s} = \|[\mathrm{diag}(\chi), \mathsf{P}_{\mathscr{D}'}]\, u\|_{H^s(M,\mathbb{C}^N)} \lesssim \|u\|_{H^{s+1}(M,\mathbb{C}^N)} = \|\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}} u\|_{\mathcal{K}^{s+1}} ,$$

and the same holds for P^*. Now, consider $s \in \mathbb{Z}$, $s \leq -1$, and $u \in \mathcal{K}^{s+2}$. Note that

$$\begin{aligned}&\|[\mathrm{diag}(\chi), \mathsf{P}_{s+1}]\, u\|_{\mathcal{K}^s}\\ &= \sup\left\{\left|\left\langle[\mathrm{diag}(\chi), \mathsf{P}_{s+1}]\, u, \iota_{\mathcal{K}_*^{-s+1}\to\mathcal{K}_*^{-s}} v\right\rangle_{\mathcal{K}^s,\mathcal{K}_*^{-s}}\right|, v \in \mathcal{K}_*^{-s+1}, \left\|\iota_{\mathcal{K}_*^{-s+1}\to\mathcal{K}_*^{-s}} v\right\|_{\mathcal{K}_*^{-s}} \leq 1\right\},\end{aligned}$$

as $\iota_{\mathcal{K}_*^{-s+1}\to\mathcal{K}_*^{-s}}$ has a dense range. For $v \in \mathcal{K}_*^{-s+1}$, using the case $s \in \mathbb{N}$, one finds

$$\begin{aligned}\left|\left\langle[\mathrm{diag}(\chi), \mathsf{P}_{s+1}]\, u, \iota_{\mathcal{K}_*^{-s+1}\to\mathcal{K}_*^{-s}} v\right\rangle_{\mathcal{K}^s,\mathcal{K}_*^{-s}}\right| &= \left|\left\langle\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}} u, \left[\mathrm{diag}(\chi), \mathsf{P}_{-s}^*\right] v\right\rangle_{\mathcal{K}^{s+1},\mathcal{K}_*^{-s-1}}\right|\\ &\lesssim \|\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}} u\|_{\mathcal{K}^{s+1}} \left\|\iota_{\mathcal{K}_*^{-s+1}\to\mathcal{K}_*^{-s}} v\right\|_{\mathcal{K}_*^{-s}} ,\end{aligned}$$

and that completes the proof of Lemma 3.A.3. □

Lemma 3.A.3 and $\mathcal{K}^s$-observability for (χ, T) yield

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+2}\times\mathcal{K}^{s+1}} \lesssim \|\mathrm{diag}(\chi) u\|_{L^2((0,T),\mathcal{K}^{s+2})} + \left\|\left(\iota_{\mathcal{K}^{s+2}\to\mathcal{K}^{s+1}} u^0, \iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{s}} u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s} ,$$

for $(u^0, u^1) \in \mathcal{K}^{s+2} \times \mathcal{K}^{s+1}$. The remainder term is compact, and $\mathcal{K}^s$-observability for (χ, T) implies that the operator $(u^0, u^1) \in \mathcal{K}^{s+2} \times \mathcal{K}^{s+1} \mapsto \mathrm{diag}(\chi) u$ is one-to-one. This proves that $\mathcal{K}^s$-observability implies $\mathcal{K}^{s+2}$-observability.

To decrease the regularity level, one relies on the following result about the shift operator of Proposition 3.1.8. We use the notation $\mathcal{S}_s^{-1} = \left(\mathcal{S}_s^1\right)^{-1}$, for $s \in \mathbb{R}$.

Lemma 3.A.4. *Consider $s \in \mathbb{R}$ and $\chi \in \mathscr{C}^\infty(M, \mathbb{C}^N)$. There exists $C > 0$ such that*

$$\left\|\left[\mathrm{diag}(\chi), \mathcal{S}_{s-1}^{-1}\right] u\right\|_{\mathcal{K}^s} \leq C \|\iota_{\mathcal{K}^{s-2}\to\mathcal{K}^{s-3}} u\|_{\mathcal{K}^{s-3}} , \quad u \in \mathcal{K}^{s-2}.$$

Sketch of proof of Lemma 3.A.4. Using Proposition 3.1.8 and Lemma 3.A.3, one finds

$$\begin{aligned}\left\|\left[\mathrm{diag}(\chi), \mathcal{S}_{s-1}^{-1}\right] u\right\|_{\mathcal{K}^s} &= \left\|\mathcal{S}_{s-1}^{-1}\left(\mathcal{S}_{s-1}^{1}\left(\mathrm{diag}(\chi)\mathcal{S}_{s-1}^{-1} u\right) - \mathrm{diag}(\chi)\mathcal{S}_{s-1}^{1}\mathcal{S}_{s-1}^{-1} u\right)\right\|_{\mathcal{K}^s}\\ &\lesssim \left\|\mathcal{S}_{s-1}^{1}\left(\mathrm{diag}(\chi)\mathcal{S}_{s-1}^{-1} u\right) - \mathrm{diag}(\chi)\mathcal{S}_{s-1}^{1}\mathcal{S}_{s-1}^{-1} u\right\|_{\mathcal{K}^{s-2}}\\ &= \left\|\mathsf{P}_{s-1}\left(\mathrm{diag}(\chi)\mathcal{S}_{s-1}^{-1} u\right) - \mathrm{diag}(\chi)\mathsf{P}_{s-1}\mathcal{S}_{s-1}^{-1} u\right\|_{\mathcal{K}^{s-2}}\\ &\lesssim \left\|\iota_{\mathcal{K}^{s}\to\mathcal{K}^{s-1}}\mathcal{S}_{s-1}^{-1} u\right\|_{\mathcal{K}^{s-1}}\\ &\lesssim \|\iota_{\mathcal{K}^{s-2}\to\mathcal{K}^{s-3}} u\|_{\mathcal{K}^{s-3}} ,\end{aligned}$$

for all $u \in \mathcal{K}^{s-2}$. □

Assume that $\mathcal{K}^s$-observability for (χ, T_1) holds. Then, Lemma 3.A.4 gives

$$\begin{aligned}\left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s-2}\times\mathcal{K}^{s-3}} \lesssim\ & \|\mathrm{diag}(\chi) u\|_{L^2((0,T_1),\mathcal{K}^{s-2})}\\ &+ \left\|\left(\iota_{\mathcal{K}^{s-2}\to\mathcal{K}^{s-3}} u^0, \iota_{\mathcal{K}^{s-3}\to\mathcal{K}^{s-4}} u^1\right)\right\|_{\mathcal{K}^{s-3}\times\mathcal{K}^{s-4}} ,\end{aligned}$$

for $(u^0,u^1)\in\mathcal{K}^{s-2}\times\mathcal{K}^{s-3}$. For $T>0$ and $s'\in\mathbb{R}$, set

$$A_{T,s'}:\begin{array}{ccc}\mathcal{K}^{s'}\times\mathcal{K}^{s'-1} & \longrightarrow & L^2((0,T),\mathcal{K}^{s'})\\ (u^0,u^1) & \longmapsto & \operatorname{diag}(\chi)u\end{array},$$

and write $\operatorname{Ker} A_{T,s'}$ for the kernel of that operator. Then $A_{T_1,s}$ is one-to-one, $\operatorname{Ker} A_{T_1,s-2}$ is finite-dimensional, and to complete the proof of Theorem 3.A.2, it suffices to prove that $A_{T_2,s-2}$ is one-to-one, for all $T_2>T_1$. For $s_1>s_2$ and $T>0$, introduce the embedding

$$\Phi_{T,s_1,s_2}:\begin{array}{ccc}\operatorname{Ker} A_{T,s_1} & \longrightarrow & \operatorname{Ker} A_{T,s_2}\\ (u^0,u^1) & \longmapsto & \left(\iota_{\mathcal{K}^{s_1}\to\mathcal{K}^{s_2}}u^0,\iota_{\mathcal{K}^{s_1-1}\to\mathcal{K}^{s_2-1}}u^1\right)\end{array}.$$

Consider $T_2>T_1$ and $\sigma\in\{0,1\}$. We prove that $\Phi_{T_2,s-\sigma,s-\sigma-1}$ is an isomorphism. Take $(u^0,u^1)\in\operatorname{Ker} A_{T_2,s-\sigma-1}$. For $t\in(0,T_2]$, set

$$V_t=\frac{1}{t}\left((u(t),\partial_t u(t))-\left(u^0,u^1\right)\right).$$

As $T_2>T_1$, one has $V_t\in\operatorname{Ker} A_{T_1,s-\sigma-1}$ for all $t>0$ sufficiently small. In addition, V_t converges to a limit as $t\to 0^+$, for one particular norm on $\operatorname{Ker} A_{T_1,s-\sigma-1}$, and hence for any norm on $\operatorname{Ker} A_{T_1,s-\sigma-1}$, as $\operatorname{Ker} A_{T_1,s-\sigma-1}$ is finite-dimensional. This gives $(u^0,u^1)\in\Phi_{T_2,s-\sigma,s-\sigma-1}(\operatorname{Ker} A_{T_2,s-\sigma})$. Hence, $\Phi_{T_2,s-\sigma,s-\sigma-1}$ is an isomorphism, implying in particular that $\operatorname{Ker} A_{T_2,s-2}$ is isomorphic to $\operatorname{Ker} A_{T_2,s}=\{0\}$. This completes the proof of Theorem 3.A.2.

3.B Proof of the results of Section 3.1

3.B.1 Proof of Proposition 3.1.8

We start by giving some details about *(i)*. If $s\geq 0$ then the map $\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}:\mathcal{K}^{s+\delta}\hookrightarrow\mathcal{K}^s$ is just a natural inclusion, and is thus one-to-one. It is an embedding, and it will often be omitted. If $s+\delta<0$, then by definition $\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}$ is the restriction operator

$$\begin{array}{ccc}\mathcal{K}^{s+\delta} & \longrightarrow & \mathcal{K}^s\\ u & \longmapsto & u_{|\mathcal{K}_*^{-s}}\end{array}.$$

In Step 5 below, we prove that $\iota_{\mathcal{K}_*^{-s}\to\mathcal{K}_*^{-s-\delta}}$ has dense range, implying that $\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}$ is one-to-one if $s+\delta<0$. By definition, if $s+\delta\geq 0>s$, one has

$$\langle\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}(u),v\rangle_{\mathcal{K}^s,\mathcal{K}_*^{-s}}=\langle u,\overline{v}\rangle_{L^2(M,\mathbb{C}^N)},$$

for $u\in\mathcal{K}^{s+\delta}$ and $v\in\mathcal{K}_*^{-s}=\mathcal{K}_*^{|s|}$. As $\mathscr{D}(M,\mathbb{C}^N)\subset\mathcal{K}_*^{-s}$, one sees that $\mathcal{K}_*^{-s}$ is dense in $L^2(M,\mathbb{C}^N)$. This implies that $\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}$ is one-to-one in the case $s+\delta\geq 0>s$.

To prove *(iii)*, one can always assume that $r=1$: the operator $\mathcal{S}_s^r$ is then defined by

$$\mathcal{S}_s^r:\mathcal{K}^{s+r}\xrightarrow{\mathcal{S}^1_{s+r-1}}\mathcal{K}^{s+r-2}\xrightarrow{\mathcal{S}^1_{s+r-3}}\cdots\xrightarrow{\mathcal{S}^1_{s-r+3}}\mathcal{K}^{s-r+2}\xrightarrow{\mathcal{S}^1_{s-r+1}}\mathcal{K}^{s-r}.$$

Note that in particular, (3.1.5) holds true by definition.

The proof is organized as follows. First, we prove *(ii)* for $s\in\mathbb{N}$, $s\geq r$, except the inequality. Second, we construct the shift operator in three steps. Third, using the shift operator, we prove *(i)*. Finally, we complete the proof of *(ii)* and *(iii)*. We often assume that $s\in\mathbb{Z}$: the case $s\in\mathbb{R}$ follows by interpolation.

Step 1: Proof of *(ii)* for $s \in \mathbb{N}$, $s \geq r$ (except the inequality). As $s - r \geq 0$ here, we know that $\mathsf{P}^r_{s-1} u = \mathsf{P}^r_{\mathscr{D}'} u$ for all $u \in \mathcal{K}^{s+r-1}$. Hence, all equations of this step can be understood in $\mathscr{D}'(M, \mathbb{C}^N)$ (or in $H^{-1}(M, \mathbb{C}^N)$), and we omit embeddings.

We prove by induction on $r \in \mathbb{N}$ that for all $s \in \mathbb{N}$, $s \geq r$, and all $u \in \mathcal{K}^{s+r-1}$, if $\mathsf{P}^r_{s-1} u \in \mathcal{K}^{s-r}$ then $u \in \mathcal{K}^{s+r}$. It is true for $r = 0$, as $\mathsf{P}^0_{s-1} = \mathrm{Id}_{\mathcal{K}^{s-1}}$. Take $r \in \mathbb{N}$ such that the result holds. Consider $s \in \mathbb{N}$, $s \geq r + 1$, and $u \in \mathcal{K}^{s+r}$ such that $\mathsf{P}^{r+1}_{s-1} u \in \mathcal{K}^{s-r-1}$. We want to show that $u \in \mathcal{K}^{s+r+1}$. By induction, we only need to prove that $v = \mathsf{P}^r_s u \in \mathcal{K}^{s+1-r}$.

We start by proving that $v \in H^{s+1-r}(M, \mathbb{C}^N)$. By definition of P^r_s, we know that $v \in \mathcal{K}^{s-r}$. One has $\mathsf{P}_{s-r-1} v = \mathsf{P}^{r+1}_{s-1} u$, so that $\mathsf{P}_{s-r-1} v \in \mathcal{K}^{s-r-1}$ by assumption. In particular, one has $v \in H^{s-r}(M, \mathbb{C}^N)$ and $\mathsf{P}_{s-r-1} v \in H^{s-r-1}(M, \mathbb{C}^N)$: this gives

$$\Delta v = \mathsf{P}_{s-r-1} v + (X + q)\, v \in H^{s-r-1}(M, \mathbb{C}^N),$$

with equality in $\mathscr{D}'(M, \mathbb{C}^N)$. As $s - r \geq 1$, one also has $v \in H^1_0(M, \mathbb{C}^N)$. Thus, by a standard elliptic regularity result, applied componentwise, one finds $v \in H^{s+1-r}(M, \mathbb{C}^N)$.

Now, we prove that $v = \mathsf{P}^r_s u \in \mathcal{K}^{s+1-r}$. Assume that $s-r$ is odd and write $s-r = 2\sigma+1$. By definition, the fact that $u \in \mathcal{K}^{s+r}$ gives $\mathsf{P}^k_{\mathscr{D}'} u \in H^1_0(M, \mathbb{C}^N)$ for $k \in [\![0, \sigma + r]\!]$. As $v = \mathsf{P}^r_{\mathscr{D}'} u$, this implies

$$\mathsf{P}^k_{\mathscr{D}'} v \in H^1_0(M, \mathbb{C}^N) \text{ for } k \in [\![0, \sigma]\!],$$

yielding $v \in \mathcal{K}^{s+1-r}$. Now, assume that $s - r$ is even and write $s - r = 2\sigma$. By definition, $u \in \mathcal{K}^{s+r}$ gives $\mathsf{P}^k_{\mathscr{D}'} u \in H^1_0(M, \mathbb{C}^N)$ for $k \in [\![0, \sigma + r - 1]\!]$, implying

$$\mathsf{P}^k_{\mathscr{D}'} v \in H^1_0(M, \mathbb{C}^N) \text{ for } k \in [\![0, \sigma - 1]\!].$$

As $\mathsf{P}_{s-r-1} v = \mathsf{P}^{r+1}_{s-1} u \in \mathcal{K}^{s-r-1}$, one also has $\mathsf{P}^\sigma_{\mathscr{D}'} v \in H^1_0(M, \mathbb{C}^N)$, so that $v \in \mathcal{K}^{s+1-r}$.

Step 2: Injectivity of the shift operator for $s \in \mathbb{N}$. Consider $\mu \in \mathbb{R}$. We show that for $|\mu|$ sufficiently large and for $s \in \mathbb{N}$, the operator

$$\begin{array}{rccc} \mathsf{P}_s + i\mu : & \mathcal{K}^{s+1} & \longrightarrow & \mathcal{K}^{s-1} \\ & u & \longmapsto & (\mathsf{P}_s + i\mu)\, u \end{array}$$

is one-to-one, where the embedding $\mathcal{K}^{s+1} \hookrightarrow \mathcal{K}^{s-1}$ has been omitted (as explained in the beginning of the proof, this embedding is indeed an embedding even if $s = 0$). Consider $s \in \mathbb{N}$ and $u = (u^1, \cdots, u^N) \in \mathcal{K}^{s+1}$ such that $(\mathsf{P}_s + i\mu)\, u = 0$. Write $(\pi_1, \cdots, \pi_N)$ for the projections associated with the canonical basis of $\mathbb{C}^N$. If $s = 0$, one has

$$\mathsf{P}_0 u = -i\mu u \in \mathcal{K}^1 \subset L^2(M, \mathbb{C}^N),$$

so that Step 1 gives $u \in \mathcal{K}^2$. Hence, $u \in \mathcal{K}^2$ for all $s \in \mathbb{N}$.

For $k \in [\![1, N]\!]$, one has

$$-\Delta u^k + \pi_k\, (Xu + qu) + i\mu u^k = 0.$$

Multiplication by $\overline{u^k}$, integration on M and an integration by parts give

$$\int_M \left(\left| \nabla u^k \right|^2 + i\mu |u^k|^2 \right) \mathrm{d}x = -\int_M \overline{u^k} \pi_k\, (Xu + qu)\, \mathrm{d}x. \tag{3.B.1}$$

Computing the real part and using the Cauchy-Schwarz inequality yields

$$\int_M \left| \nabla u^k \right|^2 \mathrm{d}x \leq \| Xu + qu \|_{L^2(M, \mathbb{C}^N)} \| u \|_{L^2(M, \mathbb{C}^N)} \lesssim \| u \|_{H^1(M, \mathbb{C}^N)} \| u \|_{L^2(M, \mathbb{C}^N)} .$$

By the Poincaré inequality, one obtains

$$\|u\|^2_{H^1(M,\mathbb{C}^N)} \lesssim \|u\|_{H^1(M,\mathbb{C}^N)} \|u\|_{L^2(M,\mathbb{C}^N)} .$$

Thus, there exists $C > 0$ depending only on X and q such that

$$\|u\|_{H^1(M,\mathbb{C}^N)} \leq C \|u\|_{L^2(M,\mathbb{C}^N)} . \tag{3.B.2}$$

Now, compute the imaginary part of (3.B.1) and use the Cauchy-Schwarz inequality to find

$$|\mu| \|u\|^2_{L^2(M,\mathbb{C}^N)} \leq \|Xu + qu\|_{L^2(M,\mathbb{C}^N)} \|u\|_{L^2(M,\mathbb{C}^N)} \lesssim \|u\|_{H^1(M,\mathbb{C}^N)} \|u\|_{L^2(M,\mathbb{C}^N)} .$$

Together with (3.B.2), this gives

$$|\mu| \|u\|_{L^2(M,\mathbb{C}^N)} \leq C \|u\|_{L^2(M,\mathbb{C}^N)} ,$$

with $C > 0$ depending only on X and q. Thus, for $|\mu|$ chosen sufficiently large, one has $u = 0$.

Step 3: Surjectivity of the shift operator for $s = 0$. We prove that

$$\begin{array}{rccc} \mathsf{P}_0 + i\mu : & H^1_0(M,\mathbb{C}^N) & \longrightarrow & H^{-1}(M,\mathbb{C}^N) \\ & u & \longmapsto & (\mathsf{P}_0 + i\mu)\, u \end{array}$$

is onto. By definition of the operator $\mathsf{P}_0 : H^1_0(M,\mathbb{C}^N) \to H^{-1}(M,\mathbb{C}^N)$, we need to show that for $v \in H^{-1}(M,\mathbb{C}^N)$, there exists $u \in H^1_0(M,\mathbb{C}^N)$ such that

$$-\langle \nabla u, \nabla \phi \rangle_{L^2(M,\mathbb{C}^N)} + \langle i\mu u - (X+q)u, \phi \rangle_{L^2(M,\mathbb{C}^N)} = \left\langle v, \overline{\phi} \right\rangle_{H^{-1},H^1_0}, \quad \phi \in H^1_0(M,\mathbb{C}^N),$$

where one has used the notation

$$\langle \nabla u, \nabla \phi \rangle_{L^2(M,\mathbb{C}^N)} = \sum_{k=1}^{N} \left\langle \nabla u^k, \nabla \phi^k \right\rangle_{L^2(M)} .$$

Using the Lax-Milgram theorem, it suffices to prove a coercivity inequality of the form

$$\left| -\|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + \langle i\mu u - (X+q)u, u \rangle_{L^2(M,\mathbb{C}^N)} \right| \gtrsim \|u\|^2_{H^1(M,\mathbb{C}^N)}, \quad u \in H^1_0(M,\mathbb{C}^N). \tag{3.B.3}$$

Take $u \in H^1_0(M,\mathbb{C}^N)$. Using the triangular inequality, one has

$$\begin{aligned} &\left| -\|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + \langle i\mu u - (X+q)u, u \rangle_{L^2(M,\mathbb{C}^N)} \right| \\ \geq &\left| -\|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + i\mu \|u\|^2_{L^2(M,\mathbb{C}^N)} \right| - \left| \langle (X+q)u, u \rangle_{L^2(M,\mathbb{C}^N)} \right| . \end{aligned} \tag{3.B.4}$$

Using the Cauchy-Schwarz and Poincaré inequalities, one obtains

$$\left| \langle (X+q)u, u \rangle_{L^2(M,\mathbb{C}^N)} \right| \lesssim \|u\|_{H^1(M,\mathbb{C}^N)} \|u\|_{L^2(M,\mathbb{C}^N)} \lesssim \|\nabla u\|_{L^2(M,\mathbb{C}^N)} \|u\|_{L^2(M,\mathbb{C}^N)} .$$

For $\varepsilon > 0$, one writes

$$\left| \langle (X+q)u, u \rangle_{L^2(M,\mathbb{C}^N)} \right| \leq c_1 \left(\varepsilon \|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + \frac{1}{\varepsilon} \|u\|^2_{L^2(M,\mathbb{C}^N)} \right) .$$

For the other term of (3.B.4), simply write

$$\left| - \|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + i\mu \|u\|^2_{L^2(M,\mathbb{C}^N)} \right| \geq c_2 \left(\|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + |\mu| \|u\|^2_{L^2(M,\mathbb{C}^N)} \right).$$

Thus, coming back to (3.B.4), one obtains

$$\begin{aligned} & \left| - \|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + \langle i\mu u - (X+q)u, u \rangle_{L^2(M,\mathbb{C}^N)} \right| \\ \geq\ & (c_2 - c_1\varepsilon) \|\nabla u\|^2_{L^2(M,\mathbb{C}^N)} + \left(c_2|\mu| - \frac{c_1}{\varepsilon} \right) \|u\|^2_{L^2(M,\mathbb{C}^N)}. \end{aligned}$$

We choose ε so that $c_2 - c_1\varepsilon > 0$. Then, for $|\mu|$ sufficiently large, the coercivity inequality (3.B.3) holds.

Step 4: Construction of the shift operator. We start by the case $s \in \mathbb{N}$. As above, we omit the embeddings. We proceed by induction on $s \in \mathbb{N}$ and prove that $\mathsf{P}_s + i\mu : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ is onto. By Step 3, it holds for $s = 0$. Assume that it holds for some fixed $s \in \mathbb{N}$. Take $v \in \mathcal{K}^s$. We show that there exists $u \in \mathcal{K}^{s+2}$ such that $v = (\mathsf{P}_{s+1} + i\mu)\, u$.

As $v \in \mathcal{K}^s \subset \mathcal{K}^{s-1}$, there exists $u \in \mathcal{K}^{s+1}$ such that $v = (\mathsf{P}_s + i\mu)\, u$. We apply Step 1. One has $u \in \mathcal{K}^{s+1}$, $v \in \mathcal{K}^s$, and $\mathsf{P}_s u = v - i\mu u \in \mathcal{K}^s$. As $s \geq 0$, this gives $u \in \mathcal{K}^{s+2}$.

Together with Step 2, this shows that $\mathsf{P}_s + i\mu : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ is an isomorphism for all $s \in \mathbb{N}$. Note that as P and P^* are of the same form, the operator $\mathsf{P}^* - i\mu : \mathcal{K}_*^{s+2} \longrightarrow \mathcal{K}_*^s$ is also an isomorphism for all $s \in \mathbb{N}$. Now, for $s \leq -1$, we define $\mathcal{S}_s^1 : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ as the adjoint of $\mathsf{P}^* - i\mu : \mathcal{K}_*^{-s+1} \longrightarrow \mathcal{K}_*^{-s-1}$: it is an isomorphism. Note that for $s \in \mathbb{Z}$, $\mathcal{S}_s^1 : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ is a continuous isomorphism between Hilbert spaces, so its inverse is continuous. As $\mathcal{S}_s^1$ is now defined for $s \in \mathbb{Z}$, we can define $\mathcal{S}_s^1 : \mathcal{K}^{s+1} \longrightarrow \mathcal{K}^{s-1}$ for $s \in \mathbb{R}$ by interpolation.

Step 5: Proof of *(i)*. Note that at this stage, the injectivity of the operator $\iota_{\mathcal{K}^{s+\delta} \to \mathcal{K}^s}$ is yet to be proven if $s + \delta < 0$. To that purpose, the commutativity property (3.1.6) is needed.

We start by proving (3.1.3). It suffices to prove it for $r = 1$, that is

$$\mathsf{P}_s \circ \iota_{\mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s+1}} = \iota_{\mathcal{K}^{s-1+\delta} \to \mathcal{K}^{s-1}} \circ \mathsf{P}_{s+\delta} : \mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s-1}. \tag{3.B.5}$$

By interpolation, it suffices to prove it for $s \in \mathbb{Z}$ and $\delta \in \mathbb{N}^*$. If $s \geq 0$, then (3.B.5) is true as it holds in $\mathscr{D}'(M, \mathbb{C}^N)$. Similarly, one has

$$\mathsf{P}_s^* \circ \iota_{\mathcal{K}_*^{s+1+\delta} \to \mathcal{K}_*^{s+1}} = \iota_{\mathcal{K}_*^{s-1+\delta} \to \mathcal{K}_*^{s-1}} \circ \mathsf{P}_{s+\delta}^* : \mathcal{K}_*^{s+1+\delta} \to \mathcal{K}_*^{s-1}, \quad s \geq 0. \tag{3.B.6}$$

Computing the adjoint of (3.B.6) and using (3.1.2) and (3.1.8), one finds

$$\iota_{\mathcal{K}^{-\tilde{s}-1} \to \mathcal{K}^{-\tilde{s}-1-\delta}} \circ \mathsf{P}_{-\tilde{s}} = \mathsf{P}_{-\tilde{s}-\delta} \circ \iota_{\mathcal{K}^{-\tilde{s}+1} \to \mathcal{K}^{-\tilde{s}+1-\delta}}, \quad \tilde{s} \geq 0,$$

and this gives (3.B.5) for $s = -\tilde{s} - \delta$, that is, in the case $s \leq -1$ and $s + \delta \leq 0$. Thus, (3.B.5) only remains to be proven for $s \leq -1$ and $s + \delta \geq 1$. In this case, for $u \in \mathcal{K}^{s+1+\delta} \subset \mathcal{K}^2$ and $v \in \mathcal{K}_*^{1-s} \subset \mathcal{K}_*^2$, one writes

$$\langle \mathsf{P}_s \circ \iota_{\mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s+1}} u, v \rangle_{\mathcal{K}^{s-1}, \mathcal{K}_*^{1-s}} = \langle \iota_{\mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s+1}} u, \mathsf{P}_{-s}^* v \rangle_{\mathcal{K}^{s+1}, \mathcal{K}_*^{-s-1}} = \left\langle u, \overline{\mathsf{P}_{-s}^* v} \right\rangle_{L^2(M,\mathbb{C}^N)},$$

as $s + 1 + \delta \geq 0 \geq s + 1$. Using $-s \geq 0$ and $s + \delta \geq 0$, one has

$$\langle \mathsf{P}_s \circ \iota_{\mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s+1}} u, v \rangle_{\mathcal{K}^{s-1}, \mathcal{K}_*^{1-s}} = \left\langle u, \overline{\mathsf{P}_{\mathscr{D}'}^* v} \right\rangle_{L^2(M,\mathbb{C}^N)} = \langle \mathsf{P}_{s+\delta} u, \overline{v} \rangle_{L^2(M,\mathbb{C}^N)}.$$

As $s-1+\delta \geq 0 \geq s-1$, one finds

$$\langle \mathsf{P}_s \circ \iota_{\mathcal{K}^{s+1+\delta}\to\mathcal{K}^{s+1}} u, v\rangle_{\mathcal{K}^{s-1},\mathcal{K}_*^{1-s}} = \langle \iota_{\mathcal{K}^{s-1+\delta}\to\mathcal{K}^{s-1}} \circ \mathsf{P}_{s+\delta} u, v\rangle_{\mathcal{K}^{s-1},\mathcal{K}_*^{1-s}},$$

and this gives (3.B.5).

Now, we prove that it implies the commutativity property (3.1.6). The case $r=1$, $s \in \mathbb{Z}$ and $\delta \in \mathbb{N}^*$ suffices, that is,

$$\mathcal{S}_s^1 \circ \iota_{\mathcal{K}^{s+1+\delta}\to\mathcal{K}^{s+1}} = \iota_{\mathcal{K}^{s-1+\delta}\to\mathcal{K}^{s-1}} \circ \mathcal{S}_{s+\delta}^1. \tag{3.B.7}$$

For $s \geq 0$, one has $\mathcal{S}_s^1 = \mathsf{P}_s + i\mu\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{s-1}}$ by definition, yielding (3.B.7) as a direct consequence of (3.B.5). For $s \leq -1$, one has

$$\mathcal{S}_s^1 = \left(\mathsf{P}_{-s}^* - i\mu\iota_{\mathcal{K}^{-s+1}\to\mathcal{K}^{-s-1}}\right)^*,$$

implying $\mathcal{S}_s^1 = \mathsf{P}_s + i\mu\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{s-1}}$ by (3.1.2) and (3.1.8). Hence, (3.B.7) is also a consequence of (3.B.5) in that case.

Now, we complete the proof of *(i)*. As $\mathscr{D}(M,\mathbb{C}^N) \subset \mathcal{K}^{1+\delta}$, $\mathcal{K}^{1+\delta}$ is dense in $\mathcal{K}^1$. Take $s \in \mathbb{N}$. If $s = 2\sigma$, then using (3.1.6), one can factor the map $\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}$ into

$$\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s} : \mathcal{K}^{s+\delta} \xrightarrow{\mathcal{S}_{\delta+\sigma}^{\sigma}} \mathcal{K}^{\delta} \xhookrightarrow{\iota_{\mathcal{K}^\delta\to\mathcal{K}^0}} \mathcal{K}^0 \xrightarrow{(\mathcal{S}_\sigma^\sigma)^{-1}} \mathcal{K}^s \tag{3.B.8}$$

and this proves the density of $\mathcal{K}^{s+\delta}$ in $\mathcal{K}^s$. Similarly, if $s = 2\sigma+1$ then one writes

$$\iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s} : \mathcal{K}^{s+\delta} \xrightarrow{\mathcal{S}_{1+\delta+\sigma}^{\sigma}} \mathcal{K}^{1+\delta} \xhookrightarrow{\iota_{\mathcal{K}^{1+\delta}\to\mathcal{K}^1}} \mathcal{K}^1 \xrightarrow{\left(\mathcal{S}_{1+\sigma}^\sigma\right)^{-1}} \mathcal{K}^s. \tag{3.B.9}$$

This proves that all the maps of *(i)* are embeddings. The rest of the proof of *(i)* is similar, using (3.B.8), (3.B.9). For the compactness property, note that for $0 < \delta < \frac{1}{2}$, one has $\mathcal{K}^\delta = H^\delta(M,\mathbb{C}^N)$, implying the compactness of $\iota_{\mathcal{K}^\delta\to\mathcal{K}^0}$ by the Rellich theorem.

Step 6: End of the proof of *(ii)*. Consider $s \in \mathbb{Z}$ and $r \in \mathbb{N}^*$ such that $s \leq r-1$, the case $s \geq r$ having been carried out in Step 1. Take $w \in \mathcal{K}^{s+r-1}$ and $v \in \mathcal{K}^{s-r}$ such that

$$\mathsf{P}_{s-1}^r w = \iota_{\mathcal{K}^{s-r}\to\mathcal{K}^{s-r-1}} v. \tag{3.B.10}$$

We seek $u \in \mathcal{K}^{s+r}$ such that

$$\iota_{\mathcal{K}^{s+r}\to\mathcal{K}^{s+r-1}} u = w \tag{3.B.11}$$

and

$$\mathsf{P}_s^r u = v. \tag{3.B.12}$$

Note that (3.B.11) implies

$$\begin{aligned}\iota_{\mathcal{K}^{s-r}\to\mathcal{K}^{s-r-1}}\left(\mathsf{P}_s^r u - v\right) &= \mathsf{P}_{s-1}^r \circ \iota_{\mathcal{K}^{s+r}\to\mathcal{K}^{s+r-1}} u - \iota_{\mathcal{K}^{s-r}\to\mathcal{K}^{s-r-1}} v \\ &= \mathsf{P}_{s-1}^r w - \iota_{\mathcal{K}^{s-r}\to\mathcal{K}^{s-r-1}} v = 0\end{aligned}$$

and this gives (3.B.12) since $\iota_{\mathcal{K}^{s-r}\to\mathcal{K}^{s-r-1}}$ is one-to-one. Hence, it suffices to find $u \in \mathcal{K}^{s+2r}$ such that (3.B.11) holds. Note that such a u is unique since $\iota_{\mathcal{K}^{s+2r}\to\mathcal{K}^{s+2r-1}}$ is one-to-one. If $s+r \geq 1$, the embedding could be omitted. If however $s+r \leq 0$, (3.B.11) means that u is an extension of w as a continuous linear form.

Take $\sigma \in \mathbb{N}$ such that $s-r-1+2\sigma \geq 0$. Applying $\left(\mathcal{S}_{s-r-1+\sigma}^\sigma\right)^{-1}$ to (3.B.10), one finds

$$\mathsf{P}_{s+2\sigma-1}^r \circ \left(\mathcal{S}_{s+r+\sigma-1}^\sigma\right)^{-1} w = \iota_{\mathcal{K}^{s-r+2\sigma}\to\mathcal{K}^{s-r+2\sigma-1}} \circ \left(\mathcal{S}_{s-r+\sigma}^\sigma\right)^{-1} v.$$

Apply Step 1 with $W = \left(\mathcal{S}^{\sigma}_{s+r+\sigma-1}\right)^{-1} w \in \mathcal{K}^{s+r+2\sigma-1}$ and $\tilde{s} = s + 2\sigma \geq r$: there exists $U \in \mathcal{K}^{s+r+2\sigma}$ such that

$$W = \iota_{\mathcal{K}^{s+r+2\sigma} \to \mathcal{K}^{s+r+2\sigma-1}} U.$$

Hence, one finds

$$w = \mathcal{S}^{\sigma}_{s+r+\sigma-1} \circ \iota_{\mathcal{K}^{s+r+2\sigma} \to \mathcal{K}^{s+r+2\sigma-1}} U = \iota_{\mathcal{K}^{s+r} \to \mathcal{K}^{s+r-1}} \circ \mathcal{S}^{\sigma}_{s+r+\sigma} U,$$

and this is (3.B.11) with $u = \mathcal{S}^{\sigma}_{s+r+\sigma} U$.

We now prove (3.1.4) by induction on $r \in \mathbb{N}^*$. We start with the case $r = 1$. Take $s \in \mathbb{R}$ and $u \in \mathcal{K}^{s+1}$, and write

$$\|u\|_{\mathcal{K}^{s+1}} = \left\| \left(\mathcal{S}^1_s\right)^{-1} \circ \mathcal{S}^1_s u \right\|_{\mathcal{K}^{s+1}} \lesssim \left\| \mathcal{S}^1_s u \right\|_{\mathcal{K}^{s-1}} .$$

Using the definition of $\mathcal{S}^1_s$ and the triangular inequality, one obtains

$$\|u\|_{\mathcal{K}^{s+1}} \lesssim \|\mathsf{P}_s u\|_{\mathcal{K}^{s-1}} + \|\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s-1}} u\|_{\mathcal{K}^{s-1}} ,$$

a better estimate than (3.1.4) in the case $r = 1$, since one has

$$\|\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s-1}} u\|_{\mathcal{K}^{s-1}} = \|\iota_{\mathcal{K}^{s} \to \mathcal{K}^{s-1}} \circ \iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s}} u\|_{\mathcal{K}^{s-1}} \lesssim \|\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s}} u\|_{\mathcal{K}^{s}} .$$

Now, we take $r \in \mathbb{N}^*$ such that (3.1.4) holds, and we prove that (3.1.4) also holds for $r + 1$. Take $s \in \mathbb{R}$ and $u \in \mathcal{K}^{s+r+1}$. We want to show that

$$\|u\|_{\mathcal{K}^{s+r+1}} \lesssim \left\| \mathsf{P}^{r+1}_s u \right\|_{\mathcal{K}^{s-r-1}} + \|\iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r}} u\|_{\mathcal{K}^{s+r}} .$$

By induction, one has

$$\|u\|_{\mathcal{K}^{s+r+1}} \lesssim \|\mathsf{P}^{r}_{s+1} u\|_{\mathcal{K}^{s+1-r}} + \|\iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r}} u\|_{\mathcal{K}^{s+r}} .$$

The case $r = 1$ gives

$$\|\mathsf{P}^{r}_{s+1} u\|_{\mathcal{K}^{s+1-r}} \lesssim \|\mathsf{P}_{s-r} \circ \mathsf{P}^{r}_{s+1} u\|_{\mathcal{K}^{s-r-1}} + \|\iota_{\mathcal{K}^{s-r+1} \to \mathcal{K}^{s-r}} \circ \mathsf{P}^{r}_{s+1} u\|_{\mathcal{K}^{s-r}} .$$

Using the definition of P^{r+1}_s and the commutativity property (3.1.3), one finds

$$\begin{aligned} \|\mathsf{P}^{r}_{s+1} u\|_{\mathcal{K}^{s+1-r}} &\lesssim \left\| \mathsf{P}^{r+1}_s u \right\|_{\mathcal{K}^{s-r-1}} + \|\mathsf{P}^{r}_s \circ \iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r}} u\|_{\mathcal{K}^{s-r}} \\ &\lesssim \left\| \mathsf{P}^{r+1}_s u \right\|_{\mathcal{K}^{s-r-1}} + \|\iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r}} u\|_{\mathcal{K}^{s+r}} . \end{aligned}$$

This completes the proof of *(ii)*.

Step 7: End of the proof of *(iii)*. Consider $s \in \mathbb{R}$. We prove (3.1.7), that is,

$$\|\left(\mathsf{P}^{r}_s - \mathcal{S}^{r}_s\right) u\|_{\mathcal{K}^{s-r}} \lesssim \|\iota_{\mathcal{K}^{s+r} \to \mathcal{K}^{s+r-1}} u\|_{\mathcal{K}^{s+r-1}} ,$$

for $r \in \mathbb{N}^*$ and $u \in \mathcal{K}^{s+r}$. Note that this is true for $r = 1$: one has

$$\left\| \left(\mathsf{P}_s - \mathcal{S}^1_s\right) u \right\|_{\mathcal{K}^{s-1}} = \|\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s-1}} u\|_{\mathcal{K}^{s-1}} \lesssim \|\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s}} u\|_{\mathcal{K}^{s}} ,$$

for all $u \in \mathcal{K}^{s+1}$. Now, assume that the result is true for some $r \in \mathbb{N}^*$, and take $u \in \mathcal{K}^{s+r+1}$. We write

$$\begin{aligned}
\mathsf{P}_s^{r+1} - \mathcal{S}_s^{r+1} &= \mathsf{P}_{s-r} \circ \mathsf{P}_{s+1}^r - \mathcal{S}_{s-r}^1 \circ \mathcal{S}_{s+1}^r \\
&= \mathsf{P}_{s-r} \circ \mathsf{P}_{s+1}^r - (\mathsf{P}_{s-r} + i\mu\iota_{\mathcal{K}^{s-r+1} \to \mathcal{K}^{s-r-1}}) \circ \mathcal{S}_{s+1}^r \\
&= \mathsf{P}_{s-r} \circ (\mathsf{P}_{s+1}^r - \mathcal{S}_{s+1}^r) - i\mu\iota_{\mathcal{K}^{s-r+1} \to \mathcal{K}^{s-r-1}} \circ \mathcal{S}_{s+1}^r \\
&= \mathsf{P}_{s-r} \circ (\mathsf{P}_{s+1}^r - \mathcal{S}_{s+1}^r) - i\mu\mathcal{S}_{s-1}^r \circ \iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r-1}}
\end{aligned}$$

and we use the continuity of P_{s-r} and $\mathcal{S}_{s-1}^r$ to find

$$\left\| \left(\mathsf{P}_s^{r+1} - \mathcal{S}_s^{r+1} \right) u \right\|_{\mathcal{K}^{s-r-1}} \lesssim \left\| \left(\mathsf{P}_{s+1}^r - \mathcal{S}_{s+1}^r \right) u \right\|_{\mathcal{K}^{s-r+1}} + \left\| \iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r-1}} u \right\|_{\mathcal{K}^{s+r-1}} .$$

By induction, we get

$$\left\| \left(\mathsf{P}_s^{r+1} - \mathcal{S}_s^{r+1} \right) u \right\|_{\mathcal{K}^{s-r-1}} \lesssim \left\| \iota_{\mathcal{K}^{s+r+1} \to \mathcal{K}^{s+r}} u \right\|_{\mathcal{K}^{s+r}} ,$$

and this completes the proof.

3.B.2 Solutions of the wave equations

Proof of Theorem 3.1.12

The proof of Theorem 3.1.12 is organized as follows. First, we check that the assumptions of the Hille-Yosida theorem are fulfilled, to construct the solution for $s \geq 0$, with the regularity result

$$u \in \bigcap_{k=0}^{s+1} \mathscr{C}^k(\mathbb{R}, \mathcal{K}^{s+1-k}).$$

Second, we construct the solution for $s < 0$ by using the shift operator of Proposition 3.1.8-*(iii)*. Third, we prove Theorem 3.1.12-*(ii)* and the regularity result

$$u \in \bigcap_{k=0}^{\infty} \mathscr{C}^k(\mathbb{R}, \mathcal{K}^{s+1-k}), \quad s \in \mathbb{R}.$$

Fourth, we construct the solution of the wave equation with a source term as is Theorem 3.1.12-*(iv)*. Finally, we prove the results about the normal derivative. Note that by interpolation, we can always assume that $s \in \mathbb{Z}$.

Step 1: Construction of the solution for $s \geq 0$ with the Hille-Yosida theorem. We write $\mathcal{X}$ for the Hilbert space $\mathcal{K}^1 \times \mathcal{K}^0$ and $A : \mathcal{X} \longrightarrow \mathcal{X}$ for the unbounded operator

$$A = \begin{pmatrix} 0 & -\mathrm{Id} \\ -\mathsf{P}_1 & 0 \end{pmatrix}$$

with domain $D(A) = \mathcal{K}^2 \times \mathcal{K}^1$. We also write $U = \binom{u}{\partial_t u}$ so that the wave equation reads (formally at first)

$$\partial_t U + AU = 0 \quad \text{with} \quad U(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}.$$

Lemma 3.B.1 (The irerated domains of A)**.** *For $k \in \mathbb{N}$, one has*

$$D(A^k) = \mathcal{K}^{k+1} \times \mathcal{K}^k. \tag{3.B.13}$$

Proof. By definition, one has $D(A^0) = \mathcal{K}^1 \times \mathcal{K}^0$, $\quad D(A^1) = \mathcal{K}^2 \times \mathcal{K}^1$, and

$$D(A^{k+1}) = \left\{U \in D(A^k), A^k U \in D(A)\right\}, \quad k \in \mathbb{N}^*.$$

Note that we can omit the embeddings here, as we are only working with subspaces of $L^2(M, \mathbb{C}^N)$. Consider $k \in \mathbb{N}$ such that (3.B.13) is true. The fact that

$$\mathcal{K}^{k+2} \times \mathcal{K}^{k+1} \subset D(A^{k+1})$$

follows from the definitions. Take $U = (u^0, u^1) \in D(A^{k+1})$. One has $U \in D(A^k)$ and $AU \in D(A^k)$. This gives $u^1 \in \mathcal{K}^{k+1}$, $u^0 \in \mathcal{K}^{k+1}$ and $\mathsf{P}u^0 \in \mathcal{K}^k$. By Proposition 3.1.8-*(ii)*, one finds $u^0 \in \mathcal{K}^{k+2}$ and so $U \in \mathcal{K}^{k+2} \times \mathcal{K}^{k+1}$. ☐

Lemma 3.B.2 (Assumptions of the Hille-Yosida theorem)**.** *The operator A is closed, and $D(A)$ is dense in $\mathcal{X}$. There exists $\omega \in \mathbb{R}$ such that the resolvent set of A contains $(-\infty, -\omega)$ and such that for all $\lambda < -\omega$ and all $k \in \mathbb{N}^*$, one has*

$$\left\|R_\lambda(A)^k\right\|_{\mathcal{L}(\mathcal{X})} \leq \frac{1}{|\omega + \lambda|^k},$$

where $R_\lambda(A) = (\lambda \mathrm{Id}_{\mathcal{X}} - A)^{-1}$. The same is true for the operator $-A$.

The proof of Lemma 3.B.2 is given below. Consider $s \in \mathbb{N}$ and

$$\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{K}^{s+1} \times \mathcal{K}^s = D(A^s).$$

Using the Hille-Yosida theorem (see for example [Vra03], Theorem 3.3.1 and Corollary 2.4.1), together with Lemma 3.B.1 and Lemma 3.B.2, one obtains that there exists a unique solution

$$U \in \bigcap_{l=0}^{s} \mathscr{C}^l(\mathbb{R}, D(A^{s-l})) = \bigcap_{l=0}^{s} \mathscr{C}^l(\mathbb{R}, \mathcal{K}^{s-l+1} \times \mathcal{K}^{s-l})$$

of

$$\left\{ \begin{array}{rcl} \partial_t U + AU &=& 0 \\ U(0) &=& (u^0, u^1) \end{array} \right. .$$

One can check that in fact, U is of the form $(u, \partial_t u)$, with

$$u \in \bigcap_{l=0}^{s+1} \mathscr{C}^l(\mathbb{R}, \mathcal{K}^{s-l+1}).$$

Note that this gives $u \in H^{s+1}((0,T) \times M, \mathbb{C}^N)$.

Proof of Lemma 3.B.2. The fact that $D(A)$ is dense in $\mathcal{X}$ is well-known. We prove that A is closed. Consider a sequence $\left((u_n^0, u_n^1)\right)_n$ of elements of $D(A)$ such that

$$\begin{pmatrix} u_n^0 \\ u_n^1 \end{pmatrix} \xrightarrow{n \to \infty} \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{X} \quad \text{and} \quad A \begin{pmatrix} u_n^0 \\ u_n^1 \end{pmatrix} \xrightarrow{n \to \infty} \begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \in \mathcal{X}.$$

By definition one has

$$u_n^1 \xrightarrow{n \to \infty} -v^0 \quad \text{in } \mathcal{K}^1 \quad \text{and} \quad \mathsf{P}_1 u_n^0 \xrightarrow{n \to \infty} -v^1 \quad \text{in } \mathcal{K}^0.$$

In particular, this gives $u^1 = -v^0 \in \mathcal{K}^1$. One also has $u_n^0 \longrightarrow u^0$ in $\mathcal{K}^1$ yielding $\mathsf{P}_0 u_n^0 \longrightarrow \mathsf{P}_0 u^0$ in $\mathcal{K}^{-1}$. This implies $\mathsf{P}_0 u^0 = -v^1 \in \mathcal{K}^0$, so the ellipticity estimate for P (Proposition 3.1.8-*(ii)*) gives $u^0 \in \mathcal{K}^2$. Thus, A is closed.

Now, we show that there exists $\omega \in \mathbb{R}$ such that if $\lambda \in \mathbb{R}$ satisfies $|\lambda| + \omega > 0$ then

$$\|(\lambda \mathrm{Id} - A)\, U\|_{\mathcal{X}} \geq (|\lambda| + \omega)\, \|U\|_{\mathcal{X}} \tag{3.B.14}$$

for all $U \in D(A)$. Recall that $\mathcal{X}$ is a Hilbert space for scalar product associated with the norm

$$\left\|\left(u^0, u^1\right)\right\|_{\mathcal{X}}^2 = \left\|\nabla u^0\right\|_{L^2(M,\mathbb{C}^N)}^2 + \left\|u^1\right\|_{L^2(M,\mathbb{C}^N)}^2 .$$

Consider $U = (u^0, u^1) \in D(A)$, and write

$$\begin{aligned} \|(\lambda \mathrm{Id} - A)\, U\|_{\mathcal{X}}^2 &= |\lambda|^2 \|U\|_{\mathcal{X}}^2 + \|AU\|_{\mathcal{X}}^2 - 2\,\mathrm{Re}\, \langle \lambda U, AU\rangle_{\mathcal{X}} \\ &\geq |\lambda|^2 \|U\|_{\mathcal{X}}^2 - 2\lambda\, \mathrm{Re}\, \langle U, AU\rangle_{\mathcal{X}} . \end{aligned} \tag{3.B.15}$$

By definition, one has

$$\begin{aligned} \mathrm{Re}\, \langle U, AU\rangle_{\mathcal{X}} &= -\,\mathrm{Re} \left\langle \nabla u^0, \nabla u^1 \right\rangle_{L^2(M,\mathbb{C}^N)} - \mathrm{Re} \left\langle \mathsf{P}_1 u^0, u^1 \right\rangle_{L^2(M,\mathbb{C}^N)} \\ &= -\,\mathrm{Re} \left\langle \nabla u^0, \nabla u^1 \right\rangle_{L^2(M,\mathbb{C}^N)} - \mathrm{Re} \left\langle (\Delta - X - q)\, u^0, u^1 \right\rangle_{L^2(M,\mathbb{C}^N)} . \end{aligned}$$

Integrating by parts, one finds

$$\mathrm{Re}\, \langle U, AU\rangle_{\mathcal{X}} = \mathrm{Re} \left\langle (X + q)\, u^0, u^1 \right\rangle_{L^2(M,\mathbb{C}^N)} ,$$

and using the Cauchy-Schwarz and Poincaré inequalities, this gives

$$|\mathrm{Re}\, \langle U, AU\rangle_{\mathcal{X}}| \lesssim \left\|\nabla u^0\right\|_{L^2(M,\mathbb{C}^N)} \left\|u^1\right\|_{L^2(M,\mathbb{C}^N)} \lesssim \|U\|_{\mathcal{X}}^2.$$

Coming back to (3.B.15), one finds that there exists $c > 0$ such that

$$\|(\lambda \mathrm{Id} - A)\, U\|_{\mathcal{X}}^2 \geq \left(|\lambda|^2 - c|\lambda|\right) \|U\|_{\mathcal{X}}^2 \geq (|\lambda| - c)^2\, \|U\|_{\mathcal{X}}^2$$

for $|\lambda| > c$. This gives (3.B.14) with $\omega = -c$.

Next, we show that the operator $\lambda \mathrm{Id} - A : D(A) \to \mathcal{X}$ is onto if $|\lambda|$ is sufficiently large. Consider $(v^0, v^1) \in \mathcal{X}$. We seek $(u^0, u^1) \in D(A)$ such that

$$(\lambda \mathrm{Id} - A) \begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \begin{pmatrix} v^0 \\ v^1 \end{pmatrix},$$

which reads

$$\left\{ \begin{array}{rcl} u^1 &=& -v^0 + \lambda u^0 \\ (\Delta - X - q)\, u^0 - \lambda^2 u^0 &=& -v^1 - \lambda v^0 \end{array} \right. .$$

Using the Lax-Milgram theorem and the ellipticity of P, it follows if the coercivity inequality

$$\left| \left\|\nabla u^0\right\|_{L^2(M,\mathbb{C}^N)}^2 + \lambda^2 \left\|u^0\right\|_{L^2(M,\mathbb{C}^N)}^2 + \left\langle (X+q) u^0, u^0 \right\rangle_{L^2(M,\mathbb{C}^N)} \right| \gtrsim \|\nabla u^0\|_{L^2(M,\mathbb{C}^N)}^2 \tag{3.B.16}$$

is proven. As above, for $\varepsilon > 0$, write

$$\left| \left\langle (X+q) u^0, u^0 \right\rangle_{L^2(M,\mathbb{C}^N)} \right| \lesssim \varepsilon \|\nabla u^0\|_{L^2(M,\mathbb{C}^N)}^2 + \frac{1}{\varepsilon} \|u^0\|_{L^2(M,\mathbb{C}^N)}^2,$$

and find, for $c_1 > 0$, $c_2 > 0$, by choosing ε sufficiently small,

$$\begin{aligned}&\left|\left\|\nabla u^0\right\|^2_{L^2(M,\mathbb{C}^N)} + \lambda^2\left\|u^0\right\|^2_{L^2(M,\mathbb{C}^N)} + \left\langle (X+q)u^0, u^0\right\rangle_{L^2(M,\mathbb{C}^N)}\right| \\ &\geq c_1\left\|\nabla u^0\right\|^2_{L^2(M,\mathbb{C}^N)} + \left(\lambda^2 - c_2\right)\left\|u^0\right\|^2_{L^2(M,\mathbb{C}^N)} \\ &\gtrsim \|\nabla u^0\|^2_{L^2(M,\mathbb{C}^N)}\end{aligned}$$

for $|\lambda|$ sufficiently large. This gives (3.B.16).

At this stage, one has proved that there exists $\omega \in \mathbb{R}$ such that $(-\infty, -\omega)$ is contained in the resolvent set of both A and $-A$, and for λ such that $|\lambda| + \omega > 0$, one has

$$\|R_\lambda(A)\|_{\mathcal{L}(\mathcal{X})} \leq \frac{1}{|\omega + |\lambda||}.$$

This completes the proof of Lemma 3.B.2. □

Step 2: Construction of the solution for $s < 0$. The idea of this step is to construct the solution with the shift operator and the solution in $\mathcal{K}^3 \times \mathcal{K}^2$ or $\mathcal{K}^2 \times \mathcal{K}^1$, depending of the parity of s. Consider $s \in \mathbb{Z}$, $s < 0$. There exist $\sigma \in \mathbb{N}^*$ and $\alpha \in \{1, 2\}$ such that $s = -2\sigma + \alpha$. Take $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. Let $(\tilde{u}^0, \tilde{u}^1)$ be the unique element of $\mathcal{K}^{\alpha+1} \times \mathcal{K}^\alpha$ such that

$$\left(u^0, u^1\right) = \left(\mathcal{S}^\sigma_{s+1+\sigma}\tilde{u}^0, \mathcal{S}^\sigma_{s+\sigma}\tilde{u}^1\right).$$

Let $\tilde{u}$ be the solution associated with $(\tilde{u}^0, \tilde{u}^1)$ defined above. Set $u(t) = \mathcal{S}^\sigma_{s+1+\sigma}\tilde{u}(t)$, for $t \in \mathbb{R}$. We will refer to u as the solution of

$$\begin{cases} \partial_t^2 u - \mathsf{P}u = 0 & \text{in } \mathbb{R} \times M, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (u^0, u^1) & \text{in } M, \\ u = 0 & \text{on } \mathbb{R} \times \partial M. \end{cases}$$

We prove that

$$u \in \mathscr{C}^0(\mathbb{R}, \mathcal{K}^{s+1}) \cap \mathscr{C}^1(\mathbb{R}, \mathcal{K}^s) \cap \mathscr{C}^2(\mathbb{R}, \mathcal{K}^{s-1}),$$

and that $\partial_t^2 u = \mathsf{P}_s u$. In particular, if $s = -1$ or -2, it implies $u \in H^{s+1}((0,T) \times M, \mathbb{C}^N)$ for $T > 0$, using the embedding

$$\mathscr{C}^0((0,T), H^{-1}(M, \mathbb{C}^N)) \longhookrightarrow H^{-1}((0,T) \times M, \mathbb{C}^N)$$

for the case $s = -2$.

Continuity. The continuity of u is a consequence of that of the shift operator $\mathcal{S}^\sigma_{s+1+\sigma}$ and that of $\tilde{u}$. In addition, for $T > 0$, one has

$$\|u\|_{L^\infty([0,T],\mathcal{K}^{s+1})} \lesssim \|\tilde{u}\|_{L^\infty([0,T],\mathcal{K}^{\alpha+1})} \lesssim \left\|\left(\tilde{u}^0, \tilde{u}^1\right)\right\|_{\mathcal{K}^{\alpha+1}\times\mathcal{K}^{\alpha+1}} \lesssim \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}.$$

First-order time-derivative. We show that

$$\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^s}\left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon}\right) \xrightarrow{\varepsilon\to 0} \mathcal{S}^\sigma_{s+\sigma}\partial_t\tilde{u}(t) \in \mathcal{K}^s, \quad t \in \mathbb{R}.$$

By Proposition 3.1.8, one has

$$\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^s} \circ \mathcal{S}^\sigma_{s+1+\sigma} = \mathcal{S}^\sigma_{s+\sigma} \circ \iota_{\mathcal{K}^{\alpha+1}\to\mathcal{K}^\alpha} = \mathcal{S}^\sigma_{s+\sigma},$$

where the last embedding can be omitted as it is just an inclusion. Hence, for $t \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$, we can write

$$\begin{aligned}\left\| \iota_{\mathcal{K}^{s+1}\to\mathcal{K}^s}\left(\frac{u(t+\varepsilon)-u(t)}{\varepsilon}\right) - \mathcal{S}^\sigma_{s+\sigma}\partial_t\tilde{u}(t)\right\|_{\mathcal{K}^s} &= \left\|\mathcal{S}^\sigma_{s+\sigma}\left(\frac{\tilde{u}(t+\varepsilon)-\tilde{u}(t)}{\varepsilon} - \partial_t\tilde{u}(t)\right)\right\|_{\mathcal{K}^s} \\ &\lesssim \left\|\frac{\tilde{u}(t+\varepsilon)-\tilde{u}(t)}{\varepsilon} - \partial_t\tilde{u}(t)\right\|_{\mathcal{K}^\alpha} \\ &\xrightarrow{\varepsilon\to 0} 0.\end{aligned}$$

As above, one also has

$$\|\partial_t u\|_{L^\infty([0,T],\mathcal{K}^s)} \lesssim \|\partial_t \tilde{u}\|_{L^\infty([0,T],\mathcal{K}^\alpha)} \lesssim \left\|\left(\tilde{u}^0,\tilde{u}^1\right)\right\|_{\mathcal{K}^{\alpha+1}\times\mathcal{K}^{\alpha+1}} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}, \quad T>0.$$

Second-order time-derivative. As above, one shows that $u \in \mathscr{C}^2(\mathbb{R},\mathcal{K}^{s-1})$, with

$$\partial_t^2 u(t) = \mathcal{S}^\sigma_{s-1+\sigma}\partial_t^2\tilde{u}(t), \quad t \in \mathbb{R},$$

and

$$\left\|\partial_t^2 u\right\|_{L^\infty([0,T],\mathcal{K}^{s-1})} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}, \quad T>0.$$

In particular, for $t \in \mathbb{R}$, one finds $\partial_t^2 u(t) = \mathcal{S}^\sigma_{s-1+\sigma}\partial_t^2\tilde{u}(t) = \mathcal{S}^\sigma_{s-1+\sigma}\mathsf{P}_\alpha\tilde{u}(t)$, and by Proposition 3.1.8-*(iii)*, this gives

$$\partial_t^2 u(t) = \mathsf{P}_s\mathcal{S}^\sigma_{s+1+\sigma}\tilde{u}(t) = \mathsf{P}_s u(t).$$

Step 3: Regularity, uniqueness and approximation. Here, we prove the uniqueness result of Theorem 3.1.12 for $s \in \mathbb{Z}$, the regularity result *(i)*, and then *(ii)* and the uniqueness result of Theorem 3.1.12 for $s \in \mathbb{R}$.

Uniqueness for $s \in \mathbb{Z}$. For $s \in \mathbb{N}$, the uniqueness result of Theorem 3.1.12 is given by the Hille-Yosida Theorem. Consider $s \in \mathbb{Z}$, $s < 0$, and $\left(u^0,u^1\right) \in \mathcal{K}^{s+1}\times\mathcal{K}^s$. Let $v \in \mathscr{C}^0(\mathbb{R},\mathcal{K}^{s+1}) \cap \mathscr{C}^1(\mathbb{R},\mathcal{K}^s) \cap \mathscr{C}^2(\mathbb{R},\mathcal{K}^{s-1})$ be such that $\partial_t^2 v(t) = \mathsf{P}_s v(t)$ for all $t \in \mathbb{R}$, and $(v(0,\cdot),\partial_t v(0,\cdot)) = \left(u^0,u^1\right)$ in M. As above, let $\sigma \in \mathbb{N}^*$ and $\alpha \in \{1,2\}$ be such that $s = -2\sigma+\alpha$. For $t \in \mathbb{R}$, define

$$\tilde{v}(t) = \left(\mathcal{S}^\sigma_{s+1+\sigma}\right)^{-1} v(t).$$

As in Step 1, one shows that $\tilde{v} \in \mathscr{C}^0(\mathbb{R},\mathcal{K}^{\alpha+1}) \cap \mathscr{C}^1(\mathbb{R},\mathcal{K}^\alpha) \cap \mathscr{C}^2(\mathbb{R},\mathcal{K}^{\alpha-1})$, with $\partial_t\tilde{v}(t) = \left(\mathcal{S}^\sigma_{s+\sigma}\right)^{-1}\partial_t v(t)$ and $\partial_t^2\tilde{v}(t) = \left(\mathcal{S}^\sigma_{s-1+\sigma}\right)^{-1}\partial_t^2 v(t)$. By Proposition 3.1.8-*(iii)*, one finds

$$\partial_t^2\tilde{v}(t) = \left(\mathcal{S}^\sigma_{s-1+\sigma}\right)^{-1}\circ\mathsf{P}_s v(t) = \mathsf{P}_\alpha\circ\left(\mathcal{S}^\sigma_{s+1+\sigma}\right)^{-1} v(t) = \mathsf{P}_\alpha\tilde{v}(t).$$

Hence, the functions $\tilde{v}$ and $\tilde{u}$ (defined in the previous step) satisfy the same wave equation. Using the uniqueness in the case $s \geq 0$, one finds $\tilde{v} = \tilde{u}$, and so

$$v(t) = \mathcal{S}^\sigma_{s+1+\sigma}\tilde{v}(t) = \mathcal{S}^\sigma_{s+1+\sigma}\tilde{u}(t) = u(t).$$

The regularity result *(i)*. Take $s \in \mathbb{Z}$ and $\left(u^0,u^1\right) \in \mathcal{K}^{s+1}\times\mathcal{K}^s$. We know that

$$u \in \mathscr{C}^0(\mathbb{R},\mathcal{K}^{s+1}) \cap \mathscr{C}^1(\mathbb{R},\mathcal{K}^s) \cap \mathscr{C}^2(\mathbb{R},\mathcal{K}^{s-1}),$$

and we show that

$$u \in \bigcap_{k\in\mathbb{N}} \mathscr{C}^k(\mathbb{R},\mathcal{K}^{s+1-k}).$$

One can prove that

$$\mathsf{P}_s u \in \mathscr{C}^0(\mathbb{R}, \mathcal{K}^{s-1}) \cap \mathscr{C}^1(\mathbb{R}, \mathcal{K}^{s-2}) \cap \mathscr{C}^2(\mathbb{R}, \mathcal{K}^{s-3}),$$

and $\partial_t^2 (\mathsf{P}_s u) = \mathsf{P}_{s-2} \circ \mathsf{P}_s u$. For example, to show that $\mathsf{P}_s u \in \mathscr{C}^1(\mathbb{R}, \mathcal{K}^{s-2})$, write

$$\begin{aligned}
&\left\| \iota_{\mathcal{K}^{s-1} \to \mathcal{K}^{s-2}} \left(\frac{\mathsf{P}_s u(t+\varepsilon) - \mathsf{P}_s u(t)}{\varepsilon} \right) - \mathsf{P}_{s-1} \partial_t u(t) \right\|_{\mathcal{K}^{s-2}} \\
&= \left\| \mathsf{P}_{s-1} \left(\iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s}} \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon} \right) - \partial_t u(t) \right) \right\|_{\mathcal{K}^{s-2}} \\
&\lesssim \left\| \iota_{\mathcal{K}^{s+1} \to \mathcal{K}^{s}} \left(\frac{u(t+\varepsilon) - u(t)}{\varepsilon} \right) - \partial_t u(t) \right\|_{\mathcal{K}^{s}} \\
&\xrightarrow{\varepsilon \to 0} 0.
\end{aligned}$$

By uniqueness, $\mathsf{P}_s u = \partial_t^2 u$ is the solution associated with the initial data $\left(\mathsf{P}_s u^0, \mathsf{P}_{s-1} u^1\right)$, implying

$$u \in \mathscr{C}^3(\mathbb{R}, \mathcal{K}^{s-2}) \cap \mathscr{C}^4(\mathbb{R}, \mathcal{K}^{s-3}).$$

One also has

$$\left\|\partial_t^3 u\right\|_{L^\infty((0,T),\mathcal{K}^{s-2})} = \left\|\mathsf{P}_{s-1} \partial_t u\right\|_{L^\infty((0,T),\mathcal{K}^{s-2})} \lesssim \left\|\partial_t u\right\|_{L^\infty((0,T),\mathcal{K}^{s})} \lesssim \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s},$$

and

$$\left\|\partial_t^4 u\right\|_{L^\infty((0,T),\mathcal{K}^{s-3})} = \left\|\mathsf{P}_{s-1}^2 u\right\|_{L^\infty((0,T),\mathcal{K}^{s-3})} \lesssim \left\|u\right\|_{L^\infty((0,T),\mathcal{K}^{s+1})} \lesssim \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s},$$

for $T > 0$. The result follows by iteration.

Proof of *(ii)*. Take $s \in \mathbb{Z}$ and $\delta > 0$. For $\left(u^0, u^1\right) \in \mathcal{K}^{s+\delta+1} \times \mathcal{K}^{s+\delta}$, if u is the solution with initial data $\left(u^0, u^1\right)$, then arguing as above, one has

$$\iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u \in \mathscr{C}^0(\mathbb{R}, \mathcal{K}^{s+1}) \cap \mathscr{C}^1(\mathbb{R}, \mathcal{K}^{s}) \cap \mathscr{C}^2(\mathbb{R}, \mathcal{K}^{s-1}),$$

with $\partial_t \left(\iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u\right) = \iota_{\mathcal{K}^{s+\delta} \to \mathcal{K}^{s}} \partial_t u$ and

$$\partial_t^2 \left(\iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u\right) = \iota_{\mathcal{K}^{s+\delta-1} \to \mathcal{K}^{s-1}} \partial_t^2 u = \iota_{\mathcal{K}^{s+\delta-1} \to \mathcal{K}^{s-1}} \circ \mathsf{P}_{s+\delta} u = \mathsf{P}_s \circ \iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u.$$

By uniqueness, one finds

$$\iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u = \tilde{u},$$

where $\tilde{u}$ is the solution associated with $\left(\iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u^0, \iota_{\mathcal{K}^{s+\delta} \to \mathcal{K}^{s}} u^1\right)$. By interpolation, this is in fact true for all $s \in \mathbb{R}$.

We prove the approximation result of *(ii)*. Consider $s \in \mathbb{R}$, $\delta > 0$, and $\left(u^0, u^1\right) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. Let u be the solution with initial data $\left(u^0, u^1\right)$. By Proposition 3.1.8-*(i)*, there exists a sequence $\left((\tilde{u}_k^0, \tilde{u}_k^1)\right)_{k \in \mathbb{N}}$ of elements of $\mathcal{K}^{s+1+\delta} \times \mathcal{K}^{s+\delta}$ such that, writing $u_k^0 = \iota_{\mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s+1}} \tilde{u}_k^0$ and $u_k^1 = \iota_{\mathcal{K}^{s+\delta} \to \mathcal{K}^{s}} \tilde{u}_k^1$, one has

$$\left(u_k^0, u_k^1\right) \xrightarrow{k \to \infty} \left(u^0, u^1\right) \text{ in } \mathcal{K}^{s+1} \times \mathcal{K}^s.$$

Denote $\tilde{u}_k \in \mathscr{C}^0(\mathbb{R}, \mathcal{K}^{s+1+\delta}) \cap \mathscr{C}^1(\mathbb{R}, \mathcal{K}^{s+\delta})$ and $u_k \in \mathscr{C}^0(\mathbb{R}, \mathcal{K}^{s+1}) \cap \mathscr{C}^1(\mathbb{R}, \mathcal{K}^{s})$ the solutions with initial data $\left(\tilde{u}_k^0, \tilde{u}_k^1\right)$ and $\left(u_k^0, u_k^1\right)$. One has $\iota_{\mathcal{K}^{s+1+\delta} \to \mathcal{K}^{s+1}} \tilde{u}_k = u_k$ for all $k \in \mathbb{N}$, and

$$\sum_{j=0}^{2} \left\|\partial_t^j (u_k - u)\right\|_{L^\infty([0,T],\mathcal{K}^{s+1-j})} \lesssim \left\|\left(u_k^0, u_k^1\right) - \left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s} \xrightarrow{k \to \infty} 0,$$

for $T > 0$. Hence, one obtains

$$\iota_{\mathcal{K}^{s+1+\delta}\to\mathcal{K}^{s+1}}\tilde{u}_k \xrightarrow{k\to\infty} u,$$

in $\mathscr{C}^0([0,T],\mathcal{K}^{s+1})\cap\mathscr{C}^1([0,T],\mathcal{K}^{s})\cap\mathscr{C}^2([0,T],\mathcal{K}^{s-1})$. In Step 5, we define the normal derivative of a solution, and we show that

$$\|\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}.$$

for all $(u^0,u^1)\in\mathcal{K}^{s+1}\times\mathcal{K}^s$. Hence, we also have

$$\|\partial_\nu (u_k) - \partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\left(u_k^0,u_k^1\right)-\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}$$

and so

$$\partial_\nu (u_k) \xrightarrow{k\to\infty} \partial_\nu u,$$

in $H^s((0,T)\times\partial M,\mathbb{C}^N)$.

Uniqueness for $s\in\mathbb{R}$. Lastly, we show that *(ii)* implies the uniqueness result of Theorem 3.1.12 for $s\in\mathbb{R}$. If u and v are two solutions of the wave equation starting from $(u^0,u^1)\in\mathcal{K}^{s+1}\times\mathcal{K}^s$, then using the uniqueness result for $\tilde{s}\in\mathbb{Z}$ such that $s>\tilde{s}$, one has

$$\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{\tilde{s}+1}}u = \iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{\tilde{s}+1}}v.$$

This gives $u=v$, as the map $\iota_{\mathcal{K}^{s+1}\to\mathcal{K}^{\tilde{s}+1}}$ is one-to-one.

Step 4: Study of the Duhamel term. In this step, we construct the solution of the wave equation with a source term. We define the solution of

$$\left\{\begin{array}{rcll} \partial_t^2 u - \mathsf{P}u &=& F & \text{in } (0,T)\times M,\\ (u(0,\cdot),\partial_t u(0,\cdot)) &=& 0 & \text{in } M,\\ u &=& 0 & \text{on } (0,T)\times\partial M, \end{array}\right. \tag{3.B.17}$$

for $F\in L^1((0,T),H_0^s(M,\mathbb{C}^N))$, $s\in\mathbb{N}$. The solution could be constructed with $F\in L^1((0,T),\mathcal{K}^s)$ instead, but this is of no use for our main results.

Consider $s\in\mathbb{N}$, $T>0$ and $F\in\mathscr{C}^0([0,T],H_0^s(M,\mathbb{C}^N))$. For $\tau\in[0,T]$, let u_τ be the solution of

$$\left\{\begin{array}{rcll} \partial_t^2 u_\tau - \mathsf{P}u_\tau &=& 0 & \text{in } (0,T)\times M,\\ (u(0,\cdot),\partial_t u(0,\cdot)) &=& (0,F(\tau)) & \text{in } M,\\ u &=& 0 & \text{on } (0,T)\times\partial M. \end{array}\right.$$

As in the classical Duhamel formula, the solution of (3.B.17) is given by

$$\Psi(t) = \int_0^t u_\tau(t-\tau)\mathrm{d}\tau,\quad t\in[0,T].$$

The function Ψ is a one-parameter integral, and as $H_0^s(M,\mathbb{C}^N)\subset\mathcal{K}^s$, one has

$$u_\tau\in\mathscr{C}^0([0,T],\mathcal{K}^{s+1})\cap\mathscr{C}^1([0,T],\mathcal{K}^{s})\cap\mathscr{C}^2([0,T],\mathcal{K}^{s-1}),\quad \tau\in[0,T].$$

Thus, the following regularity results hold. First, one has $\Psi\in\mathscr{C}^0([0,T],\mathcal{K}^{s+1})$, with

$$\begin{aligned}\|\Psi\|_{L^\infty([0,T],\mathcal{K}^{s+1})} &\le \int_0^T \sup_{t\in[0,T]}\|u_\tau(t)\|_{\mathcal{K}^{s+1}}\,\mathrm{d}\tau\\ &\lesssim \int_0^T \|(u_\tau(0),\partial_t u_\tau(0))\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}\,\mathrm{d}\tau\\ &\lesssim \|F\|_{L^1([0,T],\mathcal{K}^s)} = \|F\|_{L^1([0,T],H^s)}.\end{aligned}$$

Second, one has $\Psi \in \mathscr{C}^1([0,T],\mathcal{K}^s)$, with

$$\partial_t \Psi(t) = \int_0^t \partial_t u_\tau(t-\tau)\mathrm{d}\tau, \quad t \in [0,T],$$

and

$$\|\partial_t \Psi\|_{L^\infty([0,T],\mathcal{K}^s)} \leq \int_0^T \sup_{t\in[0,T]} \|\partial_t u_\tau(t)\|_{\mathcal{K}^s} \,\mathrm{d}\tau \lesssim \|F\|_{L^1([0,T],H^s)}\,.$$

As $\mathscr{C}^0([0,T],H_0^s(M,\mathbb{C}^N))$ is dense in $L^1((0,T),H_0^s(M,\mathbb{C}^N))$, the previous results hold for all $F \in L^1((0,T),H_0^s(M,\mathbb{C}^N))$. Third, one has $\Psi \in \mathscr{C}^2([0,T],\mathcal{K}^{s-1})$, with

$$\partial_t^2 \Psi(t) = \partial_t u_t(0) + \int_0^t \partial_t^2 u_\tau(t-\tau)\mathrm{d}\tau = F(t) + \mathsf{P}\Psi(t)$$

for $t \in [0,T]$, and

$$\begin{aligned}\left\|\partial_t^2 \Psi\right\|_{L^\infty([0,T],\mathcal{K}^{s-1})} &\leq \|F\|_{L^\infty([0,T],\mathcal{K}^{s-1})} + \int_0^T \sup_{t\in[0,T]} \left\|\partial_t^2 u_\tau(t)\right\|_{\mathcal{K}^{s-1}} \mathrm{d}\tau \\ &\lesssim \|F\|_{L^\infty([0,T],\mathcal{K}^{s-1})} + \|F\|_{L^1([0,T],H^s)}\,.\end{aligned}$$

As $\mathscr{C}^0([0,T],H_0^s(M,\mathbb{C}^N))$ is dense in

$$L^1((0,T),H_0^s(M,\mathbb{C}^N)) \cap \mathscr{C}^0([0,T],H^{s-1}(M,\mathbb{C}^N)),$$

the previous results hold for F in the latter space.

The following duality result will be useful later.

Lemma 3.B.3. *For $F_1, F_2 \in L^2((0,T)\times M,\mathbb{C}^N)$, one has*

$$\langle u, F_2\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} = \langle F_1, v\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} + \left\langle u^0, \partial_t v(T)\right\rangle_{L^2(M,\mathbb{C}^N)} - \left\langle u^1, v(T)\right\rangle_{L^2(M,\mathbb{C}^N)}, \tag{3.B.18}$$

where u and v are the solutions of

$$\begin{cases} \partial_t^2 u - \mathsf{P}u &= F_1 & \text{in } (0,T)\times M, \\ (u(T,\cdot),\partial_t u(T,\cdot)) &= (u^0,u^1) & \text{in } M, \\ u &= 0 & \text{on } (0,T)\times\partial M, \end{cases}$$

$$\begin{cases} \partial_t^2 v - \mathsf{P}^* v &= F_2 & \text{in } (0,T)\times M, \\ (v(0,\cdot),\partial_t v(0,\cdot)) &= 0 & \text{in } M, \\ v &= 0 & \text{on } (0,T)\times\partial M. \end{cases}$$

Proof. For $F_1, F_2 \in \mathscr{C}^0([0,T],\mathcal{K}^1)$, an integration by parts gives (3.B.18). Both sides of (3.B.18) are continuous with respect to the norm of $L^2((0,T)\times M,\mathbb{C}^N)$, (3.B.18) holds for all F_1 and F_2 in $L^2((0,T)\times M,\mathbb{C}^N)$ by density. □

Step 5: Regularity of the normal derivative. Consider $T>0$. If $s=0$, then the standard scalar proof works without any change (see for example [LLT86]). In addition, for $(u^0,u^1) \in \mathcal{K}^1\times\mathcal{K}^0$ and $F \in L^1((0,T),\mathcal{K}^0)$, if u is the solution of

$$\begin{cases} \partial_t^2 u - \mathsf{P}u &= F & \text{in } (0,T)\times M, \\ (u(0,\cdot),\partial_t u(0,\cdot)) &= (u^0,u^1) & \text{in } M, \\ u &= 0 & \text{on } (0,T)\times\partial M, \end{cases}$$

then one has

$$\|\partial_\nu u\|_{L^2((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^1\times\mathcal{K}^0} + \|F\|_{L^1((0,T),\mathcal{K}^0)}\,.$$

Case $s>0$. For $s\in\mathbb{N}^*$, we prove that

$$\partial_\nu u \in H^s((0,T)\times\partial M,\mathbb{C}^N)$$

with

$$\|\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s} + \|F\|_{L^1((0,T),H^s)} \tag{3.B.19}$$

for $\left(u^0,u^1\right)\in\mathcal{K}^{s+1}\times\mathcal{K}^s$ and $F\in L^1((0,T),H^s_0(M,\mathbb{C}^N))$, where u is the solution of the wave equation with initial data $\left(u^0,u^1\right)$ and with source term F.

We start with the case $F=0$. By density it suffices to prove that for $\left(u^0,u^1\right)\in\mathcal{K}^{s+2}\times\mathcal{K}^{s+1}$, one has

$$\|\partial_\nu u\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}\,.$$

Consider $k\in[\![0,s]\!]$, and let $L_1,\cdots,L_k$ be smooth vector fields on the Riemannian manifold ∂M. We prove

$$L_1\cdots L_k\partial_t^{s-k}\partial_\nu u\in L^2((0,T)\times\partial M,\mathbb{C}^N). \tag{3.B.20}$$

For $j\in[\![0,k]\!]$, there exists a smooth vector field $\tilde{L}_j$ on M such that $\tilde{L}_j=L_j$ on the boundary. Define

$$v=\tilde{L}_1\cdots\tilde{L}_k\partial_t^{s-k}u.$$

Note that $u\in\mathscr{C}^{s-k}(\mathbb{R},\mathcal{K}^{k+2})$, so that $v\in H^2(M,\mathbb{C}^N)$. As s is positive here, we can omit the subscript of P and use the usual differential operator P. We can write

$$\left(\partial_t^2-\mathsf{P}\right)v=\left[\left(\partial_t^2-\mathsf{P}\right),\tilde{L}_1\cdots\tilde{L}_k\partial_t^{s-k}\right]u=Ru$$

where R is a differential operator of order $s+1$. As $u\in H^{s+2}((0,T)\times M,\mathbb{C}^N)$, one has $Ru\in H^1((0,T)\times M,\mathbb{C}^N)$. We claim that

$$(v(0),\partial_t v(0))\in H^1_0(M)\times L^2(M). \tag{3.B.21}$$

Then, by the standard case, one has

$$\|\partial_\nu v\|_{L^2((0,T)\times\partial M,\mathbb{C}^N)}\lesssim\|(v(0),\partial_t v(0))\|_{\mathcal{K}^1\times\mathcal{K}^0}+\|Ru\|_{L^2((0,T)\times M,\mathbb{C}^N)}\lesssim\left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}$$

implying (3.B.20). Indeed, if $\mathtt{N}$ is a smooth vector field on M that coincides with the unit normal vector at the boundary, then one has

$$\partial_\nu v=(\mathtt{N}v)_{|\partial M}=\left(\tilde{L}_1\cdots\tilde{L}_k\partial_t^{s-k}(\mathtt{N}u)\right)_{|\partial M}+\left(R\partial_t^{s-k}u\right)_{|\partial M}$$

where R is a time-independent differential operator of order $k-1$. Hence, using $u\in\mathscr{C}^{s-k}((0,T),\mathcal{K}^{k+1})$ and the fact that for $j\in[\![0,k]\!]$, the vector field $\tilde{L}_j$ is tangent to the boundary, one finds (3.B.20).

One has $\partial_t^{s-k}u\in\mathscr{C}^0(\mathbb{R},\mathcal{K}^{k+2})$, and (3.B.21) follows if one proves that $w\in\mathcal{K}^{k+2}$ implies $\tilde{L}_1\cdots\tilde{L}_k w\in H^1_0(M,\mathbb{C}^N)$. For $w\in H^{k+1}(M,\mathbb{C}^N)$, one has

$$\left(\tilde{L}_1\cdots\tilde{L}_k w\right)_{|\partial M}=L_1\cdots L_k\left(w_{|\partial M}\right)\in H^{\frac{1}{2}}(\partial M,\mathbb{C}^N).$$

Indeed, it is true if $w \in \mathscr{C}^\infty(M, \mathbb{C}^N)$ and both sides are continuous with respect to the norm of $H^{k+1}(M, \mathbb{C}^N)$. Thus, for $w \in H^{k+1}(M, \mathbb{C}^N) \cap H_0^1(M, \mathbb{C}^N)$, one has

$$\left(\tilde{L}_1 \cdots \tilde{L}_k w\right)_{|\partial M} = 0 \in H^{\frac{1}{2}}(\partial M, \mathbb{C}^N)$$

and this gives (3.B.21).

Now, we prove (3.B.19) in the case $F \neq 0$. Note that the previous proof gives

$$\|\partial_\nu u\|_{H^s((0,T)\times\partial M, \mathbb{C}^N)} \lesssim \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s} + \|F\|_{H^s((0,T)\times M, \mathbb{C}^N)},$$

a weaker result. By linearity, we may assume that $(u^0, u^1) = 0$. By density, it suffices to prove that for $F \in \mathscr{C}^\infty([0,T], \mathscr{C}_c^\infty(\operatorname{Int} M, \mathbb{C}^N))$, one has

$$\|\partial_\nu \Psi\|_{H^s((0,T)\times\partial M, \mathbb{C}^N)} \lesssim \|F\|_{L^1((0,T), H^s)}$$

where Ψ is the Duhamel term defined above. Consider $k \in [\![0, s]\!]$, and let $L_1, \cdots, L_k$ be smooth vector fields on the Riemannian manifold ∂M. We prove

$$\left\|L_1 \cdots L_k \partial_t^{s-k} \partial_\nu \Psi\right\|_{L^2((0,T)\times\partial M, \mathbb{C}^N)} \lesssim \|F\|_{L^1((0,T), H^s)}.$$

For $\tau \in [0,T]$ and $k \in \mathbb{N}$, one has

$$\partial_t^{2k} u_\tau(0) = 0, \quad 2k \in [\![0, s]\!],$$

and

$$\partial_t^{2k+1} u_\tau(0) = \mathsf{P}^k F(\tau), \quad 2k+1 \in [\![0, s]\!].$$

Hence, for $k \in [\![0, s]\!]$, there exists a differential operator R_k such that

$$\partial_t^k \Psi(t) = (R_k F)(t) + \int_0^t \partial_t^k u_\tau(t-\tau)\mathrm{d}\tau.$$

As $F(t)$ is compactly supported in $\operatorname{Int} M$, this gives

$$\partial_t^k \partial_\nu \Psi(t) = \int_0^t \partial_t^k \partial_\nu u_\tau(t-\tau)\mathrm{d}\tau.$$

Hence, one has

$$\begin{aligned}
&\left\|L_1 \cdots L_k \partial_t^{s-k} \partial_\nu \Psi\right\|_{L^2((0,T)\times\partial M, \mathbb{C}^N)} \\
=&\left\|\int_0^T \mathbb{1}_{\tau\leq t} L_1 \cdots L_k \partial_t^{s-k} \partial_\nu u_\tau(t-\tau, x)\mathrm{d}\tau\right\|_{L^2((0,T)\times\partial M, \mathbb{C}^N)} \\
\leq &\int_0^T \left\|L_1 \cdots L_k \partial_t^{s-k} \partial_\nu u_\tau\right\|_{L^2((0,T)\times\partial M, \mathbb{C}^N)} \mathrm{d}\tau \\
\lesssim &\int_0^T \|\partial_\nu u_\tau\|_{H^s((0,T)\times\partial M, \mathbb{C}^N)} \mathrm{d}\tau.
\end{aligned}$$

By (3.B.19) in the case $F = 0$, we get

$$\left\|L_1 \cdots L_k \partial_t^{s-k} \partial_\nu \Psi\right\|_{L^2((0,T)\times\partial M, \mathbb{C}^N)} \lesssim \int_0^T \|F(\tau)\|_{\mathcal{K}^s} \mathrm{d}\tau = \|F\|_{L^1((0,T), H^s)}.$$

The case $s < 0$. Note that in the sense of classical trace theorem, the normal derivative of a solution does not exist in that case. Take $s \in \mathbb{Z}$, $s < 0$, and $\left(u^0, u^1\right) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. There exist $\sigma \in \mathbb{N}^*$ and $\alpha \in \{1, 2\}$ such that $s = -2\sigma + \alpha$. Let $\left(\tilde{u}^0, \tilde{u}^1\right)$ be the unique element of $\mathcal{K}^{\alpha+1} \times \mathcal{K}^\alpha$ such that

$$\left(u^0, u^1\right) = \left(\mathcal{S}^\sigma_{s+1+\sigma} \tilde{u}^0, \mathcal{S}^\sigma_{s+\sigma} \tilde{u}^1\right).$$

Recall that the solution u associated with $\left(u^0, u^1\right)$ is defined by $u = \mathcal{S}^\sigma_{s+1+\sigma} \tilde{u}$, where $\tilde{u}$ is the solution associated with $\left(\tilde{u}^0, \tilde{u}^1\right)$. Using Proposition 3.1.8, we can write

$$\begin{aligned} u(t) &= \mathcal{S}^1_{s+2} \circ \cdots \circ \mathcal{S}^1_\alpha \left(\tilde{u}(t)\right) \\ &= \left(\mathsf{P}_{s+2} + i\mu \iota_{\mathcal{K}^{s+3} \to \mathcal{K}^{s+1}}\right) \circ \cdots \circ \left(\mathsf{P}_\alpha + i\mu \iota_{\mathcal{K}^{\alpha+1} \to \mathcal{K}^{\alpha-1}}\right) \left(\tilde{u}(t)\right) \\ &= \sum_{k=0}^{\sigma} \binom{\sigma}{k} (i\mu)^{\sigma-k} \iota_{\mathcal{K}^{\alpha+1-2k} \to \mathcal{K}^{s+1}} \left(\mathsf{P}^k_{\alpha+1-k} \tilde{u}(t)\right) \end{aligned}$$

and by Theorem 3.1.12-*(i)*, we get

$$u(t) = \sum_{k=0}^{\sigma} \binom{\sigma}{k} (i\mu)^{\sigma-k} \iota_{\mathcal{K}^{\alpha+1-2k} \to \mathcal{K}^{s+1}} \left(\partial_t^{2k} \tilde{u}(t)\right)$$

for $t \in \mathbb{R}$. As $\partial_\nu \tilde{u} \in H^\alpha((0,T) \times \partial M, \mathbb{C}^N)$, we define $\partial_\nu u$ by

$$\partial_\nu u = \sum_{k=0}^{\sigma} \binom{\sigma}{k} (i\mu)^{\sigma-k} \iota_{H^{\alpha-2k} \to H^s} \left(\partial_t^{2k} \partial_\nu \tilde{u}\right)$$

where $\iota_{H^{\alpha-2k} \to H^s}$ is the embedding from $H^{\alpha-2k}((0,T) \times \partial M, \mathbb{C}^N)$ to $H^s((0,T) \times \partial M, \mathbb{C}^N)$. Clearly, one has

$$\begin{aligned} \left\|\partial_\nu u\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} &\lesssim \sum_{k=0}^{\sigma} \left\| \iota_{H^{\alpha-2k} \to H^s} \left(\partial_t^{2k} \partial_\nu \tilde{u}\right) \right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \\ &\lesssim \left\|\partial_\nu \tilde{u}\right\|_{H^\alpha((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \left\|\left(\tilde{u}^0, \tilde{u}^1\right)\right\|_{\mathcal{K}^{\alpha+1}\times\mathcal{K}^\alpha} \lesssim \left\|\left(u^0, u^1\right)\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}. \end{aligned}$$

To complete the proof, one has to show the two additional results of Theorem 3.1.12-*(iii)*.

Connection with the usual normal derivative. Here, we show that our definition of the normal derivative of a solution coincide with the usual normal derivative for a regular solution. More precisely, we prove that for all $s \in \mathbb{R}$, $\delta > 0$ and $\left(u^0, u^1\right) \in \mathcal{K}^{s+\delta+1} \times \mathcal{K}^{s+\delta}$, one has

$$\partial_\nu \left(\iota_{\mathcal{K}^{s+\delta+1} \to \mathcal{K}^{s+1}} u\right) = \iota_{H^{s+\delta} \to H^s} \partial_\nu u. \tag{3.B.22}$$

By interpolation, it suffices to prove (3.B.22) for $s \in \mathbb{Z}$.

Lemma 3.B.4. *For $s \in \mathbb{Z}$, $s \leq -1$, and $\left(u^0, u^1\right) \in \mathcal{K}^2 \times \mathcal{K}^1$, one has*

$$\partial_\nu \left(\iota_{\mathcal{K}^2 \to \mathcal{K}^{s+1}} u\right) = \iota_{H^1 \to H^s} \partial_\nu u.$$

Proof. Write $s = -2\sigma + \alpha$, with $\sigma \in \mathbb{N}^*$ and $\alpha \in \{1, 2\}$, and let $\left(\tilde{u}^0, \tilde{u}^1\right) \in \mathcal{K}^{\alpha+1} \times \mathcal{K}^\alpha$ be given by

$$\left(\iota_{\mathcal{K}^2 \to \mathcal{K}^{s+1}} u^0, \iota_{\mathcal{K}^1 \to \mathcal{K}^s} u^1\right) = \left(\mathcal{S}^\sigma_{s+1+\sigma} \tilde{u}^0, \mathcal{S}^\sigma_{s+\sigma} \tilde{u}^1\right).$$

One has $\iota_{\mathcal{K}^2 \to \mathcal{K}^{s+1}} u = \mathcal{S}^\sigma_{s+1+\sigma} \tilde{u}$, where $\tilde{u}$ is the solution with initial data $\left(\tilde{u}^0, \tilde{u}^1\right)$. By definition, one has

$$\partial_\nu \left(\iota_{\mathcal{K}^2 \to \mathcal{K}^{s+1}} u\right) = \sum_{k=0}^{\sigma} \binom{\sigma}{k} (i\mu)^{\sigma-k} \iota_{H^{\alpha-2k} \to H^s} \left(\partial_t^{2k} \partial_\nu \tilde{u}\right). \tag{3.B.23}$$

Writing

$$\begin{aligned}\left(\tilde{u}^0,\tilde{u}^1\right) &= \left(\left(\mathcal{S}^{\sigma}_{s+1+\sigma}\right)^{-1}\circ\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}\left(u^0\right),\left(\mathcal{S}^{\sigma}_{s+\sigma}\right)^{-1}\circ\iota_{\mathcal{K}^1\to\mathcal{K}^{s}}\left(u^1\right)\right)\\ &= \left(\iota_{\mathcal{K}^{2\sigma+2}\to\mathcal{K}^{\alpha+1}}\circ\left(\mathcal{S}^{\sigma}_{2+\sigma}\right)^{-1}\left(u^0\right),\iota_{\mathcal{K}^{2\sigma+1}\to\mathcal{K}^{\alpha}}\circ\left(\mathcal{S}^{\sigma}_{1+\sigma}\right)^{-1}\left(u^1\right)\right),\end{aligned}$$

one finds $\tilde{u}=\iota_{\mathcal{K}^{2\sigma+2}\to\mathcal{K}^{\alpha+1}}v$, where v is the solution with initial data

$$\left(\left(\mathcal{S}^{\sigma}_{2+\sigma}\right)^{-1}u^0,\left(\mathcal{S}^{\sigma}_{1+\sigma}\right)^{-1}u^1\right)\in\mathcal{K}^{2\sigma+2}\times\mathcal{K}^{2\sigma+1}.$$

One also has $u=\mathcal{S}^{\sigma}_{2+\sigma}v$. As $\partial_t^{2k}\partial_\nu\tilde{u}=\partial_t^{2k}\partial_\nu v$ in $\mathscr{D}'((0,T)\times\partial M,\mathbb{C}^N)$, one finds

$$\partial_t^{2k}\partial_\nu\tilde{u}=\iota_{H^{2\sigma+1-2k}\to H^{\alpha-2k}}\partial_t^{2k}\partial_\nu v$$

for all $k\in\mathbb{N}$. Coming back to (3.B.23), one obtains

$$\partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}u\right)=\sum_{k=0}^{\sigma}\binom{\sigma}{k}(i\mu)^{\sigma-k}\iota_{H^{1+2\sigma-2k}\to H^s}\left(\partial_t^{2k}\partial_\nu v\right).$$

For $k\in[\![0,\sigma]\!]$, one has $1+2\sigma-2k\geq 1$ implying

$$\partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}u\right)=\iota_{H^1\to H^s}\sum_{k=0}^{\sigma}\binom{\sigma}{k}(i\mu)^{\sigma-k}\iota_{H^{1+2\sigma-2k}\to H^1}\left(\partial_t^{2k}\partial_\nu v\right).\tag{3.B.24}$$

Omitting the embeddings in $H^1((0,T)\times\partial M,\mathbb{C}^N)$, one finds

$$\sum_{k=0}^{\sigma}\binom{\sigma}{k}(i\mu)^{\sigma-k}\iota_{H^{1+2\sigma-2k}\to H^1}\left(\partial_t^{2k}\partial_\nu v\right)=\partial_\nu\left(\mathcal{S}^{\sigma}_{2+\sigma}v\right)=\partial_\nu u.$$

Together with (3.B.24), this completes the proof. □

To prove (3.B.22), we distinguish three cases. First, if $s+1\geq 0$ then (3.B.22) is true. Second, if $s+1\leq -1$ and $s+\delta+1\geq 2$, then using Lemma 3.B.4 and the first case, one finds

$$\begin{aligned}\partial_\nu\left(\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^{s+1}}u\right) &= \partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}\circ\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^2}u\right)\\ &= \iota_{H^1\to H^s}\partial_\nu\left(\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^2}u\right)\\ &= \iota_{H^1\to H^s}\circ\iota_{H^{s+\delta}\to H^1}\left(\partial_\nu u\right)\\ &= \iota_{H^{s+\delta}\to H^s}\partial_\nu u.\end{aligned}$$

Finally, if $s+1\leq -1$ and $s+\delta+1<2$, then we consider an approximation of u: take a sequence $\left((u_k^0,u_k^1)\right)_{k\in\mathbb{N}}$ of elements of $\mathcal{K}^2\times\mathcal{K}^1$ such that

$$\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+\delta+1}}u_k^0,\iota_{\mathcal{K}^1\to\mathcal{K}^{s+\delta}}u_k^1\right)\xrightarrow{k\to\infty}\left(u^0,u^1\right).$$

For $k\in\mathbb{N}$, let u_k be the solution associated with $\left(u_k^0,u_k^1\right)$. Set

$$w_k=\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^{s+1}}u-\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}u_k.$$

As w_k is a solution of the wave equation, one has

$$\left\|\partial_\nu\left(\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^{s+1}}u\right)-\partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}u_k\right)\right\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\lesssim\left\|(w_k(0),\partial_t w_k(0))\right\|_{\mathcal{K}^{s+1}\times\mathcal{K}^s}.$$

Writing

$$(w_k(0), \partial_t w_k(0)) = \left(\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^{s+1}}\left(u^0 - \iota_{\mathcal{K}^2\to\mathcal{K}^{s+\delta+1}}u_k^0\right), \iota_{\mathcal{K}^{s+\delta}\to\mathcal{K}^s}\left(u^1 - \iota_{\mathcal{K}^1\to\mathcal{K}^{s+\delta}}u_k^1\right)\right)$$

one finds

$$\begin{aligned}&\|\partial_\nu\left(\iota_{\mathcal{K}^{s+\delta+1}\to\mathcal{K}^{s+1}}u\right) - \partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}u_k\right)\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\\ &\lesssim \left\|\left(u^0 - \iota_{\mathcal{K}^2\to\mathcal{K}^{s+\delta+1}}u_k^0, u^1 - \iota_{\mathcal{K}^1\to\mathcal{K}^{s+\delta}}u_k^1\right)\right\|_{\mathcal{K}^{s+\delta+1}\times\mathcal{K}^{s+\delta}} \xrightarrow{k\to\infty} 0.\end{aligned} \tag{3.B.25}$$

On the other hand, using Lemma 3.B.4, one has

$$\begin{aligned}&\|\iota_{H^{s+\delta}\to H^s}\partial_\nu u - \iota_{H^1\to H^s}\partial_\nu u_k\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\\ &= \|\iota_{H^{s+\delta}\to H^s}\left(\partial_\nu u - \iota_{H^1\to H^{s+\delta}}\partial_\nu u_k\right)\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\\ &= \|\iota_{H^{s+\delta}\to H^s}\left(\partial_\nu u - \partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+\delta+1}}u_k\right)\right)\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\\ &= \|\partial_\nu u - \partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+\delta+1}}u_k\right)\|_{H^{s+\delta}((0,T)\times\partial M,\mathbb{C}^N)}.\end{aligned}$$

As above, one finds

$$\|\iota_{H^{s+\delta}\to H^s}\partial_\nu u - \iota_{H^1\to H^s}\partial_\nu u_k\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \xrightarrow{k\to\infty} 0. \tag{3.B.26}$$

Lemma 3.B.4 gives $\iota_{H^1\to H^s}\partial_\nu u_k = \partial_\nu\left(\iota_{\mathcal{K}^2\to\mathcal{K}^{s+1}}u_k\right)$, and with (3.B.25) and (3.B.26), this completes the proof of the third case.

The normal derivative and the time-derivative commute. Take $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $(u^0, u^1) \in \mathcal{K}^{s+1} \times \mathcal{K}^s$. Here, we show that

$$\partial_\nu \partial_t^{2k} u = \partial_t^{2k} \partial_\nu u. \tag{3.B.27}$$

Note that the left-hand side is well-defined as we know that $\partial_t^{2k} u$ is a solution of the wave equation. By interpolation, we may assume that $s \in \mathbb{Z}$. If $s - 2k \geq 0$, then (3.B.27) holds true, so we can assume that $s - 2k \leq -1$. As above, using an approximation, it suffices to prove that

$$\partial_\nu \partial_t^{2k}\left(\iota_{\mathcal{K}^{2k+1}\to\mathcal{K}^{s+1}}u\right) = \partial_t^{2k}\partial_\nu\left(\iota_{\mathcal{K}^{2k+1}\to\mathcal{K}^{s+1}}u\right) \tag{3.B.28}$$

for all $(u^0, u^1) \in \mathcal{K}^{2k+1} \times \mathcal{K}^{2k}$. Using (3.B.22), one finds

$$\partial_\nu \partial_t^{2k}\left(\iota_{\mathcal{K}^{2k+1}\to\mathcal{K}^{s+1}}u\right) = \partial_\nu\left(\iota_{\mathcal{K}^1\to\mathcal{K}^{s-2k+1}}\partial_t^{2k}u\right) = \iota_{L^2\to H^{s-2k}}\left(\partial_\nu\partial_t^{2k}u\right).$$

Note that $\partial_\nu \partial_t^{2k} u = \partial_t^{2k}\partial_\nu u$, as $(u^0, u^1) \in \mathcal{K}^{2k+1} \times \mathcal{K}^{2k}$. One has $\partial_\nu u = \iota_{H^{2k}\to H^s}\partial_\nu u$ in $\mathscr{D}'((0,T)\times\partial M, \mathbb{C}^N)$, implying

$$\partial_t^{2k}\partial_\nu u = \partial_t^{2k}\iota_{H^{2k}\to H^s}\partial_\nu u$$

in $\mathscr{D}'((0,T)\times\partial M, \mathbb{C}^N)$. This gives

$$\iota_{L^2\to H^{s-2k}}\partial_t^{2k}\partial_\nu u = \partial_t^{2k}\iota_{H^{2k}\to H^s}\partial_\nu u.$$

Hence, one obtains

$$\partial_\nu \partial_t^{2k}\left(\iota_{\mathcal{K}^{2k+1}\to\mathcal{K}^{s+1}}u\right) = \partial_t^{2k}\iota_{H^{2k}\to H^s}\partial_\nu u.$$

Using (3.B.22) again, one finds (3.B.28).

Proof of Theorem 3.1.14 in negative regularity

Here, we prove Theorem 3.1.14 for $s \le 0$. An integration by parts gives the following identity.

Lemma 3.B.5. *For u and v in*

$$\mathscr{C}^0([0,T], H^2(M,\mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^1(M,\mathbb{C}^N)) \cap \mathscr{C}^2([0,T], L^2(M,\mathbb{C}^N))$$

one has

$$\begin{aligned} &\left\langle \left(\partial_t^2 - \mathsf{P}\right) u, v\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} - \left\langle u, \left(\partial_t^2 - \mathsf{P}^*\right) v\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} \\ &= \Big[\langle \partial_t u(t), v(t)\rangle_{L^2(M,\mathbb{C}^N)} - \langle u(t), \partial_t v(t)\rangle_{L^2(M,\mathbb{C}^N)}\Big]_0^T - \left\langle \langle X,\nu\rangle_g u, v\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} \\ &\quad - \langle \partial_\nu u, v\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} + \langle u, \partial_\nu v\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)}\,. \end{aligned}$$

Step 1: Definition of the solution. Take $f \in \mathscr{C}^\infty((0,T)\times\partial M,\mathbb{C}^N)$. If there exists a smooth solution v of (3.1.9), then for all smooth function u, one has

$$\begin{aligned} \left\langle \left(\partial_t^2 - \mathsf{P}\right) u, v\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} &= \langle \partial_t u(T), v(T)\rangle_{L^2(M,\mathbb{C}^N)} - \langle u(T), \partial_t v(T)\rangle_{L^2(M,\mathbb{C}^N)} \\ &- \left\langle \langle X,\nu\rangle_g u + \partial_\nu u, \operatorname{diag}(\Theta) f\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} + \langle u, \partial_\nu v\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)}\,. \end{aligned}$$

In particular, if u is such that $u_{|(0,T)\times\partial M} = 0$ and $(u(T), \partial_t u(T)) = 0$ then

$$\left\langle \left(\partial_t^2 - \mathsf{P}\right) u, v\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} = -\left\langle \partial_\nu u, \operatorname{diag}(\Theta) f\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)}\,.$$

Hence, if u is a smooth solution of

$$\left\{\begin{array}{rcll} \partial_t^2 u - \mathsf{P}u &=& F & \text{in } (0,T)\times M, \\ (u(T,\cdot), \partial_t u(T,\cdot)) &=& 0 & \text{in } M, \\ u &=& 0 & \text{on } (0,T)\times\partial M. \end{array}\right. \tag{3.B.29}$$

then one has

$$\langle F, v\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} = -\left\langle \partial_\nu u, \operatorname{diag}(\Theta) f\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)}\,.$$

We use this as the definition of v. More precisely, take $s \le 0$ and define

$$\begin{array}{rccc} L_s: & L^1((0,T), H_0^{-s}(M,\mathbb{C}^N)) & \longrightarrow & H_0^{-s}((0,T)\times\partial M,\mathbb{C}^N) \\ & F & \longmapsto & -\operatorname{diag}(\Theta)\partial_\nu u \end{array}$$

where u is the solution of (3.B.29). By Theorem 3.1.12, the operator L_s is well-defined and continuous. For $f \in H^s((0,T)\times\partial M,\mathbb{C}^N)$, we define the solution v of (3.1.9) by $v = L_s^* f$. One has $v \in L^\infty((0,T), H^s(M,\mathbb{C}^N))$. In the next step, we show that v is more regular.

Step 2: Regularity of the solution. Consider $s \le 0$. In this step, we show that for $f \in H^s((0,T)\times\partial M,\mathbb{C}^N)$, one has

$$v = L_s^*(f) \in \mathscr{C}^0([0,T], H^s(M,\mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^{s-1}(M,\mathbb{C}^N)) \cap \mathscr{C}^2([0,T], H^{s-2}(M,\mathbb{C}^N)) \tag{3.B.30}$$

with an inequality, and $\partial_t^2 v = \mathsf{P}^*_{\mathscr{D}'} v$ in $\mathscr{D}'((0,T)\times M,\mathbb{C}^N)$. To get (3.B.30) for all f, it suffices to show that (3.B.30) holds for f smooth, with an inequality of the form

$$\|v\|_{\mathscr{C}^0(H^s)\cap\mathscr{C}^1(H^{s-1})\cap\mathscr{C}^2(H^{s-2})} \lesssim \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}\,. \tag{3.B.31}$$

Proof of (3.B.30). Suppose $f \in \mathscr{C}^\infty((0,T)\times\partial M, \mathbb{C}^N)$, and denote by $\tilde f \in \mathscr{C}^\infty_{\mathrm{c}}((0,T)\times M, \mathbb{C}^N)$ an extension of $\operatorname{diag}(\Theta)f$. One writes $v = \tilde f + w$, where w is a solution of the wave equation with homogeneous Dirichlet boundary condition, as follows. Set

$$F = -\left(\partial_t^2 - \mathsf{P}^*\right)\tilde f \in \mathscr{C}^\infty((0,T)\times M, \mathbb{C}^N).$$

Since $F \in L^1((0,T), L^2(M,\mathbb{C}^N)) \cap \mathscr{C}^0((0,T), H^{-1}(M,\mathbb{C}^N))$, the solution w of

$$\begin{cases} \partial_t^2 w - \mathsf{P}^* w &= F & \text{in } (0,T)\times M, \\ (w(0,\cdot), \partial_t w(0,\cdot)) &= 0 & \text{in } M, \\ w &= 0 & \text{on } (0,T)\times\partial M, \end{cases} \tag{3.B.32}$$

is well-defined, and

$$w \in \mathscr{C}^0([0,T], H^1_0(M,\mathbb{C}^N)) \cap \mathscr{C}^1([0,T], L^2(M,\mathbb{C}^N)) \cap \mathscr{C}^2([0,T], H^{-1}(M,\mathbb{C}^N))$$

by Theorem 3.1.12-*(iv)*. We claim that $v = \tilde f + w$, that is,

$$\langle v, \phi\rangle_{L^\infty((0,T),H^s), L^1((0,T),H_0^{-s})} = \left\langle w + \tilde f, \overline{\phi}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} \tag{3.B.33}$$

for all $\phi \in L^1((0,T), H_0^{-s}(M))$. By density, it suffices to prove (3.B.33) for $\phi \in \mathscr{C}^\infty_{\mathrm{c}}((0,T)\times \operatorname{Int} M, \mathbb{C}^N)$. Note that Lemma 3.B.5 does not apply to $w + \tilde f$, due to the lack of regularity of w. However, it applies to $\tilde f$, and Lemma 3.B.3 can be used for w. Consider $\phi \in \mathscr{C}^\infty_{\mathrm{c}}((0,T)\times \operatorname{Int} M, \mathbb{C}^N)$, and let u be the solution of

$$\begin{cases} \partial_t^2 u - \mathsf{P} u &= \phi & \text{in } (0,T)\times M, \\ (u(T,\cdot), \partial_t u(T,\cdot)) &= 0 & \text{in } M, \\ u &= 0 & \text{on } (0,T)\times\partial M. \end{cases} \tag{3.B.34}$$

One has

$$\begin{aligned} &\left\langle w + \tilde f, \overline{\phi}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} \\ =&\left\langle w, \overline{\phi}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} + \left\langle \tilde f, \overline{\phi}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} \\ =&\left\langle F, \overline{u}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} + \left\langle \left(\partial_t^2 - \mathsf{P}^*\right)\tilde f, \overline{u}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} - \left\langle \tilde f, \partial_\nu \overline{u}\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} \\ =&-\left\langle \operatorname{diag}(\Theta) f, \partial_\nu \overline{u}\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)}, \end{aligned}$$

where one has used the fact that $\tilde f$ is compactly supported in $(0,T)$. This implies

$$\left\langle w + \tilde f, \overline{\phi}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} = \left\langle f, \overline{L_s\phi}\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} = \langle v, \phi\rangle_{L^\infty((0,T),H^s), L^1((0,T),H_0^{-s})},$$

as $v = L_s^* f$. This proves (3.B.33). In particular, this gives (3.B.30). Note that, for now, (3.B.30) has only been proved for smooth f.

Proof of (3.B.31). We prove (3.B.31) for f smooth. First, note that the operator

$$L_s^* : H^s((0,T)\times\partial M, \mathbb{C}^N) \longrightarrow L^\infty((0,T), H^s(M,\mathbb{C}^N))$$

is continuous, as L_s is continuous. This gives

$$\|v\|_{L^\infty((0,T),H^s(M,\mathbb{C}^N))} \lesssim \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)},$$

implying $v = L_s^* f \in \mathscr{C}^0([0,T], H^s(M, \mathbb{C}^N))$ if $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$.

Second, we prove

$$\|\partial_t v\|_{L^\infty((0,T),H^{s-1}(M,\mathbb{C}^N))} \lesssim \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} . \tag{3.B.35}$$

Consider $f \in \mathscr{C}^\infty((0,T) \times \partial M, \mathbb{C}^N)$, and $\phi \in \mathscr{C}_c^\infty((0,T) \times \operatorname{Int} M, \mathbb{C}^N)$. By definition, one has

$$\begin{aligned}\langle \partial_t v, \phi\rangle_{\mathscr{D}'((0,T)\times M,\mathbb{C}^N),\mathscr{D}((0,T)\times M,\mathbb{C}^N)} &= - \langle v, \partial_t \phi\rangle_{L^\infty((0,T),H^s),L^1((0,T),H_0^{-s})} \\ &= \left\langle \operatorname{diag}(\Theta) f, \overline{\partial_\nu \tilde{u}}\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)}\end{aligned}$$

where $\tilde{u}$ is the solution of

$$\begin{cases} \partial_t^2 \tilde{u} - \mathsf{P}\tilde{u} &= \partial_t \phi \qquad \text{in } (0,T)\times M, \\ (\tilde{u}(T,\cdot), \partial_t \tilde{u}(T,\cdot)) &= 0 \qquad \text{in } M, \\ \tilde{u} &= 0 \qquad \text{on } (0,T)\times\partial M. \end{cases}$$

Note that $\tilde{u} = \partial_t u$, where u is the solution of (3.B.34). Indeed, one has $\partial_t u(T) = 0$ and

$$\partial_t^2 u(T) = \mathsf{P} u(T) + \phi(T) = 0$$

as ϕ is compactly supported. Hence, one finds

$$\left|\langle \partial_t v, \phi\rangle_{\mathscr{D}'((0,T)\times M,\mathbb{C}^N),\mathscr{D}((0,T)\times M,\mathbb{C}^N)}\right| \le \|\operatorname{diag}(\Theta)\partial_\nu \partial_t u\|_{H^{-s}((0,T)\times\partial M,\mathbb{C}^N)} \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} .$$

By Theorem 3.1.12, one has

$$\|\operatorname{diag}(\Theta)\partial_\nu\partial_t u\|_{H^{-s}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \|\partial_\nu u\|_{H^{-s+1}((0,T)\times\partial M,\mathbb{C}^N)} \lesssim \|\phi\|_{L^1((0,T),H_0^{-s+1}(M,\mathbb{C}^N))} .$$

As $\mathscr{C}_c^\infty((0,T)\times M, \operatorname{Int}\mathbb{C}^N)$ is dense in $L^1((0,T), H_0^{-s+1}(M,\mathbb{C}^N))$, this gives (3.B.35).

Third, the proof of

$$\left\|\partial_t^2 v\right\|_{L^\infty((0,T),H^{s-2}(M,\mathbb{C}^N))} \lesssim \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)}$$

is similar: one writes

$$\begin{aligned}&\left|\left\langle \partial_t^2 v, \phi\right\rangle_{\mathscr{D}'((0,T)\times M,\mathbb{C}^N),\mathscr{D}((0,T)\times M,\mathbb{C}^N)}\right| \\ &\le \left\|\operatorname{diag}(\Theta)\partial_\nu\partial_t^2 u\right\|_{H^{-s}((0,T)\times\partial M,\mathbb{C}^N)} \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} \\ &\lesssim \|\phi\|_{L^1((0,T),H_0^{-s+2}(M,\mathbb{C}^N))} \|f\|_{H^s((0,T)\times\partial M,\mathbb{C}^N)} .\end{aligned}$$

Note that the equality $\partial_\nu \partial_t^2 u = \partial_t^2 \partial_\nu u$ is obvious, as ϕ is smooth.

Connection with the wave equation. We show that for $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$, one has

$$\partial_t^2 v - \mathsf{P}_{\mathscr{D}'}^* v = 0, \quad \text{in } \mathscr{D}'((0,T)\times M, \mathbb{C}^N). \tag{3.B.36}$$

As $v = L_s^* f \in L^\infty((0,T), H^s(M,\mathbb{C}^N))$ is continuous with respect to $f \in H^s((0,T)\times\partial M, \mathbb{C}^N)$, we may assume that $f \in \mathscr{C}^\infty((0,T)\times\partial M,\mathbb{C}^N)$. As above, write $v = w + \tilde{f}$, where $\tilde{f} \in \mathscr{C}_c^\infty((0,T)\times M,\mathbb{C}^N)$ is an extension of $\operatorname{diag}(\Theta) f$, and w is the solution of (3.B.32). For all $t \in [0,T]$, one has $\partial_t^2 w(t) - \mathsf{P}_0^* w(t) = -\left(\partial_t^2 - \mathsf{P}^*\right)\tilde{f}(t)$ in $H^{-1}(M,\mathbb{C}^N)$. Hence, one obtains

$$\left(\partial_t^2 - \mathsf{P}_{\mathscr{D}'}^*\right) v(t) = -\left(\partial_t^2 - \mathsf{P}^*\right)\tilde{f}(t) + \left(\partial_t^2 - \mathsf{P}^*\right)\tilde{f}(t) = 0, \quad t\in[0,T],$$

in $H^{-1}(M,\mathbb{C}^N)$. This gives (3.B.36).

Step 3: The additional regularity result. Here, we complete the proof of Theorem 3.1.14 for $s \leq 0$, by proving

$$(v(T), \partial_t v(T)) \in \mathcal{K}_*^s \times \mathcal{K}_*^{s-1},$$

and the duality equality (3.1.10), for $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$ and $s \in \mathbb{Z}$, $s \leq 0$. We start with some remarks. We know that $v(T) \in H^s(M, \mathbb{C}^N)$, that is, $v(T)$ is a continuous linear form on $H_0^{-s}(M, \mathbb{C}^N)$. As one has $H_0^{-s}(M, \mathbb{C}^N) \subset \mathcal{K}^{-s}$, we prove that $v(T)$ can be extended as a continuous linear form on $\mathcal{K}^{-s}$. Such an extension is not unique: however, we seek an extension such that (3.1.10) holds true, and

$$\|v(T)\|_{\mathcal{K}_*^s} \lesssim \|f\|_{H^s((0,T)\times\partial M, \mathbb{C}^N)} . \tag{3.B.37}$$

The same remarks can be made for $\partial_t v(T)$.

Consider $f \in \mathscr{C}^\infty((0,T) \times \partial M, \mathbb{C}^N)$, and write v, $\tilde{f}$, F and w as above. Because of the support of $\tilde{f}$, one has $(v(T), \partial_t v(T)) = (w(T), \partial_t w(T))$ in $H^s(M, \mathbb{C}^N) \times H^{s-1}(M, \mathbb{C}^N)$. Consider $(u^0, u^1) \in \mathcal{K}^2 \times \mathcal{K}^1$, and write u for the solution of

$$\begin{cases} \partial_t^2 u - \mathsf{P} u &= 0 & \text{in } (0,T) \times M, \\ (u(T,\cdot), \partial_t u(T,\cdot)) &= (u^0, u^1) & \text{in } M, \\ u &= 0 & \text{on } (0,T) \times \partial M. \end{cases} \tag{3.B.38}$$

Applying Lemma 3.B.5 to u and $\tilde{f}$, one finds

$$\left\langle u, \left(\partial_t^2 - \mathsf{P}^*\right) \tilde{f} \right\rangle_{L^2((0,T)\times M, \mathbb{C}^N)} = \langle \partial_\nu u, \operatorname{diag}(\Theta) f \rangle_{L^2((0,T)\times \partial M, \mathbb{C}^N)} .$$

By Lemma 3.B.3, one has

$$\langle u, F \rangle_{L^2((0,T)\times M, \mathbb{C}^N)} = \left\langle u^0, \partial_t w(T) \right\rangle_{L^2(M, \mathbb{C}^N)} - \left\langle u^1, w(T) \right\rangle_{L^2(M, \mathbb{C}^N)}$$

Thus, one obtains

$$\left\langle u^1, v(T) \right\rangle_{L^2(M, \mathbb{C}^N)} - \left\langle u^0, \partial_t v(T) \right\rangle_{L^2(M, \mathbb{C}^N)} = \langle \partial_\nu u, \operatorname{diag}(\Theta) f \rangle_{L^2((0,T)\times \partial M, \mathbb{C}^N)} .$$

If $s \leq -1$, then this is in particular true for all $(u^0, u^1) \in H_0^{-s+1}(M, \mathbb{C}^N) \times H_0^{-s}(M, \mathbb{C}^N)$. If $s = 0$, then this is true for all $(u^0, u^1) \in H_0^{-s+1}(M, \mathbb{C}^N) \times H_0^{-s}(M, \mathbb{C}^N)$ by density. Hence, one has

$$\left\langle v(T), u^1 \right\rangle_{H^s, H_0^{-s}} - \left\langle \partial_t v(T), u^0 \right\rangle_{H^{s-1}, H_0^{-s+1}} = \langle f, \operatorname{diag}(\Theta) \partial_\nu u \rangle_{H^s, H_0^{-s}} \tag{3.B.39}$$

for all f smooth. By density and continuity, this is true for all $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$. Yet, the right-hand side is well-defined for $(u^0, u^1) \in \mathcal{K}^{-s+1} \times \mathcal{K}^{-s}$, and one has

$$\left| \langle f, \operatorname{diag}(\Theta) \partial_\nu u \rangle_{H^s, H_0^{-s}} \right| \lesssim \left\| \left(u^0, u^1\right) \right\|_{\mathcal{K}^{s+1} \times \mathcal{K}^s} \|f\|_{H^s((0,T)\times \partial M, \mathbb{C}^N)} .$$

Hence, (3.B.39) yields a unique extension of $(v(T), \partial_t v(T))$ as a linear form on $\mathcal{K}^{-s} \times \mathcal{K}^{1-s}$, which satisfies (3.1.10) and (3.B.37).

Proof of Theorem 3.1.14 in positive regularity

Here, we prove Theorem 3.1.14 for $s > 0$. We know how to construct the solution v of (3.1.9) if $f \in L^2((0,T) \times \partial M, \mathbb{C}^N)$, and one has

$$v \in \mathscr{C}^0([0,T], L^2(M, \mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^{-1}(M, \mathbb{C}^N)).$$

If $f \in H^s((0,T) \times \partial M, \mathbb{C}^N)$ with $s > 0$, one can define v as in the case $s = 0$. We show that

$$v \in \mathscr{C}^0([0,T], H^s(M, \mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^{s-1}(M, \mathbb{C}^N)) \cap \mathscr{C}^2([0,T], H^{s-2}(M, \mathbb{C}^N)).$$

We will need the following regularity result, which is an easy consequence of the corresponding scalar result. Set $W = \left\{u \in L^2(M, \mathbb{C}^N), \mathsf{P}_{\mathscr{D}'} u \in H^{-1}(M, \mathbb{C}^N)\right\}$.

Lemma 3.B.6. *The Dirichlet trace $H^1(M, \mathbb{C}^N) \to H^{\frac{1}{2}}(M, \mathbb{C}^N)$ has a continuous extension as an operator from W to $H^{-\frac{1}{2}}(\partial M, \mathbb{C}^N)$, and there exists $C > 0$ such that*

$$\left\|u_{|\partial M}\right\|_{H^{-\frac{1}{2}}(\partial M, \mathbb{C}^N)} \le C \left(\|\mathsf{P}_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)}\right), \quad u \in W.$$

In addition, for $m \in \mathbb{N}$, if $u \in L^2(M, \mathbb{C}^N)$ satisfies $\mathsf{P}_{\mathscr{D}'} u \in H^{m-1}(M, \mathbb{C}^N)$ and $u_{|\partial M} \in H^{m+\frac{1}{2}}(M, \mathbb{C}^N)$, then $u \in H^{m+1}(M, \mathbb{C}^N)$ and

$$\|u\|_{H^{m+1}(M, \mathbb{C}^N)} \le C \left(\|\mathsf{P}_{\mathscr{D}'} u\|_{H^{m-1}(M, \mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{m+\frac{1}{2}}(\partial M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)}\right)$$

with $C > 0$ independent of u.

Proof. This result is well-know in the scalar case $N = 1$. We show that the vector-valued case is a consequence of the scalar case.

Take $u = (u^1, \cdots, u^N) \in L^2(M, \mathbb{C}^N)$ such that $\mathsf{P}_{\mathscr{D}'} u \in H^{-1}(M, \mathbb{C}^N)$. Write $(\pi_1, \cdots, \pi_N)$ for the projections associated with the canonical basis of $\mathbb{C}^N$. For $k \in [\![1, N]\!]$, one has $u^k \in L^2(M, \mathbb{C})$ and

$$\Delta_{\mathscr{D}'} u^k - \pi^k (Xu + qu) \in H^{-1}(M, \mathbb{C})$$

so that $\Delta_{\mathscr{D}'} u^k \in H^{-1}(M, \mathbb{C})$. Hence, the scalar case gives $u_{|\partial M} \in H^{-\frac{1}{2}}(M, \mathbb{C}^N)$, with

$$\left\|u_{|\partial M}\right\|_{H^{-\frac{1}{2}}(\partial M, \mathbb{C}^N)} \lesssim \|\Delta_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)}.$$

Writing

$$\begin{aligned}\|\Delta_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} &\le \|\mathsf{P}_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \|Xu + qu\|_{H^{-1}(M, \mathbb{C}^N)} \\ &\lesssim \|\mathsf{P}_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)},\end{aligned}$$

one obtains the first part of the lemma.

We prove the second part of the lemma by induction. Start with $m = 0$ and $u \in L^2(M, \mathbb{C}^N)$ such that $\mathsf{P}_{\mathscr{D}'} u \in H^{-1}(M, \mathbb{C}^N)$ and $u_{|\partial M} \in H^{\frac{1}{2}}(M, \mathbb{C}^N)$. As above, for $k \in [\![1, N]\!]$, one has $\Delta_{\mathscr{D}'} u^k \in H^{-1}(M, \mathbb{C})$, so the scalar case gives $u \in H^1(M, \mathbb{C}^N)$ and

$$\begin{aligned}\|u\|_{H^1(M, \mathbb{C}^N)} &\lesssim \|\Delta_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{\frac{1}{2}}(\partial M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)} \\ &\le \|\mathsf{P}_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \|Xu + qu\|_{H^{-1}(M, \mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{\frac{1}{2}}(\partial M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)} \\ &\lesssim \|\mathsf{P}_{\mathscr{D}'} u\|_{H^{-1}(M, \mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{\frac{1}{2}}(\partial M, \mathbb{C}^N)} + \|u\|_{L^2(M, \mathbb{C}^N)}.\end{aligned}$$

Finally, assume that the result holds for some $m \in \mathbb{N}$. Take $u \in L^2(M, \mathbb{C}^N)$ such that $\mathsf{P}_{\mathscr{D}'} u \in H^{(m+1)-1}(M, \mathbb{C}^N)$ and $u_{|\partial M} \in H^{m+1+\frac{1}{2}}(M, \mathbb{C}^N)$. By induction, $u \in H^{m+1}(M, \mathbb{C}^N)$ and

$$\|u\|_{H^{m+1}(M,\mathbb{C}^N)} \lesssim \|\mathsf{P}_{\mathscr{D}'} u\|_{H^{m-1}(M,\mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{m+\frac{1}{2}}(\partial M,\mathbb{C}^N)} + \|u\|_{L^2(M,\mathbb{C}^N)}. \tag{3.B.40}$$

Hence, for $k \in [\![1, N]\!]$, one has

$$\Delta_{\mathscr{D}'} u^k = \pi^k \left(\mathsf{P}_{\mathscr{D}'} u + Xu + qu\right) \in H^m(M, \mathbb{C})$$

so the scalar case gives $u^k \in H^{m+2}(M, \mathbb{C})$ and

$$\begin{aligned}
&\|u\|_{H^{m+2}(M,\mathbb{C}^N)} \\
&\lesssim \|\Delta_{\mathscr{D}'} u\|_{H^m(M,\mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{m+\frac{3}{2}}(\partial M,\mathbb{C}^N)} + \|u\|_{L^2(M,\mathbb{C}^N)} \\
&\leq \|\mathsf{P}_{\mathscr{D}'} u\|_{H^m(M,\mathbb{C}^N)} + \|Xu + qu\|_{H^m(M,\mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{m+\frac{3}{2}}(\partial M,\mathbb{C}^N)} + \|u\|_{L^2(M,\mathbb{C}^N)} \\
&\lesssim \|\mathsf{P}_{\mathscr{D}'} u\|_{H^m(M,\mathbb{C}^N)} + \|u\|_{H^{m+1}(M,\mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{m+\frac{3}{2}}(\partial M,\mathbb{C}^N)} + \|u\|_{L^2(M,\mathbb{C}^N)}.
\end{aligned}$$

Using (3.B.40), one finds

$$\|u\|_{H^{m+2}(M,\mathbb{C}^N)} \lesssim \|\mathsf{P}_{\mathscr{D}'} u\|_{H^m(M,\mathbb{C}^N)} + \left\|u_{|\partial M}\right\|_{H^{m+\frac{3}{2}}(\partial M,\mathbb{C}^N)} + \|u\|_{L^2(M,\mathbb{C}^N)}$$

and this completes the proof. □

We prove Theorem 3.1.14 by induction on $s \geq 1$.

Step 1: The case $s = 1$**.** Consider $f \in H^1((0,T) \times \partial M, \mathbb{C}^N)$, and write v for the associated solution. From the case $s = 0$, one has

$$v \in \mathscr{C}^0([0,T], L^2(M, \mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^{-1}(M, \mathbb{C}^N)) \cap \mathscr{C}^2([0,T], H^{-2}(M, \mathbb{C}^N))$$

and $\partial_t^2 v = \mathsf{P}^*_{\mathscr{D}'} v$ in $\mathscr{D}'((0,T) \times M, \mathbb{C}^N)$. We prove first that

$$v \in \mathscr{C}^1([0,T], L^2(M, \mathbb{C}^N)) \cap \mathscr{C}^2([0,T], H^{-1}(M, \mathbb{C}^N)) \tag{3.B.41}$$

with an inequality. Then, using Lemma 3.B.6, we show that

$$v \in \mathscr{C}^0([0,T], H^1(M, \mathbb{C}^N)), \tag{3.B.42}$$

with an inequality, and

$$v(t)_{|\partial M} = (\mathrm{diag}(\Theta) f)_{|\{t\} \times \partial M} \tag{3.B.43}$$

in $H^{\frac{1}{2}}(\partial M, \mathbb{C}^N)$ for all $t \in [0,T]$.

We prove (3.B.41). Let $\tilde{v}$ be the solution of

$$\left\{\begin{array}{rcll}
\partial_t^2 \tilde{v} - \mathsf{P}^* \tilde{v} & = & 0 & \text{in } (0,T) \times M, \\
(\tilde{v}(0,\cdot), \partial_t \tilde{v}(0,\cdot)) & = & 0 & \text{in } M, \\
\tilde{v} & = & \partial_t(\mathrm{diag}(\Theta) f) & \text{on } (0,T) \times \partial M.
\end{array}\right. \tag{3.B.44}$$

For $\tilde{\Theta} \in \mathscr{C}^\infty_{\mathrm{c}}((0,T) \times \partial M, \mathbb{C}^N)$ such that for all $k \in [\![1, N]\!]$, $\pi_k \tilde{\Theta} = 1$ in a neighbourhood of $\operatorname{supp} \pi_k \Theta$, one has $\partial_t \left(\mathrm{diag}(\Theta) f\right) = \mathrm{diag}(\tilde{\Theta}) \partial_t \left(\mathrm{diag}(\Theta) f\right)$, implying that $\tilde{v}$ is well-defined. One has

$$\partial_t \left(\mathrm{diag}(\Theta) f\right) \in L^2((0,T) \times \partial M, \mathbb{C}^N),$$

yielding $\tilde{v} \in \mathscr{C}^0([0,T], L^2(M,\mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^{-1}(M,\mathbb{C}^N))$ by Theorem 3.1.14 in the case $s = 0$. We show that $\partial_t v = \tilde{v}$. Consider $\phi \in \mathscr{C}_c^\infty((0,T) \times \operatorname{Int} M, \mathbb{C}^N)$. Let u and $\tilde{u}$ be the solutions of

$$\begin{cases} \partial_t^2 u - \mathsf{P}u &= \phi & \text{in } (0,T)\times M, \\ (u(T,\cdot), \partial_t u(T,\cdot)) &= 0 & \text{in } M, \\ u &= 0 & \text{on } (0,T)\times \partial M, \end{cases}$$

$$\begin{cases} \partial_t^2 \tilde{u} - \mathsf{P}\tilde{u} &= \partial_t \phi & \text{in } (0,T)\times M, \\ (\tilde{u}(T,\cdot), \partial_t \tilde{u}(T,\cdot)) &= 0 & \text{in } M, \\ \tilde{u} &= 0 & \text{on } (0,T)\times \partial M. \end{cases}$$

One has $\partial_t u = \tilde{u}$, yielding, by definition of v and $\tilde{v}$,

$$\begin{aligned} \langle \partial_t v, \phi\rangle_{L^\infty((0,T),H^{-1}),L^1((0,T),H_0^1)} &= -\left\langle v, \partial_t \overline{\phi}\right\rangle_{L^2((0,T)\times M,\mathbb{C}^N)} \\ &= \langle f, \operatorname{diag}(\Theta)\partial_\nu \overline{\tilde{u}}\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} \\ &= -\left\langle \partial_t \left(\operatorname{diag}(\Theta) f\right), \partial_\nu \overline{u}\right\rangle_{L^2((0,T)\times\partial M,\mathbb{C}^N)} \\ &= \langle \tilde{v}, \phi\rangle_{L^\infty((0,T),L^2),L^1((0,T),L^2)} . \end{aligned}$$

This gives (3.B.41). In addition, one has

$$\begin{aligned} \|v\|_{\mathscr{C}^1([0,T],L^2)\cap\mathscr{C}^2([0,T],H^{-1})} &\lesssim \|v\|_{\mathscr{C}^0([0,T],L^2)\cap\mathscr{C}^1([0,T],H^{-1})} + \|\tilde{v}\|_{\mathscr{C}^0([0,T],L^2)\cap\mathscr{C}^1([0,T],H^{-1})} \\ &\lesssim \|f\|_{L^2((0,T)\times\partial M,\mathbb{C}^N)} + \|\partial_t \left(\operatorname{diag}(\Theta) f\right)\|_{L^2((0,T)\times\partial M,\mathbb{C}^N)} \\ &\lesssim \|f\|_{H^1((0,T)\times\partial M,\mathbb{C}^N)}. \end{aligned}$$

Now, we prove (3.B.42). Consider $f \in \mathscr{C}^\infty((0,T)\times\partial M, \mathbb{C}^N)$, and write $v = w + \tilde{f}$ as in the proof of Theorem 3.1.14 in negative regularity. As $w \in \mathscr{C}^0([0,T], H_0^1(M,\mathbb{C}^N))$, (3.B.42) is true for f smooth. To get (3.B.42) for all f, we prove

$$\|v\|_{L^\infty((0,T),H^1(M,\mathbb{C}^N))} \lesssim \|f\|_{H^1((0,T)\times\partial M,\mathbb{C}^N)} . \tag{3.B.45}$$

For $t \in [0,T]$, one has

$$\mathsf{P}^*_{\mathscr{D}'} v(t) = \partial_t^2 v(t) \in H^{-1}(M,\mathbb{C}^N)$$

and

$$v(t)_{|\partial M} = w(t)_{|\partial M} + \tilde{f}(t)_{|\partial M} = 0 + (\operatorname{diag}(\Theta) f)_{|\{t\}\times\partial M} \tag{3.B.46}$$

in $H^{\frac{1}{2}}(\partial M, \mathbb{C}^N)$. Hence, Lemma 3.B.6 gives

$$\begin{aligned} \|v(t)\|_{H^1(M,\mathbb{C}^N)} &\lesssim \left\|\partial_t^2 v(t)\right\|_{H^{-1}(M,\mathbb{C}^N)} + \left\|(\operatorname{diag}(\Theta) f)_{|\{t\}\times\partial M}\right\|_{H^{\frac{1}{2}}(\partial M,\mathbb{C}^N)} + \|v\|_{L^\infty((0,T),L^2)} \\ &\lesssim \|f\|_{H^1((0,T)\times\partial M,\mathbb{C}^N)} . \end{aligned}$$

for all $t \in [0,T]$. This gives (3.B.45). By density, it holds for all $f \in H^1((0,T)\times\partial M,\mathbb{C}^N)$, yielding (3.B.42). Note also that (3.B.43) holds for smooth f by (3.B.46). As both sides of (3.B.43) are continuous with respect to $f \in H^1((0,T)\times\partial M,\mathbb{C}^N)$, we obtain (3.B.43) for all $f \in H^1((0,T)\times\partial M,\mathbb{C}^N)$.

Step 2: The case $s \in \mathbb{N}^*$. We show by induction on $s \in \mathbb{N}^*$ that

$$v \in \mathscr{C}^0([0,T], H^s(M,\mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^{s-1}(M,\mathbb{C}^N)) \cap \mathscr{C}^2([0,T], H^{s-2}(M,\mathbb{C}^N)), \quad (3.B.47)$$

if $f \in H^s((0,T)\times\partial M,\mathbb{C}^N)$, with an inequality, and $v(t)_{|\partial M} = (\mathrm{diag}(\Theta)f)_{|\{t\}\times\partial M}$ for all $t \in [0,T]$, with equality in $H^{s-\frac{1}{2}}(M,\mathbb{C}^N)$. Assume that the result holds for some $s \in \mathbb{N}^*$, and consider $f \in H^{s+1}((0,T)\times\partial M,\mathbb{C}^N)$.

By induction, one has $v(t)_{|\partial M} = (\mathrm{diag}(\Theta)f)_{|\{t\}\times\partial M}$ in $H^{s-\frac{1}{2}}(M,\mathbb{C}^N)$ for $t \in [0,T]$. In particular, this gives $v(t)_{|\partial M} \in H^{s+\frac{1}{2}}(M,\mathbb{C}^N)$. As in the case $s=1$, one has $\partial_t v = \tilde{v}$, where $\tilde{v}$ is the solution of (3.B.44), and this gives

$$\|v\|_{\mathscr{C}^1([0,T],H^s)\cap\mathscr{C}^2([0,T],H^{s-1})} \lesssim \|f\|_{H^{s+1}((0,T)\times\partial M,\mathbb{C}^N))},$$

since $\tilde{v}$ fulfills (3.B.47). In particular, one has $\mathsf{P}^*_{\mathscr{D}'}v(t) = \partial_t^2 v(t) \in H^{s-1}(M,\mathbb{C}^N)$ for $t \in [0,T]$. Hence, for $t \in [0,T]$, Lemma 3.B.6 gives $v(t) \in H^{s+1}(M,\mathbb{C}^N)$, and

$$\begin{aligned}
\|v(t)\|_{H^{s+1}(M,\mathbb{C}^N)} &\lesssim \|\mathsf{P}^*_{\mathscr{D}'}v(t)\|_{H^{s-1}(M,\mathbb{C}^N)} + \left\|v(t)_{|\partial M}\right\|_{H^{s+\frac{1}{2}}(\partial M,\mathbb{C}^N)} + \|v(t)\|_{L^2(M,\mathbb{C}^N)} \\
&\lesssim \left\|\partial_t^2 v(t)\right\|_{H^{s-1}(M,\mathbb{C}^N)} + \left\|(\mathrm{diag}(\Theta)f)_{|\{t\}\times\partial M}\right\|_{H^{s+\frac{1}{2}}(\partial M,\mathbb{C}^N)} + \|v\|_{L^\infty((0,T),L^2)} \\
&\lesssim \|f\|_{H^{s+1}((0,T)\times\partial M,\mathbb{C}^N)}.
\end{aligned}$$

This gives $v \in L^\infty([0,T], H^{s+1}(M,\mathbb{C}^N))$. To complete the proof, it suffices to show that

$$v \in \mathscr{C}^0([0,T], H^{s+1}(M,\mathbb{C}^N))$$

for f smooth. For such f, Lemma 3.B.6 gives

$$\begin{aligned}
\|v(t+\varepsilon)-v(t)\|_{H^{s+1}(M,\mathbb{C}^N)} \lesssim{}& \left\|\partial_t^2 v(t+\varepsilon) - \partial_t^2 v(t)\right\|_{H^{s-1}(M,\mathbb{C}^N)} \\
&+ \|f(t+\varepsilon)-f(t)\|_{H^{s+\frac{1}{2}}(\partial M,\mathbb{C}^N)} + \|v(t+\varepsilon)-v(t)\|_{L^2(M,\mathbb{C}^N)},
\end{aligned}$$

and one concludes

$$\|v(t+\varepsilon)-v(t)\|_{H^{s+1}(M,\mathbb{C}^N)} \xrightarrow{\varepsilon\to 0} 0.$$

This gives (3.B.47).

Step 3: The additional regularity result. Here, we complete the proof of Theorem 3.1.14 by proving

$$(v(T), \partial_t v(T)) \in \mathcal{K}^s_* \times \mathcal{K}^{s-1}_* \quad (3.B.48)$$

for $f \in H^s((0,T)\times\partial M,\mathbb{C}^N)$ and $s \in \mathbb{N}^*$. We will also prove the duality equality of Theorem 3.1.14. As $\mathcal{K}^s_* \times \mathcal{K}^{s-1}_*$ is a closed subspace of $H^s(M,\mathbb{C}^N)\times H^{s-1}(M,\mathbb{C}^N)$, it suffices to prove (3.B.48) for f smooth. We proceed by induction. The result is true for $s=0$, but one has to treat the case $s=1$ separately.

Case 1: $s=1$. Take $f \in \mathscr{C}^\infty((0,T)\times\partial M,\mathbb{C}^N)$, and write v for the associated solution. We prove that $v(T) \in H^1_0(M,\mathbb{C}^N)$. Write $v = w + \tilde{f}$ as above. One has $w \in \mathscr{C}^0([0,T], H^1_0(M,\mathbb{C}^N))$, implying

$$v(T) = w(T) + \tilde{f}(T) = w(T) \in H^1_0(M,\mathbb{C}^N),$$

since $\tilde{f}$ is compactly supported in $(0,T)\times M$. Note that $w \in \mathscr{C}^0([0,T],\mathcal{K}^s_*)$ for $s \geq 2$ is not always true, preventing a straightforward generalization of this argument.

Case 2: s odd. Assume that the result holds true for some odd $s \in \mathbb{N}^*$. Write $s = 2\sigma + 1$, and consider $f \in \mathscr{C}^\infty((0,T) \times \partial M, \mathbb{C}^N)$. We prove that $(v(T), \partial_t v(T)) \in \mathcal{K}_*^{2\sigma+2} \times \mathcal{K}_*^{2\sigma+1}$. By induction, one has $(v(T), \partial_t v(T)) \in \mathcal{K}_*^{2\sigma+1} \times \mathcal{K}_*^{2\sigma}$, and as $f \in H^{s+1}((0,T) \times \partial M, \mathbb{C}^N)$, we know that

$$v \in \mathscr{C}^0([0,T], H^{s+1}(M, \mathbb{C}^N)) \cap \mathscr{C}^1([0,T], H^s(M, \mathbb{C}^N)).$$

By definition, $H^{2\sigma+2}(M, \mathbb{C}^N)) \cap \mathcal{K}_*^{2\sigma+1} = \mathcal{K}_*^{2\sigma+2}$, implying $v(T) \in \mathcal{K}_*^{2\sigma+2}$. Set $\tilde{v} = \partial_t v$, solution to (3.B.44). By induction, one has $\partial_t v(T) = \tilde{v}(T) \in \mathcal{K}_*^{2\sigma+1}$, completing the proof in the case s is odd.

Case 3: s even. Assume that the result holds for some even $s \in \mathbb{N}^*$. Write $s = 2\sigma$, and consider $f \in \mathscr{C}^\infty((0,T) \times \partial M, \mathbb{C}^N)$. We show that $(v(T), \partial_t v(T)) \in \mathcal{K}_*^{2\sigma+1} \times \mathcal{K}_*^{2\sigma}$. By induction, one has $(v(T), \partial_t v(T)) \in \mathcal{K}_*^{2\sigma} \times \mathcal{K}_*^{2\sigma-1}$, and we know that $(v(T), \partial_t v(T)) \in H^{s+1}(M, \mathbb{C}^N) \times H^s(M, \mathbb{C}^N)$, yielding $\partial_t v(T) \in \mathcal{K}_*^{2\sigma}$ as above. The definition of $\mathcal{K}_*^{2\sigma+1}$ reads

$$\mathcal{K}_*^{2\sigma+1} = \left\{ u \in H^{2\sigma+1}(M, \mathbb{C}^N)) \cap \mathcal{K}_*^{2\sigma}, \mathsf{P}_{\mathscr{D}'}^{*}{}^{\sigma} u \in H_0^1(M, \mathbb{C}^N)) \right\},$$

implying that one has $v(T) \in \mathcal{K}_*^{2\sigma+1}$, if $\mathsf{P}_{\mathscr{D}'}^{*}{}^{\sigma} v(T) \in H_0^1(M, \mathbb{C}^N))$. Again, write $\tilde{v} = \partial_t v$. One has $\mathsf{P}_{\mathscr{D}'}^* v(T) = \partial_t^2 v(T) = \partial_t \tilde{v}(T)$, and by induction, one finds $\partial_t \tilde{v}(T) \in \mathcal{K}_*^{2\sigma-1}$. Thus, as $\sigma > 0$, one obtains

$$\mathsf{P}_{\mathscr{D}'}^{*}{}^{\sigma} v(T) = \mathsf{P}_{\mathscr{D}'}^{*}{}^{\sigma-1} \partial_t \tilde{v}(T) \in H_0^1(M, \mathbb{C}^N),$$

completing the proof in the case s is even.

Finally, we prove the duality equality of Theorem 3.1.14 for $s \geq 1$. Consider $s \geq 1$, $f \in H^s((0,T) \times \partial M, \mathbb{C}^{\mathbb{N}})$ and $(u^0, u^1) \in \mathcal{K}^{-s+1} \times \mathcal{K}^{-s}$. Write u for the solution of (3.B.38). We show that

$$\left\langle u^1, v(T) \right\rangle_{\mathcal{K}^{-s+1}, \mathcal{K}_*^{s-1}} - \left\langle u^0, \partial_t v(T) \right\rangle_{\mathcal{K}^{-s}, \mathcal{K}_*^{s}} = \left\langle \partial_\nu u, \mathrm{diag}(\Theta) f \right\rangle_{H^{-s}, H_0^s}. \tag{3.B.49}$$

With the approximation result of Theorem 3.1.12, consider a sequence $\left((u_k^0, u_k^1)\right)_{k\in\mathbb{N}}$ of elements of $\mathcal{K}^2 \times \mathcal{K}^1$ such that

$$\left(\iota_{\mathcal{K}^2 \to \mathcal{K}^{-s+1}} u_k^0, \iota_{\mathcal{K}^1 \to \mathcal{K}^{-s}} u_k^1 \right) \xrightarrow{k\to\infty} \left(u^0, u^1 \right) \text{ in } \mathcal{K}^{-s+1} \times \mathcal{K}^{-s}.$$

If u_k is the solution of (3.B.38) associated with (u_k^0, u_k^1), then one has

$$\iota_{\mathcal{K}^2 \to \mathcal{K}^{-s+1}} u_k \xrightarrow{k\to\infty} u \text{ in } \mathscr{C}^0([0,T], \mathcal{K}^{-s+1}) \cap \mathscr{C}^1([0,T], \mathcal{K}^{-s}),$$

and

$$\partial_\nu \left(\iota_{\mathcal{K}^2 \to \mathcal{K}^{-s+1}} u_k \right) \xrightarrow{k\to\infty} \partial_\nu u \text{ in } H^{-s}((0,T) \times \partial M, \mathbb{C}^N).$$

The duality equality of Theorem 3.1.14 for $s = 0$ gives

$$\left\langle u_k^1, v(T) \right\rangle_{L^2(M,\mathbb{C}^N)} - \left\langle u_k^0, \partial_t v(T) \right\rangle_{L^2(M,\mathbb{C}^N)} = \left\langle \mathrm{diag}(\Theta) \partial_\nu u_k, f \right\rangle_{L^2((0,T)\times\partial M, \mathbb{C}^N)}.$$

Since $(v(T), \partial_t v(T)) \in \mathcal{K}_*^s \times \mathcal{K}_*^{s-1}$, one finds

$$\begin{aligned} &\left\langle \iota_{\mathcal{K}^2 \to \mathcal{K}^{-s+1}} u_k^1, v(T) \right\rangle_{\mathcal{K}^{-s+1}, \mathcal{K}_*^{s-1}} - \left\langle \iota_{\mathcal{K}^1 \to \mathcal{K}^{-s}} u_k^0, \partial_t v(T) \right\rangle_{\mathcal{K}^{-s}, \mathcal{K}_*^{s}} \\ &= \left\langle \iota_{H^1 \to H^{-s}} \partial_\nu u_k, \mathrm{diag}(\Theta) f \right\rangle_{H^{-s}, H_0^s}. \end{aligned}$$

From Theorem 3.1.12, one has $\iota_{H^1 \to H^{-s}} \partial_\nu u_k = \partial_\nu \iota_{\mathcal{K}^2 \to \mathcal{K}^{-s+1}} u_k$, yielding (3.B.49).

Chapter 4

Uniform estimates for solutions of nonlinear focusing damped wave equations

This chapter is based on the article [Per24], which has been prepublished and will be submitted in a journal shortly.

Contents

Introduction

Setting and main result. Let Ω be a compact Riemannian manifold of dimension $d \geq 3$, with or without boundary. Let $\beta \in \mathbb{R}$ be such that the Poincaré inequality

$$\int_\Omega \left(|\nabla u|^2 + \beta |u|^2 \right) \mathrm{d}x \gtrsim \int_\Omega |u|^2 \mathrm{d}x, \quad u \in H_0^1(\Omega), \tag{4.0.1}$$

is satisfied. For real-valued initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, and $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$, consider the semilinear wave (or Klein-Gordon) equation

$$\begin{cases} \Box u + \gamma \partial_t u + \beta u &= f(u) & \text{in } \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } \mathbb{R} \times \partial\Omega. \end{cases} \tag{4.0.2}$$

Set $F(s) = \int_0^s f(\tau) \mathrm{d}\tau$. We assume that $f(0) = 0$, and that there exist $1 < p < +\infty$, $1 < p_0 < +\infty$, $C > 0$, $\varepsilon > 0$ such that

$$|f(s_1) - f(s_2)| \leq C |s_1 - s_2| \left(1 + |s_1|^{p-1} + |s_2|^{p-1} \right), \quad s_1, s_2 \in \mathbb{R}, \tag{4.0.3}$$

$$sf(s) \geq (2+\varepsilon)F(s), \quad s \in \mathbb{R}, \tag{4.0.4}$$

and

$$\|u\|_{L^2}^{p_0+1} \leq C \int_\Omega F(u) \mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{4.0.5}$$

The fact that Ω is bounded will only be used to ensure the existence of a nonlinearity satisfying (4.0.3), (4.0.4) and (4.0.5). We will always assume that Ω and f are such that the Cauchy problem (4.0.2) can be solved locally (see Theorem 4.1.1 for a precise statement).

For $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, write $\|u^0\|_{H_0^1}^2 = \int_\Omega \left(|\nabla u^0|^2 + \beta |u^0|^2\right) \mathrm{d}x$ and set

$$E\left(u^0, u^1\right) = \frac{1}{2}\left\|u^0\right\|_{H_0^1}^2 + \frac{1}{2}\left\|u^1\right\|_{L^2}^2 - \int_\Omega F(u^0) \mathrm{d}x.$$

For $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solution u of (4.0.2) exists on some interval $[0, T]$, then one has the energy equality

$$E\left(u(t_1), \partial_t u(t_1)\right) = E\left(u(t_0), \partial_t u(t_0)\right) - \int_{t_0}^{t_1} \int_\Omega \gamma \left|\partial_t u\right|^2 \mathrm{d}t\mathrm{d}x, \quad 0 \leq t_0 \leq t_1 \leq T. \tag{4.0.6}$$

The objective of this chapter is to prove the following theorem, which generalizes the results of [Caz85] to the case of a damped equation.

Theorem 4.0.1. *Assume that f satisfies* (4.0.3), *with $p < \frac{d+2}{d-2}$,* (4.0.4), *and* (4.0.5).

(i) There exists $C = C(f, \gamma) > 0$ such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solution u of (4.0.2) *exists on $\mathbb{R}_+$, then*

$$E\left(u^0, u^1\right) \geq E\left(u(t), \partial_t u(t)\right) \geq -C, \quad t \geq 0.$$

(ii) There exist $c_0 = c_0(f, \gamma) > 0$, $c_1 = c_1(f, \gamma) > 0$, and $c_2 = c_2(f) > 0$ such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solution u of (4.0.2) *exists on $\mathbb{R}_+$, then*

$$\|u(t)\|_{L^2}^2 \leq \left\|u^0\right\|_{L^2}^2 e^{-c_2 t} + \left(c_0 + c_1 \left|E\left(u^0, u^1\right)\right|\right)\left(1 - e^{-c_2 t}\right), \quad t \geq 0. \tag{4.0.7}$$

(iii) Assume, in addition, that f satisfies (4.0.3) *with $p \leq \frac{d}{d-2}$. Then, there exists $c = c(f, \gamma) > 0$ such that the function $\alpha : s \mapsto c \exp(cs)$ satisfies the following property. For all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solution u of* (4.0.2) *exists on $\mathbb{R}_+$, then*

$$\|(u(t), \partial_t u(t))\|_{H_0^1 \times L^2} \leq \alpha\left(\left\|\left(u^0, u^1\right)\right\|_{H_0^1 \times L^2}^2\right), \quad t \geq 0. \tag{4.0.8}$$

In addition, there exists $T \geq 0$, which depends on f, γ and $\|(u^0, u^1)\|_{H_0^1 \times L^2}$, such that

$$\|(u(t), \partial_t u(t))\|_{H_0^1 \times L^2} \leq \alpha\left(\left|E\left(u^0, u^1\right)\right|\right), \quad t \geq T. \tag{4.0.9}$$

All constants in this chapter may depend on Ω. Note that when we specify that a constant depends on f, it depends, in fact, only on the constants appearing in (4.0.3), (4.0.4), and (4.0.5). We refer to Section 2.2 for a discussion about Theorem 4.0.1 and its connection to the existing literature.

Examples. A typical example of f satisfying (4.0.3) for some $p > 1$, (4.0.4), and (4.0.5) for some $p_0 > 1$, is given by

$$f(s) = \lambda_1 s|s|^{\alpha_1 - 1} + \cdots + \lambda_n s|s|^{\alpha_n - 1}, \quad s \in \mathbb{R},$$

with $n \in \mathbb{N}^*$, $\lambda_i > 0$ for all $i \in [\![1, n]\!]$, and $p_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq p$. Assumption (4.0.5) is satisfied by such f since Ω is bounded. As previously mentioned, the fact that Ω is bounded will not be used elsewhere in this chapter. Theorem 4.0.1-*(i)* and Theorem 4.0.1-*(ii)* can be applied to such f if $p < \frac{d+2}{d-2}$, and Theorem 4.0.1-*(iii)* can also be applied if $p \leq \frac{d}{d-2}$. For example, Theorem 4.0.1 applies to the focusing cubic wave equation on a compact manifold of dimension 3.

Outline of the chapter. In Section 1, we recall the local Cauchy theory for (4.0.2), and the expression of the derivative of the square of the L^2-norm of a solution. In Section 2, 3 and 4, we prove respectively the energy bound (Theorem 4.0.1-*(i)*), the L^2-bound (Theorem 4.0.1-*(ii)*), and the H^1-bound (Theorem 4.0.1-*(iii)*). An appendix contains proofs of some elementary lemmas.

4.1 Preliminaries

We recall the local Cauchy theory for (4.0.2).

Theorem 4.1.1. *Consider f satisfying* (4.0.3) *for some $p < \frac{d+2}{d-2}$. For any real-valued initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exist a maximal time of existence $T \in (0, +\infty]$ and a unique solution u of* (4.0.2) *in $\mathscr{C}^0([0,T), H_0^1(\Omega)) \cap \mathscr{C}^1([0,T), L^2(\Omega))$. If $T < +\infty$, then*

$$\|(u(t), \partial_t u(t))\|_{H_0^1 \times L^2} \underset{t \to T^-}{\longrightarrow} +\infty.$$

The proof of Theorem 4.1.1 relies on Strichartz estimates and a fixed point argument (see for example [GV89]). If Ω is a manifold without boundary, then Theorem 4.1.1 follows from the Strichartz estimates proved in [KT98]. In the case of a manifold with boundary, we *assume* that f is such that Theorem 4.1.1 holds true : we refer to [BSS09] for Strichartz estimates for a large range of exponents. Ivanovici's counterexamples in [Iva12] show that Strichartz estimates are not true for the full range of exponents in the case of a manifold with boundary.

Lemma 4.1.2. *Consider u a solution of* (4.0.2) *on some interval $[0,T]$, and set $M(t) = \|u(t)\|_{L^2}^2$, for $t \in [0,T]$. Then $M \in \mathscr{C}^2([0,T], \mathbb{R})$, and for $t \in [0,T]$, one has $M'(t) = 2\int_\Omega u(t)\partial_t u(t) \mathrm{d}x$, and*

$$M''(t) = 2\|\partial_t u(t)\|_{L^2}^2 - 2\|u(t)\|_{H_0^1}^2 + 2\int_\Omega u(t) f(u(t)) \mathrm{d}x - 2\int_\Omega \gamma u(t)\partial_t u(t) \mathrm{d}x. \qquad (4.1.1)$$

Proof. First, note that $u \in \mathscr{C}^0(\mathbb{R}_+, H_0^1(\Omega)) \cap \mathscr{C}^1(\mathbb{R}_+, L^2(\Omega))$ implies $M \in \mathscr{C}^1(\mathbb{R}_+, \mathbb{R})$, with $M'(t) = 2\int_\Omega u(t)\partial_t u(t)\mathrm{d}x$ for all $t \in [0,T]$. By Theorem 4.1.1, there exists $\varepsilon > 0$ such that u is defined on $(-\varepsilon, T+\varepsilon)$. We show that (4.1.1) holds true in $\mathcal{D}'((-\varepsilon, T+\varepsilon), \mathbb{R})$. Consider $\phi \in \mathscr{C}_c^\infty((0,+\infty), \mathbb{R})$, and write

$$\int_\mathbb{R} M'(t)\phi'(t)\mathrm{d}t = 2\int_\mathbb{R} \langle \partial_t u(t), \partial_t (u\phi)(t) - \partial_t u(t)\phi(t)\rangle_{L^2} \mathrm{d}t.$$

One has $u\phi \in H_0^1((-\varepsilon, T+\varepsilon)\times\Omega)$. Consider a sequence $(u_n)_n$ of elements of $\mathscr{C}_c^\infty((-\varepsilon, T+\varepsilon)\times\Omega, \mathbb{R})$ that converges to $u\phi$ in $H^1((-\varepsilon, T+\varepsilon)\times\Omega)$. For $n\in\mathbb{N}$, using the equation satisfied by u, one obtains

$$\begin{aligned}\int_{\mathbb{R}} \langle \partial_t u(t), \partial_t u_n(t)\rangle_{L^2}\,\mathrm{d}t &= -\left\langle \partial_t^2 u, u_n\right\rangle_{\mathcal{D}'((-\varepsilon,T+\varepsilon)\times\Omega),\mathcal{D}((-\varepsilon,T+\varepsilon)\times\Omega)} \\ &= \int_{\mathbb{R}}\int_\Omega \left(\nabla u\nabla u_n + \gamma\partial_t u u_n + \beta u u_n - u^3 u_n\right)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

In the limit $n\to+\infty$, one obtains

$$\int_{\mathbb{R}} \langle \partial_t u(t), \partial_t (u\phi)(t)\rangle_{L^2}\,\mathrm{d}t = \int_{\mathbb{R}}\int_\Omega \left(\nabla u\nabla(u\phi) + \gamma u\partial_t u\phi + \beta u^2\phi - u^4\phi\right)\mathrm{d}x\mathrm{d}t$$

and as ϕ is independent of x, this gives

$$\int_{\mathbb{R}} M'\phi'\mathrm{d}t = 2\int_{\mathbb{R}}\int_\Omega \left(|\nabla u|^2 + \gamma u\partial_t u + \beta u^2 - u^4 - (\partial_t u)^2\right)\phi\mathrm{d}x\mathrm{d}t.$$

This proves that (4.1.1) holds true in $\mathcal{D}'((-\varepsilon, T+\varepsilon), \mathbb{R})$. As the right-hand side of (4.1.1) depends continuously on $t\geq 0$, it implies that $M\in\mathscr{C}^2([0,T],\mathbb{R})$, and (4.1.1) holds true in a strong sense. □

4.2 The energy bound

Here, we prove Theorem 4.0.1-*(i)*. Consider $(u^0, u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ such that the solution u of (4.0.2) exists on $\mathbb{R}_+$. By (4.0.5), there exists $C_0 = C_0(f) > 0$ such that

$$\left|2\int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x\right| \leq C_0\|\gamma\|_{L^\infty}^2 \left(\int_\Omega F(u(t))\mathrm{d}x\right)^{\frac{2}{p_0+1}} + \frac{\varepsilon}{4}\|\partial_t u(t)\|_{L^2}^2, \quad t\geq 0, \tag{4.2.1}$$

where ε is given by (4.0.4). Using (4.0.4) and (4.2.1) in (4.1.1), one finds

$$\begin{aligned}M''(t) \geq &\left(2-\frac{\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2 - 2\|u(t)\|_{H_0^1}^2 \\ &+ 2(2+\varepsilon)\int_\Omega F(u(t))\mathrm{d}x - C_0\|\gamma\|_{L^\infty}^2\left(\int_\Omega F(u(t))\mathrm{d}x\right)^{\frac{2}{p_0+1}}, \quad t\geq 0.\end{aligned} \tag{4.2.2}$$

Write $I_0 = \left(\frac{C_0\|\gamma\|_{L^\infty}^2}{\varepsilon}\right)^{\frac{p_0+1}{p_0-1}} \geq 0$, so that

$$I\geq I_0 \implies 2(2+\varepsilon)I - C_0\|\gamma\|_{L^\infty}^2 I^{\frac{2}{p_0+1}} \geq (4+\varepsilon)I.$$

Assume by contradiction that there exists $T_0\geq 0$ such that $E(u(T_0), \partial_t u(T_0)) < -I_0$. Then for $t\geq T_0$, one has $\int_\Omega F(u(t))\mathrm{d}x \geq -E(u(t), \partial_t u(t)) > I_0$, yielding

$$M''(t) \geq \left(2-\frac{\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2 - 2\|u(t)\|_{H_0^1}^2 + (4+\varepsilon)\int_\Omega F(u(t))\mathrm{d}x, \quad t\geq T_0.$$

By definition of the energy, this gives

$$M''(t) \geq \left(4+\frac{\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2 + \frac{\varepsilon}{2}\|u(t)\|_{H_0^1}^2 - (4+\varepsilon)E(u(t), \partial_t u(t)), \quad t\geq T_0. \tag{4.2.3}$$

One has $E\left(u(t),\partial_t u(t)\right)\leq 0$ for $t\geq T_0$, implying

$$M''(t)\geq\left(4+\frac{\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2,\quad t\geq T_0.$$

In particular, one obtains

$$\left(M'(t)\right)^2\leq\left(1+\frac{\varepsilon}{16}\right)^{-1}M(t)M''(t),\quad t\geq T_0.$$

We use the following lemma.

Lemma 4.2.1. *Consider $T\geq 0$, $M\in\mathscr{C}^2([T,+\infty),\mathbb{R}_+)$ and $0<\delta<1$ such that $\left(M'(t)\right)^2\leq\delta M(t)M''(t)$ for $t\geq T$. Then M is non-increasing. In particular, one has $M(t)\leq M(T)$ for $t\geq T$.*

By (4.2.3), one has $M''(t)\geq-(4+\varepsilon)E\left(u(T_0),\partial_t u(T_0)\right)>0$ for $t\geq T_0$, a contradiction with Lemma 4.2.1. Hence, one has $E\left(u(t),\partial_t u(t)\right)\geq -I_0$ for all $t\geq 0$.

Remark 4.2.2. If $\gamma=0$, then $I_0=0$: we recover the fact that a global solution of the undamped equation has a non-negative energy.

4.3 The $L^2(\Omega)$-estimate

Here, we prove Theorem 4.0.1-*(ii)*. Consider $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ such that the solution u of (4.0.2) exists on $\mathbb{R}_+$. We split the proof into 2 steps.

Step 1 : an estimate on M''. Set $r_0=\frac{2}{p_0+1}\in(0,1)$. Using (4.2.2) and the definition of the energy, one finds

$$\begin{aligned}M''(t)\geq&\left(4+\frac{3\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2+\varepsilon\|u(t)\|_{H_0^1}^2-2(2+\varepsilon)E\left(u(t),\partial_t u(t)\right)\\&-C_0\|\gamma\|_{L^\infty}^2\left(\frac{1}{2}\|\partial_t u(t)\|_{L^2}^2+\frac{1}{2}\|u(t)\|_{H_0^1}^2-E\left(u(t),\partial_t u(t)\right)\right)^{r_0},\quad t\geq 0,\end{aligned}$$

with $C_0=C_0(f)$. There exists $C_1=C_1(\gamma,f)>0$ such that

$$\begin{aligned}M''(t)\geq&\left(4+\frac{3\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2+\varepsilon\|u(t)\|_{H_0^1}^2-2(2+\varepsilon)E\left(u(t),\partial_t u(t)\right)\\&-\frac{C_1}{2}\|\partial_t u(t)\|_{L^2}^{2r_0}-C_1\left|\frac{1}{2}\|u(t)\|_{H_0^1}^2-E\left(u(t),\partial_t u(t)\right)\right|^{r_0},\quad t\geq 0.\end{aligned}$$

By Theorem 4.0.1-*(i)*, one has $\sup_{t\geq 0}\left|E\left(u(t),\partial_t u(t)\right)\right|<+\infty$. Hence, there exists

$$C_2=C_2\left(\gamma,f,\sup_{t\geq 0}\left|E\left(u(t),\partial_t u(t)\right)\right|\right)>0$$

such that

$$\varepsilon X^2-2(2+\varepsilon)E\left(u(t),\partial_t u(t)\right)-C_1\left|\frac{1}{2}X^2-E\left(u(t),\partial_t u(t)\right)\right|^{r_0}\geq\frac{\varepsilon}{2}X^2-C_2,\quad t\geq 0,\quad X\geq 0,$$

yielding

$$M''(t)\geq\left(4+\frac{3\varepsilon}{4}\right)\|\partial_t u(t)\|_{L^2}^2-\frac{C_1}{2}\|\partial_t u(t)\|_{L^2}^{2r_0}+\frac{\varepsilon}{2}\|u(t)\|_{H_0^1}^2-C_2,\quad t\geq 0.$$

Note that C_2 can be chosen of the form

$$C_2 = \tilde{C}_0 + \tilde{C}_1 \sup_{t\geq 0} |E(u(t), \partial_t u(t))|,$$

with $\tilde{C}_0 = \tilde{C}_0(\gamma, f)$ and $\tilde{C}_1 = \tilde{C}_1(\gamma, f)$, and that using Theorem 4.0.1-*(i)* and the fact that the energy is non-increasing, one has

$$\sup_{t\geq 0} |E(u(t), \partial_t u(t))| \leq \max\left(\left|E\left(u^0, u^1\right)\right|, \tilde{C}_2\right) \leq \left|E\left(u^0, u^1\right)\right| + \tilde{C}_2,$$

for some $\tilde{C}_2 = \tilde{C}_2(f, \gamma) > 0$. Finally, there exists $C_3 = C_3(\gamma, f) > 0$ such that

$$\left(4 + \frac{3\varepsilon}{4}\right) X^2 - \frac{C_1}{2} X^{2r_0} \geq \left(4 + \frac{\varepsilon}{2}\right) X^2 - C_3, \quad X \geq 0,$$

and this gives

$$M''(t) \geq \left(4 + \frac{\varepsilon}{2}\right) \|\partial_t u(t)\|_{L^2}^2 + \frac{\varepsilon}{2} \|u(t)\|_{H_0^1}^2 - C_2 - C_3, \quad t \geq 0.$$

Summarizing up, one obtains

$$M''(t) \geq \left(4 + \frac{\varepsilon}{2}\right) \|\partial_t u(t)\|_{L^2}^2 + C_4 \|u(t)\|_{H_0^1}^2 - C_5, \quad t \geq 0, \tag{4.3.1}$$

for some $C_4 = C_4(f) > 0$ and $C_5 > 0$ of the form

$$C_5 = \tilde{C}_3 + \tilde{C}_4 \left|E\left(u^0, u^1\right)\right| \tag{4.3.2}$$

with $\tilde{C}_3 = \tilde{C}_3(f, \gamma) > 0$ and $\tilde{C}_4 = \tilde{C}_4(f, \gamma) > 0$.

Step 2 : proof of (4.0.7). The proof is based on the following elementary lemma.

Lemma 4.3.1. *Consider $M_0 \in \mathscr{C}^2(\mathbb{R}_+, \mathbb{R})$ such that there exists $C > 0$ such that $M_0''(t) \geq CM_0(t)$, for all $t \geq 0$. Then either $M_0(t) \to +\infty$ as $t \to +\infty$, or $M_0(t) \leq M_0(0)e^{-\sqrt{C}t}$, for all $t \geq 0$.*

By (4.3.1) and the Poincaré inequality (4.0.1), one has

$$M''(t) \geq \left(4 + \frac{\varepsilon}{2}\right) \|\partial_t u(t)\|_{L^2}^2 + C_6 \|u(t)\|_{L^2}^2 - C_5, \quad t \geq 0, \tag{4.3.3}$$

for some $C_6 = C_6(f) > 0$. Set $M_0(t) = C_6 M(t) - C_5$, $t \geq 0$. By (4.3.3), one has $M_0''(t) \geq C_6 M_0(t)$, for all $t \geq 0$. Assume that $M_0(t) \to +\infty$ as $t \to +\infty$. Then, there exists $T \geq 0$ such that $M_0(t) \geq 0$, for $t \geq T$, implying

$$M''(t) \geq \left(4 + \frac{\varepsilon}{2}\right) \|\partial_t u(t)\|_{L^2}^2, \quad t \geq T,$$

by (4.3.3). This gives

$$\left(M'(t)\right)^2 \leq \left(1 + \frac{\varepsilon}{8}\right)^{-1} M(t) M''(t), \quad t \geq T.$$

Applying Lemma 4.2.1, one finds that M is bounded, a contradiction with $M_0(t) \to +\infty$. Hence, by Lemma 4.3.1, one obtains $M_0(t) \leq M_0(0)e^{-\sqrt{C_6}t}$ for all $t \geq 0$, that is,

$$M(t) \leq \frac{C_5}{C_6} + \left(M(0) - \frac{C_5}{C_6}\right) e^{-\sqrt{C_6}t}, \quad t \geq 0.$$

Using (4.3.2), one obtains (4.0.7).

4.4 The $H^1(\Omega)$-estimates

Here, we prove Theorem 4.0.1-*(iii)*. We first need an estimate on M'. Note that it does not require the restrictive condition $p \leq \frac{d}{d-2}$.

Lemma 4.4.1. *Assume that $d \geq 3$, and that f satisfies* (4.0.3)*, with $p < \frac{d+2}{d-2}$,* (4.0.4)*, and* (4.0.5)*. Then, there exist $c_0 = c_0(f,\gamma) > 0$, $c_1 = c_1(f,\gamma) > 0$, and $c_2 = c_2(f) > 0$ such that for all $(u^0,u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solution u of* (4.0.2) *exists on $\mathbb{R}_+$, then*

$$M'(0)e^{-c_2 t} - \left(c_0 + c_1 \left|E\left(u^0,u^1\right)\right|\right)\left(1 - e^{-c_2 t}\right) \leq M'(t) \leq c_0 + c_1 \left|E\left(u^0,u^1\right)\right|, \quad t \geq 0. \tag{4.4.1}$$

Proof. We use (4.3.2) and (4.3.3) : there exist $C_0 = C_0(f) > 0$ and $C_1 > 0$ of the form

$$\tilde{C}_0 + \tilde{C}_1 \left|E\left(u^0,u^1\right)\right|, \tag{4.4.2}$$

with $\tilde{C}_0 = \tilde{C}_0(f,\gamma) > 0$ and $\tilde{C}_1 = \tilde{C}_1(f,\gamma) > 0$, such that

$$M''(t) \geq \left(4 + \frac{\varepsilon}{2}\right) \|\partial_t u(t)\|_{L^2}^2 + C_0 \|u(t)\|_{L^2}^2 - C_1, \quad t \geq 0. \tag{4.4.3}$$

For $\eta > 0$, using (4.4.3), one finds

$$\begin{aligned}|M'(t)| &\leq \eta \|\partial_t u(t)\|_{L^2}^2 + \frac{1}{\eta}\|u(t)\|_{L^2}^2 \\ &\leq \eta\left(4 + \frac{\varepsilon}{2}\right)^{-1} M''(t) + \left(\frac{1}{\eta} - C_0\eta\left(4 + \frac{\varepsilon}{2}\right)^{-1}\right)\|u(t)\|_{L^2}^2 + C_1\eta\left(4 + \frac{\varepsilon}{2}\right)^{-1}, \quad t \geq 0.\end{aligned}$$

Choosing η such that $\frac{1}{\eta} = C_0\eta\left(4 + \frac{\varepsilon}{2}\right)^{-1}$, one obtains

$$|M'(t)| \leq C_2 M''(t) + C_3, \quad t \geq 0, \tag{4.4.4}$$

with $C_2 = C_2(f) > 0$ and $C_3 > 0$ of the form (4.4.2). Integrating (4.4.4), one finds

$$\left(M'(t_0) - C_3\right) e^{-\frac{t_0}{C_2}} \leq \left(M'(t_1) - C_3\right) e^{-\frac{t_1}{C_2}}, \quad 0 \leq t_0 \leq t_1, \tag{4.4.5}$$

and

$$M'(0) + C_3 \leq \left(M'(t_1) + C_3\right) e^{\frac{t_1}{C_2}}, \quad t_1 \geq 0. \tag{4.4.6}$$

If there exists $t_0 \geq 0$ such that $M'(t_0) - C_3 > 0$, then (4.4.5) implies that $M'(t) \to +\infty$ as $t \to +\infty$, a contradiction with Theorem 4.0.1-*(ii)*. Hence, one has $M'(t) \leq C_3$, for all $t \geq 0$. This gives the upper bound of (4.4.1). The lower bound of (4.4.1) is given by (4.4.6). □

Now, we prove Theorem 4.0.1-*(iii)*. Consider $\left(u^0,u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ such that the solution u of (4.0.2) exists on $\mathbb{R}_+$. Set

$$E_{\mathscr{L}}(t) = \frac{1}{2}\|u(t)\|_{H_0^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 = E\left(u(t), \partial_t u(t)\right) + \int_\Omega F(u(t))\mathrm{d}x, \quad t \geq 0.$$

By (4.3.1) and (4.3.2), one has

$$E_{\mathscr{L}}(t) \leq C_0 M''(t) + C_1, \quad t \geq 0, \tag{4.4.7}$$

for some $C_0 = C_0(f) > 0$, and $C_1 > 0$ of the form $\tilde{C}_0 + \tilde{C}_1 \left|E\left(u^0,u^1\right)\right|$, with $\tilde{C}_0 = \tilde{C}_0(f,\gamma) > 0$ and $\tilde{C}_1 = \tilde{C}_1(f,\gamma) > 0$.

Remark 4.4.2. Note that without the restrictive assumption of Theorem 4.0.1-*(iii)*, Lemma 4.4.1 and (4.4.7) imply

$$\int_0^T E_{\mathscr{L}}(t)\mathrm{d}t \lesssim 1+T, \quad T \geq 0,$$

with a constant depending on f, γ, and (u^0, u^1). This is an average estimate of $E_{\mathscr{L}}$. It won't be used later on : to derive a non-average estimate, it seems necessary to estimate the derivative of $E_{\mathscr{L}}$, which is undefined without the additional assumption of Theorem 4.0.1-*(iii)*.

For the rest of the proof, we write C and C' for some positive constants, that may change from line to line, and which depend on Ω, f and γ. We assume that f satisfies (4.0.3) with $p \leq \frac{d}{d-2}$. Together with the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\frac{2d}{d-2}}$, it gives $f(u(t)) \in L^2(\Omega)$, implying

$$\int_\Omega |\partial_t u(t) f(u(t))| \,\mathrm{d}x \leq C \left\|\partial_t u(t)\right\|_{L^2} \left(\|u(t)\|_{L^2} + \|u(t)\|_{H_0^1}^p\right), \quad t \geq 0. \tag{4.4.8}$$

In particular, the derivative of $E_{\mathscr{L}}$ is well-defined, and using also (4.0.6), one has

$$E'_{\mathscr{L}}(t) = -\int_\Omega \gamma \left|\partial_t u\right|^2 \mathrm{d}x + \int_\Omega \partial_t u(t) f(u(t))\mathrm{d}x \leq \int_\Omega \partial_t u(t) f(u(t))\mathrm{d}x, \quad t \geq 0.$$

Together with (4.0.1) and (4.4.8), one finds

$$E'_{\mathscr{L}}(t) \leq C \left\|\partial_t u(t)\right\|_{L^2} \|u(t)\|_{H_0^1} \left(1 + \|u(t)\|_{H_0^1}^{p-1}\right), \quad t \geq 0,$$

Using a second time the fact that $p \leq \frac{d}{d-2}$, and $d \geq 3$, one has $p-1 \leq 2$, implying

$$E'_{\mathscr{L}}(t) \leq C E_{\mathscr{L}}(t) \left(1 + E_{\mathscr{L}}(t)\right), \quad t \geq 0.$$

Integrating, this gives

$$E_{\mathscr{L}}(t_1) \leq E_{\mathscr{L}}(t_0) \exp\left(C \int_{t_0}^{t_1} \left(1 + E_{\mathscr{L}}(s)\right) \mathrm{d}s\right), \quad t_1 \geq t_0 \geq 0. \tag{4.4.9}$$

Estimate for $t \leq 1$. Choosing $t_0 = 0$ in (4.4.9) and using (4.4.7), one finds

$$E_{\mathscr{L}}(t_1) \leq E_{\mathscr{L}}(0) \exp\left(\int_0^1 \left(C + C' M''(s)\right) \mathrm{d}s\right), \quad 1 \geq t_1 \geq 0.$$

Note that $\left|E\left(u^0, u^1\right)\right| \lesssim E_{\mathscr{L}}(0)$, with a constant depending on f, and that $|M'(0)| \lesssim E_{\mathscr{L}}(0)$. Hence, using Lemma 4.4.1, one obtains

$$E_{\mathscr{L}}(t_1) \leq E_{\mathscr{L}}(0) \exp\left(C + C' E_{\mathscr{L}}(0)\right), \quad 1 \geq t_1 \geq 0,$$

We choose the function α of Theorem 4.0.1-*(iii)* so that $\alpha(s) \geq 2s \exp\left(C + 2C's\right)$, for all $s \in [0,1]$.

Estimate for $t \geq 1$. Integrating (4.4.9), and using (4.4.7), one finds

$$E_{\mathscr{L}}(t_1) \leq \left(\int_{t_1-1}^{t_1} \left(C + C' M''(t)\right) \mathrm{d}t\right) \exp\left(\int_{t_1-1}^{t_1} \left(C + C' M''(s)\right) \mathrm{d}s\right), \quad t_1 \geq 1. \tag{4.4.10}$$

Lemma 4.4.1 gives

$$\int_{t_1-1}^{t_1} M''(t)\mathrm{d}t \leq 2\left(c_0 + c_1 \left|E\left(u^0,u^1\right)\right|\right) + |M'(0)|\, e^{c_2(1-t_1)}, \quad t_1 \geq 1, \tag{4.4.11}$$

where c_0, c_1 and c_2 are the constants of Lemma 4.4.1.

Using (4.4.10) and (4.4.11), we can complete the proof of Theorem 4.0.1-*(iii)*. We first prove (4.0.8). By (4.4.11), one has

$$\int_{t_1-1}^{t_1} M''(t)\mathrm{d}t \leq C + C'E_{\mathscr{L}}(0), \quad t_1 \geq 1.$$

Hence, (4.4.10) gives

$$E_{\mathscr{L}}(t_1) \leq (C + C'E_{\mathscr{L}}(0)) \exp\left(C + C'E_{\mathscr{L}}(0)\right), \quad t_1 \geq 1,$$

We choose the function α of Theorem 4.0.1-*(iii)* so that $\alpha(s) \geq (C + 2C's)\exp(C + 2C's)$, for all $s \geq 1$. This completes the proof of (4.0.8).

Finally, we prove (4.0.9). There exists $T > 0$, which depends on f, γ, and $E_{\mathscr{L}}(0)$, such that

$$|M'(0)|\, e^{c_2(1-t_1)} \leq c_0, \quad t_1 \geq T.$$

Hence, (4.4.10) and (4.4.11) imply

$$E_{\mathscr{L}}(t_1) \leq \left(C + C'\left|E\left(u^0,u^1\right)\right|\right)\exp\left(C + C'\left|E\left(u^0,u^1\right)\right|\right), \quad t_1 \geq T.$$

Up to increasing the constant c defining the function α, this completes the proof of (4.0.9).

4.A Proof of the elementary lemmas

4.A.1 Proof of Lemma 4.2.1

Assume by contradiction that there exists $T_0 \geq T$ such that $M'(T_0) > 0$. In particular, one has $M(T_0) > 0$. We claim that $M(t) > 0$ for all $t \geq T_0$. Indeed, assume by contradiction that there exists $T_1 > T_0$ such that $M(T_1) = 0$ and $M(t) > 0$ for $t \in [T_0, T_1)$. Then, one has $M''(t) \geq 0$ for $t \in [T_0, T_1)$, implying that $M'(t) \geq M'(T_0) > 0$ for $t \in [T_0, T_1)$. In particular, this gives $0 = M(T_1) \geq M(T_0) > 0$, a contradiction.

As $M(t) > 0$ for all $t \geq T_0$, one has $M''(t) > 0$ for $t \geq T_0$, implying $M'(t) \geq M'(T_0)$, for $t \geq T_0$. In particular, $M(t) \to +\infty$ as $t \to +\infty$. Integrating two times the inequality

$$\frac{M'(t)}{M(t)} \leq \delta \frac{M''(t)}{M'(t)}, \quad t \geq T_0,$$

one obtains

$$(t - T_0)\,\frac{M'(T_0)}{M(T_0)^{\frac{1}{\delta}}} \leq \left(\frac{1}{\delta} - 1\right)^{-1}\left(\frac{1}{M(T_0)^{\frac{1}{\delta}-1}} - \frac{1}{M(t)^{\frac{1}{\delta}-1}}\right), \quad t \geq T_0.$$

For t sufficiently large, this gives a contradiction.

4.A.2 Proof of Lemma 4.3.1

Set $m(t) = M_0(t)e^{\sqrt{C}t}$, $t \geq 0$. One has $m'' \geq 2\sqrt{C}m'$, implying

$$m(t_1) \geq m(t_0) + \frac{m'(t_0)}{2\sqrt{C}}\left(e^{2\sqrt{C}(t_1-t_0)} - 1\right), \quad t_1 \geq t_0 \geq 0.$$

This gives

$$M_0(t_1) \geq m(t_0)e^{-\sqrt{C}t_1} + \frac{m'(t_0)}{2\sqrt{C}}\left(e^{\sqrt{C}(t_1-2t_0)} - e^{-\sqrt{C}t_1}\right), \quad t_1 \geq t_0 \geq 0.$$

If there exists $t_0 \geq 0$ such that $m'(t_0) > 0$, then $M_0(t) \to +\infty$. Conversely, if $m'(t) \leq 0$ for all $t \geq 0$, then one has $m(t) \leq m(0)$ for $t \geq 0$, that is, $M_0(t) \leq M_0(0)e^{-\sqrt{C}t}$, for $t \geq 0$.

Chapter 5

The damped focusing cubic wave equation on a bounded domain

This chapter is based on the article [Per23c], which has been prepublished and submitted in a journal.

Contents

Introduction

Let Ω and $\partial\Omega$ be the interior and the boundary of a smooth compact connected Riemannian manifold of dimension 3. Let $\beta \in \mathbb{R}$ be such that the Poincaré inequality

$$\int_\Omega \left(|\nabla u|^2 + \beta |u|^2\right) \mathrm{d}x \gtrsim \int_\Omega |u|^2 \mathrm{d}x$$

is satisfied, for all $u \in H_0^1(\Omega)$. This specifically requires $\beta > 0$ if $\partial\Omega$ is empty. For real-valued initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, consider the cubic wave (or Klein-Gordon) equation

$$\begin{cases} \Box u + \beta u &= u^3 & \text{in } \mathbb{R}\times\Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } \mathbb{R}\times\partial\Omega. \end{cases} \tag{5.0.1}$$

For $\left(u^0,u^1\right)\in H_0^1(\Omega)\times L^2(\Omega)$, write $\|u^0\|_{H_0^1}^2=\int_\Omega\left(\left|\nabla u^0\right|^2+\beta|u^0|^2\right)\mathrm{d}x$ and set

$$J(u^0)=\frac{1}{2}\left\|u^0\right\|_{H_0^1}^2-\frac{1}{4}\left\|u^0\right\|_{L^4}^4 \quad \text{and} \quad E\left(u^0,u^1\right)=J(u^0)+\frac{1}{2}\left\|u^1\right\|_{L^2}^2.$$

The functionals J and E are respectively called *static energy* and *energy*. The energy of a solution of (5.0.1) at time $t\in\mathbb{R}$ is defined by $E\left(u(t),\partial_t u(t)\right)$, and is conserved. Let Q be a ground state of (5.0.1), that is, a positive stationary solution of (5.0.1) of minimal energy

$$E(Q,0)=J(Q)=m_0>0.$$

See Theorem 5.1.6 for a precise definition. For $u^0\in H_0^1(\Omega)$, write

$$K(u^0)=\left\|u^0\right\|_{H_0^1}^2-\left\|u^0\right\|_{L^4}^4, \tag{5.0.2}$$

and set

$$\begin{cases}\mathscr{K}^+=\left\{\left(u^0,u^1\right)\in H_0^1(\Omega)\times L^2(\Omega), E\left(u^0,u^1\right)<m_0, K(u^0)\geq 0\right\},\\ \mathscr{K}^-=\left\{\left(u^0,u^1\right)\in H_0^1(\Omega)\times L^2(\Omega), E\left(u^0,u^1\right)<m_0, K(u^0)<0\right\}.\end{cases}$$

The following result is due to Payne and Sattinger [PS75].

Theorem 5.0.1. *The spaces $\mathscr{K}^+$ and $\mathscr{K}^-$ are stable under the flow of* (5.0.1)*. A solution starting from $\mathscr{K}^+$ is defined on $\mathbb{R}$, and a solution starting from $\mathscr{K}^-$ blows up in finite positive and negative times.*

For $\gamma\in L^\infty(\Omega,\mathbb{R}_+)$, our main focus is the damped equation

$$\begin{cases}\Box u+\gamma\partial_t u+\beta u &= u^3 & \text{in } \mathbb{R}_+\times\Omega,\\ (u(0),\partial_t u(0)) &= (u^0,u^1) & \text{in } \Omega,\\ u &= 0 & \text{on } \mathbb{R}_+\times\partial\Omega,\end{cases} \tag{5.0.3}$$

with $\left(u^0,u^1\right)\in H_0^1(\Omega)\times L^2(\Omega)$. For $t\geq 0$, the energy equality is

$$E\left(u(t),\partial_t u(t)\right)=E\left(u^0,u^1\right)-\int_0^t\int_\Omega\gamma(x)\left|\partial_t u(s,x)\right|^2\mathrm{d}x\mathrm{d}s.$$

In particular, the energy of a solution is nonincreasing. The first result of this chapter is the following theorem.

Theorem 5.0.2. *Consider $\gamma\in L^\infty(\Omega,\mathbb{R}_+)$. The spaces $\mathscr{K}^+$ and $\mathscr{K}^-$ are stable under the forward flow of* (5.0.3)*. A solution of* (5.0.3) *initiated in $\mathscr{K}^+$ is defined on $\mathbb{R}_+$, and a solution of* (5.0.3) *initiated in $\mathscr{K}^-$ blows up in finite time $t>0$.*

In particular, below the energy of the ground state, a blow-up solution of (5.0.1) cannot be stabilised by the addition of a damping term of the form $\gamma\partial_t u$.

We will use the notion of generalized geodesic, for which we refer to [MS78]. Recall that we always assume that no generalized geodesic has a contact of infinite order with $\partial\Omega$ (see [BLR92] for some details about this assumption).

Definition 5.0.3. For $\omega\subset\Omega$, we say that ω satisfies the *Geometric Control Condition* (in short, GCC) if there exists $L>0$ such that any generalized geodesic of Ω of length L meets the set ω.

The second result of this chapter is the following theorem, which can be seen as an extension of the stabilisation property proved in [JL13] to the focusing case.

Theorem 5.0.4. *Suppose $\gamma(x) \geq \alpha$ for almost all $x \in \omega$, with $\alpha > 0$ a constant and $\omega \subset \Omega$ an open set fulfilling the GCC. Then for any $E_0 \in [0, m_0)$, there exist $C > 0$ and $\lambda > 0$ such that for all $(u^0, u^1) \in \mathscr{K}^+$, the solution u of (5.0.3) satisfies*

$$\|u(t)\|_{H_0^1} + \|\partial_t u(t)\|_{L^2} \leq Ce^{-\lambda t}, \quad t \geq 0,$$

if $E(u^0, u^1) \leq E_0$.

For a solution starting from $\mathscr{K}^+$, we will see that the energy can be used almost as in the defocusing case, allowing us to adapt most of the arguments of [JL13]. Note that C and λ depend on E_0 (see Remark 5.4.2).

The proof of Theorem 5.0.4 will rely on an asymptotic compactness result, which roughly states that for all uniformly bounded sequence of global solutions of (5.0.3), one can find a subsequence which converges to a stationary solution of (5.0.1) along a time-sequence; see Proposition 5.3.1 for a precise statement. It implies the following theorem.

Theorem 5.0.5. *Consider $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$ satisfying the assumption of Theorem 5.0.4. Let $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ be such that the associated solution u of (5.0.3) exists on $\mathbb{R}_+$.*

1. *For all sequence $(T_n)_{n\in\mathbb{N}}$ such that $T_n \to +\infty$, there exist an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and a stationary solution w of (5.0.1) such that*

$$\Big(u(T_{\phi(n)} + \cdot), \partial_t u(T_{\phi(n)} + \cdot)\Big) \xrightarrow{n\to\infty} (w, 0)$$

   in $L^\infty_{\mathrm{loc}}(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$.

2. *Assume that there exists an at most countable number of stationary solution w of (5.0.1) such that*

$$J(w) = \liminf_{t\to+\infty} E(u(t), \partial_t u(t)). \tag{5.0.4}$$

   Then there exists a stationary solution w of (5.0.1) such that

$$(u(t + \cdot), \partial_t u(t + \cdot)) \xrightarrow{t\to+\infty} (w, 0)$$

   in $L^\infty_{\mathrm{loc}}(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$.

Remark 5.0.6. For example, if Ω is an open subset of $\mathbb{R}^3$ and $\liminf E(u(t), \partial_t u(t)) \leq J(Q)$, then there are at most 3 stationary solutions satisfying (5.0.4), which are 0, Q and $-Q$ (see Theorem 5.1.6).

Remark 5.0.7. The proof of Theorem 5.0.5 relies on the uniform estimates of Theorem 4.0.1.

On the one hand, observe that Theorem 5.0.5 applies to any solution defined on $\mathbb{R}_+$, while Theorem 5.0.4 only applies to solutions with energy below the energy of the ground state. On the other hand, Theorem 5.0.4 provides an exponential convergence rate, whereas Theorem 5.0.5 provides no convergence rate. We refer to Section 2.3 for a discussion about the results of this chapter and their connection to the existing literature.

Outline of the chapter. In Section 5.1, we recall some basic facts about (5.0.3) and the stabilisation of the linear version of (5.0.3), and we prove some technical results related to the ground state Q. In Section 5.2, we prove Theorem 5.0.2. In Section 5.3, we establish the asymptotic compactness result (Proposition 5.3.1), and we show that it implies Theorem 5.0.5. In Section 5.4, we first prove an easy stabilisation property in the case of a positive damping, and then we prove Theorem 5.0.4, by adapting the proof of [JL13].

5.1 Preliminaries

5.1.1 The Cauchy problem and some properties of the linear equation

In this chapter, we only consider real-valued solutions of wave equations. For $s \in [0,1]$, let $H_0^s(\Omega)$ denote the complex interpolation space between $L^2(\Omega)$ and $H_0^1(\Omega)$. Define

$$X^s = \left(H^{1+s}(\Omega) \cap H_0^1(\Omega)\right) \times H_0^s(\Omega). \tag{5.1.1}$$

For example, one has $X^0 = H_0^1(\Omega) \times L^2(\Omega)$ and $X^1 = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. For $s \geq 0$, write $D(\Delta^s)$ for the iterated domain of the Dirichlet Laplacian (defined by interpolation for $s \notin \mathbb{N}$). Then for $s \in [0,1]$, one has

$$X^s = D\left(\Delta^{\frac{1+s}{2}}\right) \times D\left(\Delta^{\frac{s}{2}}\right) \tag{5.1.2}$$

so the norm of X^s can be represented by a Fourier series. Consider the unbounded operator $A : X^0 \to X^0$ with domain $D(A) = X^1$ defined by

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - \beta & -\gamma \end{pmatrix}.$$

Write e^{tA} for the associated semi-group. The following result about the Cauchy problem corresponding to the linear version of (5.0.3) is well-known.

Theorem 5.1.1. *For $s \in [0,1]$, $(u^0, u^1) \in X^s$, $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, and $g \in L^1_{\mathrm{loc}}(\mathbb{R}, H_0^s(\Omega))$, there exists a unique solution*

$$u \in \mathscr{C}^0(\mathbb{R}, H^{1+s}(\Omega) \cap H_0^1(\Omega)) \ \cap \ \mathscr{C}^1(\mathbb{R}, H_0^s(\Omega))$$

of the linear damped wave equation

$$\begin{cases} \Box u + \gamma \partial_t u + \beta u &= g & \text{in } \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } \mathbb{R} \times \partial\Omega. \end{cases} \tag{5.1.3}$$

In addition, for $T > 0$ and $s \in [0,1]$, there exists $C_{s,T} > 0$ such that for all $(u^0, u^1) \in X^s$, $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, and $g \in L^1_{\mathrm{loc}}(\mathbb{R}, H_0^s(\Omega))$, one has

$$\|u\|_{L^\infty((0,T), H^{1+s} \cap H_0^1)} + \|\partial_t u\|_{L^\infty((0,T), H_0^s)} \leq C_{s,T} \left(\left\| \left(u^0, u^1\right) \right\|_{X^s} + \|g\|_{L^1((0,T), H_0^s)} \right).$$

Remark 5.1.2. By the energy estimate, we can assume that $C_{0,T} \lesssim C_{0,1}$ for $0 < T \leq 1$.

To construct the solution of (5.0.3), one uses Strichartz estimates (see for example Theorem 2.1 of [JL13]).

Theorem 5.1.3 (Strichartz estimates)**.** *Let $T > 0$. There exists a constant $C > 0$ such that for all (q, r) satisfying*

$$\frac{1}{q} + \frac{3}{r} = \frac{1}{2} \quad \text{and} \quad q \in \left[\frac{7}{2}, +\infty\right]$$

and for all $(u^0, u^1) \in X^0$, $g \in L^1([0,T], L^2(\Omega))$, the unique solution u of (5.1.3) *associated with (u^0, u^1) and g satisfies*

$$\|u\|_{L^q([0,T], L^r)} \leq C \left(\|u^0\|_{H_0^1} + \|u^1\|_{L^2} + \|g\|_{L^1([0,T], L^2)} \right).$$

Using those estimates, the solution of the Cauchy problem (5.0.3) can be constructed locally (see [GV89]). Unlike the defocusing case, it cannot be proven that the solution is global in time.

Theorem 5.1.4. *Consider $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$. For any (real-valued) initial data $(u^0, u^1) \in X^0$, there exist a maximal time of existence $T \in (0, +\infty]$ and a unique solution u of (5.0.3) in $\mathscr{C}^0([0,T), H_0^1(\Omega)) \cap \mathscr{C}^1([0,T), L^2(\Omega))$. If $T < +\infty$, then*

$$\|u(t)\|_{H_0^1} \xrightarrow{t \to T^-} +\infty. \tag{5.1.4}$$

Proof. We only prove (5.1.4). If $T < +\infty$, then the Cauchy theory gives

$$\|u(t)\|_{H_0^1} + \|\partial_t u(t)\|_{L^2} \xrightarrow{t \to T^-} +\infty.$$

As the energy of a solution is nonincreasing, it implies

$$\|u(t)\|_{L^4} \xrightarrow{t \to T^-} +\infty,$$

yielding (5.1.4) by the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$. □

In what follows, we refer to u as "the solution of (5.0.3) with initial data (u^0, u^1) (and damping γ)". Finally, we will need the fact that the GCC implies the stabilisation of the linear equation. More precisely, we will use the following result, which is due to Bardos, Lebeau and Rauch [BLR92].

Theorem 5.1.5. *Assume that $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$ satisfies $\gamma(x) \geq \alpha$ for almost all $x \in \omega$, with $\alpha > 0$ a constant and $\omega \subset \Omega$ an open set fulfilling the GCC. There exist $C > 0$ and $\lambda > 0$ such that for all $s \in [0,1]$, and all $(u^0, u^1) \in X^s$, if $u = e^{tA}(u^0, u^1)$ is the solution of (5.1.3) with $g = 0$, then*

$$\|(u(t), \partial_t u(t))\|_{X^s} \leq C e^{-\lambda t} \left\| \left(u^0, u^1\right) \right\|_{X^s}, \quad t \geq 0.$$

5.1.2 Ground state and some related properties

Consider $u \in H_0^1(\Omega)$, $u \neq 0$. For $\lambda \in \mathbb{R}$, set $j(\lambda) = J(\lambda u)$, that is

$$j(\lambda) = \frac{\lambda^2}{2} \|u\|_{H_0^1}^2 - \frac{\lambda^4}{4} \|u\|_{L^4}^4 .$$

Write $\lambda^* = \lambda^*(u) > 0$ for the positive argument of the maximum of j. Using $j'(\lambda^*) = 0$, one finds $K(\lambda^* u) = 0$, where K is defined by (5.0.2). Set

$$\begin{aligned} m_0 &= \inf \left\{ J\left(\lambda^*(u) u\right), u \in H_0^1(\Omega), u \neq 0 \right\} \\ &= \inf \left\{ J(w), w \in H_0^1(\Omega), w \neq 0, K(w) = 0 \right\}. \end{aligned} \tag{5.1.5}$$

The fact that equality (5.1.5) holds is clear, because for $w \in H_0^1(\Omega)$, $w \neq 0$, there exists a unique $\lambda > 0$ such that

$$\lambda \|w\|_{H_0^1}^2 - \lambda^3 \|w\|_{L^4}^4 = 0$$

so that if $K(w) = 0$ then $\lambda^*(w) = 1$. Note that if $u \in H_0^1(\Omega)$ satisfies $K(u) < 0$, then $\lambda^*(u) \in (0,1)$. Note also that one has

$$\left| \int_\Omega \left(w^4 - v^4\right) \mathrm{d}x \right| \lesssim \|w - v\|_{L^2} \left(\|w\|_{L^6(\Omega)}^3 + \|v\|_{L^6(\Omega)}^3 \right)$$

so that K and J are continuous by the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. The following result is due to Payne and Sattinger [PS75], in the case of a bounded subset of $\mathbb{R}^n$. The extension to the case of a compact manifold is straightforward, except for the uniqueness part.

Theorem 5.1.6. *One has $m_0 > 0$, and there exists $w \in H_0^1(\Omega)$ such that $w \neq 0$, $K(w) = 0$, and $J(w) = m_0$. Such a function w is a critical point of J, does not change sign, satisfies $w \in H^2(\Omega)$, and $-\Delta w + \beta w = w^3$. In addition, if Ω is a subset of $\mathbb{R}^3$, then J has exactly two critical points: w and $-w$.*

Proof. We start by proving that $m_0 > 0$. Consider $w \in H_0^1(\Omega)$ such that $w \neq 0$ and $K(w) = 0$. The Sobolev embedding $H^1(\Omega) \to L^4(\Omega)$ gives

$$\|w\|_{H_0^1}^2 = \|w\|_{L^4}^4 \lesssim \|w\|_{H_0^1}^4.$$

This yields $\|w\|_{H_0^1} \gtrsim 1$, and as $K(w) = 0$, one obtains

$$J(w) = \frac{1}{4}\|w\|_{H_0^1}^2 \gtrsim 1.$$

This proves that $m_0 > 0$.

We now prove that w exists. Consider a sequence $(w_n)_{n\in\mathbb{N}}$ of elements of $H_0^1(\Omega)$ satisfying $w_n \neq 0$ and $K(w_n) = 0$, for all $n \in \mathbb{N}$, and such that $J(w_n) \longrightarrow m_0$ as n tends to infinity. Up to a subsequence, we may assume that there exists $w \in H_0^1(\Omega)$ such that w_n converges to w weakly in $H^1(\Omega)$ and strongly in $L^4(\Omega)$. Using the fact that $K(w_n) = 0$ for all $n \in \mathbb{N}$, one finds

$$m_0 = \frac{1}{4}\lim_{n\to\infty} \|w_n\|_{H_0^1}^2 = \frac{1}{4}\lim_{n\to\infty} \|w_n\|_{L^4}^4 = \frac{1}{4}\|w\|_{L^4}^4$$

and in particular, $w \neq 0$. As w_n converges to w weakly, one has $\|w\|_{H_0^1} \leq \lim \|w_n\|_{H_0^1}$. Assume by contradiction that

$$\|w\|_{H_0^1} < \lim_{n\to\infty} \|w_n\|_{H_0^1}.$$

This implies $K(w) < 0$. Write $j(\lambda) = J(\lambda w)$, and $\lambda^* = \lambda^*(w) > 0$ for the unique $\lambda > 0$ such that $K(\lambda w) = 0$. Note that $j'(\lambda) = \frac{K(\lambda w)}{\lambda}$, $j'(\lambda) > 0$ for $\lambda \in (0, \lambda^*)$ and $j'(\lambda) < 0$ for $\lambda \in (\lambda^*, \infty)$. This yields $\lambda^* < 1$. By assumption, one has

$$J(\lambda^* w) < \frac{\lambda^*}{2}\lim_{n\to\infty} \|w_n\|_{H_0^1}^2 - \frac{\lambda^*}{4}\|w\|_{L^4}^4 = m_0\left(2(\lambda^*)^2 - (\lambda^*)^4\right).$$

As $2\lambda^2 - \lambda^4 < 1$ for $\lambda \in (0,1)$, one finds $J(\lambda^* w) < m_0$, a contradiction with the definition of m_0. Hence, one obtains

$$\|w\|_{H_0^1} = \lim_{n\to\infty} \|w_n\|_{H_0^1}$$

implying that $w_n \longrightarrow w$ in $H^1(\Omega)$ as $n \to \infty$. This gives $K(w) = 0$ and $J(w) = m_0$.

As $K(w) = 0$, one has

$$\left\langle \Delta w + \beta w - w^3, w \right\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = 0. \tag{5.1.6}$$

The differential of J and K are given by

$$\mathrm{d}J(u)(h) = \int_\Omega (\nabla u \cdot \nabla h + \beta u h)\,\mathrm{d}x - \int_\Omega u^3 h \mathrm{d}x, \quad u, h \in H_0^1(\Omega),$$

and

$$\mathrm{d}K(u)(h) = 2\int_\Omega (\nabla u \cdot \nabla h + \beta u h)\,\mathrm{d}x - 4\int_\Omega u^3 h \mathrm{d}x \quad u, h \in H_0^1(\Omega).$$

Hence, there exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that the Euler equation

$$\left(-\Delta w + \beta w - w^3\right) + \Lambda\left(-2\Delta w + 2\beta w - 4w^3\right) = 0$$

holds in $H^{-1}(\Omega)$. Rewriting that equation, one finds

$$(1 + 2\Lambda)\left(-\Delta w + \beta w - w^3\right) - 2\Lambda w^3 = 0.$$

By (5.1.6), this implies

$$\Lambda \int_\Omega w^4 \mathrm{d}x = 0.$$

As $w \neq 0$, this gives $\Lambda = 0$, yielding $-\Delta w + \beta w - w^3 = 0$. Hence, w is a critical point of J. By elliptic regularity, one has $w \in H^2(\Omega)$.

To establish the sign and uniqueness of w, we refer to [PS75]. □

In what follows, we denote by Q a nonnegative critical point of J and refer to it as a ground state of (5.0.1). To our knowledge, the uniqueness of Q has not been proven in the literature in the case of a general compact Riemannian manifold Ω. However, it will not be used anywhere in this chapter.

We prove some technical results for later use. The first one states that a function u such that $K(u) < 0$ is far from the zero function.

Lemma 5.1.7. *Let (u_n) be a sequence of elements of $H_0^1(\Omega)$ such that $K(u_n) < 0$ for all $n \in \mathbb{N}$, and $K(u_n) \xrightarrow{n\to\infty} 0$. Then, $\liminf J(u_n) \geq m_0$.*

Proof. For $n \in \mathbb{N}$, one has $J(u_n) = \frac{1}{4}K(u_n) + \frac{1}{4}\|u_n\|_{H_0^1}^2$, implying

$$\liminf J(u_n) = \frac{1}{4} \liminf \|u_n\|_{H_0^1}^2 .$$

If $\liminf J(u_n) = +\infty$, the result holds. Otherwise, up to a subsequence, one can assume that $(J(u_n))_n$ and $\left(\|u_n\|_{H_0^1}\right)_n$ converge. In particular, (u_n) is bounded and we can assume that (up to a subsequence) it converges weakly in $H_0^1(\Omega)$ and strongly in $L^4(\Omega)$. Denote by $u \in H_0^1(\Omega) \cap L^4(\Omega)$ the limit. For $n \in \mathbb{N}$, using the Sobolev embedding and the fact that $K(u_n) \leq 0$, one obtains

$$\|u_n\|_{L^4}^4 \lesssim \|u_n\|_{H_0^1}^4 \leq \|u_n\|_{L^4}^8 .$$

As $u_n \neq 0$, this gives $\|u\|_{L^4} \gtrsim 1$, and $u \neq 0$ as a result. One also has

$$\|u\|_{H_0^1}^2 \leq \lim \|u_n\|_{H_0^1}^2 \leq \|u\|_{L^4}^4,$$

yielding $K(u) \leq 0$. We split the end of the proof into two cases.

First, assume that $K(u) = 0$. Then $\|u\|_{H_0^1} = \lim \|u_n\|_{H_0^1}$, and hence, $(u_n)_n$ converges to u strongly in $H_0^1(\Omega)$. In particular, by definition of m_0, one has

$$\lim J(u_n) = J(u) \geq m_0$$

so the proof is complete in that case.

Second, assume that $K(u) < 0$, that is,

$$\|u\|_{H_0^1} < \lim \|u_n\|_{H_0^1} .$$

As above, there exists $\lambda^* \in (0,1)$ such that $K(\lambda^* u) = 0$, and $m_0 \le J(\lambda^* u)$. The fact that $K(\lambda^* u) = 0$ gives

$$J(\lambda^* u) = \frac{1}{4}\|\lambda^* u\|_{L^4}^4 .$$

Writing $J(u_n) = \frac{1}{2}K(u_n) + \frac{1}{4}\|u_n\|_{L^4}^4$, one finds $\lim J(u_n) = \frac{1}{4}\|u\|_{L^4}^4$. This gives

$$m_0 \le J(\lambda^* u) = (\lambda^*)^4 \lim J(u_n) < \lim J(u_n),$$

implying the result in the second case. □

The second result will allow us to improve inequalities of the form $K(u) \ge 0$ or $K(u) < 0$ for functions such that $J(u)$ is strictly below m_0.

Lemma 5.1.8. *Consider $\delta > 0$. There exists $c > 0$ such that for all $u \in H_0^1(\Omega)$ with $J(u) \le m_0 - \delta$, one has:*

(i) if $K(u) \ge 0$ then $K(u) \ge c\|u\|_{H_0^1}^2$.

(ii) if $K(u) < 0$ then $K(u) \le -c\|u\|_{H_0^1}^2$ and $K(u) \le -c$.

Remark 5.1.9. The proof can be carried out by contradiction, but it does not provide any information regarding the size of the constant. Our proof gives the following explicit estimates : if u is such that $J(u) \le m_0 - \delta$ then $K(u) \ge 0$ implies $K(u) \ge \sqrt{\frac{\delta}{m_0}}\|u\|_{H_0^1}^2$, and $K(u) < 0$ implies

$$K(u) \le -4\delta - 4\sqrt{m_0\delta} \quad \text{and} \quad K(u) \le -\frac{\delta + \sqrt{m_0\delta}}{m_0 + \sqrt{m_0\delta}}\|u\|_{H_0^1}^2.$$

Proof. We split the proof into 3 steps.

Step 1: an explicit Sobolev embedding. We prove that for $u \in H_0^1(\Omega)$, one has

$$\|u\|_{L^4}\|Q\|_{L^4} \le \|u\|_{H_0^1}. \tag{5.1.7}$$

Consider $u \in H_0^1(\Omega), u \neq 0$. As above, write $j(\lambda) = J(\lambda u)$ for $\lambda \in \mathbb{R}$. Then

$$\lambda^* = \frac{\|u\|_{H_0^1}}{\|u\|_{L^4}^2} \tag{5.1.8}$$

is the positive argument of the maximum of j, and one has

$$J(\lambda^* u) \ge J(Q) = m_0. \tag{5.1.9}$$

Note that $K(\lambda^* u) = K(Q) = 0$ implies

$$J(\lambda^* u) = \frac{\|\lambda^* u\|_{L^4}^4}{4} \quad \text{and} \quad J(Q) = \frac{\|Q\|_{L^4}^4}{4}.$$

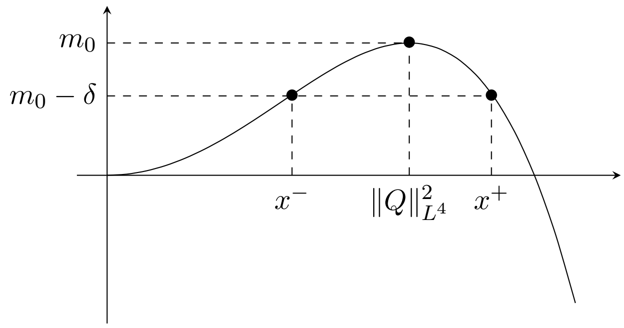
Together with (5.1.8) and (5.1.9), this gives (5.1.7).

Step 2: estimation of $\|u\|_{H_0^1}$. Here, we show that there exist x^+ and x^- such that $0 < x^- < \|Q\|_{H_0^1}^2 < x^+$, satisfying the following property: if $u \in H_0^1(\Omega)$ is such that $J(u) \leq m_0 - \delta$, then $K(u) \geq 0$ implies $\|u\|_{H_0^1} \leq x^-$ and $K(u) < 0$ implies $\|u\|_{H_0^1} \geq x^+$.

Consider $u \in H_0^1(\Omega)$. For $x \geq 0$, we put

$$\alpha(x) = \frac{x^2}{2} - \frac{x^4}{4\|Q\|_{L^4}^4}$$

so that (5.1.7) gives $\alpha\left(\|u\|_{H_0^1}\right) \leq J(u) \leq m_0 - \delta$. The graph of α is given in the following figure. Note that the maximum of α is m_0, and is reached at $x = \|Q\|_{L^4}^2$.



There exists a unique $x^+ > \|Q\|_{L^4}^2$ such that $\alpha(x^+) = m_0 - \delta$, and if $\delta \leq m_0$ then there exists a unique $x^- < \|Q\|_{L^4}^2$ such that $\alpha(x^-) = m_0 - \delta$. Explicitly, one has

$$x^\pm = 2\sqrt{m_0 \pm \sqrt{m_0\delta}}.$$

First, assume that $K(u) \geq 0$. One has

$$\frac{\|u\|_{H_0^1}^2}{4} \leq \frac{\|u\|_{H_0^1}^2}{4} + \frac{K(u)}{4} = J(u) \leq m_0 - \delta,$$

implying $\|u\|_{H_0^1}^2 \leq 4m_0 = \|Q\|_{L^4}^4$. As $\alpha\left(\|u\|_{H_0^1}\right) \leq m_0 - \delta$, this gives $\|u\|_{H_0^1} \leq x^-$.

Second, assume that $K(u) < 0$. Then, using (5.1.7), one obtains

$$\|u\|_{H_0^1}^2 < \|u\|_{L^4}^4 \leq \left(\frac{\|u\|_{H_0^1}}{\|Q\|_{L^4}}\right)^4,$$

yielding $\|u\|_{H_0^1} \geq \|Q\|_{L^4}^2$. As $\alpha\left(\|u\|_{H_0^1}\right) \leq m_0 - \delta$, this gives $\|u\|_{H_0^1} \geq x^+$.

Step 3: end of the proof. First, assume that $K(u) \geq 0$. Then using (5.1.7) and step 2, we obtain *(i)* as follows

$$K(u) \geq \|u\|_{H_0^1}^2 - \frac{\|u\|_{H_0^1}^4}{\|Q\|_{L^4}^4} \geq \|u\|_{H_0^1}^2 \left(1 - \frac{(x^-)^2}{\|Q\|_{L^4}^4}\right) \geq \sqrt{\frac{\delta}{m_0}}\|u\|_{H_0^1}^2.$$

Second, assume that $K(u) < 0$. Using step 2 and $J(u) \leq m_0 - \delta$, one obtains

$$K(u) = 4J(u) - \|u\|_{H_0^1}^2 \leq 4(m_0 - \delta) - (x^+)^2 = -4\delta - 4\sqrt{m_0\delta}.$$

That gives the second inequality of *(ii)*. For the first inequality of *(ii)*, it suffices to find a constant $C > 0$ such that

$$K(u) \leq 4(m_0 - \delta) - \|u\|^2_{H^1_0} \leq -C\|u\|^2_{H^1_0},$$

using $\|u\|_{H^1_0} \geq x^+$. It holds if one chooses

$$C \leq 1 - \frac{4(m_0 - \delta)}{(x^+)^2} = \frac{\delta + \sqrt{m_0\delta}}{m_0 + \sqrt{m_0\delta}}.$$

The proof is complete. □

5.2 The damped equation below the energy of the ground state

Here, we prove Theorem 5.0.2. To begin with, we check that the spaces $\mathscr{K}^+$ and $\mathscr{K}^-$ are invariant under the flow of the damped equation.

Lemma 5.2.1. *Suppose* $(u^0, u^1) \in \mathscr{K}^{\pm}$, $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, *and* u *is the solution of* (5.0.3). *If* $t \geq 0$ *is such that* u *exists on* $[0, t]$, *then* $(u(t), \partial_t u(t)) \in \mathscr{K}^{\pm}$.

Proof. Consider $(u^0, u^1) \in \mathscr{K}^+$, and assume by contradiction that there exists $0 < t_0 \leq t$ such that $(u(t_0), \partial_t u(t_0)) \notin \mathscr{K}^+$. As $E(u(t_0), \partial_t u(t_0)) \leq E(u^0, u^1) < m_0$, there exists a sequence $(t_n)_n$ such that $0 < t_n < t$, $K(u(t_n)) < 0$, and $K(u(t_n)) \xrightarrow{n\to\infty} 0$. Lemma 5.1.7 gives $\liminf J(u(t_n)) \geq m_0$, a contradiction with

$$J(u(t_n)) \leq E(u(t_n), \partial_t u(t_n)) \leq E\left(u^0, u^1\right) < m_0, \quad n \in \mathbb{N}.$$

The proof for $(u^0, u^1) \in \mathscr{K}^-$ is the same. □

5.2.1 Global solutions

The following result is straightforward, as in the case of the undamped equation.

Theorem 5.2.2. *Suppose* $(u^0, u^1) \in \mathscr{K}^+$, $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, *and* u *is the solution of* (5.0.3). *Then* u *is defined on* $\mathbb{R}_+$.

Proof. Using Lemma 5.2.1 and the fact that the energy is nonincreasing, one obtains

$$\begin{aligned} E\left(u^0, u^1\right) \geq E(u(t), \partial_t u(t)) &= J(u(t)) + \frac{1}{2}\|\partial_t u(t)\|^2_{L^2} \\ &= \frac{1}{4}K(u(t)) + \frac{1}{4}\|u(t)\|^2_{H^1_0} + \frac{1}{2}\|\partial_t u(t)\|^2_{L^2} \\ &\geq \frac{1}{4}\|u(t)\|^2_{H^1_0} + \frac{1}{2}\|\partial_t u(t)\|^2_{L^2}. \end{aligned}$$

Hence, there is no blow-up, and u is defined on $\mathbb{R}_+$ by the Cauchy theory (see Theorem 5.1.4). □

The previous proof also gives the following lemma.

Lemma 5.2.3 (Equivalence between the energy and the square of the $H_0^1 \times L^2$ norm for solutions in $\mathscr{K}^+$)**.** *For all $(u^0, u^1) \in \mathscr{K}^+$ and $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, one has*

$$2E\left(u(t), \partial_t u(t)\right) \leq \|u(t)\|_{H_0^1}^2 + \|\partial_t u(t)\|_{L^2}^2 \leq 4E\left(u(t), \partial_t u(t)\right), \quad t \geq 0,$$

where u is the solution of (5.0.3) *with initial data* (u^0, u^1)*.*

We will use this to prove the stabilisation of a solution starting from $\mathscr{K}^+$. one has the following source-to-solution continuity result.

Lemma 5.2.4. *Consider $T > 0$, $C_0 > 0$, and $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$. There exists $C > 0$ such that for all $(u^0, u^1), (v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solutions u and v of* (5.0.3) *with initial data (u^0, u^1) and (v^0, v^1) are defined on $[0, T]$ and satisfy*

$$\sup_{t\in[0,T]} \left(\|(u(t), \partial_t u(t))\|_{H_0^1\times L^2} + \|(v(t), \partial_t v(t))\|_{H_0^1\times L^2}\right) \leq C_0, \tag{5.2.1}$$

then one has

$$\sup_{t\in[0,T]} \|(u(t), \partial_t u(t)) - (v(t), \partial_t v(t))\|_{H_0^1\times L^2} \leq C \|(u(t_0), \partial_t u(t_0)) - (v(t_0), \partial_t v(t_0))\|_{H_0^1\times L^2}$$

for $t_0 \in \left\{0, \frac{T}{2}, T\right\}$.

Proof. It suffices to prove the result for $t_0 = 0$ and for $t_0 = T$. In both cases, we can assume that T is arbitrary small: the result for large T follows by iteration. We assume that $t_0 = 0$, the other case is similar.

Set $w = u - v$, solution of

$$\begin{cases} \Box w + \gamma \partial_t w + \beta w &= u^3 - v^3 & \text{in } \mathbb{R}_+ \times \Omega, \\ (w(0), \partial_t w(0)) &= (u^0, u^1) - (v^0, v^1) & \text{in } \Omega, \\ w &= 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Assume that $T < 1$. Then, by Theorem 5.1.1 and Remark 5.1.2, there exists a constant independent of T, (u^0, u^1) and (v^0, v^1) such that

$$\sup_{t\in[0,T]} \left(\|w(t)\|_{H_0^1} + \|\partial_t w(t)\|_{L^2}\right) \lesssim \left\|\left(u^0, u^1\right) - \left(v^0, v^1\right)\right\|_{H_0^1\times L^2} + \left\|u^3 - v^3\right\|_{L^1((0,T),L^2)}.$$

Hölder's inequality gives

$$\left\|u^3 - v^3\right\|_{L^1((0,T),L^2)} \lesssim T \|u - v\|_{L^\infty((0,T),L^6)} \left(\|u\|_{L^\infty((0,T),L^6)}^2 + \|v\|_{L^\infty((0,T),L^6)}^2\right).$$

Using (5.2.1) and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, one finds

$$\begin{aligned} \left\|u^3 - v^3\right\|_{L^1((0,T),L^2)} &\lesssim T \|w\|_{L^\infty((0,T),H_0^1)} \left(\|u\|_{L^\infty((0,T),H_0^1)}^2 + \|v\|_{L^\infty((0,T),H_0^1)}^2\right) \\ &\lesssim T \|w\|_{L^\infty((0,T),H_0^1)}. \end{aligned}$$

Hence, for T sufficiently small, one obtains

$$\sup_{t\in[0,T]} \left(\|w(t)\|_{H_0^1} + \|\partial_t w(t)\|_{L^2}\right) \lesssim \left\|\left(u^0, u^1\right) - \left(v^0, v^1\right)\right\|_{H_0^1\times L^2}$$

and this completes the proof. ∎

5.2.2 Blow-up solutions

Here, we prove that a solution initiated in $\mathscr{K}^-$ blows up in finite time.

Theorem 5.2.5. *For $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$, if u is the solution of (5.0.3) with initial data $(u^0, u^1) \in \mathscr{K}^-$, then the maximal time of existence of u is finite.*

Proof. Take $(u^0, u^1) \in \mathscr{K}^-$ and assume by contradiction that $u(t)$ exists for all $t \geq 0$. As in the original proof of Payne and Sattinger [PS75], set $M(t) = \|u(t)\|_{L^2}^2$. Then by Lemma 4.1.2, one has $M \in \mathscr{C}^2(\mathbb{R}_+, \mathbb{R})$, with

$$M'(t) = 2\int_\Omega u(t)\partial_t u(t)\mathrm{d}x. \tag{5.2.2}$$

and

$$M''(t) = -2\|u(t)\|_{H_0^1}^2 + 2\|u(t)\|_{L^4}^4 + 2\|\partial_t u(t)\|_{L^2}^2 - 2\int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x. \tag{5.2.3}$$

One has

$$M''(t) = -2K(u(t)) + 2\|\partial_t u(t)\|_{L^2}^2 - 2\int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x.$$

Write $E(t) = E(u(t), \partial_t u(t))$ for the energy. Recall that the energy equality is

$$E'(t) = -\int_\Omega \gamma |\partial_t u(t)|^2 \mathrm{d}x.$$

As the energy is nonincreasing, either it is bounded or it tends to $-\infty$ as t tends to infinity. We treat these two cases separately.

Case 1: bounded energy. For all $\varepsilon > 0$, as $\gamma \in L^\infty(\mathbb{R}_+)$, one has

$$\left|2\int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x\right| \lesssim \varepsilon\|u(t)\|_{H_0^1}^2 + \frac{1}{\varepsilon}\int_\Omega \gamma|\partial_t u(t)|^2\mathrm{d}x = \varepsilon\|u(t)\|_{H_0^1}^2 - \frac{1}{\varepsilon}E'(t).$$

This gives

$$M''(t) \geq -2K(u(t)) - C_1\varepsilon\|u(t)\|_{H_0^1}^2 + 2\|\partial_t u(t)\|_{L^2}^2 + \frac{C_1}{\varepsilon}E'(t) \tag{5.2.4}$$

for some $C_1 > 0$. Inequality (5.2.4) has two consequences.

First, we use Lemma 5.1.8 *(ii)*: for $t \geq 0$, one has

$$J(u(t)) \leq E(u(t), \partial_t u(t)) \leq E\left(u^0, u^1\right) < m_0$$

so there exists $C_2 > 0$ such that

$$K(u(t)) \leq -C_2\|u(t)\|_{H_0^1}^2 \quad \text{and} \quad K(u(t)) \leq -C_2, \quad t \geq 0. \tag{5.2.5}$$

Using this in (5.2.4) and taking ε sufficiently small, one obtains

$$M''(t) - \frac{C_1}{\varepsilon}E'(t) \gtrsim \|u(t)\|_{H_0^1}^2 + \|\partial_t u(t)\|_{L^2}^2 \geq \|u(t)\|_{H_0^1}^2.$$

Note that by (5.2.5), one has $\|u(t)\|_{H_0^1} \gtrsim 1$ for $t \geq 0$. Hence, one has $M''(t) - \frac{C_1}{\varepsilon}E'(t) \gtrsim 1$, implying

$$M'(t) - \frac{C_1}{\varepsilon}E(t) - M'(0) + \frac{C_1}{\varepsilon}E(0) \gtrsim t$$

for all $t \geq 0$. As the energy is bounded, this gives

$$M'(t) \xrightarrow{t \to +\infty} +\infty \quad \text{and} \quad M(t) \xrightarrow{t \to +\infty} +\infty.$$

Second, we use the energy equality. By definition, one has

$$K(u(t)) = 4E(t) - 2\left\|\partial_t u(t)\right\|_{L^2}^2 - \left\|u(t)\right\|_{H_0^1}^2$$

for all $t \geq 0$. Using this in (5.2.4), one obtains

$$M''(t) - \frac{C_1}{\varepsilon}E'(t) \geq 6\left\|\partial_t u(t)\right\|_{L^2}^2 + (2 - C_1\varepsilon)\left\|u(t)\right\|_{H_0^1}^2 - 8E(t).$$

Set $C = \frac{C_1}{\varepsilon}$. As the energy is bounded and as $M(t)$ tends to infinity, for ε chosen sufficiently small and t sufficiently large, one obtains

$$M''(t) - CE'(t) \geq 6\left\|\partial_t u(t)\right\|_{L^2}^2.$$

Together with the Cauchy-Schwarz inequality, this gives

$$\left|M'(t)\right|^2 \leq 4\left(\int_\Omega |u(t)|^2 \,\mathrm{d}x\right)\left(\int_\Omega |\partial_t u(t)|^2 \,\mathrm{d}x\right) \leq \frac{2}{3}M(t)\left(M''(t) - CE'(t)\right), \quad t \geq 0 \text{ large.}$$

Dividing by $M(t)M'(t)$, and using $M'(t) \to +\infty$ and $E'(t) \leq 0$, one writes

$$\frac{M'(t)}{M(t)} \leq \frac{2}{3}\left(\frac{M''(t)}{M'(t)} - C\frac{E'(t)}{M'(t)}\right) \leq \frac{2}{3}\left(\frac{M''(t)}{M'(t)} - CE'(t)\right)$$

for t large. Consider $T > 0$ such that the previous inequality holds for all $t \geq T$. Integrating, one obtains

$$\ln\left(M(t)\right) - \ln\left(M(T)\right) \leq \frac{2}{3}\left(\ln\left(M'(t)\right) - \ln\left(M'(T)\right)\right) - \frac{2C}{3}\left(E(t) - E(T)\right), \quad t \geq T.$$

As the energy is bounded and $M(t) \to +\infty$, this gives $\ln\left(M(t)\right) \leq \frac{3}{4}\ln\left(M'(t)\right)$ for t large, yielding

$$\frac{M'(t)}{M(t)^{\frac{4}{3}}} \geq 1, \quad t \geq T', \quad T' \text{ large.}$$

Integrating between t_1 and t_2 with $T' \leq t_1 \leq t_2$, one finds

$$-\frac{3}{M(t_2)^{\frac{1}{3}}} + \frac{3}{M(t_1)^{\frac{1}{3}}} \geq t_2 - t_1.$$

Letting t_2 tend to infinity gives a contradiction.

Case 2: the energy tends to $-\infty$. In that case, by definition of the energy, one has

$$\left\|u(t)\right\|_{L^4} \xrightarrow{t \to +\infty} +\infty.$$

For $\varepsilon > 0$, as $\gamma \in L^\infty(\mathbb{R}_+)$ and as Ω is bounded, one has

$$\left|2\int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x\right| \lesssim \frac{1}{\varepsilon}\left\|u(t)\right\|_{L^2}^2 + \varepsilon\left\|\partial_t u(t)\right\|_{L^2}^2 \lesssim \frac{1}{\varepsilon}\left\|u(t)\right\|_{L^4}^2 + \varepsilon\left\|\partial_t u(t)\right\|_{L^2}^2,$$

and together with (5.2.3), this gives

$$M''(t) \geq -2\|u(t)\|_{H_0^1}^2 + (2 - C\varepsilon)\|\partial_t u(t)\|_{L^2}^2 + 2\|u(t)\|_{L^4}^4 - \frac{C}{\varepsilon}\|u(t)\|_{L^4}^2$$

for some $C > 0$. Consider $\varepsilon > 0$ such that $2 - C\varepsilon \geq \frac{3}{2}$. For t sufficiently large, one has

$$2\|u(t)\|_{L^4}^4 - \frac{C}{\varepsilon}\|u(t)\|_{L^4}^2 \geq \frac{3}{2}\|u(t)\|_{L^4}^4,$$

and one obtains

$$M''(t) \geq -2\|u(t)\|_{H_0^1}^2 + \frac{3}{2}\|\partial_t u(t)\|_{L^2}^2 + \frac{3}{2}\|u(t)\|_{L^4}^4, \quad t \geq 0 \text{ large.}$$

By definition of the energy, one has

$$\|u(t)\|_{L^4}^4 = 2\|\partial_t u(t)\|_{L^2}^2 + 2\|u(t)\|_{H_0^1}^2 - 4E(t)$$

and for t sufficiently large, this gives

$$M''(t) \geq \frac{9}{2}\|\partial_t u(t)\|_{L^2}^2 + \|u(t)\|_{H_0^1}^2 - 6E(t).$$

In particular, one has $M''(t) \to +\infty$, and consequently $M(t) \to +\infty$. For t sufficiently large, one also has $M''(t) \geq \frac{9}{2}\|\partial_t u(t)\|_{L^2}^2$. Using (5.2.2) and the Cauchy-Schwarz inequality, this gives

$$M'(t)^2 \leq \frac{8}{9}M(t)M''(t), \quad t \geq 0 \text{ large.}$$

For $\alpha > 0$, set $\tilde{M}(t) = M(t)^{-\alpha}$. For t sufficiently large, one has

$$\begin{aligned}\tilde{M}''(t) &= \alpha M(t)^{-\alpha-2}\left((\alpha+1)M'(t)^2 - M(t)M''(t)\right)\\ &\leq \alpha\left(\frac{8}{9}(\alpha+1) - 1\right)M(t)^{-\alpha-1}M''(t)\end{aligned}$$

Hence, $\tilde{M}$ is a concave function for t sufficiently large if α is chosen sufficiently small. As $\tilde{M} > 0$ and $\tilde{M}(t) \to 0$, this gives a contradiction. ☐

5.3 Convergence towards a stationary solution

5.3.1 Convergence of a bounded sequence along a time sequence

Here, we show the following proposition.

Proposition 5.3.1. *Consider $\gamma \in L^\infty(\Omega, \mathbb{R}_+)$ satisfying the GCC. Let $(u_n^0, u_n^1)_{n\in\mathbb{N}}$ be a sequence of elements of $H_0^1(\Omega) \times L^2(\Omega)$, and for $n \in \mathbb{N}$, write u_n for the solution of (5.0.3) with initial data (u_n^0, u_n^1). Let $(T_n)_{n\in\mathbb{N}}$ be a time-sequence satisfying $T_n \to +\infty$. Assume that each u_n exists on $\mathbb{R}_+$, that there exists $C > 0$ such that*

$$\|u_n(t)\|_{H_0^1} + \|\partial_t u_n(t)\|_{L^2} \leq C, \quad t \geq 0, \quad n \in \mathbb{N}, \tag{5.3.1}$$

and that for all $T > 0$,

$$\int_{T_n-T}^{T_n+T}\int_\Omega \gamma\,|\partial_t u_n|^2\,\mathrm{d}x\mathrm{d}t \xrightarrow{n\to\infty} 0. \tag{5.3.2}$$

Then there exist an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and a stationary solution w of (5.0.1) such that for all $T > 0$,

$$\sup_{t\in[-T,T]}\left(\left\|u_{\phi(n)}(T_{\phi(n)} + t) - w\right\|_{H_0^1} + \left\|\partial_t u_{\phi(n)}(T_{\phi(n)} + t)\right\|_{L^2}\right) \xrightarrow{n\to\infty} 0.$$

Proof. We divide the proof into three steps.

Step 1: asymptotic compactness. We use Corollary 4.2 of [JL13] (which relies on a result from [DLZ03]), which we copy here for convenience.

Lemma 5.3.2 (Corollary 4.2 of [JL13])**.** *Take $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ such that*

$$f(0) = 0, \quad xf(x) \geq 0, \quad |f(x)| \lesssim (1 + |x|)^p, \quad |f'(x)| \lesssim (1 + |x|)^{p-1}$$

with $1 \leq p < 5$. Consider $R > 0$, $T > 0$, $0 \leq s < 1$, and set $\varepsilon = \min\left(1 - s, \frac{5-p}{2}, \frac{17-3p}{14}\right) > 0$. There exist $C > 0$ and (q, r) satisfying

$$\frac{1}{q} + \frac{3}{r} = \frac{1}{2}, \quad q \in \left[\frac{7}{2}, +\infty\right]$$

such that the following property holds: if $u \in L^\infty([0,T], H^{1+s}(\Omega) \cap H_0^1(\Omega))$ satisfies

$$\|u\|_{L^q([0,T],L^r)} \leq R,$$

then $f(u) \in L^1([0,T], H_0^{s+\varepsilon}(\Omega))$, with

$$\|f(u)\|_{L^1([0,T],H^{s+\varepsilon})} \leq C \|u\|_{L^\infty([0,T],H^{1+s})}.$$

We will use this lemma with $f(x) = x^3$. Recall that X^s is defined by (5.1.1). We prove the following corollary.

Corollary 5.3.3. *Consider $T > 0$, $0 \leq s < 1$, $C_0 > 0$, and set $\varepsilon = \min\left(1 - s, \frac{4}{7}\right)$. There exists $C > 0$ such that for all $(u^0, u^1) \in X^s$, if the solution u of (5.0.3) with initial data (u^0, u^1) exists on $[0,T]$ and satisfies $u \in L^\infty([0,T], H^{1+s}(\Omega) \cap H_0^1(\Omega))$, with*

$$\|u\|_{L^\infty([0,T],H_0^1)} + \|\partial_t u\|_{L^\infty([0,T],L^2)} \leq C_0,$$

then one has $u^3 \in L^1([0,T], H_0^{s+\varepsilon}(\Omega))$, with

$$\left\|u^3\right\|_{L^1([0,T],H_0^{s+\varepsilon})} \leq C \|u\|_{L^\infty([0,T],H^{1+s})}.$$

Proof. The function $f(x) = x^3$ clearly satisfies the assumption of the previous theorem. By Strichartz estimates (Theorem 5.1.3), one has

$$\|u\|_{L^q([0,T],L^r)} \lesssim \|u^3\|_{L^1([0,T],L^2)} + \|u^0\|_{H_0^1} + \|u^1\|_{L^2}.$$

for all (q, r) satisfying $\frac{1}{q} + \frac{3}{r} = \frac{1}{2}$ and $q \in \left[\frac{7}{2}, +\infty\right]$. Using the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, one obtains

$$\|u\|_{L^q([0,T],L^r)} \lesssim \|u\|^3_{L^\infty([0,T],H_0^1)} + \|u^0\|_{H_0^1} + \|u^1\|_{L^2} \leq C_0^3 + C_0.$$

Hence, we can apply Corollary 4.2 of [JL13]: it gives

$$\left\|u^3\right\|_{L^1([0,T],H_0^{s+\varepsilon})} \lesssim \|u\|_{L^\infty([0,T],H^{1+s})}$$

with a constant depending only on C_0. This completes the proof of Corollary 5.3.3. □

We will use this result to find a subsequence of $(u_n(T_n), \partial_t u_n(T_n))$ which converges in $H_0^1(\Omega) \times L^2(\Omega)$. Recall that

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - \beta & -\gamma \end{pmatrix}$$

is the infinitesimal generator of the linear part of (5.0.3), that e^{tA} is the associated semi-group, and put

$$U_n = (u_n, \partial_t u_n) \quad \text{and} \quad F_n = \left(0, u_n^3\right)$$

We will use the well-know fact that the GCC implies the stabilisation of the linear version of (5.0.3), as stated in Theorem 5.1.5. Using the Duhamel formula, we can write

$$\begin{aligned} U_n(T_n) &= e^{T_n A} U_n(0) + \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{kA} \int_0^1 e^{sA} F_n(T_n - k - s) \mathrm{d}s + \int_{\lfloor T_n \rfloor}^{T_n} e^{sA} F_n(T_n - s) \mathrm{d}s \\ &= e^{T_n A} U_n(0) + \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{kA} I_{n,k} + I_n. \end{aligned}$$

We show that the Duhamel term is bounded in X^ε, with $\varepsilon = \frac{4}{7}$. To do so, write

$$\begin{aligned} \|I_{n,k}\|_{X^\varepsilon} &\leq \int_0^1 \left\| e^{sA} F_n(T_n - k - s) \right\|_{X^\varepsilon} \mathrm{d}s \\ &\lesssim \int_0^1 \|F_n(T_n - k - s)\|_{X^\varepsilon} \mathrm{d}s \\ &= \int_0^1 \left\| u_n(T_n - k - s)^3 \right\|_{H_0^\varepsilon} \mathrm{d}s. \end{aligned}$$

By (5.3.1), one has

$$\|u_n(T_n - k - \cdot)\|_{L^\infty((0,1), H_0^1)} \leq \|u_n\|_{L^\infty((0,\infty), H_0^1)} \lesssim 1.$$

Hence, we can apply Corollary 5.3.3 (with $s = 0$) to find $\|I_{n,k}\|_{X^\varepsilon} \lesssim 1$. Similarly, one shows $\|I_n\|_{X^\varepsilon} \lesssim 1$. Using the linear stabilisation (Theorem 5.1.5), we see that there exists $\lambda > 0$ such that

$$\left\| \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{kA} I_{n,k} + I_n \right\|_{X^\varepsilon} \lesssim \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{-\lambda k} + 1 \lesssim 1.$$

We have proved that the sequence $\left(U_n(T_n) - e^{T_n A} U_n(0)\right)_n$ is bounded in X^ε. By Rellich's theorem, there exists $U_\infty(0) \in H_0^1(\Omega) \times L^2(\Omega)$ such that up to a subsequence, one has

$$U_n(T_n) - e^{T_n A} U_n(0) \xrightarrow{n \to \infty} U_\infty(0)$$

in $H_0^1(\Omega) \times L^2(\Omega)$. Using linear stabilisation again, one finds $U_n(T_n) \xrightarrow{n \to \infty} U_\infty(0)$ in $H_0^1(\Omega) \times L^2(\Omega)$.

Let u_∞ be the solution of (5.0.3) with initial data $U_\infty(0)$. We show that u_∞ is defined on $\mathbb{R}$. Consider $T > 0$. For n and m sufficiently large, Lemma 5.2.4 gives

$$\begin{aligned} &\sup_{t \in [-T, T]} \left(\|u_n(T_n + t) - u_m(T_m + t)\|_{H_0^1}^2 + \|\partial_t u_n(T_n + t) - \partial_t u_m(T_m + t)\|_{L^2}^2 \right) \\ &\lesssim \|u_n(T_n) - u_m(T_m)\|_{H_0^1}^2 + \|\partial_t u_n(T_n) - \partial_t u_m(T_m)\|_{L^2}^2 \end{aligned}$$

so that $(u_n(T_n+\cdot))_n$ is a Cauchy sequence in $\mathscr{C}^0([-T,T],H_0^1(\Omega))\cap\mathscr{C}^1([-T,T],L^2(\Omega))$. The limit is a solution of (5.0.3) on $[-T,T]$, and coincides with u_∞ near 0. Hence, u_∞ is defined on $\mathbb{R}$. Note that (5.3.1) gives

$$\|u_\infty(t)\|_{H_0^1}+\|\partial_t u_\infty(t)\|_{L^2}\le C,\quad t\in\mathbb{R}, \tag{5.3.3}$$

and (5.3.2) implies

$$\int_{\mathbb{R}}\int_\Omega \gamma\,|\partial_t u_\infty|^2\,\mathrm{d}x\mathrm{d}t=0.$$

Hence, one has $E\left(u_\infty(t),\partial_t u_\infty(t)\right)=E\left(u_\infty(0),\partial_t u_\infty(0)\right)$ for all $t\in\mathbb{R}$, and $\partial_t u_\infty(t,x)=0$ for all $t\in\mathbb{R}$ and almost all $x\in\omega$.

Step 2: regularity of the limit. To ease notations, write $u=u_\infty$. In this step, we will use Corollary 5.3.3 again to show that

$$(u(0),\partial_t u(0))\in H^2(\Omega)\times H_0^1(\Omega) \tag{5.3.4}$$

and we will use a result of [HR03] to prove that for all $\alpha\in\left(\frac{1}{2},1\right)$,

$$u:\mathbb{R}\longrightarrow H^{1+\alpha}(\Omega)\cap H_0^1(\Omega)\text{ is analytic.} \tag{5.3.5}$$

Finally, we will see that it implies

$$u\in\mathscr{C}^\infty(\mathbb{R}\times\overline{\Omega}). \tag{5.3.6}$$

Proof of (5.3.4). As above, set $U=(u,\partial_t u)$ and $F=(0,u^3)$. By Duhamel's formula, one has

$$\begin{aligned}U(t)&=e^{nA}U(t-n)+\int_0^n e^{sA}F(t-s)\mathrm{d}s\\&=e^{nA}U(t-n)+\sum_{k=0}^{n-1}e^{kA}\int_0^1 e^{sA}F(t-k-s)\mathrm{d}s\end{aligned}$$

for $t\in\mathbb{R}$ and $n\in\mathbb{N}$. By (5.3.3), linear stabilization gives

$$e^{nA}U(t-n)\xrightarrow{n\to\infty}0,\quad t\in\mathbb{R},$$

in X^0. Furthermore, one has

$$\begin{aligned}\left\|\int_0^1 e^{sA}F(t-k-s)\mathrm{d}s\right\|_{X^0}&\lesssim\int_0^1\|F(t-k-s)\|_{X^0}\,\mathrm{d}s\\&=\left\|u^3\right\|_{L^1([t-k-1,t-k],L^2)}\end{aligned}$$

implying

$$\left\|\int_0^1 e^{sA}F(t-k-s)\mathrm{d}s\right\|_{X^0}\lesssim\|u\|^3_{L^\infty([t-k-1,t-k],H_0^1)}\lesssim 1$$

by the Sobolev embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$. In particular, using linear stabilization again, one finds

$$\left\|e^{kA}\int_0^1 e^{sA}F(t-k-s)\mathrm{d}s\right\|_{X^0}\le Ce^{-\lambda k}$$

for some $C > 0$ and $\lambda > 0$. This gives

$$U(t) = \sum_{k=0}^{\infty} e^{kA} \int_0^1 e^{sA} F(t-k-s)\mathrm{d}s, \quad t \in \mathbb{R}, \tag{5.3.7}$$

in X^0.

Next, we prove

$$U \in L^\infty(\mathbb{R}, X^1) \tag{5.3.8}$$

by using Corollary 5.3.3 two times. First, by (5.3.3), Corollary 5.3.3 with $T = 1$ and $s = 0$ gives

$$\left\| u(t-k-\cdot)^3 \right\|_{L^1((0,1),H_0^\varepsilon)} \lesssim 1, \quad t \in \mathbb{R}, \quad k \in \mathbb{N},$$

where $\varepsilon = \frac{4}{7}$. In particular, using linear stabilisation in X^ε, one has

$$\left\| e^{kA} \int_0^1 e^{sA} F(t-k-s)\mathrm{d}s \right\|_{X^\varepsilon} \lesssim e^{-\lambda k} \left\| u(t-k-\cdot)^3 \right\|_{L^1((0,1),H_0^\varepsilon)} \lesssim e^{-\lambda k},$$

implying that equality (5.3.7) holds in X^ε, and $U \in L^\infty(\mathbb{R}, X^\varepsilon)$. Second, Corollary 5.3.3 with $T = 1$ and $s = \frac{4}{7}$ gives

$$\left\| u(t-k-\cdot)^3 \right\|_{L^1((0,1),H_0^{s+\varepsilon'})} \lesssim 1, \quad t \in \mathbb{R}, \quad k \in \mathbb{N}$$

where $\varepsilon' = \frac{3}{7}$. As above, this proves that equality (5.3.7) holds in X^1, and that (5.3.8) holds true. In particular, (5.3.4) is true. Note that using the Sobolev embedding $H^2(\Omega) \hookrightarrow \mathscr{C}^0(\overline{\Omega})$ (see for example [AF03], 4.12 Part II with $n = 3$, $m = p = 2$, $j = 0$), one finds $u \in L^\infty(\mathbb{R} \times \overline{\Omega})$.

Proof of (5.3.5). Following [JL13], we use Theorem 2.20 of [HR03] (applied with hypotheses (H3mod) and (H5)), which we copy here for convenience.

Theorem 5.3.4. *Let Y be a Banach space. Let $P_n \in \mathscr{L}(Y)$ be a sequence of continuous linear maps and let $Q_n = \mathrm{Id} - P_n$. Let $A : D(A) \to Y$ be the generator of a continuous semi-group e^{tA} and let $G \in \mathscr{C}^1(Y, Y)$. Let U be a global solution in Y of*

$$\partial_t U(t) = AU(t) + G(U(t)), \quad t \in \mathbb{R}.$$

We further assume that:

(i) $\{U(t), t \in \mathbb{R}\}$ is contained in a compact set K of Y.

(ii) For any $y \in Y$, $P_n y$ converges to y when n goes to infinity and (P_n) and (Q_n) are sequences of $\mathscr{L}(Y)$ bounded by a constant C_0.

(iii) The operator A splits as $A = A_1 + B_1$, where B_1 is bounded and A_1 commutes with P_n.

(iv) There exist M and $\lambda > 0$ such that $\left\| e^{tA} \right\|_{\mathscr{L}(Y)} \leq M e^{-\lambda t}$ for all $t \geq 0$.

(v) G is analytic in the ball $B_Y(0, r)$, where r is such that $r \geq 4C_0 \sup_{t \in \mathbb{R}} \|U(t)\|_Y$. More precisely, there exists $\rho > 0$ such that G can be extended to an holomorphic function of $B_Y(0, r) + iB_Y(0, \rho)$.

(vi) $\{DG(U(t))V, t \in \mathbb{R}, \|V\|_Y \leq 1\}$ is a relatively compact set of Y.

Then the solution $U : \mathbb{R} \to Y$ is analytic.

Consider $\alpha \in \left(\frac{1}{2}, 1\right)$. Write $(\lambda_k)_{k\in\mathbb{N}}$ for the eigenvalues of the Dirichlet Laplacian, and $(\varphi_k)_{k\in\mathbb{N}}$ for a basis of normalized eigenvectors. We will apply this theorem with $Y = X^\alpha$. For $n \in \mathbb{N}^*$, let P_n be the restriction to Y of the $L^2(M) \times L^2(M)$ orthogonal projection on the vector space generated by the vectors

$$(\varphi_1, 0), \cdots, (\varphi_n, 0), (0, \varphi_1), \cdots, \text{ and } (0, \varphi_n).$$

By (5.1.2), we can equip Y with a norm represented by a Fourier series, using the sequence $(\varphi_k)_{k\in\mathbb{N}}$. This implies $P_n \in \mathscr{L}(Y)$ and $Q_n = \mathrm{Id} - P_n \in \mathscr{L}(Y)$, with

$$\|P_n\|_{\mathscr{L}(Y)} \leq C_0 \quad \text{and} \quad \|Q_n\|_{\mathscr{L}(Y)} \leq C_0 \tag{5.3.9}$$

for some $C_0 > 0$.

Let $G : Y \to Y$ be given by $G\left(u^0, u^1\right) = \left(0, (u^0)^3\right)$. Note that G is well-defined as $H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$ is an algebra, as $\alpha > \frac{1}{2}$. We need to use a smooth version of the damping γ. Recall that ω is an open subset of Ω satisfying the GCC, and that $\gamma(x) > \alpha > 0$ for almost all $x \in \omega$. Recall also that u satisfies $\gamma\partial_t u = 0$ and $\Box u + \beta u = u^3$. By continuity of generalized geodesics (see for example Theorem 3.34 of [MS78]), there exists an open subset $\tilde{\omega}$, compactly included in ω, such that the GCC holds for $\tilde{\omega}$. Consider $\tilde{\gamma} \in \mathscr{C}^\infty_{\mathrm{c}}(\omega, [0, 1])$ such that $\tilde{\gamma} = 1$ on $\tilde{\omega}$. one has $\tilde{\gamma}\partial_t u = 0$, implying

$$\Box u + \tilde{\gamma}\partial_t u + \beta u = u^3.$$

Let $\tilde{A}$ be the infinitesimal generator of the linear part of this equation, that is

$$A = \begin{pmatrix} 0 & -1 \\ -\Delta + \beta & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\gamma} \end{pmatrix} = A_1 + B_1.$$

Now, we check that the assumptions of the previous theorem are satisfied. One has $U \in L^\infty(\mathbb{R}, X^1)$, and as $\alpha < 1$ the embedding $X^1 \to X^\alpha$ is compact by the Rellich theorem, so *(i)* is clear. The fact that for all $y \in Y$, $P_n y$ converges to y when n goes to infinity is obvious when the norm of Y is expressed as a Fourier series. Together with (5.3.9), this gives *(ii)*. As $\tilde{\gamma}$ is smooth, *(iii)* is clear. We have already used the fact that *(iv)* is true (see Theorem 5.1.5). For $(u^0, u^1) \in Y$ and $(v_0, v_1) \in Y$, the function

$$z \in \mathbb{C} \longmapsto G\left(\left(u^0, u^1\right) + z(v_0, v_1)\right) = \left(0, (u_0 + zv_0)^3\right)$$

is well-defined and analytic, as $H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$ is an algebra, so *(v)* is true.

Finally, we check that *(vi)* holds. one has

$$\{DG(U(t))V, t \in \mathbb{R}, \|V\|_Y \leq 1\} = \left\{\left(0, u(t)^2 v_0\right), t \in \mathbb{R}, \|v_0\|_{H^{1+\alpha}\cap H^1_0} \leq 1\right\}.$$

Take $t \in \mathbb{R}$ and $v_0 \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$ such that $\|v_0\|_{H^{1+\alpha}\cap H^1_0} \leq 1$. For $\varepsilon > 0$ sufficiently small, using again the fact that $H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$ is an algebra, one obtains

$$\left\|u(t)^2 v_0\right\|_{H_0^{\alpha+\varepsilon}} \leq \left\|u(t)^2 v_0\right\|_{H^{1+\alpha}\cap H^1_0} \lesssim \|u(t)\|^2_{H^{1+\alpha}\cap H^1_0} \|v_0\|_{H^{1+\alpha}\cap H^1_0} \lesssim \|u\|^2_{L^\infty(\mathbb{R}, X^1)}.$$

The embedding $H_0^{\alpha+\varepsilon}(\Omega) \to H_0^\alpha(\Omega)$ is compact by the Rellich theorem, so *(vi)* is true. The previous theorem gives (5.3.5).

Proof of (5.3.6). One has $\partial_t^k u(t) \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$ for all $k \in \mathbb{N}$ and all $\alpha \in \left(\frac{1}{2}, 1\right)$. This will give (5.3.6) by standard elliptic regularity properties and a bootstrap argument. More precisely, we use the following result, which can be found (for example) in [GT01], Theorem 9.19. Note that for $\ell \in \mathbb{N}$ and $\lambda \in (0,1)$, the norm of $\mathscr{C}^{\ell,\lambda}(\overline{\Omega})$ is given by

$$\|v\|_{\mathscr{C}^{\ell,\lambda}} = \max_{|\beta|\leq \ell}\left(\|\partial_x^\beta v\|_{L^\infty(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega},x\neq y}\frac{\left|\partial_x^\beta v(x)-\partial_x^\beta v(y)\right|}{|x-y|^\lambda}\right), \quad v\in \mathscr{C}^{\ell,\lambda}(\overline{\Omega}).$$

Theorem 5.3.5. *Let $\mathcal{O}$ be a smooth Riemannian manifold, with or without boundary. Let $u \in H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ be such that $\Delta u = f \in \mathscr{C}^{\ell,\lambda}(\overline{\mathcal{O}})$, with $\ell \in \mathbb{N}$ and $\lambda \in (0,1)$. Then $u \in \mathscr{C}^{\ell+2,\lambda}\left(\overline{\mathcal{O}}\right)$.*

Set $\alpha = \frac{3}{4}$. We will use the fact that there exists $\lambda \in (0,1)$ such that the Sobolev embedding $H^{1+\alpha}(\Omega) \hookrightarrow \mathscr{C}^{0,\lambda}(\overline{\Omega})$ holds true (see for example [AF03], 4.12, Part II with $n=3$, $p=2$ and $j=1$). We prove by induction on $\ell\in\mathbb{N}$ that for all $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ and all $k\in\mathbb{N}$,

$$\phi\partial_t^k u \in \mathscr{C}^{2\ell,\lambda}\left(\mathbb{R}\times\overline{\Omega}\right). \tag{5.3.10}$$

We start with $\ell=0$. Consider $\phi\in\mathscr{C}_c^\infty(\mathbb{R})$, $k\in\mathbb{N}$, and let $I\subset\mathbb{R}$ be a compact interval such that $\operatorname{supp}\phi\subset I$. One has $\partial_t^k u$, $\partial_t^{k+1}u \in \mathscr{C}^0(I, H^{1+\alpha}(\Omega))$, implying

$$\partial_t^k u, \partial_t^{k+1}u \in L^\infty(I,\mathscr{C}^{0,\lambda}(\overline{\Omega}))$$

by the Sobolev embedding mentioned above. For $x,y\in\overline{\Omega}$, $t,t'\in I$, one has

$$\begin{aligned}
&\left|\phi(t)\partial_t^k u(t,x) - \phi(t')\partial_t^k u(t',y)\right| \\
\leq\ & |\phi(t)|\left|\partial_t^k u(t,x)-\partial_t^k u(t,y)\right| + \left|\int_{t'}^t \partial_t\left(\phi\partial_t^k u\right)(s,y)\mathrm{d}s\right| \\
\lesssim\ & \left\|\partial_t^k u\right\|_{L^\infty(I,\mathscr{C}^{0,\lambda})}|x-y|^\lambda + \left(\left\|\partial_t^k u\right\|_{L^\infty(I,\mathscr{C}^{0,\lambda})} + \left\|\partial_t^{k+1}u\right\|_{L^\infty(I,\mathscr{C}^{0,\lambda})}\right)|t-t'|
\end{aligned}$$

yielding (5.3.10) for $\ell=0$.

Now, let $\ell\in\mathbb{N}$ be such that (5.3.10) holds true. As above, consider $\phi\in\mathscr{C}_c^\infty(\mathbb{R})$, $k\in\mathbb{N}$, and $I\subset\mathbb{R}$ a compact interval such that $\operatorname{supp}\phi\subset I$. One has

$$\left(\partial_t^2+\Delta\right)\left(\phi\partial_t^k u\right) = \partial_t^2\left(\phi\partial_t^k u\right) + \phi\partial_t^k\left(\partial_t^2 u + \beta u - u^3\right).$$

As $\mathscr{C}^{2\ell,\lambda}(\mathbb{R}\times\overline{\Omega})$ is an algebra, (5.3.10) gives $(\partial_t^2+\Delta)\left(\phi\partial_t^k u\right)\in\mathscr{C}^{2\ell,\lambda}(\mathbb{R}\times\overline{\Omega})$. One also has

$$\left(\partial_t^2+\Delta\right)\left(\phi\partial_t^k u\right)\in L^2(I\times\Omega),$$

yielding $\phi\partial_t^k u \in H^2(I\times\Omega)$. By Theorem 5.3.5, applied in a smooth open subset of $I\times\Omega$ containing $\operatorname{supp}\phi\times\Omega$, one obtains (5.3.10) for $\ell+1$. This proves (5.3.6).

Step 3: identification of the limit. Here, we complete the proof of Proposition 5.3.1 by showing that u is a stationary solution of (5.0.1). We use Corollary 3.2 of [JL13] (which is a consequence of Theorem A of [RZ98]), which we copy here for convenience.

Theorem 5.3.6 (Corollary 3.2 of [JL13]). *Let $T\in(0,+\infty]$ and let b, $(c_i)_{i=1,2,3}$ and d be coefficients in $\mathscr{C}^\infty(\Omega\times[0,T),\mathbb{R})$. Assume moreover that b, c and d are analytic in time and that u is a strong solution of*

$$\partial_t^2 u = \Delta u + b\partial_t u + c\cdot\nabla u + m_0 u, \quad (t,x)\in(-T,T)\times\Omega.$$

Let $\mathcal{O}$ be a nonempty open subset of Ω such that $u(x,t)=0$ in $\mathcal{O}\times(-T,T)$. Then $u(x,0)=0$ in

$$\mathcal{O}_T=\{x\in\Omega,\mathrm{d}(x,\mathcal{O})<T\}$$

where the distance $\mathrm{d}(x,\mathcal{O})$ is defined as the infimum of the lengths of the $\mathscr{C}^1-$paths between x and a point of $\mathcal{O}$.

We apply this theorem with $\mathcal{O}=\omega$ and $T=+\infty$: the GCC implies that $\mathcal{O}_T=\Omega$ is that case. As $u\in\mathscr{C}^\infty(\mathbb{R}\times\Omega)$, if $v=\partial_t u$ then one has

$$\partial_t^2 v=\Delta v-\beta v+3u^2 v.$$

As u is smooth and analytic in time, and as $v=0$ on ω, one obtains $v=0$. Hence, u is a stationary solution of (5.0.1), and this completes the proof of Proposition 5.3.1. □

5.3.2 Convergence of a global solution

Here, we prove Theorem 5.0.5. Let $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ be such that the solution u of (5.0.3) with initial data (u^0,u^1) exists on $\mathbb{R}_+$. In a companion paper [Per24], we prove that there exists $C>0$ such that

$$\|u(t)\|_{H_0^1}+\|\partial_t u(t)\|_{L^2}\leq C,\quad t\geq 0. \tag{5.3.11}$$

By (5.3.11), the energy of u is bounded. Hence, as the energy is nonincreasing, there exists $E_\infty\leq E\left(u^0,u^1\right)$ such that

$$E\left(u(t),\partial_t u(t)\right)\xrightarrow{t\to+\infty}E_\infty.$$

The energy equality gives

$$E\left(u^0,u^1\right)-E_\infty=\int_0^\infty\int_\Omega\gamma\left|\partial_t u\right|^2\mathrm{d}x\mathrm{d}t<\infty$$

and this allows us to use Proposition 5.3.1. More precisely, let $(T_n)_{n\in\mathbb{N}}$ be a time-sequence satisfying $T_n\to+\infty$. For all $T>0$, one has

$$\int_{T_n-T}^{T_n+T}\int_\Omega\gamma\left|\partial_t u\right|^2\mathrm{d}x\mathrm{d}t\xrightarrow{n\to\infty}0.$$

Using also (5.3.11), we see that all the assumptions of Proposition 5.3.1 are satisfied by the constant sequence $\left(u_n^0,u_n^1\right)_{n\in\mathbb{N}}=\left(u^0,u^1\right)_{n\in\mathbb{N}}$. Hence, there exist an increasing function $\phi:\mathbb{N}\to\mathbb{N}$ and a stationary solution w of (5.0.1), both depending on the sequence (T_n), such that for all $T>0$,

$$\sup_{t\in[-T,T]}\left(\left\|u(T_{\phi(n)}+t)-w\right\|_{H_0^1}+\left\|\partial_t u(T_{\phi(n)}+t)\right\|_{L^2}\right)\xrightarrow{n\to\infty}0.$$

In particular, this implies

$$J(w)=\lim_{n\to\infty}E\left(u(T_{\phi(n)}),\partial_t u(T_{\phi(n)})\right)=E_\infty.$$

Now, we assume that

$$\text{there is an at most countable number of stationary solution } w \text{ such that } J(w)=E_\infty. \tag{5.3.12}$$

We show that this implies that w is, in fact, independent of (T_n). Assume by contradiction that there exist two sequences (T_n) and (T_n') such that for all $T>0$, one has

$$\sup_{t\in[-T,T]}\left(\|u(T_n+t)-w\|_{H_0^1}+\|\partial_t u(T_n+t)\|_{L^2}\right)\xrightarrow{n\to\infty}0$$

and

$$\sup_{t\in[-T,T]}\left(\|u(T_n'+t)-w'\|_{H_0^1}+\|\partial_t u(T_n'+t)\|_{L^2}\right)\xrightarrow{n\to\infty}0$$

where w and w' are two distinct stationary solutions of (5.0.1). Consider $\varphi\in L^2(\Omega)$ such that

$$\int_\Omega w\varphi \mathrm{d}x<\int_\Omega w'\varphi \mathrm{d}x.$$

For $t\geq 0$, set $\alpha(t)=\int_\Omega u(t,x)\varphi(x)\mathrm{d}x$. The function α is real and continuous. One has

$$\lim_{n\to\infty}\alpha(T_n)=\int_\Omega w\varphi \mathrm{d}x<\int_\Omega w'\varphi \mathrm{d}x=\lim_{n\to\infty}\alpha(T_n')$$

so by the intermediate value theorem, for all $\ell\in[\int_\Omega w\varphi \mathrm{d}x,\int_\Omega w'\varphi \mathrm{d}x]$, there exists a sequence $(t_n)_n$ such that $t_n\to+\infty$ and

$$\alpha(t_n)\xrightarrow{n\to\infty}\ell.$$

Hence, for all such ℓ, there exists a stationary solution w'' such that $J(w'')=E_\infty$ and

$$\ell=\int_\Omega w''\varphi \mathrm{d}x.$$

This is a contradiction with (5.3.12).

Summarizing, we have proved that there exists a stationary solution w such that for all sequence $(T_n)_n$ such that $T_n\to+\infty$, there exists a subsequence $(T_{\phi(n)})_n$ such that for all $T>0$, one has

$$\sup_{t\in[-T,T]}\left(\left\|u(T_{\phi(n)}+t)-w\right\|_{H_0^1}+\left\|\partial_t u(T_{\phi(n)}+t)\right\|_{L^2}\right)\xrightarrow{n\to\infty}0.$$

A basic contradiction argument ends the proof of Theorem 5.0.5.

5.4 Stabilisation of global solutions below the ground state energy

5.4.1 Case of a positive damping

In the case of a positive damping, it is possible to make a short proof of the stabilization of solutions of (5.0.3) initiated in $\mathscr{K}^+$. That proof is based on the arguments of the proof of Proposition 2.5 of [JL13] and on Lemma 5.1.8.

Theorem 5.4.1. *Assume that $\gamma(x)\geq\alpha>0$ on Ω. Then for any $E_0\in[0,m_0)$, there exist $C>0$ and $\lambda>0$ such that for all $(u^0,u^1)\in\mathscr{K}^+$ such that $E(u^0,u^1)\in[0,E_0]$, if u is the solution of* (5.0.3) *with initial data (u^0,u^1), then*

$$\|u(t)\|_{H_0^1}+\|\partial_t u(t)\|_{L^2}\leq Ce^{-\lambda t},\quad t\geq 0.$$

Remark 5.4.2. Note that C and λ depend on E_0. The following result is false: there exist $C > 0$ and $\lambda > 0$ such that for $(u^0, u^1) \in \mathscr{K}^+$ and $t \geq 0$, one has

$$\|u(t)\|_{H_0^1} + \|\partial_t u(t)\|_{L^2} \leq Ce^{-\lambda t}. \tag{5.4.1}$$

Indeed, for $n \in \mathbb{N}^*$, consider $(u_n^0, u_n^1) = \left(\left(1 - \frac{1}{n}\right) Q, 0\right)$ and write u_n for the solution with initial data (u_n^0, u_n^1). Recall that if $j(\lambda) = J(\lambda Q)$, then one has $j'(\lambda) > 0$ for $\lambda \in (0, 1)$ and $j'(\lambda) < 0$ for $\lambda > 1$. In particular, this implies $K(u_n^0) > 0$ and $(u_n^0, u_n^1) \in \mathscr{K}^+$. Hence, if (5.4.1) is true, then there exists $T > 0$ such that for all $n \in \mathbb{N}^*$, one has

$$\|u_n(T)\|_{H_0^1} \leq \frac{1}{2} \|Q\|_{H_0^1} .$$

By continuity of the source-to-solution operator (see Lemma 5.2.4), one has

$$\|u_n(T) - Q\|_{H_0^1} \xrightarrow{n\to\infty} 0$$

and this is a contradiction.

Proof. Consider $E_0 \in [0, m_0)$, and $(u^0, u^1) \in \mathscr{K}^+$ such that $E(u^0, u^1) \leq E_0$. For $\varepsilon > 0$, we define

$$E_\varepsilon(t) = E(u(t), \partial_t u(t)) + \varepsilon \int_\Omega u(t, x)\partial_t u(t, x)\mathrm{d}x.$$

Recall that as $K(u(t)) \geq 0$, one has

$$E(u(t), \partial_t u(t)) \geq \frac{1}{4} \|u(t)\|_{H_0^1}^2 + \frac{1}{2} \|\partial_t u(t)\|_{L^2}^2 , \quad t \geq 0.$$

For $\varepsilon > 0$ sufficiently small, one finds

$$\varepsilon \left| \int_\Omega u(t, x)\partial_t u(t, x)\mathrm{d}x \right| \leq \frac{1}{2} E(u(t), \partial_t u(t))$$

implying

$$\frac{1}{2} E(u(t), \partial_t u(t)) \leq E_\varepsilon(t) \leq \frac{3}{2} E(u(t), \partial_t u(t)), \quad t \geq 0. \tag{5.4.2}$$

Using the energy equality, for $\varepsilon > 0$ and $t \geq 0$, one obtains

$$E'_\varepsilon(t) = \int_\Omega (\varepsilon - \gamma) |\partial_t u(t)|^2 \mathrm{d}x - \varepsilon \|u(t)\|_{H_0^1}^2 + \varepsilon \|u(t)\|_{L^4}^4 - \varepsilon \int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x.$$

As $\gamma \geq \alpha$, this gives

$$E'_\varepsilon(t) \leq (\varepsilon - \alpha) \|\partial_t u(t)\|_{L^2}^2 - \varepsilon \|u(t)\|_{H_0^1}^2 + \varepsilon \|u(t)\|_{L^4}^4 - \varepsilon \int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x.$$

Note that for $t \geq 0$, one has

$$J(u(t)) \leq E(u(t), \partial_t u(t)) \leq E\left(u^0, u^1\right) \leq E_0 < m_0$$

so by Lemma 5.1.8 *(i)*, there exists $C > 0$ such that $K(u(t)) \geq C \|u(t)\|_{H_0^1}^2$ for all $t \geq 0$. This gives

$$E'_\varepsilon(t) \leq (\varepsilon - \alpha) \|\partial_t u(t)\|_{L^2}^2 - \varepsilon C \|u(t)\|_{H_0^1}^2 - \varepsilon \int_\Omega \gamma u(t)\partial_t u(t)\mathrm{d}x.$$

Writing

$$\varepsilon \left| \int_\Omega u(t,x)\partial_t u(t,x)\mathrm{d}x \right| \lesssim \varepsilon^{\frac{3}{2}} \|u(t)\|^2_{H^1_0} + \varepsilon^{\frac{1}{2}} \|\partial_t u(t)\|^2_{L^2}$$

one finds

$$E'_\varepsilon(t) \le (\varepsilon - \alpha + C'\sqrt{\varepsilon}) \|\partial_t u(t)\|^2_{L^2} + \varepsilon \left(C'\sqrt{\varepsilon} - C\right) \|u(t)\|^2_{H^1_0}$$

for some $C' > 0$. Consider $T > 0$. For $\varepsilon > 0$ sufficiently small, one has

$$E_\varepsilon(T) - E_\varepsilon(0) \lesssim -\int_0^T \left(\|\partial_t u(t)\|^2_{L^2} + \|u(t)\|^2_{H^1_0} \right) \mathrm{d}t,$$

and using (5.4.2), one finds

$$E\left(u(T), \partial_t u(T)\right) - 3E\left(u(0), \partial_t u(0)\right) \lesssim -\int_0^T E\left(u(t), \partial_t u(t)\right) \mathrm{d}t.$$

As the energy is decreasing, this gives

$$E\left(u(T), \partial_t u(T)\right) - 3E\left(u(0), \partial_t u(0)\right) \lesssim -TE\left(u(T), \partial_t u(T)\right).$$

Choosing T sufficiently large, one obtains

$$E\left(u(T), \partial_t u(T)\right) \le \mu E\left(u(0), \partial_t u(0)\right)$$

for some $\mu \in (0,1)$. As $(u(t), \partial_t u(t)) \in \mathscr{K}^+$ for all $t \ge 0$, we can iterate this process to get

$$E\left(u(nT), \partial_t u(nT)\right) \le \mu^n E\left(u^0, u^1\right), \quad n \in \mathbb{N}.$$

This completes the proof. □

5.4.2 Case of a damping satisfying the GCC

Here, we prove Theorem 5.0.4, using Proposition 5.3.1. We spit the proof into two steps.

Step 1: observability estimate and application of Proposition 5.3.1. To prove Theorem 5.0.4, it suffices to show the following observability inequality: there exist $C > 0$ and $T > 0$ such that for all $(u^0, u^1) \in \mathscr{K}^+$ with $E(u^0, u^1) \le E_0$, the solution u of (5.0.3) with initial data (u^0, u^1) satisfies

$$E\left(u^0, u^1\right) \le C \int_0^T \int_\Omega \gamma(x) \left|\partial_t u(t,x)\right|^2 \mathrm{d}x\mathrm{d}t.$$

Indeed, if that observability inequality holds, then using the energy equality, one obtains

$$E\left(u(T), \partial_t u(T)\right) \le \left(1 - \frac{1}{C}\right) E\left(u^0, u^1\right).$$

As $(u(t), \partial_t u(t)) \in \mathscr{K}^+$ for all $t \ge 0$, we can iterate this process to get

$$E\left(u(nT), \partial_t u(nT)\right) \le \left(1 - \frac{1}{C}\right)^n E\left(u^0, u^1\right), \quad n \in \mathbb{N}.$$

This proves that $t \mapsto E\left(u(t), \partial_t u(t)\right)$ decays exponentially. As $K(u(t)) \ge 0$, one has

$$E\left(u(t), \partial_t u(t)\right) \ge \frac{1}{4} \|u(t)\|^2_{H^1_0} + \frac{1}{2} \|\partial_t u(t)\|^2_{L^2}$$

and this gives the conclusion.

We prove the observability inequality by contradiction. We assume that there exists a sequence $(T_n)_{n\geq 1}$, satisfying $T_n \to +\infty$, such that for all $n \in \mathbb{N}$, $n \geq 1$, there exists $(u_n^0, u_n^1) \in \mathscr{K}^+$ such that $E(u_n^0, u_n^1) \leq E_0$ and

$$\int_0^{T_n} \int_\Omega \gamma(x) \left|\partial_t u_n(t,x)\right|^2 \mathrm{d}x\mathrm{d}t < \frac{1}{n} E\left(u_n^0, u_n^1\right). \tag{5.4.3}$$

For $n \geq 1$ and $t \geq 0$, as $K(u_n(t)) \geq 0$, one has

$$\|u_n(t)\|_{H_0^1}^2 + \|\partial_t u_n(t)\|_{L^2}^2 \lesssim E\left(u_n(t), \partial_t u_n(t)\right) \leq E\left(u_n^0, u_n^1\right) \leq E_0.$$

Write $T_n' = \frac{T_n}{2}$. For all $T > 0$, if $n \in \mathbb{N}$ is sufficiently large, then

$$\int_{T_n'-T}^{T_n'+T} \int_\Omega \gamma \left|\partial_t u_n\right|^2 \mathrm{d}x\mathrm{d}t \leq \int_0^{T_n} \int_\Omega \gamma \left|\partial_t u_n\right|^2 \mathrm{d}x\mathrm{d}t.$$

This gives

$$\int_{T_n'-T}^{T_n'+T} \int_\Omega \gamma \left|\partial_t u_n\right|^2 \mathrm{d}x\mathrm{d}t \xrightarrow{n\to\infty} 0, \quad T > 0.$$

Hence, we can apply Proposition 5.3.1: there exist an increasing function $\phi : \mathbb{N} \to \mathbb{N}$ and a stationary solution w of (5.0.1) such that

$$\sup_{t\in[-T,T]} \left(\left\| u_{\phi(n)}(T'_{\phi(n)} + t) - w \right\|_{H_0^1} + \left\| \partial_t u_{\phi(n)}(T'_{\phi(n)} + t) \right\|_{L^2} \right) \xrightarrow{n\to\infty} 0, \quad T > 0.$$

As the energy is nondecreasing, one has

$$J(w) = \lim_{n\to\infty} E\left(u_{\phi(n)}(T'_{\phi(n)}), \partial_t u_{\phi(n)}(T'_{\phi(n)})\right) \leq E_0 < m_0.$$

Hence, w is a stationary solution satisfying $J(w) \in [0, m_0)$: this gives $w = 0$. Using the energy equality, one has

$$E\left(u_n(T_n'), \partial_t u_n(T_n')\right) \geq E\left(u_n(T_n), \partial_t u_n(T_n)\right) = E\left(u_n^0, u_n^1\right) - \int_0^{T_n} \int_\Omega \gamma(x) \left|\partial_t u_n(t,x)\right|^2 \mathrm{d}x\mathrm{d}t$$

so that (5.4.3) gives

$$E\left(u_n(T_n'), \partial_t u_n(T_n')\right) \geq \left(1 - \frac{1}{n}\right) E\left(u_n^0, u_n^1\right).$$

In particular, one obtains

$$E\left(u_{\phi(n)}^0, u_{\phi(n)}^1\right) \xrightarrow{n\to\infty} 0.$$

For $n \in \mathbb{N}^*$, set $\alpha_n = \sqrt{E\left(u_n^0, u_n^1\right)}$. In the next step, we will consider the equation scaled by α_n to get a contradiction.

Step 2: the scaled equation. For $n \in \mathbb{N}$, set $w_n = \frac{u_n}{\alpha_n}$. For $n \in \mathbb{N}^*$, w_n is the solution of

$$\Box w_n + \beta w_n = \alpha_n^2 w_n^3$$

and one has

$$\|w_n(T_n)\|_{H_0^1}^2 + \|\partial_t w_n(T_n)\|_{L^2}^2 = \frac{1}{\alpha_n^2} \left(\|u_n(T_n)\|_{H_0^1}^2 + \|\partial_t u_n(T_n)\|_{L^2}^2 \right) \gtrsim \frac{E\left(u_n(T_n), \partial_t u_n(T_n)\right)}{\alpha_n^2}.$$

Using the energy equality and (5.4.3), one finds

$$\|w_n(T_n)\|_{H_0^1}^2 + \|\partial_t w_n(T_n)\|_{L^2}^2 \gtrsim \frac{1}{\alpha_n^2}\left(E\left(u_n^0, u_n^1\right) - \frac{1}{n}E\left(u_n^0, u_n^1\right)\right) = 1 - \frac{1}{n} \geq \frac{1}{2}, \quad n \geq 2. \tag{5.4.4}$$

Recall that A denotes the infinitesimal generator of the linear part of (5.0.3), that e^{tA} is the associated semi-group, and set

$$W_n = (w_n, \partial_t w_n) \quad \text{and} \quad F_n = \left(0, \alpha_n^2 w_n^3\right).$$

Using the Duhamel formula, we can write

$$\begin{aligned} W_n(T_n) &= e^{T_n A} W_n(0) + \int_0^{T_n} e^{(T_n - s)A} F_n(s)\mathrm{d}s \\ &= e^{T_n A} W_n(0) + \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{(T_n - k)A} \int_0^1 e^{-sA} F_n(k+s)\mathrm{d}s + \int_{\lfloor T_n \rfloor}^{T_n} e^{(T_n - s)A} F_n(s)\mathrm{d}s. \end{aligned} \tag{5.4.5}$$

As $K(u_n(0)) \geq 0$, one has

$$\|w_n(0)\|_{H_0^1}^2 + \|\partial_t w_n(0)\|_{L^2}^2 = \frac{1}{\alpha_n^2}\left(\|u_n(0)\|_{H_0^1}^2 + \|\partial_t u_n(0)\|_{L^2}^2\right) \lesssim \frac{E\left(u_n(0), \partial_t u_n(0)\right)}{\alpha_n^2} = 1.$$

Hence, the sequence $(W_n(0))$ is bounded, and linear stabilisation (Theorem 5.1.5) gives

$$e^{T_n A} W_n(0) \xrightarrow{n \to \infty} 0$$

in $H_0^1(\Omega) \times L^2(\Omega)$. Next, write

$$\left\|\int_0^1 e^{-sA} F_n(k+s)\mathrm{d}s\right\|_{H_0^1 \times L^2} \lesssim \|F_n(k+\cdot)\|_{L^1((0,1), H_0^1 \times L^2)} = \left\|\alpha_n^2 w_n^3\right\|_{L^1((k,k+1),L^2)}.$$

A Sobolev embedding gives

$$\left\|\int_0^1 e^{-sA} F_n(k+s)\mathrm{d}s\right\|_{H_0^1 \times L^2} \lesssim \frac{1}{\alpha_n}\|u_n\|_{L^3((k,k+1),L^6)}^3 \lesssim \frac{1}{\alpha_n}\|(u_n, \partial_t u_n)\|_{L^\infty((k,k+1),H_0^1 \times L^2)}^3.$$

Using $K(u_n) \geq 0$ and the fact that the energy is nonincreasing, one obtains

$$\left\|\int_0^1 e^{-sA} F_n(k+s)\mathrm{d}s\right\|_{H_0^1 \times L^2} \lesssim \frac{E\left(u_n^0, u_n^1\right)^{\frac{3}{2}}}{\alpha_n} = \sqrt{\alpha_n}.$$

Hence linear stabilisation gives

$$\left\|e^{(T_n - k)A}\int_0^1 e^{-sA} F_n(k+s)\mathrm{d}s\right\|_{H_0^1 \times L^2} \lesssim e^{\lambda(k - T_n)}\sqrt{\alpha_n}$$

where $\lambda > 0$ is a constant, implying

$$\left\|\sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{(T_n - k)A}\int_0^1 e^{-sA} F_n(k+s)\mathrm{d}s\right\|_{H_0^1 \times L^2} \lesssim \sum_{k=0}^{\lfloor T_n \rfloor - 1} e^{\lambda(k - T_n)}\sqrt{\alpha_n} \lesssim \sqrt{\alpha_n}.$$

Similarly, one has

$$\left\|\int_{\lfloor T_n \rfloor}^{T_n} e^{(T_n - s)A} F_n(s)\mathrm{d}s\right\|_{H_0^1 \times L^2} \lesssim \sqrt{\alpha_n}.$$

As $\alpha_{\phi(n)} \to 0$, coming back to (5.4.5), one finds

$$W_{\phi(n)}(T_{\phi(n)}) \xrightarrow{n \to \infty} 0$$

in $H_0^1(\Omega) \times L^2(\Omega)$, a contradiction with (5.4.4). This completes the proof of Theorem 5.0.4.

Chapter 6

Local controllability around a regular solution and null-controllability of scattering solutions for semilinear wave equations

This chapter is based on the article [Per23b], which has been prepublished and will be submitted in a journal shortly.

Contents

Introduction

Setting and main results. Let $2 \leq d \leq 5$. Let Ω be the interior of a smooth d-dimensional Riemannian manifold, with or without boundary, which is either a compact Riemannian manifold, or a compact perturbation of $\mathbb{R}^d$, that is, the complement in $\mathbb{R}^d$ of a smooth bounded (possibly empty) open set, with a metric equal to the Euclidean one outside a ball. In short, we write $\partial\Omega = \emptyset$ if Ω is either $\mathbb{R}^d$ or a compact Riemannian manifold without boundary, and $\partial\Omega \neq \emptyset$ if Ω is either a compact perturbation of $\mathbb{R}^d$ (with $\Omega \neq \mathbb{R}^d$) or a compact Riemannian manifold with nonempty boundary. If $\partial\Omega \neq \emptyset$, then we denote by $\partial\Omega$ the boundary of Ω, and we write $\overline{\Omega} = \Omega \cup \partial\Omega$, and if $\partial\Omega = \emptyset$, then we write $\overline{\Omega} = \Omega$. In addition, we say that Ω is unbounded if Ω is a compact perturbation of $\mathbb{R}^d$ (or if $\Omega = \mathbb{R}^d$).

Write $H_0^1(\Omega)$ for the closure of $\mathscr{C}_c^\infty(\Omega)$ in $H^1(\Omega)$. Let $\beta \in \mathbb{R}$ be such that the Poincaré inequality

$$\int_\Omega \left(|\nabla u|^2 + \beta|u|^2\right) \mathrm{d}x \gtrsim \int_\Omega |u|^2 \mathrm{d}x, \quad u \in H_0^1(\Omega),$$

is satisfied. This specifically requires $\beta > 0$ if $\partial\Omega = \emptyset$ or if Ω is unbounded. For $u \in H_0^1(\Omega)$, we write $\|u\|_{H_0^1(\Omega)}^2 = \int_\Omega \left(|\nabla u|^2 + \beta|u|^2\right) \mathrm{d}x$.

This chapter contains a local controllability and a null-controllability result. We consider a power-like nonlinearity $f \in \mathscr{C}^2(\mathbb{R},\mathbb{R})$ satisfying $f(0) = f'(0) = 0$, and the following assumptions. For the local controllability result, we assume that there exist $C_0 > 0$ and α such that

$$\begin{gathered} |f''(s)| \leq C_0 \left(1 + |s|\right)^{\alpha-2} \text{ for all } s \in \mathbb{R}, \\ 2 \leq \alpha < \tfrac{d+2}{d-2}, \quad \text{ and } \quad (\alpha = 2 \text{ if } d = 5 \text{ and } \partial\Omega \neq \emptyset)\,. \end{gathered} \tag{6.0.1}$$

For the null-controllability result, we assume that Ω is unbounded, and that there exist $C_0 > 0$ and $\alpha_0 \leq \alpha_1$ such that

$$\begin{gathered} |f''(s)| \leq C_0 \left(|s|^{\alpha_0-2} + |s|^{\alpha_1-2}\right) \text{ for all } s \in \mathbb{R}, \\ 2 < \alpha_0 \leq \alpha_1 < \tfrac{d+2}{d-2}, \quad \text{ and } \quad (d \neq 5 \text{ if } \partial\Omega \neq \emptyset)\,. \end{gathered} \tag{6.0.2}$$

Note that α, α_0 and α_1 can be arbitrarily large if $d = 2$, and that (6.0.2) implies $f''(0) = 0$. Note also that no assumption is made on the sign of f. A typical example of such a nonlinearity f is given by

$$f(s) = \sum_{j=0}^n \lambda_j s|s|^{\alpha_j-1}, \quad s \in \mathbb{R},$$

with $n \in \mathbb{N}$, $\lambda_0, \cdots, \lambda_n \in \mathbb{R}$, and $2 \leq \alpha_0 \leq \cdots \leq \alpha_n$ such that (6.0.1) or (6.0.2) holds.

Remark 6.0.1. If f satisfies (6.0.2) for some $\alpha_0 \leq \alpha_1$, then it also satisfies (6.0.1) for $\alpha = \alpha_1$. Hence, any result stated with f satisfying (6.0.2) can be applied with f satisfying (6.0.1).

For a nonlinearity f satisfying (6.0.1), we consider the semilinear wave (or Klein-Gordon) equation

$$\left\{\begin{array}{rcll} \Box u + \beta u &=& f(u) & \text{in } \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) &=& (u^0, u^1) & \text{in } \Omega, \\ u &=& 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{array}\right. \tag{6.0.3}$$

with real-valued initial data $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$. If $\partial\Omega = \emptyset$, then the Dirichlet boundary condition can be removed. The local Cauchy theory for (6.0.3) is well-known. We say that $u \in H_0^1(\Omega)$ is a stationary solution of (6.0.3) if u is the solution of (6.0.3) with initial data $(u, 0)$ and if u is time-independent. If $sf(s) \leq 0$ for $s \in \mathbb{R}$, then (6.0.3) is said to be defocusing. In this case, solutions of (6.0.3) are globally defined, and the only stationary solution is 0. If $sf(s) \geq 0$ for $s \in \mathbb{R}$, then (6.0.3) is said to be focusing, and blow-up solutions and non-zero stationary solutions may exist (see, for example, [PS75]).

Consider $T > 0$, $a \in \mathscr{C}^\infty(\Omega, \mathbb{R})$ and $\left(\mathbf{u}^0, \mathbf{u}^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ such that the solution $\mathbf{u}$ of (6.0.3) with initial data $\left(\mathbf{u}^0, \mathbf{u}^1\right)$ exists on the interval $[0, T]$.

Definition 6.0.2 (Local controllability around $\mathbf{u}$ at time T)**.** We say that local controllability around $\mathbf{u}$ at time T holds if there exists $\delta > 0$ such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$\left\| \left(u^0, u^1\right) - \left(\mathbf{u}^0, \mathbf{u}^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \delta,$$

there exists $g \in L^1((0,T), L^2(\Omega))$ such that the solution u of

$$\begin{cases} \Box u + \beta u &= f(u) + ag & \text{in } (0,T)\times\Omega, \\ (u(T), \partial_t u(T)) &= (\mathbf{u}(T), \partial_t \mathbf{u}(T)) & \text{in } \Omega, \\ u &= 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

satisfies $(u(0), \partial_t u(0)) = (u^0, u^1)$.

We will use the notion of generalized bicharacteristic, for which we refer to [MS78].

Definition 6.0.3. For $\omega \subset \Omega$ open, we say that (ω, T) satisfies the *Geometric Control Condition* (in short, GCC) if for every generalized bicharacteristic $s \mapsto (t(s), x(s), \tau(s), \xi(s))$, there exists $s \in \mathbb{R}$ such that $t(s) \in (0,T)$ and $x(s) \in \omega$.

Recall that a smooth function $F : (0,T)\times\Omega \to \mathbb{R}$ is said to be *analytic with respect to t* if for all $(t_0, x_0) \in (0,T)\times\Omega$, there exists a neighbourhood $\mathcal{O} \subset (0,T)\times\Omega$ of (t_0,x_0) such that

$$F(t,x) = \sum_{k=0}^{\infty} \partial_t^k F(t_0, x) \frac{(t-t_0)^k}{k!}, \quad (t,x) \in \mathcal{O}.$$

The first result of this chapter is the following.

Theorem 6.0.4 (Local controllability around a trajectory)**.** *Assume that f satisfies* (6.0.1), *and consider $(\mathbf{u}^0, \mathbf{u}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that the solution $\mathbf{u}$ of* (6.0.3) *with initial data $(\mathbf{u}^0, \mathbf{u}^1)$ exists on the interval $[0,T]$. We make the following assumptions.*

(i) Assume that there exist $\omega \subset \Omega$ open and $c > 0$ such that $a \geq c$ on ω and such that (ω, T) satisfies the GCC. In addition, if Ω is unbounded, assume that there exists $R_0 > 0$ such that $\mathbb{R}^d \backslash B(0, R_0) \subset \omega$.

(ii) Assume that $f'(\mathbf{u}) \in L^\infty((0,T)\times\Omega)$, and that $f'(\mathbf{u})$ is smooth, and analytic with respect to t. In addition, if Ω is unbounded, assume that for all $t \in [0,T]$,

$$|\nabla f'(\mathbf{u}(t,x))| + |f'(\mathbf{u}(t,x))| \xrightarrow{|x|\to\infty} 0,$$

where ∇ is the gradient with respect to the space variable x.

Then, local controllability around $\mathbf{u}$ at time T holds.

In particular, if there exists a sequence $(u_n^0, u_n^1) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\left\| \left(u_n^0, u_n^1\right) - \left(\mathbf{u}^0, \mathbf{u}^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} \xrightarrow{n\to\infty} 0,$$

and such that for all $n \in \mathbb{N}$, the solution of (6.0.3) with initial data $(u_n^0, u_n^1) \in H_0^1(\Omega) \times L^2(\Omega)$ blows up in finite time, then Theorem 6.0.4 contains a controllability result for some blow-up solutions. An example of a solution $\mathbf{u}$ satisfying this condition is given below.

The second result of this chapter concerns the null-controllability in a long time of scattering solutions, in the case of an unbounded domain satisfying the non-trapping condition.

Definition 6.0.5 (Null-controllability in a long time)**.** We say that null-controllability in a long time for $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ holds if there exist $T > 0$ and $g \in L^1((0,T), L^2(\Omega))$ such that the solution u of

$$\begin{cases} \Box u + \beta u &= f(u) + ag \quad \text{in } (0,T) \times \Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } (0,T) \times \partial\Omega, \end{cases}$$

satisfies $(u(T), \partial_t u(T)) = 0$.

A domain is said to be non-trapping if all generalized geodesics leave any compact set in finite time (see for example [Mel79] and [MRS77] for a precise definition). When this condition is satisfied, resolvent estimates can be proven (see [Bur03], Remark 2.6, and references therein). For simplicity, we adopt these resolvent estimates as our definition of the non-trapping condition.

Definition 6.0.6. Assume that Ω is unbounded. We say that Ω is non-trapping if for all $\chi \in \mathscr{C}_c^\infty(\Omega)$, there exists $C > 0$ such that

$$\sqrt{1 + |\lambda|} \left\| \chi \left(-\Delta + \lambda\right)^{-1} \chi u \right\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)}, \quad u \in L^2(\Omega), \quad \operatorname{Im} \lambda \neq 0.$$

We recall the definition of a scattering solution.

Definition 6.0.7. Consider $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We say the solution u_{NL} of

$$\begin{cases} \Box u_{\mathrm{NL}} + \beta u_{\mathrm{NL}} &= f(u_{\mathrm{NL}}) \quad \text{in } \mathbb{R}_+ \times \Omega, \\ (u_{\mathrm{NL}}(0), \partial_t u_{\mathrm{NL}}(0)) &= (u^0, u^1) \quad \text{in } \Omega, \\ u_{\mathrm{NL}} &= 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{6.0.4}$$

is scattering if u_{NL} exists on the whole interval $\mathbb{R}_+$ and satisfies

$$\|(u_{\mathrm{NL}}(t), \partial_t u_{\mathrm{NL}}(t)) - (u_{\mathrm{L}}(t), \partial_t u_{\mathrm{L}}(t))\|_{H_0^1(\Omega) \times L^2(\Omega)} \xrightarrow{t \to +\infty} 0$$

for some solution u_{L} of the linear equation

$$\begin{cases} \Box u_{\mathrm{L}} + \beta u_{\mathrm{L}} &= 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u_{\mathrm{L}} &= 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

The second result of this chapter is the following.

Theorem 6.0.8 (Null-controllability of scattering solutions)**.** *Assume that Ω is a non-trapping unbounded domain, and that f satisfies* (6.0.2)*. Consider $\omega \subset \Omega$ open such that there exist $R_0 > 0$ and $T > 0$ such that (ω, T) satisfies the GCC, and $\mathbb{R}^d \backslash B(0, R_0) \subset \omega$. Assume that there exists $c > 0$ such that $a \geq c$ on ω. For $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, if the solution u_{NL} of* (6.0.4) *is scattering, then null-controllability in a long time for (u^0, u^1) holds.*

The proof is based on a local energy decay result (Theorem 6.3.1), and global-in-time Strichartz estimates (Theorem 6.1.3), both of which can be of their own interest.

We refer to Section 2.4 for a discussion about the results of this chapter and their connection to the existing literature.

Consequences. An immediate corollary of Theorem 6.0.4 is the following. For shortness, once a function a is fixed, we refer to the solution u of

$$\begin{cases} \Box u + \beta u &= f(u) + ag \quad \text{in } (0,+\infty)\times\Omega, \\ (u(0),\partial_t u(0)) &= (u^0,u^1) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \end{cases} \tag{6.0.5}$$

as the solution of (6.0.5) with initial data (u^0,u^1) and with control g.

Corollary 6.0.9. *Consider f satisfying* (6.0.1), $\mathbf{u}\in H_0^1(\Omega)$ *a stationary solution of* (6.0.3), *and $a\in\mathscr{C}^\infty(\Omega,\mathbb{R})$. Assume that assumption (i) of Theorem 6.0.4 is fulfilled for some $T>0$, that assumption (ii) is fulfilled (for all $T>0$), and that the two following conditions are satisfied.*

(i) For $n\in\mathbb{N}$, there exist $(v_n^0,v_n^1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $g_n\in L^1_{\text{loc}}((0,+\infty),L^2(\Omega))$ satisfying

$$\left\|\left(v_n^0,v_n^1\right)-(\mathbf{u},0)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\xrightarrow{n\to\infty}0,$$

and such that the solution v_n of (6.0.5) *with initial data (v_n^0,v_n^1) and with control g_n exists on $(0,+\infty)$, and satisfies*

$$\|(v_n(t),\partial_t v_n(t))\|_{H_0^1(\Omega)\times L^2(\Omega)}\xrightarrow{t\to\infty}0,\quad n\in\mathbb{N}.$$

(ii) For $n\in\mathbb{N}$, there exist $(w_n^0,w_n^1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $\tilde g_n\in L^1_{\text{loc}}((0,+\infty),L^2(\Omega))$ satisfying

$$\left\|\left(w_n^0,w_n^1\right)-(\mathbf{u},0)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}\xrightarrow{n\to\infty}0,$$

and such that the solution w_n of (6.0.5) *with initial data (w_n^0,w_n^1) and with control $\tilde g_n$ blows up in finite time.*

Then, there exist a neighbourhood $\mathcal{O}_0$ of 0 in $H_0^1(\Omega)\times L^2(\Omega)$ and a neighbourhood $\mathcal{O}_1$ of $(\mathbf{u},0)$ in $H_0^1(\Omega)\times L^2(\Omega)$ satisfying the following properties.

(i) For $j=0,1$, $(u^0,u^1)\in\mathcal{O}_j$ and $(\tilde u^0,\tilde u^1)\in\mathcal{O}_{1-j}$, there exist a time $T'>0$ and a control $g\in L^1((0,T'),L^2(\Omega))$ such that the solution u of (6.0.5) *with initial data (u^0,u^1) and with control g satisfies*

$$(u(T'),\partial_t u(T'))=\left(\tilde u^0,\tilde u^1\right).$$

(ii) For $j=0,1$ and $(u^0,u^1)\in\mathcal{O}_j$, there exists a control $g\in L^1_{\text{loc}}((0,+\infty),L^2(\Omega))$ such that the solution u of (6.0.5) *with initial data (u^0,u^1) and with control g blows up in finite time.*

(iii) If n is sufficiently large, then for $j=0,1$, and $(u^0,u^1)\in\mathcal{O}_j$, there exist $T'=T'(n)>0$ and $g=g(n)\in L^1((0,T'),L^2(\Omega))$ such that the solution u_n of (6.0.5) *with initial data (w_n^0,w_n^1) and with control g satisfies*

$$(u_n(T'),\partial_t u_n(T'))=\left(u^0,u^1\right).$$

Note that *(iii)* contains a null-controllability property for some blow-up solutions. The proof of Corollary 6.0.9 is straightforward, using Theorem 6.0.4 multiple times and the time-reversibility of (6.0.3).

We give two examples of applications of Corollary 6.0.9. If f is a focusing nonlinearity satisfying (6.0.1) or (6.0.2), then one can prove that there exists a special stationary solution Q of (6.0.3), called the ground state (see for example [IMN11], [PS75]). The ground state is smooth, and decays at infinity if Ω is unbounded.

First, we consider the case $\Omega = \mathbb{R}^d$, $\beta = 1$, with a nonlinearity f satisfying the H^1-subcritical case of [IMN11], and satisfying (6.0.2). An explicit example is $f(s) = s^3$, with $d = 3$. It is shown in [IMN11] that the set of initial data with energy strictly below the energy of the ground state can be partitioned into two disjoint non-empty sets, $\mathcal{K}^+$ and $\mathcal{K}^-$, such that a solution initiated in $\mathcal{K}^+$ is globally defined and is scattering, while a solution initiated in $\mathcal{K}^-$ blows up in finite time. One can check that $((1 \pm \varepsilon)Q, 0) \in \mathcal{K}^\mp$ if $\varepsilon > 0$ is sufficiently small. Hence, assumption *(ii)* of Corollary 6.0.9 is satisfied, for $(w_n^0, w_n^1) = \left((1 + \frac{1}{n})Q, 0\right)$ and $\tilde{g}_n = 0$. By Theorem 6.0.8, for n sufficiently large, there exists a control g_n such that the solution v_n of (6.0.5), with initial data $\left((1 - \frac{1}{n})Q, 0\right)$ and with control g_n, is equal to 0 for t sufficiently large. This implies that assumption *(ii)* of Corollary 6.0.9 is satisfied. Hence, Corollary 6.0.9 can be applied in that case, with $(\mathbf{u}^0, \mathbf{u}^1) = (Q, 0)$. Another way to see this is to use the existence of a heteroclinic solution in the spirit of [DM08], that is, a solution W which is scattering (for positive time), and satisfies

$$\|(W(t), \partial_t W(t)) - (Q, 0)\|_{H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)} \xrightarrow{t\to-\infty} 0.$$

The existence of such a solution W is proved in [NS11a] and [NS12], in the case $f(s) = s^3$, $d = 3$.

Secondly, we consider the case of a bounded domain Ω, with $d = 3$ and $f(s) = s^3$. In this case, the sets $\mathcal{K}^+$ and $\mathcal{K}^-$ are defined by

$$\begin{cases} \mathscr{K}^+ = \left\{(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega), E\left(u^0, u^1\right) < E\left(Q, 0\right), \|u^0\|^2_{H_0^1(\Omega)} \geq \|u^0\|^4_{L^4(\Omega)}\right\}, \\ \mathscr{K}^- = \left\{(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega), E\left(u^0, u^1\right) < E\left(Q, 0\right), \|u^0\|^2_{H_0^1(\Omega)} < \|u^0\|^4_{L^4(\Omega)}\right\}, \end{cases}$$

where the energy is given by $E\left(u^0, u^1\right) = \frac{1}{2}\|u^0\|^2_{H_0^1} - \frac{1}{4}\|u^0\|^4_{L^4} + \frac{1}{2}\|u^1\|^2_{L^2}$. A solution initiated in $\mathcal{K}^+$ is globally defined, and a solution initiated in $\mathcal{K}^-$ blows up in finite time (see [PS75]). In [Per23c], a stabilisation property under the GCC is shown for solutions initiated in $\mathcal{K}^+$. As above, using the fact $((1 \pm \varepsilon)Q, 0) \in \mathcal{K}^\mp$ if ε is sufficiently small, one concludes that the assumptions of Corollary 6.0.9 are satisfied.

We give a second immediate corollary of Theorem 6.0.4.

Corollary 6.0.10. *Consider f satisfying* (6.0.1)*, and $a \in \mathscr{C}^\infty(\Omega, \mathbb{R})$ such that assumption (i) of Theorem 6.0.4 is fulfilled for some $T > 0$. Consider $\mathcal{O} \subset H_0^1(\Omega) \times L^2(\Omega)$ satisfying the following condition : for all $(u^0, u^1) \in \mathcal{O}$, there exists $g \in L^1_{\text{loc}}((0, +\infty), L^2(\Omega))$ such that the solution u of* (6.0.5) *with initial data (u^0, u^1) and with control g exists on $(0, +\infty)$, and satisfies*

$$\|(u(t), \partial_t u(t))\|_{H_0^1(\Omega)\times L^2(\Omega)} \xrightarrow{t\to\infty} 0.$$

Then, for all $(u^0, u^1), (\tilde{u}^0, \tilde{u}^1) \in \mathcal{O}$, there exist $T' > 0$ and $g \in L^1((0, T'), L^2(\Omega))$ such that the solution u of (6.0.5) *with initial data (u^0, u^1) and with control g satisfies*

$$(u(T'), \partial_t u(T')) = \left(\tilde{u}^0, \tilde{u}^1\right).$$

In short, if $\mathcal{O} \subset H_0^1(\Omega) \times L^2(\Omega)$ satisfies the conclusion of Corollary 6.0.10, we say that exact controllability in $\mathcal{O}$ in a long time holds. If $\mathcal{O} = H_0^1(\Omega) \times L^2(\Omega)$, we simply say that exact controllability in a long time holds. We give three examples of applications of Corollary 6.0.10.

First, as above, consider the case of a bounded domain Ω, with $d = 3$ and $f(s) = s^3$. The stabilisation result of [Per23c], together with Corollary 6.0.10, implies that exact controllability in $\mathcal{K}^+$ in a long time holds.

Secondly, consider the case of an unbounded domain with a defocusing nonlinearity, such that all (finite-energy) solutions are scattering (see [Bre84], [GV89], [Nak01]). Assuming, in addition, that Ω is nontrapping, that f satisfies (6.0.2), and using Theorem 6.0.8, one concludes that the assumption of Corollary 6.0.10 is satisfied for $\mathcal{O} = H_0^1(\Omega) \times L^2(\Omega)$. Hence, in this case, exact controllability in a long time holds.

Thirdly, consider the case of a domain Ω and a defocusing nonlinearity f such that a stabilisation property holds, as in [AIN10], [DLZ03], [JL13] for example. Then Corollary 6.0.10 implies that exact controllability in a long time holds.

Outline of the chapter. In Section 1, we recall local-in-time Strichartz estimates, we prove some basic inequalities which follow from (6.0.1) and (6.0.2), and we construct the solutions of (6.0.3) and of some time-dependent equation. In Section 2, we prove Theorem 6.0.4, relying on an exact controllability result for a linear wave equation with partially analytic coefficients. In Section 3, we establish the local decay of the energy and the global-in-time Strichartz estimates, and we show that they imply Theorem 6.0.8.

6.1 Preliminaries

6.1.1 Strichartz estimates

Definition 6.1.1 (Local admissible exponents)**.** Consider $p, q \in \mathbb{R}$.

- Assume that $d \geq 3$ and $\partial\Omega = \emptyset$. Then, we say that (p, q) is a pair of local admissible exponents for Ω, and we write $(p, q) \in \Lambda_\Omega$, if $1 \leq p \leq \infty$, $2 \leq q \leq \frac{2d}{d-3}$, $(q, d) \neq (+\infty, 3)$, and $\frac{1}{p} + \frac{d}{q} \geq \frac{d}{2} - 1$.
- Assume that $d \geq 3$ and $\partial\Omega \neq \emptyset$. Then, we say that (p, q) is a pair of local admissible exponents for Ω, and we write $(p, q) \in \Lambda_\Omega$, if $1 \leq p \leq \infty$,
$$\frac{1}{p} + \frac{d}{q} \geq \frac{d}{2} - 1 \quad \text{and} \quad \begin{cases} 2 \leq q \leq 14 & \text{if } d = 3 \\ 2 \leq q \leq \frac{2(d-1)}{d-3} & \text{if } d \geq 4 \end{cases}.$$
- If $d = 2$, then we say that (p, q) is a pair of local admissible exponents for Ω, and we write $(p, q) \in \Lambda_\Omega$, if $1 \leq p \leq \infty$ and $1 \leq q < \infty$.

A picture of the local admissible exponents can be found in Figure 6.1.

Theorem 6.1.2 (Local-in-time Strichartz estimates)**.** *Consider $T > 0$. There exists $C > 0$ such that for all $(p, q) \in \Lambda_\Omega$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $g \in L^1([0, T], L^2(\Omega))$, the unique solution u of*
$$\begin{cases} \Box u + \beta u = g & \text{in } \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) = (u^0, u^1) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{cases}$$
satisfies
$$\|u\|_{L^p([0,T],L^q)} \leq C \left(\left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|g\|_{L^1([0,T],L^2)} \right). \tag{6.1.1}$$

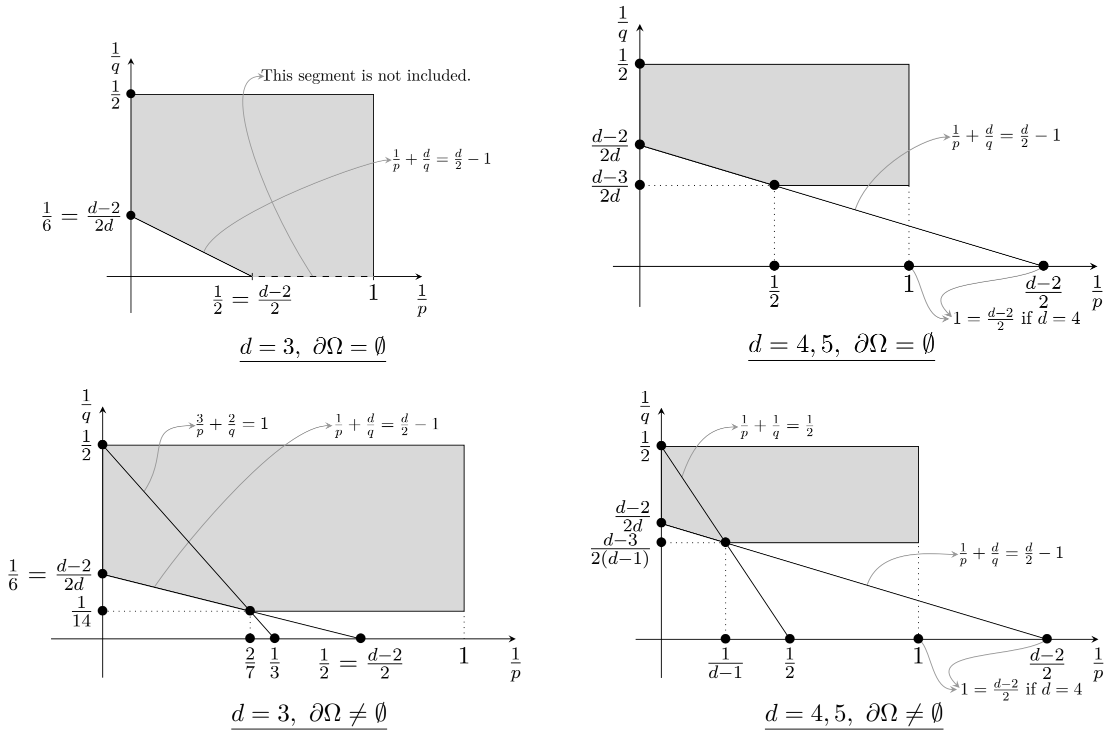


Figure 6.1: The local admissible exponents Λ_Ω, in gray.

Proof. If $d=2$, then the Sobolev embedding $H^1(\Omega)\hookrightarrow L^q(\Omega)$ holds true for $1\le q<+\infty$ (see for example [AF03], 4.12 Part I Case B, with $n=p=2$ and $m=1$). Hence, in that case, one has

$$\|u\|_{L^p([0,T],L^q)}\lesssim\|u\|_{L^\infty([0,T],H_0^1)}\lesssim\left\|\left(u^0,u^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}+\|g\|_{L^1([0,T],L^2)}$$

for all $(p,q)\in\Lambda_\Omega$.

Now, we assume that $d\ge 3$. Note that an estimate for the wave equation can be used for the Klein-Gordon equation, as one can absorb the low-order term for T sufficiently small, and iterate to get the result for large T.

Case 1 : $\Omega=\mathbb{R}^d$. Here, we rely on [KT98] (which is a generalisation of [Kap90], [MSS93] and [LS95]). See [Tat02] for the case of non-smooth coefficients. The original result in $\mathbb{R}^3$ was proved by Strichartz [Str77]. By Corollary 1.3 of [KT98] with $(\tilde q,\tilde r,\gamma)=(+\infty,2,1)$, (6.1.1) holds true if $2\le p\le\infty$, $2\le q<\infty$, $(p,q,d)\neq(2,\infty,3)$,

$$\frac{1}{p}+\frac{d}{q}=\frac{d}{2}-1\quad\text{and}\quad\frac{2}{p}+\frac{d-1}{q}\le\frac{d-1}{2}.$$

Note that if $\frac{1}{p}+\frac{d}{q}=\frac{d}{2}-1$, then $\frac{2}{p}+\frac{d-1}{q}\le\frac{d-1}{2}$ is equivalent with $\frac{1}{p}\le\frac{d-1}{d+1}$, a condition which is weaker than $p\ge 2$. Hence, (6.1.1) holds true if $2\le p\le\infty$, $2\le q<\infty$, $(p,q,d)\neq(2,\infty,3)$, and $\frac{1}{p}+\frac{d}{q}=\frac{d}{2}-1$. It is well-known that

$$\|u\|_{L^\infty([0,T],H^1)}\le C\left(\left\|\left(u^0,u^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}+\|g\|_{L^1([0,T],L^2)}\right).$$

Using also the Sobolev embedding $H^1(\Omega)\hookrightarrow L^{\frac{2d}{d-2}}(\Omega)$, one finds (6.1.1) with $(p,q)=(+\infty,2)$ and $(p,q)=\left(+\infty,\frac{2d}{d-2}\right)$. By interpolation, (6.1.1) holds true with $(p,q)=(+\infty,q)$, for all

$q \in \left[2, \frac{2d}{d-2}\right]$. Finally, as the estimate is local in time, if (6.1.1) holds true for some (p,q), then it also holds true for (p_0, q), for all $p_0 \in [1,p]$.

Case 2 : Ω is a compact manifold with boundary. In that case, we refer to the results of [BSS09] (which extend those of [BLP08]). Note that Ivanovici's counterexamples in [Iva12] show that Strichartz estimates are not true for the full range of exponents in the case of a manifold with boundary. By Corollary 1.2 of [BSS09], applied with $(r,s,\gamma) = (1,2,1)$, (6.1.1) holds true if $2 < p \leq \infty$, $2 \leq q < \infty$,

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - 1 \quad \text{and} \quad \begin{cases} \frac{3}{p} + \frac{2}{q} \leq 1 & \text{if } d = 3 \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} & \text{if } d \geq 4 \end{cases}.$$

If $\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - 1$ and $d = 3$, then $\frac{3}{p} + \frac{2}{q} \leq 1$ is equivalent with $q \leq 14$. Similarly, if $\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - 1$ and $d \geq 4$, then $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$ is equivalent with $q \leq \frac{2(d-1)}{d-3}$. As above, (6.1.1) holds true with $(p,q) = (+\infty, q)$, for all $q \in \left[2, \frac{2d}{d-2}\right]$, and if (6.1.1) holds true for some (p,q), then it also holds true for (p_0, q), for all $p_0 \in [1,p]$. This proves that (6.1.1) holds true for all $(p,q) \in \Lambda_\Omega$.

Case 3 : Next, assume that Ω is a compact manifold without boundary. Let $(O_j)_{j\in J}$ be a finite family of open subsets of Ω such that each O_j is included in a coordinate chart of Ω, and such that

$$\Omega = \bigcup_{j \in J} O_j.$$

Let $(\psi_j)_{j\in J}$ be such that $\psi_j \in \mathscr{C}_c^\infty(O_j, [0,1])$ for $j \in J$, and $\sum_{j\in J} \psi_j = 1$ on Ω. For $j \in J$, let u_j be the solution of

$$\begin{cases} \Box u_j + u_j &= \psi_j g & \text{in } \mathbb{R} \times \Omega, \\ (u_j(0), \partial_t u_j(0)) &= (\psi_j u^0, \psi_j u^1) & \text{in } \Omega. \end{cases}$$

One has $u = \sum_{j\in J} u_j$. For $(p,q) \in \Lambda_\Omega$ and $T > 0$, write

$$\|u\|_{L^p([0,T],L^q)} \leq \sum_{j \in J} \|u_j\|_{L^p([0,T],L^q)}.$$

If T is sufficiently small, then by finite speed of propagation, u_j is supported in O^j for all $j \in J$. As O^j is supported in a coordinate chart of Ω, we can apply Strichartz estimate in the case of $\mathbb{R}^d$ (with variable coefficients), to find

$$\begin{aligned} \|u\|_{L^p([0,T],L^q)} &\lesssim \sum_{j\in J} \left(\left\| \left(\psi_j u^0, \psi_j u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|\psi_j g\|_{L^1([0,T],L^2)} \right) \\ &\lesssim \left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|g\|_{L^1([0,T],L^2)}. \end{aligned}$$

Case 4 : Finally, assume that Ω is a compact perturbation of $\mathbb{R}^d$, and write $\Omega = \mathbb{R}^d \backslash U$. Fix $R > 0$ such that $U \subset B(0,R)$ and such that the metric of $\Omega \cap B(0,R)^\complement$ is equal to the Euclidean one. Let $(O_j)_{j\in J}$ be a finite family of open subsets of $\Omega \cap B(0, R+2)$ such that

$$\Omega \cap B(0, R+1) \subset \bigcup_{j\in J} O_j.$$

There exist ψ_0 and $(\psi_j)_{j\in J}$, satisfying the following properties : $\psi_j \in \mathscr{C}_c^\infty(O_j,[0,1])$ for $j\in J$, $\psi_0 \in \mathscr{C}^\infty(\mathbb{R}^d,[0,1])$, with $\psi_0 = 0$ on $B(0,R)$ and $\psi_0 = 1$ on $\mathbb{R}^d\backslash B(0,R+1)$, and

$$\psi_0 + \sum_{j\in J} \psi_j = 1$$

on Ω. Write u_0 and $(u_j)_{j\in J}$ as in the case of a manifold without boundary. Note that $\Lambda_\Omega \subset \Lambda_{\mathbb{R}^d}$. Hence, applying Strichartz estimates in the case of a compact manifold for the functions u_j, $j\in J$, and Strichartz estimates in the case of $\mathbb{R}^d$ for u_0, one completes the proof as in the case of a manifold without boundary. □

Theorem 6.1.3 (Global-in-time Strichartz estimates)**.** *Assume that Ω is either a non-trapping exterior domain in $\mathbb{R}^d$, with $d=3,4$, or $\Omega=\mathbb{R}^d$, with $3\le d\le 5$. Consider $2<\alpha<\frac{d+2}{d-2}$. There exists $C>0$ such that for all $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ and $F\in L^1(\mathbb{R},L^2(\Omega))$, the solution u of*

$$\begin{cases} \Box u + \beta u &= F & \text{in } \mathbb{R}\times\Omega, \\ (u(0),\partial_t u(0)) &= (u^0,u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{6.1.2}$$

satisfies

$$\|u\|_{L^\alpha(\mathbb{R},L^{2\alpha})} \le C\left(\left\|\left(u^0,u^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|F\|_{L^1(\mathbb{R},L^2)}\right). \tag{6.1.3}$$

A proof of Theorem 6.1.3 is given in Section 3.

6.1.2 Basic nonlinear estimates

Here, we gather some basic estimates involving nonlinearities f satisfying (6.0.1) or (6.0.2), which essentially result from Hölder's inequality. For $\mathbf{u},h:(0,T)\times\Omega\to\mathbb{R}$, set

$$\mathrm{NL}_{\mathbf{u}}(h) = f(\mathbf{u}+h) - f(\mathbf{u}) - f'(\mathbf{u})h. \tag{6.1.4}$$

Lemma 6.1.4 (Basic nonlinear estimates - 1)**.** *Consider f satisfying* (6.0.1) *for some α. For $T>0$, set*

$$X_T = \mathscr{C}^0([0,T],H_0^1(\Omega)) \ \cap\ \mathscr{C}^1([0,T],L^2(\Omega)) \ \cap\ L^\alpha((0,T),L^{2\alpha}(\Omega)) \tag{6.1.5}$$

with

$$\|u\|_{X_T} = \max\left(\|u\|_{L^\infty([0,T],H_0^1)}, \|\partial_t u\|_{L^\infty([0,T],L^2)}, \|u\|_{L^\alpha((0,T),L^{2\alpha})}\right), \quad u\in X_T.$$

(i) There exists $C>0$ such that for all $T>0$, and all $u,v\in X_T$, one has

$$\|f(u)-f(v)\|_{L^1((0,T),L^2)} \le C\|u-v\|_{X_T}\left(T + \|u\|^{\alpha-1}_{L^\alpha((0,T),L^{2\alpha})} + \|v\|^{\alpha-1}_{L^\alpha((0,T),L^{2\alpha})}\right).$$

(ii) Consider $T>0$ and $\mathbf{u}\in L^\alpha((0,T),L^{2\alpha}(\Omega))$. There exists $C=C(T,\mathbf{u})>0$ such that for $u,v\in X_T$, one has

$$\|\mathrm{NL}_{\mathbf{u}}(u)-\mathrm{NL}_{\mathbf{u}}(v)\|_{L^1((0,T),L^2)} \le C\|u-v\|_{X_T}\left(\|u\|_{X_T} + \|v\|_{X_T} + \|u\|^{\alpha-1}_{X_T} + \|v\|^{\alpha-1}_{X_T}\right).$$

(iii) There exists $C>0$ such that for $T>0$ and $\mathbf{u}\in L^\alpha((0,T),L^{2\alpha}(\Omega))$, one has

$$\|f'(\mathbf{u})u\|_{L^1((0,T),L^2)} \le C\|u\|_{X_T}\left(T + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,T),L^{2\alpha})}\right),$$

for all $u\in X_T$.

Remark 6.1.5. If $d = 2$, then one has $X_T = \mathscr{C}^0([0,T], H^1(\Omega)) \cap \mathscr{C}^1([0,T], L^2(\Omega))$, and

$$\|u\|_{X_T} \lesssim \max\left(\|u\|_{L^\infty([0,T],H^1)}, \|\partial_t u\|_{L^\infty([0,T],L^2)}\right), \quad u \in X_T,$$

by the Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $1 \le p < +\infty$ (see the beginning of the proof of Theorem 6.1.2).

Proof. As $f'(0) = 0$ and $|f''(s)| \le C_0(1+|s|)^{\alpha-2}$, one has

$$|f(s_1) - f(s_2)| \lesssim |s_1 - s_2| \left(1 + |s_1|^{\alpha-1} + |s_2|^{\alpha-1}\right), \quad s_1, s_2 \in \mathbb{R},$$

implying

$$\begin{aligned}\|f(u) - f(v)\|_{L^1((0,T),L^2)} \lesssim T\|u-v\|_{L^\infty((0,T),L^2)} &+ \left\||u-v||u|^{\alpha-1}\right\|_{L^1((0,T),L^2)} \\ &+ \left\||u-v||v|^{\alpha-1}\right\|_{L^1((0,T),L^2)}\end{aligned}$$

for $u, v \in X_T$. Let α' be given by $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Applying Hölder's inequality twice, one obtains

$$\begin{aligned}\left\||u-v||u|^{\alpha-1}\right\|_{L^1((0,T),L^2)} &\le \|u-v\|_{L^\alpha((0,T),L^{2\alpha})} \|u\|^{\alpha-1}_{L^{\alpha'(\alpha-1)}((0,T),L^{2\alpha'(\alpha-1)})} \\ &= \|u-v\|_{L^\alpha((0,T),L^{2\alpha})} \|u\|^{\alpha-1}_{L^\alpha((0,T),L^{2\alpha})}\end{aligned}$$

and this gives *(i)*.

Next, we prove *(ii)*. For $s, h_1, h_2 \in \mathbb{R}$ such that $h_2 \le h_1$, one has

$$\begin{aligned}&\left|f(s+h_1) - f(s+h_2) - f'(s)(h_1-h_2)\right| \\ =&\left|\int_{s+h_2}^{s+h_1}\int_s^t f''(\tau)\mathrm{d}\tau\mathrm{d}t\right| \\ \lesssim&\left(1+|s+h_1|^{\alpha-2}+|s+h_2|^{\alpha-2}\right)\int_{s+h_2}^{s+h_1}|t-s|\mathrm{d}t \\ \lesssim&\left(1+|s+h_1|^{\alpha-2}+|s+h_2|^{\alpha-2}\right)|h_1-h_2|\left(|h_1|+|h_2|\right),\end{aligned} \tag{6.1.6}$$

and (6.1.6) also holds true if $h_1 \le h_2$. It implies

$$\begin{aligned}&\|\mathrm{NL}_{\mathbf{u}}(u) - \mathrm{NL}_{\mathbf{u}}(v)\|_{L^1((0,T),L^2)} \\ \lesssim\ &\|(u-v)u\|_{L^1((0,T),L^2)} + \|(u-v)v\|_{L^1((0,T),L^2)} \\ &+ \left\|(u-v)u|\mathbf{u}+u|^{\alpha-2}\right\|_{L^1((0,T),L^2)} + \left\|(u-v)u|\mathbf{u}+v|^{\alpha-2}\right\|_{L^1((0,T),L^2)} \\ &+ \left\|(u-v)v|\mathbf{u}+u|^{\alpha-2}\right\|_{L^1((0,T),L^2)} + \left\|(u-v)v|\mathbf{u}+v|^{\alpha-2}\right\|_{L^1((0,T),L^2)}\end{aligned} \tag{6.1.7}$$

for $u, v \in X_T$.

One the one hand, Hölder's inequality gives

$$\|(u-v)u\|_{L^1((0,T),L^2)} \le \|u-v\|_{L^\alpha((0,T),L^{2\alpha})} \|u\|_{L^{\alpha'}((0,T),L^{2\alpha'})}.$$

One has $1 \le \alpha' \le \alpha$, implying

$$\|u\|_{L^{\alpha'}((0,T),L^{2\alpha'})} \le \|u\|^{\theta_1}_{L^1((0,T),L^2)} \|u\|^{1-\theta_1}_{L^\alpha((0,T),L^{2\alpha})},$$

where $\theta_1 = \frac{\alpha-2}{\alpha-1}$ is given by $\frac{1}{\alpha'} = \theta_1 + \frac{1-\theta_1}{\alpha}$. This gives

$$\|(u-v)u\|_{L^1((0,T),L^2)} \lesssim \|u-v\|_{X_T} \|u\|_{X_T}. \tag{6.1.8}$$

On the other hand, as above, one has

$$\left\|(u-v)u|\mathbf{u}+v|^{\alpha-2}\right\|_{L^1((0,T),L^2)} \le \|u-v\|_{L^\alpha((0,T),L^{2\alpha})} \left\|u|\mathbf{u}+v|^{\alpha-2}\right\|_{L^{\alpha'}((0,T),L^{2\alpha'})}. \tag{6.1.9}$$

Note that $\theta_2 = \frac{1}{\alpha-1}$ satisfies $\frac{2\alpha'(\alpha-2)}{1-\theta_2} = \frac{2\alpha'}{\theta_2} = 2\alpha$. Hence, applying Hölder's inequality with $1 = \frac{1}{1/\theta_2} + \frac{1}{1/(1-\theta_2)}$, one obtains

$$\left\|u|\mathbf{u}+v|^{\alpha-2}\right\|_{L^{\alpha'}((0,T),L^{2\alpha'})} \le \|u\|_{L^\alpha((0,T),L^{2\alpha})} \|\mathbf{u}+v\|^{\alpha-2}_{L^\alpha((0,T),L^{2\alpha})}. \tag{6.1.10}$$

Using $\mathbf{u} \in L^\alpha((0,T), L^{2\alpha}(\Omega))$, (6.1.9) and (6.1.10), one finds

$$\left\|(u-v)u|\mathbf{u}+v|^{\alpha-2}\right\|_{L^1((0,T),L^2)} \lesssim \|u-v\|_{L^\alpha((0,T),L^{2\alpha})} \|u\|_{L^\alpha((0,T),L^{2\alpha})} \left(1 + \|v\|^{\alpha-2}_{L^\alpha((0,T),L^{2\alpha})}\right). \tag{6.1.11}$$

Combining (6.1.7), (6.1.8) and (6.1.11), one obtains

$$\begin{aligned} &\|\mathrm{NL}_{\mathbf{u}}(u) - \mathrm{NL}_{\mathbf{u}}(v)\|_{L^1((0,T),L^2)} \\ &\lesssim \|u-v\|_{X_T} \left(\|u\|_{X_T} + \|v\|_{X_T} + \|u\|_{X_T}\|v\|^{\alpha-2}_{X_T} + \|v\|_{X_T}\|u\|^{\alpha-2}_{X_T} + \|v\|^{\alpha-1}_{X_T} + \|v\|^{\alpha-1}_{X_T}\right), \end{aligned}$$

implying *(ii)*.

Finally, we prove *(iii)*. Using $|f'(s)| \lesssim 1 + |s|^{\alpha-1}$, $s \in \mathbb{R}$, one finds

$$\|f'(\mathbf{u})u\|_{L^1((0,T),L^2)} \lesssim \|u\|_{L^1((0,T),L^2)} + \left\||\mathbf{u}|^{\alpha-1} u\right\|_{L^1((0,T),L^2)},$$

for all $u \in X_T$. As above, using Hölder's inequality, this gives *(iii)*. □

Lemma 6.1.6 (Basic nonlinear estimates - 2)**.** *Consider f satisfying* (6.0.2) *for some $\alpha_0 \le \alpha_1$.*

(i) There exists $C > 0$ such that for all $T > 0$, one has

$$\begin{aligned} &\|f(u) - f(v)\|_{L^1((0,T),L^2)} \\ &\le C\|u-v\|_{L^{\alpha_0}((0,T),L^{2\alpha_0})} \left(\|u\|^{\alpha_0-1}_{L^{\alpha_0}((0,T),L^{2\alpha_0})} + \|v\|^{\alpha_0-1}_{L^{\alpha_0}((0,T),L^{2\alpha_0})}\right) \\ &+C\|u-v\|_{L^{\alpha_1}((0,T),L^{2\alpha_1})} \left(\|u\|^{\alpha_1-1}_{L^{\alpha_1}((0,T),L^{2\alpha_1})} + \|v\|^{\alpha_1-1}_{L^{\alpha_1}((0,T),L^{2\alpha_1})}\right), \end{aligned}$$

for all $u, v \in L^{\alpha_0}((0,T), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T), L^{2\alpha_1}(\Omega))$.

(ii) Consider $T > 0$ and $\mathbf{u} \in L^{\alpha_0}((0,T), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T), L^{2\alpha_1}(\Omega))$. Recall that $\mathrm{NL}_{\mathbf{u}}$ is defined by (6.1.4)*. There exists $C = C(T, \mathbf{u}) > 0$ such that*

$$\begin{aligned} \|\mathrm{NL}_{\mathbf{u}}(u) - \mathrm{NL}_{\mathbf{u}}(v)\|_{L^1((0,T),L^2)} \le C \sum_{i=0,1} \|u-v\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})} \Big(\|u\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})} \\ + \|v\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})} + \|u\|^{\alpha_i-1}_{L^{\alpha_i}((0,T),L^{2\alpha_i})} + \|v\|^{\alpha_i-1}_{L^{\alpha_i}((0,T),L^{2\alpha_i})}\Big), \end{aligned}$$

for all $u, v \in L^{\alpha_0}((0,T), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T), L^{2\alpha_1}(\Omega))$. In addition, one can assume that $C(T, \mathbf{u}) \le C(1, \mathbf{u})$ if $T \in (0,1]$.

(iii) There exists $C > 0$ such that for all $T > 0$ and

$$\mathbf{u} \in L^{\alpha_0}((0,T), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T), L^{2\alpha_1}(\Omega)),$$

one has

$$\|f'(\mathbf{u})u\|_{L^1((0,T),L^2)} \leq C \sum_{i=0,1} \|\mathbf{u}\|^{\alpha_i - 1}_{L^{\alpha_i}((0,T),L^{2\alpha_i})} \|u\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})},$$

for all $u \in L^{\alpha_0}((0,T), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T), L^{2\alpha_1}(\Omega))$.

Proof. Using

$$|f(s_1) - f(s_2)| \lesssim |s_1 - s_2| \left(|s_1|^{\alpha_0 - 1} + |s_2|^{\alpha_0 - 1} + |s_1|^{\alpha_1 - 1} + |s_2|^{\alpha_1 - 1}\right), \quad s_1, s_2 \in \mathbb{R},$$

one obtains

$$\|f(u) - f(v)\|_{L^1((0,T),L^2)} \lesssim \sum_{i=0,1} \left(\left\| |u-v||u|^{\alpha_i - 1} \right\|_{L^1((0,T),L^2)} + \left\| |u-v||v|^{\alpha_i - 1} \right\|_{L^1((0,T),L^2)} \right).$$

As in the proof of Lemma 6.1.4, Hölder's inequality gives

$$\left\| |u-v||u|^{\alpha_i - 1} \right\|_{L^1((0,T),L^2)} \leq \|u - v\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})} \|u\|^{\alpha_i - 1}_{L^{\alpha_i}((0,T),L^{2\alpha})}$$

implying *(i)*.

Consider $u, v \in L^{\alpha_0}((0,T), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T), L^{2\alpha_1}(\Omega))$. For $s, h_1, h_2 \in \mathbb{R}$, one has

$$\begin{aligned} &|f(s + h_1) - f(s + h_2) - f'(s)(h_1 - h_2)| \\ \lesssim &\left(|s + h_1|^{\alpha_0 - 2} + |s + h_2|^{\alpha_0 - 2} + |s + h_1|^{\alpha_1 - 2} + |s + h_2|^{\alpha_1 - 2}\right) |h_1 - h_2| \left(|h_1| + |h_2|\right), \end{aligned}$$

implying

$$\begin{aligned} &\|\mathrm{NL}_{\mathbf{u}}(u) - \mathrm{NL}_{\mathbf{u}}(v)\|_{L^1((0,T),L^2)} \\ \lesssim &\sum_{i=0,1} \Big(\left\| (u-v)u|\mathbf{u} + u|^{\alpha_i - 2} \right\|_{L^1((0,T),L^2)} + \left\| (u-v)u|\mathbf{u} + v|^{\alpha_i - 2} \right\|_{L^1((0,T),L^2)} \\ &\qquad \left\| (u-v)v|\mathbf{u} + u|^{\alpha_i - 2} \right\|_{L^1((0,T),L^2)} + \left\| (u-v)v|\mathbf{u} + v|^{\alpha_i - 2} \right\|_{L^1((0,T),L^2)} \Big). \end{aligned}$$

As in the proof of Lemma 6.1.4, one has

$$\begin{aligned} &\left\| (u-v)u|\mathbf{u} + v|^{\alpha_i - 2} \right\|_{L^1((0,T),L^2)} \\ \lesssim &\|u - v\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})} \|u\|_{L^{\alpha_i}((0,T),L^{2\alpha_i})} \left(1 + \|v\|^{\alpha_i - 2}_{L^{\alpha_i}((0,T),L^{2\alpha_i})}\right), \end{aligned}$$

and this gives the inequality of *(ii)*. The constant can be expressed as an nondecreasing function of $\|\mathbf{u}\|_{L^{\alpha_i}((0,T),L^{2\alpha_0})} + \|\mathbf{u}\|_{L^{\alpha_i}((0,T),L^{2\alpha_1})}$, implying the remark of *(ii)* about the constant.

Finally, one has

$$\|f'(\mathbf{u})u\|_{L^1((0,T),L^2)} \lesssim \left\| u|\mathbf{u}|^{\alpha_0 - 1} \right\|_{L^1((0,T),L^2)} + \left\| u|\mathbf{u}|^{\alpha_1 - 1} \right\|_{L^1((0,T),L^2)},$$

and as above, it implies *(iii)*. □

6.1.3 Solutions of the wave equations

We only consider real-valued solutions of wave equations. We start with the case of linear wave equations with time-dependent potential.

Proposition 6.1.7 (Solution of a linear wave equation with time-dependent potential).

(i) *Consider f satisfying* (6.0.1) *for some α, $T > 0$. Write X_T for the set defined by* (6.1.5)*, and consider $\mathbf{u} \in X_T$. Then, for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $g \in L^1((0,T), L^2(\Omega))$, there exists a unique solution $u \in X_T$ of*

$$\begin{cases} \Box u + \beta u &= f'(\mathbf{u})u + g & \text{in } (0,T)\times\Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

In addition, there exists $C > 0$, independent of (u^0, u^1) and g, such that

$$\|u\|_{X_T} \leq C \left(\left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((0,T),L^2)} \right).$$

(ii) *Consider f satisfying* (6.0.2) *for some $\alpha_0 \leq \alpha_1$, and $T_1, T_2 \in \mathbb{R}$, with $T_1 \leq T_2$. Set*

$$\begin{aligned} Y_{[T_1,T_2]} = \mathscr{C}^0([T_1,T_2], H_0^1(\Omega)) \cap \mathscr{C}^1([T_1,T_2], L^2(\Omega)) \\ \cap L^{\alpha_0}((T_1,T_2), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((T_1,T_2), L^{2\alpha_1}(\Omega)), \end{aligned} \tag{6.1.12}$$

with

$$\begin{aligned} \|u\|_{Y_{[T_1,T_2]}} = \max\Big(&\|u\|_{L^\infty([T_1,T_2],H_0^1)}, \|\partial_t u\|_{L^\infty([T_1,T_2],L^2)}, \\ &\|u\|_{L^{\alpha_0}((T_1,T_2),L^{2\alpha_0})}, \|u\|_{L^{\alpha_1}((T_1,T_2),L^{2\alpha_1})} \Big). \end{aligned}$$

Fix $\mathbf{u} \in Y_{[T_1,T_2]}$. For all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $g \in L^1((T_1,T_2), L^2(\Omega))$, there exists a unique solution $u \in Y_{[T_1,T_2]}$ of

$$\begin{cases} \Box u + \beta u &= f'(\mathbf{u})u + g & \text{in } (T_1,T_2)\times\Omega, \\ (u(T_1), \partial_t u(T_1)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } (T_1,T_2)\times\partial\Omega. \end{cases}$$

In addition, there exists $C > 0$, independent of (u^0, u^1) and g, such that

$$\|u\|_{Y_{[T_1,T_2]}} \leq C \left(\left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((T_1,T_2),L^2)} \right).$$

(iii) *Consider $V \in L^\infty((0,T), \mathscr{C}^1(\Omega))$ satisfying*

$$\|V\|_{L^\infty((0,T)\times\Omega)} + \sum_{j=1}^d \|\partial_{x^j} V\|_{L^\infty((0,T)\times\Omega)} < +\infty. \tag{6.1.13}$$

Then for all $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $g \in L^1((0,T), H^{-1}(\Omega))$, there exists a unique solution $u \in \mathscr{C}^0([0,T], L^2(\Omega)) \cap \mathscr{C}^1([0,T], H^{-1}(\Omega))$ of

$$\begin{cases} \Box u + \beta u &= Vu + g & \text{in } (0,T)\times\Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

In addition, there exists $C > 0$, independent of (u^0, u^1) and g, such that

$$\|u\|_{L^\infty([0,T],L^2)} + \|\partial_t u\|_{L^\infty([0,T],H^{-1})} \leq C \left(\left\| \left(u^0, u^1\right) \right\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \|g\|_{L^1((0,T),H^{-1})} \right).$$

Proof. We start with the proof of *(i)* in the case $\mathbf{u} = 0$. In that case, classical semi-group theory gives $u \in \mathscr{C}^0([0,T], H_0^1(\Omega)) \cap \mathscr{C}^1([0,T], L^2(\Omega))$, so we only need to prove

$$\|u\|_{L^\alpha((0,T),L^{2\alpha})} \lesssim \left(\left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((0,T),L^2)} \right).$$

By Theorem 6.1.2, it suffices to show that $(\alpha, 2\alpha) \in \Lambda_\Omega$. That is true if $d = 2$. Assume that $d \geq 3$. One has $\alpha < \frac{d+2}{d-2}$, implying

$$\frac{1}{\alpha} + \frac{d}{2\alpha} = \frac{d+2}{2\alpha} > \frac{d-2}{2}. \tag{6.1.14}$$

If $\partial\Omega = \emptyset$ and $3 \leq d \leq 5$, then $\alpha \leq \frac{d+2}{d-2}$ implies $2\alpha \leq \frac{2d}{d-3}$, yielding $(\alpha, 2\alpha) \in \Lambda_\Omega$. Now, assume that $\partial\Omega \neq \emptyset$. If $d = 3$, then $\alpha \leq \frac{d+2}{d-2} = 5$ implies $2\alpha \leq 14$. If $d = 4$, then $\alpha \leq \frac{d+2}{d-2} = 3$ implies $2\alpha \leq \frac{2(d-1)}{d-3} = 6$. If $d = 5$, then $\alpha = 2$ implies $2\alpha \leq \frac{2(d-1)}{d-3} = 4$. Hence, in any case, $(\alpha, 2\alpha) \in \Lambda_\Omega$.

Now, we prove *(i)* in the case $\mathbf{u} \neq 0$, using the case $\mathbf{u} = 0$ and Picard's fixed point theorem in X_ε, for $\varepsilon > 0$ sufficiently small. Consider $0 < \varepsilon < 1$, $\varepsilon \leq T$. Set

$$\delta = \left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((0,T),L^2)}.$$

For $U \in X_\varepsilon$, write $u = L(U)$ for the solution of

$$\begin{cases} \Box u + \beta u &= f'(\mathbf{u})U + g & \text{in } (0,\varepsilon)\times\Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } (0,\varepsilon)\times\partial\Omega. \end{cases}$$

Using the case $\mathbf{u} = 0$, one finds

$$\|L(U)\|_{X_\varepsilon} \lesssim \left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|f'(\mathbf{u})U + g\|_{L^1((0,\varepsilon),L^2)}.$$

Lemma 6.1.4 *(iii)* gives

$$\|f'(\mathbf{u})U\|_{L^1((0,\varepsilon),L^2)} \leq C\,\|U\|_{X_\varepsilon} \left(\varepsilon + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,\varepsilon),L^{2\alpha})}\right),$$

yielding

$$\|L(U)\|_{X_\varepsilon} \leq C\left(\delta + \left(\varepsilon + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,\varepsilon),L^{2\alpha})}\right)\|U\|_{X_\varepsilon}\right) \tag{6.1.15}$$

for some $C > 0$.

Consider $U, \tilde{U} \in X_\varepsilon$ and write $u = L(U)$ and $\tilde{u} = L\left(\tilde{U}\right)$. One has

$$\begin{cases} \Box(u-\tilde{u}) + (u-\tilde{u}) &= f'(\mathbf{u})(U-\tilde{U}) & \text{in } (0,\varepsilon)\times\Omega, \\ ((u-\tilde{u})(0), \partial_t(u-\tilde{u})(0)) &= 0 & \text{in } \Omega, \\ u-\tilde{u} &= 0 & \text{on } (0,\varepsilon)\times\partial\Omega. \end{cases}$$

Hence, the case $\mathbf{u} = 0$ gives

$$\left\|L(U) - L\left(\tilde{U}\right)\right\|_{X_\varepsilon} \lesssim \left\|f'(\mathbf{u})(U-\tilde{U})\right\|_{L^1((0,\varepsilon),L^2)},$$

and as above, it implies

$$\left\|L(U) - L\left(\tilde{U}\right)\right\|_{X_\varepsilon} \leq C'\left(\varepsilon + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,\varepsilon),L^{2\alpha})}\right)\left\|U - \tilde{U}\right\|_{X_\varepsilon}. \tag{6.1.16}$$

The constants in (6.1.15) and (6.1.16) do not depend on ε, and up to increasing C or C', we can assume that $C = C'$. To apply Picard's fixed point theorem in a ball of radius $R > 0$ in X_ε, one needs

$$\begin{cases} C\left(\delta + \left(\varepsilon + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,\varepsilon),L^{2\alpha})}\right) R\right) \leq R. \\ \quad C\left(\varepsilon + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,\varepsilon),L^{2\alpha})}\right) < 1 \end{cases}$$

We choose ε such that $C\left(\varepsilon + \|\mathbf{u}\|^{\alpha-1}_{L^\alpha((0,\varepsilon),L^{2\alpha})}\right) \leq \frac{1}{2}$. Then, we can simply chose $R = 2C\delta$. By Picard's fixed point theorem, the solution u is constructed on $[0,\varepsilon]$, and one has

$$\|u\|_{X_\varepsilon} \leq 2C\left(\left\|\left(u^0,u^1\right)\right\|_{H^1_0(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((0,\varepsilon),L^2)}\right).$$

In particular, this implies

$$\|(u(\varepsilon),\partial_t u(\varepsilon))\|_{H^1_0(\Omega)\times L^2(\Omega)} \leq 2C\left(\left\|\left(u^0,u^1\right)\right\|_{H^1_0(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((0,T),L^2)}\right).$$

Hence, this process can be iterated to construct the solution u on $[\varepsilon, 2\varepsilon]$. After a finite number of iterations, the solution u is constructed in the space X_T, and satisfies

$$\|u\|_{X_T} \leq (2C)^m\left(\left\|\left(u^0,u^1\right)\right\|_{H^1_0(\Omega)\times L^2(\Omega)} + \|g\|_{L^1((0,T),L^2)}\right)$$

for some $m \in \mathbb{N}$. This completes the proof of *(i)*.

Now, we prove *(ii)*. By a basic time-translation, we can assume that $[T_1,T_2] = [0,T]$. Note that both α_0 and α_1 satisfy the conditions of α in *(i)*, so that *(ii)* in the case $\mathbf{u} = 0$ is a direct consequence of *(i)* in the case $\mathbf{u} = 0$. To prove that *(ii)* is a consequence of *(ii)* in the case $\mathbf{u} = 0$, one argue as above, by constructing the solution in $Y_{[0,\varepsilon]}$ if $\varepsilon > 0$ is sufficiently small. For $U \in Y_{[0,\varepsilon]}$, by Lemma 6.1.6 *(iii)*, one has

$$\|f'(\mathbf{u})U\|_{L^1((0,\varepsilon),L^2)} \lesssim \left(\|\mathbf{u}\|^{\alpha_0-1}_{L^{\alpha_0}((0,\varepsilon),L^{2\alpha_0})} + \|\mathbf{u}\|^{\alpha_1-1}_{L^{\alpha_1}((0,\varepsilon),L^{2\alpha_1})}\right)\|U\|_{Y_{[0,\varepsilon]}}.$$

Set $\eta(\varepsilon) = \|\mathbf{u}\|^{\alpha_0-1}_{L^{\alpha_0}((0,\varepsilon),L^{2\alpha_0})} + \|\mathbf{u}\|^{\alpha_1-1}_{L^{\alpha_1}((0,\varepsilon),L^{2\alpha_1})}$. To apply Picard's fixed point theorem in a ball of radius $R > 0$ in $Y_{[0,\varepsilon]}$, one needs

$$\begin{cases} C\left(\delta + \eta(\varepsilon)R\right) \leq R, \\ \quad C\eta(\varepsilon) < 1 \end{cases}$$

where $\delta > 0$ is defined as above. If ε is sufficiently small, then one has $C\eta(\varepsilon) \leq \frac{1}{2}$. If $R = 2C\delta$, then the previous conditions are satisfied. By Picard's fixed point theorem, the solution u is constructed on $[0,\varepsilon]$. One can iterate this process as above.

Finally, we prove *(iii)*. The case $V = 0$ is well-known, and as above, using Picard's fixed point theorem, we prove that it implies the case $V \neq 0$. Write $Z_\varepsilon = \mathscr{C}^0([0,\varepsilon],L^2(\Omega)) \cap \mathscr{C}^1([0,\varepsilon],H^{-1}(\Omega))$. By (6.1.13), one has

$$\|VU\|_{L^1((0,\varepsilon),H^{-1})} \lesssim \varepsilon\,\|U\|_{L^\infty((0,\varepsilon),H^{-1})} \lesssim \varepsilon\|U\|_{Z_\varepsilon}, \quad U \in Z_\varepsilon.$$

Using this estimate, the rest of the proof of *(iii)* is similar to the proof of *(i)*. □

Remark 6.1.8. Note that if $C(T_1,T_2)$ denotes the constant of *(ii)*, then one can assume that $C(T_1,T_2) \leq C(T_1,T_1+1)$, for $T_1 \leq T_2 \leq T_1+1$.

We recall the local Cauchy theory for (6.0.3).

Theorem 6.1.9. *Consider f satisfying* (6.0.1)*. For any (real-valued) initial data* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, there exist a maximal time of existence* $T \in (0, +\infty]$ *and a unique solution* u *of* (6.0.3) *in*

$$\mathscr{C}^0([0,T), H_0^1(\Omega)) \cap \mathscr{C}^1([0,T), L^2(\Omega)).$$

If $T < +\infty$*, then*

$$\|(u(t), \partial_t u(t))\|_{H_0^1(\Omega)\times L^2(\Omega)} \xrightarrow{t\to T^-} +\infty.$$

For $T' < T$*, if* f *satisfies* (6.0.1) *for some* α*, then* $u \in L^\alpha((0,T'), L^{2\alpha}(\Omega))$*, and if* f *satisfies* (6.0.2) *for some* $\alpha_0 \leq \alpha_1$*, then*

$$u \in L^{\alpha_0}((0,T'), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,T'), L^{2\alpha_1}(\Omega)). \tag{6.1.17}$$

Proof. We only recall that the solution exists on $[0, \varepsilon]$ if ε is sufficiently small, in the case $d \geq 3$. Set $q = 2\alpha$. By (6.1.14), there exists $p > \alpha$ such that $(p,q) \in \Lambda_\Omega$. One has

$$\|u\|_{L^\alpha((0,\varepsilon),L^{2\alpha})} \leq \varepsilon^\theta \|u\|_{L^p((0,\varepsilon),L^q)}, \quad u \in L^p((0,\varepsilon), L^q(\Omega)), \tag{6.1.18}$$

for some $\theta > 0$. Write

$$\tilde{X}_\varepsilon = \mathscr{C}^0([0,\varepsilon], H_0^1(\Omega)) \cap \mathscr{C}^1([0,\varepsilon], L^2(\Omega)) \cap L^p((0,\varepsilon), L^q(\Omega)).$$

Note that (6.1.18) implies that $\|u\|_{X_\varepsilon} \lesssim \|u\|_{\tilde{X}_\varepsilon}$ for $u \in \tilde{X}_\varepsilon$, with a constant independent of $\varepsilon \in (0,1)$. Together with Lemma 6.1.4 *(i)* and (6.1.18), this gives

$$\|f(u) - f(v)\|_{L^1((0,\varepsilon),L^2)} \lesssim \|u - v\|_{\tilde{X}_\varepsilon} \left(\varepsilon + \varepsilon^{\theta(\alpha-1)} \|u\|_{\tilde{X}_\varepsilon}^{\alpha-1} + \varepsilon^{\theta(\alpha-1)} \|v\|_{\tilde{X}_\varepsilon}^{\alpha-1}\right), \quad u, v \in \tilde{X}_\varepsilon.$$

Using this estimate, Theorem 6.1.2, and Picard's fixed point theorem, one can construct the solution in $\tilde{X}_\varepsilon$ if $\varepsilon \in (0,1)$ is sufficiently small. Note that if f satisfies (6.0.2) for some $\alpha_0 \leq \alpha_1$, then (6.1.18) holds for α_0 and α_1, implying (6.1.17). □

6.2 Local controllability around a trajectory

The proof of Theorem 6.0.4 is organized as follows. First, we prove that local controllability around $\mathbf{u}$ can be reduced to local controllability around 0 for a modified nonlinear equation, and we check that the solution of the controlled equation exists if the control is sufficiently small. Secondly, we show an exact controllability result for the linearized equation. Third, we complete the proof of local controllability.

Consider f, a, T and $\mathbf{u}$ satisfying the assumptions of Theorem 6.0.4.

6.2.1 The linearized equation

Local controllability around $\mathbf{u}$ can be reformulated as follows. Consider $g \in L^1((0,T), L^2(\Omega))$, and denote by u the solution of

$$\begin{cases} \Box u + \beta u = f(u) + ag & \text{in } (0,T)\times\Omega, \\ (u(T), \partial_t u(T)) = (\mathbf{u}(T), \partial_t \mathbf{u}(T)) & \text{in } \Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

We prove below that u exists on $[0,T]$ if g is sufficiently small. Set $h = u - \mathbf{u}$. Then $(u(0), \partial_t u(0)) = (u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ if and only if $(h(0), \partial_t h(0)) = (u^0, u^1) - (\mathbf{u}^0, \mathbf{u}^1)$, and h solves

$$\begin{cases} \Box h + \beta h = f'(\mathbf{u})h + \mathrm{NL}_{\mathbf{u}}(h) + ag & \text{in } (0,T)\times\Omega, \\ (h(T), \partial_t h(T)) = (0,0) & \text{in } \Omega, \\ h = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{6.2.1}$$

where $\mathrm{NL}_{\mathbf{u}}(h)$ is defined by (6.1.4). Hence, local controllability around $\mathbf{u}$ is equivalent to local controllability around zero for this modified Klein-Gordon equation. We prove that h (and so u) exists on $[0,T]$ if g is sufficiently small.

Lemma 6.2.1. *If $g \in L^1((0,T), L^2(\Omega))$ is sufficiently small, then the solution h of* (6.2.1) *is well-defined on* $[0,T]$.

Proof. We use Picard's fixed point theorem in X_T (defined by (6.1.5)). For $H \in X_T$, write $h = L(H)$ for the solution of

$$\begin{cases} \Box h + \beta h &= f'(\mathbf{u})h + \mathrm{NL}_{\mathbf{u}}(H) + ag & \text{in } (0,T)\times\Omega, \\ (h(T), \partial_t h(T)) &= 0 & \text{in } \Omega, \\ h &= 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

By Proposition 6.1.7 *(i)*, one has

$$\|L(H)\|_{X_T} \lesssim \|\mathrm{NL}_{\mathbf{u}}(H) + ag\|_{L^1((0,T),L^2)} \lesssim \|\mathrm{NL}_{\mathbf{u}}(H)\|_{L^1((0,T),L^2)} + \|g\|_{L^1((0,T),L^2)},$$

and by Lemma 6.1.4 *(ii)*, this gives

$$\|L(H)\|_{X_T} \lesssim \|H\|_{X_T}^2 + \|H\|_{X_T}^{\alpha} + \|g\|_{L^1((0,T),L^2)}.$$

Similarly, for $H, \tilde{H} \in X_T$, one finds

$$\left\|L(H) - L(\tilde{H})\right\|_{X_T} \lesssim \left\|H - \tilde{H}\right\|_{X_T} \left(\|H\|_{X_T} + \left\|\tilde{H}\right\|_{X_T} + \|H\|_{X_T}^{\alpha-1} + \left\|\tilde{H}\right\|_{X_T}^{\alpha-1}\right).$$

To apply Picard's fixed point theorem in a ball of radius $R \in (0,1)$ in X_T, as $\alpha > 2$, one needs

$$\begin{cases} C\left(\|g\|_{L^1((0,T),L^2)} + R^2\right) \leq R \\ CR < 1 \end{cases}$$

for some $C > 0$. We choose $R = 2C\|g\|_{L^1((0,T),L^2)}$. If $\|g\|_{L^1((0,T),L^2)}$ is sufficiently small, then the previous conditions are satisfied. This completes the proof. $\square$

6.2.2 Exact controllability for the linearized equation

Here, we prove the following exact controllability result.

Proposition 6.2.2. *There exists a continuous linear operator*

$$\begin{array}{rccc} \mathsf{g}: & H_0^1(\Omega)\times L^2(\Omega) & \longrightarrow & L^1((0,T),L^2(\Omega)). \\ & (u^0,u^1) & \longmapsto & \mathsf{g}\left(u^0,u^1\right) \end{array}$$

such that for $(u^0, u^1) \in H_0^1(\Omega)\times L^2(\Omega)$, the solution of

$$\begin{cases} \Box u + \beta u &= f'(\mathbf{u})u + a\mathsf{g}\left(u^0,u^1\right) & \text{in } (0,T)\times\Omega, \\ (u(T), \partial_t u(T)) &= (0,0) & \text{in } \Omega, \\ u &= 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{6.2.2}$$

satisfies $(u(0), \partial_t u(0)) = (u^0, u^1)$.

Remark 6.2.3. The two main difficulties of Proposition 6.2.2 are the following.

(i) As the potential $f'(\mathbf{u})$ is time-dependent, we need unique continuation for wave equations with partially analytic coefficients, to prove that there is no nonzero solution of the dual equation which is equal to zero on the support of a.

(ii) As the domain Ω may be unbounded, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ may fail to be compact. In that case, our proof relies on the assumptions that $|\nabla f'(\mathbf{u}(t,x))| + |f'(\mathbf{u}(t,x))| \longrightarrow 0$ as $|x| \to \infty$, and that $a \geq c > 0$ on the complement of a bounded region.

Proof. We show that the operator

$$\begin{array}{rccc} L: & L^2((0,T)\times\Omega) & \longrightarrow & H^1_0(\Omega)\times L^2(\Omega). \\ & g & \longmapsto & (u(0),\partial_t u(0)) \end{array}$$

is onto, where u is the solution of (6.2.2).

Step 1 : the dual problem. For the duality between $H^1_0(\Omega)\times L^2(\Omega)$ and $L^2(\Omega)\times H^{-1}(\Omega)$, we choose

$$\left\langle (v^0,v^1),\left(u^0,u^1\right)\right\rangle_{L^2(\Omega)\times H^{-1}(\Omega),H^1_0(\Omega)\times L^2(\Omega)} = \left\langle v^1,u^0\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} - \left\langle v^0,u^1\right\rangle_{L^2(\Omega)}.$$

Fix $(v^0,v^1)\in L^2(\Omega)\times H^{-1}(\Omega)$. By definition, $L^*(v^0,v^1)$ satisfies

$$\left\langle L^*(v^0,v^1),g\right\rangle_{L^2((0,T)\times\Omega)} = \left\langle (v^0,v^1),L(g)\right\rangle_{L^2(\Omega)\times H^{-1}(\Omega),H^1_0(\Omega)\times L^2(\Omega)}, \quad g\in L^2((0,T)\times\Omega).$$

In particular, if v is a function such that $(v(0),\partial_t v(0)) = (v^0,v^1)$, then one has

$$\left\langle L^*(v^0,v^1),g\right\rangle_{L^2((0,T)\times\Omega)} = \int_0^T \partial_t\left(-\langle\partial_t v(t),u(t)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \langle v(t),\partial_t u(t)\rangle_{L^2(\Omega)}\right)\mathrm{d}t$$

for all $g\in L^2((0,T)\times\Omega)$. If, in addition, v is smooth, then

$$\begin{aligned} &\left\langle L^*(v^0,v^1),g\right\rangle_{L^2((0,T)\times\Omega)} \\ &= \int_0^T\left(\left\langle -\partial_t^2 v(t)+\Delta v(t)-\beta v(t)+f'(\mathbf{u}(t))v(t),u(t)\right\rangle_{L^2(\Omega)} + \langle av(t),g(t)\rangle_{L^2(\Omega)}\right)\mathrm{d}t \end{aligned}$$

for all $g\in L^2((0,T)\times\Omega)$. This shows that for $(v^0,v^1)\in\mathscr{C}^\infty_{\mathrm{c}}(\Omega)^2$, one has

$$L^*(v^0,v^1) = av \tag{6.2.3}$$

where v is the solution of

$$\left\{\begin{array}{rcll} \Box v+\beta v & = & f'(\mathbf{u})v & \text{in } (0,T)\times\Omega, \\ (v(0),\partial_t v(0)) & = & (v^0,v^1) & \text{in } \Omega, \\ v & = & 0 & \text{on } (0,T)\times\partial\Omega. \end{array}\right. \tag{6.2.4}$$

By definition, L^* is a continuous operator from $L^2(\Omega)\times H^{-1}(\Omega)$ to $L^2((0,T)\times\Omega)$. Thus, for $(v^0,v^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, one has

$$L^*(v^0,v^1) = \lim_{n\to\infty} L^*(v^0_n,v^1_n)$$

where $\left((v^0_n,v^1_n)\right)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathscr{C}^\infty_{\mathrm{c}}(\Omega)^2$ converging to (v^0,v^1) in $L^2(\Omega)\times H^{-1}(\Omega)$. This proves that (6.2.3) holds for all $(v^0,v^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, where v is the solution of (6.2.4) given by Proposition 6.1.7 *(iii)*.

Step 2 : a compactness property. Write $L^* = A + K$, where A is the operator from $L^2(\Omega) \times H^{-1}(\Omega)$ to $L^2((0,T) \times \Omega)$ defined by $A(v^0, v^1) = a\phi$, where ϕ is the solution of

$$\begin{cases} \Box\phi + \beta\phi &= 0 & \text{in } (0,T) \times \Omega, \\ (\phi(0), \partial_t\phi(0)) &= (v^0, v^1) & \text{in } \Omega, \\ \phi &= 0 & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

By definition, $K(v^0, v^1) = aw$, where w is the solution of

$$\begin{cases} \Box w + \beta w &= f'(\mathbf{u})v & \text{in } (0,T) \times \Omega, \\ (w(0), \partial_t w(0)) &= 0 & \text{in } \Omega, \\ w &= 0 & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

and v is the solution of (6.2.4).

We show that K is compact. Let $\left((v_n^0, v_n^1)\right)_{n\in\mathbb{N}}$ be a bounded sequence of elements of $L^2(\Omega) \times H^{-1}(\Omega)$. We want to show that there exists a subsequence of $\left(K(v_n^0, v_n^1)\right)_{n\in\mathbb{N}}$ which converges in $L^2((0,T) \times \Omega)$. If Ω is compact, then the proof is a consequence of Rellich's theorem. Indeed, in that case, we can assume that $\left((v_n^0, v_n^1)\right)_{n\in\mathbb{N}}$ converges in $H^{-1}(\Omega) \times H^{-2}(\Omega)$ up to a subsequence. One has

$$\begin{aligned} \|aw_n - aw_m\|_{L^2((0,T)\times\Omega)} &\lesssim \|f'(\mathbf{u})(v_n - v_m)\|_{L^1((0,T),H^{-1})} \\ &\lesssim \left\|(v_n^0, v_n^1) - (v_m^0, v_m^1)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)}, \end{aligned}$$

implying that the sequence $(aw_n)_{n\in\mathbb{N}}$ converges.

Now, assume that Ω is not compact. We use the following extension of Rellich's theorem.

Lemma 6.2.4. *Consider U a (possibly empty) smooth bounded open subset of $\mathbb{R}^d$ and $s \in \mathbb{R}$. Let $V \in \mathscr{C}^\infty(\mathbb{R}^d \backslash U)$ be such that*

$$\sum_{|\beta| \le |s-1|} \left|\partial_x^\beta V(x)\right| \xrightarrow{|x|\to\infty} 0.$$

Then the operator

$$\begin{array}{ccc} H^s(\mathbb{R}^d \backslash U) & \longrightarrow & H^{s-1}(\mathbb{R}^d \backslash U) \\ u & \longmapsto & Vu \end{array}$$

is compact.

A proof of Lemma 6.2.4 can be found in Appendix A. We apply Ascoli's theorem to the sequence $(f'(\mathbf{u})v_n)_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, one has $f'(\mathbf{u})v_n \in \mathscr{C}^0([0,T], H^{-1}(\Omega))$, and

$$\|\partial_t \left(f'(\mathbf{u})v_n\right)\|_{L^\infty([0,T],H^{-1})} \lesssim \left\|(v_n^0, v_n^1)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

Hence, the sequence $(f'(\mathbf{u})v_n)_{n\in\mathbb{N}}$ is equicontinuous. Applying Lemma 6.2.4 with $s = 0$, one finds that for all $t \in [0,T]$, the set

$$\{f'(\mathbf{u}(t))v_n(t), n \in \mathbb{N}\}$$

is relatively compact in $H^{-1}(\Omega)$. Hence, by Ascoli's theorem, the sequence $(f'(\mathbf{u})v_n)_{n\in\mathbb{N}}$ converges in $L^\infty([0,T], H^{-1}(\Omega))$, up to a subsequence. Then, as

$$\|aw_n - aw_m\|_{L^2((0,T)\times\Omega)} \lesssim \|f'(\mathbf{u})(v_n - v_m)\|_{L^1((0,T),H^{-1})} \lesssim \|f'(\mathbf{u})(v_n - v_m)\|_{L^\infty((0,T),H^{-1})},$$

one finds that the sequence $(aw_n)_{n\in\mathbb{N}}$ converges. Hence, K is compact.

Step 3 : observability for a wave equation with constant coefficients. One has

$$\left\|(v^0,v^1)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \lesssim \left\|A(v^0,v^1)\right\|_{L^2((0,T)\times\Omega)},$$

by the following theorem.

Theorem 6.2.5. *Assume that there exist $\omega\subset\Omega$ and $c>0$ such that $a\geq c$ on ω, and such that (ω,T) satisfies the GCC. In addition, if Ω is unbounded, assume that there exists $R_0>0$ such that $\mathbb{R}^d\backslash B(0,R_0)\subset\omega$. Then, there exists $C>0$ such that for all $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, the solution u of*

$$\begin{cases} \Box u+\beta u &= \quad 0 & \text{in } (0,T)\times\Omega,\\ (u(0),\partial_t u(0)) &= \quad (u^0,u^1) & \text{in } \Omega,\\ u &= \quad 0 & \text{on } (0,T)\times\partial\Omega,\end{cases}$$

given by Proposition 6.1.7 (iii) *(with $V=0$), satisfies*

$$\left\|\left(u^0,u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \|au\|_{L^2((0,T)\times\Omega)}.$$

In the case of a compact domain Ω, it is well-known that the GCC implies Theorem 6.2.5, since the work of Bardos, Leabeau and Rauch (see [BLR92], Theorem 3.8). If Ω is not compact, we give two proofs of Theorem 6.2.5 in Appendix B. A stabilisation property in a similar context can be found in [JL13].

Step 4 : invisible solutions of the dual of the linearized equation. In that step, we prove that the operator L^* is one-to-one. Let $(v^0,v^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ be such that $L^*(v^0,v^1)=av=0$. One has $v(t)=0$ on ω, for all $t\in[0,T]$, and by assumption, (ω,T) fulfils the GCC. By the theorem of propagation of singularities of Melrose and Sjöstrand (see [MS78]), v is smooth. In particular, we can use Theorem 6.1 of [LL15], which we copy here for convenience. We write d for the geodesic distance on a Riemannian manifold M, and

$$\mathrm{d}(x_1,\omega)=\inf_{x_0\in\omega}\mathrm{d}(x_0,x_1),\quad x_1\in M,$$

for the distance to a subset $\omega\subset M$.

Theorem 6.2.6 (Theorem 6.1 of [LL15])**.** *Let (M,g) be a compact Riemannian manifold with (or without) boundary and write Δ_g for the Laplace-Beltrami operator on M. Let ω be an open subset of M, and consider $T>0$ such that*

$$T>\sup_{x_1\in M}\mathrm{d}(x_1,\omega). \tag{6.2.5}$$

Set $P=\partial_t^2-\Delta_g+V$, where $V\in\mathscr{C}^\infty([-T,T]\times\Omega)$ depends analytically on the variable t. There exist C, κ, $\mu_0>0$ such that for any $(u^0,u^1)\in H_0^1(M)\times L^2(M)$, if u is the solution of

$$\begin{cases} Pu &= \quad 0 & \text{in } (-T,T)\times M,\\ (u(0),\partial_t u(0)) &= \quad (u^0,u^1) & \text{in } M,\\ u &= \quad 0 & \text{on } (-T,T)\times\partial M,\end{cases}$$

then for any $\mu\geq\mu_0$, one has

$$\left\|\left(u^0,u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leq Ce^{\kappa\mu}\|u\|_{L^2((-T,T),H^1(\omega))}+\frac{C}{\mu}\left\|\left(u^0,u^1\right)\right\|_{H^1(\Omega)\times L^2(\Omega)}.$$

If Ω is compact, then this theorem immediately gives $(v^0,v^1)=0$. Note that (6.2.5) is a consequence of the fact that (ω,T) satisfies the GCC : see, for example, Lemma B.4 of [LL16]. If Ω is not compact, then by assumption there exists $R_0>0$ such that $a>0$ on $\mathbb{R}^d\backslash B(0,R_0)$. Hence, v is the solution of (6.2.4) on the compact domain $\Omega\cap B(0,R_0)$, and we can also apply the previous theorem. That proves that L^* is one-to-one in all cases.

Step 5 : conclusion. By Step 3, one has

$$\left\|(v^0, v^1)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \lesssim \left\|L^*(v^0, v^1)\right\|_{L^2((0,T)\times\Omega)} + \left\|K(v^0, v^1)\right\|_{L^2((0,T)\times\Omega)}.$$

By Step 2 and Step 4, K is compact and L^* is one-to-one. We apply the following classical result, which proves that L is onto.

Theorem 6.2.7. *Let X, Y and Z be Hilbert spaces, and $L : X \to Y$ and $K : Y \to Z$ be linear continuous operators. Assume that L^* is one-to-one and K is compact, and that there exists $C > 0$ such that*

$$\|y\|_Y \leq C\left(\|L^* y\|_X + \|Ky\|_Z\right), \quad y \in Y.$$

Then, L is onto.

Proof. By Lemma 3.3.1, one has

$$\|y\|_Y \lesssim \|L^* y\|_X, \quad y \in Y.$$

Then, Theorem 3.1.15 implies that L is onto. □

Then, for any linear subspace E of $L^2((0,T)\times\Omega)$ such that

$$E \oplus \operatorname{Ker} L = L^2((0,T)\times\Omega),$$

the operator g can be constructed as a continuous linear operator from $H_0^1(\Omega)\times L^2(\Omega)$ to E. This completes the proof of Proposition 6.2.2. □

6.2.3 Local controllability for the non-linear equation

Here, we prove Theorem 6.0.4. Fix $(h^0, h^1) \in H_0^1(\Omega)\times L^2(\Omega)$. Let $X = X_T$ be the space defined by (6.1.5). Write g for the operator of Proposition 6.2.2. For $H \in X$, write ϕ_H for the solution of

$$\begin{cases} \Box\phi_H + \beta\phi_H &= f'(\mathbf{u})\phi_H + \mathrm{NL}_{\mathbf{u}}(H) & \text{in } (0,T)\times\Omega, \\ (\phi_H(T), \partial_t\phi_H(T)) &= 0 & \text{in } \Omega, \\ \phi_H &= 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

given by Proposition 6.1.7 *(i)*. We claim that the solution $h = \Gamma(H)$ of

$$\begin{cases} \Box h + \beta h &= f'(\mathbf{u})h + \mathrm{NL}_{\mathbf{u}}(H) + a\mathsf{g}\left((h^0, h^1) - (\phi_H(0), \partial_t\phi_H(0))\right) \\ (h(T), \partial_t h(T)) &= 0 \\ h &= 0 \end{cases}$$

satisfies $(h(0), \partial_t h(0)) = (h^0, h^1)$. Indeed, $w = h - \phi_H$ solves

$$\begin{cases} \Box w + \beta w &= f'(\mathbf{u})w + a\mathsf{g}\left((h^0, h^1) - (\phi_H(0), \partial_t\phi_H(0))\right) & \text{in } (0,T)\times\Omega, \\ (w(T), \partial_t w(T)) &= 0 & \text{in } \Omega, \\ w &= 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

implying $(h(0), \partial_t h(0)) - (\phi_H(0), \partial_t\phi_H(0)) = (w(0), \partial_t w(0)) = (h^0, h^1) - (\phi_H(0), \partial_t\phi_H(0))$, by definition of g.

We show that if

$$\delta = \left\|(h^0, h^1)\right\|_{H_0^1(\Omega)\times L^2(\Omega)}$$

is sufficiently small, then Γ has a unique fixed-point in a small neighbourhood of zero in X. By Proposition 6.1.7 *(i)* (applied to $h = \Gamma(H)$), one has

$$\|\Gamma(H)\|_X \lesssim \left\|\mathrm{NL}_{\mathbf{u}}(H) + a\mathsf{g}\left((h^0, h^1) - (\phi_H(0), \partial_t\phi_H(0))\right)\right\|_{L^1((0,T),L^2)}.$$

Using the continuity of g (see Proposition 6.2.2), one finds

$$\|\Gamma(H)\|_X \lesssim \|\mathrm{NL}_{\mathbf{u}}(H)\|_{L^1((0,T),L^2)} + \left\|(h^0, h^1)\right\|_{H^1_0(\Omega)\times L^2(\Omega)} + \|(\phi_H(0), \partial_t\phi_H(0))\|_{H^1_0(\Omega)\times L^2(\Omega)}.$$

By Proposition 6.1.7 *(i)* (applied to ϕ_H), one has

$$\|(\phi_H(0), \partial_t\phi_H(0))\|_{H^1_0(\Omega)\times L^2(\Omega)} \lesssim \|\mathrm{NL}_{\mathbf{u}}(H)\|_{L^1((0,T),L^2)}$$

implying

$$\|\Gamma(H)\|_X \lesssim \|\mathrm{NL}_{\mathbf{u}}(H)\|_{L^1((0,T),L^2)} + \left\|(h^0, h^1)\right\|_{H^1_0(\Omega)\times L^2(\Omega)}.$$

Thus, using Lemma 6.1.4 *(ii)*, one obtains

$$\|\Gamma(H)\|_X \lesssim \|H\|_X^2 + \|H\|_X^\alpha + \delta.$$

Similarly, for $H, \tilde{H} \in X$, one has

$$\begin{aligned}
&\left\|\Gamma(H) - \Gamma(\tilde{H})\right\|_X \\
\lesssim &\left\|\mathrm{NL}_{\mathbf{u}}(H) - \mathrm{NL}_{\mathbf{u}}(\tilde{H}) + a\mathsf{g}\left((\phi_H(0), \partial_t\phi_H(0)) - (\phi_{\tilde{H}}(0), \partial_t\phi_{\tilde{H}}(0))\right)\right\|_{L^1((0,T),L^2)} \\
\lesssim &\left\|\mathrm{NL}_{\mathbf{u}}(H) - \mathrm{NL}_{\mathbf{u}}(\tilde{H})\right\|_{L^1((0,T),L^2)} + \|(\phi_H(0), \partial_t\phi_H(0)) - (\phi_{\tilde{H}}(0), \partial_t\phi_{\tilde{H}}(0))\|_{H^1_0(\Omega)\times L^2(\Omega)} \\
\lesssim &\left\|\mathrm{NL}_{\mathbf{u}}(H) - \mathrm{NL}_{\mathbf{u}}(\tilde{H})\right\|_{L^1((0,T),L^2)} \\
\lesssim &\left\|H - \tilde{H}\right\|_X \left(\|H\|_X + \|H\|_X^{\alpha-1} + \left\|\tilde{H}\right\|_X + \left\|\tilde{H}\right\|_X^{\alpha-1}\right).
\end{aligned}$$

To apply Picard's fixed point theorem in a ball of radius $R > 0$ in X, one needs

$$\begin{cases} C\left(\delta + R^2 + R^\alpha\right) \leq R \\ C\left(R + R^{\alpha-1}\right) < 1 \end{cases}$$

where $C > 0$ is a constant. We choose $R = 2C\delta$. As $\alpha > 1$, the previous conditions are satisfied if δ is sufficiently small. This completes the proof.

6.3 Null-controllability of a scattering solution in a long time

In this section, we prove Theorem 6.0.8. In particular, we only consider f satisfying (6.0.2), implying that Ω is unbounded, and that $3 \leq d \leq 5$. Note also that this requires $\beta > 0$. The proof of Theorem 6.0.8 is organized as follows. First, we prove a local energy decay result for solutions of the linear equation. Second, we prove that together with local-in-time Strichartz estimates and global-in-time Strichartz estimates on $\mathbb{R}^d$, it implies global-in-time Strichartz estimates on Ω (Theorem 6.1.3). Finally, using local energy decay, global-in-time Strichartz estimates, and local controllability around zero, we prove Theorem 6.0.8.

6.3.1 Local energy decay

Here, we prove the following result.

Theorem 6.3.1. *Assume that Ω is unbounded and non-trapping, with $d \geq 3$, and consider $\chi \in \mathscr{C}_c^\infty(\Omega)$ and $R_0 > 0$. There exists $C > 0$ such that for all $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ and $F \in L^2(\mathbb{R} \times \Omega)$ supported in $\mathbb{R} \times (\Omega \cap B(0, R_0))$, the solution u of*

$$\left\{ \begin{array}{rclll} \Box u + \beta u & = & F & & \text{in } \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) & = & (u^0, u^1) & & \text{in } \Omega, \\ u & = & 0 & & \text{on } \mathbb{R} \times \partial\Omega, \end{array} \right. \tag{6.3.1}$$

satisfies $(\chi u, \chi \partial_t u) \in L^2(\mathbb{R}, H_0^1(\Omega) \times L^2(\Omega))$, with

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}, H_0^1 \times L^2)} \leq C \left(\left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|F\|_{L^2(\mathbb{R} \times \Omega)} \right).$$

Proof. The proof is based on [Bur03] and [Bur]. There is a small mistake in the TT^* argument in [Bur03] : formula (2.6) is incorrect, because the operator

$$\begin{array}{ccc} H^s(\Omega) & \longrightarrow & H^s(\Omega) \\ u & \longmapsto & \chi u \end{array}$$

is not self-adjoint when $s \neq 0$. Carrying out the argument with the adjoint of this operator requires the use of more complicated resolvent estimates than those employed in [Bur03]. Instead of doing that, we rely on [Bur] : we use two TT^* arguments, at two different levels of regularity, and we conclude using interpolation.

We split the proof in 4 steps.

Step 1 : a first TT^* argument. Here, we prove that

$$\|\chi \partial_t u\|_{L^2(\mathbb{R} \times \Omega)} \leq C \left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega) \times L^2(\Omega)}, \quad \left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega), \tag{6.3.2}$$

where u is the solution of (6.3.1) with $F = 0$. Write $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$, which is a Hilbert space for the scalar product

$$\begin{aligned} \left\langle \left(u^0, u^1\right), \left(v^0, v^1\right) \right\rangle_{\mathcal{H}} &= \left\langle u^0, v^0 \right\rangle_{H_0^1(\Omega)} + \left\langle u^1, v^1 \right\rangle_{L^2(\Omega)} \\ &= \left\langle \nabla u^0, \nabla v^0 \right\rangle_{L^2(\Omega)} + \beta \left\langle u^0, v^0 \right\rangle_{L^2(\Omega)} + \left\langle u^1, v^1 \right\rangle_{L^2(\Omega)}, \end{aligned}$$

and $S(t) : \mathcal{H} \to \mathcal{H}$ for the linear semi-group associated with (6.3.1), of infinitesimal generator

$$A = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta - \beta & 0 \end{pmatrix} : D(A) \subset \mathcal{H} \to \mathcal{H}, \quad D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega).$$

By conservation of the energy, one has

$$\left\| S(t) \left(u^0, u^1\right) \right\|_{\mathcal{H}} = \left\| \left(u^0, u^1\right) \right\|_{\mathcal{H}}, \quad t \in \mathbb{R}, \quad \left(u^0, u^1\right) \in \mathcal{H}. \tag{6.3.3}$$

Denote by $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ the projection on the second coordinate. For $t \in \mathbb{R}$, consider the linear continuous operator

$$T(t) : \begin{array}{ccc} \mathcal{H} & \longrightarrow & L^2(\Omega) \\ (u^0, u^1) & \longmapsto & \chi \partial_t u(t) = \chi \pi_1 S(t) \left(u^0, u^1\right) \end{array}.$$

We start with the computation of $T(t)^*$. One has

$$\left\langle T(t)\left(u^0,u^1\right),v^1\right\rangle_{L^2(\Omega)}=\left\langle\left(u^0,u^1\right),S(t)^*\left(0,\chi v^1\right)\right\rangle_{\mathcal{H}},\quad t\in\mathbb{R},\quad v^1\in L^2(\Omega).$$

By Corollary 10.6 of [Paz83], the adjoint semigroup of S is a C_0-semigroup, generated by A^*. As all functions considered are real-valued, an integration by parts gives

$$\left\langle A\left(u^0,u^1\right),\left(v^0,v^1\right)\right\rangle_{\mathcal{H}}=\left\langle u^1,v^0\right\rangle_{H_0^1(\Omega)}-\left\langle u^0,v^1\right\rangle_{H_0^1(\Omega)}=-\left\langle\left(u^0,u^1\right),A\left(v^0,v^1\right)\right\rangle_{\mathcal{H}},$$

for $\left(u^0,u^1\right),\left(v^0,v^1\right)\in D(A)$, yielding $A^*=-A$. Hence, one finds

$$T(t)^*v^1=S(-t)\left(0,\chi v^1\right),\quad t\in\mathbb{R},\quad v^1\in L^2(\Omega).$$

Fix $\left(u^0,u^1\right)\in\mathcal{H}$ and $\phi\in\mathscr{C}_c^\infty(\mathbb{R}\times\Omega)$. Note that (6.3.2) is equivalent to

$$\left|\int_{\mathbb{R}}\left\langle T(t)\left(u^0,u^1\right),\phi(t)\right\rangle_{L^2(\Omega)}\mathrm{d}t\right|\le C\left\|\phi\right\|_{L^2(\mathbb{R}\times\Omega)}\left\|\left(u^0,u^1\right)\right\|_{\mathcal{H}},$$

for some $C>0$ independent of $\left(u^0,u^1\right)$ and ϕ. The Cauchy-Schwarz inequality gives

$$\left|\int_{\mathbb{R}}\left\langle T(t)\left(u^0,u^1\right),\phi(t)\right\rangle_{L^2(\Omega)}\mathrm{d}t\right|\le\left\|\left(u^0,u^1\right)\right\|_{\mathcal{H}}\left\|\int_{\mathbb{R}}T(t)^*\phi(t)\mathrm{d}t\right\|_{\mathcal{H}},$$

and

$$\begin{aligned}\left\|\int_{\mathbb{R}}T(t)^*\phi(t)\mathrm{d}t\right\|_{\mathcal{H}}^2&=\left\langle\int_{\mathbb{R}}T(t)^*\phi(t)\mathrm{d}t,\int_{\mathbb{R}}T(s)^*\phi(s)\mathrm{d}s\right\rangle_{\mathcal{H}}\\&=\int_{\mathbb{R}}\left\langle\phi(t),\int_{\mathbb{R}}T(t)T(s)^*\phi(s)\mathrm{d}s\right\rangle_{L^2(\Omega)}\mathrm{d}t\\&\le\left\|\phi\right\|_{L^2(\mathbb{R}\times\Omega)}\left\|T_0(\phi)\right\|_{L^2(\mathbb{R}\times\Omega)},\end{aligned}$$

where $T_0:L^2(\mathbb{R}\times\Omega)\to L^2(\mathbb{R}\times\Omega)$ is the operator given by

$$T_0(\phi):t\longmapsto\int_{\mathbb{R}}T(t)T(s)^*\phi(s)\mathrm{d}s=\int_{\mathbb{R}}\chi\pi_1S(t-s)\left(0,\chi\phi(s)\right)\mathrm{d}s,\quad\phi\in L^2(\mathbb{R}\times\Omega).$$

Write $T_0^\pm$ for the operator

$$T_0^\pm(\psi):t\longmapsto\int_{\mathbb{R}}\mathbb{1}_{t-s\in\mathbb{R}_\pm}\pi_1S(t-s)\left(0,\psi(s)\right)\mathrm{d}s,\quad\psi\in L^2(\mathbb{R}\times\Omega),$$

so that $T_0\phi=\chi T_0^+(\chi\phi)+\chi T_0^-(\chi\phi)$. To prove (6.3.2), we show that

$$\left\|\chi T_0^\pm(\chi\phi)\right\|_{L^2(\mathbb{R}\times\Omega)}\lesssim\left\|\phi\right\|_{L^2(\mathbb{R}\times\Omega)},\quad\phi\in L^2(\mathbb{R}\times\Omega).\tag{6.3.4}$$

We start with the contribution of $\chi T_0^+(\chi\phi)$. Set

$$U(t)=\int_{\mathbb{R}}\mathbb{1}_{t-s>0}S(t-s)\left(0,\chi\phi(s)\right)\mathrm{d}s,\quad t\in\mathbb{R},$$

so that $\pi_1U=T_0^+(\chi\phi)$. Let $R_1>0$ be such that $\operatorname{supp}\phi\subset(-R_1,R_1)\times\Omega$. One has $U(t)\in D(A)$ for $t\in\mathbb{R}$, and

$$\left\{\begin{array}{rcll}\partial_tU&=&AU+(0,\chi\phi)&\text{in }\mathbb{R}\times\Omega,\\U&=&0&\text{in }(-\infty,-R_1)\times\Omega.\end{array}\right.\tag{6.3.5}$$

By (6.3.3), one has $\sup_{t\in\mathbb{R}} \|U(t)\|_{\mathcal{H}} < \infty$, yielding

$$\int_{\mathbb{R}} \left\|U(t)e^{-i\tau t}\right\|_{\mathcal{H}} \mathrm{d}t \lesssim \int_{-R_1}^{+\infty} e^{\mathrm{Im}(\tau)t}\mathrm{d}t < \infty, \quad \mathrm{Im}\,\tau < 0.$$

This implies that the Fourier transform of U with respect to t, defined by

$$\widehat{U}(\tau) = \int_{\mathbb{R}} U(t)e^{-i\tau t}\mathrm{d}t, \quad \mathrm{Im}\,\tau < 0,$$

is holomorphic in the half-plane $\{\mathrm{Im}\,\tau < 0\}$. Using (6.3.5), one obtains

$$(i\tau - A)\,\widehat{U}(\tau) = \left(0, \chi\widehat{\phi}(\tau)\right), \quad \mathrm{Im}\,\tau < 0.$$

If $\tau^2 \in \mathbb{C}\backslash\,[\beta, +\infty)$, then the operator $i\tau - A$ is invertible, with

$$(i\tau - A)^{-1} = \begin{pmatrix} i\tau\left(-\Delta + \beta - \tau^2\right)^{-1} & \left(-\Delta + \beta - \tau^2\right)^{-1} \\ (\Delta - \beta)\left(-\Delta + \beta - \tau^2\right)^{-1} & i\tau\left(-\Delta + \beta - \tau^2\right)^{-1} \end{pmatrix}.$$

Note that for $\tau = \tau_0 + i\tau_1$ with $\tau_1 < 0$, one has $\tau^2 = \tau_0^2 - \tau_1^2 + 2i\tau_0\tau_1$, implying $\tau^2 \notin [\beta, +\infty)$ as $\beta \geq 0$. In particular, one has

$$\widehat{U}(\tau) = (i\tau - A)^{-1}\left(0, \chi\widehat{\phi}(\tau)\right), \quad \mathrm{Im}\,\tau < 0,$$

yielding

$$\left\|\chi\pi_1\widehat{U}(\tau)\right\|_{L^2(\Omega)} = \left\|\chi\tau\left(-\Delta + \beta - \tau^2\right)^{-1}\chi\widehat{\phi}(\tau)\right\|_{L^2(\Omega)}, \quad \mathrm{Im}\,\tau < 0. \tag{6.3.6}$$

We use the following lemma.

Lemma 6.3.2. *Assume that $\beta > 0$. Then there exists $C > 0$ such that*

$$(1 + |\tau|)\left\|\chi\left(-\Delta + \beta - \tau^2\right)^{-1}\chi w\right\|_{L^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}, \quad \mathrm{Im}\,\tau \neq 0, \quad w \in L^2(\Omega).$$

Proof. We prove

$$(1 + |\tau_0 + i\tau_1|)\left\|\chi\left(-\Delta + \beta - \tau_0^2 + \tau_1^2 - 2i\tau_0\tau_1\right)^{-1}\chi w\right\|_{L^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}, \tag{6.3.7}$$

for $w \in L^2(\Omega)$, $\tau_0 \in \mathbb{R}$ and $\tau_1 \neq 0$. We start with the case $\tau_0 = 0$. For $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\tau_1 \in \mathbb{R}$, integrating by parts, one finds

$$\left\|\left(-\Delta + \beta + \tau_1^2\right)u\right\|_{L^2(\Omega)}^2 = \|(-\Delta + \beta)\,u\|_{L^2(\Omega)}^2 + \tau_1^4\,\|u\|_{L^2(\Omega)}^2 + 2\tau_1^2\,\|u\|_{H_0^1(\Omega)}^2\,,$$

implying

$$\left\|\left(-\Delta + \beta + \tau_1^2\right)u\right\|_{L^2(\Omega)}^2 \gtrsim \|u\|_{L^2(\Omega)}^2 + \tau_1^2\,\|u\|_{L^2(\Omega)}^2\,,$$

by the Poincaré inequality and the ellipticity of $-\Delta + \beta$. This gives

$$(1 + |\tau_1|)\left\|\left(-\Delta + \beta + \tau_1^2\right)^{-1}w\right\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)}, \quad \tau_1 \in \mathbb{R}, \quad w \in L^2(\Omega),$$

implying (6.3.7) in the case $\tau_0 = 0$.

For $\tau = \tau_0 + i\tau_1 \in \mathbb{C}$, with $\tau_0 \neq 0$ and $\tau_1 \neq 0$, one has $\mathrm{Im}\left(\beta - \tau^2\right) \neq 0$, implying

$$\sqrt{1 + |\beta - \tau^2|}\left\|\chi\left(-\Delta + \beta - \tau^2\right)^{-1}\chi w\right\|_{L^2(\Omega)} \leq C\|w\|_{L^2(\Omega)}, \quad w \in L^2(\Omega),$$

as Ω is non-trapping (see Definition 6.0.6). As $\sqrt{1 + |\beta - \tau^2|} \gtrsim 1 + |\tau|$ for $\tau \in \mathbb{C}$, this completes the proof. ☐

Coming back to (6.3.6) and using Lemma 6.3.2, one finds

$$\left\|\chi\pi_1\widehat{U}(\tau)\right\|_{L^2(\Omega)} \lesssim \left\|\widehat{\phi}(\tau)\right\|_{L^2(\Omega)}, \quad \operatorname{Im}\tau < 0.$$

Writing $\tau = \tau_0 + i\tau_1$ and letting τ_1 tends to zero, one obtains

$$\left\|\chi\pi_1\widehat{U}(\tau_0)\right\|_{L^2(\Omega)} \lesssim \left\|\widehat{\phi}(\tau_0)\right\|_{L^2(\Omega)}, \quad \tau_0 \in \mathbb{R}.$$

The Plancherel theorem gives

$$\left\|\chi T_0^+(\chi\phi)\right\|_{L^2(\mathbb{R}\times\Omega)}^2 = \|\chi\pi_1 U\|_{L^2(\mathbb{R}\times\Omega)}^2 \lesssim \|\phi\|_{L^2(\mathbb{R}\times\Omega)}^2. \tag{6.3.8}$$

To estimate the contribution of $\chi T_0^-(\chi\phi)$, one argues similarly. Set

$$V(t) = \int_{\mathbb{R}} \mathbb{1}_{t-s<0} S(t-s)\,(0,\chi\phi(s))\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

One has $\pi_1 V = T_0^-(\chi\phi)$, and

$$\begin{cases} \partial_t V &= AV - (0,\chi\phi) & \text{in } \mathbb{R}\times\Omega, \\ V &= 0 & \text{in } (R_1,+\infty)\times\Omega. \end{cases}$$

Arguing as above, one finds

$$\widehat{V}(\tau) = -(i\tau - A)^{-1}\left(0,\chi\widehat{\phi}(\tau)\right), \quad \operatorname{Im}\tau > 0,$$

and with Lemma 6.3.2 and the Plancherel theorem, this implies

$$\left\|\chi T_0^-(\chi\phi)\right\|_{L^2(\mathbb{R}\times\Omega)}^2 = \|\chi\pi_1 V\|_{L^2(\mathbb{R}\times\Omega)}^2 \lesssim \|\phi\|_{L^2(\mathbb{R}\times\Omega)}^2.$$

Together with (6.3.8), this gives (6.3.4), completing the proof of (6.3.2).

Step 2 : a second TT^* argument. Here, we prove that

$$\|\chi u\|_{L^2(\mathbb{R}\times\Omega)} \leq C\left\|\left(u^0,u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)}, \quad \left(u^0,u^1\right) \in L^2(\Omega)\times H^{-1}(\Omega), \tag{6.3.9}$$

where u is the solution of (6.3.1) with $F = 0$, at another level of regularity (see Proposition 6.1.7 *(iii)* with $V = 0$). We define the scalar product on $H^{-1}(\Omega)$ as

$$\left\langle u^1, v^1\right\rangle_{H^{-1}(\Omega)} = \left\langle(-\Delta+\beta)^{-\frac{1}{2}}u^1, (-\Delta+\beta)^{-\frac{1}{2}}v^1\right\rangle_{L^2(\Omega)}, \quad u^1, v^1 \in H^{-1}(\Omega),$$

and we write $\mathcal{K} = L^2(\Omega)\times H^{-1}(\Omega)$, which is a Hilbert space for the scalar product

$$\left\langle\left(u^0,u^1\right),\left(v^0,v^1\right)\right\rangle_{\mathcal{K}} = \left\langle u^0,v^0\right\rangle_{L^2(\Omega)} + \left\langle u^1,v^1\right\rangle_{H^{-1}(\Omega)}.$$

The semi-group $\mathsf{S}(t) : \mathcal{K}\to\mathcal{K}$ associated with (6.3.1) at the level of regularity $L^2(\Omega)\times H^{-1}(\Omega)$ is generated by

$$\mathsf{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta-\beta & 0 \end{pmatrix} : D(\mathsf{A}) \subset \mathcal{K}\to\mathcal{K}, \quad D(\mathsf{A}) = H_0^1(\Omega)\times L^2(\Omega).$$

One has

$$\left\langle \mathsf{A}\left(u^0,u^1\right),\left(v^0,v^1\right)\right\rangle_{\mathcal{K}} = \left\langle u^1,v^0\right\rangle_{L^2(\Omega)} - \left\langle u^0,v^1\right\rangle_{L^2(\Omega)} = -\left\langle \left(u^0,u^1\right),\mathsf{A}\left(v^0,v^1\right)\right\rangle_{\mathcal{K}},$$

for $\left(u^0,u^1\right),\left(v^0,v^1\right) \in D(\mathsf{A})$, that is, $\mathsf{A}^* = -\mathsf{A}$. As $\mathsf{S}(t) = (-\Delta+\beta)^{\frac{1}{2}} S(t) (-\Delta+\beta)^{-\frac{1}{2}}$, (6.3.3) implies

$$\left\|\mathsf{S}(t)\left(u^0,u^1\right)\right\|_{\mathcal{K}} \lesssim \left\|\left(u^0,u^1\right)\right\|_{\mathcal{K}}, \quad t\in\mathbb{R}, \quad \left(u^0,u^1\right)\in\mathcal{K}.$$

Denote by $\pi_0 : \mathbb{R}^2 \to \mathbb{R}$ the projection on the first coordinate, and for $t\in\mathbb{R}$, consider

$$\begin{array}{llll}\mathsf{T}(t): & \mathcal{K} & \longrightarrow & L^2(\Omega) \\ & (u^0,u^1) & \longmapsto & \chi u(t) = \chi\pi_0\mathsf{S}(t)\left(u^0,u^1\right)\end{array}.$$

One has $\mathsf{T}(t)^* v^0 = \mathsf{S}(-t)\left(\chi v^0, 0\right)$, for $t\in\mathbb{R}$ and $v^0\in L^2(\Omega)$. As above, (6.3.9) will follow from

$$\|\mathsf{T}_0(\phi)\|_{L^2(\mathbb{R}\times\Omega)} \lesssim \|\phi\|_{L^2(\mathbb{R}\times\Omega)}, \quad \phi\in L^2(\mathbb{R}\times\Omega),$$

where T_0 is given by

$$\mathsf{T}_0(\phi) : t\longmapsto \int_{\mathbb{R}} \chi\pi_0\mathsf{S}(t-s)\left(\chi\phi(s),0\right)\mathrm{d}s, \quad \phi\in L^2(\mathbb{R}\times\Omega).$$

One has $\mathsf{T}_0\phi = \chi\mathsf{T}_0^+(\chi\phi) + \chi\mathsf{T}_0^-(\chi\phi)$, with

$$\mathsf{T}_0^{\pm}(\psi) : t\longmapsto \int_{\mathbb{R}} \mathbb{1}_{t-s\in\mathbb{R}_\pm}\pi_0\mathsf{S}(t-s)\left(\psi(s),0\right)\mathrm{d}s, \quad \psi\in L^2(\mathbb{R}\times\Omega).$$

We only estimate the contribution of $\chi\mathsf{T}_0^+(\chi\phi)$, the corresponding estimate for $\chi\mathsf{T}_0^-(\chi\phi)$ being similar. As above, set

$$\mathsf{U}(t) = \int_{\mathbb{R}} \mathbb{1}_{t-s>0}\mathsf{S}(t-s)\left(\chi\phi(s),0\right)\mathrm{d}s, \quad t\in\mathbb{R},$$

so that $\pi_0\mathsf{U} = \mathsf{T}_0^+(\chi\phi)$, and let $R_1>0$ be such that $\operatorname{supp}\phi\subset(-R_1,R_1)\times\Omega$. One has

$$\left\{\begin{array}{lllll}\partial_t\mathsf{U} & = & \mathsf{A}\mathsf{U}+(\chi\phi,0) & & \text{in } \mathbb{R}\times\Omega, \\ \mathsf{U} & = & 0 & & \text{in } (-\infty,-R_1)\times\Omega.\end{array}\right.$$

One has $\widehat{\mathsf{U}}(\tau) = (i\tau-\mathsf{A})^{-1}\left(\chi\widehat{\phi}(\tau),0\right)$ for $\operatorname{Im}\tau<0$, implying

$$\left\|\chi\pi_0\widehat{\mathsf{U}}(\tau)\right\|_{L^2(\Omega)} = \left\|\chi\tau\left(-\Delta+\beta-\tau^2\right)^{-1}\chi\widehat{\phi}(\tau)\right\|_{L^2(\Omega)}, \quad \operatorname{Im}\tau<0.$$

As above, Lemma 6.3.2 gives

$$\left\|\chi\pi_0\widehat{\mathsf{U}}(\tau)\right\|_{L^2(\Omega)} \lesssim \left\|\widehat{\phi}(\tau)\right\|_{L^2(\Omega)}, \quad \operatorname{Im}\tau<0.$$

Letting $\operatorname{Im}\tau$ tends to zero, and using the Plancherel theorem as above, one obtains

$$\left\|\chi\mathsf{T}_0^+(\chi\phi)\right\|^2_{L^2(\mathbb{R}\times\Omega)} = \|\chi\pi_0\mathsf{U}\|^2_{L^2(\mathbb{R}\times\Omega)} \lesssim \|\phi\|^2_{L^2(\mathbb{R}\times\Omega)}.$$

This proves (6.3.9).

Step 3 : interpolation. Here, we prove that

$$\|\chi u\|_{L^2(\mathbb{R},H_0^1(\Omega))} \le C \left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega)\times L^2(\Omega)}, \quad \left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega), \tag{6.3.10}$$

where u is the solution of (6.3.1) with $F = 0$. By interpolation, (6.3.10) follows from Step 2 and

$$\|\chi u\|_{L^2(\mathbb{R},H^2\cap H_0^1)} \le C \left\| \left(u^0, u^1\right) \right\|_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)}, \quad \left(u^0, u^1\right) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega), \tag{6.3.11}$$

where the norm of $H^2(\Omega) \cap H_0^1(\Omega)$ is given by

$$\left\|u^0\right\|_{H^2\cap H_0^1(\Omega)} = \left\|(-\Delta + \beta)\, u^0\right\|_{L^2(\Omega)}, \quad u^0 \in H^2(\Omega) \cap H_0^1(\Omega),$$

and where u is the solution of (6.3.1) with $F = 0$.

We use the following elementary lemma.

Lemma 6.3.3. *There exist* $\tilde{\chi} \in \mathscr{C}_c^\infty(\Omega)$ *and* $C > 0$ *such that*

$$\|\chi u\|_{H^2\cap H_0^1(\Omega)} \le C \left(\|\tilde{\chi}\,(-\Delta + \beta)\, u\|_{L^2(\Omega)} + \|\tilde{\chi} u\|_{L^2(\Omega)} \right), \quad u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Proof. Fix $u \in H^2(\Omega) \cap H_0^1(\Omega)$, and let $\chi_1 \in \mathscr{C}_c^\infty(\Omega)$ be such that $\chi_1\chi = \chi$. One has

$$\begin{aligned}\|\chi u\|_{H^2\cap H_0^1(\Omega)} &\le \|\chi\,(-\Delta + \beta)\, u\|_{L^2(\Omega)} + \|(\Delta\chi)\, u\|_{L^2(\Omega)} + 2\,\|\nabla\chi \cdot \nabla u\|_{L^2(\Omega)} \\ &\lesssim \|\chi\,(-\Delta + \beta)\, u\|_{L^2(\Omega)} + \|(\Delta\chi)\, u\|_{L^2(\Omega)} + \|\chi_1 u\|_{H_0^1(\Omega)}.\end{aligned}$$

We only need to estimate the last term. Integrating by parts, one finds

$$\begin{aligned}\|\chi_1 u\|_{H_0^1(\Omega)}^2 &= \langle(-\Delta + \beta)\,(\chi_1 u)\,, \chi_1 u\rangle_{L^2(\Omega)} \\ &= \langle\chi_1\,(-\Delta + \beta)\, u, \chi_1 u\rangle_{L^2(\Omega)} - 2\,\langle\nabla\chi_1 \cdot \nabla u, \chi_1 u\rangle_{L^2(\Omega)} - \langle(\Delta\chi_1)\, u, \chi_1 u\rangle_{L^2(\Omega)}.\end{aligned} \tag{6.3.12}$$

Another integration by parts gives

$$-2\,\langle\nabla\chi_1 \cdot \nabla u, \chi_1 u\rangle_{L^2(\Omega)} = \frac{1}{2}\int_\Omega \Delta\left(\chi_1^2\right) u^2 \mathrm{d}x, \tag{6.3.13}$$

and together with (6.3.12) and the Cauchy-Schwarz inequality, this yields

$$\|\chi_1 u\|_{H_0^1(\Omega)}^2 \lesssim \|\chi_2\,(-\Delta + \beta)\, u\|_{L^2(\Omega)}^2 + \|\chi_2 u\|_{L^2(\Omega)},$$

for some $\chi_2 \in \mathscr{C}_c^\infty(\Omega)$ satisfying $\chi_2\chi_1 = \chi_1$. This completes the proof. ☐

Now, we prove (6.3.11). Consider $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, and write u for the solution of (6.3.1) of initial data (u^0, u^1), with $F = 0$. Lemma 6.3.3 gives

$$\|\chi u\|_{L^2((-T,T),H^2\cap H_0^1)} \lesssim \|\tilde{\chi}\,(-\Delta + \beta)\, u\|_{L^2((-T,T)\times\Omega)} + \|\tilde{\chi} u\|_{L^2((-T,T)\times\Omega)}, \quad T > 0,$$

for some $\tilde{\chi} \in \mathscr{C}_c^\infty(\Omega)$. Note that $v = (-\Delta + \beta)\, u$ is the solution of (6.3.1) of initial data

$$\left(v^0, v^1\right) = \left((-\Delta + \beta)\, u^0, (-\Delta + \beta)\, u^1\right) \in L^2(\Omega) \times H^{-1}(\Omega),$$

and with $F = 0$. Applying the estimate of Step 2 (with $\tilde{\chi}$ instead of χ) to u and v, one finds

$$\begin{aligned}\|\chi u\|_{L^2((-T,T),H^2\cap H_0^1)} &\lesssim \left\|\left(v^0, v^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \left\|\left(u^0, u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \\ &\lesssim \left\|\left(u^0, u^1\right)\right\|_{H^2\cap H_0^1(\Omega)\times H_0^1(\Omega)},\end{aligned}$$

for $T > 0$. This gives (6.3.11). As explained above, this proves that (6.3.10) holds true.

Step 4 : the inhomogeneous estimate. Here, we prove

$$\|(\chi u, \chi\partial_t u)\|_{L^2(\mathbb{R},H_0^1\times L^2)} \lesssim \|F\|_{L^2(\mathbb{R}\times\Omega)}, \quad F\in L^2(\mathbb{R}\times B(0,R_0)), \tag{6.3.14}$$

where u is the solution of (6.3.1) of initial data $\left(u^0,u^1\right)=0$, with source term F. By linearity, together with Step 1 and Step 3, this will complete the proof of Theorem 6.3.1. Writing $F = F\mathbb{1}_{[0,+\infty)} + F\mathbb{1}_{(-\infty,0)}$ and using the linearity and the time-reversibility of (6.3.1), it suffices to prove (6.3.14) for F supported in $\mathbb{R}_+$. By density, one can also assume that F is smooth and compactly supported.

Consider $F\in\mathscr{C}_c^\infty(\mathbb{R}_+\times B(0,R_0))$, and write $U=(u,\partial_t u)$, where u is the solution of (6.3.1) of initial data $\left(u^0,u^1\right)=0$ and with source term F. By the Duhamel formula, one has

$$U(t) = \int_0^t S(t-s)\,(0,F(s))\,\mathrm{d}s, \quad t\in\mathbb{R},$$

yielding

$$\|U(t)\|_{H_0^1(\Omega)\times L^2(\Omega)} \lesssim \int_0^t \|F(s)\|_{L^2(\Omega)}\,\mathrm{d}s \lesssim 1, \quad t\in\mathbb{R},$$

by (6.3.3). As $U(t)=0$ for $t\le 0$, this implies that the Fourier transform of U is holomorphic in the half-plane $\{\operatorname{Im}\tau<0\}$. As u is a solution of (6.3.1), one finds

$$(i\tau - A)\,\widehat{U}(\tau) = \left(0,\widehat{F}(\tau)\right), \quad \operatorname{Im}\tau<0.$$

As in Step 1, one has

$$\begin{aligned}\widehat{U}(\tau) &= (i\tau-A)^{-1}\left(0,\widehat{F}(\tau)\right)\\ &= \left(\left(-\Delta+\beta-\tau^2\right)^{-1}\widehat{F}(\tau), i\tau\left(-\Delta+\beta-\tau^2\right)^{-1}\widehat{F}(\tau)\right), \quad \operatorname{Im}\tau<0.\end{aligned}$$

Let $\chi_0\in\mathscr{C}_c^\infty(\Omega)$ be such that $\chi_0\chi=\chi$ and $\chi_0 F = F$. One has

$$\begin{aligned}\left\|\chi\widehat{U}(\tau)\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} &\lesssim \left\|\chi_0\left(-\Delta+\beta-\tau^2\right)^{-1}\chi_0\widehat{F}(\tau)\right\|^2_{H_0^1(\Omega)}\\ &\quad+\left\|\chi_0\tau\left(-\Delta+\beta-\tau^2\right)^{-1}\chi_0\widehat{F}(\tau)\right\|^2_{L^2(\Omega)}, \quad \operatorname{Im}\tau<0.\end{aligned} \tag{6.3.15}$$

Fix $\tau\in\mathbb{C}$, $\operatorname{Im}\tau<0$. On the one hand, Lemma 6.3.2 gives

$$\left\|\chi_0\tau\left(-\Delta+\beta-\tau^2\right)^{-1}\chi_0\widehat{F}(\tau)\right\|_{L^2(\Omega)} \lesssim \left\|\widehat{F}(\tau)\right\|_{L^2(\Omega)}. \tag{6.3.16}$$

On the other hand, to estimate the other term of (6.3.15), we use (6.3.12) and (6.3.13). It gives

$$\|\chi_0 w\|^2_{H_0^1(\Omega)} = \langle\chi_0\,(-\Delta+\beta)\,w,\chi_0 w\rangle_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \Delta\left(\chi_0^2\right)w^2\mathrm{d}x - \langle(\Delta\chi_0)\,w,\chi_0 w\rangle_{L^2(\Omega)}.$$

with $w=\left(-\Delta+\beta-\tau^2\right)^{-1}\chi_0\widehat{F}(\tau)$. One has

$$\begin{aligned}\langle\chi_0\,(-\Delta+\beta)\,w,\chi_0 w\rangle_{L^2(\Omega)} &= \left\langle\chi_0\left(-\Delta+\beta-\tau^2\right)w,\chi_0 w\right\rangle_{L^2(\Omega)} + \tau^2\langle\chi_0 w,\chi_0 w\rangle_{L^2(\Omega)}\\ &= \left\langle\chi_0^2\widehat{F}(\tau),\chi_0 w\right\rangle_{L^2(\Omega)} + \tau^2\|\chi_0 w\|^2_{L^2(\Omega)}\end{aligned}$$

implying

$$\|\chi_0 w\|^2_{H^1_0(\Omega)} \lesssim \left\|\widehat{F}(\tau)\right\|^2_{L^2(\Omega)} + \left(1+|\tau|^2\right)\left\|\chi_1\left(-\Delta+\beta-\tau^2\right)^{-1}\chi_1\chi_0\widehat{F}(\tau)\right\|^2_{L^2(\Omega)},$$

for some $\chi_1 \in \mathscr{C}^\infty_c(\Omega)$. By Lemma 6.3.2, this gives

$$\|\chi_0 w\|^2_{H^1_0(\Omega)} \lesssim \left\|\widehat{F}(\tau)\right\|^2_{L^2(\Omega)}. \tag{6.3.17}$$

Using (6.3.15), (6.3.16), and (6.3.17), one obtains

$$\left\|\chi\widehat{U}(\tau)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)} \lesssim \left\|\widehat{F}(\tau)\right\|_{L^2(\Omega)}, \quad \operatorname{Im}\tau < 0.$$

Using the Plancherel theorem as in Step 1, one finds (6.3.14). This completes the proof of Theorem 6.3.1. ☐

6.3.2 Global Strichartz estimates for a non-trapping exterior domain

Here, we prove Theorem 6.1.3. As in [Bur03], Theorem 6.1.3 will be a consequence of the local energy decay (Theorem 6.3.1) and the global-in-time Strichartz estimate in the case $\Omega = \mathbb{R}^d$. The latter is derived from the following result of [GV89]. A definition of the Besov spaces can be found, for example, in [AF03], paragraph 7.32.

Theorem 6.3.4 (Proposition 2.2 of [GV89])**.** *Consider $d \geq 3$, $2 \leq r \leq \infty$, $\rho \in \mathbb{R}$, $1 \leq m \leq \infty$, and write*

$$\delta(r) = \frac{d}{2} - \frac{d}{r}, \quad \gamma(r) = \frac{d-1}{2} - \frac{d-1}{r}, \quad \sigma = \rho + \delta(r) - 1, \quad \textit{and} \quad \frac{1}{q} = \max(\sigma, 0).$$

Assume

$$\sigma < \frac{1}{2}, \quad 2\sigma \leq \gamma(r) \tag{6.3.18}$$

and

$$\begin{cases} \frac{1}{m} = \min\left(\frac{1}{2},\ \frac{\delta(r)}{2},\ \gamma(r)-\sigma\right) & \textit{if} \quad \min\left(\frac{\delta(r)}{2},\ \gamma(r)-\sigma\right) \neq \frac{1}{2}, \\ \frac{1}{m} < \frac{1}{2} & \textit{if} \quad \min\left(\frac{\delta(r)}{2},\ \gamma(r)-\sigma\right) = \frac{1}{2}. \end{cases} \tag{6.3.19}$$

Then there exists a constant $C > 0$ such that for all (u^0, u^1) in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, the solution u of

$$\begin{cases} \Box u + \beta u &= 0 & \textit{in } \mathbb{R}\times\mathbb{R}^d, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \textit{in } \mathbb{R}^d, \end{cases}$$

satisfies

$$\left(\sum_{z\in\mathbb{Z}}\left(\int_{z-\frac{1}{2}}^{z+\frac{1}{2}}\|u(t)\|^q_{B^\rho_{r,2}}\,\mathrm{d}t\right)^{\frac{m}{q}}\right)^{\frac{1}{m}} \leq C\left\|\left(u^0, u^1\right)\right\|_{H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)}. \tag{6.3.20}$$

We prove the following corollary.

Corollary 6.3.5. *Consider $3 \leq d \leq 5$. For $2 < \alpha \leq \frac{d+2}{d-2}$, there exists $C > 0$ such that for all (u^0, u^1) in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, and all $F \in L^1(\mathbb{R}, L^2(\mathbb{R}^d))$, the solution u of*

$$\begin{cases} \Box u + \beta u &= F & \textit{in } \mathbb{R}\times\mathbb{R}^d, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \textit{in } \mathbb{R}^d, \end{cases} \tag{6.3.21}$$

satisfies

$$\|u\|_{L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d))} \leq C\left(\left\|\left(u^0, u^1\right)\right\|_{H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d}+\|F\|_{L^1(\mathbb{R}, L^2(\mathbb{R}^d))}\right).$$

Proof. We start with the case $F = 0$. The following choices are motivated by the case $d = \alpha = 3$, which can be found in [NS11b] (see in particular (2.115) and (2.121)). We choose $m = q = \alpha$, so that the left-hand side of (6.3.20) is $\|u\|_{L^\alpha(\mathbb{R},B^\rho_{r,2})}$. Then, we choose $\gamma(r) - \sigma = \frac{1}{m} = \frac{1}{\alpha}$, and together with $\frac{1}{\alpha} = \frac{1}{q} = \sigma$, this gives

$$\frac{1}{r} = \frac{1}{2} - \frac{2}{\alpha(d-1)} \quad \text{and} \quad \rho = \frac{1}{\alpha} - \frac{d}{2} + 1 + \frac{d}{r}.$$

Using $\alpha > 2$, one can verify that (6.3.18) and (6.3.19) are satisfied.

Next, we prove that $B^\rho_{r,2} \hookrightarrow L^{2\alpha}(\mathbb{R}^d)$. One has $B^{\rho_0}_{r_0,2} \hookrightarrow L^p(\mathbb{R}^d)$ for $p = \frac{r_0 d}{d-\rho_0 r_0}$ (see for example (2.121) of [NS11b]). In particular, one has $B^{\rho_0}_{r,2} \hookrightarrow L^{2\alpha}(\mathbb{R}^d)$ for $\rho_0 = \frac{d}{r} - \frac{d}{2\alpha}$. As $\alpha \le \frac{d+2}{d-2}$, one has $\rho \ge \rho_0$, implying

$$B^\rho_{r_0,2} \hookrightarrow B^{\rho_0}_{r_0,2} \hookrightarrow L^{2\alpha}(\mathbb{R}^d).$$

This completes the proof of Corollary 6.3.5 in the case $F = 0$.

Lastly, by linearity, it suffices to prove Corollary 6.3.5 in the case $(u^0, u^1) = 0$ and $F \neq 0$. Writing $F = F\mathbb{1}_{[0,+\infty)} + F\mathbb{1}_{(-\infty,0)}$ and using the linearity and the time-reversibility of (6.3.21), we can assume that F is supported in $\mathbb{R}_+$. Consider $F \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^d))$, and let u be the solution of (6.3.21) associated with F and with $(u^0, u^1) = 0$. The Duhamel formula gives

$$(u(t), \partial_t u(t)) = \int_0^t S(t-s)\,(0, F(s))\,\mathrm{d}s, \quad t \in \mathbb{R},$$

where S is the semi-group associated with (6.3.21). Set

$$\begin{array}{rccc} T: & L^1(\mathbb{R}, L^2(\mathbb{R}^d)) & \longrightarrow & L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d)) \\ & F & \longmapsto & \left(t \mapsto \int_0^\infty \pi_0 S(t-s)\,(0, F(s))\,\mathrm{d}s\right) \end{array},$$

where $\pi_0 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the first coordinate, and denote by

$$\tilde{T} : L^1(\mathbb{R}, L^2(\mathbb{R}^d)) \to L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d))$$

the operator defined in the Christ-Kiselev lemma (Lemma 6.A.1). One has $u = \tilde{T}F$, implying that Corollary 6.3.5 follows from Lemma 6.A.1 and the continuity of T.

Consider $F \in L^1(\mathbb{R}, L^2(\mathbb{R}^d))$, and write $u = TF$. Then u is the solution of

$$\begin{cases} \Box u + \beta u &= 0 & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \mathbb{R}^d, \end{cases}$$

with

$$\left(u^0, u^1\right) = \int_0^\infty S(-s)\,(0, F(s))\,\mathrm{d}s.$$

Using (6.3.3), one finds

$$\left\|\left(u^0, u^1\right)\right\|_{H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)} \lesssim \|F\|_{L^1(\mathbb{R},L^2(\mathbb{R}^d))}.$$

Hence, Corollary 6.3.5 in the case $F = 0$ gives

$$\|u\|_{L^\alpha(\mathbb{R},L^{2\alpha}(\mathbb{R}^d))} \lesssim \|F\|_{L^1(\mathbb{R},L^2(\mathbb{R}^d))}.$$

This proves that $T : L^1(\mathbb{R}, L^2(\mathbb{R}^d)) \to L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d))$ is well-defined and continuous, completing the proof of Corollary 6.3.5. □

Now, following the strategy of [Bur03] (and [SS00]), we use Theorem 6.3.1 and Corollary 6.3.5 to prove Theorem 6.1.3.

Proof of Theorem 6.1.3. We split the proof in 2 steps.

Step 1 : the homogeneous estimate. Consider $\left(u^0, u^1\right) \in H^1_0(\Omega) \times L^2(\Omega)$, and write u for the solution u of (6.1.2) with initial data $\left(u^0, u^1\right)$ and with $F = 0$. Consider $\chi \in \mathscr{C}^\infty_{\mathrm{c}}(\Omega)$ such that $\chi = 1$ on $B(0,R)$, with $R > 0$ such that $\mathbb{R}^d \backslash B(0,R) \subset \Omega$, and such that the metric of $\left(\mathbb{R}^d \backslash B(0,R)\right) \cap \Omega$ is the Euclidean metric. To show that (6.1.3) holds true, we estimate separately the contribution of $v = \chi u$ and of $w = (1-\chi)u$.

Contribution of w. One has $w = 0$ in $B(0,R)$, and w is the solution of

$$\left\{ \begin{array}{rcll} \Box w + \beta w &=& 2\nabla\chi \cdot \nabla u + \Delta\chi u & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ (w(0), \partial_t w(0)) &=& ((1-\chi)u^0, (1-\chi)u^1) & \text{in } \mathbb{R}^d. \end{array} \right.$$

Write $w = w_0 + w_1$, where w_0 is the solution of

$$\left\{ \begin{array}{rcll} \Box w_0 + \beta w_0 &=& 0 & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ (w_0(0), \partial_t w_0(0)) &=& ((1-\chi)u^0, (1-\chi)u^1) & \text{in } \mathbb{R}^d, \end{array} \right.$$

and w_1 is the solution of

$$\left\{ \begin{array}{rcll} \Box w_1 + \beta w_1 &=& 2\nabla\chi \cdot \nabla u + \Delta\chi u & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ (w_1(0), \partial_t w_1(0)) &=& 0 & \text{in } \mathbb{R}^d. \end{array} \right. \tag{6.3.22}$$

The global-in-time Strichartz estimate in $\mathbb{R}^d$ (Corollary 6.3.5) gives

$$\begin{aligned} \|w_0\|_{L^\alpha(\mathbb{R}, L^{2\alpha}(\Omega))} &\leq \|w_0\|_{L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d))} \\ &\lesssim \left\| \left((1-\chi)u^0, (1-\chi)u^1\right) \right\|_{H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \\ &\lesssim \left\| \left(u^0, u^1\right) \right\|_{H^1_0(\Omega) \times L^2(\Omega)}. \end{aligned}$$

Next, we estimate the contribution of w_1, using twice the local energy decay (Theorem 6.3.1). Write $F_1 = 2\nabla\chi \cdot \nabla u + \Delta\chi u$. Note that one has

$$\|F_1\|_{L^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|\chi_1 u\|_{L^2(\mathbb{R}, H^1_0(\Omega))}$$

for some $\chi_1 \in \mathscr{C}^\infty_{\mathrm{c}}(\Omega)$, implying

$$\|F_1\|_{L^2(\mathbb{R} \times \mathbb{R}^d)} \lesssim \left\| \left(u^0, u^1\right) \right\|_{H^1_0(\Omega) \times L^2(\Omega)} \tag{6.3.23}$$

by Theorem 6.3.1.

We prove

$$\|w_1\|_{L^\alpha((0,+\infty), L^{2\alpha}(\Omega))} \lesssim \|F_1\|_{L^2(\mathbb{R} \times \mathbb{R}^d)}. \tag{6.3.24}$$

By the Duhamel formula, one has

$$(w_1(t), \partial_t w_1(t)) = \int_0^t S(t-s)\,(0, F_1(s))\,\mathrm{d}s, \quad t \in \mathbb{R},$$

where S is the semi-group associated with equation (6.3.22). Consider $\chi_2 \in \mathscr{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ such that $\chi_2 \chi = \chi$, set

$$\begin{array}{rccc} T: & L^1(\mathbb{R}, L^2(\mathbb{R}^d)) & \longrightarrow & L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d)) \\ & F & \longmapsto & \left(t \mapsto \int_0^\infty \pi_0 S(t-s)\,(0, \chi_2 F(s))\,\mathrm{d}s\right) \end{array},$$

where $\pi_0 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the first coordinate, and denote by

$$\tilde{T} : L^1(\mathbb{R}, L^2(\mathbb{R}^d)) \to L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d))$$

the operator defined in the Christ-Kiselev lemma (Lemma 6.A.1). As $\alpha > 1$, we can apply Lemma 6.A.1 : to prove that $\tilde{T}$ is well-defined and continuous, it suffices to prove that T is well-defined and continuous. As $\mathbb{1}_{(0,+\infty)} w_1 = \tilde{T} F_1$, this will imply (6.3.24).

By definition, one has

$$TF(t) = \int_0^\infty \pi_0 S(t-s)\,(0, \chi_2 F(s))\,\mathrm{d}s, \quad F \in L^2(\mathbb{R}, L^2(\mathbb{R}^d)), \quad t \in \mathbb{R}.$$

Write $T = T_1 \circ T_0$, with

$$\begin{array}{rccc} T_0: & L^2(\mathbb{R}, L^2(\mathbb{R}^d)) & \longrightarrow & H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \\ & F & \longmapsto & \int_0^\infty S(-s)\,(0, \chi_2 F(s))\,\mathrm{d}s \end{array},$$

and

$$\begin{array}{rccc} T_1: & H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) & \longrightarrow & L^\alpha(\mathbb{R}, L^{2\alpha}(\mathbb{R}^d)) \\ & (u^0, u^1) & \longmapsto & (t \mapsto \pi_0 S(t)\,(u^0, u^1)) \end{array}.$$

The operator T_1 is continuous by the global Strichartz estimate in the case $\Omega = \mathbb{R}^d$ (Theorem 6.3.5). By Theorem 6.3.1 (applied with $\Omega = \mathbb{R}^d$ and χ_2), the operator

$$\begin{array}{rccc} T_2: & H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) & \longrightarrow & L^2(\mathbb{R}, L^2(\mathbb{R}^d)) \\ & (u^0, u^1) & \longmapsto & (s \mapsto \chi_2 \pi_1 S(s)\,(u^0, u^1)) \end{array}$$

is well-defined and continuous, where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the second coordinate. We prove that $T_0 = T_2^*$, implying that T_0 is well-defined and continuous. Consider $F \in \mathscr{C}_c^\infty(\mathbb{R} \times \mathbb{R}^d)$, $(u^0, u^1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, and write

$$\left\langle T_2 \left(u^0, u^1\right), F \right\rangle_{L^2(\mathbb{R}, L^2(\mathbb{R}^d))} = \int_{\mathbb{R}} \left\langle S(s) \left(u^0, u^1\right), (0, \chi_2 F(s)) \right\rangle_{H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)} \mathrm{d}s.$$

As explained in the proof of Theorem 6.3.1, one has $S(s)^* = S(-s)$, yielding

$$\left\langle T_2 \left(u^0, u^1\right), F \right\rangle_{L^2(\mathbb{R}, L^2(\mathbb{R}^d))} = \left\langle \left(u^0, u^1\right), \int_{\mathbb{R}} S(-s)\,(0, \chi_2 F(s))\,\mathrm{d}s \right\rangle_{H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)}.$$

This proves that $T_0 = T_2^*$, and completes the proof of (6.3.24).

Using (6.3.24), and also (6.3.24) applied to $t \mapsto w_1(-t)$, one obtains

$$\|w_1\|_{L^\alpha(\mathbb{R}, L^{2\alpha}(\Omega))} \lesssim \|F_1\|_{L^2(\mathbb{R} \times \mathbb{R}^d)}.$$

Together with (6.3.23), this gives

$$\|w\|_{L^\alpha(\mathbb{R}, L^{2\alpha}(\Omega))} \lesssim \left\| \left(u^0, u^1\right) \right\|_{H_0^1(\Omega) \times L^2(\Omega)}.$$

Contribution of v. By definition, v is the solution of

$$\left\{ \begin{array}{rcll} \Box v + \beta v & = & -2\nabla\chi \cdot \nabla u - \Delta\chi u & \text{in } \mathbb{R} \times \Omega, \\ (v(0), \partial_t v(0)) & = & (\chi u^0, \chi u^1) & \text{in } \Omega, \\ v & = & 0 & \text{on } \mathbb{R} \times \partial\Omega. \end{array} \right.$$

Consider $\phi \in \mathscr{C}_c^\infty((0,1))$ such that $\phi = 1$ on $\left[\frac{1}{4}, \frac{3}{4}\right]$, and set $v_n(t) = \phi\left(t - \frac{n}{2}\right) v(t)$, for $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. One has

$$\left\{ \begin{array}{rcll} \Box v_n + \beta v_n & = & F_n & \text{in } \mathbb{R} \times \Omega, \\ (v_n(t), \partial_t v_n(t)) & = & 0 & \text{in } \left(\mathbb{R} \backslash \left(\frac{n}{2}, \frac{n}{2} + 1\right)\right) \times \Omega, \\ v_n & = & 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{array} \right.$$

with

$$F_n = -\phi\left(\cdot - \frac{n}{2}\right)(2\nabla\chi \cdot \nabla u + \Delta\chi u) + 2\phi'\left(\cdot - \frac{n}{2}\right)\chi\partial_t u + \phi''\left(\cdot - \frac{n}{2}\right)\chi u.$$

Consider $N \in \mathbb{N}$. Using

$$\sum_{n\in\mathbb{Z}} \phi\left(t - \frac{n}{2}\right)^\alpha \gtrsim 1, \quad t \in \mathbb{R},$$

one finds

$$\begin{aligned}\|v\|^\alpha_{L^\alpha\left(\left(-\frac{N}{2},\frac{N}{2}+1\right),L^{2\alpha}(\Omega)\right)} &\lesssim \int_{-\frac{N}{2}}^{\frac{N}{2}+1} \sum_{n\in\mathbb{Z}} \phi\left(t - \frac{n}{2}\right)^\alpha \|v(t)\|^\alpha_{L^{2\alpha}(\Omega)}\, \mathrm{d}t \\ &= \sum_{|n|\le N+1} \|v_n\|^\alpha_{L^\alpha\left(\left(\frac{n}{2},\frac{n}{2}+1\right),L^{2\alpha}(\Omega)\right)}.\end{aligned}$$

Using the local-in-time Strichartz estimate given by Proposition 6.1.7 *(ii)* (with $\mathbf{u} = 0$), this gives

$$\|v\|^\alpha_{L^\alpha\left(\left(-\frac{N}{2},\frac{N}{2}+1\right),L^{2\alpha}(\Omega)\right)} \lesssim \sum_{|n|\le N+1} \|F_n\|^\alpha_{L^1\left(\left(\frac{n}{2},\frac{n}{2}+1\right),L^2(\Omega)\right)},$$

implying

$$\|v\|^\alpha_{L^\alpha\left(\left(-\frac{N}{2},\frac{N}{2}+1\right),L^{2\alpha}(\Omega)\right)} \lesssim \left(\sum_{|n|\le N+1} \|F_n\|^2_{L^1\left(\left(\frac{n}{2},\frac{n}{2}+1\right),L^2(\Omega)\right)}\right)^{\frac{\alpha}{2}}, \tag{6.3.25}$$

as $\alpha \ge 2$. One has

$$\begin{aligned}\sum_{|n|\le N+1} \|F_n\|^2_{L^1\left(\left(\frac{n}{2},\frac{n}{2}+1\right),L^2(\Omega)\right)} &\le \sum_{|n|\le N+1} \|F_n\|^2_{L^2\left(\left(\frac{n}{2},\frac{n}{2}+1\right),L^2(\Omega)\right)} \\ &\lesssim \sum_{|n|\le N+1} \|(\chi_2 u, \chi_2\partial_t u)\|^2_{L^2\left(\left(\frac{n}{2},\frac{n}{2}+1\right),H^1_0\times L^2\right)} \\ &\lesssim \|(\chi_2 u, \chi_2\partial_t u)\|^2_{L^2(\mathbb{R},H^1_0\times L^2)}\end{aligned}$$

for some $\chi_2 \in \mathscr{C}^\infty_c(\Omega)$. Hence, Theorem 6.3.1 implies

$$\sum_{|n|\le N+1} \|F_n\|^2_{L^1\left(\left(\frac{n}{2},\frac{n}{2}+1\right),L^2(\Omega)\right)} \lesssim \left\|\left(u^0, u^1\right)\right\|^2_{H^1_0(\Omega)\times L^2(\Omega)}.$$

Together with (6.3.25), this gives

$$\|v\|_{L^\alpha(\mathbb{R},L^{2\alpha}(\Omega))} \lesssim \left\|\left(u^0, u^1\right)\right\|_{H^1_0(\Omega)\times L^2(\Omega)},$$

completing the proof of (6.1.3) in the case $F = 0$.

Step 2 : the inhomogeneous estimate. Here, we prove (6.1.3) in the case $F \neq 0$ and $(u^0, u^1) = 0$. The proof is similar to that of the inhomogeneous estimate of Corollary 6.3.5, so we only sketch it. Using the Duhamel formula and the Christ-Kiselev lemma (Lemma 6.A.1), it suffices to prove that the operator

$$\begin{aligned} T:\ L^1(\mathbb{R}, L^2(\Omega)) &\longrightarrow L^\alpha(\mathbb{R}, L^{2\alpha}(\Omega)) \\ F &\longmapsto \left(t \mapsto \int_0^\infty \pi_0 S(t-s)\,(0, F(s))\,\mathrm{d}s\right)\end{aligned}$$

is well-defined and continuous.

Consider $F \in L^1(\mathbb{R}, L^2(\Omega))$, and write $u = TF$. Then u is the solution of

$$\begin{cases} \Box u + \beta u &= 0 & \text{in } \mathbb{R} \times \Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{cases}$$

with

$$\left(u^0, u^1\right) = \int_0^\infty S(-s)\,(0, F(s))\,\mathrm{d}s.$$

One has

$$\left\|\left(u^0, u^1\right)\right\|_{H^1(\Omega)\times L^2(\Omega)} \lesssim \|F\|_{L^1(\mathbb{R},L^2(\Omega))},$$

implying

$$\|v\|_{L^\alpha(\mathbb{R},L^{2\alpha}(\Omega))} \lesssim \|F\|_{L^1(\mathbb{R},L^2(\Omega))}$$

by Step 1. This yields (6.1.3) in the case $F \neq 0$. By linearity, this completes the proof of Theorem 6.1.3. □

Note that a global-in-time Strichartz estimates implies a local-in-time Strichartz estimate with a constant independent of the time. More precisely, one has the following corollary.

Corollary 6.3.6 (Local-in-time Strichartz estimates with a time-independent constant)**.** *Consider Ω and α satisfying the assumptions of Theorem 6.1.3. There exists $C > 0$ such that for $T_1 < T_2$, $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $F \in L^1([T_1, T_2], L^2(\Omega))$, the solution u of*

$$\begin{cases} \Box u + \beta u &= F & \textit{in } [T_1, T_2] \times \Omega, \\ (u(T_1), \partial_t u(T_1)) &= (u^0, u^1) & \textit{in } \Omega, \\ u &= 0 & \textit{on } [T_1, T_2] \times \partial\Omega, \end{cases}$$

satisfies

$$\|u\|_{L^\alpha([T_1,T_2],L^{2\alpha})} \leq C\left(\left\|\left(u^0, u^1\right)\right\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|F\|_{L^1([T_1,T_2],L^2)}\right). \tag{6.3.26}$$

Proof. Write v for the solution of

$$\begin{cases} \Box v + \beta v &= F\mathbb{1}_{[T_1,T_2]} & \text{in } \mathbb{R} \times \Omega, \\ (v(T_1), \partial_t v(T_1)) &= (u^0, u^1) & \text{in } \Omega, \\ v &= 0 & \text{on } \mathbb{R} \times \partial\Omega. \end{cases}$$

One has $u = v$ on $[T_1, T_2]$, implying $\|u\|_{L^\alpha([T_1,T_2],L^{2\alpha})} \leq \|v\|_{L^\alpha(\mathbb{R},L^{2\alpha})}$. Hence, using Theorem 6.1.3 and a basic time-translation, one finds

$$\|u\|_{L^\alpha([T_1,T_2],L^{2\alpha})} \leq \|(v(T_1), \partial_t v(T_1))\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|F\mathbb{1}_{[T_1,T_2]}\|_{L^1(\mathbb{R},L^2)}.$$

The gives (6.3.26). □

6.3.3 Proof of the null-controllability of a scattering solution

Here, we prove Theorem 6.0.8. We start by proving that a scattering solution is bounded in the energy space and has a finite Strichartz norm.

Lemma 6.3.7. *Assume that Ω is a non-trapping unbounded domain, and consider f satisfying* (6.0.2) *for some $\alpha_0 \leq \alpha_1$. Let $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ be such that the solution u_{NL} of*

$$\begin{cases} \Box u_{\mathrm{NL}} + \beta u_{\mathrm{NL}} &= f(u_{\mathrm{NL}}) & \text{in } \mathbb{R}_+ \times \Omega, \\ (u_{\mathrm{NL}}(0), \partial_t u_{\mathrm{NL}}(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u_{\mathrm{NL}} &= 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

is scattering. Then $(u_{\mathrm{NL}}, \partial_t u_{\mathrm{NL}}) \in L^\infty((0,+\infty), H_0^1(\Omega) \times L^2(\Omega))$ and

$$u_{\mathrm{NL}} \in L^{\alpha_0}((0,+\infty), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,+\infty), L^{2\alpha_1}(\Omega)). \tag{6.3.27}$$

Proof. First, we prove (6.3.27). As u_{NL} is scattering, there exists a solution u_{L} of the linear equation

$$\begin{cases} \Box u_{\mathrm{L}} + \beta u_{\mathrm{L}} &= 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u_{\mathrm{L}} &= 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{6.3.28}$$

such that

$$\|(u_{\mathrm{NL}}(t), \partial_t u_{\mathrm{NL}}(t)) - (u_{\mathrm{L}}(t), \partial_t u_{\mathrm{L}}(t))\|_{H_0^1(\Omega)\times L^2(\Omega)} \xrightarrow{t\to+\infty} 0. \tag{6.3.29}$$

Consider $\varepsilon > 0$. By Theorem 6.1.3, one has

$$u_{\mathrm{L}} \in L^{\alpha_0}((0,+\infty), L^{2\alpha_0}(\Omega)) \cap L^{\alpha_1}((0,+\infty), L^{2\alpha_1}(\Omega)).$$

Hence, using also (6.3.29), there exists $T = T(\varepsilon)$ such that

$$\|(u_{\mathrm{NL}}(t), \partial_t u_{\mathrm{NL}}(t)) - (u_{\mathrm{L}}(t), \partial_t u_{\mathrm{L}}(t))\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon, \quad t \geq T, \tag{6.3.30}$$

and

$$\|u_{\mathrm{L}}\|_{L^{\alpha_0}((T,+\infty),L^{2\alpha_0})} + \|u_{\mathrm{L}}\|_{L^{\alpha_1}((T,+\infty),L^{2\alpha_1})} \leq \varepsilon. \tag{6.3.31}$$

For $T' \geq T$, set

$$\eta(T') = \|u_{\mathrm{NL}}\|_{L^{\alpha_0}((T,T'),L^{2\alpha_0})} + \|u_{\mathrm{NL}}\|_{L^{\alpha_1}((T,T'),L^{2\alpha_1})}.$$

Note that $\eta(T') < +\infty$ for all $T' \geq T$ by Theorem 6.1.9, and that η is a continuous real function satisfying $\eta(T) = 0$. Set $v = u_{\mathrm{NL}} - u_{\mathrm{L}}$. Then v is the solution of

$$\begin{cases} \Box v + \beta v &= f(u_{\mathrm{NL}}) & \text{in } \mathbb{R}_+ \times \Omega, \\ (v(T), \partial_t v(T)) &= (u_{\mathrm{NL}}(T), \partial_t u_{\mathrm{NL}}(T)) - (u_{\mathrm{L}}(T), \partial_t u_{\mathrm{L}}(T)) & \text{in } \Omega, \\ v &= 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

By (6.3.31), one has

$$\eta(T') \leq \varepsilon + \|v\|_{L^{\alpha_0}((T,T'),L^{2\alpha_0})} + \|v\|_{L^{\alpha_1}((T,T'),L^{2\alpha_1})}, \quad T' \geq T.$$

Using Strichartz estimates with a time-independent constant (Corollary 6.3.6), together with (6.3.30), one finds

$$\begin{aligned} \eta(T') &\lesssim \varepsilon + \|(v(T), \partial_t v(T))\|_{H_0^1(\Omega)\times L^2(\Omega)} + \|f(u_{\mathrm{NL}})\|_{L^1((T,T'),L^2)} \\ &\lesssim \varepsilon + \|f(u_{\mathrm{NL}})\|_{L^1((T,T'),L^2)}, \\ &\lesssim \varepsilon + \|f(u_{\mathrm{L}})\|_{L^1((T,T'),L^2)} + \|f(u_{\mathrm{NL}}) - f(u_{\mathrm{L}})\|_{L^1((T,T'),L^2)}, \quad T' \geq T. \end{aligned}$$

Applying Lemma 6.1.6 *(i)* twice, assuming that $\varepsilon < 1$, and using (6.3.31) again, one obtains

$$\begin{aligned}\eta(T') &\lesssim \varepsilon + \|u_{\mathrm{L}}\|^{\alpha_0}_{L^{\alpha_0}((T,T'),L^{2\alpha_0})} + \|u_{\mathrm{L}}\|^{\alpha_1}_{L^{\alpha_1}((T,T'),L^{2\alpha_1})} \\ &\quad + \|v\|_{L^{\alpha_0}((T,T'),L^{2\alpha_0})}\left(\|u_{\mathrm{L}}\|^{\alpha_0-1}_{L^{\alpha_0}((T,T'),L^{2\alpha_0})} + \|u_{\mathrm{NL}}\|^{\alpha_0-1}_{L^{\alpha_0}((T,T'),L^{2\alpha_0})}\right) \\ &\quad + \|v\|_{L^{\alpha_1}((T,T'),L^{2\alpha_1})}\left(\|u_{\mathrm{L}}\|^{\alpha_1-1}_{L^{\alpha_1}((T,T'),L^{2\alpha_1})} + \|u_{\mathrm{NL}}\|^{\alpha_1-1}_{L^{\alpha_1}((T,T'),L^{2\alpha_1})}\right) \\ &\lesssim \varepsilon + (\varepsilon + \eta(T'))\left(\varepsilon + \eta(T')^{\alpha_0-1} + \eta(T')^{\alpha_1-1}\right), \\ &\lesssim \varepsilon + \varepsilon\eta(T') + \eta(T')^{\alpha_0} + \eta(T')^{\alpha_1}, \qquad T' \geq T.\end{aligned}$$

Hence, for ε sufficiently small, one finds

$$\eta(T') \lesssim \varepsilon + \eta(T')^{\alpha_0} + \eta(T')^{\alpha_1}, \quad T' \geq T.$$

By the mean value theorem, this implies that there exists $c = c(\varepsilon)$ such that either $\eta(T') < c$ for all $T' \geq T$, or $\eta(T') > c$ for all $T' \geq T$. As $\eta(T) = 0$, this proves that η is bounded, yielding (6.3.27).

Second, we prove

$$(u_{\mathrm{NL}}, \partial_t u_{\mathrm{NL}}) \in L^\infty((0,+\infty), H^1_0(\Omega) \times L^2(\Omega)). \tag{6.3.32}$$

The Duhamel formula gives

$$(u_{\mathrm{NL}}(T), \partial_t u_{\mathrm{NL}}(T)) = S(T)\,(u_{\mathrm{NL}}(0), \partial_t u_{\mathrm{NL}}(0)) + \int_0^T S(T-t)\,(0, f(u_{\mathrm{NL}}(t)))\,\mathrm{d}t, \quad T \geq 0,$$

where S is the semi-group associated with (6.3.28). Using (6.3.3), one finds

$$\|(u_{\mathrm{NL}}(T), \partial_t u_{\mathrm{NL}}(T))\|_{H^1_0(\Omega)\times L^2(\Omega)} \lesssim 1 + \|f(u_{\mathrm{NL}})\|_{L^1((0,T),L^2)}, \quad T \geq 0.$$

By Lemma 6.1.6 *(i)*, this implies

$$\|(u_{\mathrm{NL}}(T), \partial_t u_{\mathrm{NL}}(T))\|_{H^1_0(\Omega)\times L^2(\Omega)} \lesssim 1 + \|u_{\mathrm{NL}}\|^{\alpha_0}_{L^{\alpha_0}((0,T)),L^{2\alpha_0})} + \|u_{\mathrm{NL}}\|^{\alpha_1}_{L^{\alpha_1}((0,T),L^{2\alpha_1})}, \quad T \geq 0.$$

Hence, (6.3.32) is a consequence of (6.3.27). This completes the proof. □

Now, we prove Theorem 6.0.8.

Proof of Theorem 6.0.8. Consider $(u^0, u^1) \in H^1_0(\Omega) \times L^2(\Omega)$ such that the solution u_{NL} of

$$\begin{cases} \Box u_{\mathrm{NL}} + \beta u_{\mathrm{NL}} = f(u_{\mathrm{NL}}) & \text{in } \mathbb{R}_+ \times \Omega, \\ (u_{\mathrm{NL}}(0), \partial_t u_{\mathrm{NL}}(0)) = (u^0, u^1) & \text{in } \Omega, \\ u_{\mathrm{NL}} = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases}$$

is scattering. Using local controllability around 0 (Theorem 6.0.4), it suffices to show that for all $\varepsilon > 0$, there exist T and g such that $\|(u(T), \partial_t u(T))\|_{H^1_0(\Omega)\times L^2(\Omega)} \leq \varepsilon$, where u is the solution of

$$\begin{cases} \Box u + \beta u = f(u) + g & \text{in } \mathbb{R}_+ \times \Omega, \\ (u(0), \partial_t u(0)) = (u^0, u^1) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Consider $\varepsilon \in (0,1)$. As u_{NL} is scattering, there exist $T > 0$ and $u_{\mathrm{L}} \in \mathscr{C}^0(\mathbb{R}, H^1_0(\Omega)) \cap \mathscr{C}^1(\mathbb{R}, L^2(\Omega))$, satisfying $\Box u_{\mathrm{L}} + \beta u_{\mathrm{L}} = 0$, such that

$$\|(u_{\mathrm{NL}}(t), \partial_t u_{\mathrm{NL}}(t)) - (u_{\mathrm{L}}(t), \partial_t u_{\mathrm{L}}(t))\|_{H^1_0(\Omega)\times L^2(\Omega)} \leq \varepsilon, \quad t \geq T. \tag{6.3.33}$$

Recall that $a \geq c > 0$ on $\mathbb{R}^d \backslash B(0, R_0)$. Hence, there exists a bounded function $\chi \in \mathscr{C}^\infty(\Omega)$ such that $\chi = \frac{1}{a}$ on $\mathbb{R}^d \backslash B(0, R_0+1)$. Up to increasing T, we can assume that

$$\|(1-a\chi)u_{\mathrm{L}}\|_{L^1((T,T+1),L^2)} + \|(1-a\chi)\partial_t u_{\mathrm{L}}\|_{L^1((T,T+1),L^2)} \leq \varepsilon \tag{6.3.34}$$

by the local energy decay (Theorem 6.3.1). Up to increasing T again, we can assume that

$$\|u_{\mathrm{NL}}\|^{\alpha_0}_{L^{\alpha_0}((T,T+1),L^{2\alpha_0})} + \|u_{\mathrm{NL}}\|^{\alpha_1}_{L^{\alpha_1}((T,T+1),L^{2\alpha_1})} \leq \varepsilon, \tag{6.3.35}$$

by Lemma 6.3.7, as u_{NL} is scattering.

Let $\varphi \in \mathscr{C}^\infty(\mathbb{R}, [0,1])$ be such that $\varphi(t) = 1$ for $t \leq T$ and $\varphi(t) = 0$ for $t \geq T+1$. Set $v(t,x) = \varphi(t) u_{\mathrm{NL}}(t,x)$ and $g = \Box v + \beta v - f(v)$. By definition, g is supported in $[T, T+1]$, and one has

$$g = u_{\mathrm{NL}} \partial_t^2 \varphi + 2\partial_t u_{\mathrm{NL}} \partial_t \varphi + \varphi f(u_{\mathrm{NL}}) - f(\varphi u_{\mathrm{NL}}).$$

To complete the proof, we prove that the solution u of

$$\begin{cases} \Box u + \beta u &= f(u) + a\chi g & \text{in } (0, T+1) \times \Omega, \\ (u(0), \partial_t u(0)) &= (u^0, u^1) & \text{in } \Omega, \\ u &= 0 & \text{on } (0, T+1) \times \partial\Omega, \end{cases}$$

satisfies

$$\|(u(T+1), \partial_t u(T+1))\|_{H_0^1(\Omega) \times L^2(\Omega)} \lesssim \varepsilon. \tag{6.3.36}$$

As $g = 0$ on $[0,T]$, one has $u = u_{\mathrm{NL}}$ on $[0,T]$. Note that (6.3.36) is equivalent to

$$\|(h(T+1), \partial_t h(T+1))\|_{H_0^1(\Omega) \times L^2(\Omega)} \lesssim \varepsilon, \tag{6.3.37}$$

where $h = u - v$ is the solution of

$$\begin{cases} \Box h + \beta h &= f(v+h) - f(v) + (a\chi - 1) g & \text{in } (T, T+1) \times \Omega, \\ (h(T), \partial_t h(T)) &= 0 & \text{in } \Omega, \\ h &= 0 & \text{on } (T, T+1) \times \partial\Omega. \end{cases}$$

Now, we prove

$$\|(1-a\chi)g\|_{L^1((T,T+1),L^2)} \lesssim \varepsilon. \tag{6.3.38}$$

The triangular inequality gives

$$\begin{aligned} \|(1-a\chi)g\|_{L^1((T,T+1),L^2)} \lesssim\ & \|(1-a\chi)u_{\mathrm{L}}\|_{L^1((T,T+1),L^2)} + \|(1-a\chi)\partial_t u_{\mathrm{L}}\|_{L^1((T,T+1),L^2)} \\ & + \|(u_{\mathrm{NL}}, \partial_t u_{\mathrm{NL}}) - (u_{\mathrm{L}}, \partial_t u_{\mathrm{L}})\|_{L^\infty((T,T+1), H_0^1 \times L^2)} \\ & + \|f(u_{\mathrm{NL}})\|_{L^1((T,T+1),L^2)} + \|f(\varphi u_{\mathrm{NL}})\|_{L^1((T,T+1),L^2)}. \end{aligned} \tag{6.3.39}$$

Using Lemma 6.1.6 *(i)*, together with (6.3.35), one finds

$$\begin{aligned} & \|f(u_{\mathrm{NL}})\|_{L^1((T,T+1),L^2)} + \|f(\varphi u_{\mathrm{NL}})\|_{L^1((T,T+1),L^2)} \\ \lesssim\ & \|u_{\mathrm{NL}}\|^{\alpha_0}_{L^{\alpha_0}((T,T+1),L^{2\alpha_0})} + \|u_{\mathrm{NL}}\|^{\alpha_1}_{L^{\alpha_1}((T,T+1),L^{2\alpha_1})} \\ \lesssim\ & \varepsilon. \end{aligned} \tag{6.3.40}$$

Coming back to (6.3.39), and using (6.3.33), (6.3.34) and (6.3.40), one obtains (6.3.38).

To complete the proof, we show that (6.3.38) implies (6.3.37). For $\tau \in [0,1]$, set $Y_\tau = Y_{[T,T+\tau]}$, where $Y_{[T,T+\tau]}$ is defined by (6.1.12). Note that h is the solution of

$$\begin{cases} \Box h + \beta h &= f'(v)h + \mathrm{NL}_v(h) + (a\chi - 1) g & \text{in } (T, T+1) \times \Omega, \\ (h(T), \partial_t h(T)) &= 0 & \text{in } \Omega, \\ h &= 0 & \text{on } (T, T+1) \times \partial\Omega. \end{cases}$$

where $\mathrm{NL}_v(h)$ is defined by (6.1.4). By Proposition 6.1.7 *(ii)* and Remark 6.1.8, there exists a constant independent of $\tau \in [0,1]$ such that

$$\|h\|_{Y_\tau} \lesssim \|\mathrm{NL}_v(h) + (a\chi - 1)g\|_{L^1((T,T+\tau),L^2)} .$$

Using Lemma 6.1.6 *(ii)* and (6.3.38), one obtains

$$\|h\|_{Y_\tau} \leq c\left(\varepsilon + \|h\|_{Y_\tau}^2 + \|h\|_{Y_\tau}^{\alpha_1}\right), \tag{6.3.41}$$

for some $c > 0$ independent of ε and of $\tau \in [0,1]$. Set $\theta(s) = c\left(s^2 + s^{\alpha_1}\right) - s$, for $s \geq 0$. If ε is sufficiently small, then by the mean value theorem, either $\theta'\left(\|h\|_{Y_\tau}\right) \leq 0$ for all $\tau \in [0,1]$, or $\theta'\left(\|h\|_{Y_\tau}\right) \geq 0$ for all $\tau \in [0,1]$. As

$$\|h\|_{Y_\tau} \xrightarrow{\tau \to 0^+} 0,$$

this latter case cannot occur. Hence, one has

$$\|h\|_{Y_\tau} \geq c\left(2\|h\|_{Y_\tau}^2 + \alpha_1 \|h\|_{Y_\tau}^{\alpha_1}\right) \geq 2c\left(\|h\|_{Y_\tau}^2 + \|h\|_{Y_\tau}^{\alpha_1}\right),$$

for $\tau \in [0,1]$. In particular, (6.3.41) gives

$$\|h\|_{Y_1} \leq 2c\varepsilon,$$

yielding (6.3.37). This completes the proof. □

6.A Statement of the Christ-Kiselev lemma

We recall the statement of the Christ-Kiselev lemma (see [CK01]).

Lemma 6.A.1. *Let X and Y be Banach spaces. Consider $1 \leq p < q \leq \infty$, and let $T : L^p(\mathbb{R}, X) \to L^q(\mathbb{R}, Y)$ be a continuous linear operator. Then, the operator*

$$\begin{array}{cccc} \tilde{T}: & L^p(\mathbb{R},X) & \longrightarrow & L^q(\mathbb{R},Y) \\ & F & \longmapsto & \left(t \longmapsto T\left(\mathbb{1}_{(-\infty,t)}F\right)\right) \end{array}$$

is well-defined and continuous.

6.B An extension of Rellich's theorem

Here, we prove Lemma 6.2.4. Consider $s \in \mathbb{R}$, U a (possibly empty) smooth bounded open subset of $\mathbb{R}^d$, $V \in \mathscr{C}^\infty(\mathbb{R}^d \backslash U)$ such that

$$\sum_{|\beta| \leq |s-1|} \left|\partial_x^\beta V(x)\right| \xrightarrow{|x| \to \infty} 0,$$

and $\chi \in \mathscr{C}^\infty(\mathbb{R}^d, [0,1])$ such that $\chi = 1$ on $B(0,1)$ and $\chi = 0$ on $\mathbb{R}^d \backslash B(0,2)$. Write $\Omega = \mathbb{R}^d \backslash U$.

Let $(u_n)_n$ be a bounded sequence in $H^s(\Omega)$. For all $k \in \mathbb{N}$ sufficiently large, by the usual Rellich theorem, there exists a subsequence of $\left(\chi\left(\frac{\cdot}{k}\right) u_n\right)_n$ converging in $H^{s-1}(\Omega \cap B(0,2k))$. Using a diagonal argument, one proves that up to a subsequence, there exists $u_\infty \in H^{s-1}_{\mathrm{loc}}(\Omega)$ such that

$$\chi\left(\frac{\cdot}{k}\right) u_n \xrightarrow{n \to \infty} u_\infty \tag{6.B.1}$$

in $H^{s-1}_{\mathrm{loc}}(\Omega)$, for all $k \in \mathbb{N}$ sufficiently large.

We show that $(Vu_n)_n$ is a Cauchy sequence in $H^{s-1}(\Omega)$. Consider $\varepsilon > 0$, and let $k \in \mathbb{N}$ be such that

$$\sum_{|\beta| \leq |s-1|} \left|\partial_x^\beta V(x)\right| \leq \varepsilon \tag{6.B.2}$$

for all $x \in \Omega \backslash B(0,k)$. Write

$$\begin{aligned} &\|Vu_n - Vu_m\|_{H^{s-1}(\Omega)} \\ &\leq \left\|\chi\left(\frac{\cdot}{k}\right)(Vu_n - Vu_m)\right\|_{H^{s-1}(\Omega)} + \left\|\left(1-\chi\left(\frac{\cdot}{k}\right)\right)(Vu_n - Vu_m)\right\|_{H^{s-1}(\Omega)}. \end{aligned}$$

By (6.B.1), one has

$$\left\|\chi\left(\frac{\cdot}{k}\right)(Vu_n - Vu_m)\right\|_{H^{s-1}(\Omega)} \lesssim \left\|\chi\left(\frac{\cdot}{k}\right)(u_n - u_m)\right\|_{H^{s-1}(\Omega)} \leq \varepsilon$$

for n and m sufficiently large. For $\phi \in H^{s-1}(\Omega)$, one can prove that

$$\|V\phi\|_{H^{s-1}(\Omega)} \lesssim \sum_{|\beta| \leq |s-1|} \left\|\partial_x^\beta V\right\|_{L^\infty(\Omega)} \|\phi\|_{H^{s-1}(\Omega)}.$$

As $(u_n)_n$ is bounded in $H^{s-1}(\Omega)$, this implies

$$\left\|\left(1-\chi\left(\frac{\cdot}{k}\right)\right)(Vu_n - Vu_m)\right\|_{H^{s-1}(\Omega)} \lesssim \sum_{|\beta| \leq |s-1|} \left\|\partial_x^\beta \left(\left(1-\chi\left(\frac{\cdot}{k}\right)\right)V\right)\right\|_{L^\infty(\Omega)}.$$

As $\chi\left(\frac{x}{k}\right) = 1$ for $x \in B(0,k)$, (6.B.2) gives

$$\left\|\left(1-\chi\left(\frac{\cdot}{k}\right)\right)(Vu_n - Vu_m)\right\|_{H^{s-1}(\Omega)} \lesssim \varepsilon,$$

implying that $(Vu_n)_n$ is a Cauchy sequence in $H^{s-1}(\Omega)$. This completes the proof of Lemma 6.2.4.

6.C Observability on an unbounded domain

Here, we give two proofs of Theorem 6.2.5, in the case of an unbounded domain Ω. The first one is elementary, but relies on Theorem 6.2.5 in the case of a compact domain. The second one is based on the propagation of singularities, with microlocal defect measures.

6.C.1 First proof

We start with the case $\Omega = \mathbb{R}^d$ and $a = 1$, in which Theorem 6.2.5 can be proved by a direct Fourier computation. Consider $(u^0, u^1) \in L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ and write u for the solution of

$$\left\{\begin{array}{rclll} \Box u + \beta u & = & 0 & & \text{in } (0,T) \times \mathbb{R}^d, \\ (u(0), \partial_t u(0)) & = & (u^0, u^1) & & \text{in } \mathbb{R}^d. \end{array}\right.$$

Write $\widehat{u}$ for the Fourier transform of u with respect to the space variable x. There exist some complex functions A and B such that

$$\widehat{u}(t,\xi) = A(\xi)e^{i\langle\xi\rangle t} + B(\xi)e^{-i\langle\xi\rangle t}$$

for $t \in [0,T]$ and $\xi \in \mathbb{R}^d$, where $\langle \xi \rangle^2 = 1 + |\xi|^2$. On the one hand, one has

$$\begin{aligned}\left\|\left(u^0, u^1\right)\right\|^2_{L^2(\mathbb{R}^d)\times H^{-1}(\mathbb{R}^d)} &\lesssim \int_{\mathbb{R}^d} \left(\left|\widehat{u^0}(\xi)\right|^2 + \langle \xi \rangle^{-2}\left|\widehat{u^1}(\xi)\right|^2\right) \mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} \left(|A(\xi) + B(\xi)|^2 + |A(\xi) - B(\xi)|^2\right) \mathrm{d}\xi \\ &= 2\int_{\mathbb{R}^d} \left(|A(\xi)|^2 + |B(\xi)|^2\right) \mathrm{d}\xi.\end{aligned}$$

On the other hand, a direct computation gives

$$\begin{aligned}\|u\|^2_{L^2((0,T)\times\mathbb{R}^d)} &\gtrsim \int_{\mathbb{R}^d} \left(T\,|A(\xi)|^2 + T\,|B(\xi)|^2 + 2\,\mathrm{Re}\left(A(\xi)\overline{B(\xi)}\frac{e^{2i\langle\xi\rangle T} - 1}{2i\langle \xi \rangle}\right)\right) \mathrm{d}\xi \\ &\geq \int_{\mathbb{R}^d} \left(|A(\xi)|^2 + |B(\xi)|^2\right)\left(T - \frac{|\sin\left(\langle \xi \rangle T\right)|}{\langle \xi \rangle}\right) \mathrm{d}\xi.\end{aligned}$$

For all $T > 0$, there exists a constant $C = C(T)$ such that for all $\alpha \geq 1$, one has

$$T - \frac{|\sin(\alpha T)|}{\alpha} \geq C.$$

This completes the proof of Theorem 6.2.5 in the case $\Omega = \mathbb{R}^d$ and $a = 1$. Note that in that particular case, T can be arbitrary small.

Now, we prove Theorem 6.2.5 in the case of an unbounded domain Ω. Write U for the (possibly empty) smooth bounded open subset of $\mathbb{R}^d$ such that $\Omega = \mathbb{R}^d \backslash U$. Let $\chi \in \mathscr{C}^\infty(\Omega, [0,1])$ be such that $\chi = 1$ on $B(0, R_0 + T)$ and $\chi = 0$ on $\mathbb{R}^d \backslash B(0, R_0 + 2T)$. Consider $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and write u for the solution of

$$\left\{\begin{array}{rcll} \Box u + \beta u &=& 0 & \text{in } (0,T)\times\Omega, \\ (u(0), \partial_t u(0)) &=& (u^0, u^1) & \text{in } \Omega, \\ u &=& 0 & \text{on } (0,T)\times\partial\Omega. \end{array}\right.$$

One has $u = \chi u + (1-\chi)u$, $\chi u = v + \tilde{v}$ and $(1-\chi)u = w + \tilde{w}$, with

$$\left\{\begin{array}{rcll} \Box v + \beta u &=& -2\nabla\chi\nabla u - \Delta\chi u & \text{in } (0,T)\times\Omega, \\ (v(0), \partial_t v(0)) &=& 0 & \text{in } \Omega, \\ v &=& 0 & \text{on } (0,T)\times\partial\Omega, \end{array}\right.$$

$$\left\{\begin{array}{rcll} \Box \tilde{v} + \beta \tilde{v} &=& 0 & \text{in } (0,T)\times\Omega, \\ (\tilde{v}(0), \partial_t \tilde{v}(0)) &=& (\chi u^0, \chi u^1) & \text{in } \Omega, \\ \tilde{v} &=& 0 & \text{on } (0,T)\times\partial\Omega, \end{array}\right.$$

$$\left\{\begin{array}{rcll} \Box w + \beta w &=& 2\nabla\chi\nabla u + \Delta\chi u & \text{in } (0,T)\times\Omega, \\ (w(0), \partial_t w(0)) &=& 0 & \text{in } \Omega, \\ w &=& 0 & \text{on } (0,T)\times\partial\Omega, \end{array}\right.$$

$$\left\{\begin{array}{rcll} \Box \tilde{w} + \beta \tilde{w} &=& 0 & \text{in } (0,T)\times\Omega, \\ (\tilde{w}(0), \partial_t \tilde{w}(0)) &=& ((1-\chi)u^0, (1-\chi)u^1) & \text{in } \Omega, \\ \tilde{w} &=& 0 & \text{on } (0,T)\times\partial\Omega. \end{array}\right.$$

As $(1-\chi)u^0$ and $(1-\chi)u^1$ are supported in $\mathbb{R}^d \backslash B(0, R_0 + 2T)$, $\tilde{w}$ is the solution of

$$\left\{\begin{array}{rcll} \Box \tilde{w} + \beta \tilde{w} &=& 0 & \text{in } (0,T)\times\mathbb{R}^d, \\ (\tilde{w}(0), \partial_t \tilde{w}(0)) &=& ((1-\chi)u^0, (1-\chi)u^1) & \text{in } \mathbb{R}^d, \end{array}\right.$$

by finite speed of propagation. Hence, the case $\Omega = \mathbb{R}^d$ and $a = 1$ treated above gives

$$\begin{aligned}\left\|\left((1-\chi)u^0,(1-\chi)u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} &= \left\|\left((1-\chi)u^0,(1-\chi)u^1\right)\right\|_{L^2(\mathbb{R}^d)\times H^{-1}(\mathbb{R}^d)}\\ &\lesssim \|\tilde{w}\|_{L^2((0,T)\times\mathbb{R}^d)}\\ &= \|a\tilde{w}\|_{L^2((0,T)\times\Omega)}\,. \end{aligned} \tag{6.C.1}$$

By finite speed of propagation again, $\tilde{v}$ is the solution of

$$\left\{\begin{array}{rcll} \Box\tilde{v}+\beta\tilde{v} &=& 0 & \text{in } (0,T)\times(\Omega\cap B(0,R_0+2T))\,,\\ (\tilde{v}(0),\partial_t\tilde{v}(0)) &=& (\chi u^0,\chi u^1) & \text{in } \Omega\cap B(0,R_0+2T),\\ \tilde{v} &=& 0 & \text{on } (0,T)\times\partial(\Omega\cap B(0,R_0+2T))\,. \end{array}\right.$$

Hence, Theorem 6.2.5 in the case of a compact domain gives

$$\left\|\left(\chi u^0,\chi u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \lesssim \|a\tilde{v}\|_{L^2((0,T)\times\Omega)}\,. \tag{6.C.2}$$

Using (6.C.1) and (6.C.2), together with the continuity estimate of Proposition 6.1.7 *(iii)* (with $V=0$), one obtains

$$\begin{aligned}\left\|\left(u^0,u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} &\leq \left\|\left(\chi u^0,\chi u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} + \left\|\left((1-\chi)u^0,(1-\chi)u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)}\\ &\lesssim \|au\|_{L^2((0,T)\times\Omega)} + \|av\|_{L^2((0,T)\times\Omega)} + \|aw\|_{L^2((0,T)\times\Omega)}\\ &\lesssim \|au\|_{L^2((0,T)\times\Omega)} + \|2\nabla\chi\nabla u + \Delta\chi u\|_{L^1((0,T),H^{-1})}\,. \end{aligned}$$

Consider $\phi \in H_0^1(\Omega)$, with $\|\phi\|_{H_0^1(\Omega)} \leq 1$. As $\nabla\chi$ and $\Delta\chi$ are supported in $B(0,R_0+2T)\backslash B(0,R_0+T)$, one has

$$\left|\langle 2\nabla\chi\nabla u(t)+\Delta\chi u(t),\phi\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)}\right| = \left|\langle au(t),\phi\Delta\chi - 2\operatorname{div}(\phi\nabla\chi)\rangle_{L^2(\Omega)}\right| \lesssim \|au(t)\|_{L^2(\Omega)}$$

for all $t\in(0,T)$. Hence, the Cauchy-Schwarz inequality gives

$$\left\|\left(u^0,u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \lesssim \|au\|_{L^2((0,T)\times\Omega)}$$

and this completes the proof.

6.C.2 Second proof

The proof is decomposed into two steps. For an example of the use of microlocal defect measures in a similar context to prove a stabilization property, see [JL13].

Step 1 : a weak observability inequality. Let $V \in \mathscr{C}^\infty(\Omega,(0,+\infty))$ be such that

$$\sum_{|\beta|\leq 1}\left|\partial_x^\beta V(x)\right| \xrightarrow{|x|\to\infty} 0.$$

We prove that there exists a constant $C>0$ such that for all $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, if u is the solution of

$$\left\{\begin{array}{rcll} \Box u+\beta u &=& 0 & \text{in } (0,T)\times\Omega,\\ (u(0),\partial_t u(0)) &=& (u^0,u^1) & \text{in } \Omega,\\ u &=& 0 & \text{on } (0,T)\times\partial\Omega, \end{array}\right. \tag{6.C.3}$$

then

$$\left\|\left(u^0,u^1\right)\right\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C\left(\|au\|_{L^2((0,T)\times\Omega)} + \left\|\left(Vu^0,Vu^1\right)\right\|_{H^{-1}(\Omega)\times H^{-2}(\Omega)}\right). \tag{6.C.4}$$

Assume by contradiction that there exists a sequence $\left((u^0,u^1)\right)_n$ of elements of $L^2(\Omega)\times H^{-1}(\Omega))$ such that $\|(u_n^0,u_n^1)\|_{L^2(\Omega)\times H^{-1}(\Omega)} = 1$ for all $n\in\mathbb{N}$, and

$$\|au_n\|_{L^2((0,T)\times\Omega)} + \left\|\left(Vu_n^0,Vu_n^1\right)\right\|_{H^{-1}(\Omega)\times H^{-2}(\Omega)} \xrightarrow{n\to\infty} 0. \tag{6.C.5}$$

Consider $\chi\in\mathscr{C}_c^\infty(\Omega,\mathbb{R})$. There exists a constant such that for all $n\in\mathbb{N}$,

$$\left\|\partial_t\begin{pmatrix}\chi u_n\\ \chi\partial_t u_n\end{pmatrix}\right\|_{L^\infty([0,T],H^{-1}(\Omega)\times H^{-2})} \lesssim 1,$$

and by the Rellich theorem, the set

$$\left\{\begin{pmatrix}\chi u_n(t)\\ \chi\partial_t u_n(t)\end{pmatrix}, n\in\mathbb{N}\right\}$$

is relatively compact in $H^{-1}(\Omega)\times H^{-2}(\Omega)$, as χ is compactly supported. Hence, by Ascoli's theorem, there exists a subsequence of $((\chi u_n,\chi\partial_t u_n))_n$ which converges to some limit in $L^\infty([0,T],H^{-1}(\Omega)\times H^{-2}(\Omega))$. Using a diagonal argument, one proves that there exists a solution

$$u_\infty\in\mathscr{C}^0([0,T],H^{-1}_{\mathrm{loc}}(\Omega))\cap\mathscr{C}^1([0,T],H^{-2}_{\mathrm{loc}}(\Omega))$$

of $\Box u_\infty+\beta u_\infty=0$ such that, up to a subsequence,

$$(u_n,\partial_t u_n)\xrightarrow{n\to\infty}(u_\infty,\partial_t u_\infty)$$

in $L^\infty([0,T],H^{-1}_{\mathrm{loc}}(\Omega)\times H^{-2}_{\mathrm{loc}}(\Omega))$. For all $\chi\in\mathscr{C}_c^\infty(\Omega,\mathbb{R})$, one has

$$\left(V\chi u_n^0,V\chi u_n^1\right)\xrightarrow{n\to\infty}\left(V\chi u_\infty(0),V\chi\partial_t u_\infty(0)\right).$$

As V is positive, this gives $u_\infty(0)=\partial_t u_\infty(0)=0$, implying

$$(u_n,\partial_t u_n)\xrightarrow{n\to\infty}0$$

in $L^\infty([0,T],H^{-1}_{\mathrm{loc}}(\Omega)\times H^{-2}_{\mathrm{loc}}(\Omega))$.

Up to a subsequence, we can assume that the sequence $(u_n)_n$ converges weakly to zero in $L^2((0,T)\times\Omega)$. Let μ be a microlocal defect measure associated with $(u_n)_n$ (for the definition of μ, see [Gér91], [Tar90], [Bur97b]). For $\chi\in\mathscr{C}_c^\infty((0,T)\times\Omega,\mathbb{R})$, one has

$$\|au_n\|^2_{L^2((0,T)\times\Omega)}\gtrsim\|a\chi u_n\|^2_{L^2((0,T)\times\Omega)}\xrightarrow{n\to\infty}\int_{S^*((0,T)\times\Omega)}a(x)^2\chi(t,x)^2\mathrm{d}\mu(t,x,\tau,\xi).$$

Hence, (6.C.5) gives

$$\int_{S^*((0,T)\times\Omega)}a(x)^2\chi(t,x)^2\mathrm{d}\mu(t,x,\tau,\xi)=0$$

implying

$$\int_{S^*((0,T)\times\Omega)}a(x)^2\mathrm{d}\mu(t,x,\tau,\xi).$$

Recall that $a \geq c > 0$ on an open set ω such that (ω, T) satisfies the GCC. Using the propagation of the measure along the generalized bicharacteristic flow of Melrose–Sjöstrand, one obtains $\mu = 0$, implying that $(u_n)_n$ converges strongly to zero in $L^2_{\text{loc}}((0,T)\times\Omega)$. Together with (6.C.5) and the fact that $\mathbb{R}^d \backslash B(0, R_0) \subset \omega$, this gives

$$u_n \xrightarrow{n\to\infty} 0 \quad \text{in } L^2((0,T)\times\Omega). \tag{6.C.6}$$

We prove

$$\partial_t u_n \xrightarrow{n\to\infty} 0 \quad \text{in } L^2\left(\left(\frac{T}{4}, \frac{3T}{4}\right), H^{-1}(\Omega)\right). \tag{6.C.7}$$

Note that it suffices to prove (6.C.4) for smooth compactly supported initial data, so we can assume that $u_n^0, u_n^1 \in \mathscr{C}_c^\infty(\Omega)$ for all $n \in \mathbb{N}$. Let $\phi \in \mathscr{C}_c^\infty((0,T), \mathbb{R})$ be such that $\phi = 1$ on $\left[\frac{T}{4}, \frac{3T}{4}\right]$. Recall that all functions considered here are real-valued. Starting from

$$0 = \int_0^T \int_\Omega \left(\Box u_n + \beta u_n\right)(t,x)\phi(t)^2 \left(-\Delta + \beta\right)^{-1}\left(u_n(t)\right)(x)\mathrm{d}x\mathrm{d}t,$$

and integrating by parts, one finds

$$\begin{aligned}
0 &= \|\phi u_n\|^2_{L^2((0,T)\times\Omega)} - \int_0^T \int_\Omega \partial_t u_n \left(\partial_t\left(\phi^2\right)\left(-\Delta+\beta\right)^{-1}(u_n) + \phi^2\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\right)\mathrm{d}x\mathrm{d}t \\
&= \|\phi u_n\|^2_{L^2((0,T)\times\Omega)} - \int_0^T \int_\Omega \partial_t u_n \phi^2\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega u_n \left(\partial_t^2\left(\phi^2\right)\left(-\Delta+\beta\right)^{-1}(u_n) + \partial_t\left(\phi^2\right)\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\right)\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{6.C.8}$$

One has

$$\begin{aligned}
&\left|\int_0^T \int_\Omega u_n \left(\partial_t^2\left(\phi^2\right)\left(-\Delta+\beta\right)^{-1}(u_n) + \partial_t\left(\phi^2\right)\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\right)\mathrm{d}x\mathrm{d}t\right| \\
&\lesssim \|u_n\|_{L^2((0,T)\times\Omega)}\left(\left\|\left(-\Delta+\beta\right)^{-1}(u_n)\right\|_{L^2((0,T)\times\Omega)} + \left\|\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\right\|_{L^2((0,T)\times\Omega)}\right) \\
&\lesssim \|u_n\|_{L^2((0,T)\times\Omega)}\left(\left\|\left(-\Delta+\beta\right)^{-1}(u_n)\right\|_{L^\infty((0,T),H_0^1)} + \left\|\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\right\|_{L^\infty((0,T),H_0^1)}\right) \\
&\lesssim \|u_n\|_{L^2((0,T)\times\Omega)}\left(\|u_n\|_{L^\infty((0,T),H^{-1})} + \|\partial_t u_n\|_{L^\infty((0,T),H^{-1})}\right) \\
&\lesssim \|u_n\|_{L^2((0,T)\times\Omega)},
\end{aligned}$$

implying that the first and second terms of (6.C.8) converge to zero as $n \to +\infty$. In particular, (6.C.8) gives

$$\int_0^T \int_\Omega \partial_t u_n \phi^2\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\mathrm{d}x\mathrm{d}t \xrightarrow{n\to\infty} 0.$$

One has

$$\begin{aligned}
\|\phi\partial_t u_n\|^2_{L^2((0,T),H^{-1})} &\lesssim \left\|\phi\left(-\Delta+\beta\right)^{-\frac{1}{2}}(\partial_t u_n)\right\|^2_{L^2((0,T)\times\Omega)} \\
&= \int_0^T \int_\Omega \partial_t u_n \phi^2\left(-\Delta+\beta\right)^{-1}(\partial_t u_n)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

implying (6.C.7).

Now, we complete the proof of (6.C.4). Fatou's lemma, together with (6.C.6) and (6.C.7), gives

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \liminf_{n\to\infty}\left(\|\partial_t u_n(t)\|_{H^{-1}(\Omega)} + \|u_n(t)\|_{L^2(\Omega)}\right) = 0.$$

In particular, for almost all $t \in \left(\frac{T}{4}, \frac{3T}{4}\right)$, one has

$$\|\partial_t u_n(t)\|_{H^{-1}(\Omega)} + \|u_n(t)\|_{L^2(\Omega)} \xrightarrow{n\to\infty} 0. \tag{6.C.9}$$

Let t_0 be such that (6.C.9) holds true for t_0. There exists a constant independent of $\left((u^0,u^1)\right)_n$ such that

$$1 = \|(u_n(0), \partial_t u_n(0))\|_{L^2(\Omega)\times H^{-1}(\Omega)} \lesssim \|(u_n(t_0), \partial_t u_n(t_0))\|_{L^2(\Omega)\times H^{-1}(\Omega)}\,, \quad n \in \mathbb{N},$$

and that is a contradiction.

Step 2 : removing the compact term. If a solution u of (6.C.3) with initial data $\left(u^0,u^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ satisfies $au = 0$, then our assumption on a implies that $u = 0$ on $\mathbb{R}^d \backslash B(0,R_0)$: in particular, u is the solution of (6.C.3) on a bounded domain, and $u = 0$ on an open set ω such that (ω, T) satisfies the GCC, implying that $\left(u^0,u^1\right) = 0$ (see for example [BLR92]). This proves that the operator

$$\begin{array}{ccc} L^2(\Omega) \times H^{-1}(\Omega) & \longrightarrow & L^2((0,T)\times\Omega) \\ (u^0,u^1) & \longmapsto & au \end{array}$$

is one-to-one. By Lemma 6.2.4, the operator

$$\begin{array}{ccc} L^2(\Omega) \times H^{-1}(\Omega) & \longrightarrow & H^{-1}(\Omega)\times H^{-2}(\Omega) \\ (u^0,u^1) & \longmapsto & (Vu^0, Vu^1) \end{array}$$

is compact. By Lemma 3.3.1, there exists a constant such that (6.C.4) holds true without the additional term on the right-hand side. This completes the proof of Theorem 6.2.5.

Bibliography

[AF03] R. A. Adams and J. F. Fournier. *Sobolev spaces.* English. 2nd ed. Vol. 140. Pure Appl. Math., Academic Press. New York, NY: Academic Press, 2003. ISBN: 0-12-044143-8.

[AIN10] L. Aloui, S. Ibrahim, and K. Nakanishi. “Exponential Energy Decay for Damped Klein- Gordon Equation with Nonlinearities of Arbitrary Growth”. In: *Communications in Partial Differential Equations* 36.5 (2010), pp. 797–818. DOI: 10.1080/03605302.2010.534684.

[BG97] N. Burq and P. Gérard. “A necessary and sufficient condition for the exact controllability of the wave equation”. French. In: *C. R. Acad. Sci., Paris, Sér. I, Math.* 325.7 (1997), pp. 749–752. ISSN: 0764-4442. DOI: 10.1016/S0764-4442(97)80053-5.

[BL01] N. Burq and G. Lebeau. “Microlocal defect measures, application to the Lamé system”. French. In: *Ann. Sci. Éc. Norm. Supér. (4)* 34.6 (2001), pp. 817–870. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(01)01078-3.

[BL76] J. Bergh and J. Löfström. *Interpolation Spaces: An Introduction.* Springer Berlin Heidelberg, 1976. ISBN: 9783642664519. DOI: 10.1007/978-3-642-66451-9. URL: http://dx.doi.org/10.1007/978-3-642-66451-9.

[BLP08] N. Burq, G. Lebeau, and F. Planchon. “Global existence for energy critical waves in 3-D domains”. English. In: *J. Am. Math. Soc.* 21.3 (2008), pp. 831–845. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-08-00596-1.

[BLR92] C. Bardos, G. Lebeau, and J. Rauch. “Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary”. In: *SIAM J. Control Optim.* 30.5 (1992), pp. 1024–1065. DOI: 10.1137/0330055.

[Bre84] P. Brenner. “On space-time means and everywhere defined scattering operators for nonlinear Klein-Gordon equations”. In: *Math. Z.* 186 (1984), pp. 383–391. ISSN: 0025-5874. DOI: 10.1007/BF01174891.

[BRS17] N. Burq, G. Raugel, and W. Schlag. “Long time dynamics for damped Klein-Gordon equations”. In: *Annales scientifiques de l’École normale supérieure* 50.6 (2017), pp. 1447–1498. DOI: 10.24033/asens.2349.

[BRS18] N. Burq, G. Raugel, and W. Schlag. *Long time dynamics for weakly damped nonlinear Klein-Gordon equations.* 2018. URL: https://arxiv.org/abs/1801.06735.

[BSS09] M. D. Blair, H. F. Smith, and C. D. Sogge. “Strichartz estimates for the wave equation on manifolds with boundary”. English. In: *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 26.5 (2009), pp. 1817–1829. ISSN: 0294-1449. DOI: 10.1016/j.anihpc.2008.12.004.

[Bur] N. Burq. Personnal communication.

[Bur03] N. Burq. “Global Strichartz Estimates for Nontrapping Geometries: About an Article by H. Smith and C. Sogge”. In: *Communications in Partial Differential Equations* 28.9-10 (2003), pp. 1675–1683. DOI: 10.1081/PDE-120024528.

[Bur97a] N. Burq. “Exact controlability of waves in somewhat regular domains”. French. In: *Asymptotic Anal.* 14.2 (1997), pp. 157–191. ISSN: 0921-7134.

[Bur97b] N. Burq. “Semi-classical measures and defect measures”. French. In: *Séminaire Bourbaki. Volume 1996/97. Exposés 820–834*. Paris: Société Mathématique de France, 1997, 167–195, ex.

[Caz85] T. Cazenave. “Uniform estimates for solutions of nonlinear Klein-Gordon equations”. In: *Journal of Functional Analysis* 60 (1985), pp. 36–55. URL: https://api.semanticscholar.org/CorpusID:121770305.

[CK01] M. Christ and A. Kiselev. “Maximal Functions Associated to Filtrations”. In: *Journal of Functional Analysis* 179.2 (2001), pp. 409–425. ISSN: 0022-1236. DOI: https://doi.org/10.1006/jfan.2000.3687.

[CLW20] Y. Cui, C. Laurent, and Z. Wang. “On the observability inequality of coupled wave equations: the case without boundary”. In: *ESAIM: COCV* 26 (2020), p. 14. DOI: 10.1051/cocv/2020004.

[Cof72] C. V. Coffman. “Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions”. English. In: *Arch. Ration. Mech. Anal.* 46 (1972), pp. 81–95. ISSN: 0003-9527. DOI: 10.1007/BF00250684.

[DL09] B. Dehman and G. Lebeau. “Analysis of the HUM Control Operator and Exact Controllability for Semilinear Waves in Uniform Time”. In: *SIAM J. Control and Optimization* 48 (Jan. 2009), pp. 521–550. DOI: 10.1137/070712067.

[DLZ03] B. Dehman, G. Lebeau, and E. Zuazua. “Stabilization and control for the subcritical semilinear wave equation”. In: *Annales scientifiques de l’École Normale Supérieure* Ser. 4, 36.4 (2003), pp. 525–551. DOI: 10.1016/S0012-9593(03)00021-1.

[DM08] T. Duyckaerts and F. Merle. “Dynamics of Threshold Solutions for Energy-Critical Wave Equation”. In: *International Mathematics Research Papers* 2008 (Jan. 2008), rpn002. ISSN: 1687-3017. DOI: 10.1093/imrp/rpn002.

[DR77] S. Dolecki and D. L. Russell. “A General Theory of Observation and Control”. In: *SIAM Journal on Control and Optimization* 15.2 (1977), pp. 185–220. DOI: 10.1137/0315015.

[DS11] R. Donninger and W. Schlag. “Numerical study of the blowup/global existence dichotomy for the focusing cubic nonlinear Klein–Gordon equation”. In: *Nonlinearity* 24.9 (2011), p. 2547. DOI: 10.1088/0951-7715/24/9/009. URL: https://dx.doi.org/10.1088/0951-7715/24/9/009.

[EZ10] S. Ervedoza and E. Zuazua. “A Systematic method for building smooth controls for smooth data”. In: *Discrete and Continuous Dynamical Systems. Series B* 4 (Nov. 2010). DOI: 10.3934/dcdsb.2010.14.1375.

[Fei98] F. Feireisl. “Finite Energy Travelling Waves for Nonlinear Damped Wave Equations”. In: *Quarterly of Applied Mathematics* 56.1 (1998), pp. 55–70. ISSN: 0033569X, 15524485. URL: http://www.jstor.org/stable/43638210.

[Gér91] P. Gérard. “Microlocal defect measures”. English. In: *Commun. Partial Differ. Equations* 16.11 (1991), pp. 1761–1794. ISSN: 0360-5302. DOI: 10.1080/03605309108820822.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin Heidelberg, 2001. DOI: 10.1007/978-3-642-61798-0.

[GV89] J. Ginibre and G. Velo. "The global Cauchy problem for the nonlinear Klein-Gordon equation. II". English. In: *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 6.1 (1989), pp. 15–35. ISSN: 0294-1449. DOI: 10.1016/S0294-1449(16)30329-8.

[Hör07] L. Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Pseudo-differential operators, Reprint of the 1994 edition. Springer, Berlin, 2007.

[HR03] J. K. Hale and G. Raugel. "Regularity, determining modes and Galerkin methods". In: *J. Math. Pures Appl. (9)* 82.9 (2003), pp. 1075–1136. DOI: 10.1016/S0021-7824(03)00045-X.

[IMN11] S. Ibrahim, N. Masmoudi, and K. Nakanishi. "Scattering threshold for the focusing nonlinear Klein–Gordon equation". In: *Analysis & PDE* 4.3 (Dec. 2011), pp. 405–460. DOI: 10.2140/apde.2011.4.405. URL: https://doi.org/10.2140/apde.2011.4.405.

[Iva12] O. Ivanovici. "Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary". In: *Journal of the European Mathematical Society* 014.5 (2012), pp. 1357–1388.

[JL13] R. Joly and C. Laurent. "Stabilization for the semilinear wave equation with geometric control condition". In: *Analysis & PDE* 6.5 (2013), pp. 1089–1119. DOI: 10.2140/apde.2013.6.1089.

[JL14] R. Joly and C. Laurent. "A note on the semiglobal controllability of the semilinear wave equation". English. In: *SIAM J. Control Optim.* 52.1 (2014), pp. 439–450. ISSN: 0363-0129. DOI: 10.1137/120891174.

[Kap90] L. V. Kapitanski. "Some generalizations of the Strichartz-Brenner inequality". In: *Leningr. Math. J.* 1.3 (1990), pp. 693–726. ISSN: 1048-9924.

[KM08] C. E. Kenig and F. Merle. "Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation". In: *Acta Mathematica* 201.2 (2008), pp. 147–212. ISSN: 0001-5962. DOI: 10.1007/s11511-008-0031-6. URL: http://dx.doi.org/10.1007/s11511-008-0031-6.

[KT98] M. Keel and T. Tao. "Endpoint Strichartz Estimates". In: *American Journal of Mathematics* 120.5 (1998), pp. 955–980.

[KX20] J. Krieger and S. Xiang. "Boundary Stabilization of Focusing NLKG near Unstable Equilibria: Radial Case". In: *Pure and Applied Analysis* (2020). 2010 Mathematics Subject Classification. 35B34, 35B37, 35B40. URL: http://infoscience.epfl.ch/record/277137.

[Lau10] C. Laurent. "Global Controllability and Stabilization for the Nonlinear Schrödinger Equation on Some Compact Manifolds of Dimension 3". In: *SIAM Journal on Mathematical Analysis* 42.2 (2010), pp. 785–832. DOI: 10.1137/090749086.

[Lio88] J-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*. Vol. 1 et 2. Paris: Masson, RMA, 1988.

[LL15] C. Laurent and M. Léautaud. "Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves". In: *Journal of the European Mathematical Society* 21 (June 2015). DOI: 10.4171/JEMS/854.

[LL16] C. Laurent and M. Léautaud. "Uniform observability estimates for linear waves". In: *ESAIM: Control, Optimisation and Calculus of Variations* 22.4 (2016). DOI: 10.1051/cocv/2016046.

[LLT86] I. Lasiecka, J-L. Lions, and R. Triggiani. "Non homogeneous boundary value problems for second order hyperbolic operators". In: *Journal de Mathématiques Pures et Appliquées* 65 (1986), pp. 149–192.

[LS95] H. Lindblad and C. D. Sogge. "On Existence and Scattering with Minimal Regularity for Semilinear Wave Equations". In: *Journal of Functional Analysis - J FUNCT ANAL* 130 (June 1995), pp. 357–426. DOI: 10.1006/jfan.1995.1075.

[LT05] I. Lasiecka and R. Triggiani. "Global exact controllability of semilinear wave equations by a double compactness/uniqueness argument". In: *Conference Publications* 2005.Special (2005), pp. 556–565. ISSN: 0133-0189. DOI: 10.3934/proc.2005.2005.556.

[LT91] I. Lasiecka and R. Triggiani. "Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems". In: *Applied Mathematics Optimization* 23.1 (1991), pp. 109–154. ISSN: 1432-0606. DOI: 10.1007/bf01442394.

[Mel79] R. B. Melrose. "Singularities and energy decay in acoustical scattering". In: *Duke Mathematical Journal* 46.1 (1979), pp. 43–59. DOI: 10.1215/S0012-7094-79-04604-0. URL: https://doi.org/10.1215/S0012-7094-79-04604-0.

[MRS77] C. S. Morawetz, J. V. Ralston, and W. A. Strauss. "Decay of solutions of the wave equation outside nontrapping obstacles". In: *Communications on Pure and Applied Mathematics* 30.4 (1977), pp. 447–508. DOI: https://doi.org/10.1002/cpa.3160300405.

[MS78] R. B. Melrose and J. Sjöstrand. "Singularities of boundary value problems. I". In: *Communications on Pure and Applied Mathematics* 31.5 (1978), pp. 593–617. DOI: https://doi.org/10.1002/cpa.3160310504. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160310504. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.3160310504.

[MS82] R. B. Melrose and J. Sjöstrand. "Singularities of boundary value problems. II". In: *Communications on Pure and Applied Mathematics* 35.2 (1982), pp. 129–168. DOI: https://doi.org/10.1002/cpa.3160350202. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/cpa.3160350202. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/cpa.3160350202.

[MSS93] G. Mockenhaupt, A. Seeger, and C. D. Sogge. "Local Smoothing of Fourier Integral Operators and Carleson-Sjolin Estimates". In: *Journal of The American Mathematical Society - J AMER MATH SOC* 6 (Jan. 1993), pp. 65–65. DOI: 10.2307/2152795.

[Nak01] K. Nakanishi. "Remarks on the energy scattering for nonlinear Klein-Gordon and Schrödinger equations". In: *Tôhoku Math. J. (2)* 53.2 (2001), pp. 285–303. ISSN: 0040-8735. DOI: 10.2748/tmj/1178207482.

[NS11a] K. Nakanishi and W. Schlag. "Global dynamics above the ground state energy for the focusing nonlinear Klein–Gordon equation". In: *Journal of Differential Equations* 250.5 (2011), pp. 2299–2333. ISSN: 0022-0396. DOI: https://doi.org/10.1016/j.jde.2010.10.027. URL: https://www.sciencedirect.com/science/article/pii/S0022039610004213.

[NS11b] K. Nakanishi and W. Schlag. *Invariant manifolds and dispersive Hamiltonian evolution equations*. English. Zur. Lect. Adv. Math. Zürich: European Mathematical Society (EMS), 2011. ISBN: 978-3-03719-095-1. DOI: 10.4171/095.

[NS12] K. Nakanishi and W. Schlag. "Global dynamics above the ground state for the nonlinear Klein–Gordon equation without a radial assumption". In: *Arch. Ration. Mech. Anal.* 203.3 (2012), pp. 809–851. DOI: 10.1007/s00205-011-0462-7.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer New York, 1983. DOI: 10.1007/978-1-4612-5561-1. URL: https://doi.org/10.1007/978-1-4612-5561-1.

[Per23a] T. Perrin. *Change of regularity in controllability and observability of systems of wave equations*. 2023. arXiv: 2312.06311 [math.AP]. URL: arxiv.org/abs/2312.06311.

[Per23b] T. Perrin. *Local controllability around a regular solution and null-controllability of scattering solutions for semilinear wave equations*. 2023. arXiv: 2312.06373 [math.AP]. URL: arxiv.org/abs/2312.06373.

[Per23c] T. Perrin. *The damped focusing cubic wave equation on a bounded domain*. 2023. arXiv: 2310.12644 [math.AP].

[Per24] T. Perrin. *Uniform estimates for solutions of nonlinear focusing damped wave equations*. 2024. arXiv: 2403.06541 [math.AP].

[PS75] L. E. Payne and D. H. Sattinger. "Saddle points and instability of nonlinear hyperbolic equations". In: *Israel Journal of Mathematics* 22 (1975), pp. 273–303. DOI: 10.1007/BF02761595.

[RT74] J. Rauch and M. Taylor. "Exponential decay of solutions to hyperbolic equations in bounded domains". In: *Indiana University Mathematics Journal* 24.1 (1974), pp. 79–86. URL: http://www.jstor.org/stable/24890762.

[RZ09] L. Rosier and B-Y. Zhang. "Exact boundary controllability of the nonlinear Schrödinger equation". In: *Journal of Differential Equations* 246.10 (2009), pp. 4129–4153. ISSN: 0022-0396. DOI: https://doi.org/10.1016/j.jde.2008.11.004.

[RZ98] L. Robbiano and C. Zuily. "Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients". In: *Invent. Math.* 131.3 (1998), pp. 493–539. DOI: 10.1007/s002220050212.

[SS00] H. F. Smith and C. D. Sogge. "Global strichartz estimates for nonthapping perturbations of the laplacian". In: *Communications in Partial Differential Equations* 25.11-12 (2000), pp. 2171–2183. DOI: 10.1080/03605300008821581.

[Str77] R. S. Strichartz. "Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations". In: *Duke Mathematical Journal* 44.3 (1977), pp. 705–714.

[Tar90] L. Tartar. "H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations". English. In: *Proc. R. Soc. Edinb., Sect. A, Math.* 115.3-4 (1990), pp. 193–230. ISSN: 0308-2105. DOI: 10.1017/S0308210500020606.

[Tat02] D. Tataru. "Strichartz Estimates for Second Order Hyperbolic Operators with Nonsmooth Coefficients III". In: *Journal of the American Mathematical Society* 15.2 (2002), pp. 419–442.

[Ton19] B. A. Ton. “Exact controllability of a strongly nonlinear wave equation with boundary controls”. English. In: *Adv. Math. Sci. Appl.* 28.2 (2019), pp. 213–234. ISSN: 1343-4373.

[Vra03] I. I. Vrabie. *C_0-semigroups and applications*. English. Vol. 191. North-Holland Math. Stud. Amsterdam: North-Holland, 2003. ISBN: 0-444-51288-8.

[ZL07] Y. Zhou and Z. Lei. “Local Exact Boundary Controllability for Nonlinear Wave Equations”. In: *SIAM Journal on Control and Optimization* 46.3 (2007), pp. 1022–1051. DOI: 10.1137/060650222.

[Zua90] E. Zuazua. “Exact controllability for the semilinear wave equation”. English. In: *J. Math. Pures Appl. (9)* 69.1 (1990), pp. 1–31. ISSN: 0021-7824.