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Quelques résultats sur des champs des (φ, Γ)-modules

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Résumé

Cette thèse est composée de trois parties essentiellement indépendantes, obtenues dans les articles [Pha22], [Pha23a], [Pha23b] de l'auteur. Dans la première partie, nous donnons une nouvelle preuve plus simple de la description explicite de la champ des (φ, Γ)-modules de rang un d'Emerton–Gee. Dans la deuxième partie, nous définissons et étudions des champs paramétrant les (φ, Γ)-modules de Lubin–Tate. En particulier, nous les comparons avec les champs des (φ, Γ)-modules cyclotomiques d'Emerton–Gee. En conséquence, nous déduisons la perfection du complexe de Herr dans le cadre Lubin–Tate. Dans la troisième partie, nous étendons partiellement l'équivalence de Bhatt–Scholze entre les F-cristaux prismatiques et les réseaux dans les représentations cristallines de Galois au "contexte de Lubin–Tate". En cours de route, nous prouvons un résultat général sur la pleine fidélité d'un foncteur de changement de base sur certains fibrés vectoriels sur le site prismatique, ce qui simplifie et raffine l'étape clé dans l'approche de Bhatt–Scholze sans invoquer la séquence de fibres de Beilinson.

Mots clés : rang un, (φ, Γ)-modules, champs, Lubin–Tate, F-cristaux prismatiques, représentations Galoisiennes.

Abstract

This thesis is composed of three essentially independent parts, obtained in the articles [Pha22], [Pha23a], [Pha23b] by the author. In the first part, we give a new and simpler proof of the explicit description of the Emerton–Gee stack of rank one (φ, Γ)-modules. In the second part, we define and study stacks parametrizing Lubin–Tate (φ, Γ)-modules. In particular, we compare these with the Emerton–Gee stacks of cyclotomic (φ, Γ)-modules. As a consequence, we deduce perfectness of the Herr complex in the Lubin–Tate setting. In the third part, we extend partially the equivalence of Bhatt–Scholze between prismatic F-crystals and lattices in crystalline Galois representations to the "Lubin–Tate context". Along the way, we prove a general full faithfulness result for certain vector bundles on the prismatic site, which simplifies and refines the key descent step in the approach of Bhatt–Scholze without invoking the Beilinson fibre sequence.

Keywords: rank one, (φ, Γ)-modules, moduli stacks, Lubin–Tate, prismatic F-crystals, Galois representations.

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The mathematical debt that this thesis owes to the work of Emerton and Gee [EG23] will be obvious to the reader, and I would like to thank them for their pioneering and stimulating work, as well as for helpful discussions related to Chapter 5 of *loc. cit.*

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Contents

Chapter 0

General introduction

0.1 Context

Galois representations and p-adic Hodge theory

Traditionally, number theory is the branch of mathematics that studies questions related to the integers. A typical problem in number theory is to determine the set of integers satisfying a system of polynomial equations with rational coefficients. To such a system of equations, one can associate a geometric object, called an algebraic variety, over the field of rational numbers $\mathbf{Q}$. This change of perspective is particularly fruitful as it allows the introduction of tools and intuitions from geometry to the study of algebraic equations. Thanks to the work of Grothendieck and his school, one can then extract from such an algebraic variety X certain algebraic invariant known as its étale cohomology groups $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$, one for each prime p. By general results, it is known that for $X/\mathbf{Q}$ proper these groups are finite dimensional $\mathbf{Q}_p$-vector spaces equipped with a continuous linear action of $G_{\mathbf{Q}}$, the absolute Galois group of $\mathbf{Q}$. This is an example of a p-adic Galois representation. Furthermore, the operation $X \mapsto H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ is not too drastic in the sense that very often many geometric properties of the former are already encoded in the latter. As an example, the Néron–Ogg–Shafarevich criterion allows one to read off the reduction type of an elliptic curve over $\mathbf{Q}$ at a prime p from its associated Galois representations.

Thus, in this optic, modern (algebraic) number theory can be essentially regarded as the study of p-adic representations of the Galois group $G_{\mathbf{Q}}$ of $\mathbf{Q}$, or more generally of a number field. Moreover, as should be clear from our preceding discussion, it is important to understand the representations coming from geometry, which is to say, those arising as a subquotient of $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ for some X and i. Now the representation theory of $G_{\mathbf{Q}}$ is most interesting when one regards it not merely as an abstract (topological) group but as a group equipped with additional local structure at each prime number ℓ. More precisely, for each ℓ, the choice of an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$ gives rise to a closed embedding $G_{\mathbf{Q}_\ell} \hookrightarrow G_{\mathbf{Q}}$ which is determined up to conjugation. In particular, given a p-adic representation of $G_{\mathbf{Q}}$, one obtains by restriction, a p-adic representation of $G_{\mathbf{Q}_\ell}$ for each ℓ. Conversely, in view of the Brauer–Nesbitt theorem, a representation of $G_{\mathbf{Q}}$ coming from geometry is uniquely determined by its various restrictions to $G_{\mathbf{Q}_\ell}$. Thus, we may reduce our task to study p-adic representations of the local Galois groups $G_{\mathbf{Q}_\ell}$ which are structurally much simpler than

$G_{\mathbf{Q}}$.

Now if $\ell \neq p$, then because of the incompatibility between the ℓ-adic and the p-adic topologies, the resulting category of representations of $G_{\mathbf{Q}_\ell}$ is relatively simple. In fact, Grothendieck's monodromy theorem states that any such representation is automatically "potentially unipotent". On the other hand, if $\ell = p$, i.e. if one is interested in p-adic representations of $G_{\mathbf{Q}_p}$, then the situation is much more complicated. The main difference is that now wild inertia can act in a highly nontrivial fashion, and there is no naive analogue of Grothendieck's result (although see Theorem 0.1.2 below). In this context, an approach introduced by Fontaine, which has proved to be very successful, is to consider certain topological $\mathbf{Q}_p$-algebras B equipped with an action of $G_{\mathbf{Q}_p}$ and certain additional structures preserved by this action (e.g. a filtration, a Frobenius or a monodromy operator). For such algebra B, Fontaine then indicated how to single out a particular class of representations which are called B-admissible. By definition, a representation V of $G_{\mathbf{Q}_p}$ is called B-admissible if it becomes trivial (as a B-semilinear representation) after extending scalars along $\mathbf{Q}_p \to B$. In this case, the 'Dieudonnée module' $D_B(V) := (V \otimes_{\mathbf{Q}_p} B)^{G_{\mathbf{Q}_p}}$ is naturally a module over the invariant subring $B^{G_{\mathbf{Q}_p}}$ and is equipped with additional structures coming from B. If the structures imposed on B are sufficiently fine, then one can hope to recover V from the linear algebraic object $D_B(V)$.

The key part of the theory is therefore to define such algebras B. Most important examples of these so-called period rings include $B_{\mathrm{dR}}, B_{\mathrm{cris}}$ and B_{st}. The resulting subcategories of admissible representations are called de Rham, crystalline, and semistable respectively. For a detailed account of the construction of these rings, we refer the reader to [BC09]. Here we only mention that B_{dR} is a complete discrete valuation ring with residue field $C_p := \widehat{\overline{\mathbf{Q}}}_p$ equipped with its maximal adic filtration; B_{st} is endowed with a Frobenius φ and a monodromy operator N satisfying the relation $N\varphi = p\varphi N$; and finally B_{cris} can be recovered as $(B_{\mathrm{st}})^{N=0}$. The relation between these classes of representations is given by the following hierachy:

$$\{\text{crystalline}\} \quad \subsetneq \quad \{\text{semistable}\} \quad \subsetneq \quad \{\text{de Rham}\}.$$

The following famous result of Faltings (see e.g. [Ill90]) in particular provides a natural source of de Rham representations:

Theorem 0.1.1 (Faltings)**.** *Let X be a proper smooth scheme over $\mathbf{Q}_p$. Then for any i, there is a canonical isomorphism*

$$H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}} \simeq H^i_{\mathrm{dR}}(X/\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}}$$

respecting all structures. In particular each $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ is a de Rham representation with $D_{\mathrm{dR}}(H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)) \simeq H^i_{\mathrm{dR}}(X/\mathbf{Q}_p)$.

In the same way, the property of being crystalline (resp. semistable) is meant to capture those representations arising from p-adic étale cohomology of varieties over p-adic fields with good (resp. semistable) reduction.

As mentioned above, there is no simple analogue of Grothendieck's monodromy theorem in the p-adic setting. However, if one restricts to the subclass of de Rham representations, then one has the following celebrated result thanks to the work of André, Berger, Kedlaya and Mebkhout. See Colmez's Bourbaki talk [Col03] for an excellent account on this topic.

Theorem 0.1.2 (p-adic local monodromy theorem)**.** *Any de Rham representation is potentially semistable.*

Combining with our previous discussion, the above theorem can be thought of as a Galois-theoretic shadow of the semistable reduction theorem for abelian varieties.

After Theorem 0.1.1, we see in particular that for a smooth projective variety $X/\mathbf{Q}$, the $G_{\mathbf{Q}}$-representation $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ is de Rham when restricted to the Galois group at p. Moreover, as X is easily seen to have good reduction outside finitely many primes, standard results in étale cohomology imply that the $G_{\mathbf{Q}}$-action on $H^i_{\text{ét}}(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)$ is unramified almost everywhere. At this point, we cannot resist the pleasure of stating the following beautiful conjecture of Fontaine and Mazur, which says that these are in fact the only obstructions for a *global* Galois representation to come from geometry.

Conjecture 0.1.3 (Fontaine–Mazur)**.** Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a continuous, absolutely irreducible representation. Then ρ 'comes from geometry' if and only if it is (i) unramified almost everywhere, and (ii) de Rham at p.

Fontaine's theory of (φ, Γ)-modules

Let $K/\mathbf{Q}_p$ be a finite extension. There is another (but related) approach, again introduced by Fontaine, which aims to describe the category of *all* p-adic representations of G_K. In this approach, the basic idea is to consider "deeply ramified" subextensions $K \subseteq K_\infty \subseteq \overline{K}$ which encodes interesting ramifications of $\overline{K}/K$, yet whose Galois group $\mathrm{Gal}(K_\infty/K)$ is simply enough to control. The difficult ramified part $\overline{K}/K_\infty$ is then subsumed into certain complicated coefficient rings, which will then also remember the "easy" Galois action coming from K_∞/K. Thus, the key part of the theory once again lies in the construction of these coefficient rings! When K_∞/K is the cyclotomic extension, this idea was realized by Fontaine in [Fon90] via his theory of étale (φ, Γ)-modules. Before recalling his result, we need to introduce some notation. For simplicity assume $K/\mathbf{Q}_p$ is unramified. Consider the ring $\mathbf{A}_K := \widehat{W(k)((T))}$ where the hat denotes the p-adic completion. This ring is endowed with two commuting actions of φ and $\Gamma := \mathrm{Gal}(K(\zeta_{p^\infty})/K) \simeq \mathbf{Z}_p^\times$ given respectively by $\varphi(T) = (1+T)^p - 1$ and $\gamma(T) = (1+T)^\gamma - 1$ for $\gamma \in \Gamma$. Then by definition, an étale (φ, Γ)-module (with $\mathbf{Z}_p$-coefficients) is a finite $\mathbf{A}_K$-module endowed with two semilinear commuting actions φ and γ with the property that the linearization of φ is an isomorphism. The following classical result of Fontaine underlies the relevance of the notion of (φ, Γ)-modules in the study of p-adic local Galois representations.

Theorem 0.1.4 (Fontaine)**.** *There is a natural equivalence between the category of étale (φ, Γ)-modules and the category of representations of G_K on finite $\mathbf{Z}_p$-modules.*

Moduli spaces of Galois representations

Putting objects in families has proved to be a useful technique in algebraic geometry for a long time. Partially inspired by Hida's theory of ordinary families of p-adic modular forms, the theory of Galois deformation was initiated by Mazur in the late 1980s. The work of Mazur can be thought of as the first systematic introduction of moduli spaces to the study of Galois representations. Given a mod p representation $\overline{\rho} : G_K \to \mathrm{GL}_d(\mathbf{F}_p)$, Mazur proposed to consider the space of lifts of $\overline{\rho}$ to local Artinian rings with residue field $\mathbf{F}_p$. Under certain rigidity condition on $\overline{\rho}$, Mazur showed that the space of such lifts is represented by a formal scheme $\mathrm{Spf}(R_{\overline{\rho}})$ for some complete Noetherian local ring $R_{\overline{\rho}}$. The study of such lifts is very important in the Langlands program; for instance, after the work of Wiles ([Wil95]), it became clear that for the purpose of proving modularity lifting theorems it was important to understand loci in $\mathrm{Spf}(R_{\overline{\rho}})$ cut out by various p-adic Hodge theoretic conditions.

Given the work of Mazur, it is natural to ask if one can construct a more global moduli space of p-adic Galois representations in which the residual representation $\overline{\rho}$ is allowed to vary. While one can naively define a stack parametrizing literal Galois representations $G_K \to \mathrm{GL}_d(A)$ for varying p-complete test rings A (cf. the work [Wan18]), the resulting stack is not truly global in the sense that the families of mod p Galois representations appearing on each of its connected components will have constant semisimplification. In a recent advance ([EG23]), Emerton and Gee have realized how to overcome this problem, and thereby obtaining the desired globalization of Mazur's deformation rings. The key insight of Emerton–Gee is to work instead with families of étale (φ, Γ)-modules in the sense of Fontaine. More precisely, for each $d \geq 1$, the Emerton–Gee stack $\mathcal{X}_{K,d}$ is by definition the functor defined on p-complete algebras, taking such ring A to the groupoid of rank d projective étale (φ, Γ)-modules with A-coefficients. Here the category of étale (φ, Γ)-modules with A-coefficients is defined in exactly the same way as before except that the coefficient ring $\mathbf{A}_K$ is now replaced by the completed tensor product $\mathbf{A}_{K,A} := \mathbf{A}_K \widehat{\otimes}_{\mathbf{Z}_p} A$.

The following summarizes the main geometric properties of the Emerton–Gee stacks.

Theorem 0.1.5 ([EG23, Thm. 1.2.1])**.** *$\mathcal{X}_{K,d}$ is a Noetherian formal algebraic stack over $\mathbf{Z}_p$. The underlying reduced substack $(\mathcal{X}_{K,d})_{\mathrm{red}}$ is an algebraic stack of finite type over $\mathbf{F}_p$, which is equidimensional of dimension $[K : \mathbf{Q}_p]d(d-1)/2$. Moreover, the irreducible components of $(\mathcal{X}_{K,d})_{\mathrm{red}}$ admits a natural labeling by Serre weights of GL_d.*

Using the preceding correspondence between étale (φ, Γ)-modules and Galois representations, one sees easily that these stacks admit universal lifting rings as versal rings at finite type points, and thus, as mentioned above, can be thought of as an algebraization of Mazur's formal Galois deformation rings. The existence of such stacks therefore opens up the possibility of employing (global) geometric techniques in the study of p-adic Galois representations and their deformations. As a first application, Emerton and Gee gave the first (and so far the only) proof of the existence of crystalline lifts of a mod p Galois representation. The idea of using the global nature of the Emerton–Gee stacks to obtain pointwise information by interpolation from "easy" points has subsequently also been exploited in some other problems. For instance, in [LLLM23] the authors proved a generic case of the geometric Breuil–Mézard conjecture by spreading the result from the tame case, using crucially the preceding mechanism.

The fact that the Emerton–Gee stack is defined in terms of (φ, Γ)-modules also makes it germane to the p-adic local Langlands program. Indeed, at the heart of the construction of the correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ is Colmez's Montréal functor, which produces a (φ, Γ)-modules out of a representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. It has also become more clear in recent years that it is the Emerton–

Gee stack $\mathcal{X}_d^{\mathrm{EG}}$, rather than its substack of literal Galois representations, that is the natural geometric object for the emerging categorical p-adic Langlands program.

Towards extending Colmez's work beyond $\mathrm{GL}_2(\mathbf{Q}_p)$, there have been now several proposed generalizations of (φ, Γ)-modules, including the Lubin–Tate (φ, Γ)-modules, the multivariable (φ, Γ)-modules of Carter–Kedlaya–Zábrádi ([CKZ21]), and the recent construction of [BHHMS23], which can be seen roughly as a hybrid of the previous two. The precise relation between these notions is still not yet clear, as predicted by the very mysterious nature of the p-adic Langlands correspondence outside the case $\mathrm{GL}_2(\mathbf{Q}_p)$. The theme of my thesis is to study these various flavors of families of (φ, Γ)-modules, establish their geometric properties as well as the relation between them.

0.2 Organization of the thesis

We now discuss in more detail the content of this thesis. As the main results in each chapter are essentially independent of each other, the title of each subsection below is simply taken to be that of the corresponding chapter. Furthermore, for convenience of the reader, each chapter will begin with essentially a reprise of the introductions below.

0.2.1 The Emerton–Gee stack of rank one (φ, Γ)-modules

In the first chapter, we give a new and simpler proof of the following explicit description of the Emerton–Gee stack of rank one étale (φ, Γ)-modules, first obtained in [EG23].

Theorem 0.2.1 ([EG23, Prop. 7.2.17])**.** *There is an isomorphism*

$$\left[\left(\mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbf{G}}_m\right) / \widehat{\mathbf{G}}_m\right] \xrightarrow{\sim} \mathcal{X}_{K,1},$$

where, in the formation of the quotient stack, the action of $\widehat{\mathbf{G}}_m$ is taken to be trivial.

We remark that after choosing an isomorphism $W_K^{\mathrm{ab}} \simeq I_K^{\mathrm{ab}} \times \mathbf{Z}$, this result shows in particular that $\mathcal{X}_{K,1}$ can be regarded as the moduli stack of continuous characters of the Weil group W_K of K. This again illustrates the difference between the Emerton–Gee stack $\mathcal{X}_1$ and its substack parametrizing literal Galois representations: while the universal unramified character does not correspond to a G_K-representation, it does give rise to a representation of the Weil group.

We now explain our approach to the above theorem. To begin with, we need to construct a morphism between the two given stacks, which we will then show is an isomorphism. This amounts to extending Fontaine's construction of rank one étale (φ, Γ)-modules from Galois characters to the case of an arbitrary p-complete test ring, which in turn reduces to the construction of the universal unramified character. We refer the reader to Section 1.2 below for more detail. Now in order to show that the resulting map is indeed an isomorphism, Emerton and Gee proved a general criterion for a morphism of stacks to be an isomorphism, and then showed that it indeed applies to the present situation by carefully analyzing the ramification of certain families of characters valued in Artinian algebras. Although the general strategy for this last step is rather standard, the actual argument is quite involved; in particular one needs to be slightly careful in the case when K is not unramified.

In contrast, our approach avoids the need of such a ramification bound of characters, and in particular works uniformly for all K. First, by combining a standard inductive argument and the Herr complex, we show that it is in fact enough to show that the two stacks are isomorphic on reduced test algebras. For this, it in turn suffices to show that the embedding

$$\mathrm{ur}_x : [\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow \mathcal{X}_1$$

induced by the universal unramified character is a closed immersion. While one can prove this by identifying the source with the crystalline substack with Hodge type 0 of the target (using again a general criterion for a morphism of stacks to be an isomorphism), we are able to give a more elementary argument (without making use of the existence of the crystalline substacks of $\mathcal{X}_1$) as follows. The key observation is that as crystalline representations (e.g. unramified characters) are of finite height, the composition $[\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\mathrm{ur}_x} \mathcal{X}_1 \to \mathcal{R}_1$ should factor through the map $\mathcal{C}_{1,0} \to \mathcal{R}_1$. Here $\mathcal{C}_{1,0}$ (resp. $\mathcal{R}_1$) denotes the stack of Breuil–Kisin modules of *height* 0 (resp. of étale φ-modules), and $\mathcal{X}_1 \to \mathcal{R}_1$ is the natural map given by “restriction to $G_{K_\infty} \subseteq G_K$”. Having guessed this, we show that there is in fact an isomorphism $[\mathbf{G}_m/\mathbf{G}_m] \simeq \mathcal{C}_{1,0}$ making the diagram

$$\begin{array}{ccc} [\mathbf{G}_m/\mathbf{G}_m] & \xrightarrow{\simeq} & \mathcal{C}_{1,0} \\ {\scriptstyle \mathrm{ur}_x}\downarrow & & \downarrow \\ \mathcal{X}_1 & \longrightarrow & \mathcal{R}_1 \end{array}$$

commute. As the right vertical map is known to be proper and the diagonal of the bottom horizontal map is a closed immersion, the result now follows by the usual graph argument and the standard fact that proper monomorphisms are closed immersions.

0.2.2 Moduli stacks of Lubin–Tate (φ, Γ)-modules

As mentioned above, with an eye toward realizing a p-adic local Langlands correspondence for fields other than $\mathbf{Q}_p$, there has been a growing interest in studying the analogue of Fontaine's notion in which the cyclotomic extension K_∞/K is replaced by a Lubin–Tate extension (in what follows, we will often refer to these objects simply as Lubin–Tate (φ, Γ)-modules). We will not try to survey these results, but instead refer the reader, for instance, to [KR09], [Sch17], and [KV22].

Our goal in the second chapter is to extend the aforementioned construction of Emerton–Gee to the Lubin–Tate setting. Before stating our main results, we need to introduce some notation. Let $\pi \in K$ be a fixed uniformizer, and let $\mathcal{F}_\phi$ be a Lubin–Tate group for π, with Frobenius power series $\phi(T) \in \mathcal{O}_K[[T]]$. The corresponding ring map $\mathcal{O}_K \to \mathrm{End}(\mathcal{F}_\phi)$ is denoted by $a \mapsto [a](T)$; in particular $[\pi](T) = \phi(T)$. Let K_∞/K be the extension generated by the torsion points of $\mathcal{F}_\phi$ and let $\chi : \Gamma := \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathcal{O}_K^\times$ be the resuting Lubin–Tate character. For the purpose of this introduction, we simply let

$$\mathbf{A}_K := \widehat{\mathcal{O}_K((T))} = \left\{ \sum_{n \in \mathbf{Z}} a_n T^n \mid a_n \in \mathcal{O}_K \text{ and } a_n \to 0 \text{ as } n \to -\infty \right\},$$

where the hat denotes π-adic completion. The ring $\mathbf{A}_K$ is further endowed with a Frobenius $\varphi_q : f(T) \mapsto f([\pi](T))$ and an action of Γ given by $(g, f(T)) \mapsto f([\chi(g)](T))$ for $g \in \Gamma$; as the notation suggests, φ_q is a lift of the q-power Frobenius modulo π. An étale (φ_q, Γ)-module (over $\mathbf{A}_K$) is then, by definition, a finite $\mathbf{A}_K$-module endowed with commuting continuous semilinear actions of φ_q and Γ such that the linearization of φ_q an isomorphism. There is again a natural equivalence between étale (φ_q, Γ)-modules and representations of G_K on finite $\mathcal{O}_K$-modules. (One can be slightly more general by allowing also representations on $\mathcal{O}_F$-modules with F being a finite subextension of $K/\mathbf{Q}_p$. See Section 2.2 below.)

Fix now an integer $d \geq 1$. By definition, our stack $\mathcal{X}_{K,d}^{\mathrm{LT}}$ takes a π-adically complete $\mathcal{O}_K$-algebra A to the groupoid of rank d projective étale (φ_q, Γ)-modules over $\mathbf{A}_{K,A} := \mathbf{A}_K \widehat{\otimes}_{\mathcal{O}_K} A$.

Theorem 0.2.2 (Proposition 2.3.24). *$\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a limit preserving Ind-algebraic stack over $\mathrm{Spf}\mathcal{O}_K$, with finitely presented affine diagonal.*

The proof of Theorem 2.1.1 follows closely the argument used in [EG23] for the stack $\mathcal{X}_{K,d}^{\mathrm{EG}}$ of rank d projective *cyclotomic* étale (φ, Γ)-modules. Namely, we will deduce the claimed properties for $\mathcal{X}_{K,d}^{\mathrm{LT}}$ from the corresponding properties of the stack of étale φ_q-modules.

The next result gives a comparison between $\mathcal{X}_{K,d}^{\mathrm{LT}}$ and the stack $\mathcal{X}_{K,d}^{\mathrm{EG}}$ in [EG23].

Theorem 0.2.3 (Corollary 2.4.2). *There is an isomorphism*

$$\mathcal{X}_{K,d}^{\mathrm{LT}} \xrightarrow{\sim} \mathcal{X}_{K,d}^{\mathrm{EG}}.$$

The proof proceeds by using the descent results in [EG23] to reduce the statement to a comparison between étale φ_p-modules over $W(\mathbf{C}^\flat)\widehat{\otimes}_{\mathbf{Z}_p} A$ and étale φ_q-modules over $W_{\mathcal{O}_K}(\mathbf{C}^\flat)\widehat{\otimes}_{\mathcal{O}_K} A$.

As a consequence of Theorem 2.1.2 and the results in [EG23], we deduce the following refinement of Theorem 2.1.1 regarding the geometry of the stack of $\mathcal{X}_{K,d}^{\mathrm{LT}}$.

Corollary 0.2.4 (Corollary 2.4.3). *$\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a Noetherian formal algebraic stack over $\mathrm{Spf}(\mathcal{O}_K)$. The underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}}$ is an algebraic stack of finite presentation over $\mathbf{F}$. Moreover, the irreducible components of $\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}}$ admits a natural labeling by Serre weights.*

We also introduce a version of the Herr complex ([Her98]) in the Lubin–Tate setting with coefficients, and give a new proof of the fact that this complex computes Galois cohomology (Theorem 2.4.15). We refer the reader to Subsection 2.4.2 for the definition of this complex. Finally, by using again the above comparison (Theorem 2.1.2), we are able to deduce the following result, which may be of independent interest.

Theorem 0.2.5 (Theorem 2.4.12). *Let A is a finite type π-nilpotent $\mathcal{O}_K$-algebra, and let M be a finite projective étale (φ_q, Γ)-module with A-coefficients. Then the Lubin–Tate Herr complex associated to M is a perfect complex of A-modules, whose formation commutes with arbitrary finite type base change in A.*

0.2.3 Prismatic F-crystals and "Lubin–Tate" crystalline Galois representations

Let $K/\mathbf{Q}_p$ be a completed discrete valued extension with perfect residue field k, fixed completed algebraic closure C, and absolute Galois group G_K. We have seen that the category of crystalline $\mathbf{Q}_p$-linear representations of G_K can be effectively described using filtered étale φ-modules which are objects of linear algebraic nature. However, for certain purposes such as Mazur's theory of Galois deformations, it is useful to have an integral theory in which p-adic vector spaces are replaced with lattices or even torsion modules. The study of such integral structures inside crystalline (or more generally, semistable) representations itself form an important part of p-adic Hodge theory, known as 'Integral p-adic Hodge theory'.

There have been various (partial) classifications of such lattices, including Fontaine–Laffaille's theory [FL82], Breuil's theory of strongly divisible S-lattices [Bre02], and Kisin's theory of Breuil–Kisin modules [Kis06]. In [BS23], Bhatt and Scholze give a site-theoretic description of such lattices, which unifies many of the previous classifications, and in fact can recover them by "evaluating" suitably. To recall their result, let $(\mathcal{O}_K)_{\Delta}$ denote the absolute prismatic site of $\mathcal{O}_K$; this comes equipped with a structure sheaf $\mathcal{O}_{\Delta}$, a "Frobenius" $\varphi : \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}$, and an invertible ideal sheaf $\mathcal{I}_{\Delta} \subseteq \mathcal{O}_{\Delta}$.

Definition 0.2.6. A prismatic F-crystal on $\mathcal{O}_K$ is a crystal of vector bundles on the ringed site $((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$ equipped with an isomorphism $(\varphi^*\mathcal{E})[1/\mathcal{I}_{\Delta}] \simeq \mathcal{E}[1/\mathcal{I}_{\Delta}]$; denote by $\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$ the resulting category. Similarly, we obtain the category $\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)$ of so-called Laurent F-crystals.

In the statement below, $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(G_K)$ denotes the category of Galois stable lattices in crystalline $\mathbf{Q}_p$-representations of G_K.

Theorem 0.2.7 ([BS23])**.** *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}) & \xrightarrow{\simeq} & \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(G_K) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p) & \xrightarrow{\simeq} & \mathrm{Rep}_{\mathbf{Z}_p}(G_K). \end{array}$$

Here the vertical embeddings are given by the obvious maps; the horizontal equivalences are induced by evaluating on the Fontaine's prism $\mathbf{A}_{\mathrm{inf}}$,the so-called étale realization functor.

(We note that the bottom horizontal equivalence was also obtained independently by Zhiyou Wu [Wu21].) Motivated by the study of the stacks of Lubin–Tate (φ, Γ)-modules in Chapter 2, it is natural to ask if there is a variant of Theorem 3.1.2 for coefficient rings other than $\mathbf{Z}_p$. More specifically, let E be a finite extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_q$ and a fixed uniformizer π; we are interested in crystalline representations of G_K on finite dimensional E-vector spaces (or rather, G_K-stable $\mathcal{O}_E$-lattices in such).

Hypothesis 0.2.8. We assume throughout that there is an embedding $\tau_0 : E \hookrightarrow K$, which we will fix once and for all.

Definition 0.2.9 ([KR09])**.** An E-linear representation V of G_K is called E-crystalline if it is crystalline (as a $\mathbf{Q}_p$-representation), and the C-semilinear representation $\bigoplus_{\tau\neq\tau_0} V\otimes_{E,\tau} C$ is trivial[1].

A natural source of such representations comes from the rational Tate modules of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$; see Lemma 3.4.23. Later, we will show that, just as in the case $E=\mathbf{Q}_p$, E-crystalline representations can be classified using weakly admissible filtered φ_q-modules over K. In fact, the above notion is indeed a natural extension of the usual notion in the sense that there is a natural period ring $B_{\mathrm{cris},E}$ with the property that an E-linear representation V is E-crystalline if and only if $V\otimes_E B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-semilinear representation (cf. Theorem 3.2.4).

Using the theory of Lubin–Tate (φ,Γ)-modules, Kisin–Ren give a classification of the category of Galois stable lattices in E-crystalline representations of G_K (under a condition on the ramification of K) [KR09, Theorem (0.1)], generalizing the earlier classification in terms of Wach modules by Wach, Colmez and Berger (cf. [Ber04]).

In another direction, in [Mar23] Marks defines a variant of the (absolute) prismatic site $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$ of $\mathcal{O}_K$ "relative to $\mathcal{O}_E$", using the notion of $\mathcal{O}_E$-prisms[2], a mild generalization of prisms: they are roughly $\mathcal{O}_E$-algebras A equipped with a lift $\varphi_q : A\to A$ of the q-power Frobenius modulo π together with a Cartier divisor I of $\mathrm{Spec}(A)$ satisfying $\pi\in(I,\varphi_q(I))$ (cf. [Mar23, §3]). Furthermore, it is shown in *loc. cit.* that the étale realization functor again defines an equivalence

$$T:\mathrm{Vect}^{\varphi_q}((\mathcal{O}_K)_{\Delta,\mathcal{O}_E},\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)\simeq \mathrm{Rep}_{\mathcal{O}_E}(G_K),$$

generalizing the aforementioned result of Wu and Bhatt–Scholze in the case $E=\mathbf{Q}_p$. In this chapter, we push this analogy further by showing the following extension of Theorem 3.1.2.

Theorem 0.2.10 (Theorem 3.4.7)**.** *The étale realization functor induces an equivalence*

$$T:\mathrm{Vect}^{\varphi_q}((\mathcal{O}_K)_{\Delta,\mathcal{O}_E},\mathcal{O}_{\Delta})\simeq \mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K),$$

where the target denotes the category of Galois stable $\mathcal{O}_E$-lattices in E-crystalline representations of G_K.

As will be explained in §3.4.5 below, by evaluating at a suitable prism in $(\mathcal{O}_K)_{\Delta}$, Theorem 3.1.5 recovers the main equivalence from Kisin–Ren's work [KR09].

Finally, by combining Theorem 3.1.5 with a key result from [AL23] (generalized to the "$\mathcal{O}_E$-context" in [Ito23]), we deduce the following classification of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$.

Theorem 0.2.11 (Theorem 3.4.24)**.** *There is a natural equivalence between the category of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ and the category of minuscule Breuil–Kisin modules over $\mathfrak{S}_E$.*

[1] This is not quite the original definition in [KR09], but can be easily seen to be equivalent to it (see Lemma 3.2.2 below).

[2] These are called E-typical prisms in [Mar23].

0.2.3.1 Sketch of the proof of Theorem 3.1.5

Let us now briefly discuss the proof of Theorem 3.1.5. As alluded above, an important observation is that the condition of being E-crystalline can be characterized in a manner similar to the usual notion for $E = \mathbf{Q}_p$. Namely, there is a natural period ring $B_{\mathrm{cris},E}$ with the property that an E-linear representation V is E-crystalline if and only if $V \otimes_E B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-representation; see Theorem 3.2.4. Once this is justified, that the étale realization functor is well-defined and fully faithful can be proved in exactly the same way as [BS23]. For essential surjectivity, we again follow largely the proof in [BS23]; the main difference here is that instead of invoking the Beilinson fibre sequence from [AMMN22] for the key descent step, we are able to prove the following more general result by adapting a key lemma from [DL22].

Proposition 0.2.12 (Theorem 3.4.15)**.** *Let $(A,(d))$ be a transversal $\mathcal{O}_E$-prism. Then the base change*

$$\mathrm{Vect}^{\varphi_q}(A)[1/\pi] \to \mathrm{Vect}^{\varphi_q}(A\langle d/\pi\rangle[1/\pi])$$

is fully faithful; here the source denotes the isogeny category of $\mathrm{Vect}^{\varphi_q}(A)$.

We regard Proposition 3.1.7 as a result of independent interest. For instance, as alluded above, by specializing to the prism $A = \Delta_{\mathcal{O}_C\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_C}$, this recovers (and refines) Proposition 6.10 in [BS23]. Furthermore, by specializing to a Breuil–Kisin prism $(\mathfrak{S}, I)$, this recovers the embedding

$$\mathrm{Vect}^{(\varphi,N)}(\mathfrak{S})[1/p] \hookrightarrow \mathrm{Vect}^{(\varphi,N)}(\mathcal{O})$$

from [Kis06, Lemma 1.3.13] without using Kedlaya's results on slope filtrations; here $\mathcal{O}$ denotes the ring of functions on the rigid open unit disk over K_0.

The proof of Proposition 3.1.7 proceeds by first reducing to the case of finite free modules (which is the only case we need in proving essential surjectivity). In this case, by working with matrices for the φ_q-actions, we reduce to showing that if

$$d^h Y = B\varphi_q(Y)C$$

with $h \geq 0$, $Y \in M_d(A\langle d/\pi\rangle)$ and $B, C \in M_d(A)$, then $Y \in M_d(A[1/\pi])$. Here the idea is to approximate d-adically Y by matrices in $M_d(A)$. This is possible thanks to the following variant of [DL22, Lemma 2.2.10] on the contracting effect of the Frobenius on the d-adic filtration on $A\langle d/\pi\rangle$.

Lemma 0.2.13 (Lemma 3.4.12)**.** *Let $(A,(d))$ be a transversal $\mathcal{O}_E$-prism. Then given any $h \geq 0$,*

$$\varphi_q(d^m A\langle d/\pi\rangle) \subseteq A + d^{m+h}A\langle d/\pi\rangle$$

for all $m \gg 0$ (depending only on h).

We now detail the organization of the chapter. In Section 3.2, we recall the definition of E-crystalline representations from [KR09], and then in the appendix, following [FF18, Chapitre 10], we interpret it in terms of vector bundles on the Fargues–Fontaine curve (Proposition 3.A.13). In particular, we show that the category of E-crystalline representations of G_K is equivalent to the category of weakly admissible filtered φ_q-modules over K, and moreover that being E-crystalline is equivalent to being $B_{\mathrm{cris},E}$-admissible for a natural period ring $B_{\mathrm{cris},E}$. In Section 3.3, we adapt some constructions from [Kis06] to the present context. Next, in Section 3.4, we review briefly the notion of $\mathcal{O}_E$-prisms and define the étale realization functor in Theorem 3.1.2. Full faithfulness is then addressed in Subsection 3.4.3. Subsection 3.4.4 begins with some further ring-theoretic properties of transversal prisms, culminating with the proof of Proposition 3.4.15, which is then used in the proof of essential surjectivity. Finally, in the last two subsections, we briefly discuss an application of Theorem 3.1.5 to the theory of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ as well as its relation with Kisin–Ren's classification in [KR09].

Bibliography

[AL23] Johannes Anschütz and Arthur-César Le Bras. "Prismatic Dieudonné theory". In: *Forum Math. Pi* 11 (2023), Paper No. e2, 92.

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. "On the Beilinson fiber square". In: *Duke Math. J.* 171.18 (2022), pp. 3707–3806.

[Ber04] Laurent Berger. "Limites de représentations cristallines". In: *Compos. Math.* 140.6 (2004), pp. 1473–1498.

[BS23] Bhargav Bhatt and Peter Scholze. "Prismatic F-crystals and crystalline Galois representations". In: *Camb. J. Math.* 11.2 (2023), pp. 507–562.

[Bre02] Christophe Breuil. "Integral p-adic Hodge theory". In: *Algebraic geometry 2000, Azumino (Hotaka)*. Vol. 36. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2002, pp. 51–80.

[BHHMS23] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen. *Multivariable (φ,$\mathcal{O}_K^\times$)-modules and local-global compatibility*. 2023. arXiv: 2211.00438.

[BC09] Olivier Brion and Brian Conrad. *CMI summer school notes on p-adic Hodge theory*. 2009. Avaliable at: http://math.stanford.edu/~conrad/papers/notes.pdf.

[CKZ21] Annie Carter, Kiran S. Kedlaya, and Gergely Zábrádi. "Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules". In: *Doc. Math.* 26 (2021), pp. 1329–1393.

[Col03] Pierre Colmez. "Les conjectures de monodromie p-adiques". In: 290. Séminaire Bourbaki. Vol. 2001/2002. 2003, Exp. No. 897, vii, 53–101.

[DL22] Heng Du and Tong Liu. *A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals*. 2022. arXiv: 2107.12240.

[EG23] Matthew Emerton and Toby Gee. *Moduli stacks of Étale (ϕ, Γ)-modules and the existence of crystalline lifts*. Vol. 215. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2023, pp. ix+298.

[FF18] Laurent Fargues and Jean-Marc Fontaine. "Courbes et fibrés vectoriels en théorie de Hodge p-adique". In: *Astérisque* 406 (2018). With a preface by Pierre Colmez, pp. xiii+382.

[Fon90] Jean-Marc Fontaine. "Représentations p-adiques des corps locaux. I". In: *The Grothendieck Festschrift, Vol. II*. Vol. 87. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[FL82] Jean-Marc Fontaine and Guy Laffaille. "Construction de représentations p-adiques". In: *Ann. Sci. École Norm. Sup. (4)* 15.4 (1982), 547–608 (1983).

[Her98] Laurent Herr. "Sur la cohomologie galoisienne des corps p-adiques". In: *Bull. Soc. Math. France* 126.4 (1998), pp. 563–600.

[Ill90] Luc Illusie. "Cohomologie de de Rham et cohomologie étale p-adique (d'après G. Faltings, J.-M. Fontaine et al.)" In: 189-190. Séminaire Bourbaki, Vol. 1989/90. 1990, Exp. No. 726, 325–374.

[Ito23] Kazuhiro Ito. *Prismatic G-display and descent theory*. 2023. arXiv: 2303.15814.

[Kis06] Mark Kisin. "Crystalline representations and F-crystals". In: *Algebraic geometry and number theory*. Vol. 253. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 459–496.

[KR09] Mark Kisin and Wei Ren. "Galois representations and Lubin-Tate groups". In: *Doc. Math.* 14 (2009), pp. 441–461.

[KV22] Benjamin Kupferer and Otmar Venjakob. "Herr-complexes in the Lubin-Tate setting". In: *Mathematika* 68.1 (2022), pp. 74–147.

[LLLM23] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra. "Local models for Galois deformation rings and applications". In: *Invent. Math.* 231.3 (2023), pp. 1277–1488.

[Mar23] Samuel Marks. *Prismatic F-crystals and Lubin-Tate (φ_q, Γ)-modules*. 2023. arXiv: 2303.07620.

[Pha22] Dat Pham. *The Emerton–Gee stack of rank one (φ, Γ)-modules*. To appear in Documenta Mathematica. 2022. arXiv: 2206.02888.

[Pha23a] Dat Pham. *Moduli stacks of Lubin–Tate (φ, Γ)-modules*. 2023. arXiv: 2302.09534.

[Pha23b] Dat Pham. *Prismatic F-crystals and analytic crystalline Galois representations*. 2023. arXiv: 2306.13205.

[Sch17] Peter Schneider. *Galois representations and (φ, Γ)-modules*. Vol. 164. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017, pp. vii+148.

[Wan18] Carl Wang-Erickson. "Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings". In: *Math. Ann.* 371.3-4 (2018), pp. 1615–1681.

[Wil95] Andrew Wiles. "Modular elliptic curves and Fermat's last theorem". In: *Ann. of Math. (2)* 141.3 (1995), pp. 443–551.

[Wu21] Zhiyou Wu. "Galois representations, (φ, Γ)-modules and prismatic F-crystals". In: *Doc. Math.* 26 (2021), pp. 1771–1798.

Chapter 1

The Emerton–Gee stack of rank one (φ, Γ)-modules

1.1 Introduction

Our goal in this chapter is to give a new and simpler proof of the following explicit description of the Emerton–Gee stack of rank one étale (φ, Γ)-modules, first obtained in [EG23].

Theorem 1.1.1 ([EG23, Prop. 7.2.17])**.** *There is an isomorphism*

$$\left[\left(\mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbf{G}}_m\right)/\widehat{\mathbf{G}}_m\right] \xrightarrow{\sim} \mathcal{X}_1,$$

where, in the formation of the quotient stack, the action of $\widehat{\mathbf{G}}_m$ is taken to be trivial.

To begin with, we need to construct a morphism between the two given stacks, which we will then show is an isomorphism. This amounts to extending Fontaines' construction of Galois characters from rank one (φ, Γ)-modules to the case of an arbitrary p-complete test ring, and is reviewed in detail in Section 1.2 below. Now in order to show that the resulting map is indeed an isomorphism, Emerton and Gee proved a general criterion for a morphism of stacks to be an isomorphism, and then showed that it indeed applies to the present situation by carefully analyzing the ramification of certain families of characters valued in Artinian algebras. Although the general strategy for this last step is standard, the actual argument is rather involved; in particular one needs to be slightly careful in the case when K is not unramified.

In contrast, our approach avoids the need of such a ramification bound of characters, and in particular works uniformly for all K. First, by combining a standard inductive argument and the Herr complex, we show that it is in fact enough to show that the two stacks are isomorphic on reduced test algebras. For this, it in turn suffices to show that the embedding

$$\mathrm{ur}_x : [\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow \mathcal{X}_1$$

induced by the universal unramified character is a closed immersion. While one can prove this by identifying the source with the crystalline substack with Hodge type 0 of the target (using again a general criterion for a morphism of stacks to be an isomorphism), we are able to give a more elementary argument (without making use of the existence of the crystalline substacks of $\mathcal{X}_1$) as follows. The key observation is that as crystalline representations (e.g. unramified characters) are of finite height, the composition $[\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\mathrm{ur}_x} \mathcal{X}_1 \to \mathcal{R}_1$ should factor through the map $\mathcal{C}_{1,0} \to \mathcal{R}_1$. Here $\mathcal{C}_{1,0}$ (resp. $\mathcal{R}_1$) denotes the stack of Breuil–Kisin modules of *height* 0 (resp. of étale φ-modules), and $\mathcal{X}_1 \to \mathcal{R}_1$ is the natural map given by "restriction to $G_{K_\infty} \subseteq G_K$". Having guessed this, we show that there is in fact an isomorphism $[\mathbf{G}_m/\mathbf{G}_m] \simeq \mathcal{C}_{1,0}$ making the diagram

$$\begin{array}{ccc} [\mathbf{G}_m/\mathbf{G}_m] & \xrightarrow{\simeq} & \mathcal{C}_{1,0} \\ {\scriptstyle \mathrm{ur}_x}\downarrow & & \downarrow \\ \mathcal{X}_1 & \longrightarrow & \mathcal{R}_1 \end{array}$$

commute. As the right vertical map is known to be proper, and proper monomorphisms are closed immersions, the result now follows by the usual graph argument.

Remark 1.1.2. The method above is also suited in other situations. For instance, it also applies to give explicit descriptions of the stacks of rank one étale φ-modules (i.e. in the absence of a Γ-action), generalizing another result of Emerton–Gee to a large class of coefficient rings; see Subsection 1.3.3. Furthermore, we expect that the same method also applies in the context of rank one multivariable (φ, Γ)-modules of Carter–Kedlaya–Zábrádi ([CKZ21]).

Notation 1.1.3. We mostly follow the notation in [EG23]. In particular, we fix a finite extension $K/\mathbf{Q}_p$ with residue field k and inertia degree f. Fix also an algebraic closure $\overline{K}$ of K, with absolute Galois group G_K, Weil group W_K, and inertia group I_K. As usual, W_K^{ab} denotes the abelianization of W_K, while I_K^{ab} denotes the image of I_K in W_K^{ab}. We denote by $\mathbf{C}^\flat$ the tilt of the completion $\mathbf{C} := \widehat{\overline{K}}$, by K_{cyc} the cyclotomic $\mathbf{Z}_p$-extension of K and by k_∞ its residue field. We also fix a finite extension $E/\mathbf{Q}_p$ with ring of integers $\mathcal{O}$, which will serve as the base of our coefficients. As usual, ϖ (resp. $\mathbf{F}$) denotes a uniformizer (resp. the residue field) of $\mathcal{O}$. We will fix throughout an embedding $k \hookrightarrow \mathbf{F}$.

We refer the reader to [EG23, §2.2] for the definition of the coefficient rings $\mathbf{A}_{K,A}$ of our (φ, Γ)-modules. Finally, as the field K is fixed throughout, we will often drop K from the notation in what follows.

1.2 (φ, Γ)-modules associated to characters of the Weil group

In this section, we explain how to associate a free étale (φ, Γ)-module of rank 1 to any character of W_K.

First recall the following result of Dee, which is a generalization of Fontaine's equivalence between Galois representations on finite $\mathbf{Z}_p$-modules and étale (φ, Γ)-modules to the context with coefficients. For simplicity, we only state the result for Artinian coefficients.

Theorem 1.2.1 ([Dee01]). *Let A be a finite Artinian local $\mathcal{O}$-algebra, and let $W(\mathbf{C}^\flat)_A := W(\mathbf{C}^\flat)\otimes_{\mathbf{Z}_p} A$. Then the functor*

$$M \mapsto T_A(M) := (W(\mathbf{C}^\flat)_A \otimes_{\mathbf{A}_{K,A}} M)^{\varphi=1}$$

defines an equivalence between the category of finite projective étale (φ,Γ)-modules with A-coefficients, and the category of finite free A-modules with a continuous action of G_K.

We want to extend the above construction of rank one étale (φ,Γ)-modules from Galois characters to the case where A is an arbitrary ϖ-adically complete $\mathcal{O}$-algebra. We begin with the case of unramified characters.

Lemma 1.2.2. *Let A be an $\mathcal{O}$-algebra, and let $a \in A^\times$. Then, up to isomorphism, there is a unique free étale φ-module $D_{k,a}$ of rank one over $W(k)\otimes_{\mathbf{Z}_p} A$ with the property that $\varphi^f = 1\otimes a$ on $D_{k,a}$.*

Proof. We need to show that the norm map $(W(k)\otimes_{\mathbf{Z}_p} A)^\times \to A^\times, x \mapsto N(x) := x\varphi(x)\ldots\varphi^{f-1}(x)$ is surjective with kernel given by the set $\{\varphi(y)/y \mid y \in (W(k)\otimes_{\mathbf{Z}_p} A)^\times\}$. Since $\mathbf{F}$ is assumed to contain k, $\mathcal{O}$ (and hence A) is naturally a $W(k)$-algebra. In particular, we have an A-algebra isomorphism $W(k)\otimes_{\mathbf{Z}_p} A \to \prod A, x\otimes 1 \mapsto (x, \varphi(x), \ldots, \varphi^{f-1}(x))$. The lemma then follows easily using this isomorphism. □

Definition 1.2.3. Let A be a ϖ-adically complete $\mathcal{O}$-algebra, and let $a \in A^\times$. Define $\mathbf{A}_{K,A}(\mathrm{ur}_a) := D_{k,a}\otimes_{W(k)\otimes_{\mathbf{Z}_p} A} \mathbf{A}_{K,A}$. This is a rank one étale (φ,Γ)-module with A-coefficients, where we let φ act diagonally, and Γ act on the second factor.

Lemma 1.2.4. *Let A be a finite Artinian local $\mathcal{O}$-algebra, and let $a \in A^\times$. Then, under Dee's equivalence (1.2.1), $\mathbf{A}_{K,A}(\mathrm{ur}_a)$ corresponds to the unramified character ur_a of G_K sending geometric Frobenii to a.*

Proof. By definition, the rank one A-representation of G_K corresponding to $\mathbf{A}_{K,A}(\mathrm{ur}_a)$ is given by

$$\begin{aligned} V &:= (W(\mathbf{C}^\flat)_A \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\mathrm{ur}_a))^{\varphi=1} \\ &\cong \{hv \mid h \in W(\mathbf{C}^\flat)\otimes_{\mathbf{Z}_p} A \text{ such that } \varphi(hv) = hv\}, \end{aligned}$$

where v is a basis of $D_{k,a}$. Assume V has a basis hv with $h \in W(\overline{\mathbf{F}}_p)\otimes_{\mathbf{Z}_p} A$. We verify that G_K acts on this basis via the unramified character taking geometric Frobenius to a. First, for $\sigma \in I_K$, we have $\sigma(hv) = \sigma(h)v = hv$ (note that $\sigma(h) = h$ as $h \in W(\overline{\mathbf{F}}_p)\otimes_{\mathbf{Z}_p} A$). Now, let $\sigma = \varphi_q^{-1}$ be an arithmetic Frobenius. From the relation $\varphi(hv) = hv$ and the fact that $\varphi^f(v) = av$, we obtain $\sigma(h)av = \varphi^f(hv) = hv$ whence $\sigma(hv) = \sigma(h)v = a^{-1}(hv)$, as desired.

It remains to find a basis as stated. If A is a field, say $A = \mathbf{F}_q$ for some finite extension $\mathbf{F}_q/\mathbf{F}$, this can be done by using the ring isomorphism $\mathbf{C}^\flat\otimes_{\mathbf{F}_p}\mathbf{F}_q \xrightarrow{\sim} \prod \mathbf{C}^\flat$. Indeed, h is a vector in $\prod \mathbf{C}^\flat$ whose coordinates satisfy a finite set of polynomial equations with coefficients in $\overline{\mathbf{F}}_p$, so it necessarily lies in $\prod \overline{\mathbf{F}}_p$. In general, by factoring $A \twoheadrightarrow A/\mathfrak{m}_A$ as a chain of square-zero thickenings, we may assume that, for some ideal I with $I^2 = 0$, there is a basis $\overline{hv}$ of V/IV with $\overline{h} \in W(\overline{\mathbf{F}}_p)\otimes_{\mathbf{Z}_p}(A/I)$. Let $h \in W(\overline{\mathbf{F}}_p)\otimes_{\mathbf{Z}_p} A$ be a lift of $\overline{h}$. Then $\varphi(hv) = g(hv)$ for some

$g \in 1 + W(\overline{\mathbf{F}}_p) \otimes I$, say $g = 1 + g_1 \otimes m_1 + \ldots + g_n \otimes m_n$ with $g_i \in W(\overline{\mathbf{F}}_p)$ and $m_i \in I$. For each i, choose $h_i \in W(\overline{\mathbf{F}}_p)$ such that $\varphi(h_i) - h_i = -g_i$. Let $f := 1 + h_1 \otimes m_1 + \ldots h_n \otimes m_n \in 1 + W(\overline{\mathbf{F}}_p) \otimes I$. Then $\varphi(f) = f/g$, and hence $\varphi(hfv) = hfv$. Finally, as $(hf)v$ lifts a basis of V/IV, it is a basis of V by Nakayama's lemma. □

Recall that $\widehat{\mathbf{G}}_m$ denotes the ϖ-adic completion of $\mathbf{G}_{m,\mathcal{O}}$. We denote the resulting map $\widehat{\mathbf{G}}_m \to \mathcal{X}_1, a \mapsto \mathbf{A}_{K,A}(\mathrm{ur}_a)$ by ur_x (where we can think of x as the coordinate on $\widehat{\mathbf{G}}_m = \mathrm{Spf}(\mathcal{O}[x, x^{-1}])$), and refer to it simply as the universal unramified character[1].

We now consider the case of a general character of the Weil group W_K. It is convenient to introduce some notation.

Definition 1.2.5. Let X^{an} be the functor on ϖ-adically complete $\mathcal{O}$-algebras taking A to the set of (continuous) characters $\delta : W_K \to A^\times$.

As a first remark, we note that fixing a geometric Frobenius $\sigma \in G_K$ (or equivalently, an isomorphism $W_K^{\mathrm{ab}} \cong I_K^{\mathrm{ab}} \times \mathbf{Z}$) is equivalent to fixing an isomorphism of (Noetherian) affine formal schemes

$$X^{\mathrm{an}} \simeq \mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbf{G}}_m$$

over $\mathrm{Spf}(\mathcal{O})$. Concretely, at the level of A-valued points (with A a ϖ-adically complete $\mathcal{O}$-algebra), this is given by the assignment $(\delta : W_K \to A^\times) \mapsto (\delta|_{I_K^{\mathrm{ab}}}, \delta(\sigma))$.

In what follows, we will always endow X^{an} with the trivial action of $\widehat{\mathbf{G}}_m$.

Lemma 1.2.6. *There is a morphism $X^{\mathrm{an}} \to \mathcal{X}_1, \delta \mapsto \mathbf{A}_{K,A}(\delta)$ with the property that for any finite Artinian $\mathcal{O}$-algebra A, the (φ, Γ)-module $\mathbf{A}_{K,A}(\delta)$ corresponds, under Dee's equivalence (1.2.1), to the character δ.*

Proof. Fix a geometric Frobenius $\sigma \in G_K$, and hence an isomorphism $X^{\mathrm{an}} \cong \mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbf{G}}_m$. For each finite Artinian quotient A of $\mathcal{O}[[I_K^{\mathrm{ab}}]]$, we extend the map $I_K^{\mathrm{ab}} \to A^\times$ to a character $G_K \to A^\times$ by taking σ to 1. Under Dee's equivalence (1.2.1), this gives rise to a rank one étale (φ, Γ)-module with A-coefficients, i.e. an object of $\mathcal{X}_1(A)$. As $\mathcal{O}[[I_K^{\mathrm{ab}}]]$ is the inverse limit of all such quotients A, we obtain a map $\mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \to \mathcal{X}_1$. We can now define $X^{\mathrm{an}} \to \mathcal{X}_1$ as the composite

$$X^{\mathrm{an}} \cong \mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbf{G}}_m \to \mathcal{X}_1 \times \widehat{\mathbf{G}}_m \to \mathcal{X}_1,$$

where the last map is given by taking tensor product with the universal unramified character ur_x. That the map is compatible with Dee's equivalence in case of Artinian coefficients follows immediately from construction and Lemma (1.2.4). □

It is easy to check that the construction $\delta \mapsto \mathbf{A}_{K,A}(\delta)$ is independent on our choice of σ (see also Remark (1.3.8) below), and that $\mathbf{A}_{K,A}(\delta_1\delta_2) \cong \mathbf{A}_{K,A}(\delta_1) \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\delta_2)$ for all δ_i. Of course, in case $\delta = \mathrm{ur}_a$ is an unramified character, this agrees with the notation introduced earlier. For a (φ, Γ)-module M with A-coefficients, we set $M(\delta) := M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\delta)$, equipped with the obvious (diagonal) (φ, Γ)-structure.

[1]Note that the definition given in [EG23, §5.3] of the map ur_x is slightly incorrect in the sense that it does not agree with the construction of Dee for Artinian coefficients.

1.3 Explicit descriptions of the rank one stacks

1.3.1 Statement

Our main result in this text is the following.

Theorem 1.3.1. *Let A be a ϖ-adically complete $\mathcal{O}$-algebra. Let M be a rank one étale (φ, Γ)-module with A-coefficients. Then there exist a unique continuous character $\delta : W_K \to A^\times$ and a unique (up to isomorphism) invertible A-module L, such that $M \cong \mathbf{A}_{K,A}(\delta) \otimes_A L$.*

Corollary 1.3.2 ([EG23, Prop. 7.2.17])**.** *The map $X^{\mathrm{an}} \to \mathcal{X}_1, \delta \mapsto \mathbf{A}_{K,A}(\delta)$ induces an isomorphism*

$$\left[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m\right] \xrightarrow{\sim} \mathcal{X}_1.$$

Proof of Corollary (1.3.2). Since X^{an} is endowed with the trivial action of $\widehat{\mathbf{G}}_m$, the quotient stack $[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m]$ is naturally identified with $[\mathrm{Spf}(\mathcal{O})/\widehat{\mathbf{G}}_m] \times_{\mathrm{Spf}(\mathcal{O})} X^{\mathrm{an}}$. In other words, for any ϖ-adically complete $\mathcal{O}$-algebra A, its groupoid of A-valued points is equivalent to the groupoid of pairs (L, δ) consisting of an invertible A-module L, and a character $\delta \in X^{\mathrm{an}}(A)$. Via this identification, the map $X^{\mathrm{an}} \to \mathcal{X}_1$ factors through the map

$$\left[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m\right] \to \mathcal{X}_1$$

defined by $(L, \delta) \mapsto \mathbf{A}_{K,A}(\delta) \otimes_A L$. The result now follows from Theorem (1.3.1), and the fact that the automorphism group of any rank one étale (φ, Γ)-module is given simply by the scalars:

$$\mathrm{Aut}_{\mathbf{A}_{K,A},\varphi,\Gamma}(M) = ((M \otimes M^\vee)^{\varphi,\Gamma=1})^\times = (\mathbf{A}_{K,A}^{\varphi,\Gamma=1})^\times = A^\times;$$

here we have used [EG23, Lem. 2.2.19 and Prop. 2.2.12] for the last equality. □

1.3.2 Proof

This subsection is devoted to proving Theorem (1.3.1). In order to streamline the arguments, we postpone the proof of one key result (Lemma (1.3.6)) to §1.3.2.1 below.

We begin with the uniqueness statement.

Lemma 1.3.3. *The uniqueness part of Theorem* (1.3.1) *holds.*

Proof. As $(\mathbf{A}_{K,A})^{\varphi=1} = A$, we necessarily have $L \cong M(\delta^{-1})^{\varphi=1}$. It remains to show that if $\mathbf{A}_{K,A}(\delta) \cong \mathbf{A}_{K,A}(\delta')$ as (φ, Γ)-modules, then $\delta = \delta'$. Using that X^{an} and $\mathcal{X}_1$ are both limit preserving (this is easy for X^{an} by its definition; for $\mathcal{X}_1$, see [EG23, Lem. 3.2.19]), we may assume that A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$. In this case, A embeds naturally into the product of its Artinian quotients, and so we may assume further that A is a finite Artinian $\mathcal{O}$-algebra. The lemma now follows since the map $\delta \mapsto \mathbf{A}_{K,A}(\delta)$ recovers the equivalence between rank one étale (φ, Γ)-modules and Galois characters for Artinian coefficients (Lemma (1.2.6)). □

Remark 1.3.4. By Lemma (1.3.3) and the fact that the automorphism group of any rank one (φ, Γ) module is simply $\widehat{\mathbf{G}}_m$, we see that the map $[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ is at least a monomorphism. Showing that it is in fact essentially surjective (i.e. an isomorphism) is equivalent to showing the existence part of Theorem (1.3.1).

The next lemma allows us to reduce to the case where our test object A is a reduced $\mathbf{F}$-algebra.

Lemma 1.3.5. *If Theorem (1.3.1) holds for reduced finite type $\mathbf{F}$-algebras A, then it holds for any ϖ-adically complete $\mathcal{O}$-algebra A.*

Proof. Let A be an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let M be a rank one étale (φ, Γ)-module with A-coefficients. We want to show that $M \cong \mathbf{A}_{K,A}(\delta) \otimes_A L$ for some δ and L. As $\mathcal{X}_1$ is limit preserving, we may assume A is of finite type over $\mathcal{O}/\varpi^a$. We will induct on the nilpotency index e of the nilradical $A^{\circ\circ}$. The case $e = 1$ is just our assumption. Assume now that $e \geq 2$. Let $I := (A^{\circ\circ})^{e-1}$ and $\bar{A} := A/I$. By the induction hypothesis, $M_{\bar{A}} := M \otimes_A \bar{A}$ has the form $\mathbf{A}_{K,\bar{A}}(\bar{\delta}) \otimes_{\bar{A}} \tilde{L}$ for some character $\bar{\delta}$ and some invertible $\bar{A}$-module $\tilde{L}$. Lifting $\bar{\delta}$ to a character $\delta \in X^{\mathrm{an}}(A)$, $\tilde{L}$ to an invertible A-module L (recall that finite projective modules always deform uniquely through nilpotent thickenings), and replacing M with $M(\delta^{-1}) \otimes_A L^\vee$, we may assume that $M_{\bar{A}}$ is trivial. By [EG23, Prop. 5.1.33], the set of isomorphism classes of such M is given by $H^1(\mathcal{C}^\bullet(\mathbf{A}_{K,I}))$, where $\mathcal{C}^\bullet(\mathbf{A}_{K,I})$ is the Herr complex of $\mathbf{A}_{K,I} := I\mathbf{A}_{K,A} \cong (W(k_\infty) \otimes_{\mathbf{Z}_p} I)((t))$. Namely, given such M, there is an $\mathbf{A}_{K,A}$-basis v of M so that $\varphi(v) = fv$ and $\gamma(v) = gv$ for some $f, g \in \ker((\mathbf{A}_{K,A})^\times \to (\mathbf{A}_{K,\bar{A}})^\times) = 1 + \mathbf{A}_{K,I}$, where γ is a fixed topological generator of $\Gamma_K \cong \mathbf{Z}_p$. The corresponding cohomology class is then given by $[(f-1, g-1)]$.

Let $(\mathcal{O}/\varpi^a)[I]$ be the usual square-zero thickening defined by I. Using the above description in terms of $H^1(\mathcal{C}^\bullet(\mathbf{A}_{K,I}))$, we see that M arises as the base change of some rank one (φ, Γ)-module with $(\mathcal{O}/\varpi^a)[I]$-coefficients via the natural map $(\mathcal{O}/\varpi^a)[I] \to A$. Thus we may reduce to the case $A = (\mathcal{O}/\varpi^a)[I]$. By writing I as the colimit of its finite sub-$\mathcal{O}$-modules and using again the fact that $\mathcal{X}_1$ is limit preserving, we may assume further that I is finite over $\mathcal{O}$. But in this case $(\mathcal{O}/\varpi^a)[I]$ is a finite Artinian $\mathcal{O}$-algebra, so we are done by using (again) the fact that the construction $\delta \mapsto \mathbf{A}_{K,A}(\delta)$ recovers the equivalence between Galois representations and (φ, Γ)-modules for Artinian coefficients. □

Lemma 1.3.6. *The map $\mathrm{ur}_x : \widehat{\mathbf{G}}_m \to \mathcal{X}_1$ induces a closed immersion $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$.*

The proof of Lemma (1.3.6) takes up §1.3.2.1 below. Armed with this crucial result, we now finish our proof of Theorem (1.3.1).

Proof of Theorem (1.3.1). It remains to show that the monomorphism $[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ is an isomorphism. In view of Lemma (1.3.5), it suffices to show that the induced map between underlying reduced substacks

$$\left[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m\right]_{\mathrm{red}} \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$$

is an isomorphism. As our stacks will all live over $\mathrm{Spec}(\mathbf{F})$ in the rest of this proof, we will drop the subscript $\mathbf{F}$ for ease of notation. For each "Serre weight" $\delta : I_K^{\mathrm{ab}} \to \mathbf{F}^\times$ (recall that $\mathbf{F}$ is assumed to contain k), we abusively denote also by δ a fixed choice of an extension of it to G_K.

By twisting $\delta : \mathrm{Spec}(\mathbf{F}) \to X^{\mathrm{an}}$ by unramified characters, we obtain a map $\mathbf{G}_m \to X^{\mathrm{an}}$. The induced map $\coprod_\delta \mathbf{G}_m \to (X^{\mathrm{an}})_{\mathrm{red}}$ is then easily seen to be an isomorphism. In particular, we have an isomorphism $\coprod_\delta [\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\sim} [X^{\mathrm{an}}/\widehat{\mathbf{G}}_m]_{\mathrm{red}}$. Thus, it suffices to show that the map

$$\coprod_\delta [\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$$

is an isomorphism. Of course, by construction the component $[\mathbf{G}_m/\mathbf{G}_m] \to (\mathcal{X}_1)_{\mathrm{red}}$ indexed by δ is just obtained by twisting the residual gerbe $[\mathrm{Spec}(\mathbf{F})/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$ associated to δ by unramified characters. In particular, for $\delta = 1$, we recover the map $[\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$ induced by the universal unramified character ur_x.

By Lemma (1.3.6), this last map is a closed immersion. After twisting by δ, we see that the same is true of the map $[\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$ indexed by δ. As any character $\delta : G_K \to \overline{\mathbf{F}}_p^\times$ is an unramified twist of exactly one of the δ, it is now straightforward (see Lemma (1.3.7) below) to deduce that the map $\coprod_\delta [\mathbf{G}_m/\mathbf{G}_m] \hookrightarrow (\mathcal{X}_1)_{\mathrm{red}}$ is indeed an isomorphism, as desired. □

Lemma 1.3.7. *Let $\mathcal{Z}$ be a reduced algebraic stack locally of finite type over a field k. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ be a family of closed algebraic substacks of $\mathcal{Z}$ with the property that every $\bar{k}$-point of $\mathcal{Z}$ factors through exactly one of the $\mathcal{Z}_i$. Then the natural map $\coprod_i \mathcal{Z}_i \to \mathcal{Z}$ is an isomorphism.*

Proof. As usual, we denote by $|\mathcal{Z}|$ the underlying topological space of $\mathcal{Z}$, and similarly for $|\mathcal{Z}_i|$. We first show that $|\mathcal{Z}| = \coprod_i |\mathcal{Z}_i|$ set-theoretically. Let $\mathcal{Z}'$ be the scheme-theoretic image of the map $\coprod_i \mathcal{Z}_i \to \mathcal{Z}$. Then $\mathcal{Z}'$ is a closed algebraic substack of $\mathcal{Z}$ with $\mathcal{Z}'(\bar{k}) = \mathcal{Z}(\bar{k})$. Since $\mathcal{Z}$ is reduced, this forces $\mathcal{Z}' = \mathcal{Z}$ (this can be checked after passing to a smooth cover of $\mathcal{Z}$ by a reduced scheme, where the result is standard). In particular, we have $|\mathcal{Z}| = |\mathcal{Z}'|$. As $|\mathcal{Z}'|$ is the closure of $\bigcup_i |\mathcal{Z}_i|$ in $|\mathcal{Z}|$ (cf. [Sta23, Tag 0CML]), and each $|\mathcal{Z}_i|$ is a closed subset of $|\mathcal{Z}|$, we see that $|\mathcal{Z}| = \bigcup_i |\mathcal{Z}_i|$. Now for each $i \neq j$, $\mathcal{Z}_i \times_{\mathcal{Z}} \mathcal{Z}_j$ is an algebraic stack locally of finite type over k with $(\mathcal{Z}_i \times_{\mathcal{Z}} \mathcal{Z}_j)(\bar{k}) = \emptyset$ by our assumption. This forces $|\mathcal{Z}_i| \cap |\mathcal{Z}_j| = |\mathcal{Z}_i \times_{\mathcal{Z}} \mathcal{Z}_j| = \emptyset$. Thus we have a disjoint decomposition $|\mathcal{Z}| = \coprod_i |\mathcal{Z}_i|$, and hence each $|\mathcal{Z}_i|$ is also an open subset of $|\mathcal{Z}|$. Let $\mathcal{U}_i$ be the unique *open* substack of $\mathcal{Z}$ with underlying space $|\mathcal{U}_i| = |\mathcal{Z}_i|$ (cf. [Sta23, Tag 06FJ]). In particular, we have a decomposition $\mathcal{Z} = \coprod_i \mathcal{U}_i$ into open substacks. Now for each i, the map $\mathcal{Z}_i \hookrightarrow \mathcal{Z}$ necessarily factors through a closed immersion $\mathcal{Z}_i \hookrightarrow \mathcal{U}_i$. Since $\mathcal{U}_i$ is reduced (being an open substack of $\mathcal{Z}$) and $|\mathcal{Z}_i| \xrightarrow{\sim} |\mathcal{U}_i|$ by construction, this closed immersion is necessarily an isomorphism. □

Remark 1.3.8. Assume $f : X^{\mathrm{an}} \to \mathcal{X}_1$ is a morphism of stacks over $\mathrm{Spf}(\mathcal{O})$ with the property that $f(\delta) \cong \mathbf{A}_{K,A}(\delta)$ for all characters δ valued in a finite Artinian $\mathcal{O}$-algebra. We claim that in fact $f(\delta) \cong \mathbf{A}_{K,A}(\delta)$ everywhere, or equivalently, that the map $g : X^{\mathrm{an}} \to \mathcal{X}_1, \delta \mapsto f(\delta)\mathbf{A}_{K,A}(\delta)^{-1}$ satisfies $g(\delta) \cong \mathbf{A}_{K,A}$ for any δ. Indeed, by [EG23, Lem. 7.1.14], our assumption on f implies that g factors through the closed immersion $[\mathrm{Spf}(\mathcal{O})/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ (induced by the map $\mathrm{Spf}(\mathcal{O}) \hookrightarrow X^{\mathrm{an}}$ classifying trivial characters). It therefore suffices to show that any map $X^{\mathrm{an}} \to [\mathrm{Spf}(\mathcal{O})/\widehat{\mathbf{G}}_m]$ is "trivial", i.e. that any line bundle on the Noetherian affine formal scheme X^{an} is trivial. For this, it suffices to check the same result for the underlying reduced subscheme $(X^{\mathrm{an}})_{\mathrm{red}}$. But we have seen that $(X^{\mathrm{an}})_{\mathrm{red}}$ is just a disjoint union of finitely many copies of $\mathbf{G}_{m,\mathbf{F}} = \mathrm{Spec}(\mathbf{F})[x, x^{-1}]$, and hence (as $\mathbf{F}[x, x^{-1}]$ is a PID), we have the claimed result. Thus we see that there is a unique functorial way to extend the construction $\delta \mapsto \mathbf{A}_{K,A}(\delta)$ appearing in Dee's equivalence (1.2.1) from Artinian coefficients to all ϖ-adically complete $\mathcal{O}$-algebras A.

1.3.2.1 Proof of Lemma (1.3.6)

First proof of Lemma (1.3.6). As in Remark (1.3.4), the map $\mathrm{ur}_x : \widehat{\mathbf{G}}_m \to \mathcal{X}_1$ induces a monomorphism

$$\mathrm{ur}_x : [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1.$$

(More formally, this map is given by composing the monomorphism $[X^{\mathrm{an}}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ with the closed immersion $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow [X^{\mathrm{an}}/\widehat{\mathbf{G}}_m]$ induced by the closed immersion $\widehat{\mathbf{G}}_m \hookrightarrow X^{\mathrm{an}}$ classifying unramified characters.) We want to show that this is in fact a closed immersion. As proper monomorphisms are closed immersions, it suffices to show that ur_x is (representable by algebraic spaces and) proper. Note also that it suffices to work over $\mathcal{O}/\varpi^a$ for some $a \geq 1$ (for our proof of Theorem (1.3.1) it suffices to take $a = 1$, but this will not simplify the argument).

Let $\mathcal{C}_{1,0}$ be the stack of rank 1 projective Breuil–Kisin modules over $\mathfrak{S}_A$ of *height at most* 0 (in the terminology of [EG23]). Concretely, objects of $\mathcal{C}_{1,0}$ are rank 1 projective $\mathfrak{S}_A$-modules $\mathfrak{M}$ equipped with an *isomorphism* $\varphi^*\mathfrak{M} \simeq \mathfrak{M}$; here $\mathfrak{S}_A$ denotes the ring $(W(k) \otimes_{\mathbf{Z}_p} A)[[t]]$, equipped with the A-linear Frobenius φ taking $t \mapsto t^p$ (and restricting to the natural Frobenius on $W(k)$). Let $\mathcal{R}_1$ be the the corresponding stack of rank 1 projective étale φ-modules over $\mathcal{O}_{\mathcal{E},A}$, where $\mathcal{O}_{\mathcal{E},A} := \mathfrak{S}_A[1/t]$. Our strategy is to relate ur_x to the natural map $\mathcal{C}_{1,0} \to \mathcal{R}_1, \mathfrak{M} \mapsto \mathfrak{M}[1/t]$, which is known to be proper. Intuitively, as the source of ur_x classifies unramified characters, and as crystalline representations (e.g. unramified characters) are of finite height, it is natural to guess that the composition $[\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\mathrm{ur}_x} \mathcal{X}_1 \to \mathcal{R}_1$ factors through the map $\mathcal{C}_{1,0} \to \mathcal{R}_1$. Here $\mathcal{X}_1 \to \mathcal{R}_1$ is the natural map given by "restriction to $G_{K_\infty} \subseteq G_K$", where $K_\infty := K(\pi^{1/p^\infty})$ for a compatible system π^{1/p^∞} of p-power roots of some fixed uniformizer π of K (see [EG23, §3.7]). We claim that this is indeed the case. More precisely, we will show that there is an isomorphism $[\mathbf{G}_m/\mathbf{G}_m] \simeq \mathcal{C}_{1,0}$ making the diagram

$$\begin{array}{ccc} [\mathbf{G}_m/\mathbf{G}_m] & \xrightarrow{\simeq} & \mathcal{C}_{1,0} \\ {\scriptstyle \mathrm{ur}_x}\downarrow & & \downarrow \\ \mathcal{X}_1 & \longrightarrow & \mathcal{R}_1 \end{array} \qquad (1.3.8.1)$$

commute. Indeed, as finite projective modules over $\mathfrak{S}_A$ are Zariski locally free on $\mathrm{Spec}(A)$ (see e.g. [EG21, Prop. 5.1.9]), we see that

$$\mathcal{C}_{1,0} \simeq [LG^+/_\varphi]$$

where LG^+ denotes the functor $A \mapsto \mathfrak{S}_A^\times$ and the quotient $/_\varphi$ is via the action of $LG^+(A)$ on itself by φ-twisted conjugation: $g \cdot M := gM\varphi(g)^{-1}$. We have $\mathfrak{S}_A = (W(k) \otimes_{\mathbf{Z}_p} A)[[t]] \simeq \prod_{0 \leq j \leq f-1} A[[t]], x \otimes a \mapsto (\varphi^j(x)a)_j$. Under this identification, the action of φ on $\mathfrak{S}_A$ is given by $\varphi : (h_j)_j \mapsto (\varphi(h_{j+1}))_j$; here we also denote by φ the A-linear action on $A[[t]]$ taking $t \mapsto t^p$. It is then easy to see that the map $\prod_j A[[t]]^\times \to A[[t]]^\times, (h_j)_j \mapsto \prod_j \varphi^j(h_j)$ induces an equivalence of groupoids

$$LG^+(A)/_\varphi \simeq A[[t]]^\times/_{\varphi^f}.$$

Now since $A[[t]]^\times = A^\times \times (1 + tA[[t]])$ and elements in $1 + tA[[t]]$ are "killed" in the quotient $/_{\varphi^f}$ (given any $h \in 1 + tA[[t]]$, the series $g := \prod_{n\geq 0} \varphi^{fn}(h) \in 1 + tA[[t]]$ is well-defined and satisfies $h = g/\varphi^f(g)$), we obtain $A[[t]]^\times/_{\varphi^f} \simeq A^\times/A^\times$ with $A^\times$ acting trivially on itself, whence $\mathcal{C}_{1,0} \simeq [\mathbf{G}_m/\mathbf{G}_m]$. Unwinding definitions, one checks that a quasi-inverse is given by $a \in A^\times \mapsto D_{k,a} \otimes_{W(k)\otimes_{\mathbf{Z}_p} A} \mathfrak{S}_A$, where $D_{k,a}$ is the rank one φ-module from Lemma (1.2.2). This description also implies that the diagram (1.3.8.1) commutes (again by unwinding definitions), as claimed.

In conclusion, we deduce that the composition $[\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\mathrm{ur}_x} \mathcal{X}_1 \to \mathcal{R}_1$ is proper by [EG21, Thm. 5.4.11 (1)][2]. As the diagonal of the second map is a closed immersion by [EG23, Prop. 3.7.4], the usual graph argument shows that the first map is proper, as wanted. □

The following proof was actually our first approach; it is somewhat more complicated than the previous proof in that it makes use of the existence of a certain crystalline substack of $\mathcal{X}_1$.

Second proof of Lemma (1.3.6). We want to show that the monomorphism $\mathrm{ur}_x : [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ is a closed immersion. First recall that by [EG23, Thm. 4.8.12], $\mathcal{X}_1$ admits a closed $\mathcal{O}$-flat p-adic formal algebraic substack $\mathcal{X}_1^{\mathrm{ur}}$, which is uniquely characterized by the property that, for any finite flat $\mathcal{O}$-algebra Λ, $\mathcal{X}_1^{\mathrm{ur}}(\Lambda)$ is the subgroupoid of $\mathcal{X}_1(\Lambda)$ consisting of characters $G_K \to \Lambda^\times$ which are (after inverting p) crystalline of Hodge–Tate weights 0, or equivalently, unramified characters $G_K \to \Lambda^\times$. (Note that we are free to enlarge the field of coefficients E; in particular we may assume that it contains the Galois closure of K so that the running assumption of [EG23, Thm. 4.8.12] is satisfied.) We will show that the map ur_x factors through an isomorphism $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \simeq \mathcal{X}_1^{\mathrm{ur}}$, proving the lemma. Before continuing, we note that here it will be crucial to work directly over $\mathcal{O}$ (as opposed to the previous proof where we work modulo ϖ^a for some $a \geq 1$).

We begin by showing that ur_x factors through the closed substack $\mathcal{X}_1^{\mathrm{ur}}$, or equivalently, that the closed immersion

$$\mathcal{X}_1^{\mathrm{ur}} \times_{\mathcal{X}_1} [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m]$$

is an isomorphism. As the target is a p-adic formal algebraic stack of finite type and flat over $\mathrm{Spf}(\mathcal{O})$ (since it admits a smooth cover by the p-adic formal algebraic space $\widehat{\mathbf{G}}_m$, which is of finite type and flat over $\mathrm{Spf}(\mathcal{O})$), it follows from [LLLM23, Lem. 7.2.6 (3)] that it suffices to show that for any morphism $\mathrm{Spf}(\Lambda) \to [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m]$ whose source is a finite flat $\mathcal{O}$-algebra (endowed with the p-adic topology), the composite $\mathrm{Spf}(\Lambda) \to [\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$ factors through $\mathcal{X}_1^{\mathrm{ur}}$. This is clear as $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m](\Lambda)$ is also equivalent (via the monomorphism $[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1$) to the subgroupoid of $\mathcal{X}_1(\Lambda)$ consisting of unramified characters $G_K \to \Lambda^\times$.

[2] In this context (of usual Breuil–Kisin modules), a similar properness is first proved in [PR09, Cor. 2.6]. Note however that the definition of étale φ-modules in [EG21] (and [EG23]) is slightly less restrictive than that in [PR09]: namely, in *loc. cit.*, the authors demand furthermore that such a module M is free *fpqc* locally on $\mathrm{Spec}(A)$. In the rank 1 case, it follows *a posteriori* from the explicit description in Corollary (1.3.12) below that this local freeness is a consequence of M being projective (and even holds Zariski locally), and hence the two definitions yield the same stack; in general we do not know whether or not this is true.

It remains to show that the induced monomorphism

$$[\widehat{\mathbf{G}}_m/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{X}_1^{\mathrm{ur}}.$$

is in fact an isomorphism. As the source and target are both p-adic formal algebraic stacks which are of finite type and flat over $\mathrm{Spf}(\mathcal{O})$, and moreover $\mathcal{X}_1^{\mathrm{ur}}$ is analytically unramified by Lemma (1.3.10) below, it suffices, by [LLLM23, Lem. 7.2.6 (1)], to check that the above map induces an isomorphism on any finite flat $\mathcal{O}$-algebra Λ. This follows again from the fact that both sides admit the same moduli description (namely, as unramified characters) on these points. □

The following lemma is presumably standard, but for lack of a reference, we include a proof here.

Lemma 1.3.9. *Let $\mathcal{X}$ be a p-adic formal algebraic stack locally of finite type over* $\mathrm{Spf}(\mathcal{O})$*. If $\mathcal{X}$ admits reduced Noetherian versal rings at all finite type points, then $\mathcal{X}$ is (residually Jacobson and) analytically unramified in the sense of [Eme, Rmk. 8.23].*

Proof. Let $\mathcal{X}'$ be the associated reduced formal algebraic substack of $\mathcal{X}$, as defined in [Eme, Ex. 9.10]. We claim that $\mathcal{X}'$ is analytically unramified. Choose a smooth (in particular, representable by algebraic spaces) surjection $\coprod_i \mathrm{Spf}(B_i) \to \mathcal{X}$ where each B_i is a p-adically complete $\mathcal{O}$-algebra. By construction of $\mathcal{X}'$, we know that the base change $\mathcal{X}' \times_{\mathcal{X}} \mathrm{Spf}(B_i)$ is identified with $\mathrm{Spf}(B_i)_{\mathrm{red}}$ (see [Eme, Ex. 9.10]), and furthermore that each $\mathrm{Spf}(B_i)_{\mathrm{red}}$ is analytically unramified (e.g. by [Eme, Cor. 8.25], applied to the finite type adic map $\mathrm{Spf}(B_i)_{\mathrm{red}} \to \mathrm{Spf}(\mathcal{O})$). As $\mathcal{X}'$ receives a smooth surjection from the disjoint union $\coprod_i \mathrm{Spf}(B_i)_{\mathrm{red}}$ of analytically unramified affine formal algebraic spaces, it is analytically unramified by definition.

We now show that the closed immersion $\mathcal{X}' \hookrightarrow \mathcal{X}$ is in fact an isomorphism (this will imply that $\mathcal{X}$ is analytically unramified, as desired). As this can be checked at the level of Artinian points (e.g. by [EG23, Lem. 7.1.14]), it in turn suffices to work with versal rings. More precisely, let $x \in \mathcal{X}(A)$ be a point valued in a finite Artinian local $\mathcal{O}$-algebra A. By assumption, $\mathcal{X}$ admits a reduced Noetherian versal ring $\mathrm{Spf}(B)$ at the finite type point induced by x. By versality, the map $x : \mathrm{Spec}(A) \to \mathcal{X}$ factors through $\mathrm{Spf}(B) \to \mathcal{X}$, so it suffices to show that the latter map factors through $\mathcal{X}'$. Choose a smooth cover $\coprod_i \mathrm{Spf}(B_i) \to \mathcal{X}$ as before. It suffices to show that each base change $\mathrm{Spf}(B) \times_{\mathcal{X}} \mathrm{Spf}(B_i) \to \mathrm{Spf}(B_i)$ factors through $\mathrm{Spf}(B_i)_{\mathrm{red}}$, which in turn will follow once we show that given any smooth morphism $\mathrm{Spf}(C) \to \mathrm{Spf}(B) \times_{\mathcal{X}} \mathrm{Spf}(B_i)$, the composite $\mathrm{Spf}(C) \to \mathrm{Spf}(B) \times_{\mathcal{X}} \mathrm{Spf}(B_i) \to \mathrm{Spf}(B_i)$ factors through $\mathrm{Spf}(B_i)_{\mathrm{red}}$. As any ring map from B_i to a reduced ring necessarily factors through $(B_i)_{\mathrm{red}}$, it suffices to show that C is reduced. As B is complete local Noetherian and reduced, it is analytically unramified by definition. The upshot is that we have a smooth morphism $\mathrm{Spf}(C) \to \mathrm{Spf}(B)$ whose target is analytically unramified (and residually Jacobson, as $(\mathrm{Spf}(B))_{\mathrm{red}} = \mathrm{Spec}(B/\mathfrak{m}_B)$ is just a point). It then follows from [Eme, Lem. 8.20] that the source $\mathrm{Spf}(C)$ is also analytically unramified, and in particular that C is reduced, as required. □

Lemma 1.3.10. *The stack $\mathcal{X}_1^{\mathrm{ur}}$ is analytically unramified.*

Proof. This follows from Lemma (1.3.9), [EG23, Prop. 4.8.10] and [Kis08, Thm. 3.3.8]. □

1.3.3 The case of étale φ-modules

We end this note by briefly explaining how our method can also be used to give explicit descriptions of the stacks of rank one étale φ-modules (i.e. in the absence of a Γ-action), generalizing [EG23, Prop. 7.2.11] to a large class of coefficient rings. We will put ourselves in the context of Situation 2.2.15 in [EG23]. Namely, fix a finite field $k \simeq \mathbf{F}_{p^f}$ and write $\mathbf{A}^+ := W(k)[[t]]$. Let $\mathbf{A}$ be the p-adic completion of $\mathbf{A}^+[1/t]$. Let $\varphi : \mathbf{A} \to \mathbf{A}$ be a ring map lifting the Frobenius modulo p (we do not assume that φ preserves the subring $\mathbf{A}^+$). We note that φ necessarily induces the natural Frobenius on $W(k)$. If A is a p-adically complete $\mathbf{Z}_p$-algebra, we write $\mathbf{A}_A^+ := (W(k) \otimes_{\mathbf{Z}_p} A)[[t]]$ and let $\mathbf{A}_A$ be the p-adic completion of $\mathbf{A}_A^+[1/t]$, equipped with the A-linear extension of φ. Let $\mathcal{R}_1$ denote the stack of rank 1 projective étale φ-modules over $\mathbf{A}_A$, as defined in [EG23, §3.1] (we continue to assume that the coefficient ring $\mathcal{O}$ is large enough so that $k \hookrightarrow \mathbf{F}$). By $W_{k((t))}$ (resp. $G_{k((t))}$), we will mean the Weil group (resp. the Galois group) of the local field $k((t))$.

Proposition 1.3.11. *There is a natural isomorphism*

$$\left[X^{\mathrm{an}}_{k((t))}/\widehat{\mathbf{G}}_m\right] \xrightarrow{\sim} \mathcal{R}_1;$$

here $X^{\mathrm{an}}_{k((t))}$ denotes the functor on p-adically complete $\mathcal{O}$-algebras taking A to the set of continuous characters $\delta : W_{k((t))} \to A^\times$, and in the formation of the quotient stack, the $\widehat{\mathbf{G}}_m$-action is taken to be trivial.

Proof. Since the proof is very similar to that of Theorem (1.3.1), we will content ourselves with indicating the main steps.

- Recall firstly that Dee's equivalence (1.2.1) also admits a variant for étale φ-modules: if A is a finite Artinian $\mathcal{O}$-algebra, then there is a natural equivalence from the category of finite projective étale φ-modules over $\mathbf{A}_A$, to the category of finite free A-modules with a continuous action of $G_{k((t))}$ (see [Dee01, Thm. 2.1.27]).

- Construct a map $X^{\mathrm{an}}_{k((t))} \to \mathcal{R}_1, \delta \mapsto \mathbf{A}_A(\delta)$ extending Dee's equivalence from Artinian coefficients to all p-adically complete $\mathcal{O}$-algebras A; the main point is again the construction of the universal unramified character, which can be done similarly as in Definition (1.2.4) (namely, given $a \in A^\times$, we define $\mathbf{A}_A(\mathrm{ur}_a) := D_{k,a} \otimes_{W(k)\otimes_{\mathbf{Z}_p} A} \mathbf{A}_A$, where $D_{k,a}$ is the φ-module from Lemma (1.2.2)).

- Show that the automorphisms of a rank one étale φ-modules are simply the scalars. This amounts to showing that $(\mathbf{A}_A)^{\varphi=1} = A$ for any $\mathcal{O}/\varpi^a$-algebra A. The case $a = 1$ is clear as then $\varphi(t) = t^p$. The general case then follows easily from this by dévissage.

  Combining with the previous item, one obtains a natural monomorphism

  $$[X^{\mathrm{an}}_{k((t))}/\widehat{\mathbf{G}}_m] \hookrightarrow \mathcal{R}_1, \tag{1.3.11.1}$$

  which we want to show is an isomorphism.

- Reduce to proving essential surjectivity of (1.3.11.1) for reduced test rings. The proof is similar to that of Lemma (1.3.5), except that we have to be slightly more careful as we do not know a priori if $\mathcal{R}_1$ is limit preserving in general (recall that we do not assume that $\mathbf{A}^+$ is φ-stable). To overcome this, the idea is to reduce first to the case A is an $\mathbf{F}$-algebra by mimicking the argument there for the ideal ϖA. Note that the set of isomorphism classes of objects in $\mathcal{R}_1(A)$ lifting the trivial object in $\mathcal{R}_1(A/I)$ (I being a square-zero ideal) is now given by $\mathbf{A}_I/(\varphi - 1)$, or more precisely, by H^1 of the complex $[\mathbf{A}_I \xrightarrow{\varphi-1} \mathbf{A}_I]$ (concretely, any such object admits an $\mathbf{A}_A$-basis v so that $\varphi(v) = fv$ for some $f \in \ker(\mathbf{A}_A^\times \to \mathbf{A}_{A/I}^\times) = 1 + \mathbf{A}_I$; the corresponding cohomology class is then given by $[f-1]$). Again, here we cannot immediately reduce to the case where I is a finite $\mathcal{O}$-module as in the proof of Lemma (1.3.5). Instead, we will show directly that if $\varpi I = 0$ (which is the only case that we need), then $\mathbf{A}_I/(\varphi - 1) = \varinjlim \mathbf{A}_{I_i}/(\varphi - 1)$ where $\{I_i\}$ is the system of the finite sub-$\mathbf{F}$-modules of I. As $k \otimes_{\mathbf{F}_p} \mathbf{F} \simeq \prod \mathbf{F}$, we have $\mathbf{A}_I/(\varphi - 1) \simeq I((t))/(\varphi^f - 1)$ (here $\varphi(t) = t^p$ as $\varpi I = 0$). As elements in $tI[[t]]$ are killed in the quotient (given $h \in tI[[t]]$, the series $g := \sum_{n\geq 0} \varphi^{fn}(h) \in tI[[t]]$ is well-defined and satisfies $h = (1-\varphi^f)(g)$), it suffices to prove the analogous claim for $I[1/t]$, which is obvious (as we are working with polynomials, as opposed to (infinite) series). Finally, over $\mathbf{F}$, the subring $\mathbf{A}^+$ is φ-stable (as $\varphi(t) = t^p$), and we can invoke [EG21, Thm. 5.4.11 (3)] to deduce that $(\mathcal{R}_1)_{\mathbf{F}}$ is limit preserving, and hence reduce to the case A is a finite type $\mathbf{F}$-algebra. We now run the same argument, but for the (nilpotent) ideal A^{00}.

- By the previous item, it suffices to show that the map (1.3.11.1) induces an isomorphism on underlying reduced substacks. Again as there are only finitely many mod p characters of $G_{k((t))}$ up to unramified twist, we are reduced to showing that the map
$$[\mathbf{G}_m/\mathbf{G}_m]_{\mathbf{F}} \hookrightarrow (\mathcal{R}_1)_{\mathbf{F}} \tag{1.3.11.2}$$
induced by the universal unramified character is a closed immersion (see Lemma (1.3.7)). Again, over $\mathbf{F}$, the subring $\mathbf{A}^+$ is φ-stable (this is automatic in our first proof of Lemma (1.3.6)), and we can invoke [EG21, Thm. 5.4.11 (1)] to see that the natural map $\mathcal{C}_{1,0} \to \mathcal{R}_1$ is proper, where the source denotes the stack (over $\mathbf{F}$) of rank 1 projective φ-modules over $\mathbf{A}_A^+$ of height at most 0. The rest of the proof is now as before: namely, we show that the map (1.3.11.2) factors as $[\mathbf{G}_m/\mathbf{G}_m] \xrightarrow{\simeq} \mathcal{C}_{1,0} \to \mathcal{R}_1$ with the first map being given by $a \in A^\times \mapsto D_{k,a} \otimes_{W(k)\otimes_{\mathbf{Z}_p} A} \mathbf{A}_A^+$.

□

Specializing to the case $\mathbf{A} = \mathbf{A}_K$, and combing with the Fontaine–Wintenberger isomorphism $G_{K_{\mathrm{cyc}}} \simeq G_{k_\infty((t))}$ (which restricts to an isomorphism of Weil subgroups), we recover the following result.

Corollary 1.3.12 ([EG23, Prop. 7.2.11]). *There is an isomorphism*
$$\left[\left(\mathrm{Spf}(\mathcal{O}[[I_{K_{\mathrm{cyc}}}^{\mathrm{ab}}]]) \times \widehat{\mathbf{G}}_m\right)/\widehat{\mathbf{G}}_m\right] \xrightarrow{\sim} \mathcal{R}_{K,1}$$
(again, in the formation of the quotient stack, the $\widehat{\mathbf{G}}_m$-action is taken to be trivial).

Bibliography

[CKZ21] Annie Carter, Kiran S. Kedlaya, and Gergely Zábrádi. "Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules". In: *Doc. Math.* 26 (2021), pp. 1329–1393.

[Dee01] Jonathan Dee. "Φ-Γ modules for families of Galois representations". In: *J. Algebra* 235.2 (2001), pp. 636–664.

[Eme] Matthew Emerton. *Formal algebraic stacks*. Avaliable at: http://www.math.uchicago.edu/~emerton/pdffiles/formal-stacks.pdf.

[EG21] Matthew Emerton and Toby Gee. "'Scheme-theoretic images' of morphisms of stacks". In: *Algebr. Geom.* 8.1 (2021), pp. 1–132.

[EG23] Matthew Emerton and Toby Gee. *Moduli stacks of Étale (ϕ, Γ)-modules and the existence of crystalline lifts*. Vol. 215. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2023, pp. ix+298.

[Kis08] Mark Kisin. "Potentially semi-stable deformation rings". In: *J. Amer. Math. Soc.* 21.2 (2008), pp. 513–546.

[LLLM23] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra. "Local models for Galois deformation rings and applications". In: *Invent. Math.* 231.3 (2023), pp. 1277–1488.

[PR09] G. Pappas and M. Rapoport. "Φ-modules and coefficient spaces". In: *Mosc. Math. J.* 9.3 (2009), 625–663, back matter.

[Sta23] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.

Chapter 2

Moduli stacks of Lubin–Tate (φ, Γ)-modules

2.1 Introduction

Let $K/\mathbf{Q}_p$ be a finite extension with algebraic closure $\overline{K}$. A basic idea in p-adic Hodge theory is to consider "deeply ramified" subextensions $K \subseteq K_\infty \subseteq \overline{K}$ which encodes most interesting ramifications of $\overline{K}/K$, yet whose Galois group is simply enough to control. When K_∞/K is the cyclotomic extension, this idea was realized in [Fon90], where Fontaine introduced the notion of étale (φ, Γ)-modules, and showed that they are naturally equivalent to continuous representations of G_K on finite $\mathbf{Z}_p$-modules. Since its introduction, the concept of (φ, Γ)-modules has proved to be a very powerful tool in the study of p-adic Galois representations.

In [EG23], Emerton and Gee define and study stacks which interpolate Fontaine's (φ, Γ)-modules in families. More precisely, for each integer $d \geq 1$, they consider the stack $\mathcal{X}_d$ over $\mathrm{Spf}\mathbf{Z}_p$ whose groupoid of A-valued points, for any p-adically complete $\mathbf{Z}_p$-algebra A, is given by the groupoid of rank d projective étale (φ, Γ)-modules with A-coefficients. The geometry of $\mathcal{X}_d$ has been studied extensively in [EG23]. In particular, the authors show that $\mathcal{X}_d$ is a Noetherian formal algebraic stack, and moreover, its underlying reduced substack is an algebraic stack of finite type over $\mathbf{F}_p$, whose irreducible components admit a natural labelling by Serre weights. The stack $\mathcal{X}_d$ is also expected to play a critical role in the emerging categorical p-adic local Langlands program (cf. [EGH22]).

With an eye toward realizing a p-adic local Langlands correspondence for fields other than $\mathbf{Q}_p$, there has been a growing interest in studying the analogue of Fontaine's notion in which the cyclotomic extension K_∞/K is replaced by a Lubin–Tate extension (in what follows, we will often refer to these objects simply as Lubin–Tate (φ, Γ)-modules). We will not try to survey these results, but instead refer the reader, for instance, to [KR09], [Sch17], and [KV22].

The goal of this chapter is to extend the aforementioned construction of Emerton–Gee to the Lubin–Tate setting. Before stating our main results, we need to introduce some notation. Let $\pi \in K$ be a fixed uniformizer, and let $\mathcal{F}_\phi$ be a Lubin-Tate group for π, with Frobenius power series $\phi(T) \in \mathcal{O}_K[[T]]$. The corresponding ring map $\mathcal{O}_K \to \mathrm{End}(\mathcal{F}_\phi)$ is denoted by $a \mapsto [a](T)$; in particular $[\pi](T) = \phi(T)$. Let K_∞/K be the extension generated by the torsion points of $\mathcal{F}_\phi$ and let $\chi : \Gamma := \mathrm{Gal}(K_\infty/K) \xrightarrow{\sim} \mathcal{O}_K^\times$ be the resuting Lubin–Tate character. For the purpose of this introduction, we simply let

$$\mathbf{A}_K := \widehat{\mathcal{O}_K((T))} = \left\{ \sum_{n \in \mathbf{Z}} a_n T^n \mid a_n \in \mathcal{O}_K \text{ and } a_n \to 0 \text{ as } n \to -\infty \right\},$$

where the hat denotes π-adic completion. The ring $\mathbf{A}_K$ is further endowed with a Frobenius $\varphi_q : f(T) \mapsto f([\pi](T))$ and an action of Γ given by $(g, f(T)) \mapsto f([\chi(g)](T))$ for $g \in \Gamma$; as the notation suggests, φ_q is a lift of the q-power Frobenius modulo π. An étale (φ_q, Γ)-module (over $\mathbf{A}_K$) is then, by definition, a finite $\mathbf{A}_K$-module endowed with commuting continuous semilinear actions of φ_q and Γ such that the linearization of φ_q an isomorphism. There is again a natural equivalence between étale (φ_q, Γ)-modules and representations of G_K on finite $\mathcal{O}_K$-modules. (One can be slightly more general by allowing also representations on $\mathcal{O}_F$-modules with F being a finite subextension of $K/\mathbf{Q}_p$. See Section 2.2 below.)

Fix now an integer $d \geq 1$. By definition, our stack $\mathcal{X}_{K,d}^{\mathrm{LT}}$ takes a π-adically complete $\mathcal{O}_K$-algebra A to the groupoid of rank d projective étale (φ_q, Γ)-modules over $\mathbf{A}_{K,A} := \mathbf{A}_K \widehat{\otimes}_{\mathcal{O}_K} A$.

Theorem 2.1.1 (Proposition 2.3.24)**.** *$\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a limit preserving Ind-algebraic stack over* $\mathrm{Spf}\mathcal{O}_K$*, with finitely presented affine diagonal.*

The proof of Theorem 2.1.1 follows closely the argument used in [EG23] for the stack $\mathcal{X}_{K,d}^{\mathrm{EG}}$ of rank d projective *cyclotomic* étale (φ, Γ)-modules. Namely, we will deduce the claimed properties for $\mathcal{X}_{K,d}^{\mathrm{LT}}$ from the corresponding properties of the stack of étale φ_q-modules.

The next result gives a comparison between $\mathcal{X}_{K,d}^{\mathrm{LT}}$ and the stack $\mathcal{X}_{K,d}^{\mathrm{EG}}$ in [EG23].

Theorem 2.1.2 (Corollary 2.4.2)**.** *There is an isomorphism*

$$\mathcal{X}_{K,d}^{\mathrm{LT}} \xrightarrow{\sim} \mathcal{X}_{K,d}^{\mathrm{EG}}.$$

The proof proceeds by using the descent results in [EG23] to reduce the statement to a comparison between étale φ_p-modules over $W(\mathbf{C}^\flat)\widehat{\otimes}_{\mathbf{Z}_p} A$ and étale φ_q-modules over $W_{\mathcal{O}_K}(\mathbf{C}^\flat)\widehat{\otimes}_{\mathcal{O}_K} A$.

As a consequence of Theorem 2.1.2 and the results in [EG23], we deduce the following refinement of Theorem 2.1.1 regarding the geometry of the stack of $\mathcal{X}_{K,d}^{\mathrm{LT}}$.

Corollary 2.1.3 (Corollary 2.4.3)**.** *$\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a Noetherian formal algebraic stack over* $\mathrm{Spf}\mathcal{O}_K$*. The underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}}$ is an algebraic stack of finite presentation over* $\mathbf{F}$*. Moreover, the irreducible components of $\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}}$ admits a natural labeling by Serre weights.*

We also introduce a version of the Herr complex ([Her98]) in the Lubin–Tate setting with coefficients, and give a new proof of the fact that this complex computes Galois cohomology (Theorem 2.4.15). We refer the reader to Subsection 2.4.2 for the definition of this complex. Finally, by using again the above comparison (Theorem 2.1.2), we are able to deduce the following result, which may be of independent interest.

Theorem 2.1.4 (Theorem 2.4.12)**.** *Let A is a finite type π-nilpotent $\mathcal{O}_K$-algebra, and let M be a finite projective étale (φ_q, Γ)-module with A-coefficients. Then the Lubin–Tate Herr complex associated to M is a perfect complex of A-modules, whose formation commutes with arbitrary finite type base change in A.*

2.2 Generalities on φ_q-modules

2.2.1 Setup

Fix a complete algebraic closure $\mathbf{C} := \widehat{\overline{\mathbf{Q}}}_p$ of $\mathbf{Q}_p$. All algebraic extensions of $\mathbf{Q}_p$ will be regarded as subfields of $\mathbf{C}$. Let $F/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$, uniformizer π_F, residue field k of cardinality q.

Let $\phi \in \mathcal{O}_F[[T]]$ be a Frobenius power series for π (i.e. $\phi(T) \equiv \pi T \bmod T^2$ and $\phi(T) \equiv T^q \bmod \pi$) and let $\mathcal{F}_\phi$ be the corresponding Lubin-Tate formal group law over $\mathcal{O}_F$. As usual, denote by $[\cdot]_\phi : \mathcal{O}_F \to \mathrm{End}_{\mathcal{O}_F}(\mathcal{F}_\phi)$ the unique ring homomorphism satisfying $[a]_\phi(T) \equiv aT \bmod T^2$ for all $a \in \mathcal{O}_F$, and $[\pi]_\phi = \phi$. We have a natural $\mathcal{O}_F$-module structure on $\mathfrak{m}_{\overline{F}}$ given by $x + y := \mathcal{F}_\phi(x, y)$ and $a.x := [a]_\phi(x)$ for $x, y \in \mathfrak{m}_{\overline{F}}$ and $a \in \mathcal{O}_F$. Let $F_n := F(\mathcal{F}_\phi[\pi^n])$ be the field obtained by adjoining to F all the π^n-torsion points of $\mathfrak{m}_{\overline{F}}$, and set $F_\infty := \bigcup_{n\geq 0} F_n$ (note that the extensions F_n/F (and hence F_∞/F) only depend on π and not on the choice of ϕ).

We write $\chi : G_F \twoheadrightarrow \widetilde{\Gamma}_F \xrightarrow{\sim} \mathcal{O}_F^\times$ for the Lubin-Tate character associated to $\mathcal{F}$, where $\widetilde{\Gamma}_F := \mathrm{Gal}(F_\infty/F)$[1]. We also let $T\mathcal{F}_\phi := \varprojlim_n \mathcal{F}_\phi[\pi^n]$ be the Tate module of $\mathcal{F}$. This is a free $\mathcal{O}_F$-module of rank 1. We have a G_F-equivariant map $\iota : T\mathcal{F}_\phi \to \mathcal{O}^\flat_{\widehat{F_\infty}} \subseteq \mathcal{O}^\flat_{\mathbf{C}}$ given by mapping $v = (v_0, v_1, \ldots) \in T\mathcal{F}_\phi$ to $\iota(v) := (v_0 \bmod \pi, v_1 \bmod \pi, \ldots)$. This map is indeed well-defined, since $\mathcal{O}^\flat_{\mathbf{C}}$ is naturally identified with $\varprojlim_{x\mapsto x^q} \mathcal{O}_{\mathbf{C}}/\pi$, and if $[\pi](v_{i+1}) = v_i$, then $v_{i+1}^q \equiv v_i \bmod \pi$ by the defining property of ϕ. Note also that the image of ι is in fact contained in the maximal ideal $\mathfrak{m}_{\widehat{F_\infty}^\flat}$ (see [Sch17, Rem. 2.1.1]).

Finally, if K is a finite extension of F, we let $K_n := KF_n, K_\infty := KF_\infty$, and $\widetilde{\Gamma}_K := \mathrm{Gal}(K_\infty/K)$.

2.2.2 Rings

We now introduce the various coefficient rings for our Lubin–Tate $(\varphi_q, \widetilde{\Gamma}_K)$-modules. In what follows, $W_{\mathcal{O}_F}(A) := W(A) \otimes_{W(\mathbf{F}_q)} \mathcal{O}_F$ will denote the ring of ramified Witt vectors with coefficients in a perfect $\mathbf{F}_q$-algebra A. We also denote by φ_q the natural Frobenius on $W_{\mathcal{O}_F}(A)$ (induced by functoriality and the q-power Frobenius on A).

Lemma 2.2.1. *There is a unique set-theoretic map $\{\cdot\} : \mathfrak{m}_{\mathbf{C}^\flat} \to W_{\mathcal{O}_F}(\mathcal{O}^\flat_{\mathbf{C}})$ lifting the inclusion $\mathfrak{m}_{\mathbf{C}^\flat} \subseteq \mathcal{O}^\flat_{\mathbf{C}}$, such that $[\pi]_\phi(\{x\}) = \varphi_q(\{x\})$ for all $x \in \mathfrak{m}_{\mathbf{C}^\flat}$. Moreover, $\{\cdot\}$ respects the action of G_F (where G_F acts on $W_{\mathcal{O}_F}(\mathcal{O}^\flat_{\mathbf{C}})$ by functoriality), and*

(i) *For all $a \in \mathcal{O}_F$ and $v \in T\mathcal{F}_\phi$, the series $[a]_\phi(\{\iota(v)\})$ converges and we have $[a]_\phi(\{\iota(v)\}) = \{\iota(av)\}$.*

(ii) *The action of G_F on $\{\iota(T\mathcal{F}_\phi)\} \subseteq W_{\mathcal{O}_F}(\mathcal{O}^\flat_{\mathbf{C}})$ factors through $\widetilde{\Gamma}_F$ and for $g \in \widetilde{\Gamma}_F$*

$$[\chi(g)]_\phi(\{\iota(v)\}) = \{\iota(gv)\} = \{g \cdot \iota(v)\} = g \cdot \{\iota(v)\}.$$

[1] There is a slight conflict with the notation Γ used in the Introduction; this is however to make it compatible with those in [EG23]. We hope that this won't cause any confusion.

.

Proof. This is [Col02, Lem. 9.3]. Concretely, the map $x \mapsto \{x\}$ is given by

$$\{x\} := \lim_n [\pi^n]_\phi([x^{q^{-n}}])$$

(recall that we always endow $W_{\mathcal{O}_K}(\mathcal{O}_{\mathbf{C}}^\flat)$ with the *weak topology*, i.e. the (π, u)-adic topology, where $u \in \mathbf{C}^\flat$ is an arbitrary pseudo-uniformizer). (Strictly speaking, unlike *loc. cit.*, here $\{\cdot\}$ is only defined on $\mathfrak{m}_{\mathbf{C}^\flat}$. This is because we are not assuming that ϕ is actually a polynomial; consequently, restricting the domain to $\mathfrak{m}_{\mathbf{C}^\flat}$ is necessary to ensure that the limit defining $\{x\}$ indeed exists. See [Sch17, Lem. 2.1.11] and the surrounding material.) □

We begin with the case where $K = F$. Let $v \in T\mathcal{F}_\phi$ be an $\mathcal{O}_F$-generator. There is an embedding $k[[T]] \hookrightarrow \mathcal{O}_{\widehat{F_\infty}}^\flat \subseteq \mathcal{O}_{\mathbf{C}}^\flat$ sending $T \mapsto \iota(v)$, whose image is identified with the ring of integers of the imperfect norm field $\mathbf{E}'_F := X_F(F_\infty)$ (cf. [Win83, §2]) via a canonical embedding $X_F(F_\infty) \hookrightarrow \widehat{F_\infty}^\flat \subseteq \mathbf{C}^\flat$ (this embedding identifies the target with the completion of the perfect closure of the source, see [Win83, Cor. 4.3.4]). Since $\{\iota(v)\}$ is a lift of $\iota(v)$ by construction, we obtain an embedding

$$\mathcal{O}_F[[T]] \hookrightarrow W_{\mathcal{O}_F}(\mathcal{O}_{\widehat{F_\infty}}^\flat) \subseteq W_{\mathcal{O}_F}(\mathcal{O}_{\mathbf{C}}^\flat)$$

sending $T \mapsto \{\iota(v)\}$ (the source being endowed with the (π, T)-topology, and the target with its weak topology), which extends further to a map

$$\widehat{\mathcal{O}_F((T))} \hookrightarrow W_{\mathcal{O}_F}(\widehat{F_\infty}^\flat) \subseteq W_{\mathcal{O}_F}(\mathbf{C}^\flat),$$

whose image we denote by $\mathbf{A}'_F$ (where the hat denotes the π-adic completion). It is a complete discrete valuation ring with uniformizer π and residue field $X_F(F_\infty) \cong k((T))$, which is in fact independent on the choice of $v \in T\mathcal{F}_\phi$ ([Sch17, Rem. 2.1.17]). By Lemma 2.2.1, $\mathbf{A}'_F$ are stable by φ_q and by the natural action of G_F on $W_{\mathcal{O}_F}(\mathcal{O}_{\mathbf{C}}^\flat)$. Moreover, the G_F-action on $\mathbf{A}'_F$ factors through $\widetilde{\Gamma}_F$.

To summarize, $\mathbf{A}'_F$ is $(\varphi_q, \widetilde{\Gamma}_F)$-equivariantly isomorphic to the ring

$$\widehat{\mathcal{O}_F((T))} \cong \left\{ \sum_{n \in \mathbf{Z}} a_n T^n \mid a_n \in \mathcal{O}_F \text{ and } a_n \to 0 \text{ as } n \to -\infty \right\},$$

where the actions of φ_q and $\widetilde{\Gamma}_F$ on the latter ring are given by $\varphi_q : f(T) \mapsto f([\pi]_\phi(T))$ and $g : f(T) \mapsto f([\chi(g)]_\phi(T))$ for $g \in \widetilde{\Gamma}_F$.

We now return to the case of a general finite extension K/F. Since $K_\infty = KF_\infty$ by definition, the norm field $\mathbf{E}'_K := X_K(K_\infty)$ associated to the extension K_∞/K is a finite separable extension of $\mathbf{E}'_F$ (despite the notation, the field $X_K(K_\infty)$ depends only on K_∞ and not on K itself, see [Win83, Rem. 2.1.4]). As $\mathbf{B}'_F := \mathbf{A}'_F[1/p]$ is a discrete valued field with uniformizer π and residue field $\mathbf{E}'_F$, it follows that there is a unique finite unramified extension of $\mathbf{B}'_F$ contained in the field $W(\mathbf{C}^\flat)[1/p]$ with residue field $\mathbf{E}'_K$. We denote this extension by $\mathbf{B}'_K$, and by $\mathbf{A}'_K$ its ring of integers, or equivalently, $\mathbf{A}'_K := \mathbf{B}'_K \cap W(\mathbf{C}^\flat)$. Thus, we see that $\mathbf{A}'_K$ is a discrete valuation ring, admitting π as a uniformizer, and that $\mathbf{A}'_K/\pi\mathbf{A}'_K = \mathbf{E}'_K$. In order to emphasize the dependence on F, we will sometimes denote $\mathbf{A}'_K$ by $\mathbf{A}'_{K|F}$, and similarly for the related notation. There is a natural lift of the q-power Frobenius φ_q from $\mathbf{E}'_K$ to $\mathbf{A}_K$. Furthermore the action of $\widetilde{\Gamma}_K$ on $\mathbf{E}'_K = X_K(K_\infty)$ induces an action of $\widetilde{\Gamma}_K$ on $\mathbf{A}'_K$, and this action commutes with that of φ_q.

As in [EG23, §2.1.9], it will be convenient for us to introduce a variant of the ring $\mathbf{A}'_K$. Let Δ_K be the torsion subgroup of $\widetilde{\Gamma}_K$. As $\widetilde{\Gamma}_K$ can be identified with an open subgroup of $\mathcal{O}_F^\times$ (via the Lubin–Tate character), we have an isomorphism $\widetilde{\Gamma}_K \cong \Gamma_K \times \Delta_K$, where $\Gamma_K \cong \mathbf{Z}_p^{\oplus[F:\mathbf{Q}_p]}$. We now let $\mathbf{A}_K := (\mathbf{A}'_K)^{\Delta}_K$.

Lemma 2.2.2. $\mathbf{A}_K$ *is again is a complete discrete valuation ring with uniformizer* π *and residue field* $\mathbf{E}_K := (\mathbf{E}'_K)^{\Delta_K}$.

Proof. Only the statement on residue field needs a proof. We claim that $\widetilde{\Gamma}_K$ (hence Δ_K) acts faithfully on $\mathbf{E}'_K$ (hence on $\mathbf{A}'_K$). Equivalently, we need to show that if $g \in G_K$ acts trivially on $\mathbf{E}'_K$, then necessarily $g \in G_{K_\infty}$. Indeed, in this case we can view g as an element in $G_{\mathbf{E}'_K}$. We can then find some $g' \in G_{K_\infty}$ which maps into g under the Fontaine–Wintenberger isomorphism $G_{K_\infty} \xrightarrow{\sim} G_{\mathbf{E}'_K}$. This means that the actions of g and g' on $(\mathbf{E}'_K)^{\mathrm{sep}}$ agree. As $(\mathbf{E}'_K)^{\mathrm{sep}}$ is dense in $\mathbf{C}^\flat$ (cf. [Sch17, Prop. 1.4.27]), they in fact agree on $\mathbf{C}^\flat$, hence $g = g' \in G_{K_\infty}$, as claimed. (We can also prove the claim by first reducing to the case $K = F$, and then use the explicit description of $\mathbf{E}'_F$ as $\mathbf{E}'_F = k((\iota(v))$, where recall that v is a generator of the Tate module $T\mathcal{F}_\phi$.)

It now follows from Artin's lemma in Galois theory that $\mathbf{A}'_K$ is finite free of rank $\#\Delta_K$ over $\mathbf{A}_K = (\mathbf{A}'_K)^{\Delta_K}$ (e.g. after passing to their fraction fields). In particular, the same is true for the induced extension of residue fields. Similarly, we see that $\mathbf{E}'_K$ has degree $\#\Delta_K$ over $(\mathbf{E}'_K)^{\Delta_K} =: \mathbf{E}_K$, and so the latter must coincide with the residue field of $\mathbf{A}_K = (\mathbf{A}'_K)^{\Delta_K}$. □

Clearly, $\mathbf{A}_K$ is φ_q-stable and equipped with an induced action of $\Gamma_K = \widetilde{\Gamma}_K/\Delta_K$. We let $\mathbf{B}_K := \mathbf{A}_K[1/\pi]$ be its fraction field, so that $\mathbf{B}_K = (\mathbf{B}'_K)^{\Delta_K}$.

If T'_K is any lift of a uniformizer of $\mathbf{E}'_K$ to $\mathbf{A}'_K$, then there is an identification $W_{\mathcal{O}_F}\widehat{(k'_\infty)((T'_K))} \xrightarrow{\sim} \mathbf{A}'_K$ (again the hat denotes the π-adic completion), where $k'_{K,\infty}$ denotes the residue field of K_∞. Similarly, for any lift T_K of a uniformizer of $\mathbf{E}_K$, we have an isomorphism $W_{\mathcal{O}_F}\widehat{(k_{K,\infty})((T_K))} \xrightarrow{\sim} \mathbf{A}_K$, where $k_{K,\infty}$ denotes the residue field of the subfield of K_∞ corresponding to the subgroup $\Delta_K \subseteq \widetilde{\Gamma}_K$.

In general, it seems difficult to explicitly write down formulas for the actions of φ_q and $\widetilde{\Gamma}_K$ on $\mathbf{A}'_K$. This is possible however in the case where K is unramified over F (equivalently, $K = K_0F$). Indeed, we claim that in this case the element $\iota(v)$ defined earlier is also a uniformizer of $\mathbf{E}'_K$. We have seen that it is a uniformizer of $\mathbf{E}'_F$, hence it suffices to show that $\mathbf{E}'_K$ is unramified over $\mathbf{E}'_F$, which is true because $[\mathbf{E}'_K : \mathbf{E}'_F] = [K_\infty : F_\infty] = [K : K \cap F_\infty] = [K : F] = [k_K : k_F] \leq [k_{K_\infty} : K_F]$. Thus, if we choose $T'_K := \{\iota(v)\}$, then $\mathbf{A}'_K = \widehat{\mathcal{O}_{K_0F}((T'_K))}$, and the actions of φ_q and $\widetilde{\Gamma}_K = \widetilde{\Gamma}_F$ on T'_K are given exactly as before. We furthermore let $T_K := \prod_{g\in\Delta_F} g \cdot \{\iota(v)\}$ so that $T_K \in \mathbf{A}_K$ (as it is fixed by $\Delta_F = \Delta_K$). By Lemma 2.2.3 below, we have $\mathbf{A}_K = \widehat{\mathcal{O}_{K_0F}((T_K))}$, and furthermore, the "integral" subring $\mathbf{A}_K^+ := \mathcal{O}_{K_0F}[[T_K]]$ is stable under the actions of φ_q and Γ_K.

Lemma 2.2.3. *Assuming K/F is unramified. Then $\mathbf{A}_K = \widehat{\mathcal{O}_{K_0F}((T_K))}$. Furthermore, the subring $\mathbf{A}_K^+ := \mathcal{O}_{K_0F}[[T_K]]$ is (φ_q,Γ_K)-stable; in fact, $\varphi_q(T_K) \in T_K\mathbf{A}_K^+$ and $g(T_K) \in T_K\mathbf{A}_K^+$ for all $g \in \Gamma_K$.*

Proof. For the first statement, it suffices to show that the image $\overline{T}_K$ of T_K in $\mathbf{E}'_K = k_K((\iota(v)))$ is a uniformizer of the residue field $\mathbf{E}_K := (\mathbf{E}'_K)^{\Delta_K}$ of $\mathbf{A}_K$. Since $\mathbf{E}'_K/(\mathbf{E}'_K)^{\Delta_K}$ is a totally ramified extension of degree $\#\Delta$, we only need to show that $|\overline{T}_K|_\flat = |\iota(v)|_\flat^{\#\Delta}$. To see this, first recall that for each $n \geq 1$, there is a canonical isomorphism $(\mathcal{O}_F/\pi^n)^\times \xrightarrow{\sim} \mathrm{Gal}(F_n/F)$ defined by $a \mapsto \sigma_a := (x \mapsto [a]_\phi(x), x \in F_\phi[\pi^n])$. We can now compute

$$\begin{aligned}
|\overline{T}_K|_\flat &= \prod_{a\in\Delta} |\iota(av)|_\flat \\
&= \prod_{a\in\Delta} |([a]_\phi(v_0) \bmod \pi, [a]_\phi(v_1) \bmod \pi, \ldots)|_\flat \\
&\overset{\text{def}}{=} \prod_{a\in\Delta} \lim_n |[a]_\phi(v_n)|^{q^n} \\
&= \prod_{a\in\Delta} \lim_n |\sigma_a(v_n)|^{q^n} \\
&= \prod_{a\in\Delta} \lim_n |v_n|^{q^n} \\
&\overset{\text{def}}{=} |(v_0 \bmod \pi, v_1 \bmod \pi, \ldots)|_\flat^{\#\Delta} = |\iota(v)|_\flat^{\#\Delta},
\end{aligned}$$

as desired. Set $\mathbf{A}_K^+ := \mathcal{O}_{K_0F}[[T_K]]$ with T_K defined as above. We check that $\mathbf{A}_K^+$ is stable under the actions of φ_q and Γ_K. As the ring $\mathcal{O}_{K_0F}[[T'_K]]$ is visibly $(\varphi_q,\widetilde{\Gamma}_K)$-stable, it suffices to show that $\mathbf{A}_K \cap \mathcal{O}_{K_0F}[[T'_K]] = \mathbf{A}_K^+$. To see this, let $f \in \mathbf{A}_K \cap \mathcal{O}_{K_0F}[[T'_K]] \subseteq \mathcal{O}_{K_0F}[[T'_K]]$. Then $f \bmod \pi \in \mathbf{E}_K \cap \mathcal{O}_{\mathbf{E}'_K} = \mathcal{O}_{\mathbf{E}_K} = k_K[[\overline{T}_K]]$ whence $f = f_0 + \pi g_1$ for some $f_0 \in \mathbf{A}_K^+$ and $g_1 \in \mathcal{O}_{K_0F}[[T'_K]]$. Since f and f_0 are both fixed by Δ_K, the same is true for g_1. Thus, by induction, we can find a sequence $(f_n) \subseteq \mathbf{A}_K^+$ such that

$$f \equiv f_0 + \pi f_1 + \ldots + \pi^n f_n \pmod{\pi^{n+1}\mathcal{O}_{K_0F}[[T'_K]]}$$

for all $n \geq 0$. Since $\mathcal{O}_{K_0F}[[T_K]]$ and $\mathcal{O}_{K_0F}[[T'_K]]$ are both π-adically completed, this implies $f \in \mathbf{A}_K^+$, as required. By using the same argument (i.e. consider the reductions modulo π), we see that $\varphi_q(T_K) \in T_K\mathbf{A}_K^+$ and $g(T_K) \in T_K\mathbf{A}_K^+$ for all $g \in \Gamma_K$. □

For each finite extension K/F, we let K_{cyc} denote the subfield of K_∞ corersponding to the torsion subgroup Δ_K of $\widetilde{\Gamma}_K$ (it can also be characterized as the unique subextension of K_∞/K whose Galois groups is isomorphic to $\mathbf{Z}_p^{\oplus[F:\mathbf{Q}_p]}$). The following definition is modeled on [EG23, Defn. 2.1.12].

Definition 2.2.4. We say that K is F-*basic* if it is contained in $(K_0F)_{\mathrm{cyc}}$, or equivalently, $K_{\mathrm{cyc}} = (K_0F)_{\mathrm{cyc}}$ (to see this equivalence, it suffices to note that any open subgroup of $\mathbf{Z}_p^{\oplus[F:\mathbf{Q}_p]}$ is itself isomorphic to $\mathbf{Z}_p^{\oplus[F:\mathbf{Q}_p]}$).

Thus, if K is F-basic, then $\mathbf{A}_K = \mathbf{A}_{K_0F}$ and the action of Γ_K is just the restriction of the action of Γ_{K_0F} om $\mathbf{A}_{K_0F}$. Thus by Lemma 2.2.3, we can choose $T_K := T_{K_0F}$ so that the integral subring $\mathbf{A}_K^+ := \mathcal{O}_{K_0F}[[T_K]]$ is (φ_q, Γ_K)-stable, and moreover $\varphi_q(T_K) \in T_K\mathbf{A}_K^+$ and $g(T_K) \in T_K\mathbf{A}_K^+$ for all $g \in \Gamma_K$ (and similarly for $\mathbf{A}'_K$ and $(\mathbf{A}'_K)^+$). In general, as explained above, we can still choose some T_K so that $\mathbf{A}_K = W_{\mathcal{O}_F}(\widehat{k_{K,\infty})}((T_K))$; once this is done, we will set $\mathbf{A}_K^+ := W_{\mathcal{O}_F}(k_{K,\infty})[[T_K]]$. We endow $\mathbf{A}_K^+$ with the (π, T_K)-adic topology, and $\mathbf{A}_K$ with the unique topology for which a fundamental system of open neighborhoods of $0 \in \mathbf{A}_K$ is given by the sets $\pi^n\mathbf{A}_K + T_K^m\mathbf{A}_K^+, n, m \geq 0$[2] (one can check easily that the subspace topology on $\mathbf{A}_A^+$ is indeed the (π, T_K)-adic topology).

Remark 2.2.5. We have seen that if $u \in \mathbf{A}_K$ is any element lifting a uniformizer of the residue field of $\mathbf{A}_K$, then $\mathbf{A}_K$ can be identified with $W_{\mathcal{O}_F}(\widehat{k_{K,\infty})}((u))$. However, for a general choice of u, the integral subring $\mathbf{A}_K^+ := W_{\mathcal{O}_F}(k_{K,\infty})[[u]]$ may not be φ_q-stable (cf. [Her98, §1.1.2.2]). As in [EG23], the advantage of having φ_q-stability for $\mathbf{A}_K^+$ is that it allows us to invoke the geometric properties proved in [EG21] of (a variant of) the stack $\mathcal{R}_d$ of étale φ_q-modules over $\mathbf{A}_K$ (see Theorem 2.3.7 below).

2.2.3 Coefficients

As we will be interested in moduli stacks parametrizing famillies of Lubin–Tate (φ_q, Γ_K)-modules (or related variants), it is necessary to introduce the version of the various rings considered in Subsection 2.2.2 relative to a varying coefficient ring A. As the stacks we consider will liver over the p-adic formal scheme $\mathrm{Spf}\mathbf{Z}_p$, the test ring A will be always assumed to be a p-adically complete. In fact, as in [EG23], it will be convenient to introduce an auxiliary base ring over which A lives. More precisely, we will fix a finite extension E/F with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field $\mathbf{F}$; accordingly, A will be taken to be a ϖ-adically complete $\mathcal{O}$-algebra. (We will sometimes need to assume E is large enough, e.g. so that it contains all the images of all embeddings $K \hookrightarrow \overline{\mathbf{Q}}_p$, but for now this is irrelevant to us.)

For A as above, we will set $\mathbf{A}_{K,A}^+ := \mathbf{A}_K^+\widehat{\otimes}_{\mathcal{O}_F}A$, where the completed tensor product is taken with respect to the ϖ-adic topology on A and the (π, T_K)-adic topology topology on $\mathbf{A}_K^+$ (where as above T_K is a lift of a uniformizer of $\mathbf{E}_K$ to $\mathbf{A}_K^+$ so that we have $\mathbf{A}_K^+ = W_{\mathcal{O}_F}(k_{K,\infty})[[T_K]]$), i.e.

$$\mathbf{A}_{K,A}^+ := \varprojlim_n \mathbf{A}_K^+/(p, T_K)^n \otimes_{\mathcal{O}_F} A = \varprojlim_m(\varprojlim_n \mathbf{A}_K^+/(p^m, T_K^n) \otimes_{\mathcal{O}_F} A).$$

Similarly, we define $\mathbf{A}_{K,A}$ to be the completed tensor product $\mathbf{A}_K\widehat{\otimes}_{\mathcal{O}_F}A$ taken with respect to the π-adic topology on A and the natural topology on $\mathbf{A}_K$ (cf. the discussion following Definition 2.2.4). Concretely, we have

$$\mathbf{A}_{K,A} := \varprojlim_m((\varprojlim_n \mathbf{A}_K^+/(p^m, T_K^n) \otimes_{\mathcal{O}_F} A)[1/T_K]).$$

Thus, we see that $\mathbf{A}_{K,A}^+ \cong (W_{\mathcal{O}_F}(k_{K,\infty}) \otimes_{\mathcal{O}_F} A)[[T_K]]$, whereas $\mathbf{A}_{K,A}$ is the p-adic completion of $\mathbf{A}_{K,A}^+[1/T_K]$. Similarly, we can define the rings $(\mathbf{A}'_{K,A})^+$ and $\mathbf{A}'_{K,A}$ (use T'_K in place of T_K).

[2]This is designed so that the quotient topology on the local field $\mathbf{A}_K/\pi \cong k_{K,\infty}((T_K))$ is its valuation topology.

Recall that we have the actions of φ_q and Γ_K on $\mathbf{A}_K$. We can extend these actions on each $\mathbf{A}_{K,A}$ by decreeing that they are A-linear. In case $\mathbf{A}_K^+$ is (φ_q, Γ)-stable (e.g. when K is F-basic), the same is true of the ring $\mathbf{A}_{K,A}^+$ for each A. For a more detailed discussion on these actions, see [EG23, Prop. 2.2.17] (which in turn rests on [EG23, Lem. B.31] and [EG23, Lem. B.34]).

2.2.4 The relationship with Galois representations

Denote by $\mathbf{A}_K^{\mathrm{ur}} \subseteq W_{\mathcal{O}_F}(\mathbf{C}^\flat)$ the π-adic completion of the maximal unramified extension of $\mathbf{A}'_K$ (or equivalently of $\mathbf{A}'_F$) in $W_{\mathcal{O}_F}(\mathbf{C}^\flat)$. As $\mathbf{A}'_K$ is preserved by the natural actions of φ_q and G_K on $W_{\mathcal{O}_K}(\mathcal{O}_{\mathbf{C}}^\flat)$, the same is true for the ring $\mathbf{A}_K^{\mathrm{ur}}$ by functoriality.

Let $(\mathbf{E}'_K)^{\mathrm{sep}}$ be the separable closure of $\mathbf{E}'_K$ inside $\mathbf{C}^\flat$. It is clear that the natural action of G_K on $\mathbf{C}^\flat$ preserves $(\mathbf{E}'_K)^{\mathrm{sep}}$. If, in addition, $g \in \mathrm{Gal}(\overline{K}/K_\infty)$, then g fixes $\widehat{K_\infty}^\flat$, and in particular, $\mathbf{E}'_K$. In this way, we obtain a natural homomorphism

$$G_{K_\infty} \to G_{\mathbf{E}'_K}$$

which turns out to be an isomorphism by the theory of norm fields of Fontaine–Wintenberger. From this, we can deduce the following result.

Theorem 2.2.6. *The functor*

$$V \mapsto D(V) := (V \otimes_{\mathcal{O}_F} \mathbf{A}_K^{\mathrm{ur}})^{G_{K_\infty}}$$

gives an equivalence of categories between the category $\mathrm{Rep}_{\mathcal{O}_F}(G_K)$ of continuous G_K-representations on finite $\mathcal{O}_F$-modules and the category of finite étale $(\varphi_q, \widetilde{\Gamma}_K)$-modules over $\mathbf{A}'_K$. A quasi-inverse functor is given by $M \mapsto T(M) := (\mathbf{A}_K^{\mathrm{ur}} \otimes_{\mathbf{A}'_K} M)^{\varphi_q = 1}$.

Furthermore, V is free of rank d over $\mathcal{O}_F$ if and only if $D(V)$ is free of rank d over $\mathbf{A}'_K$.

Proof. This is proved in [KR09, Thm. 1.6], see also [Sch17, Thm. 3.3.10] for a more detailed exposition. (Strictly speaking, the latter reference only considers the case $K = F$, but the proof for general K follows the same strategy.) □

In fact, there is also an analogous equivalence of categories using (φ_q, Γ_K)-modules, and taking Δ_K-invariants gives an equivalence of categories between $(\varphi_q, \widetilde{\Gamma}_K)$-modules and (φ_q, Γ_K)-modules (see Lemma 2.2.8 below). Since it is the version with $\mathbf{A}_K$-coefficients that we will work mostly with, let us record the equivalence for (φ_q, Γ_K)-modules over $\mathbf{A}_K$ separately below.

Theorem 2.2.7. *(Recall that K_{cyc} denotes the subextension of K_∞ corresponding to the subgroup Δ_K of $\widetilde{\Gamma}_K$.) The functor*

$$V \mapsto D(V) := (V \otimes_{\mathcal{O}_F} \mathbf{A}_K^{\mathrm{ur}})^{G_{K_{\mathrm{cyc}}}}$$

gives an equivalence of categories between the category $\mathrm{Rep}_{\mathcal{O}_F}(G_K)$ of continuous G_K-representations on finite $\mathcal{O}_F$-modules and the category of finite étale (φ_q, Γ_K)-modules over $\mathbf{A}_K$. A quasi-inverse functor is given by $M \mapsto T(M) := (\mathbf{A}_K^{\mathrm{ur}} \otimes_{\mathbf{A}_K} M)^{\varphi_q = 1}$.

Furthermore, V is free of rank d over $\mathcal{O}_F$ if and only if $D(V)$ is free of rank d over $\mathbf{A}_K$.

Lemma 2.2.8. *The functor*

$$M \mapsto M' := \mathbf{A}'_K \otimes_{\mathbf{A}_K} M$$

gives an equivalence of categories between finite étale (φ_q, Γ_K)-module over $\mathbf{A}_K$ and finite étale $(\varphi_q, \widetilde{\Gamma}_K)$-module over $\mathbf{A}'_K$. A quasi-inverse functor is given by $M' \mapsto (M')^{\Delta}$.

Again, M is free of rank d over $\mathbf{A}_K$ if and only if M' is free of rank d over $\mathbf{A}'_K$.

Proof. We first show that the natural map $(M')^{\Delta} \otimes_{\mathbf{A}_K} \mathbf{A}'_K \to M'$ is an isomorphism for any finite étale $(\varphi_q, \widetilde{\Gamma}_K)$-module M' over $\mathbf{A}'_K$ (the induced φ_q-action on M^{Δ} is then automatically étale as its linearization becomes an isomorphism after base changing along the faithfully flat map $\mathbf{A}_K \hookrightarrow \mathbf{A}'_K$.)

As $M' \xrightarrow{\sim} \varprojlim_n M'/\pi^n$, we may reduce to the case where M' is killed by π^n for some $n \geq 1$ (as $\mathbf{A}'_K$ is finite free over $\mathbf{A}_K$, tensoring with $\mathbf{A}'_K$ commutes with inverse limits). The case $n = 1$ follows directly from Hilbert's 90 theorem. Note also that $H^1(\Delta, M'') = 0$ in this case. Now assume $n \geq 1$ and M' is killed by π^{n+1}. Let $M'' := \pi^n M'$ and $M''' := M'/M''$ so that we have an exact sequence $0 \to M'' \to M' \to M''' \to 0$. As M'' is killed by π, $H^1(\Delta, M'') = 0$, and the sequence $0 \to (M'')^{\Delta} \to (M')^{\Delta} \to (M''')^{\Delta}_K \to 0$ remains exact. Thus, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{A}'_K \otimes_{\mathbf{A}_K} (M'')^{\Delta} & \longrightarrow & \mathbf{A}'_K \otimes_{\mathbf{A}_K} (M')^{\Delta} & \longrightarrow & \mathbf{A}'_K \otimes_{\mathbf{A}_K} (M''')^{\Delta} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M'' & \longrightarrow & M' & \longrightarrow & M''' & \longrightarrow & 0. \end{array}$$

By induction, the outer vertical maps are isomoprhisms, and hence the same is true of the middle map.

Using a similar inductive argument, we can show that for any finite étale (φ_q, Γ_K)-module M over $\mathbf{A}_K$, the natural injection $M \hookrightarrow M' := \mathbf{A}'_K \otimes_{\mathbf{A}_K} M$ identifies M with $(M')^{\Delta}$ (injectivity follows from the fact that $\mathbf{A}'_K$ is faithfully flat over $\mathbf{A}_K$). The last statement is clear as any finite π-torsion free module over the discrete valuation ring $\mathbf{A}_K$ is necessarily free. □

2.3 Moduli stacks of φ_q-modules and Lubin-Tate (φ_q, Γ_K)-modules

2.3.1 Moduli stacks of φ_q-modules

In this subsection, we briefly define the moduli stacks of étale φ_q-modules, and show in particular that they are Ind-algebraic stacks (Theorem 2.3.7).

Setup 2.3.1. Fix a finite extension $l \supseteq k \supseteq \mathbf{F}_p$. let $\mathbf{A}^+ := W(l)[[T]]$, and let $\mathbf{A}$ be the p-adic completion of $\mathbf{A}^+[1/T]$. Let φ be a ring endomorphism of $\mathbf{A}$ which is trivial on the subring $W(k) \subseteq W(l)$, and moreover congruent to the q-power Frobenius modulo p for some fixed power q of p.

As before, we will fix a finite extension $E/W(k)[1/p]$ with uniformizer ϖ and ring of integers $\mathcal{O}$, which will serve as the base for our test rings. If A is a ϖ-adically complete $\mathcal{O}$-algebra, we set $\mathbf{A}_A^+ := (W(l) \otimes_{W(k)} A)[[T]]$, equipped with the (p,T)-adic topology. We also let $\mathbf{A}_A$ be the p-adic completion of $\mathbf{A}_A^+[1/T]$, and endow it with the unique topology for which a fundamental system of neighborhoods of 0 is given by the sets $p^n\mathbf{A}_A + T^m\mathbf{A}_A^+, n, m \geq 0$ (again one checks easily that the subspace topology on $\mathbf{A}_A^+$ is indeed the (p,T)-adic topology). In particular, we see that the rings $\mathbf{A}_{K,A}^+, \mathbf{A}_{K,A}$ introduced earlier are special case of the this construction (with $l = k_{K,\infty}$ being the residue field of K_{cyc} and k being the residue field of F).

Again, as in [EG23, Prop. 2.2.17], we can extend φ to a continuous A-linear action on $\mathbf{A}_A$. In case $\mathbf{A}^+$ is moreover φ-stable, the same is true of each $\mathbf{A}_A^+$.

Remark 2.3.2. Our setting here is slightly more general than that considered in [EG21, Chap. 5] as we are considering power series (or Laurent series) over rings of the form $W(l) \otimes_{W(k)} A$ (as opposed to the ring $W(l) \otimes_{\mathbf{Z}_p} A$ in that reference). Nevertheless, as we will see shortly, we can deduce various results about our stacks from the corresponding results in that paper.

Definition 2.3.3. Let A be a ϖ-adically complete $\mathcal{O}$-algebra. An étale φ-module with A-coefficients is a finitely generated $\mathbf{A}_A$-module, equipped with a φ-semilinear morphism $\varphi_M : M \to M$ for which the linearized map $\Phi_M : \varphi^*M \to M$ is an isomorphism.

Definition 2.3.4. Fix an integer $d \geq 1$. For each $a \geq 1$, let $\mathcal{R}_d^a$ be the fibered category over $\mathrm{Spec}(\mathcal{O}/\varpi^a)$ taking an $\mathcal{O}/\varpi^a$-algebra A to the groupoid of rank d projective étale φ-modules with A-coefficients. It follows from Drinfeld's descent results (cf. [EG21, Thm. 5.1.18]) that $\mathcal{R}_d^a$ is in fact an *fpqc* (hence *fppf*) stack over $\mathcal{O}/\varpi^a$. As usual, we may regard $\mathcal{R}_d^a$ as an *fppf* stack over $\mathrm{Spec}\,\mathcal{O}$ (equipped with a map to $\mathrm{Spec}(\mathcal{O}/\varpi^a)$). We then define $\mathcal{R}_d$ as the colimit of the stacks $\mathcal{R}_d^a$ over all $a \geq 1$. Thus $\mathcal{R}_d$ is an *fppf* stack over $\mathrm{Spec}\,\mathcal{O}$ which furthermore admits a (necessarily unique) morphism to $\mathrm{Spf}\mathcal{O}$. Moreover, using [GD71, Prop. 0.7.2.10(ii)] we see that for any ϖ-adically complete $\mathcal{O}$-algebra A (not just those living over some $\mathcal{O}/\varpi^a$), $\mathcal{R}_d(A)$ (which, by definition, is the groupoid of morphisms $\mathrm{Spf}A \to \mathcal{R}_d$) is equivalent to the groupoid of rank d projective étale φ-modules with A-coefficients.

We now define the moduli stacks of so-called finite height φ-modules. For this, we need to assume that $\mathbf{A}^+$ is φ-stable. We also need to fix a polynomial F in $(W(l) \otimes_{W(k)} \mathcal{O})[[T]]$ which is congruent to a positive power of T modulo φ (e.g. F can be T itself, or an Eisenstein polynomial).

Definition 2.3.5. Assume $\mathbf{A}^+$ is φ-stable. Let h be a non-negative integer, and let A be a ϖ-adically complete $\mathcal{O}$-algebra. A φ-module of F-height $\leq h$ with A-coefficients is a finitely generated T-torsion free $\mathbf{A}_A^+$-module $\mathfrak{M}$, equipped with a φ-semilinear morphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ for which the linearized map $\Phi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective, and has cokernel killed by F^h.

Definition 2.3.6. Fix $d \geq 1$ and $h \geq 0$. For each $a \geq 1$, let $\mathcal{C}_{d,h}^a$ be the *fppf* stack over $\mathrm{Spec}(\mathcal{O}/\varpi^a)$ taking an $\mathcal{O}/\varpi^a$-algebra A to the groupoid of rank d projective φ-modules of F-height $\leq h$ with A-coefficients. Again, the colimit $\mathcal{C}_{d,h} := \varinjlim_a \mathcal{C}_{d,h}^a$ is an *fppf* stack over $\mathrm{Spf}\mathcal{O}$, whose groupoid of A-valued points, for any ϖ-adically complete $\mathcal{O}$-algebra A, is equivalent to the groupoid of rank d projective φ-modules of F-height $\leq h$ with A-coefficients.

For each $a \geq 1$, as T is invertible in $\mathbf{A}_A$, the same is true of F; thus, we have a natural morphism $\mathcal{C}^a_{d,h} \to \mathcal{R}^a_d$ taking $\mathfrak{M} \mapsto \mathfrak{M}[1/T]$. Taking the colimit over $a \geq 1$, we obtain a map $\mathcal{C}_{d,h} \to \mathcal{R}_d$.

Theorem 2.3.7. *Assume that* $\mathbf{A}^+$ *is* φ*-stable.*

(1) $\mathcal{C}_{d,h}$ *is a* p*-adic formal algebraic stack of finite presentation over* $\operatorname{Spf}\mathcal{O}$*, with affine diagonal.*

(2) *The morphism* $\mathcal{C}_{d,h} \to \mathcal{R}_d$ *is representable by algebraic spaces, proper, and of finite presentation.*

(3) $\mathcal{R}_d$ *is a limit preserving Ind-algebraic stack, whose diagonal is representable by algebraic spaces, affine, and of finite presentation.*

Proof. By [EG23, Prop. A.13], it suffices to show that $\mathcal{C}^a_{d,h}$ is an algebraic stack of finite presentation over $\operatorname{Spec}\mathcal{O}/\varpi^a$, with affine diagonal. That $\mathcal{C}^a_{d,h}$ is an algebraic stack of finite type over $\operatorname{Spec}\mathcal{O}/\varpi^a$ is [PR09, Thm. 2.1 (a)]. More precisely, the arguments in that reference remain valid in our setting provided that we replace the group G there by $\operatorname{Res}_{W(l)/W(k)}\mathrm{GL}_d$, that we replace [PR09, Prop. 2.2] by [EG21, Lem. 5.2.9], and accordingly, that in [PR09, §3] we replace eah by the quantity $n(a,h)$ in [EG21, Lem. 5.2.9]. The point of [EG21, Lem. 5.2.9] is that when $n > n(a,h)$, one can replace the φ-conjugation action of $U_n := 1 + T^n M_d(\mathbf{A}^+_A)$ on $LG^{\leq h} := \{A \in \mathrm{GL}_d(\mathbf{A}_A) \mid A^{\pm 1} \in T^{-h}M_d(\mathbf{A}^+_A)\}$ by the *free* action of U_n given by left translations. Together with the fact that any object in $\mathcal{C}^a_{d,h}(M)$ admits a basis locally on $\operatorname{Spec} A$ (cf. [EG21, Lem. 5.1.9 (1)]), one can then show that $\mathcal{C}^a_{d,h}$ can be written as the quotient stack of a finite type $\mathcal{O}/\varpi^a$-scheme by the action of a smooth finite type group scheme over $\mathcal{O}/\varpi^a$, and so is an algebraic stack of finite type over $\mathcal{O}/\varpi^a$ (e.g. by [Sta23, Tag 06FI]).

With the same modifications, the arguments in [PR09, Thm. 2.5 (b)] show that the natural morphism $\mathcal{C}^a_{d,h} \to \mathcal{R}^a_{d,fpqc-\mathrm{free}}, \mathfrak{M} \mapsto \mathfrak{M}[1/T]$ is representable by algebraic spaces, proper, and of finite presentation, where $\mathcal{R}^a_{d,fpqc-\mathrm{free}}$ is the substack of $\mathcal{R}^a_d$ classifying those objects which are furthermore free *fpqc* locally on $\operatorname{Spec} A$. Combining this with the arguments in [EG21, Lem. 5.4.10], [EG21, Thm. 5.4.11], we deduce that the map $\mathcal{C}^a_{d,h} \to \mathcal{R}^a_d$ is representable by algebraic spaces, proper, and of finite presentation. At this point, we have shown (1) and (2), except the claim that $\mathcal{C}_{d,h}$ has affine diagonal (indeed, once this is done, it will follow that $\mathcal{C}_{d,h}$ has quasi-compact and quasi-separated diagonal, hence is quasi-separated and then of finite presentation over $\mathcal{O}/\varpi^a$). As the map $\mathcal{C}^a_{d,h} \to \mathcal{R}^a_d$ is representable by algebraic spaces, and proper (hence separated), its diagonal is a proper monomorphism, hence a closed immersion. Using this, we are reduced to show that the diagonal of $\mathcal{R}^a_d$ is affine. This will be done in part (3) below.

(3) When $W(k) = \mathbf{Z}_p$, this is one of the main results of [EG21]. As alluded earlier, our strategy for the general case is to relate $\mathcal{R}^a_d$ with the "corresponding" stack introduced in [EG21]. More precisely, let $\mathcal{R}^{\mathrm{EG},a}_d$ be the *fppf* stack over $\operatorname{Spec}\mathcal{O}/\varpi^a$, taking an $\mathcal{O}/\varpi^a$-algebra A to the groupoid of rank d projective étale φ-modules over $(W(l) \otimes_{\mathbf{Z}_p} A)((T))$. Clearly, we have $\mathcal{R}^{\mathrm{EG},a}_d(A) \cong \mathcal{R}^a_d(W(k) \otimes_{\mathbf{Z}_p} A)$. Combining with the A-algebra isomorphism $W(k) \otimes_{\mathbf{Z}_p} A \cong \prod_{\overline{\sigma}:k \hookrightarrow \overline{\mathbf{F}}_p} A$, it follows that there is an isomorphism $\mathcal{R}^{\mathrm{EG},a}_d \cong \prod_{\overline{\sigma}} \mathcal{R}^a_d$ of stacks over $\mathcal{O}/\varpi^a$.

Assume for the moment that the diagonal of $\mathcal{R}^a_d$ is affine, and of finite presentation. By induction and the standard graph argument, we deduce that the diagonal $\Delta_n : \mathcal{R}^a_d \to \prod_{1 \leq i \leq n} \mathcal{R}^a_d$ has the same properties for all $n \geq 1$ (factor Δ_n as Δ_{n+1} followed by the projection $\prod_{n+1} \mathcal{R}_d^a \twoheadrightarrow \prod_n \mathcal{R}_d^a$ onto the first n components, and note that the latter map, being a base change of the structure map $\mathcal{R}_d^a \to \mathrm{Spec}(\mathcal{O}/\varpi^a)$, also has finitely presented affine diagonal). In particular, we obtain a morphism $\mathcal{R}_d^a \to \mathcal{R}_d^{\mathrm{EG},a}$ which is affine, and of finite presentation. Now it follows from [EG23, Cor. 3.2.9], [EG21, Thm. 5.4.11] and [EG21, Thm. 5.4.20] that $\mathcal{R}_d^a$ has the claimed properties.

Thus, it remains to show that the diagonal of $\mathcal{R}_d^a$ is affine, and of finite presentation. *Mutatis mutandis*, this is proved in [EG21, Prop. 5.4.8]. □

2.3.2 Moduli stacks of Lubin-Tate (φ_q, Γ_K)-modules

We can now define our main objects of interest. We will keep the notation as in Setup 2.2.1; in particular we will fix a finite extension $F/\mathbf{Q}_p$, a Lubin–Tate formal group law associated to a uniformizer π of F, and a finite extension K of F. As in Subsection 2.2.3, we will also fix throughout a finite extension E/F with uniformizer ϖ and ring of integers $\mathcal{O}$, which will serve as the base of our coefficients A. Recall also that by an étale (φ_q, Γ_K)-module with A-coefficients, we mean an étale (φ_q, Γ_K)-module over $\mathbf{A}_{K,A}$ in the usual sense.

Definition 2.3.8. Fix an integer $d \geq 1$. We let $\mathcal{X}_{K,d}^{\mathrm{LT}}$ denote the *fppf* stack over $\mathrm{Spf}\mathcal{O}$, whose groupoid of A-valued points, for any ϖ-adically complete $\mathcal{O}$-algebra A, is equivalent to the groupoid of rank d étale (φ_q, Γ_K)-modules with A-coefficients. (That this is well-defined follows exactly as in the definitions of the stacks $\mathcal{R}_d$ and $\mathcal{C}_{d,h}$; in particular we have implicitly used Drinfeld's descent results for verifying the stack property.)

2.3.2.1 Basic geometric properties of $\mathcal{X}_{K,d}^{\mathrm{LT}}$

We want to show the following preliminary result regarding the geometry of the stack $\mathcal{X}_{K,d}^{\mathrm{LT}}$. A more detailed study of its geometric properties will be given in Section 2.4; in particular, we will show that it is in fact a Noetherian formal algebraic stack (Corollary 2.4.3).

Theorem 2.3.9. *$\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a limit preserving Ind-algebraic stack whose diagonal is affine (in particular, representable by schemes), and of finite presentation.*

For this we will follow closely the argument used in [EG23] for the stack $\mathcal{X}_{K,d}$ of usual (cyclotomic) étale (φ, Γ)-modules; in particular, the strategy is to deduce the claimed properties for $\mathcal{X}_{K,d}^{\mathrm{LT}}$ from the corresponding properties of the stack of étale φ_q-modules. Thus, let us first define the latter.

Definition 2.3.10. We let $\mathcal{R}_{K,d}$ be the moduli stack of étale φ_q-modules over $\mathbf{A}_K$; in other words, it is the stack $\mathcal{R}_d$ defined in Subsection 2.3.1, with $\mathbf{A}$ there taken to be $\mathbf{A}_K$.

In case the ring $\mathbf{A}_{K,A}^+$ is φ_q-stable (e.g. if K is F-basic in the sense of Definition 2.2.4), we can apply Theorem 2.3.7 to use various properties of the stack $\mathcal{R}_{K,d}$. Our first goal is to show that these properties still hold for general K (i.e. without assuming that $\mathbf{A}_K^+$ is φ_q-stable).

Definition 2.3.11 ([EG23, Defn. 3.2.3])**.** We set $K^{\mathrm{basic}} := K \cap (K_0 F)_{\mathrm{cyc}}$.

It is easy to see that K^{basic} is indeed F-basic and $(K^{\text{cyc}})_{\text{cyc}} = (K_0F)_{\text{cyc}}$, and that the natural map $\Gamma_{K^{\text{basic}}} \hookrightarrow \Gamma_K$ is an isomorphism. Furthermore, as $\mathbf{A}_K$ is finite free of rank $[K_{\text{cyc}} : K_{\text{cyc}}^{\text{basic}}] = [K : K^{\text{basic}}]$ over $\mathbf{A}_{K^{\text{basic}}}$, we have a natural forgetful map $\mathcal{R}_{K,d} \to \mathcal{R}_{K^{\text{basic}},d[K:K^{\text{basic}}]}$.

Lemma 2.3.12. *The natural map $\mathcal{R}_{K,d} \to \mathcal{R}_{K^{\text{basic}},d[K:K^{\text{basic}}]}$ is affine, and of finite presentation.*

Proof. Again the proof of [EG23, Lem. 3.2.5] works *mutatis mutandis* in our setting. □

Corollary 2.3.13. *For any K/F, $\mathcal{R}_{K,d}$ is a limit preserving Ind-algebraic stack, whose diagonal is affine, and of finite presentation.*

Proof. Combine Theorem 2.3.7, Lemma 2.3.12, and [EG23, Cor. 3.2.9]. □

Having shown that $\mathcal{R}_{K,d}$ has the claimed properties, the next step is to define and study an “intermediate" stack sitting between $\mathcal{R}_{K,d}$ and $\mathcal{X}_{K,d}$, namely, it will be a stack of étale φ_q-modules which are furthermore equipped with an action of certain “discretization" of Γ_K. More precisely, let $\gamma_1, \ldots, \gamma_{[F:\mathbf{Q}_p]}$ be a $\mathbf{Z}_p$-free basis for $\Gamma_K \cong \mathbf{Z}_p^{\oplus[F:\mathbf{Q}_p]}$, and accordingly, $\Gamma_{\text{disc}} \subseteq \Gamma_K$ be the sub-$\mathbf{Z}$-module generated by $\gamma_1, \ldots, \gamma_{[F:\mathbf{Q}_p]}$. Let $\mathcal{R}_d^{\Gamma_{\text{disc}}}$ be the moduli stack of étale φ_q-modules over $\mathbf{A}_{K,A}$ equipped with a (not necessarily continuous!) semilinear action of Γ_{disc} that commutes with φ.

Lemma 2.3.14. *The natural morphism $\mathcal{R}_d^{\Gamma_{\text{disc}}} \to \mathcal{R}_d$ given by forgetting the Γ_{disc}-action, is affine, and of finite presentation.*

Proof. We can prove the statement after pulling back along a morphism $\operatorname{Spec} A \to \mathcal{R}_d$ where A is a $\mathbf{Z}/p^a$-algebra for some $a \geq 1$. Denote by M the étale φ_q-module over $\mathbf{A}_{K,A}$ corresponding to this morphism. Note firstly that the data of a commuting semi-linear action of Γ_{disc} on M is precisely the data of isomorphisms $\alpha_i : \gamma_i^* M \xrightarrow{\sim} M$ of φ_q-modules for $1 \leq i \leq [F : \mathbf{Q}_p]$ such that $\alpha_i \circ \gamma_i^* \alpha_j = \alpha_j \circ \gamma_j^* \alpha_i$ for all i, j. Thus we need to show that the functor on A-algebras sending B to the set

$$\left\{ (\alpha_i)_i \in \prod_i \operatorname{Isom}_{\mathbf{A}_{K,B},\varphi_q}(\gamma_i^* M_B, M_B) \mid \alpha_i \circ \gamma_i^* \alpha_j = \alpha_j \circ \gamma_j^* \alpha_i \text{ for all } i, j \right\}$$

is represented by an affine A-scheme of finite presentation. By [EG21, Prop. 5.4.8], the functor on A-algebras sending B to $\prod_i \operatorname{Isom}_{\mathbf{A}_{K,B},\varphi_q}(\gamma_i^* M_B, M_B)$ is represented by an affine scheme of finite presentation over A. More precisely, as in the proof of that proposition, we may reduce to the case where M_B is finite free over $\mathbf{A}_{K,B}$; then after choosing bases, any morphism $\alpha_i : \gamma_i^* M \xrightarrow{\sim} M$ of φ_q-modules is determined by the coefficients of finitely many powers of T in the Laurent series expansions of the entries of the matrix representing α_i. To conclude, it suffices to note that, the conditions $\alpha_i \circ \gamma_i^* \alpha_j = \alpha_j \circ \gamma_j^* \alpha_i$ are evidently given by finitely many equations in these coefficients. □

Corollary 2.3.15. *$\mathcal{R}_d^{\Gamma_{\text{disc}}}$ is a limit preserving Ind-algebraic stack whose diagonal is affine, and of finite presentation.*

Proof. This follows from Lemma 2.3.14 and [EG23, Lem. 3.2.9]. □

Restricting the Γ_K-action from an étale (φ_q, Γ_K)-module to Γ_{disc} defines a fully faithful morphism $\mathcal{X}_{K,d}^{\mathrm{LT}} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$. Indeed, by construction, any linear map between two finite projective $\mathbf{A}_{K,A}$-modules is continuous with respect to the canonical topology (cf. [Dri06, Exam. 3.2.2], [EG23, Rem. D.2]). As Γ_{disc} is dense in Γ_K, it follows that any Γ_{disc}-equivariant morphism between two objects of $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is automatically Γ_K-equivariant.

Corollary 2.3.16. *The diagonal of $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is representable by algebraic spaces, affine, and of finite presentation.*

Proof. As the map $\mathcal{X}_{K,d}^{\mathrm{LT}} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is a monomorphism, its diagonal is an isomorphism. Using the cartesian diagram

$$\begin{array}{ccc} \mathcal{X}_{K,d}^{\mathrm{LT}} \times_{\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}} \mathcal{X}_{K,d}^{\mathrm{LT}} & \longrightarrow & \mathcal{X}_{K,d}^{\mathrm{LT}} \times_{\mathrm{Spf}\,\mathcal{O}} \mathcal{X}_{K,d}^{\mathrm{LT}} \\ \downarrow & & \downarrow \\ \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} & \longrightarrow & \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathrm{Spf}\,\mathcal{O}} \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \end{array}$$

we see that the diagonal of $\mathcal{X}_d^{\mathrm{LT}}$ is a base change of that of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$, which has the claimed properties by Corollary 2.3.15. $\square$

The next lemma allows us to reduce to the case where K is F-basic (in proving Theorem 2.3.9).

Lemma 2.3.17. *We have a Cartesian diagram*

$$\begin{array}{ccc} \mathcal{X}_{K,d}^{\mathrm{LT}} & \longrightarrow & \mathcal{X}_{K^{\mathrm{basic}},d[K:K^{\mathrm{basic}}]}^{\mathrm{LT}} \\ \downarrow & & \downarrow \\ \mathcal{R}_{K,d}^{\Gamma_{\mathrm{disc}}} & \longrightarrow & \mathcal{R}_{K^{\mathrm{basic}},d[K:K^{\mathrm{basic}}]}^{\Gamma_{\mathrm{disc}}}, \end{array}$$

where the horizontal arrows are the natural forgetful maps, and the vertical arrows are the monomorphisms given by restricting the action of Γ_K to Γ_{disc}.

Proof. This is clear because Γ_{disc} acts continuously on an object M of $\mathcal{R}_{K,d}^{\Gamma_{\mathrm{disc}}}$ only if it does so when M is regarded as an object of $\mathcal{R}_{K^{\mathrm{basic}},d[K:K^{\mathrm{basic}}]}^{\Gamma_{\mathrm{disc}}}$. $\square$

Corollary 2.3.18. *The natural map $\mathcal{X}_{K,d}^{\mathrm{LT}} \to \mathcal{X}_{K^{\mathrm{basic}},d[K:K^{\mathrm{basic}}]}^{\mathrm{LT}}$ is affine, and of finite presentation.*

Proof. This follows easily from Lemmas 2.3.14, 2.3.12, and 2.3.17. $\square$

In view of Corollaries 2.3.16, 2.3.18 and [EG23, Cor. 3.2.9], it remains to show that if $K = K^{\mathrm{basic}}$ is F-basic, then $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a limit preserving Ind-algebraic stack. For this we will need to understand the continuity condition in the definition of an object of $\mathcal{X}_{K,d}^{\mathrm{LT}}$ more carefully.

The following is an analogue of [EG23, Lem. 3.2.18] in our setting.

Lemma 2.3.19. *For any $\gamma \in \Gamma_K$, we have $\gamma(T) - T \in \pi\mathbf{A}_{K,A} + T^2\mathbf{A}_{K,A}^+$. If K is F-basic, then $\gamma(T) - T \in (\pi, T)T\mathbf{A}_{K,A}^+$.*

Proof. The proof is identical to that of [EG23, Lem. 3.2.18], except we consider the reduction modulo π instead of modulo p (recall also that in case K is F-basic, we have chosen T so that $\gamma(T) \in T\mathbf{A}^+_{K,A}$ for all $\gamma \in \Gamma_K$). □

Using Lemma 2.3.19, we can obtain (among other things) an analogue of [EG23, Lem. D.28] in our setting. This will be worked out in Appendix 2.A linear below. In particular, we can now apply these results to study the action of Γ_K on the objects of $\mathcal{X}^{\mathrm{LT}}_{K,d}$.

Lemma 2.3.20. *The stack $\mathcal{X}^{\mathrm{LT}}_{K,d}$ is limit preserving.*

Proof. We may assume that K is F-basic. As the natural map $\mathcal{X}^{\mathrm{LT}}_{K,d} \hookrightarrow \mathcal{R}^{\Gamma_{\mathrm{disc}}}_{K,d}$ is fully faithful with limit preserving target, it suffices to show that the map is limit preserving on objects. In other words, given any direct limit $A = \varinjlim_{i \geq i_0} A_i$ of $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$ and any object M of $\mathcal{R}^{\Gamma_{\mathrm{disc}}}_{K,d}(A_{i_0})$ such that $M_A := M \otimes_{\mathbf{A}_{K,A_{i_0}}} \mathbf{A}_{K,A}$ lies in the subgroupoid $\mathcal{X}^{\mathrm{LT}}_{K,d}(A)$, we need to find some $i \geq i_0$ such that $M_{A_i} \in \mathcal{X}^{\mathrm{LT}}_{K,d}(A_i)$, i.e. that the action of Γ_{disc} on M_{A_i} extends (uniquely) to a continuous action of Γ_K. This follows from the Lemma 2.A.8 below and (the proof of) [EG23, Lem. D.31] (note also the obvious observation that the action of Γ_{disc} is continuous if and if its restriction to each $\langle \gamma_i \rangle$ is continuous). □

$\mathcal{X}^{\mathrm{LT}}_{K,d}$ is an Ind-algebraic stack

We now turn to showing that $\mathcal{X}^{\mathrm{LT}}_{K,d}$ is an Ind-algebraic stack (thereby completing our proof of Theorem 2.3.9). As in [EG23], we will need to relate $\mathcal{X}^{\mathrm{LT}}_{K,d}$ with the moduli stacks of weak Wach modules. We will assume throughout that K is F-basic.

Definition 2.3.21 ([EG23, Def. 3.3.2])**.** (Assume K is F-basic.) A rank d projective weak Wach module of T-height $\leq h$ with A-coefficients is a rank d projective φ_q-module $\mathfrak{M}$ over $\mathbf{A}^+_{K,A}$ of T-height $\leq h$, such that $\mathfrak{M}[1/T]$ is equipped with a continuous semi-linear action of Γ_K.

If $s \geq 0$, then we say that $\mathfrak{M}$ has level $\leq s$ if $(\gamma_i^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$ for all $1 \leq i \leq n$.

Denote by $\mathcal{W}_{d,h}$ the moduli stack of weak Wach modules which are of T_K-height $\leq h$. For each $s \geq 0$, denote by $\mathcal{W}_{d,h,s}$ the substack of those weak Wach modules of level $\leq s$. (Again, by using Drinfeld's descent results, we see that these are *fppf* stacks over $\mathrm{Spf}\mathcal{O}$.) By Lemma 2.A.8 below, any projective weak Wach module is of level $\leq s$ for some $s \gg 0$.

Let $\mathcal{C}_{d,h}$ denote the moduli stack of rank d projective φ_q-modules of T-height $\leq h$ over $\mathbf{A}^+_K$ (that is, it is the stack $\mathcal{C}_{d,h}$ defined in Subsection 2.3.1 with $\mathbf{A}$ taken to be $\mathbf{A}_{K,A}$). By Theorem 2.3.7, this is a p-adic formal algebraic stack of finite presentation over $\mathrm{Spf}\mathcal{O}$. Consider the fiber product $\mathcal{R}^{\Gamma_{\mathrm{disc}}}_d \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$ where the map $\mathcal{R}^{\Gamma_{\mathrm{disc}}}_d \to \mathcal{R}_d$ is given by forgetting the Γ_{disc}-action. This is the moduli stack of rank d projective φ_q-modules $\mathfrak{M}$ of T-height $\leq h$, equipped with a semilinear action of Γ_{disc} on $\mathfrak{M}[1/T]$. As the map $\mathcal{R}^{\Gamma_{\mathrm{disc}}}_d \to \mathcal{R}_d$ is representable by algebraic spaces, and of finite presentation (by Lemma 2.3.14), the same is true of the map $\mathcal{R}^{\Gamma_{\mathrm{disc}}}_d \times_{\mathcal{R}_d} \mathcal{C}_{d,h} \to \mathcal{C}_{d,h}$. It follows that $\mathcal{R}^{\Gamma_{\mathrm{disc}}}_d \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$ is also a p-adic formal algebraic stack of finite presentation over $\mathrm{Spf}\mathcal{O}$.

By restricting the Γ_K-action to Γ_{disc}, we obtain a natural morphism $\mathcal{W}_{d,h} \to \mathcal{R}^{\Gamma_{\mathrm{disc}}}_d \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$ which is again fully faithful since Γ_{disc} is dense Γ_K.

Proposition 2.3.22. *For each $s \geq 0$, the natural morphism*

$$\mathcal{W}_{d,h,s} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}} \times_{\mathcal{R}_d} \mathcal{C}_{d,h}$$

is a closed immersion of finite presentation. In particular, each $\mathcal{W}_{d,h,s}$ is a p-adic formal algebraic stack of finite presentation over $\operatorname{Spf}\mathcal{O}$*, and $\mathcal{W}_{d,h} = \varinjlim_s \mathcal{W}_{d,h,s}$ is an Ind-algebraic stack.*

Proof. See the proof of [EG23, Lem. 3.3.5]. □

For each $a \geq 1$, let $\mathcal{W}_{d,h,s}^a := \mathcal{W}_{d,h,s} \times_{\operatorname{Spf}\mathcal{O}} \operatorname{Spec}\mathcal{O}/\varpi^a$, an algebraic stack of finite presentation over $\mathcal{O}/\varpi^a$. As $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is an Ind-algebraic stack, it makes sense to consider the scheme-theoretic image $\mathcal{X}_{d,h,s}^a$ of the composite

$$\mathcal{W}_{d,h,s}^a \hookrightarrow \mathcal{W}_{d,h,s} \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}},$$

As the map $\mathcal{W}_{d,h,s}^a \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ is representable by algebraic spaces, proper and of finite presentation by Proposition 2.3.22 (recall that the natural map $\mathcal{C}_{d,h} \to \mathcal{R}_d$ has the same properties by Theorem 2.3.7), we find that $\mathcal{X}_{d,h,s}^a$ is a closed algebraic substack of $\mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ which is of finite presentation over $\operatorname{Spec}\mathcal{O}/\varpi^a$, and moreover, the induced map $\mathcal{W}_{d,h,s}^a \to \mathcal{X}_{d,h,s}^a$ is proper, scheme-theoretically dominant and surjective.

Proposition 2.3.23. *Each $\mathcal{X}_{d,h,s}^a$ is a closed (algebraic) substack of $\mathcal{X}_{K,d}^{\mathrm{LT}}$. Moreover, the induced map $\varinjlim_{a,h,s} \mathcal{X}_{d,h,s}^a \to \mathcal{X}_{K,d}^{\mathrm{LT}}$ is an isomorphism. Thus $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is an Ind-algebraic stack, and may in fact be written as the inductive limit of algebraic stacks of finite presentation over* $\operatorname{Spec}\mathcal{O}$*, with the transition maps being closed immersions.*

Proof. The arguments given in [EG23, Lem. 3.4.8, 3.4.9 and 3.4.10] go through provided one replaces the condition "$(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$" with "$(\gamma_i^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$ for all i" everywhere. For convenience of the reader, we sketch the proof here. We begin with the first assertion. It suffices to show that $\mathcal{X}_{d,h,s}^a$ is a substack of $\mathcal{X}_{K,d}^{\mathrm{LT}}$. By definition, this amounts to showing that if M is an étale φ-module with coefficients in a finite type $\mathcal{O}/\varpi^a$-algebra A, equipped with a Γ_{disc}-action for which the map $\operatorname{Spec} A \to \mathcal{R}_d^{\Gamma_{\mathrm{disc}}}$ classifying M factors through $\mathcal{X}_{d,h,s}^a$, then the Γ_{disc}-action on M is continuous. In view of Lemma 2.A.8, we need to produce a lattice $\mathfrak{M} \subseteq M$ such that $(\gamma_i^{p^s} - 1)\mathfrak{M} \subseteq T\mathfrak{M}$ for all i. By embedding A into the product of its Artinian quotients, we may reduce to the case where A is a finite Artinian $\mathcal{O}/\varpi^a$-algebra. In this case, the existence of the desired lattice is established in [EG23, Lem. 3.4.8] provided one modifies the definition of the subfunctor D' there by imposing the condition that $(\gamma_i^{p^s} - 1)\mathfrak{M}_\Lambda \subseteq T\mathfrak{M}_\Lambda$ for all i.

For the second assertion, we need to show that any morphism $\operatorname{Spec} A \to \mathcal{X}_{K,d}^{\mathrm{LT}}$ whose source is a Noetherian $\mathcal{O}/\varpi^a$-algebra necessarily factors through some $\mathcal{X}_{d,h,s}^a$. It suffices to do this after replacing A by some algebra B for which the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is scheme-theoretically dominant. Thus, by [EG21, Lem. 5.4.7], we may assume that M is free, where M is the étale φ-module over $\mathbf{A}_{K,A}$ classified by the map $\operatorname{Spec} A \to \mathcal{R}_d$. In particular, we can choose a free φ-stable lattice $\mathfrak{M}$ inside M, say of height $\leq h$ for some h large enough. As the Γ-action on M is continuous by definition, Lemma 2.A.8 implies that for all s sufficiently large, we have $(\gamma_i^{p^s} - 1)\mathfrak{M} \subseteq T\mathfrak{M}$ for all i. It follows that the map $\operatorname{Spec} A \to \mathcal{X}_{K,d}^{\mathrm{LT}}$ classifying M factors through $\mathcal{W}_{d,h,s}^a$, and hence through $\mathcal{X}_{d,h,s}^a$, as required. □

As mentioned before, using Corollary 2.3.18, we can now drop the assumption that K is F-basic.

Proposition 2.3.24. *Let K be an arbitrary finite extension of F. Then $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a limit preserving Ind-algebraic stack, which can in fact be written as the inductive limit of a sequence of algebraic stacks of finite presentations over $\operatorname{Spec}\mathcal{O}$ with transition maps being closed immersions. Furthermore the diagonal of $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is affine (in particular, representable by schemes), and of finite presentation.*

2.3.3 Galois representations with coefficients

As one might expect from the case of cyclotomic (φ, Γ)-modules, there is also an analogue of the equivalence in Theorem 2.2.6 in the presence of coefficients.

To this end, let A be a complete local Noetherian $\mathcal{O}$-algebra with residue field. As in [EG23, §3.6.1], we denote by $\widehat{\mathbf{A}}_{K,A}$ the $\mathfrak{m}_A$-adic completion of $\mathbf{A}_{K,A}$, and we define a formal étale (φ_q, Γ_K)-module with A-coefficients to be an étale (φ_q, Γ_K)-module over $\widehat{\mathbf{A}}_{K,A}$ in the obvious sense. We also let $\widehat{\mathbf{A}}_{K,A}^{\mathrm{ur}}$ be the $\mathfrak{m}_A$-adic completion of the tensor product $\mathbf{A}^{\mathrm{ur}} \otimes_{\mathcal{O}_F} A$.

Theorem 2.3.25. *The functor*

$$V \mapsto D_A(V) := (V \otimes_A \widehat{\mathbf{A}}_{K,A}^{\mathrm{ur}})^{G_{K_\infty}}$$

gives an equivalence of categories between the category of continuous G_K-representations on finite free A-modules and the category of finite projective formal étale (φ_q, Γ_K)-modules with A-coefficients. A quasi-inverse functor is given by $M \mapsto T_A(M) := (\widehat{\mathbf{A}}_{K,A}^{\mathrm{ur}} \otimes_{\widehat{\mathbf{A}}_{K,A}} M)^{\varphi_q = 1}$.

Proof. This is a generalization of of [Dee01, Thm. 2.2.1] to the setting of Lubin–Tate (φ_q, Γ_K)-modules. The idea is to first consider the case where V has finite length. In this case $D_A(V)$ simplifies to $D_{\mathcal{O}_F}(V)$ where in the latter we view V as a module over $\mathcal{O}_F$; in particular, we can make use of the already-established case of $\mathcal{O}_F$-linear representations (Theorem 2.2.6). Once this is done, we pass to the limit to deduce the general case (after checking that everything behave well under taking limits). A detailed proof has been recently worked out in [AK19, Thm. 7.18]. □

The above equivalence also holds in the case where $A = \overline{\mathbf{F}}_p$ (this follows easily from the fact that $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is limit preserving). In other words, we see that the groupoid of $\overline{\mathbf{F}}_p$-points of $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is equivalent to the groupoid of representations $G_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$. Using this equivalence, we can deduce easily that taking versal rings of the stack $\mathcal{X}_{K,d}^{\mathrm{LT}}$ at finite type points recovers the usual framed Galois deformation rings.

2.4 Finer geometric properties of $\mathcal{X}_{K,d}^{\mathrm{LT}}$

2.4.1 Relation with the Emerton–Gee stack

Having established that $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is an Ind-algebraic stack, our goal in this subsection is to make a more detailed study of its geometry. More precisely, we will show that our stack of (rank d) Lubin–Tate étale (φ_q, Γ_K)-modules is in fact isomorphic to the Emerton-Gee stack of (rank d) cyclotomic (φ, Γ)-modules. As an immediate consequence, we deduce that $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is in fact a Noetherian formal algebraic stack.

We maintain our notation from the previous section. Our main result is the following.

Proposition 2.4.1. *Assume A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$. Then there is an exact tensor-compatible rank-preserving equivalence between the category of finite projective étale (φ_q, Γ_K)-modules over $\mathbf{A}_{K,A}$, and the category of finite projective étale $(\varphi, \Gamma_K^{\mathrm{EG}})$-modules over $\mathbf{A}_{K,A}^{\mathrm{EG}}$.*

Here, as the notation suggests, $\mathbf{A}_{K,A}^{\mathrm{EG}}$ is the coefficient ring for the cyclotomic (φ, Γ)-modules considered in [EG23] (it is denoted by $\mathbf{A}_{K,A}$ in *loc. cit.*). Also, exactness here means that both the equivalence and its inverses are exact.

Proof. By [EG23, Prop. 2.7.8], if A is as in the statement, then the functor $M \mapsto M \otimes_{\mathbf{A}_{K,A}^{\mathrm{EG}}} (A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat))$ defines an exact equivalence between finite projective étale $(\varphi, \Gamma_K^{\mathrm{EG}})$-modules over $\mathbf{A}_{K,A}^{\mathrm{EG}}$, and finite projective étale (φ, G_K)-modules over $A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat)$ (exactness follows from fully faithfulness of the map $\mathbf{A}_{K,A}^{\mathrm{EG}} \to A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat)$, cf. [EG23, Prop. 2.2.12]). In the same way, extending scalars along $\mathbf{A}_{K,A} = A\widehat{\otimes}_{\mathcal{O}_F} \mathbf{A}_K \to A\widehat{\otimes}_{\mathcal{O}_F} W_{\mathcal{O}_F}(\mathbf{C}^\flat)$ defines an exact equivalence between the category of finite projective étale (φ_q, Γ_K)-modules over $\mathbf{A}_{K,A}$, and finite projective étale (φ_q, G_K)-modules over $A\widehat{\otimes}_{\mathcal{O}_F} W_{\mathcal{O}_F}(\mathbf{C}^\flat)$ (*loc. cit.* is written only for the ring of unramified Witt vectors, but the proof works in general).

It thus suffices to relate étale φ-modules over $A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat)$, and étale φ_q-modules over $A\widehat{\otimes}_{\mathcal{O}_F} W_{\mathcal{O}_F}(\mathbf{C}^\flat)$. To this end, let $\sigma_0 : k_F \hookrightarrow \mathbf{C}^\flat$ be the canonical embedding. For each $j \in \mathbf{Z}/f\mathbf{Z}$, let $\sigma_j := \sigma_0 \circ \varphi^j$, where φ is the p-power Frobenius on k_F. As $W(k_F) \otimes_{\mathbf{Z}_p} W(\mathbf{C}^\flat) \xrightarrow{\sim} \prod_j W(\mathbf{C}^\flat), a \otimes x \mapsto (\sigma_j(a)x)_j$, we obtain an isomorphism

$$\begin{aligned} A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat) &= A\widehat{\otimes}_{W(k_F)}(W(k_F) \otimes_{\mathbf{Z}_p} W(\mathbf{C}^\flat)) \\ &\xrightarrow{\sim} \prod_j A\widehat{\otimes}_{W(k_F),\sigma_j} W(\mathbf{C}^\flat) \\ &\xrightarrow{\sim} \prod_j A\widehat{\otimes}_{\mathcal{O}_F,\sigma_j} W_{\mathcal{O}_F}(\mathbf{C}^\flat). \end{aligned}$$

Thus, any module M over $A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat)$ decomposes as $M = \prod_j M_j$, where M_j is the base change of M along the map $p_j : A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat) \twoheadrightarrow A\widehat{\otimes}_{\mathcal{O}_F,\sigma_j} W_{\mathcal{O}_F}\mathbf{C}^\flat)$. We claim that the functor $M \mapsto M_0$ defines an equivalence between the category of étale φ-modules over $A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat)$, and the category of étale φ_q-module over $A\widehat{\otimes}_{\mathcal{O}_F,\sigma_0} W_{\mathcal{O}_F}(\mathbf{C}^\flat)$ (as the map p_0 is clearly Γ_K-equivariant, we then obtain an equivalence between the corresponding categories of étale (φ, G_K)-modules).

Let us abusively denote also by φ the ring map $A\widehat{\otimes}_{\mathcal{O}_F,\sigma_j} W_{\mathcal{O}_F}(\mathbf{C}^\flat) \to A\widehat{\otimes}_{\mathcal{O}_F,\sigma_{j+1}} W_{\mathcal{O}_F}(\mathbf{C}^\flat)$ induced by the p-power Frobenius on $\mathbf{C}^\flat$. As $\varphi \circ p_j = p_{j+1} \circ \varphi$, the isomorphism $\Phi_M : \varphi^* M \xrightarrow{\sim} M$ induces an isomorphism $\varphi^* M_j \xrightarrow{\sim} M_{j+1}$ for each j. In particular, we have an isomorphism $\varphi_q^* M_0 = (\varphi^f)^* M_0 \xrightarrow{\sim} M_0$, i.e. M_0 is an étale φ_q-module. Conversely, given such M_0, we can define M to be the module over $A\widehat{\otimes}_{\mathbf{Z}_p} W(\mathbf{C}^\flat)$ corresponding to the tuple $((\varphi^j)^* M_0)_j$. By design, there is a linear isomorphism $\varphi^*M \xrightarrow{\sim} M$ as $(\varphi^*M)_{j+1} = \varphi^*(M_j) = \varphi^*((\varphi^j)^*M_0) = (\varphi^{j+1})^*M_0 = M_{j+1}$ for each j. In other words, M is an étale φ-module, as desired. □

Corollary 2.4.2. *There is an isomorphism*

$$\mathcal{X}_{K,d}^{\mathrm{LT}} \xrightarrow{\sim} \mathcal{X}_{K,d}^{\mathrm{EG}}.$$

Proof. This follows immediately from Proposition 2.4.1 as $\mathcal{X}_{K,d}^{\mathrm{LT}}$ and $\mathcal{X}_{\mathrm{LT}}^{\mathrm{EG}}$ are both limit preserving by Lemma 2.3.20 and [EG23, Lem. 3.2.19]. □

As an immediate consequence of the above comparison result, we obtain the following refinement of Theorem 2.3.9 on the geometry of the stack $\mathcal{X}_{K,d}^{\mathrm{LT}}$. See also Subsection 2.4.3 below for a related discussion.

Corollary 2.4.3. $\mathcal{X}_{K,d}^{\mathrm{LT}}$ *is a Noetherian formal algebraic stack over* $\mathrm{Spf}\,\mathcal{O}$*. The underlying reduced substack* $\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}}$ *is an algebraic stack of finite presentation over* $\mathbf{F}$*. Moreover, the irreducible components of* $\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}}$ *admits a natural labeling by Serre weights.*

Proof. This follows from Corollary 2.4.2, and the corresponding properties of the stack $\mathcal{X}_{K,d}^{\mathrm{EG}}$, see [EG23, Cor. 5.5.18] and [EG23, Thm. 6.5.1]. □

2.4.2 The Lubin–Tate Herr complex

In this subsection we introduce a version of the Herr complex for Lubin–Tate (φ_q, Γ_K)-modules with coefficients; the goal is to show that it is a perfect complex. Although this is not strictly needed in our proof that $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a Noetherian formal algebraic stack, the result itself may be of independent interest.

Again, we will keep the notation in Section 2.2. In particular, we fix a set $\{\gamma_1, \dots, \gamma_n\}$ of topological generators of the group $\Gamma_K \cong \mathbf{Z}_p^{\oplus[F:\mathbf{Q}_p]}$.

Let A be a ϖ-adically complete $\mathcal{O}$-algebra, and let M be a finite projective étale (φ_q, Γ_K)-module with A-coefficients. The Herr complex $\mathcal{C}^\bullet(M)$ of M is by definition the cohomological Koszul complex of M with respect to the commuting operators $\varphi_q - 1, \gamma_1 - 1, \dots, \gamma_n - 1$. Concretely, if we let $\gamma_0 := \varphi_q$, then $\mathcal{C}^\bullet(M)$ is by the complex

$$M \xrightarrow{d^0} \bigoplus_{0\le i_1\le n} M \xrightarrow{d^1} \bigoplus_{0\le i_1<i_2\le n} M \to \dots \bigoplus_{0\le i_1<\ldots<i_n\le n} \xrightarrow{d^n} M$$

sitting in (cohomological) degrees $0, 1, \dots, n+1$, where for each $0 \le r \le n$, the component $d^r|_{i_1<\ldots<i_r}^{j_1<\ldots<j_{r+1}} : M \to M$ of the rth differential d^r is the multiplication by

$$\begin{cases} 0 & \text{if } \{i_1, \dots, i_r\} \not\subseteq \{j_1, \dots, j_{r+1}\} \\ (-1)^s(\gamma_j - 1) & \text{if } \{j_1, \dots, j_{r+1}\} = \{i_1, \dots, i_r\} \coprod \{j\}, \end{cases}$$

where s is the number of elements in the set $\{i_1 < \ldots < i_r\}$, which are smaller than j.

Example 2.4.4. If $n=2$, then $\mathcal{C}^\bullet(M)$ can be identified with the complex

$$[M \xrightarrow{\begin{pmatrix}\varphi_q-1\\ \gamma_1-1\\ \gamma_2-1\end{pmatrix}} M^{\oplus 3} \xrightarrow{\begin{pmatrix}-(\gamma_1-1) & \varphi_q-1 & 0\\ -(\gamma_2-1) & 0 & \varphi_q-1\\ 0 & -(\gamma_2-1) & (\gamma_1-1)\end{pmatrix}} M^{\oplus 3} \xrightarrow{(\gamma_2-1\ \ -(\gamma_1-1)\ \ \varphi_q-1)} M].$$

In what follows, unless otherwise stated, we assume that A is an $\mathcal{O}/\varpi^a$-algebra for some $a\geq 1$. We have the following generalization of [EG23, Lem. 5.1.2], which allows us to interpret the cohomology groups of the Herr complex entirely in terms of the ambient category of étale (φ_q,Γ_K)-modules.

Proposition 2.4.5. *There is a natural isomorphism of cohomological δ-functors $H^i(\mathcal{C}^\bullet(\cdot))) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},\cdot)$.*

Let us first recall the definition of the extension groups $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},M)$. Let $\mathcal{M}^{\text{ét},\mathrm{LT}}_{\varphi,\Gamma}$ be the category of finite projective étale (φ_q,Γ_K)-modules over $\mathbf{A}_{K,A}$. This is an exact full additive subcategory of the abelian category of all *finite* étale (φ_q,Γ_K)-modules over $\mathbf{A}_{K,A}$ (in the sense that if $0\to M\to N\to P\to 0$ is a short exact sequence of finite étale (φ_q,Γ_K)-module with M,P finite projective, then the same is true for N).

As is the case for any exact category, given any object M in $\mathcal{M}^{\text{ét},\mathrm{LT}}_{\varphi,\Gamma}$ and any $i\geq 1$, we have the abelian group $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},M)$ of equivalence classes of so-called degree i Yoneda extensions of $\mathbf{A}_{K,A}$ by M in $\mathcal{M}^{\text{ét},\mathrm{LT}}_{\varphi,\Gamma}$. By definition, such an extension is an exact sequence

$$E: 0\to M\to M_{i-1}\to M_{i-2}\to\ldots\to M_0\to \mathbf{A}_{K,A}\to 0$$

in $\mathcal{M}^{\text{ét},\mathrm{LT}}_{\varphi,\Gamma}$[3], and the relevant equivalence relation is generated by the relation that identifies two extensions E and E' whenever there is a map of extensions $E\to E'$, i.e. a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & M_{i-1} & \longrightarrow & \ldots & \longrightarrow & M_0 & \longrightarrow & \mathbf{A}_{K,A} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & M & \longrightarrow & M'_{i-1} & \longrightarrow & \ldots & \longrightarrow & M'_0 & \longrightarrow & \mathbf{A}_{K,A} & \longrightarrow & 0.
\end{array}$$

Concretely, E and E' are equivalent if and only if there is some extension E'' together with maps of extensions $E\leftarrow E''\to E'$ as above.

It is easy to see that the abelian group $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},M)$ is naturally an A-module with $a\in A$ acting via the map $a:\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},M)\to\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},M)$ induced by the multiplication by a on M (or on $\mathbf{A}_{K,A}$). As usual, for $i=0$, we define $\mathrm{Ext}^0_{\mathrm{LT}}(\mathbf{A}_{K,A},M)$ to be the A-module of morphisms $\mathbf{A}_{K,A}\to M$ in $\mathcal{M}^{\text{ét},\mathrm{LT}}_{\varphi,\Gamma}$.

The collection of functors $(\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A},\cdot))_{i\geq 0}$ forms a (cohomological) δ-functor from the exact category $\mathcal{M}^{\text{ét},\mathrm{LT}}_{\varphi,\Gamma}$ to the abelian category of A-modules. More precisely, given any short

[3] This is to say that the sequence is exact when viewed as a sequence in the *abelian* category of all finite étale (φ_q,Γ_K)-modules over $\mathbf{A}_{K,A}$.

exact sequence $0 \to M \to N \to P \to 0$, there is an associated long exact sequence for Yoneda extensions

$$\ldots \to \mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, M) \to \mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, N) \to \mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, P) \xrightarrow{\delta} \mathrm{Ext}^{i+1}_{\mathrm{LT}}(\mathbf{A}_{K,A}, M) \to \ldots, \tag{2.4.5.1}$$

where the connecting map δ is defined by "splicing" a degree i extension $[0 \to P \to Z_{i-1} \to \ldots \to Z_0 \to \mathbf{A}_{K,A} \to 0]$ in $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, P)$ with the given short exact sequence $0 \to M \to N \to P \to 0$. A proof of this can be found in [Mit65, Chap. VII, Thm. 5.1] (strictly speaking, the reference works throughout with an abelian category, but the proof goes over unchanged to any exact category).

Similarly, as the association $M \mapsto \mathcal{C}^\bullet(M)$ is clearly functorial and exact in M, we see easily that the collection $(H^i(\mathcal{C}^\bullet(\cdot)))_{i\geq 0}$ also forms a δ-functor between the same categories.

In what follows by an embedding $M \hookrightarrow N$ between objects in $\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$, we will mean an equivariant injective map $M \to N$ which splits on the level of underlying $\mathbf{A}_{K,A}$-modules (so that the quotient $P := N/M$ will also be an object in $\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$[4]).

Lemma 2.4.6. *For each $i \geq 1$, the functor $M \mapsto \mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, M)$ is effaceable. In particular, the collection $(\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, \cdot))_{i\geq 0}$ forms a universal δ-functor.*

Proof. Given any $M \in \mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$, and any class $E \in \mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, M)$, we need to find an embedding $M \hookrightarrow N$ in $\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$ so that $E = 0$ in $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, N)$. If $[0 \to M \to N \to Z_{i-2} \to \ldots \to Z_0 \to \mathbf{A}_{K,A} \to 0]$ is an extension representing E, then $E = \delta(E')$ by construction of the connecting map δ, where E' is the class in $\mathrm{Ext}^{i-1}_{\mathrm{LT}}(\mathbf{A}_{K,A}, P)$ represented by the extension $[0 \to P \to Z_{i-2} \to \ldots \to Z_0 \to \mathbf{A}_{K,A} \to 0]$ (as above, we set $P := \mathrm{Im}(N \to Z_{i-2})$, an object in $\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$). Using the long exact sequence (2.4.5.1), we find that $E = 0$ in $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, N)$, as desired. □

Lemma 2.4.7. *For each $i \geq 1$, the functor $M \mapsto H^i(\mathcal{C}^\bullet(M))$ is effaceable. In particular, the collection $(H^i(\mathcal{C}^\bullet(\cdot)))_{i\geq 0}$ forms a universal δ-functor.*

Proof. Given any class c in $H^i(\mathcal{C}^\bullet(M))$, we need to find an embedding $M \hookrightarrow N$ in $\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$ so that c becomes zero in $H^i(\mathcal{C}^\bullet(N))$. For brevity, in this proof and the next two lemmas, by "enlarging" M, we mean replacing M by an appropriate object N in $\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}$ for which there is an embedding $M \hookrightarrow N$ as in the statement.

There is nothing to prove if $i > n+1$. Now consider the case $i = 1$. We need to show that given any tuple $(x_0, x_1, \ldots, x_n) \in M^{\oplus\binom{n+1}{1}}$ such that $(\gamma_j - 1)(x_k) = (\gamma_k - 1)(x_j)$ for all $0 \leq j < k \leq n$, where $\gamma_0 := \varphi_q$, we can find N and some $x \in N$ so that $(\gamma_j - 1)(x) = x_j$ for all $0 \leq j \leq n$. This is clear: we can just set $N := M \oplus \mathbf{A}_{K,A}x$ with $\gamma_j x := x + x_j$ for each $0 \leq j \leq n$. We can check easily that our assumption on the tuple $(x_j)_j$ guarantees precisely that these actions pairwise commute, and so define an étale (φ_q, Γ_K)-module structure on N, as wanted.

[4] In a general exact category, such embedding is often called a strict (or admissible) monomorphism.

The case $i = n+1$ follows directly from Lemma 2.4.9 below. It remains to treat the case where $2 \le i \le n$. In this case, given any tuple $(y_{j_0 j_1 \dots j_{i-1}})_{0 \le j_0 < \dots < j_{i-1} \le n} \in M^{\oplus \binom{n+1}{i}}$ such that

$$\sum_{k=0}^{i} (-1)^k (\gamma_k - 1) y_{j_0 \dots \hat{j}_k \dots j_i} = 0 \tag{2.4.7.1}$$

for all $(i+1)$-tuples $0 \le j_0 < \dots < j_i \le n$ (where, as usual, by $\hat{j}_k$ we indicate that we have omitted j_k from the tuple $(j_0, \dots, j_k, \dots, j_i)$), we need to find $M \hookrightarrow N$, and a tuple $(x_{j_0 \dots j_{i-2}})_{0 \le j_0 < \dots < j_{i-2} \le n} \in N^{\oplus \binom{n+1}{i-1}}$ such that

$$y_{j_0 j_1 \dots j_{i-1}} = \sum_{k=0}^{i-1} (-1)^k (\gamma_k - 1) x_{j_0 \dots \hat{j}_k j_{i-1}} \tag{2.4.7.2}$$

for all i-tuples $0 \le j_0 < \dots < j_{i-1} \le n$. By Lemma 2.4.9, after enlarging M, we may define the tuple $(x_{j_0 \dots j_{i-2}})_{0 \le j_0 < \dots < j_{i-2} \le n}$ as follows. Set $x_{j_0 \dots j_{i-2}} := 0$ if $j_0 = 0$, otherwise set $x_{j_0 \dots j_{i-2}}$ to be any element in N for which $(\varphi_q - 1)(x_{j_0 \dots j_{i-2}}) = y_{0 j_0 \dots j_{i-2}}$. Then (2.4.7.2) holds for all tuples $0 \le j_0 < \dots < j_{i-1} \le n$ with $j_0 = 0$. Assume that we have chosen $(x_{j_0 \dots j_{i-2}})_{0 \le j_0 < \dots < j_{i-2} \le n}$ so that (2.4.7.2) is true whenever $0 \le j_0 < j_0'$. It suffices to show that we may modify the tuple $(x_{j_0 \dots j_{i-2}})_{0 \le j_0 < \dots < j_{i-2} \le n}$ further so that (2.4.7.2) in fact holds true for all $0 \le j_0 \le j_0'$. Indeed, by the inductive hypothesis, and our assumption (2.4.7.1) (applied to the $(i+1)$-tuples $0 \le j_0 < j_0' < \dots < j_{i-1} \le n$ with $0 \le j_0 < j_0'$), we obtain $(\gamma_{j_0} - 1) z_{j_0' j_1 \dots j_{i-1}} = 0$ for all $0 \le j_0 < j_0'$, or equivalently, $z_{j_0' j_1 \dots j_{i-1}} \in M^{\varphi_q = 1, \gamma_1 = \dots = \gamma_{j_0'-1} = 1}$, where

$$z_{j_0' j_1 \dots j_{i-1}} := y_{j_0' j_1 \dots j_{i-1}} - \sum_{k=0}^{i-1} (-1)^k (\gamma_k - 1) x_{j_0' \dots \hat{j}_k j_{i-1}}.$$

By Lemma 2.4.8 (applied to $i = j_0' - 1 \le n-2$), we may thus enlarge N further so that $z_{j_0' j_1 \dots j_{i-1}} = (\gamma_{j_0'} - 1) z_{j_1 \dots j_{i-1}}$ for some $z_{j_1 \dots j_{i-1}} \in M^{\varphi_q = 1, \gamma_1 = \dots = \gamma_{j_0'-1} = 1}$. By replacing $x_{j_1 \dots j_{i-1}}$ with $x_{j_1 \dots j_{i-1}} + z_{j_1 \dots j_{i-1}}$, we see that (2.4.7.2) now holds true for all $0 \le j_0 \le j_0'$, as desired. □

Lemma 2.4.8. *Given any $M \in \mathcal{M}_{\varphi,\Gamma}^{\text{ét,LT}}$, and any $x \in M^{\varphi_q = 1, \gamma_1 = \dots = \gamma_i = 1}$ with $0 \le i \le n-2$, we can find an embedding $M \hookrightarrow N$ in $\mathcal{M}_{\varphi,\Gamma}^{\text{ét,LT}}$ so that $x \in (\gamma_n - 1)(N^{\varphi_q = 1, \gamma_1 = \dots = \gamma_i = 1})$.*

Proof. We will use descending induction on $0 \le i \le n-2$. Assume first that $i = n-2$. We can pick $r \ge 0$ large enough so that $(\gamma_{n-1} - 1)^r(x) = 0$. To see this, let U be an φ_q-stable open subgroup of M: such U exists by Lemma 2.4.10 below. As the map $\gamma_{n-1} - 1 : M \to M$ is topologically nilpotent, we have $(\gamma_{n-1} - 1)^r(x) \in U$ for all $r \ge 0$ sufficiently large. But this in fact forces $(\gamma_{n-1} - 1)^r(x) = 0$ by injectivity of the map $1 - \varphi_q : U \to U$.

We claim that for each $0 \le k \le r$, we can choose N so that $(\gamma_{n-1} - 1)^k(x) \in (\gamma_n - 1)(N^{\varphi_q = 1, \gamma_1 = \dots = \gamma_{n-2} = 1})$. This is clear for $k = r$ by our choice of r. Assume the claim is true for $k+1$ (with $0 \le k \le r-1$). Then we can choose $x' \in N^{\varphi_q = 1, \gamma_1 = \dots = \gamma_{n-2} = 1}$ so that $(\gamma_{n-1} - 1)^{k+1}(x) = (\gamma_n - 1)(x')$. Now applying the construction in the case of H^1 for the tuple $(0, \dots, 0, x', (\gamma_{n-1} -$

$1)^k(x))$, we can enlarge N further so that $(\gamma_{n-1}-1)^k(x) \in (\gamma_n-1)(N^{\varphi_q=1,\gamma_1=\ldots\gamma_{n-2}=1})$. In particular, we obtain the result for $i=n-2$ by setting $k=0$.

Assume that $0 \le i \le n-3$ and that the result have been proved for all $i+1 \le j \le n-2$. Again, we can pick $r \ge 0$ large enough so that $(\gamma_{i+1}-1)^{a_{i+1}}\ldots(\gamma_{n-1}-1)^{a_{n-1}}(x) = 0$ for all nonnegative integers $a_{i+1},\ldots,a_{n-1}$ with sum $a_{i+1}+\ldots+a_{n-1}=r$. In particular, we trivially have $(\gamma_{i+1}-1)^{a_{i+1}}\ldots(\gamma_{n-1}-1)^{a_{n-1}}(x) \in (\gamma_n-1)M^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=1}$. If $r \ge 1$, then we claim that the same property holds for $r-1$ after possibly enlarging M, i.e. that we can choose N so that $(\gamma_{i+1}-1)^{a'_{i+1}}\ldots(\gamma_{n-1}-1)^{a'_{n-1}}(x) \in (\gamma_n-1)N^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=1}$ for all tuples $(a'_{i+1},\ldots,a'_{n-1})$ of nonnegative integers of sum $r-1$. Indeed, for such tuple, we can write

$$\begin{cases} (\gamma_{i+1}-1)^{1+a'_{i+1}}\ldots(\gamma_{n-1}-1)^{a'_{n-1}}(x) & = (\gamma_n-1)(x_{i+1}) \\ & \ldots \\ (\gamma_{i+1}-1)^{a'_{i+1}}\ldots(\gamma_{n-1}-1)^{1+a'_{n-1}}(x) & = (\gamma_n-1)(x_{n-1}) \end{cases}$$

for some $x_{i+1},\ldots,x_{n-1} \in N^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=1}$. In particular, we have $x_{jk} := (\gamma_j-1)x_k-(\gamma_k-1)x_j \in M^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=\gamma_n=1}$ for every $i+1 \le j < k \le n-1$.

We will now modify the elements $x_{i+1},\ldots,x_{n-1}$ suitably so that in fact $x_{jk}=0$ for all such (j,k). More precisely, we will show by ascending induction on $i+1 \le j < n-1$ that we can modify the x_k with $k > j$ so that $x_{jk}=0$ for all such k. Assume first that $j=i+1$. Using the inductive hypothesis for the case $i+1 \le n-2$, after possibly enlarging M, we may choose for each $j < k \le n-1$ an element $y_k \in M^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=\gamma_n=1}$ so that $x_{jk}=(\gamma_j-1)(y_k)$. We now simply replace each x_k by x_k-y_k. Next assume that we have modified so that $x_{j'k}=0$ for all $i+1 \le j' < k \le n-1$ with $j' < j$. In particular, as $x_{j'j}=0=x_{j'k}$, we have

$$\begin{aligned}(\gamma_{j'}-1)x_{jk} &= (\gamma_j-1)(\gamma_{j'}-1)x_k-(\gamma_k-1)(\gamma_{j'}-1)x_j \\ &= (\gamma_j-1)(\gamma_k-1)x_{j'}-(\gamma_k-1)(\gamma_j-1)x_{j'} \\ &= 0.\end{aligned}$$

In other words, we in fact have $x_{jk} \in M^{\varphi_q=1,\gamma_1=\ldots=\gamma_{j-1}=\gamma_n=1}$. Thus, by applying the inductive hypothesis for $i+1 \le j \le n-2$, we can find (after possibly further enlarging N), for each $k > j$ an element $z_k \in M^{\varphi_q=1,\gamma_1=\ldots=\gamma_{j-1}=\gamma_n=1}$ so that $x_{jk}=(\gamma_j-1)z_k$. Again, we can now replace each x_k by x_k-z_k so that in fact $x_{jk}=0$, as wanted (note that as we have designed so that z_k are fixed by $\gamma_{j'}$ for all $j' < j$, this does not affect the modifcation that we made earlier on the $x_{j'k}$ with $j' < k$ (i.e. they are still zero)).

The upshot is that now the tuple $(0,\ldots,0,x_{i+1},\ldots,x_{n-1},(\gamma_{i+1}-1)^{a'_{i+1}}\ldots(\gamma_{n-1}-1)^{a'_{n-1}}(x))$ is a 1-cocycle, and so by using the construction in the case of H^1, we can further enlarge N so that $(\gamma_{i+1}-1)^{a'_{i+1}}\ldots(\gamma_{n-1}-1)^{a'_{n-1}}(x) \in (\gamma_n-1)N^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=1}$ for all tuples $(a'_{i+1},\ldots,a'_{n-1})$ of nonnegative integers of sum $r-1$, as claimed. Again by continuing this procedure, we may arrive to the case $r=0$, where we clearly have $x \in (\gamma_n-1)N^{\varphi_q=1,\gamma_1=\ldots=\gamma_i=1}$, as wanted. □

Lemma 2.4.9. *Given any $M \in \mathcal{M}_{\varphi,\Gamma}^{\text{ét,LT}}$, and any $x \in M$, we can find an embedding $M \hookrightarrow N$ in $\mathcal{M}_{\varphi,\Gamma}^{\text{ét,LT}}$ so that $x \in (\varphi_q-1)(N)$.*

Proof. It follows from Lemma 2.4.10 that $(\varphi_q - 1)(M)$ is an open subgroup of M. We may thus pick $r \geq 0$ large enough so that $(\gamma_1 - 1)^{a_1} \dots (\gamma_n - 1)^{a_n}(x) \in (\varphi_q - 1)(M)$ for all tuples $(a_1, \dots, a_n)$ of nonnegative integers with sum r. As in the proof of Lemma 2.4.8, we claim that if $r \geq 1$, then the same property holds for $r - 1 \geq 0$. Indeed, given any tuple $(a'_1, \dots, a'_n)$ with sum $r - 1$, we can write

$$\begin{cases} (\gamma_1 - 1)^{1+a'_1} \dots (\gamma_n - 1)^{a'_n}(x) & = (\varphi_q - 1)(x_1) \\ & \dots \\ (\gamma_1 - 1)^{a'_1} \dots (\gamma_n - 1)^{1+a'_n}(x) & = (\varphi_q - 1)(x_n) \end{cases}$$

for some $x_1, \dots, x_n \in M$. Again, we have $x_{jk} := (\gamma_j - 1)x_k - (\gamma_k - 1)x_j \in M^{\varphi_q=1}$ for all $1 \leq j < k \leq n$. By applying Lemma 2.4.8 and its proof, we may modify the elements $x_1, \dots, x_n$ inductively so that in fact $x_{jk} = 0$ for all such $j < k$. Indeed, assume first that $j = 1$. By the case $i = 0$ of the that lemma, we can find N so that $x_{1k} = (\gamma_1 - 1)y_k$ for some $y_k \in N^{\varphi_q=1}$. Then by replacing each x_k with $x_k - y_k$, we obtain $x_{1k} = 0$ for all $k > 1$. Assume we have $x_{j'k} = 0$ for all $1 \leq j' < k \leq n$ with $j' < j$. Arguing as the proof of *loc. cit.*, we see that in fact $x_{jk} \in M^{\varphi_q=1, \gamma_1=\dots=\gamma_{j-1}=1}$. Then by the case $i = j - 1 \leq n - 2$ of *loc. cit.*, we may choose $z_k \in M^{\varphi_q=1, \gamma_1=\dots=\gamma_{j-1}=1}$ so that $x_{jk} = (\gamma_j - 1)(z_k)$. Again by replacing each x_k with $x_k - z_k$, we may then assume that $x_{jk} = 0 = x_{j'k}$ for all $k > j > j'$, as desired.

Now by using the construction in the case of H^1 for the 1-cocycle $((\gamma_1 - 1)^{a'_1} \dots (\gamma_n - 1)^{a'_n}(x), x_1, \dots, x_n)$, we may further enlarge N so that $(\gamma_1 - 1)^{a'_1} \dots (\gamma_n - 1)^{a'_n}(x) \in (\varphi_q - 1)(N)$, as claimed. Again, in the case $r = 0$, we obtain $x \in (\varphi_q - 1)(N)$, as desired. □

Lemma 2.4.10. *Let A be a Noetherian $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let M be a finite (not necessarily étale) φ_q-module over $\mathbf{A}_{K,A}$. Then M admits an open φ_q-stable subgroup U on which the map $\varphi_q - 1$ is bijective.*

Proof. By considering M as a finite étale φ_q-module over the subring $\mathbf{A}_{F,A} \subseteq \mathbf{A}_{K,A}$, we may assume that $K = F$ (note that the canonical topology on M does not depend on whether we view M as a finite module over $\mathbf{A}_{K,A}$ or $\mathbf{A}_{F,A}$). In this case the subring $\mathbf{A}^+_{F,A}$ of $\mathbf{A}_{F,A}$ is φ_q-stable, and we can in fact choose U to be a φ_q-stable lattice in M. To see this, pick any lattice $\mathfrak{M}$ in M, and pick $n > 0$ large enough so that $\Phi_M(\varphi_q^*\mathfrak{M}) \subseteq T^{-n}\mathfrak{M}$. As $\pi^a = 0$ in A and $\varphi_q(T) \equiv T^q \bmod \pi\mathbf{A}^+_{F,A}$, it follows easily from the binomial theorem that $\varphi_q(T^{M+a-1})$ is divisible by T^{Mq} in $\mathbf{A}^+_{F,A}$. In particular, we have $\Phi_M(\varphi_q^*(T^{M+a-1}\mathfrak{M})) = \varphi_q(T^{M+a-1})\Phi_M(\varphi_q^*\mathfrak{M}) \subseteq T^{Mq-n}\mathfrak{M}$. If we pick M large enough so that $Mq - n \geq M + a - 1 + q^{a-1}$, then $\mathfrak{M}' := T^{M+a-1}\mathfrak{M}$ is a φ_q-stable lattice in M satisfying $\varphi_q(\mathfrak{M}') \subseteq T^{q^{a-1}}\mathfrak{M}'$. As $\varphi_q(T) \equiv T^q \bmod \pi$, we have $\varphi_q(T^{q^{a-1}}) \equiv T^{q^a} \bmod \pi^a$. Thus for any $t \geq 0$, we have $\varphi_q(T^{tq^{a-1}}\mathfrak{M}') \subseteq T^{(1+tq)q^{a-1}}\mathfrak{M}'$. As $\mathfrak{M}'$ is T-adically complete (being finite over the Noetherian T-adically complete ring $\mathbf{A}_{F,A}$), it follows that for each $x \in \mathfrak{M}'$, the series $\sum_{l \geq 0} \varphi_q^l(x)$ converges in $\mathfrak{M}'$. We deduce that the map $1 - \varphi_q : \mathfrak{M}' \to \mathfrak{M}'$ is bijective with inverse given by $\sum_{l \geq 0} \varphi_q^l$, as desired. □

Proof of Proposition 2.4.5. This follows from Lemmas 2.4.6 and 2.4.7, uniqueness of universal δ-functors, and the fact that we have a natural identification between $H^0(\mathcal{C}^\bullet(M)) = M^{\varphi_q=1, \Gamma_K=1}$ and $\mathrm{Hom}_{\mathcal{M}^{\text{ét,LT}}_{\varphi,\Gamma}}(\mathbf{A}_{K,A}, M) = \mathrm{Ext}^0_{\mathrm{LT}}(\mathbf{A}_{K,A}, M)$. □

Corollary 2.4.11. *Assume A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$. Let M be a finite projective étale (φ_q, Γ_K)-module with A-coefficients, and let M^{EG} denote the $(\varphi, \Gamma_K^{\mathrm{EG}})$-module corresponding to M under the equivalence in Proposition 2.4.1. Then $H^i(\mathcal{C}^\bullet(M)) \cong H^i_{\mathrm{EG}}(\mathcal{C}^\bullet(M^{\mathrm{EG}}))$ for each integer i.*

Proof. As the equivalence in *loc. cit.* is exact and clearly takes $\mathbf{A}_{K,A}$ to $\mathbf{A}_{K,A}^{\mathrm{EG}}$, we see that $\mathrm{Ext}^i_{\mathrm{LT}}(\mathbf{A}_{K,A}, M) \cong \mathrm{Ext}^i_{\mathrm{EG}}(\mathbf{A}_{K,A}^{\mathrm{EG}}, M^{\mathrm{EG}})$ for each $i \in \mathbf{Z}$. The result now follows from Proposition 2.4.5 (applied to both M and M^{EG}). □

Theorem 2.4.12. *Let A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let M be a finite projective étale (φ_q, Γ_K)-module with A-coefficients.*

(1) *The Herr complex $\mathcal{C}^\bullet(M)$ is a perfect complex of A-modules, with tor-amplitude in* $[0, 2]$.

(2) *If B is a finite type A-algebra, then there is natural isomorphism in the derived category*

$$\mathcal{C}^\bullet(M) \otimes_A^{\mathbf{L}} B \xrightarrow{\sim} \mathcal{C}^\bullet(M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,B}).$$

In particular, there is a natural isomorphism

$$H^2(C^\bullet(M)) \otimes_A B \xrightarrow{\sim} H^2(\mathcal{C}^\bullet(M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,B}).$$

Proof. Combining Corollary 2.4.11 with the analogous result in [EG23] for the cyclotomic Herr complex, we see that the cohomology groups $H^i(\mathcal{C}^\bullet(M))$ are finitely generated, and moreover vanish unless $i \in [0, 2]$. The theorem now follows by the same argument as in the proofs of [EG23, Thm. 5.1.22, Cor. 5.1.25]. □

Remark 2.4.13. 1. For showing that the Herr complex is perfect, it suffices by [Sta23, Tag 07LU] to consider the case where A is a finite type $\mathbf{F}$-algebra. Thus, in view of the above proof, it is in fact enough to invoke the comparison 2.4.1 only for such algebras; in other words, we need the isomorphism $\mathcal{X}_{K,d}^{\mathrm{LT}} \xrightarrow{\sim} \mathcal{X}_{K,d}^{\mathrm{EG}}$ only on special fibers. On the other hand, it would be desirable to have a proof purely in the world of Lubin–Tate (φ_q, Γ_K)-modules.

2. Our first approach on showing perfectness of the Herr complex was to proceed along the lines of [EG23, §5.1] by using a kind of ψ-operator on Lubin–Tate (φ_q, Γ_K)-modules. However, this turned out to be not possible (as far as we know) since there is no Γ_K-equivariant left inverse of $\varphi_q : M \to M$ if $F \neq \mathbf{Q}_p$, cf. [BR22, Prop. 3.2.6].

Corollary 2.4.14. *Assume A is a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$. Then the Herr complex $\mathcal{C}^\bullet(M)$ can be represented by a complex $[C^0 \to C^1 \to C^2]$ of finite locally free A-modules in degrees* $[0, 2]$.

Proof. This follows from Theorem 2.4.12, and [Sta23, Tag 0658]. □

2.4.2.1 Galois cohomology via the Herr complex

As another consequence of the effaceablity of the functors $M \mapsto H^i(\mathcal{C}^\bullet(M)), i \geq 1$ (Lemma 2.4.7), we recover the following comparison result. The approach employed here is similar to that in [Her98]; see also [AK19] for a different approach.

Theorem 2.4.15. *Let A be a complete local Noetherian $\mathcal{O}$-algebra with finite residue field of characteristic p, and let V be a finite A-module equipped with a continuous linear representation of G_K. Then there are isomorphism of A-modules*

$$H^i(G_K, V) \xrightarrow{\sim} H^i(\mathcal{C}^\bullet(D_A(V))).$$

which are functorial in A.

Proof. First assume that $\mathfrak{m}_A^n V = 0$ for some n. The functor D_A extends formally to an equivalence between the category $\mathrm{Rep}_A^{\mathrm{tor}}(G_K)$ of $\mathfrak{m}_A$-power torsion linear G_K-representations, and the category $\mathcal{M}_{\varphi,\Gamma}^{\text{Ind-ét, tor}}$ of direct colimits of finite $\mathfrak{m}_A$-power torsion étale (φ, Γ)-modules over $\mathbf{A}_{K,A}$ (the point being that, unlike its subcategory of *finite* G_K-representations, $\mathrm{Rep}_A^{\mathrm{tor}}(G_K)$ has enough injective objects). Clearly, if M is an object in $\mathcal{M}_{\varphi,\Gamma}^{\text{Ind-ét, tor}}$, then M has a natural (φ_q, Γ_K)-module structure, and we can define the Herr complex of M exactly as for finite étale (φ_q, Γ_K)-modules. We can check easily that the arguments in Lemmas 2.4.7, 2.4.8 and 2.4.9 still make sense, and thus show that the functors $M \mapsto H^i(\mathcal{C}^\bullet(M)), i \geq 1$ remain effaceable on the category $\mathcal{M}_{\varphi,\Gamma}^{\mathrm{Ind-ét,tor}}$. Now as the cohomological δ-functor $(V \mapsto H^i(G_K, V))_{i\geq 0}$ is universal on $\mathrm{Rep}_A^{\mathrm{tor}}(G_K)$, we obtain the result in this case.

For a general V, consider the inverse system $(\mathcal{C}^\bullet(G_K, V/\mathfrak{m}_A^n V))_n$ of cochain complexes. As the transition maps are surjective, its derived limit can be computed by taking inverse limits termwise, i.e.

$$R\lim \mathcal{C}^\bullet(G_K, V/\mathfrak{m}_A^n V) = \varprojlim \mathcal{C}^\bullet(G_K, V/\mathfrak{m}_A^n V) = C^\bullet_{\mathrm{cts}}(G_K, V).$$

In particular, there is a Milnor short exact sequence

$$0 \to R^1 \lim H^{i-1}(G_K, V/\mathfrak{m}_A^n V) \to H^i(G_K, V) \to \varprojlim H^i(G_K, V/\mathfrak{m}_A^n V) \to 0$$

for each $i \in \mathbf{Z}$. By what we have just seen, each $H^{i-1}(G_K, V/\mathfrak{m}_A^n V)$ is computed by $H^{i-1}(\mathcal{C}^\bullet(D_A(V/\mathfrak{m}_A^n V))$, and hence finite by Theorem 2.4.12. It follows that the $R^1 \lim$ on the left vanishes, and we obtain

$$H^i(G_K, V) = \varprojlim H^i(G_K, V/\mathfrak{m}_A^n V).$$

Arguing similarly for the inverse system $(\mathcal{C}^\bullet(D_A(V/\mathfrak{m}_A^n V)))_n$ (and recalling that $D_A(V) = \varprojlim D_A(V/\mathfrak{m}_A^n V)$), we obtain

$$H^i(\mathcal{C}^\bullet(D_A(V))) = \varprojlim H^i(\mathcal{C}^\bullet(D_A(V/\mathfrak{m}_A^n V))),$$

hence the desired result. □

2.4.2.2 Obstruction theory

As in [EG23, §5.1.33], one can use perfectness of the Herr complex to show that the stack $\mathcal{X}_{K,d}^{\mathrm{LT}}$ admits a nice obstruction theory in the sense of [EG23, Defn. A.34].

Let A be a finite type $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let M be a rank d projective étale (φ_q, Γ_K)-module with A-coefficients, classified by a map $x : \operatorname{Spec} A \to \mathcal{X}_{K,d}^{\mathrm{LT}}$. Given a square-zero extension

$$0 \to I \to A' \to A \to 0$$

in which A' is a finite type $\mathcal{O}/\varpi^a$-algebra, we want to consider the problem of deforming M to a projective étale (φ_q, Γ_K)-module M' with A'-coefficients. More formally, let $\mathrm{Lift}(x, A')$ be the set of isomorphism classes of pairs (M', ι) where M' is a projective étale (φ_q, Γ_K)-module M' with A'-coefficients, and ι is an isomorphism $M' \otimes_{A'} A \xrightarrow{\sim} M$ of (φ_q, Γ_K)-module with A-coefficients.

First, we can lift M uniquely to a rank d projective $\mathbf{A}_{K,A'}$-module $\widetilde{M}$ (as a finite projective module always deforms uniquely along a nilpotent thickening). By viewing $\varphi : M \to M$ as a linear map $\varphi_q^* M \to M$, we see that φ also lifts to a semilinear map $\widetilde{\varphi} : \widetilde{M} \to \widetilde{M}$. Note that the linearization $\Phi_{\widetilde{\varphi}}$ of $\widetilde{\varphi}$ is then automatically an isomorphism as it is a surjective map between finitely projective modules of the same rank (surjectivity follows from Nakayama's lemma and the fact that $\Phi_{\widetilde{\varphi}}$ is surjective modulo a nilpotent ideal). Similarly, we can also lift each γ_i to a bijective semilinear map $\widetilde{\gamma}_i : \widetilde{M} \to \widetilde{M}$. However, the problem is that the actions of $\widetilde{\varphi}$ and $\widetilde{\gamma}_i$ on $\widetilde{M}$ may not commute with each other (the set $\mathrm{Lift}(x, A')$ is therefore empty in this case). On the other hand, as the following lemma shows, this is the only obstruction to lift M to an A'-point of $\mathcal{X}_{K,d}^{\mathrm{LT}}$.

Lemma 2.4.16. *Assume there exist pairwise commuting lifts $\widetilde{\varphi}, \widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n$ of $\varphi, \gamma_1, \dots, \gamma_n$ respectively to $\widetilde{M}$. Then the induced semilinear action of $\Gamma_{K,\mathrm{disc}}$ on $\widetilde{M}$ is continuous. In other words, these lifts makes $\widetilde{M}$ into an étale (φ_q, Γ_K)-module with A'-coefficients which lifts M.*

Proof. By assumption, $\widetilde{M}$ is an object of $\mathcal{R}_{K,d}^{\Gamma_{K,\mathrm{disc}}}(A')$, and we want to show that $\widetilde{M}$ in fact comes from an object of $\mathcal{X}_{K,d}^{\mathrm{LT}} \hookrightarrow \mathcal{R}_{K,d}^{\Gamma_{K,\mathrm{disc}}}$. Using Lemma 2.3.17, we may reduce to the case K is F-basic. In particular, we can apply the material on T-quasi-linear endomorphisms. More precisely, by Lemma 2.A.8, it suffices to show that the action of $\gamma_i - 1$ on $\widetilde{M}$ is topologically nilpotent for each $1 \leq i \leq [K : \mathbf{Q}_p]$. Arguing in as the proof of [EG23, Lem. D.31], we may reduce to the case when $\widetilde{M}$ is free. In particular, we may choose a *free* latice $\widetilde{\mathfrak{M}}$ in $\widetilde{M}$. Then $\mathfrak{M} := \widetilde{\mathfrak{M}} \otimes_{A'} A$ is also a lattice in $M := \widetilde{M} \otimes_{A'} A$. As the action of $\Gamma_{K,\mathrm{disc}}$ on M is continuous by assumption, we can use Lemma 2.A.8 again to find $m \gg 0$ large enough so that $(\gamma_i - 1)^m \mathfrak{M} \subseteq T\mathfrak{M}$. Equivalently, we have $(\gamma_i - 1)^m \widetilde{\mathfrak{M}} \subseteq T\widetilde{\mathfrak{M}} + I\widetilde{\mathfrak{M}}$. Repeating the same argument for the *free* lattice $T\widetilde{\mathfrak{M}}$ in $\widetilde{\mathfrak{M}}$, we can choose some $m' \geq m$ large enough so that $(\gamma_i - 1)^{m'} T\widetilde{\mathfrak{M}} \subseteq T^2\widetilde{\mathfrak{M}} + I(T\widetilde{\mathfrak{M}})$. As $I^2 = 0$, it follows that $(\gamma_i - 1)^{m+m'}\widetilde{\mathfrak{M}} \subseteq T^2\widetilde{\mathfrak{M}} + I(T\widetilde{\mathfrak{M}}) + I(T\widetilde{\mathfrak{M}} + I\widetilde{\mathfrak{M}}) = T^2\widetilde{\mathfrak{M}} + IT\widetilde{\mathfrak{M}} \subseteq T\widetilde{\mathfrak{M}}$, as desired. □

Let $\operatorname{ad} M := \operatorname{Hom}_{\mathbf{A}_{K,A}}(M, M)$ be the adjoint of M, endowed with its natural structure of an étale (φ_q, Γ_K)-module with A-coefficients.

We are now ready to measure the obstruction for lifting M to A'.

Lemma 2.4.17. *There is a functorial obstruction element $o_x(A') \in H^2(\mathcal{C}^\bullet(\mathrm{ad}\ M) \otimes^{\mathbf{L}}_A I)$, which vanishes precisely when x can be lifted to* $\operatorname{Spec} A'$.

Proof. As above, we can choose semilinear lifts $\widetilde{\gamma}_i$ of the γ_i to $\widetilde{M}$ (again, we will denote φ by γ_0 for ease of notation). We first check that the tuple

$$(\widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1} - 1)_{0\le i<j\le n} \in \mathcal{C}^2(\mathrm{ad}\ M \otimes_A I) \qquad (2.4.17.1)$$

is a cocyle, i.e. that $(\gamma_k - 1)(\widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1} - 1) - (\gamma_j - 1)(\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1} - 1) + (\gamma_i - 1)(\widetilde{\gamma}_j\widetilde{\gamma}_k\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1} - 1) = 0$ in $\mathrm{ad}\ M \otimes_A I$ for all $0 \le i < j < k \le n$. It is convenient to consider this as an equation in $\mathrm{ad}_{\mathbf{A}_{K,A'}} \widetilde{M} \supseteq \mathrm{ad}\ M \otimes_A I$; in particular, for each $0 \le i \le n$, the action of γ_i on $\mathrm{ad}\ M \otimes_A I$ is now nothing but the conjugation by $\widetilde{\gamma}_i : \widetilde{M} \to \widetilde{M}$. Now fix $0 \le i < j < k \le n$, and write $Y := \widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1} - 1$, $Z := \widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1} - 1$. Then $\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_i^{-1} = (1+Y)\widetilde{\gamma}_k$ and $\widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1} = (1+Z)\widetilde{\gamma}_j$. Multiplying these equations gives $\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1} = \widetilde{\gamma}_k\widetilde{\gamma}_j + \widetilde{\gamma}_k Z\widetilde{\gamma}_j + Y\widetilde{\gamma}_k\widetilde{\gamma}_j$ whence $\widetilde{\gamma}_k Z\widetilde{\gamma}_k^{-1} + Y = \widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1} - 1$. Similarly, we have $\widetilde{\gamma}_j Y\widetilde{\gamma}_j^{-1} + Z = \widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_k\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1} - 1$. It follows that $(\gamma_k - 1)Z - (\gamma_i - 1)Y = \widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1} - \widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_k\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}$. We are thus reduced to showing that $\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1} - \widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_k\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1} + \widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_k\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_i^{-1} - \widetilde{\gamma}_j\widetilde{\gamma}_k\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1} = 0$. We can rewrite the left hand side as $\widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_k(\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_i^{-1} - \widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}) + (\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1} - \widetilde{\gamma}_j\widetilde{\gamma}_k)(\widetilde{\gamma}_k\widetilde{\gamma}_j)^{-1} = \widetilde{\gamma}_i(\widetilde{\gamma}_j\widetilde{\gamma}_k - \widetilde{\gamma}_k\widetilde{\gamma}_j)(\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_i^{-1} - \widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}) + (\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1} - \widetilde{\gamma}_j\widetilde{\gamma}_k)((\widetilde{\gamma}_k\widetilde{\gamma}_j)^{-1} - (\widetilde{\gamma}_j\widetilde{\gamma}_k)^{-1}) = \widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_k(1 - \widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k\widetilde{\gamma}_j)(1 - \widetilde{\gamma}_i^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j)\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_i^{-1} + \widetilde{\gamma}_j\widetilde{\gamma}_k(\widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_i\widetilde{\gamma}_k\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1} - 1)(1 - \widetilde{\gamma}_k^{-1}\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k\widetilde{\gamma}_j)\widetilde{\gamma}_j^{-1}\widetilde{\gamma}_k^{-1}$. The last expression is now clearly zero as $I^2 = 0$.

We can now define $o_x(A')$ to be the image in $H^2(\mathcal{C}^\bullet(\mathrm{ad}\ M \otimes_A I))$ of the tuple 2.4.17.1. It remains to check that $o_x(A')$ is independent of the choice of the lifts $\widetilde{\gamma}_i$, and vanishes if and only if $\mathrm{Lift}(x, A') \neq \emptyset$. Indeed, any other choice of lifts $\widetilde{\gamma}'_i$ is of the form $\widetilde{\gamma}'_i = (1 + X_i)\widetilde{\gamma}_i$ for some $X_i \in \mathrm{ad}\ M \otimes_A I$. From this, one can check easily that

$$(\widetilde{\gamma}'_i\widetilde{\gamma}'_j(\widetilde{\gamma}'_i)^{-1}(\widetilde{\gamma}'_j)^{-1} - 1) - (\widetilde{\gamma}_i\widetilde{\gamma}_j\widetilde{\gamma}_i^{-1}\widetilde{\gamma}_j^{-1} - 1) = (\gamma_i - 1)X_j - (\gamma_j - 1)X_i.$$

This shows that the class $o_x(A') \in H^2(\mathcal{C}^\bullet(\mathrm{ad}\ M \otimes_A I))$ is well-defined, and vanishes if and only if we can choose commuting lifts $\widetilde{\gamma}_i$. By Lemma 2.4.16, the latter is equivalent to the condition that $\mathrm{Lift}(x, A') \neq \emptyset$, as desired (note also that as $\mathcal{C}^\bullet(\mathrm{ad}\ M)$ is a complex of flat A-modules, the derived tensor $\mathcal{C}^\bullet(\mathrm{ad}\ M) \otimes^{\mathbf{L}}_A I$ simplifies to $\mathcal{C}^\bullet(\mathrm{ad}\ M \otimes_A I)$). □

Lemma 2.4.18. *Let F be a finitely generated A-module. Then there is a natural isomorphism $\mathrm{Lift}(x, A[F]) \xrightarrow{\sim} H^1(\mathcal{C}^\bullet(\mathrm{ad}\ M) \otimes^{\mathbf{L}}_A F)$ of A-modules.*

Proof. As before, a lift of M to $A[F]$ is determined by lifts $\widetilde{\gamma}_i$ of the γ_i on M. Given any such lifts, we can write $\widetilde{\gamma}_i = (1 + X_i)\gamma_i$ for some $X_i \in \mathrm{ad}\ M \otimes_A F$ (note that we have abusively denoted also by γ_i the lifts corresponding to the trivial lift $M \otimes_A A[F]$). It is easy to check that the lifts $\widetilde{\gamma}_i$ commute with each other (or equivalently, define an element of $\mathrm{Lift}(x, A[F])$) if and only if the tuple $(X_0, \ldots, X_n)$ lies $Z^1(\mathcal{C}^\bullet(\mathrm{ad}\ M \otimes_A F))$. Furthermore, as the endomorphisms of the trivial lifting $M \otimes_A A[F]$ are given by $1 + X$ for $X \in \mathrm{ad}\ M \otimes_A F$, we see that the lift determined by a tuple $(X_0, \ldots, X_n) \in Z^1(\mathcal{C}^\bullet(\mathrm{ad}\ M \otimes_A F))$ is trivial if and only if there exists $X \in \mathrm{ad}\ M \otimes_A F$ such that $\widetilde{\gamma}_i = (1 + X)\gamma(1 + X)^{-1}$ for all $0 \le i \le n$. It is easy to check that this is equivalent to saying that $(X_0, \ldots, X_n) = d^0(-X)$ is a 1-coboundary. Thus, we obtain a bijection $H^1(\mathcal{C}^\bullet(\mathrm{ad}\ M \otimes_A F)) \xrightarrow{\sim} \mathrm{Lift}(x, A[F])$ at the level of sets. That it is also A-linear can be done exactly as in the proof of [EG23, Lem. 5.1.35]. □

Corollary 2.4.19. *$\mathcal{X}_{K,d}^{\mathrm{LT}}$ admits a nice obstruction theory in the sense of [EG23, Defn. A.34].*

Proof. This follows by combining Theorem 2.4.12, Lemmas 2.4.18 and 2.4.17. □

2.4.3 Families of extensions

We now briefly indicate how to proceed along the lines of the arguments in [EG23, Chap. 5] to give an alternative proof that $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is a Noetherian formal algebraic stack using only perfectness of the Lubin–Tate Herr complex (in particular, we are not using directly the comparison in Corollary 2.4.2 between $\mathcal{X}_{K,d}^{\mathrm{LT}}$ and the Emerton–Gee stack $\mathcal{X}_{K,d}^{\mathrm{EG}}$). See also Remark 2.4.21 below.

Roughly speaking, saying that $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is in fact a formal algebraic stack means that the transition maps in the colimit defining its Ind-algebraic structure are actually thickenings (rather than just closed immersions), and this amounts to showing that the underlying reduced substack $(\mathcal{X}_{K,d}^{\mathrm{LT}})_{\mathrm{red}}$ is an actual algebraic stack (see [Eme, Cor. 6.6] for a more precise statement). For this, it suffices to construct a *finite* collection of morphisms $\mathcal{Z} \to (\mathcal{X}_{K,d}^{\mathrm{LT}})_{\mathrm{red}}$ whose source is an algebraic stack, and the union of whose images exhaust all the $\overline{\mathbf{F}}_p$-points of $(\mathcal{X}_{K,d}^{\mathrm{LT}})_{\mathrm{red}}$. As any mod p Galois representation can be written as an iterated extension of irreducible subquotients, the idea is therefore to use induction on the dimension d and inductively construct spaces of extensions of some families of representations of dimension $< d$ by some given irreducible representations, which will ultimately give the desired cover. However since $\mathcal{X}_{K,d}^{\mathrm{LT}}$ is really a stack of (φ, Γ)-modules, in order to carry out the above strategy, one needs a way to algebraize the various Ext^1 spaces so as to be able to talk about families of extensions of (φ, Γ)-modules. This is where we need the Herr complex and its finiteness properties. (With the construction just outlined above, it is not clear if the *irreducible* representations can be covered at all. It is however indeed true and is in fact one of the distinctive features of these stacks of (φ, Γ)-modules (compared to the case of literal Galois representations): the representations occurring are generically reducible, but can specialise to irreducible representations.)

More precisely, starting with a family $\overline{\rho}_T : T \to (\mathcal{X}_{d,\mathrm{red}}^{\mathrm{LT}})_{\overline{\mathbf{F}}_p}$ on a reduced affine $\overline{\mathbf{F}}_p$-scheme of finite type, and any representation $\overline{\alpha} : G_K \to \mathrm{GL}_a(\overline{\mathbf{F}}_p)$ for which $\mathrm{Ext}^2_{G_K}(\overline{\alpha}, \overline{\rho}_t)$ is of constant rank for varying $t \in T(\overline{\mathbf{F}}_p)$, one can construct a vector bundle $V \to T$ together with a morphism

$$V \to (\mathcal{X}_{d+a,\mathrm{red}}^{\mathrm{LT}})_{\overline{\mathbf{F}}_p}$$

parametrizing a universal family of extensions

$$0 \to \overline{\rho}_T \otimes_{\mathcal{O}_T} \mathcal{O}_V \to \mathcal{E}_V \to \overline{\alpha} \otimes_{\overline{\mathbf{F}}_p} \mathcal{O}_V \to 0.$$

One can then follow the proof of [EG23, Thm. 5.5.12] to obtain the following result. (As explained above, for the algebraicity part, the rough idea is that by iterating the above construction, we obtain families of étale (φ_q, Γ_K)-modules parametrized by the various vector bundles V appearing; this will ultimately gives a cover of $\mathcal{X}^{\mathrm{LT}}_{K,d,\mathrm{red}}$. For obtaining the desired dimension, we will need to actually compute the dimensions of the various families of extensions arising from this construction, cf. [EG23, Prop. 5.4.4].)

Theorem 2.4.20. *$\mathcal{X}^{\mathrm{LT}}_{d,\mathrm{red}}$ is a finitely presented algebraic stack over $\mathbf{F}$ of dimension $[K : \mathbf{Q}_p]d(d-1)/2$.*

In particular, by combining algebraicity of $\mathcal{X}_{d,\mathrm{red}}$ with Corollary 2.4.19, we obtain another proof that $\mathcal{X}^{\mathrm{LT}}_{K,d}$ is a Noetherian algebraic stack (see the proof of [EG23, Cor. 5.5.18], which ultimately relies on the criteria of [Eme, Cor. 6.6 and Thm. 11.13]).

Remark 2.4.21. In fact, the inductive argument in [EG23, Thm. 5.5.12] also allows us to construct, for each Serre weight $\underline{k}$, an irreducible component $\mathcal{X}^{\underline{k},\mathrm{LT}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ of $(\mathcal{X}^{\mathrm{LT}}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$ of dimension $[K : \mathbf{Q}_p]d(d-1)/2$, whose generic $\overline{\mathbf{F}}_p$-points are maximally nonsplit of niveau 1 and weight $\underline{k}$ in the sense of [EG23, Defn. 5.5.1], and that the union of the $\mathcal{X}^{\underline{k},\mathrm{LT}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$, together with a closed substack of dimension strictly smaller than $[K : \mathbf{Q}_p]d(d-1)/2$, covers $(\mathcal{X}^{\mathrm{LT}}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$. In order to show that the $\mathcal{X}^{\underline{k},\mathrm{LT}}_{d,\mathrm{red},\overline{\mathbf{F}}_p}$ exhaust the irreducible components of $(\mathcal{X}^{\mathrm{LT}}_{d,\mathrm{red}})_{\overline{\mathbf{F}}_p}$, we need to show that the latter is equidimensional. In the cyclotomic case, this is achieved by first constructing closed substacks of $\mathcal{X}^{\mathrm{LT}}_d$ corresponding to crystalline representations, computing their versal rings at finite type points, and then using the existence of crystallne lifts of mod p representations. In order to adapt this strategy to the Lubin–Tate setting, it seems necessary to first have at one's disposal a description of G_K-stable $\mathcal{O}_F$-lattices in crystalline F-linear representations of G_K in terms of Breuil–Kisin–Fargues modules over $\mathbf{A}_{\mathrm{inf},F} := W_{\mathcal{O}_F}(\mathcal{O}^\flat_{\mathbf{C}})$. In Chapter 3 below, we will obtain such a description for a special class of crystalline $\mathcal{O}_F$-lattices. As the discussion there suggests, it seems that in order to obtain a description valid in general, one should consider modifications of vector bundles at several points (as opposed to just one point) on the Fargues–Fontaine curve X_F associated to F. We hope to discuss this in more details in a subsequent work.

Appendices

2.A T-quasi-linear endomorphisms

We recall some material on T-quasi-linear endomorphisms, adapted to our slightly more general setting. We will be content with giving only the results that we need in our proof of Ind-algebraicity of $\mathcal{X}_{K,d}^{\mathrm{LT}}$; for a more detailed account, we refer the reader to Appendix D of [EG23].

Fix a finite extension $F/\mathbf{Q}_p$ with uniformizer π. Fix also a finite unramified extension K/F, and a finite extension E/F with uniformizer ϖ and ring of integers $\mathcal{O}$. If A is ϖ-adically complete $\mathcal{O}$-algebra, we let $\mathbf{A}_A^+ := (\mathcal{O}_K \otimes_{\mathcal{O}_F} A)[[T]]$, and let $\mathbf{A}_A$ be the p-adic completion of $\mathbf{A}_A^+[1/T]$. As usual, finite projective modules over $\mathbf{A}_A$ or $\mathbf{A}_A^+$ will be endowed with their canonical topology (cf. [EG23, Rem. D.2]).

Definition 2.A.1. Let A be an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let M be a finite projective $\mathbf{A}_A$-module. A T-quasi-linear endomorphism of M is a continuous $\mathcal{O}_K \otimes_{\mathcal{O}_F} A$-linear morphism $f : M \to M$ which furthermore satisfies: there exist $a(T) \in (\mathbf{A}_A^+)^\times$ and $b(T) \in (\pi, T)\mathbf{A}_A^+$ such that

$$f(Tm) = a(T)Tf(m) + b(T)Tm$$

for every $m \in M$.

Remark 2.A.2. The above definition recovers [EG23, Defn. D.17] in case $F = \mathbf{Q}_p$.

Lemma 2.A.3. *Assume f is T-quasi-linear, then for all $n \in \mathbf{Z}$, we can write*

$$f(T^n m) = a(T)^n T^n f(m) + b_n(T)T^n m$$

for all $m \in M$, where $a(T) \in (\mathbf{A}_A^+)^\times$ and $b_n(T) \in (\pi, T)\mathbf{A}_A^+$.

Proof. See the proof of [EG23, Lem. D.18]. □

Lemma 2.A.4. *Let A be an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and let M be a finite projective $\mathbf{A}_A$-module. Let f be a T-quasi-linear endomorphism of M, and let $\mathfrak{M}$ be a lattice in M. Then there is an integer $m \geq 0$ such that $f^n(T^s\mathfrak{M}) \subseteq T^{s-mn}\mathfrak{M}$ for all $s \in \mathbf{Z}$ and $n \geq 0$.*

Proof. As f is continuous *by definition*, we can choose $m \geq 0$ large enough so that $f(T^m\mathfrak{M}) \subseteq \mathfrak{M}$. From this and Lemma 2.A.3, it is easy to see that $f(T^s\mathfrak{M}) \subseteq T^{s-m}\mathfrak{M}$ for all $s \in \mathbf{Z}$. The lemma now follows by an evident induction. □

Lemma 2.A.5. *Let A be an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$. Let M be a finite projective $\mathbf{A}_A$-module, and let f be a T-quasi-linear endomorphism of M. The following are equivalent:*

(1) *f is topologically nilpotent.*

(2) *There exists a lattice $\mathfrak{M} \subseteq M$ and some $n \geq 1$ such that $f^n(\mathfrak{M}) \subseteq (\pi, T)\mathfrak{M}$.*

(3) *For any lattice $\mathfrak{M} \subseteq M$ and any $m \geq 1$, we have $f^n(\mathfrak{M}) \subseteq T^m\mathfrak{M}$ for all sufficiently large n.*

Proof. The proof is identical to that of [EG23, Lem. D.21], except that we replace p by π everywhere, and appeal to Lemma 2.A.3 in place of [EG23, Lem. D.18]. □

We now specialize to our main case of interest. To this end, assume that $\mathbf{A}_A$ is endowed with a continuous action of $\mathbf{Z}_p$ by $\mathcal{O}_K \otimes_{\mathcal{O}_F} A$-algebra automorphisms, which preserves $\mathbf{A}_A^+$, and moreover satisfies

$$\gamma(T) - T \in (\pi, T)T\mathbf{A}_A^+ \tag{2.A.5.1}$$

for some topological generator γ of $\mathbf{Z}_p$.

Lemma 2.A.6. *Let M be a finite projective $\mathbf{A}_A$-module which is endowed with a semilinear action of the subgroup $\langle\gamma\rangle$ of $\mathbf{Z}_p$, then $f := \gamma^n - 1$ is a T-quasi-linear endomorphism of M for any $n \geq 1$.*

Proof. By induction, it is easy to see that $\gamma^n(T) - T \in (\pi, T)T\mathbf{A}_A^+$ for all $n \geq 1$. As $(\gamma^n - 1)(Tm) = \gamma^n(T)(\gamma^n - 1)(m) + (\gamma^n(T) - T)m$, we have $f(Tm) = a(T)^n T^n f(m) + b_n(T)Tm$ where $a(T) := \gamma(T)/T \in (\mathbf{A}_A^+)^\times$, and $b_n(T) := (\gamma^n(T) - T)/T \in (\pi, T)\mathbf{A}_A^+$. As we already require γ to be $\mathcal{O}_K \otimes_{\mathcal{O}_F} A$-linear, it remains to check that $f : M \to M$ is continuous. To see this, let $\mathfrak{M}$ be an arbitrary lattice in M, say generated by $m_1, \ldots, m_r \in M$. Let $\mathfrak{N}$ be the sub-$\mathbf{A}_A^+$-module of M generated by $\gamma^{-n}(m_1), \ldots, \gamma^{-n}(m_r)$. As γ acts on $\mathbf{A}_A$ and M by automorphisms, and moreover preserves $\mathbf{A}_A^+$, it is clear that $\mathfrak{N}$ is a lattice in M verifying $\gamma^n(\mathfrak{N}) \subseteq \mathfrak{M}$. Thus, γ^n (hence f) is continuous, as desired. □

Remark 2.A.7. We have seen in Lemma 2.3.19 that in case K is F-basic, the action of $\gamma \in \Gamma_K$ on $\mathbf{A}_{K|F,A}$ indeed satisfies 2.A.5.1. Accordingly[5], we can apply Lemma 2.A.6 in the case where (K is F-basic and) M is an object of $\mathcal{R}_{K,d}^{\Gamma_{\mathrm{disc}}}$, and the semilinear action of $\langle\gamma\rangle$ is given by restricting the action of Γ_{disc} on M.

Lemma 2.A.8. *Assume that A is an $\mathcal{O}/\varpi^a$-algebra for some $a \geq 1$, and that $\mathbf{A}_A$ is endowed with an action of $\mathbf{Z}_p$ as above. Let M be a finite projective $\mathbf{A}_A$-module, equipped with a semi-linear action of $\langle\gamma\rangle \subseteq \mathbf{Z}_p$. Then the following are equivalent:*

(1) *The action of $\langle\gamma\rangle$ extends (necessarily uniquely) to a continuous action of $\mathbf{Z}_p$.*

(2) *The action of $\langle\gamma\rangle$ on M is continuous.*

[5] We should also check that γ fixes the subring $W_{\mathcal{O}_F}(k_{K,\infty}) \otimes_{\mathcal{O}_F} A$, but this is clear because $k_{K,\infty} = k_K$ if K is F-basic.

(3) *For any lattice* $\mathfrak{M} \subseteq M$ *and any* $n \geq 1$*, there exists* $s \geq 0$ *such that* $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T^n\mathfrak{M}$*.*

(4) *For some lattice* $\mathfrak{M} \subseteq M$ *and some* $s \geq 0$*, we have* $(\gamma^{p^s} - 1)(\mathfrak{M}) \subseteq T\mathfrak{M}$*.*

(5) *The action of* $\gamma - 1$ *on* M *is topologically nilpotent.*

Proof. The proof is essentially identical to that of [EG23, Lem. D.28], but to make sure that everything works out as expected, we will provide the details. By [EG23, Lem. D.15], (1) and (2) are equivalent. Clearly, (3) $\implies$ (4). Now if (4) holds, then as $(\gamma^{p^s} - 1) \equiv (\gamma - 1)^{p^s} \bmod p$, we deduce that $(\gamma - 1)^{p^s}(\mathfrak{M}) \subseteq (p, T)\mathfrak{M} \subseteq (\pi, T)\mathfrak{M}$, and so $\gamma - 1$ is topologically nilpotent by Lemma 2.A.5 (2). Conversely, assume (5) holds. By Lemma 2.A.5 (3), we may choose $s \geq 0$ large enough so that $(\gamma - 1)^{p^s}\mathfrak{M} \subseteq T^n\mathfrak{M}$. As $(\gamma - 1)^{p^s} \equiv (\gamma^{p^s} - 1) \bmod p$, we see that $(\gamma^{p^s} - 1)\mathfrak{M} \subseteq (p, T^n)\mathfrak{M}$; in particular that (3) holds when $a = 1$. Assume now that (3) is true up to $a - 1 \geq 1$. By the inductive hypothesis (applied to the lattice $(\mathfrak{M} + \varpi^{a-1}M)/\varpi^{a-1}M$ inside $M/\varpi^{a-1}M$), we can choose s large enough so that $(\gamma^{p^s} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M} + \varpi^{a-1}M$. In particular, we have $p(\gamma^{p^s} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$. Now as $\gamma^{p^s} - 1$ is topologically nilpotent, we can use Lemma 2.A.5 again to pick t large enough so that $(\gamma^{p^s} - 1)^{p^t}\mathfrak{M} \subseteq T^n\mathfrak{M}$. As $(\gamma^{p^{s+t}} - 1) = ((\gamma^{p^s} - 1) + 1)^{p^t} - 1 \equiv (\gamma^{p^s} - 1)^{p^t} \bmod p(\gamma^{p^s} - 1)$, it follows that $(\gamma^{p^{s+t}} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$, as desired.

Assume now that (1) and (2) hold. Let $\mathfrak{M} \subseteq M$ be an arbitrary lattice, say with generators $m_1, \ldots, m_r$ of $\mathfrak{M}$. By [EG23, Lem. D.13 (2)(a)], we can choose $s \geq 0$ sufficiently large so that $(\gamma^{p^s} - 1)m_i \in T\mathfrak{M}$ for all i. Thus, for any $\lambda_i \in \mathbf{A}_A^+$, we have

$$(\gamma^{p^s} - 1)(\sum_i \lambda_i m_i) = \sum_i \gamma^{p^s}(\lambda_i)(\gamma^{p^s} - 1)m_i + \sum_i (\gamma^{p^s}(\lambda_i) - \lambda_i)m_i.$$

As $\gamma^{p^s}(\lambda_i) - \lambda_i \in T\mathbf{A}_A^+$ for all i (recall that γ fixes $\mathcal{O}_K \otimes_{\mathcal{O}_F} A$, and preserves $T\mathbf{A}_A^+$), it follows that $(\gamma^{p^s} - 1)\mathfrak{M} \subseteq T\mathfrak{M}$, hence (4).

Finally assume that the equivalent conditions (3)–(4)–(5) hold. To show that (1) and (2) holds, it suffices to verify the conditions of [EG23, Lem. D.13 (2)]. That [EG23, Lem. D.13 (2)(b)] holds, i.e. each $g \in \langle\gamma\rangle$ acts continuously on M follows by the same argument used in Lemma 2.A.6.

We now check [EG23, Lem. D.13 (2)(c)]. So let $\mathfrak{M} \subseteq M$ be a lattice. We need to show that $\langle\gamma^{p^s}\rangle\mathfrak{M} \subseteq \mathfrak{M}$ for all $s \geq 0$ sufficient large. As we are assuming (3), we may choose s so that $(\gamma^{p^s} - 1)\mathfrak{M} \subseteq T\mathfrak{M}$. In particular, $\gamma^{p^s}\mathfrak{M} \subseteq \mathfrak{M}$, and by induction we have $\gamma^{rp^s}\mathfrak{M} \subseteq \mathfrak{M}$ for all $r \geq 0$. To handle the case $r < 0$, it suffices (by descending induction) to check that $\gamma^{-p^s}\mathfrak{M} \subseteq \mathfrak{M}$. This follows from the fact that $\gamma^{p^s} - 1$ acts topologically nilpotent on M. Indeed, for any $m \in \mathfrak{M}$ we have

$$\gamma^{-p^s}(m) = (1 - (1 - \gamma^{p^s}))^{-1}m = m + (1 - \gamma^{p^s})m + (1 - \gamma^{p^s})^2 m + \ldots \in \mathfrak{M}.$$

It remains to check [EG23, Lem. D.13 (2)(a)]. In other words, we need to check that for any $m \in M$, the orbit map $\langle\gamma\rangle \to M, g \mapsto gm$ is continuous at the identity of $\langle\gamma\rangle$, or equivalently, for any lattice $\mathfrak{M} \subseteq M$, we can find $s \geq 0$ sufficiently large so that $gm \in m + \mathfrak{M}$ for all $g \in \langle\gamma^{p^s}\rangle$. By picking n large enough so that $m \in T^{-n}\mathfrak{M}$ and replacing $\mathfrak{M}$ by $T^{-n}\mathfrak{M}$, we are reduced to choosing s so that $(g - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$ for all $g \in \langle\gamma^{p^s}\rangle$. Again, by (3) we can choose s so that

$(\gamma^{p^s} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$. As in preceding paragraph, this implies in particular that $\gamma^{rp^s}\mathfrak{M} \subseteq \mathfrak{M}$ for all $r \in \mathbf{Z}$. It follows that $(\gamma^{rp^s} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$ for all $r \in \mathbf{Z}$. Indeed, if $r \geq 1$, then $(\gamma^{rp^s} - 1)\mathfrak{M} = (\gamma^{p^s} - 1)(\gamma^{(r-1)p^s} + \ldots + \gamma^{p^s} + 1)\mathfrak{M} \subseteq (\gamma^{p^s} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$, while if $r < 0$, then $(\gamma^{rp^s} - 1)\mathfrak{M} = -(\gamma^{-rp^s} - 1)\gamma^{rp^s}\mathfrak{M} \subseteq (\gamma^{-rp^s} - 1)\mathfrak{M} \subseteq T^n\mathfrak{M}$. □

Bibliography

[AK19] Chandrakant Aribam and Neha Kwatra. *Galois Cohomology for Lubin-Tate* $(\varphi_q, \Gamma_{\mathrm{LT}})$*-modules over Coefficient rings*. 2019.

[BR22] Laurent Berger and Sandra Rozensztajn. *Super-Hölder vectors and the field of norms*. 2022.

[Col02] Pierre Colmez. "Espaces de Banach de dimension finie". In: *J. Inst. Math. Jussieu* 1.3 (2002), pp. 331–439.

[Dee01] Jonathan Dee. "Φ-Γ modules for families of Galois representations". In: *J. Algebra* 235.2 (2001), pp. 636–664.

[Dri06] Vladimir Drinfeld. "Infinite-dimensional vector bundles in algebraic geometry: an introduction". In: *The unity of mathematics*. Vol. 244. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 263–304.

[Eme] Matthew Emerton. *Formal algebraic stacks*. Avaliable at: http://www.math.uchicago.edu/~emerton/pdffiles/formal-stacks.pdf.

[EG21] Matthew Emerton and Toby Gee. "'Scheme-theoretic images' of morphisms of stacks". In: *Algebr. Geom.* 8.1 (2021), pp. 1–132.

[EG23] Matthew Emerton and Toby Gee. *Moduli stacks of Étale* (ϕ, Γ)*-modules and the existence of crystalline lifts*. Vol. 215. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2023, pp. ix+298.

[EGH22] Matthew Emerton, Toby Gee, and Eugen Hellmann. *An introduction to the categorical p-adic Langlands program*. 2022. arXiv: 2210.01404.

[Fon90] Jean-Marc Fontaine. "Représentations p-adiques des corps locaux. I". In: *The Grothendieck Festschrift, Vol. II*. Vol. 87. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[GD71] A. Grothendieck and J. A. Dieudonné. *Éléments de géométrie algébrique. I*. Vol. 166. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1971, pp. ix+466.

[Her98] Laurent Herr. "Sur la cohomologie galoisienne des corps p-adiques". In: *Bull. Soc. Math. France* 126.4 (1998), pp. 563–600.

[KR09] Mark Kisin and Wei Ren. "Galois representations and Lubin-Tate groups". In: *Doc. Math.* 14 (2009), pp. 441–461.

[KV22] Benjamin Kupferer and Otmar Venjakob. "Herr-complexes in the Lubin-Tate setting". In: *Mathematika* 68.1 (2022), pp. 74–147.

[Mit65] Barry Mitchell. *Theory of categories*. Pure and Applied Mathematics, Vol. XVII. Academic Press, New York-London, 1965, pp. xi+273.

[PR09] G. Pappas and M. Rapoport. "Φ-modules and coefficient spaces". In: *Mosc. Math. J.* 9.3 (2009), 625–663, back matter.

[Sch17] Peter Schneider. *Galois representations and* (φ, Γ)*-modules*. Vol. 164. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017, pp. vii+148.

[Sta23] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.

[Win83] Jean-Pierre Wintenberger. "Le corps des normes de certaines extensions infinies de corps locaux; applications". In: *Ann. Sci. École Norm. Sup. (4)* 16.1 (1983), pp. 59–89.

Chapter 3

Prismatic F-crystals and "Lubin–Tate" crystalline Galois representations

3.1 Introduction

Let $K/\mathbf{Q}_p$ be a completed discrete valued extension with perfect residue field k of characteristic $p > 0$, fixed completed algebraic closure C, and absolute Galois group G_K. An important aspect of integral p-adic Hodge theory is the study of lattices in crystalline (or more generally, semistable) G_K-representations. There have been various (partial) classifications of such lattices, including Fontaine–Laffaille's theory [FL82], Breuil's theory of strongly divisible S-lattices [Bre02], and Kisin's theory of Breuil–Kisin modules [Kis06]. In [BS23], Bhatt and Scholze give a site-theoretic description of such lattices, which unifies many of the previous classifications, and in fact can recover them by "evaluating" suitably. To recall their result, let $(\mathcal{O}_K)_{\mathbb{\Delta}}$ denote the absolute prismatic site of $\mathcal{O}_K$; this comes equipped with a structure sheaf $\mathcal{O}_{\mathbb{\Delta}}$, a "Frobenius" $\varphi : \mathcal{O}_{\mathbb{\Delta}} \to \mathcal{O}_{\mathbb{\Delta}}$, and an invertible ideal sheaf $\mathcal{I}_{\mathbb{\Delta}} \subseteq \mathcal{O}_{\mathbb{\Delta}}$.

Definition 3.1.1. A prismatic F-crystal on $\mathcal{O}_K$ is a crystal of vector bundles on the ringed site $((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ equipped with an isomorphism $(\varphi^*\mathcal{E})[1/\mathcal{I}_{\mathbb{\Delta}}] \simeq \mathcal{E}[1/\mathcal{I}_{\mathbb{\Delta}}]$; denote by $\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ the resulting category. Similarly, we obtain the category $\mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}]^{\wedge}_p)$ of so-called Laurent F-crystals.

In the statement below, $\mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(G_K)$ denotes the category of Galois stable lattices in crystalline $\mathbf{Q}_p$-representations of G_K.

Theorem 3.1.2 ([BS23])**.** *There is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) & \xrightarrow{\simeq} & \mathrm{Rep}^{\mathrm{cris}}_{\mathbf{Z}_p}(G_K) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Vect}^{\varphi}((\mathcal{O}_K)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\mathcal{I}_{\mathbb{\Delta}}]^{\wedge}_p) & \xrightarrow{\simeq} & \mathrm{Rep}_{\mathbf{Z}_p}(G_K). \end{array}$$

Here the vertical embeddings are given by the obvious maps; the horizontal equivalences are induced by evaluating on the Fontaine's prism $\mathbf{A}_{\mathrm{inf}}$*,the so-called étale realization functor.*

(We note that the bottom horizontal equivalence was also obtained independently by Zhiyou Wu [Wu21].) Motivated by the study of the stacks of Lubin–Tate (φ, Γ)-modules in Chapter 2, it is natural to ask if there is a variant of Theorem 3.1.2 for coefficient rings other than $\mathbf{Z}_p$. More specifically, let E be a finite extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_q$ and a fixed uniformizer π; we are interested in crystalline representations of G_K on finite dimensional E-vector spaces (or rather, G_K-stable $\mathcal{O}_E$-lattices in such).

Hypothesis 3.1.3. We assume throughout that there is an embedding $\tau_0 : E \hookrightarrow K$, which we will fix once and for all.

Definition 3.1.4 ([KR09])**.** An E-linear representation V of G_K is called E-crystalline if it is crystalline (as a $\mathbf{Q}_p$-representation), and the C-semilinear representation $\bigoplus_{\tau \neq \tau_0} V \otimes_{E,\tau} C$ is trivial[1].

A natural source of such representations comes from the rational Tate modules of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$; see Lemma 3.4.23. Later, we will show that, just as in the case $E = \mathbf{Q}_p$, E-crystalline representations can be classified using weakly admissible filtered φ_q-modules over K. In fact, the above notion is indeed a natural extension of the usual notion in the sense that there is a natural period ring $B_{\mathrm{cris},E}$ with the property that an E-linear representation V is E-crystalline if and only if $V \otimes_E B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-semilinear representation (cf. Theorem 3.2.4).

Using the theory of Lubin–Tate (φ, Γ)-modules, Kisin–Ren give a classification of the category of Galois stable lattices in E-crystalline representations of G_K (under a condition on the ramification of K) [KR09, Theorem (0.1)], generalizing the earlier classification in terms of Wach modules by Wach, Colmez and Berger (cf. [Ber04]).

In another direction, in [Mar23] Marks defines a variant of the (absolute) prismatic site $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$ of $\mathcal{O}_K$ "relative to $\mathcal{O}_E$", using the notion of $\mathcal{O}_E$-prisms[2], a mild generalization of prisms: they are roughly $\mathcal{O}_E$-algebras A equipped with a lift $\varphi_q : A \to A$ of the q-power Frobenius modulo π together with a Cartier divisor I of $\mathrm{Spec}(A)$ satisfying $\pi \in (I, \varphi_q(I))$ (cf. [Mar23, §3]). Furthermore, it is shown in *loc. cit.* that the étale realization functor again defines an equivalence

$$T : \mathrm{Vect}^{\varphi_q}((\mathcal{O}_K)_{\Delta,\mathcal{O}_E}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]_p^{\wedge}) \simeq \mathrm{Rep}_{\mathcal{O}_E}(G_K),$$

generalizing the aforementioned result of Wu and Bhatt–Scholze in the case $E = \mathbf{Q}_p$. In this chapter, we push this analogy further by showing the following extension of Theorem 3.1.2.

Theorem 3.1.5 (Theorem 3.4.7)**.** *The étale realization functor induces an equivalence*

$$T : \mathrm{Vect}^{\varphi_q}((\mathcal{O}_K)_{\Delta,\mathcal{O}_E}, \mathcal{O}_{\Delta}) \simeq \mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K),$$

where the target denotes the category of Galois stable $\mathcal{O}_E$-lattices in E-crystalline representations of G_K.

[1] This is not quite the original definition in [KR09], but can be easily seen to be equivalent to it (see Lemma 3.2.2 below).

[2] These are called E-typical prisms in [Mar23].

As will be explained in §3.4.5 below, by evaluating at a suitable prism in $(\mathcal{O}_K)_{\Delta}$, Theorem 3.1.5 recovers the main equivalence from Kisin–Ren's work [KR09].

Finally, by combining Theorem 3.1.5 with a key result from [AL23] (generalized to the "$\mathcal{O}_E$-context" in [Ito23]), we deduce the following classification of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$.

Theorem 3.1.6 (Theorem 3.4.24)**.** *There is a natural equivalence between the category of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ and the category of minuscule Breuil–Kisin modules over $\mathfrak{S}_E$.*

3.1.1 Sketch of the proof of Theorem 3.1.5

Let us now briefly discuss the proof of Theorem 3.1.5. As alluded above, an important observation is that the condition of being E-crystalline can be characterized in a manner similar to the usual notion for $E = \mathbf{Q}_p$. Namely, there is a natural period ring $B_{\mathrm{cris},E}$ with the property that an E-linear representation V is E-crystalline if and only if $V \otimes_E B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-representation; see Theorem 3.2.4. Once this is justified, that the étale realization functor is well-defined and fully faithful can be proved in exactly the same way as [BS23]. For essential surjectivity, we again follow largely the proof in [BS23]; the main difference here is that instead of invoking the Beilinson fibre sequence from [AMMN22] for the key descent step, we are able to prove the following more general result by adapting a key lemma from [DL22].

Proposition 3.1.7 (Theorem 3.4.15)**.** *Let $(A, (d))$ be a transversal $\mathcal{O}_E$-prism. Then the base change*

$$\mathrm{Vect}^{\varphi_q}(A)[1/\pi] \to \mathrm{Vect}^{\varphi_q}(A\langle d/\pi\rangle[1/\pi])$$

is fully faithful; here the source denotes the isogeny category of $\mathrm{Vect}^{\varphi_q}(A)$.

We regard Proposition 3.1.7 as a result of independent interest. For instance, as alluded above, by specializing to the prism $A = \Delta_{\mathcal{O}_C \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C}$, this recovers (and refines) Proposition 6.10 in [BS23]. Furthermore, by specializing to a Breuil–Kisin prism $(\mathfrak{S}, I)$, this recovers the embedding

$$\mathrm{Vect}^{(\varphi,N)}(\mathfrak{S})[1/p] \hookrightarrow \mathrm{Vect}^{(\varphi,N)}(\mathcal{O})$$

from [Kis06, Lemma 1.3.13] without using Kedlaya's results on slope filtrations; here $\mathcal{O}$ denotes the ring of functions on the rigid open unit disk over K_0.

The proof of Proposition 3.1.7 proceeds by first reducing to the case of finite free modules (which is the only case we need in proving essential surjectivity). In this case, by working with matrices for the φ_q-actions, we reduce to showing that if

$$d^h Y = B\varphi_q(Y)C$$

with $h \geq 0$, $Y \in M_d(A\langle d/\pi\rangle)$ and $B, C \in M_d(A)$, then $Y \in M_d(A[1/\pi])$. Here the idea is to approximate d-adically Y by matrices in $M_d(A)$. This is possible thanks to the following variant of [DL22, Lemma 2.2.10] on the contracting effect of the Frobenius on the d-adic filtration on $A\langle d/\pi\rangle$.

Lemma 3.1.8 (Lemma 3.4.12)**.** *Let $(A, (d))$ be a transversal $\mathcal{O}_E$-prism. Then given any $h \geq 0$,*

$$\varphi_q(d^m A\langle d/\pi\rangle) \subseteq A + d^{m+h} A\langle d/\pi\rangle$$

for all $m \gg 0$ (depending only on h).

We now detail the organization of the chapter. In Section 3.2, we recall the definition of E-crystalline representations from [KR09], and following [FF18, Chapitre 10], interpret it in terms of vector bundles on the Fargues–Fontaine curve (Proposition 3.A.13). In particular, we show that the category of E-crystalline representations of G_K is equivalent to the category of weakly admissible filtered φ_q-modules over K, and moreover that being E-crystalline is equivalent to being $B_{\mathrm{cris},E}$-admissible for a natural period ring $B_{\mathrm{cris},E}$. In Section 3.3, we adapt some constructions from [Kis06] to the present context. Next, in Section 3.4, we review briefly the notion of $\mathcal{O}_E$-prisms and define the étale realization functor in Theorem 3.1.2. Full faithfulness is then addressed in Subsection 3.4.3. Subsection 3.4.4 begins with some further ring-theoretic properties of transversal prisms, culminating with the proof of Proposition 3.4.15, which is then used in the proof of essential surjectivity. Finally, in the last two subsections, we briefly discuss an application of Theorem 3.1.5 to the theory of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ as well as its relation with Kisin–Ren's classification in [KR09].

Notation 3.1.9. Throughout this chapter, K denotes a complete discrete valued extension of $\mathbf{Q}_p$ with perfect residue field, complete algebraic closure C, and absolute Galois G_K. We will also let E denote a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$ (which serves as the coefficient ring for our G_K-representations), fixed uniformizer π_E, and residue field $\mathbf{F}_q$. We will fix throughout an embedding $\tau_0 : E \hookrightarrow K$; a general embedding $E \hookrightarrow C$ will be typically denoted by τ.

3.2 E-crystalline Galois representations

3.2.1 Definition

Let V be an E-linear representation of G_K which is crystalline (in the usual sense, i.e., as a $\mathbf{Q}_p$-linear representation). Then

$$D_{\mathrm{dR}}(V) := (V \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})^{G_K}$$

is naturally a finite free $E \otimes_{\mathbf{Q}_p} K$-module equipped with a (finite, separated, exhausted) filtration by $E \otimes_{\mathbf{Q}_p} K$-submodules (whose associated graded pieces are finite projective, but not necessarily of constant rank, or equivalently, free). In particular, there is a decomposition

$$D_{\mathrm{dR}}(V) = \prod_{\mathfrak{m}} D_{\mathrm{dR}}(V)_{\mathfrak{m}} \tag{3.2.0.1}$$

where $\mathfrak{m}$ runs over the (finite) set of maximal ideals in $E \otimes_{\mathbf{Q}_p} K$.

The following definition was given by Kisin–Ren [KR09].

Definition 3.2.1 ([KR09])**.** V is called E-crystalline if (it is crystalline and) the induced filtration on $D_{\mathrm{dR}}(V)_{\mathfrak{m}}$ is trivial for all $\mathfrak{m} \neq \mathfrak{m}_0$, where $\mathfrak{m}_0$ denotes the maximal ideal corresponding to the multiplication map $K \otimes_{\mathbf{Q}_p} E \twoheadrightarrow K$ defined by the embedding τ_0.

Lemma 3.2.2. *Let V be a crystalline E-linear representation of G_K. Then V is E-crystalline if and only if $\bigoplus_{\tau\neq\tau_0} V\otimes_{E,\tau} C$ is trivial as a C-semilinear representation of G_K.*

Here the action of $g\in G_K$ on $\bigoplus_{\tau\ ne\tau_0} V\otimes_{E,\tau} C$ is given by the maps $1\otimes g: V\otimes_{E,\tau} C\to V\otimes_{E,g\tau} C$. (Note that the diagonal action of G_K on $V\otimes_{E,\tau} C$ is not well-defined in general: the map $\tau: E\hookrightarrow C$ is not G_K-equivariant unless $\tau(E)\subseteq K$.)

Proof. Recall that the filtration on $D_{\mathrm{dR}}(V)$ satisfies $\mathrm{gr}^i D_{\mathrm{dR}}(V)\simeq (V\otimes_{\mathbf{Q}_p} C(i))^{G_K}$ for each i. Thus using the decomposition $V\otimes_{\mathbf{Q}_p} C=\prod_\tau V\otimes_{E,\tau} C$, we see that $\mathrm{gr}^i D_{\mathrm{dR}}(V)_{\mathfrak{m}}=0$ for all $i\neq 0$ and $\mathfrak{m}\neq\mathfrak{m}_0$ if and only if the C-semilinear representation $W:=\bigoplus_{\tau\neq\tau_0} V\otimes_{E,\tau} C$ satisfies $(W\otimes_C C(i))^{G_K}=0$ for all $i\neq 0$. As W is Hodge–Tate (being a quotient of the Hodge–Tate representation $V\otimes_{\mathbf{Q}_p} C$), this amounts precisely to saying that W is trivial. □

3.2.2 Relation with filtered isocrystals

For our purpose of proving Theorem 3.1.5, the following equivalent characterizations of the category of E-crystalline Galois representations will be fundamental. Before stating the result, we introduce the crysalline period ring in our context.

Notation 3.2.3. Let $B_{\mathrm{cris},E}$ denote Fontaine's crystalline period ring defined using E and $\tau_0: E\hookrightarrow K\subseteq C$. More precisely, let $A_{\mathrm{inf},E}:=W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)$ (defined using the embedding τ_0), and let $A_{\mathrm{cris},E}$ be the π-completed $\mathcal{O}_E$-PD envelope of $A_{\mathrm{inf},E}$ with respect to the kernel of the Fontaine's map $\theta_E: A_{\mathrm{inf},E}\twoheadrightarrow\mathcal{O}_C$, i.e. $A_{\mathrm{cris},E}$ is the π-adic completion of the subring

$$A_{\mathrm{inf},E}[\xi^{q^n}/\pi^{1+q+\ldots+q^{n-1}}, n\geq 1]\subseteq A_{\mathrm{inf},E}[1/\pi],$$

where ξ is one (or, any) generator of $\ker(\theta_E)$. We then let $B^+_{\mathrm{cris},E}:=A_{\mathrm{cris},E}[1/\pi]$ and $B_{\mathrm{cris},E}:=B^+_{\mathrm{cris},E}[1/t_E]$[3]. These will play the roles of the usual crystalline period rings in the story over $\mathbf{Q}_p$. (As the notation suggests t_E is the analogue of the usual element $t=\log([\epsilon])$ (the "$2\pi i$ of Fontaine") in our "$\mathcal{O}_E$-context"; see Appendix 3.A for more details.)

Theorem 3.2.4 (Remark 3.A.15, Theorem 3.A.19)**.** (1) *Let $V\in\mathrm{Rep}_E(G_K)$. Then V is E-crystalline if and only if $V\otimes_E B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-semilinear representation of G_K.*

(2) *The functor*

$$D_{\mathrm{cris},E}: V\mapsto (V\otimes_E B_{\mathrm{cris},E})^{G_K}$$

defines an equivalence from the category of E-crystalline representations of G_K onto the category of weakly admissible filtered φ_q-modules.

[3]In [KR09], $B_{\mathrm{cris},E}$ denotes the ring $B_{\mathrm{cris}}\otimes_{E_0} E$; these are in general different rings.

(We refer the reader to Appendix 3.A for the notion of (weakly admissible) filtered φ_q-modules, which is simply a straightforward extension of the usual notion.) Our proof of Theorem 3.2.4 rests on the realization of the relevant objects as certain vector bundles on the Fargues–Fontaine curve (associated to E and the embedding τ_0). Since this will require a digression on the Fargues–Fontaine curve which has little do to with the rest of the chapter, we defer the proof to Appendix 3.A.

Remark 3.2.5. The above equivalence between E-crystalline representations of G_K and weakly admissible filtered φ_q-modules is presumably expected, although we cannot find a reference which explicitly states it. We can deduce it more directly (as an abstract equivalence) by combining the usual equivalence for $E = \mathbf{Q}_p$ with a standard passage from φ-modules to φ_q-modules (cf. [KR09, §3.3]). Here we prefer the more geometric perspective via the Fargues–Fontaine curve as it gives the explicit recipe for the equivalence as above, and also suggests the idea that in order to treat all E-linear crystalline (i.e. not necessarily E-crystalline) representations of G_K, one needs to study modifications of vector bundles on the Fargues–Fontaine curve X_E at finitely many points (rather than just one point as in the case $E = \mathbf{Q}_p$).

3.3 Theory of Breuil–Kisin modules

In this section, we adapt some constructions from [Kis06] to the "$\mathcal{O}_E$-context". We note that in [CL16], Cais and Liu have extended a large part of the theory in [Kis06] to accommodate more general coefficient rings and lifts of Frobenius. However, in our present context (corresponding to the Frobenius lift $f(u) = u^q$) much of the discussion in [CL16] simplifies, and we can present the material largely in parallel with [Kis06]. The only difference is that it will be important in some of our arguments to be able to work entirely with Frobenius structures: it is not clear what should be the correct analogue of the key monodrommy operator N_∇ from *loc. cit.* in our setting (cf. Proposition 3.3.11).

3.3.1 Preliminaries

We fix once and for all a uniformizer $\pi_K \in K$. Let $E(u) \in W_{\mathcal{O}_E}(k)[u]$ be the Eisenstein polynomial of π_K over $K_{0,E}$. Here $K_{0,E} := W_{\mathcal{O}_E}(k)[1/\pi]$ is naturally identified with the maximal unramified extension of E in K. Let $\mathfrak{S}_E := W_{\mathcal{O}_E}(k)[[u]]$ and let $\varphi_q : \mathfrak{S}_E \to \mathfrak{S}_E$ be the ring map that extends the Frobenius in $W_{\mathcal{O}_E}(k)$ and sends u to u^q. The map $\tilde{\theta} : \mathfrak{S}_E \twoheadrightarrow \mathcal{O}_K, u \mapsto \pi_K$ is surjective with kernel $I = (E(u))$.

Let $\Delta := \mathrm{Spa}(\mathfrak{S}_E) - \{\pi = 0\}$ be the adic generic fiber of $\mathrm{Spa}(\mathfrak{S}_E)$ over $K_{0,E}$; this is the adic open unit disk over $K_{0,E}$. Let $\mathcal{O}$ be the ring of functions on X. As Δ is an increasing union of the rational opens $U_n := \{|\varphi_q^n(E(u))| \leq |\pi| \neq 0\} = \{|u^{eq^n}| \leq |\pi| \neq 0\}$ for $n \geq 0$, we have

$$\mathcal{O} = \varprojlim_n \mathfrak{S}_E \left\langle \frac{\varphi_q^n(E(u))}{\pi} \right\rangle [1/\pi] \tag{3.3.0.1}$$

$$= \bigcap_{n\geq 0} K_{0,E} \left\langle \frac{u^{eq^n}}{\pi} \right\rangle \text{ inside } K_{0,E}[[u]]. \tag{3.3.0.2}$$

Recall that by the work of Lazard [Laz62] that each $K_{0,E}\langle u^{eq^n}/\pi\rangle$ is a PID and $\mathcal{O}$ is a Bézout domain. In particular, finite projective modules over $\mathcal{O}$ are free (see e.g. [Ked04, Proposition 2.5]). Moreover, base change defines an equivalence

$$\mathrm{Vect}(\mathcal{O}) \simeq \lim_n \mathrm{Vect}(\mathfrak{S}_E\langle u^{eq^n}/\pi\rangle[1/\pi]).$$

Denote again by $\varphi_q : \mathcal{O} \to \mathcal{O}$ the map induced by φ_q on $\mathfrak{S}_E$. For $n \geq 0$, let $x_n \in \Delta$ be the unique point where $\varphi_q^n(E(u))$ vanishes, i.e. $x_n : \mathfrak{S}_E \to \mathfrak{S}_E/\varphi_q^n(E(u))[1/\pi] =: K_n$. Let $\widehat{\mathfrak{S}}_n$ denote the complete local ring of Δ at x_n; this is a complete DVR with uniformizer $\varphi_q^n(E(u))$, residue field K_n, and fraction field $\mathrm{Fr}(\widehat{\mathfrak{S}}_n) = \widehat{\mathfrak{S}}_n[1/\varphi_q^n(E(u))]$.

As $E(u)/E(0) \in \mathfrak{S}_E[1/\pi]$ has constant term 1, the infinite product

$$\lambda := \prod_{n\geq 0} \varphi_q^n(E(u)/E(0)).$$

is well-defined as an element in $\mathcal{O}$. By design, λ has a simple root at each x_n.

Definition 3.3.1. A φ_q-module (of finite height) over $\mathfrak{S}_E$ is a finite free $\mathfrak{S}_E$-module $\mathfrak{M}$ equipped with an isomorphism

$$(\varphi_q^*\mathfrak{M})[1/E(u)] \simeq \mathfrak{M}[1/E(u)].$$

We denote by $\mathrm{Vect}^{\varphi_q}(\mathfrak{S})$ the category of φ-modules over $\mathfrak{S}$. In analogy with the case $E = \mathbf{Q}_p$, we will often refer to objects in this category as Breuil–Kisin modules over $\mathfrak{S}_E$.

Similarly, we let $\mathrm{Vect}^{\varphi_q}(\mathcal{O})$ denote the category of φ_q-modules over $\mathcal{O}$, i.e. finite free $\mathcal{O}$-modules $\mathcal{M}$ equipped with an isomorphism $(\varphi_q^*\mathcal{M})[1/E(u)] \simeq \mathcal{M}[1/E(u)]$.

Lemma 3.3.2 (Analytic continuation of φ-modules to the open unit disk)**.** *Base change defines an equivalence of categories*

$$\mathrm{Vect}^{\varphi_q}(\mathcal{O}) \simeq \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E\langle E(u)/\pi\rangle[1/\pi]).$$

Proof. This is an application of the Frobenius pullback trick. More precisely, as explained in [BS23, Rem. 6.6], by using the contracting property of φ_q, any object in $\mathrm{Vect}^{\varphi_q}\langle E(u)/\pi\rangle[1/\pi]$ extends uniquely to an object in $\mathrm{Vect}^{\varphi_q}\langle\varphi_q^n(E(u))/\pi\rangle[1/\pi]$ for any $n \geq 1$. It now suffices to show that base change gives an equivalence

$$\mathrm{Vect}^{\varphi}(\mathcal{O}) \simeq \lim_n \mathrm{Vect}^{\varphi}(\mathfrak{S}_E\langle\varphi_q^n(E(u))/\pi\rangle[1/\pi]).$$

This follows formally from the analogous equivalence at the level of vector bundles, and the equality

$$\mathcal{O}\left[\frac{1}{E(u)}\right] = \bigcap_{n\geq 0}\left(\mathfrak{S}_E\left\langle\frac{\varphi_q^n(E(u))}{\pi}\right\rangle\left[\frac{1}{\pi}\right]\left[\frac{1}{E(u)}\right]\right)^{4}.$$

□

3.3.2 Filtered isocrystals and φ-modules on the open unit disk

In this section, we explain the construction of a natural fully faithful functor

$$\mathrm{MF}^{\varphi_q}(K) \hookrightarrow \mathrm{Vect}^{\varphi_q}(\mathcal{O}).$$

In fact, inspired by [GL12] and [Kim09], we can do slightly better, namely we will construct a commutative diagram

$$\begin{array}{ccc} \mathrm{MF}^{\varphi}(K) & \hookrightarrow & \mathrm{Vect}^{\varphi_q}(\mathcal{O}) \\ \big\downarrow & \nearrow_{\simeq} & \\ \mathrm{HP}^{\varphi_q}(K), & & \end{array}$$

where $\mathrm{HP}^{\varphi_q}(K)$ denotes the category of φ_q-modules with Hodge–Pink structure over K, whose definition is recalled next.

Definition 3.3.3. A φ_q-module with Hodge–Pink structure (or simply a Hodge–Pink isocrystal) over K is a triple (D, φ_q, Λ) where (D, φ_q) is a φ_q-module over $K_{0,E}$, and Λ is a $\widehat{\mathfrak{S}}_0$-lattice in $D \otimes_{K_{0,E}} \mathrm{Fr}(\widehat{\mathfrak{S}}_0)$.

The two categories $\mathrm{MF}^{\varphi_q}(K)$ and $\mathrm{HP}^{\varphi_q}(K)$ are related in the following way. First, given a filtration $\mathrm{Fil}^\bullet D_K$, we get an associated Hodpe–Pink structure using the lattice $\Lambda := \mathrm{Fil}^0(D_K \otimes_K \mathrm{Fr}(\widehat{\mathfrak{S}}_0))$. Conversely, given a Hodge–Pink lattice Λ, one gets a filtration on on D_K by first restricting the $E(u)$-adic filtration $\{E(u)^i\Lambda\}_{i\in\mathbf{Z}}$ on $D \otimes_{K_{0,E}} \mathrm{Fr}(\widehat{\mathfrak{S}}_0)$ to the tautological lattice $\widehat{D}_0 := D \otimes_{K_{0,E}} \widehat{\mathfrak{S}}_0$, and then a filtration on D_K by pushing forward along the natural map

$$\tilde{\theta} : D \otimes_{K_{0,E}} \widehat{\mathfrak{S}}_0 \twoheadrightarrow D \otimes_{K_{0,E}} K = D_K.$$

Lemma 3.3.4. *The resulting functors*

$$\mathrm{MF}^{\varphi_q}(K) \underset{F}{\overset{P}{\rightleftarrows}} \mathrm{HP}^{\varphi_q}(K).$$

satisfy $F \circ P \simeq \mathrm{id}$. *In particular,* P *is fully faithful.*

[4] This follows e.g. from the equality 3.3.0.1 and Lemma 3.4.11 below.

Proof. Given a filtered isocrystal $(D, \varphi, \mathrm{Fil}^\bullet D_K)$, we need to show that $\tilde{\theta}(\widehat{D}_0 \cap E(u)^i\Lambda) = \mathrm{Fil}^i D_K$ for each $i \in \mathbf{Z}$, where $\Lambda := \mathrm{Fil}^0(D_K \otimes_K \mathrm{Fr}(\widehat{\mathfrak{S}}_0))$. By shifting, we may assume $i = 0$. This desired equality then follows e.g. by choosing a splitting of the given filtration:

$$D_K = \bigoplus_{j \in \mathbf{Z}} V^j, \quad \mathrm{Fil}^i D_K = \bigoplus_{j \geq i} V^j.$$

The second statement follows formally from the first and the fact that F is faithful (as it does nothing on the underlying isocrystals). □

3.3.2.1 Construction

Thus, combining with Lemma 3.3.2, it remains to construct mutually inverse equivalences

$$\mathrm{HP}^{\varphi_q}(K) \underset{D}{\overset{\mathcal{M}}{\rightleftarrows}} \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E\langle I/\pi\rangle[1/\pi]).$$

For $\mathcal{M}$, we will follow the presentation of [BS23, Construction 6.5], which is based on Beauville–Laszlo glueing; one can check that this agrees with Kisin's rather more concrete construction in [Kis06] (when $E = \mathbf{Q}_p$); see Remark 3.3.6. The idea is to define $\mathcal{M}(D)$ as a modification of the constant φ_q-module $\mathcal{M}' := D \otimes_{K_{0,E}} \mathfrak{S}_E\langle E(u)/\pi\rangle[1/\pi]$ at the Cartier divisor $\{E(u) = 0\}$ using the given Hodge–Pink lattice Λ. More precisely, the underlying module of $\mathcal{M}(D)$ is obtained by applying Beauville–Laszlo glueing to the vector bundles

- $\mathcal{M}'[1/I] \in \mathrm{Vect}(\mathfrak{S}_E\langle I/\pi\rangle[1/\pi][1/I])$
- $\Lambda \in \mathrm{Vect}(\mathfrak{S}_I\langle E/\pi\rangle[1/\pi]^\wedge_I)$

along the obvious isomorphism; here we implicitly identify $\widehat{\mathfrak{S}}_0$ with $\mathfrak{S}_E\langle E/\pi\rangle[1/\pi]^\wedge_I$ via the natural map (see Lemma 3.4.11 (3) for a more general statement). The φ_q-structure on $\mathcal{M}(D)$ is then defined as the composition

$$(\varphi_q^*\mathcal{M}(D))[1/I] \simeq (\varphi_q^*\mathcal{M}')[1/I] \overset{\varphi_{\mathcal{M}'}}{\simeq} \mathcal{M}'[1/I] \simeq \mathcal{M}(D)[1/I];$$

here for the first identification, we use that $\varphi_q(I)$ is a unit in $\mathfrak{S}_E\langle I/\pi\rangle[1/\pi]$.

Next, we define the functor D. Given $\mathcal{M} \in \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E\langle I/\pi\rangle[1/\pi])$, set $D(\mathcal{M}) := \mathcal{M} \otimes_{\mathfrak{S}_E\langle I/\pi\rangle[1/\pi]} K_{0,E}$, equipped with the natural (diagonal) φ_q-action; here $\mathfrak{S}_E\langle I/\pi\rangle \twoheadrightarrow K_{0,E}$ is the natural (φ_q-equivariant) map $u \mapsto 0$. This gives the isocrystal structure on $D(\mathcal{M})$; it remains to define the Hodge–Pink lattice. First, the standard Frobenius trick shows that there is a unique φ-equivariant map

$$\xi : D(\mathcal{M}) \otimes_{K_{0,E}} \mathfrak{S}_E\langle I/\pi\rangle[1/\pi][1/I] \to \mathcal{M}[1/I]$$

lifting the identity modulo u. See [Kis06, Lemma 1.2.6] or [GL12, Lemma 3.5]. In particular, ξ realizes $\mathcal{M}$ as a modification of $D(\mathcal{M}) \otimes_{K_{0,E}} \mathfrak{S}_E\langle I/\pi\rangle[1/\pi]$ at the divisor $\{I = 0\}$, and hence $\mathcal{M}^\wedge_I$ gives rise to the desired lattice inside $\mathfrak{M}^\wedge_I[1/I] \simeq D \otimes_{K_{0,E}} \mathrm{Fr}(\widehat{\mathfrak{S}}_0)$. While this already finishes the construction of the functor $\mathcal{M} \mapsto D(\mathcal{M})$, we note that the associated filtration on $D(\mathcal{M})_K$ can be maded slightly more explicit as follows. Indeed, as $\varphi(I)$ is a unit in $\mathfrak{S}_E\langle I/\pi\rangle[1/\pi]$ (as was already mentioned), $\varphi_q^*\xi$ is an isomorphism, fitting in the following commutative square

$$\begin{array}{ccc} \varphi_q^* D(\mathcal{M}) \otimes_{K_{0,E}} \mathfrak{S}_E\langle I/\pi\rangle[1/\pi] & \xrightarrow[\simeq]{\varphi_q^*\xi} & \varphi_q^*\mathcal{M} \\ \downarrow{\scriptstyle \varphi_{D(\mathcal{M})}} & & \downarrow{\scriptstyle \varphi_{\mathcal{M}}} \\ D(\mathcal{M}) \otimes_{K_{0,E}} \mathfrak{S}_E\langle I/\pi\rangle[1/\pi][1/I] & \xrightarrow[\xi]{\simeq} & \mathcal{M}[1/I]. \end{array}$$

In particular, we see that $D(\mathcal{M}) \otimes_{K_{0,E}} \widehat{\mathfrak{S}}_0 \simeq \varphi_q^* D(\mathcal{M}) \otimes_{K_{0,E}} \widehat{\mathfrak{S}}_0 \simeq (\varphi_q^*\mathcal{M})_I^\wedge$. The filtration on $D(\mathcal{M})_K = D(\mathcal{M}) \otimes_{K_{0,E}} K$ is simply the image of the filtration $\mathrm{Fil}^\bullet(\varphi_q^*\mathcal{M}) := \varphi_{\mathcal{M}}^{-1}(E(u)^\bullet\mathcal{M})$ on $\varphi_q^*\mathcal{M}$. This agrees with the filtration constructed in [Kis06, §1.2.7] (when $E = \mathbf{Q}_p$).

Theorem 3.3.5. *The functors*

$$\mathrm{HP}^{\varphi_q}(K) \underset{D(\cdot)}{\overset{\mathcal{M}(\cdot)}{\rightleftarrows}} \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E\langle I/\pi\rangle[1/\pi]).$$

are mutually inverse equivalences of categories.

Proof. This follows readily from the construction of the functors. Here we will only explain the proof that $\mathcal{M}\circ D \simeq \mathrm{id}$; that $D\circ\mathcal{M} \simeq \mathrm{id}$ can be proved similarly. Fix $\mathcal{M} \in \mathrm{Vect}^{\varphi}(\mathfrak{S}_E\langle I/\pi\rangle[1/\pi])$. As $\mathcal{M}_I^\wedge = \Lambda_{D(\mathcal{M})}$ by construction of $D(\mathcal{M})$, we see that $\mathcal{M}(D(\mathcal{M})) \simeq \mathcal{M}$ on underlying modules. To compare the φ-structures, recall that by construction of $\mathcal{M}(\cdot)$, $\varphi_{\mathcal{M}(D(\mathcal{M}))}$ is the unique map $\varphi_q^*\mathcal{M}(D(\mathcal{M})) \to \mathcal{M}(D(\mathcal{M}))[1/I]$ making the diagram

$$\begin{array}{ccc} \varphi_q^* D(\mathcal{M}) \otimes_{K_{0,E}} \mathfrak{S}_E\langle I/\pi\rangle[1/\pi] & \xrightarrow{\simeq} & \varphi_q^*\mathcal{M}(D(\mathcal{M})) \\ \downarrow{\scriptstyle \varphi_{D(\mathcal{M})}} & & \downarrow{\scriptstyle \varphi_{\mathcal{M}(D(\mathcal{M}))}} \\ D(\mathcal{M}) \otimes_{K_{0,E}} \mathfrak{S}_E\langle I/\pi\rangle[1/\pi][1/I] & \xrightarrow[\xi]{\simeq} & \mathcal{M}(D(\mathcal{M}))[1/I]. \end{array}$$

commute, but $\varphi_{\mathcal{M}}$ is also such a map (by φ-equivariance of ξ, as we explained above), so they coincide, as wanted. ☐

Remark 3.3.6 (Comparison with [Kis06])**.** In this remark, we briefly check that the compositions $\mathcal{M}\circ P$ and $F\circ D$ agree with the ones from [Kis06] (in case $E = \mathbf{Q}_p$), up to composing with the natural equivalence from Lemma 3.3.2. Denote the latters by $\mathcal{M}'$ and D'. That $D' \simeq F\circ D$ is already explained in the paragraph above Theorem 3.3.5. We now show that $\mathcal{M}' \simeq \mathcal{M}\circ P$. Fix $D := (D, \varphi, \mathrm{Fil}^\bullet D_K) \in \mathrm{MF}^{\varphi_q}(K)$. Note firstly that the ring $\widehat{\mathfrak{S}}_n$ from [Kis06, §1.1.1] is not our $\widehat{\mathfrak{S}}_n$, rather it is $\varphi_W^{-n}(\widehat{\mathfrak{S}}_n)$, where $\varphi_W : \mathcal{O} \to \mathcal{O}$ is the automorphism given by Frobenius on $W(k)$ (and $u \mapsto u$). Using this observation, it is easy to rewrite the definition of $\mathcal{M}'(D)$ in [Kis06, §1.2] as

$$\mathcal{M}'(D) := \{x \in D \otimes_{K_0} \mathcal{O}[1/\lambda] \mid \iota_n(x) \in \mathrm{Fil}^0(D_K \otimes_K \mathrm{Fr}(\widehat{\mathfrak{S}}_n)) \text{ for all } n \geq 0\},$$

where ι_n is the natural map $D \otimes_{K_0} \mathcal{O}[1/\lambda] \to D \otimes_{K_0} \mathrm{Fr}(\widehat{\mathfrak{S}}_n)$ (which indeed makes sense as λ has a simple root at each x_n). Then $\mathcal{M}'(D) \subseteq D \otimes_{K_0} [1/\lambda]$ is a finite free sub-$\mathcal{O}$-module with

$\mathcal{M}'(D)[1/\lambda] = D\otimes_{K_0}\mathcal{O}[1/\lambda]$; moreover the isomorphism $\varphi_D : \varphi^* D\otimes_{K_0}\mathcal{O}[1/\lambda] \simeq D\otimes_{K_0}\mathcal{O}[1/\lambda]$ restricts to an isomorphism $(\varphi^*\mathcal{M}'(D))[1/E(u)] \simeq \mathcal{M}'(D)[1/E(u)]$, making $\mathcal{M}'(D)$ an object in $\mathrm{Vect}^\varphi(\mathcal{O})$. See [Kis06, Lemma 1.2.2]. In particular, by base change along the natural map $\mathcal{O}[1/\lambda] \to \mathfrak{S}_E\langle I/p\rangle[1/p][1/I]$ (which makes sense as $\varphi^n(I)$ is invertible in $\mathfrak{S}_E\langle I/p\rangle[1/p]$ for all $n \geq 1$), we obtain

$$(\mathcal{M}'(D) \otimes_{\mathcal{O}} \mathfrak{S}_E\langle I/p\rangle[1/p])[1/I] \simeq D \otimes_{K_0} \mathfrak{S}_E\langle I/p\rangle[1/p][1/I].$$

Moreover, the description 3.3.6 also shows that $\mathcal{M}'(D)^\wedge_I$ identifies with $\mathrm{Fil}^0(D_K \otimes_K \mathrm{Fr}(\widehat{\mathfrak{S}}_0)) =: \Lambda_{F(D)}$ (e.g. by applying φ_W^n to [Kis06, Lemma 1.2.1 (2)]). This shows that

$$\mathcal{M}'(D) \otimes_{\mathcal{O}} \mathfrak{S}_E\langle I/p\rangle[1/p] \simeq \mathcal{M}(F(D)) \text{ in } \mathrm{Vect}^\varphi(\mathcal{O}),$$

as claimed.

3.3.3 Slopes and weak admissibility

Similarly to [Kis06] and [Kim09], in this subsection we relate, following Berger's oberservation [Ber08], weakly admissbility of Hodge–Pink isocrystals, and the "pure of slope 0" condition for φ-modules on the open unit disk. As many of the arguments are identical to those in [Kis06], we often sketch only the proofs.

Recall firstly the notion of weak admissibility for Hodge–Pink isocrystals. Let $D := (D, \varphi, \Lambda) \in \mathrm{HP}^{\varphi_q}(K)$. The Newton number t_N of D is defined exactly as before (i.e. using the underlying isocrystal). For defining the Hodge number, again by passing to the determinant, we may assume D is 1-dimensional; in this case we set $t_H(D) := h$, where h is the unique integer such that $\Lambda = (E(u))^{-h}(D \otimes_{K_{0,E}} \widehat{\mathfrak{S}}_0)$ (so $t_H(D)$ only depends on the Hodge–Pink structure of D).

Definition 3.3.7. A Hodge–Pink φ_q-module $D = (D, \varphi_q, \Lambda)$ is called weakly admissibile if $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all subojects $D' \subseteq D$ in $\mathrm{HP}^{\varphi_q}(K)$[5].

Lemma 3.3.8. *The (fully faithful) functor*

$$\mathrm{MF}^{\varphi_q}(K) \xhookrightarrow{P} \mathrm{HP}^{\varphi_q}(K)$$

preserves weak admissibility. More precisely, an object $D \in \mathrm{MF}^{\varphi_q}(K)$ is weakly admissible if and only if its image $P(D)$ is.

Proof. First, F and P both preserve the Newton numbers t_N as they do nothing on the underlying isocrystals. They also preserve the Hodge numbers: for this we may reduce to the rank 1 case, where the result follows by a direct computation. It follows that an object $D \in \mathrm{MF}^{\varphi_q}(K)$ (resp. $D' \in \mathrm{HP}^{\varphi_q}(K)$) is weakly admissible if $P(D)$ (resp. $F(D')$) is so. Moreover, as $F \circ P \simeq \mathrm{id}$, it then follows that $P(D)$ is weakly admissible whenever D is so. □

[5]The Hodge–Pink lattice on a subobject $D' \subseteq D$ is by definition given by $\Lambda_{D'} := \Lambda_D \cap (D' \otimes_{K_{0,E}} \mathrm{Fr}(\widehat{\mathfrak{S}}_0))$ (just as a subobject in $\mathrm{MF}^{\varphi_q}(K)$ is endowed with the subspace fitration). However, the notion of weak admissibility does not change if we weaken this into the weaker condition that $\Lambda_{D'} \subseteq \Lambda_D$.

Remark 3.3.9. It is however *not* true that F preserves weakly admissible objects. The following example is taken from [GL12]. Consider the object $D = (D, \varphi, \Lambda) \in \mathrm{HP}^{\varphi_q}(K)$ with $D = K_0e_1 \oplus K_0e_2, \varphi(e_i) = e_i$, and Hodge–Pink lattice

$$\Lambda := E(u)^{-1}\widehat{\mathfrak{S}}_0 e_1 \oplus \widehat{\mathfrak{S}}_0 e_2.$$

One can check directly that (D, φ, Λ) is weakly admissible. On the other hand, the associated filtered isocrystal is given by

$$\mathrm{Fil}^0 D_K = D_K, \quad \mathrm{Fil}^1 D_K = Ke_1, \quad \mathrm{Fil}^2 D_K = 0,$$

which is not weakly admissible as the submodule $D' := K_0e_1$ has $t_N(D', \varphi) = 0$ but $t_H(\mathrm{Fil}^\bullet D'_K) = 1$. In particular, we see that F and P are not equivalences of categories (though they are on rank 1 objects).

3.3.3.1 Kedlaya's slope filtration

Let

$$\mathcal{R} := \varinjlim_{r \to 1^-} \mathcal{O}_{(r,1)}$$

be the Robba ring over $K_{0,E}$; here $\mathcal{O}_{(r,1)}$ denotes the ring of rigid analytic functions on the open annulus $\{r < |u| < 1\}$. By work of Lazard, $\mathcal{R}$ is a Bézout domain containing $\mathcal{O}$ as a subring. Again, there is a natural Frobenius map $\varphi_q : \mathcal{R} \to \mathcal{R}$ extending φ_q on $\mathcal{O}$. Inside $\mathcal{R}$, there is the subring $\mathcal{R}^b$ formed of functions which are bounded. This is a Henselian discrete valued field with uniformizer π, and ring of integers

$$\mathcal{R}^{\mathrm{int}} = \{\sum_{n \in \mathbf{Z}} a_n u^n \in \mathcal{R} \mid u_n \in W_{\mathcal{O}_E}(k) \text{ for all } n \in \mathbf{Z}\}.$$

In particular, the p-adic completion of R^{int} identifies with $\mathfrak{S}_E[1/E(u)]_p^\wedge =: \mathcal{O}_{\mathcal{E}}$. Clearly, both $\mathcal{R}^b$ and $\mathcal{R}^{\mathrm{int}}$ are φ_q-stable inside $\mathcal{R}$.

We denote by $\mathrm{Vect}^{\varphi_q}(\mathcal{R})$ the category of φ-modules over $\mathcal{R}$, i.e. finite free $\mathcal{R}$-modules $\mathcal{M}$ equipped with an isomorphism $\varphi_q^*\mathcal{M} \simeq \mathcal{M}$. A φ-module $\mathcal{M}$ is called étale or pure of slope 0 if it contains a φ_q-stable $\mathcal{R}^{int}$-lattice $\mathcal{N}$ for which the map $\varphi_q^*\mathcal{N} \to \mathcal{N}$ is an isomorphism. By twisting suitably with a rank 1 module, one can then define the subcategory $\mathrm{Vect}^{\varphi_q,s}(\mathcal{R})$ of objects pure of slope s for any $s \in \mathbf{Q}$; see [Ked08, Definition 1.6.1]. Similarly, we denote by $\mathrm{Vect}^{\varphi_q,s}(\mathcal{R}^b)$ the subcategory of φ_q-modules over $\mathcal{R}^b$ which are pure of slope s in the sense of Dieudonné–Manin theory (recall that $\mathcal{R}^b$ is a discrete valued field). Finally, a φ_q-module $\mathcal{M}$ over $\mathcal{O}$ is called pure of slope 0 if $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ is.

Theorem 3.3.10. (1) *Base change defines an equivalence of categories*

$$\mathrm{Vect}^{\varphi_q,s}(\mathcal{R}^b) \simeq \mathrm{Vect}^{\varphi_q,s}(\mathcal{R}).$$

(2) *For any $\mathcal{M} \in \mathrm{Vect}^{\varphi_q}(\mathcal{R})$, there exists a unique filtration*

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{M}_r = \mathcal{M},$$

in $\mathrm{Vect}^{\varphi_q}(\mathcal{R})$*, called the slope filtration, such that the quotient* $\mathcal{M}_i/\mathcal{M}_{i-1}$ *is (finite free and) pure of slope* $s_i \in \mathbf{Q}$ *and* $s_1 < s_2 < \ldots < s_r$*.*

Proof. See [Ked08, Theorem 1.6.5] and [Ked08, Theorem 1.7.1]. □

Proposition 3.3.11. *Let* $\mathcal{M} \in \mathrm{Vect}^{\varphi_q}(\mathcal{O})$*. Then the slope filtration on* $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{R}$ *descends uniquely to a filtration on* $\mathcal{M}$ *by (saturated[6]) subobjects in* $\mathrm{Vect}^{\varphi_q}(\mathcal{O})$*.*

Proof. The arguments in [Kim09, §4.2] carry over to our setting. □

Remark 3.3.12. Note that, unlike the proof of [Kis06, Proposition 1.3.7], which relies crucially on a monodromy operator, the proof of [Kim09] is intrinsic in the world of φ-modules.

We can now translate the weak admissibility condition for Hodge–Pink isocrystals across the equivalence of categories in Theorem 3.3.5.

Theorem 3.3.13. *Let* $D := (D, \varphi_q, \Lambda) \in \mathrm{HP}^{\varphi_q}(K)$*. Then* D *is weakly admissible if and only if* $\mathcal{M}(D)$ *is pure of slope* 0*.*

Here in the statement we implicitly identify $\mathcal{M}(D)$ with the corresponding φ_q-module over $\mathcal{O}$ (Lemma 3.3.2).

Proof. Assume first that D has rank 1. In this case

$$\mathcal{M}(D) = D \otimes_{K_{0,E}} \lambda^{-t_H(D)}\mathcal{O};$$

see e.g. Remark 3.3.6. Pick a basis $e \in D$ and write $\varphi_D(e) = \alpha e$ for some $\alpha \in K_0$. Then

$$\varphi_q(e \otimes \lambda^{-t_H(D)}) = (E(u)/E(0))^{t_H(D)}\alpha(e \otimes \lambda^{-t_H(D)});$$

as $E(u)$ is a unit in $\mathcal{R}^{int}$, we see that $\mathcal{M}(D)$ has slope $t_N(D) - t_H(D)$. This proves the theorem for rank 1 objects. The general case then follows by the same argument as in [Kis06, Theorem 1.3.8], using the equivalence in Theorem 3.3.5 and Proposition 3.3.11 in place of [Kis06, Proposition 1.3.7]. □

It will be convenient to state the following lemma separately.

Lemma 3.3.14. *Base change defines an equivalence of categories*

$$\mathrm{Vect}(\mathfrak{S}_E) \simeq \mathrm{Vect}(\mathfrak{S}_E[1/p]) \times_{\mathrm{Vect}(\mathcal{E})} \mathrm{Vect}(\mathcal{O}_{\mathcal{E}}),$$

where $\mathcal{O}_{\mathcal{E}} := \mathfrak{S}_E[1/E(u)]^{\wedge}_p$ *and* $\mathcal{E} := \mathcal{O}_{\mathcal{E}}[1/p]$*. Moreover, this induces an equivalence[7]*

$$\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E) \simeq \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E[1/p]) \times_{\mathrm{Vect}^{\varphi_q}(\mathcal{E})} \mathrm{Vect}^{\varphi_q}(\mathcal{O}_{\mathcal{E}}).$$

[6] A submodule $\mathcal{N} \subseteq \mathcal{M}$ is called saturated if it is a direct summand of $\mathcal{M}$.

[7] Using a result of Kedlaya [BMS18, Lemma 4.6], one sees by the same argument that the lemma also holds for the perfectoid variant $A_{\mathrm{inf},E} := W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)$ of $\mathfrak{S}_E$.

Proof. It suffices to show the first assertion, which follows from Beauville–Laszlo glueing, and the fact that restricting gives an equivalence

$$\mathrm{Vect}(\mathrm{Spec}(\mathfrak{S}_E)) \simeq \mathrm{Vect}(\mathrm{Spec}(\mathfrak{S}_E) - \{\mathfrak{m}\}).$$

□

Proposition 3.3.15 ([Kis06, Lemma 1.3.13]). *Base change defines an equivalence*

$$\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E)[1/p] \simeq \mathrm{Vect}^{\varphi_q,0}(\mathcal{O}),$$

where the source denotes the isogeny category of $\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E)$.

Proof. See the proof of [Kis06, Lemma 1.3.13]. □

Corollary 3.3.16. *There is a natural fully faithful functor*

$$\mathrm{MF}^{\varphi_q,w.a}(K) \xhookrightarrow{\mathfrak{M}(\cdot)} \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E)[1/p].$$

Proof. This follows by combining Lemma 3.3.8, Theorem 3.3.13, and Proposition 3.3.15. □

Proposition 3.3.17 ([Kis06, Proposition 2.1.12]). *The base change functor*

$$\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E) \to \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E[1/E(u)]_p^\wedge)$$

is fully faithful.

Proof. With the previous results in place, the proof is similar to that of [Kis06, Proposition 2.1.12]. Namely, as in *loc. cit.*, it suffices to show that if $h : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a map in $\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E)$ such that $h[1/E(u)]_p^\wedge$ is an isomorphism, then h is an isomorphism. We may assume $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are free of rank 1. By Lemma 3.3.14, it suffices to show that $h[1/p]$ is an isomorphism. This follows by using the embedding $\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E)[1/p] \hookrightarrow \mathrm{HP}^{\varphi_q,w.a.}(K)$, and the fact that a map of *weakly admissible* objects in $\mathrm{HP}^{\varphi_q}(K)$ which is an isomrphism on underlying modules (e.g. a nonzero map between rank 1 objects) is necessarily an isomorphism. □

3.4 Prismatic F-crystals and E-crystalline Galois representations

3.4.1 Preliminaries on $\mathcal{O}_E$-prisms

We keep the notation as before. In particular, we fix throughout a uniformizer π of E, and an embedding $\tau_0 : E \hookrightarrow K \subseteq C$.

Definition 3.4.1 ([Mar23, Defn. 3.1]). Let X be a π-adic formal scheme over $\mathrm{Spf}\mathcal{O}_E$. The (absolute) prismatic site $(X)_{\Delta,\mathcal{O}_E}$ is by definition the site with whose objects are bounded $\mathcal{O}_E$-prisms (A, I) with a map $\mathrm{Spec}(A/I) \to X$ of π-adic $\mathcal{O}_E$-formal schemes, with coverings given by maps of prisms whose underlying ring map is (π, I)-adically completely faithfully flat.

For the precise definition of $\mathcal{O}_E$-prisms, we refer the reader to [Mar23] (see also [Ito23]). As we will work entirely with $\mathcal{O}_E$-prisms in what follows, we will typically drop $\mathcal{O}_E$ from the notation. We mention here some examples of $\mathcal{O}_E$-prisms that are most relevant for our purpose.

Example 3.4.2. (1) (Breuil–Kisin prisms) Choose a uniformizer $\pi_K \in K$. As $\mathfrak{S}_E := W_{\mathcal{O}_E}(k)[[u]]$, endowed with the δ_E-structure given by $\delta_E(u) = 0$ (or equivalently, $\varphi_q(u) = u^q$). Let $E(u) \in W_{\mathcal{O}_E}(k)[u]$ be the Eisenstein polynomial of π_K over $K_{0,E}$. As the map $\widetilde{\theta} : \mathfrak{S}_E \twoheadrightarrow \mathcal{O}_K, u \mapsto \pi_K$ is surjective with kernel $I = (E(u))$, the pair $(\mathfrak{S}_E, I)$ gives an object in $(\mathcal{O}_K)_{\mathbb{A},\mathcal{O}_E}$, which we will refer to as the Breuil–Kisin prism associated to the chosen uniformizer π_K.

(2) (The $A_{\mathrm{inf},E}$-prism) Recall that C denotes a fixed completed algebraic closure of K. Set $A_{\mathrm{inf},E} := W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)$, equipped with the natural Frobenius φ_q. As usual, the Fontaine's theta map $\theta_E : A_{\mathrm{inf},E} \to \mathcal{O}_C$ is surjective with kernel generated by a nonzero-divisor ξ. The twisted map $\widetilde{\theta}_E := \theta_E \circ \varphi_q^{-1}$ is thus also surjective with kernel $(\widetilde{\xi})$ where $\widetilde{\xi} := \varphi_q(\xi)$. In particular, the pair $(A_{\mathrm{inf},E}, (\widetilde{\xi}))$ defines an object in $(\mathcal{O}_K)_{\mathbb{A},\mathcal{O}_E}$, and is the $\mathcal{O}_E$-prism corresponding to the perfectoid $\mathcal{O}_E$-algebra $\mathcal{O}_C$, i.e. $A_{\mathrm{inf},E} = \mathbb{A}_{\mathcal{O}_C}$. For later use, we give here an explicit choice of ξ. Let $v = (v_0, v_1, \ldots) \in T\mathcal{G}$ be a generator of the Tate module of the Lubin–Tate group $\mathcal{G}$ of E associated to some Frobenius polynomial $Q \in \mathcal{O}_E[T]$ for π. As $v_{n+1}^q \equiv v_n \bmod \pi$ for all n, we obtain an element

$$\underline{v} := (v_0 \bmod \pi, v_1 \bmod \pi, \ldots) \in \varprojlim_{x \mapsto x^q} \mathcal{O}_C/\pi \simeq \mathcal{O}_{C^\flat}.$$

(The last identification is given by $(a_n)_n \mapsto (a_0, a_1^{p^{f-1}}, \ldots, a_1^p, a_1, a_2^{p^{f-1}}, \ldots)$.) Following [FF18, Proposition 1.2.7], for a perfect $\mathbf{F}_q$-algebra A, we denote by $[\cdot]_Q$ (or $[\cdot]_{\mathcal{G}}$) the unique map $A \to W_{\mathcal{O}_E}(A)$ satisfying $[x]_Q \equiv x \bmod \pi$ and $Q([x]_Q) = \varphi_q([x]_Q)$. When $E = \mathbf{Q}_p, \pi = p$ and $Q(T) = (1+T)^p - 1$, $[x]_Q$ is nothing but $[x+1] - 1$ and hence $[\underline{v}]_Q = [\epsilon] - 1$ is the usual element μ; accordingly, we will also write $\mu := [\underline{v}]_Q$ here. One can then check that $\xi := \mu/\varphi_q^{-1}(\mu)$ is a generator of $\ker(\theta_E)$.

Exactly as in [BS23, Example 2.6], one can show that both examples above give covers of the final object in the topos $\mathrm{Shv}((\mathcal{O}_K)_{\mathbb{A}})$. Moreover, there is a map $\mathfrak{S}_E \to \mathbb{A}_{\mathcal{O}_C}$ in $(\mathcal{O}_K)_{\mathbb{A},\mathcal{O}_E}$, defined by $u \mapsto [\pi_K^\flat]$, where $\pi_K^\flat := (\pi_K, \pi_K^{1/q}, \ldots) \in \mathcal{O}_C^\flat$ is a compatible system of q-power roots of the fixed uniformizer π_K.

(3) (The $A_{\mathrm{cris},E}$-prism) Recall that $A_{\mathrm{cris},E}$ denotes the π-completed $\mathcal{O}_E$-PD envelope of $A_{\mathrm{inf},E}$ with respect to the kernel of θ_E, $B_{\mathrm{cris},E}^+ := A_{\mathrm{cris},E}[1/\pi]$, and $B_{\mathrm{cris},E} = B_{\mathrm{cris},E}^+[1/t_E] = A_{\mathrm{cris},E}[1/\mu]$. By [Ito23, Proposition 2.6.5], the pair $(A_{\mathrm{cris},E}, (\pi))$ identifies with the prismatic envelope $\mathbb{A}_{\mathcal{O}_C}\{\widetilde{\xi}/\pi\}$, hence also defines an object in $(\mathcal{O}_K)_{\mathbb{A},\mathcal{O}_E}$, which we denote by $\mathbb{A}_{\mathcal{O}_C/\pi}$ in analogy with the usual case[8]; of course there is a natural map $\mathbb{A}_{\mathcal{O}_C} \to \mathbb{A}_{\mathcal{O}_C/\pi}$.

Remark 3.4.3 ($W_{\mathcal{O}_E}(k)$**-algebra structure on objects in** $(\mathcal{O}_K)_{\mathbb{A},\mathcal{O}_E}$**).** Fix an object $(A, I) \in (\mathcal{O}_K)_{\mathbb{A},\mathcal{O}_E}$ with structure map $\mathcal{O}_K \to A/I$. By standard deformation theory, the composition $W_{\mathcal{O}_E}(k) \to$

[8] While it is reasonable to define a notion of qrsp rings and their associated prisms in the "$\mathcal{O}_E$-context", for our purpose it suffices to see this as a purely suggestive notation.

$\mathcal{O}_K \to A/I$ lifts uniquely to an $\mathcal{O}_E$-algebra map $W_{\mathcal{O}_E}(k) \to A$. In what follows, we will always regard an object in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$ as an $W_{\mathcal{O}_E}(k)$-algebra via this map. (Note that for the prism $\Delta_{\mathcal{O}_C} = W_{\mathcal{O}_E}(\mathcal{O}_C^\flat)$, this is not the "natural" structure (induced by the canonical section $k \to \mathcal{O}_C^\flat$) but a φ_q-twist of it: the point is that we are taking into account not only the underlying ring, but also the invertible ideal defining the prism structure). By uniqueness, morphisms in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$ automatically respect this algebra structure.

Notation 3.4.4 (Some period sheaves on the prismatic site)**.** We consider the following period sheaves on X_Δ:

- The prismatic structure sheaf $\mathcal{O}_\Delta : (A, I) \mapsto A$; this comes equipped with an ideal sheaf $I_\Delta : (A, I) \mapsto I$ and a "Frobenius" $\varphi_q : \mathcal{O}_\Delta \to \mathcal{O}_\Delta$.
- The étale structure sheaf $\mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_\pi$:
$$(A, I) \mapsto A[1/I]^\wedge_\pi.$$
- The rational localization $\mathcal{O}_\Delta\langle \mathcal{I}_\Delta/\pi\rangle$:
$$(A, I) \mapsto A[I/\pi]^\wedge_\pi.$$
- The de Rham period sheaves:
$$\mathbb{B}^+_{\mathrm{dR}} := (\mathcal{O}_\Delta[1/\pi])^\wedge_{\mathcal{I}_\Delta} \quad \text{and} \quad \mathbb{B}_{\mathrm{dR}} := \mathbb{B}^+_{\mathrm{dR}}[1/\mathcal{I}_\Delta].$$

It is easy to see that the Frobenius on $\mathcal{O}_\Delta$ extends naturally to the sheaves $\mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_\pi$ and $\mathcal{O}_\Delta\langle \mathcal{I}_\Delta/\pi\rangle$. (Note again that in case $X = \mathrm{Spf}(\mathcal{O}_K)$, the value $\mathbb{B}^+_{\mathrm{dR}}(\Delta_{\mathcal{O}_C}) =: B^+_{\mathrm{dR}}$ is a φ_q-twist of the ring denoted by the same notation in the previous sections.)

Definition 3.4.5. A prismatic F-crystal on X is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ where $\mathcal{E}$ is a crystal of vector bundles on the ringed site $(X_\Delta, \mathcal{O}_\Delta)$, and $\varphi_{\mathcal{E}}$ is an isomorphism $(\varphi_q^*\mathcal{E})[1/\mathcal{I}_\Delta] \cong \mathcal{E}[1/\mathcal{I}_\Delta]$. The resulting category is denoted by $\mathrm{Vect}^{\varphi_q}(X_\Delta, \mathcal{O}_\Delta)$.

More generally, for a sheaf $\mathcal{O}'$ of $\mathcal{O}_\Delta$-algebras equipped with a compatible Frobenius, we define similarly the category $\mathrm{Vect}^{\varphi_q}(X_\Delta, \mathcal{O}')$ of F-crystals over $\mathcal{O}'$ on X. Similarly, if (A, I) is an $\mathcal{O}_E$-prism, we define in the same way the category $\mathrm{Vect}^{\varphi_q}(A, I)$ (or simply $\mathrm{Vect}^{\varphi_q}(A)$) of F-crystals (or Breuil–Kisin modules) over A.

Remark 3.4.6. As descent for vector bundles is effective for the flat topology, to give a crystal of vector bundles on $(X_\Delta, \mathcal{O}_\Delta)$ is to give for each object (A, I) in X_Δ, a finite projective A-module M_A, and for each map $(A, I) \to (B, J)$ in X_Δ, an isomorphism $M_A \otimes_A B \xrightarrow{\sim} M_B$ compatible with compositions. In other words,
$$\mathrm{Vect}^{(\varphi_q)}(X_\Delta, \mathcal{O}_\Delta) \simeq \lim_{(A,I)\in X_\Delta} \mathrm{Vect}^{(\varphi_q)}(A, I).$$

A similar result holds for the sheaves $\mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_\pi$ and $\mathcal{O}_\Delta\langle \mathcal{I}_\Delta/\pi\rangle[1/\pi]$ (for the first see [BS23, Proppositon 2.7]; for the second see the proof of [BS23, Corollary 7.17]).

3.4.2 Formulation of the main theorem

We now restrict ourselves to the case $X = \mathrm{Spf}(\mathcal{O}_K)$, viewed as an $\mathcal{O}_E$-formal scheme using the fixed embedding $\tau_0 : E \hookrightarrow K$.

Recall from [Mar23] that there is a natural equivalence

$$T : \mathrm{Vect}^{\varphi_q}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_{\pi}) \simeq \mathrm{Rep}_{\mathcal{O}_E}(G_K)$$
$$\mathcal{E} \mapsto (\mathcal{E}(\Delta_{\mathcal{O}_C}))^{\varphi_q = 1}.$$

In particular, by extending scalars along $\mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_{\pi}$, we obtain a functor

$$T : \mathrm{Vect}^{\varphi_q}(X_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Rep}_{\mathcal{O}_E}(G_K),$$

which we again refer to as the étale realization functor. We can now state our main result.

Theorem 3.4.7. *The étale realization functor gives rise to an equivalence of categories*

$$T : \mathrm{Vect}^{\varphi_q}(X_{\Delta}, \mathcal{O}_{\Delta}) \to \mathrm{Rep}^{\mathrm{crys}}_{\mathcal{O}_E}(G_K)$$

where the target denotes the category of finite free $\mathcal{O}_E$-modules T equipped with a continuous linear G_K-action such that $T[1/\pi]$ is E-crystalline.

Proof. We first show that the étale realization $T(\mathcal{E})$ of a prismatic F-crystal on $\mathcal{O}_K$ is indeed an object in the target; full faithfulness and essential surjectivity will be dealt separately below. We will follow the proof of [BS23, Proposition 5.3]. First by the crystal structure of $\mathcal{E}$, we have a natural isomorphism

$$\mathcal{E}(\Delta_{\mathcal{O}_C}) \otimes_{\Delta_{\mathcal{O}_C}} \Delta_{\mathcal{O}_C/\pi} \to \mathcal{E}(\Delta_{\mathcal{O}_C/\pi}).$$

Also, by Lemma 3.4.8 below, we have a natural identification

$$T(\mathcal{E}) \otimes_{\mathcal{O}_E} \Delta_{\mathcal{O}_C}[1/\mu] = \mathcal{E}(\Delta_{\mathcal{O}_C})[1/\mu]$$

Pick $n \gg 0$ so that the map $\varphi_q^n : \mathcal{O}_K/\pi \to \mathcal{O}_K/\pi$ factors through the natural reduction map $\mathcal{O}_K/\pi \twoheadrightarrow k$ (where k denotes the residue field of k). In particular, we see that the natural map $k \to \mathcal{O}_C/\pi$ is $\mathcal{O}_K$-linear when the target is now regarded as an $\mathcal{O}_K$-algebra via the map $\varphi_q^n : \mathcal{O}_K/\pi \to \mathcal{O}_C/\pi$. This lifts to a map $W_{\mathcal{O}_E}(k) \to \Delta'_{\mathcal{O}_C/\pi}$ in X_{Δ}, where the target denotes the object with underlying prism $\Delta_{\mathcal{O}_C/\pi}$ but the map to $\mathrm{Spf}(\mathcal{O}_K)$ is being twisted by φ_q^n (i.e. we have a map $\varphi_q^n : \Delta_{\mathcal{O}_C/\pi} \to \Delta'_{\mathcal{O}_C/\pi}$ in X_{Δ}). Thus, by using the crystal and Frobenius structures of $\mathcal{E}$, we obtain a natural isomorphism

$$\begin{aligned}\mathcal{E}(W_{\mathcal{O}_E}(k)) \otimes_{W_{\mathcal{O}_E}(k)} \Delta_{\mathcal{O}_C/\pi}[1/\pi] &\simeq \mathcal{E}(\Delta'_{\mathcal{O}_C/\pi})[1/\pi] \\ &\simeq (\varphi_q)^* \mathcal{E}(\Delta_{\mathcal{O}_C/\pi})[1/\pi] \\ &\simeq \mathcal{E}(\Delta_{\mathcal{O}_C/\pi})[1/\pi].\end{aligned}$$

Note also that as $W_{\mathcal{O}_E}(k)$ is fixed by the G_K-action on $\Delta'_{\mathcal{O}_C/\pi}$ under the map $W_{\mathcal{O}_E}(k) \to \Delta'_{\mathcal{O}_C/\pi}$, the crystal property of $\mathcal{E}$ again implies that G_K acts trivially on $\mathcal{E}(W_{\mathcal{O}_E}(k))$. Putting things together, we obtain a G_K-equivariant isomorphism

$$T(\mathcal{E}) \otimes_{\mathcal{O}_E} B_{\mathrm{cris},E} \simeq \mathcal{E}(W_{\mathcal{O}_E}(k)) \otimes_{W_{\mathcal{O}_E}(k)} B_{\mathrm{cris},E},$$

with G_K acting trivially on $\mathcal{E}(W_{\mathcal{O}_E}(k))$. By Theorem 3.2.4 (1), this means that the G_K-representation $T(\mathcal{E})[1/\pi]$ is E-crystalline, as desired. □

Lemma 3.4.8. *Fix an object $M \in \mathrm{Vect}^{\varphi_q}(\Delta_{\mathcal{O}_C})$ with étale realization $T := T(M)$. Then*

$$T \otimes_{\mathcal{O}_E} A_{\mathrm{inf},E}\left[\frac{1}{\mu}\right] = M\left[\frac{1}{\mu}\right] \tag{3.4.8.1}$$

as submodules of $T \otimes_{\mathcal{O}_E} W_{\mathcal{O}_E}(C^\flat) = M \otimes_{A_{\mathrm{inf},E}} W_{\mathcal{O}_E}(C^\flat)$.

Proof. The proof is similar to that of [BMS18, Lem. 4.26]. In fact, we will only explain the reduction to the case φ_M^{-1} maps M into M; the rest of the argument is identical to that in *loc. cit.* For this, we need a suitable variant of the usual Tate twist. Let $A_{\mathrm{inf},E}\{1\} \in \mathrm{Vect}^{\varphi_q}(\Delta_{\mathcal{O}_C})$ be the rank one object with a basis e, and $\varphi_q(e) = \frac{1}{\xi}e$. For an integer n, we set $M\{n\} := M \otimes A_{\mathrm{inf},E}\{1\}^{\otimes n}$. The étale realization of $A_{\mathrm{inf},E}\{1\}$ is

$$\{xe \mid x \in W_{\mathcal{O}_E}(C^\flat), \varphi_q(x) = \widetilde{\xi}x\} = \{xe \mid \varphi_q(x/\mu) = x/\mu\} = \mathcal{O}_E\mu e.$$

In particular, we see that the module $A_{\mathrm{inf},E}\{n\}$ satisfies 3.4.8.1 for any integer n. Now pick $n \gg 0$ so that $\varphi_M^{-1}(M) \subseteq \frac{1}{\xi^n}M$. Then $\varphi_M^{-1}(M\{n\}) \subseteq M\{n\}$, and we can replace M by this $M\{n\}$ (as the functor $M \mapsto T(M)$ is tensor-compatible in M). □

3.4.3 Full faithfulness

We now prove the full faithfulness of the étale realization function in Theorem 3.4.7. With everything in place, we can follow the proof of the corresponding statement in [BS23].

Proof of full faithfulness in Theorem 3.4.7. Choose a Breuil–Kisin prism $(\mathfrak{S}_E, I)$ in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$; this gives a cover of the final object in the associated topos. In particular, faithfulness of the étale realization functor T on X reduces to the analogous statement over $(\mathfrak{S}_E, I)$, which in turn follows from injectivity of the map $\mathfrak{S}_E \to \mathfrak{S}_E[1/I]^\wedge_\pi$. The same argument also shows that T is faithful over $\mathfrak{S}_E^{(1)}$[9], the self-coproduct of $\mathfrak{S}_E$ with itself in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$.

Fullness of T now reduces formally to the analogous statement over $(\mathfrak{S}_E, I)$, i.e. that the base change functor

$$\mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E) \to \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E[1/I]^\wedge_\pi)$$

is fully faithful. This is Proposition 3.3.17. □

[9]By construction of $\mathfrak{S}_E^{(1)}$ as a suitable prismatic envelope, $\mathfrak{S}_E^{(1)}$ is (π, I)-completely flat over $\mathfrak{S}_E$ (via either structure map). As $\mathfrak{S}_E$ is transversal, the same holds for $\mathfrak{S}_E^{(1)}$ (see e.g. [BMS19, Lemma 4.7]). In fact, as $\mathfrak{S}_E$ is Noetherian, $\mathfrak{S}_E^{(1)}$ is even classically (faithfully) flat over $\mathfrak{S}_E$, but we will not need this.

The following lemma was used above.

Lemma 3.4.9. *Let (A, I) be a transversal $\mathcal{O}_E$-prism. The natural map $A \to A[1/I]^\wedge_\pi$ is injective.*

Here, following the terminology in [AL23], an $\mathcal{O}_E$-prism (A, I) is called transversal if A/I is π-torsion free; in this case A is itself π-torsion free.

Proof. As the source is π-separated and the target is π-torsion free, it suffices to show that the map is injective modulo π, which follows directly from transversality of (A, I). □

Remark 3.4.10. As in [BS23], instead of working with a Breuil–Kisin prism, one can also work directly with the prism $\Delta_{\mathcal{O}_C}$, and deduce the desired full faithfulness from (the easy direction of) Fargues' classification of F-crystals over $\mathrm{Vect}^{\varphi_q}(\Delta_{\mathcal{O}_C})$.

3.4.4 Essential surjectivity

The goal of this subsection is to prove the essential surjectivity part of Theorem 3.4.7. As explained in the introduction, the general strategy of our proof follows that of [BS23, §6]; the main difference is that instead of using inputs from [AMMN22] to prove the desired boundedness of descent data (Proposition 6.10 in [BS23]), we adapt a key lemma from [DL22], which will in fact allows us to prove a more general result (Proposition 3.4.15).

We first collect some further ring-theoretic properties on transversal prisms, which will be used repeatedly in what follows.

Lemma 3.4.11. *Let $(A, (d))$ be a transversal $\mathcal{O}_E$-prism.*

(1) *The ring $A\langle d/\pi\rangle[1/\pi]$ is d-torsion free and d-separated.*

(2) *$A \cap d^n A\langle d/\pi\rangle[1/\pi] = d^n A$ and $A\langle d/\pi\rangle \cap d^n A\langle d/\pi\rangle[1/\pi] = (d/\pi)^n A\langle d/\pi\rangle$ for each $n \geq 0$.*

(3) *For each $n \geq 0$, the natural map*

$$A[1/\pi]^\wedge_d \to A\langle \varphi_q^n(d)/\pi\rangle[1/\pi]^\wedge_d$$

is an isomorphism. Moreover, the natural maps $A \to A\langle\varphi^n(d)/\pi\rangle[1/\pi] \to A\langle d/\pi\rangle[1/\pi]$ are injective.

As the proof shows, parts (1) and (2) in fact hold for any pair (A, d) such that: (i) (π, d) forms a regular sequence, (ii) A is (classically) (π, d)-complete; moreover part (3) needs additionally only the underlying δ_E-ring structure (rather than the full prism structure) of A.

Proof. (1) Recall that $A\langle d/\pi\rangle$ is by definition the π-adic completion of the A-subalgebra $A[d/\pi]$ of $A[1/\pi]$ generated by d/π. We note in particular that it is π-torsion free. We claim that the natural map $A[x]/(\pi x - d) \to A[d/\pi], x \mapsto d/\pi$ is an isomorphism. As it is clearly an isomorphism after inverting π, it suffices to show that the source is π-torsion free, which in turn follows formally from the facts that $A[x]$ is π-torsion free and that $\pi x - d = -d$ is regular on $(A/\pi)[x]$. Thus

$$A\langle d/\pi\rangle = A[x]/(\pi x - d)^\wedge_p = A\langle x\rangle/J,$$

where $A\langle x\rangle := A[x]_p^\wedge$ and $J := \overline{(\pi x - d)} = \cap_{n\geq 0}(\pi x - d, \pi^n)A\langle x\rangle$ is the closure of $(\pi x - d)$ in $A\langle x\rangle$ for the π-adic topology. We need to show that if $f \in A\langle x\rangle$ satisfies $df \in (\pi x - d, \pi^n)$ for all $n \gg 0$, then the same holds for f. Write $df = (\pi x - d)g + \pi^n h$ for some $g, h \in A\langle x\rangle$. As $d(f+g) = \pi(xg+\pi^{n-1}h)$, we can write $f+g = \pi k, xg+\pi^{n-1}h = dk$ for some $k \in A\langle x\rangle$. Now as $A\langle x\rangle/(x, \pi^{n-1}) = A[x]/(x, \pi^{n-1}) = A/\pi^{n-1}$ is d-torsion free, $k \in (x, \pi^{n-1})$, say $k = xa + \pi^{n-1}b$ for some $a, b \in A\langle x\rangle$. Then $x(g-da) = \pi^{n-1}(db-h)$ whence $g-da = \pi^{n-1}a'$ for some $a' \in A\langle x\rangle$ whence $f = \pi k-g = \pi(xa+\pi^{n-1}b)-(da+\pi^{n-1}a') = (\pi x-d)a+\pi^{n-1}(\pi b-a') \in (\pi x-d, \pi^{n-1})$, as wanted.

For the last statement of (1), it suffices to show that

$$A\langle d/\pi\rangle \cap \cap_{n\geq 0} d^n A\langle d/\pi\rangle[1/\pi] = 0.$$

We first check that $A\langle d/\pi\rangle \cap (d/\pi)^n A\langle d/\pi\rangle[1/\pi] = (d/\pi)^n A\langle d/\pi\rangle$. As d (hence d/π) is regular on $A\langle d/\pi\rangle$ by (1), we may assume $n = 1$. We need to show that $A\langle d/\pi\rangle/(d/\pi)$ is π-torsion free. We claim that $A/\langle d/\pi\rangle/(d/\pi) \simeq A/d$ via the natural map. Indeed, the map $A\langle x\rangle \twoheadrightarrow A, x \mapsto 0$ maps $\overline{(\pi x - d)}$ into $dA \subseteq A$ (as dA is closed for the π-topology on A: $A/dA = \mathrm{coker}(A \xrightarrow{\times d} A)$ is derived π-complete, hence classically π-complete as it is π-torsion free by our assumptions on A). We thus get a surjection $A\langle x\rangle/(\overline{(\pi x - d)}, x) \twoheadrightarrow A/d, x \mapsto 0$. But $A/d = A\langle x\rangle/(\pi x - d, x)$ also surjects naturally onto $A\langle x\rangle/(\overline{(\pi x - d)}, x)$ whence $A/d = A\langle x\rangle/(\overline{(\pi x - d)}, x)$:

$$A/d \twoheadrightarrow A\langle x\rangle/(\overline{(\pi x - d)}, x) \twoheadrightarrow A/d.$$

Thus, $A\langle d/\pi\rangle/(d/\pi) = A/d$ is π-torsion free, as wanted. Note that the equality $A\langle d/\pi\rangle/(d/\pi) = A/d$ also implies that $A \cap (d/\pi)A\langle d/\pi\rangle = dA$. By induction (using that (π, d) is a regular sequence), we see that in fact $A \cap (d/\pi)^n A\langle d/\pi\rangle = d^n A$ for all $n \geq 0$.

We thus need to show that $A\langle d/\pi\rangle$ is (d/π)-separated. Write $\overline{x} := d/\pi$. Assume $f \in \cap_{n\geq 0}\overline{x}^n A\langle\overline{x}\rangle$. We can write $f = a_0 + a_1\overline{x} + \ldots$ for a (not necessarily unique) sequence (a_n) in A with $a_n \to 0$ π-adically. In particular, $a_0 \in \overline{x}A\langle\overline{x}\rangle \cap A = dA$ (by the preceding paragraph), say $a_0 = da_0' = \pi a_0'\overline{x}$. As $\overline{x}$ is regular in $A\langle\overline{x}\rangle$, we still have $f/\overline{x} = (\pi a_0' + a_1) + \ldots \in \cap_{n\geq 0}(\overline{x}^n)$. Similarly we find that $a_1 + \pi a_0' \in (\overline{x}) \cap A = dA$, say $a_1 + \pi a_0' = da_1'$. Next, we have $0 = (\pi\overline{x} - d)(a_0' + a_1'\overline{x})$, so $a_0 + a_1\overline{x} = da_0' + (da_1' - \pi a_0')\overline{x} = \pi a_1'\overline{x}^2$. Again, as $\overline{x}$ is regular, we deduce that $a_2 + \pi a_1' \in dA$ say $a_2 + \pi a_1' = da_2'$. Repeating this argument, we can inductively find a sequence (a_n') in A such that $a_n = da_n' - \pi a_{n-1}'$ for all $n \geq 0$. We claim that the sequence (a_n') also tends to 0 π-adically. Once this is done, $a_0' + a_1'\overline{x} + \ldots$ makes sense as an element in $A\langle\overline{x}\rangle$ and we have $f = a_0 + a_1\overline{x} + \ldots = (d - \pi\overline{x})(a_0' + a_1'\overline{x} + \ldots) = 0$, as wanted. We will show by induction on n that $a_m' \in \pi^n A$ for all $m \gg 0$ (depending on n). So assume that $a_m' \in \pi^n A$ for all $m \geq m_0$. Enlarging m_0 if necessary, we may also assume that $a_m \in \pi^{n+1}A$ for all $m > m_0$ (as $a_m \to 0$ π-adically by assumption). Then $da_m' = a_m + \pi a_{m-1}' \in \pi^{n+1}A$, and so, as (π, d) is a regular sequence, $a_m' \in \pi^{n+1}A$ for all $m > m_0$, as claimed.

(2) This is contained in the proof of (1) above.

(3) For the first statement, as the sources and target are both d-complete and d-torison free (by part (1) as $A\langle\varphi_q^n(d)/\pi\rangle = A\langle d^{q^n}/\pi\rangle$), it suffices to show that the map

$$A[1/\pi]/d^{q^n} \to A\langle d^{q^n}/\pi\rangle[1/\pi]/d^{q^n}$$

is an isomorphism, which in turn follows from the proof of (1). (This part is also proved in [BS23, Lemma 6.7].) The second statement can be proved similarly (i.e. by reducing modulo d (or d^{q^n})). □

The following lemma, whose statement is inspired by [DL22, Lemma 2.2.10], records the contracting effect of the Frobenius on the d-adic filtration on $A\langle d/\pi\rangle$.

Lemma 3.4.12. *Let $(A,(d))$ be a transversal $\mathcal{O}_E$-prism. Then given any $h \geq 0$,*

$$\varphi_q(d^m A\langle d/\pi\rangle) \subseteq A + d^{m+h}A\langle d/\pi\rangle$$

for all $m \gg 0$ (depending only on h).

Proof. We will show more generally that

$$\varphi_q(d^m A\langle d/\pi\rangle) \subseteq A + \frac{d^{q(m+1)}}{\pi}A\langle d^q/\pi\rangle \quad \text{for all } m \geq 0.$$

This easily implies the lemma. Write $\varphi_q(d) = d^q + \pi a$ with $a \in A$. As $\varphi_q(A\langle d/\pi\rangle) \subseteq A\langle d^q/\pi\rangle$, it suffices by the binomial theorem to show that

$$d^{q(m-k)}\pi^k A\langle d^q/\pi\rangle \subseteq A + \frac{d^{q(m+1)}}{\pi}A\langle d^q/\pi\rangle \quad \text{for all } 0 \leq k \leq m.$$

This follows immediately from the inclusion $A\langle d^q/\pi\rangle \subseteq (1/\pi^k)A + (d^q/\pi)^{k+1}A\langle d^q/\pi\rangle$. □

Proposition 3.4.13. *Let $(A,(d))$ be a transversal $\mathcal{O}_E$-prism. Let $h \geq 0$. Assume*

$$d^h Y = B\varphi_q(Y)C$$

with matrices $Y \in M_d(A\langle d/\pi\rangle[1/\pi])$ and $B, C \in M_d(A)$. Then $Y \in M_d(A[1/\pi])$.

Proof. The argument here is inspired by the proof of [DL22, Proposition 2.2.11][10]. Replacing Y by $\pi^k Y$ for some $k \gg 0$, we may assume that

$$Y = Y_{m_0} + Y'$$

for some $Y_{m_0} \in M_d(A)$ and $X \in M_d(d^{m_0}R)$, where m_0 satisfies $\varphi_q(d^m R) \subseteq A + d^{h+m+1}R$ for all $m \geq m_0$ (m_0 exists by Lemma 3.4.12); here $R := A\langle d/\pi\rangle$. We will show that $Y \in M_d(A)$. The idea is to d-adically approximate Y by matrices in $M_d(A)$. More precisely, we will construct

[10] It is not clear to us if the various rings in [DL22] indeed agree with the more standard rings denoted by the same notation; for instance, we do not know if the ring $A^{(2)}_{\max}$ there equals literally to $A^{(2)}\langle I/p\rangle$.

inductively a sequence $(Y_m)_{m\geq m_0}$ in $M_d(A)$ with the property that $Y_{m+1} \equiv Y_m \bmod d^m A$ and $Y_m \equiv Y \bmod d^m R$ for all $m \geq m_0$. As both A and R are d-complete, this implies that $Y \in M_d(A)$, as wanted.

Assume Y_m has been constructed. Write $Y = Y_m + X$ with $X \in M_d(d^m R)$. By assumption,

$$d^h(Y_m + X) = B\varphi_q(Y_m)C + B\varphi_q(X)C.$$

By our choice of m_0 we can write $B\varphi_q(X)C = Z + d^h X'$ for some $Z \in M_d(A)$ and $X' \in M_d(d^{m+1}R)$. Then $B\varphi_q(Y_m)C + Z - d^h Y_m = d^h(X - X')$ has entries in $A \cap d^{h+m}R = d^{h+m}A$ by Lemma 3.4.11 (2), say $d^{h+m}X''$ with $X'' \in M_d(A)$. As d is regular on R, we obtain $X - X' = d^m X''$. Now set $Y_{m+1} := Y_m + d^m X''$. □

Remark 3.4.14. Note that one cannot weaken the conditions on B, C into $B, C \in M_d(A[1/\pi])$. For instance, the infinite product

$$\lambda := \prod_{n\geq 0} \varphi^n(E(u)/E(0)) \in \mathcal{O}$$

satisfies $\lambda = (E(u)/E(0))\varphi(\lambda) \in \mathfrak{S}[1/p] \cdot \varphi(\lambda)$, but $\lambda \notin \mathfrak{S}[1/p]$[11].

Proposition 3.4.15. *Let $(A, (d))$ be a transversal $\mathcal{O}_E$-prism. Then the base change*

$$\mathrm{Vect}^{\varphi_q}(A)[1/\pi] \to \mathrm{Vect}^{\varphi_q}(A\langle d/\pi\rangle[1/\pi])$$

is fully faithful; here the source denotes the isogeny category of $\mathrm{Vect}^{\varphi_q}(A)$.

Proof. Given objects $\mathfrak{M}_i$ in $\mathrm{Vect}^{\varphi_q}(A)$, we need to show that any φ_q-equivariant map

$$\alpha : \mathfrak{M}_1 \otimes_A A\langle d/\pi\rangle[1/\pi] \to \mathfrak{M}_2 \otimes_A A\langle d/\pi\rangle[1/\pi]$$

extends to a map $\mathfrak{M}_1[1/\pi] \to \mathfrak{M}_2[1/\pi]$. (By injectivity of $A[1/\pi] \to A\langle d/\pi\rangle[1/\pi]$ (Lemma 3.4.11 (3)), such an extension is necessarily uniquely and φ_q-equivariant.) By Lemma 3.4.16 below $\mathfrak{M}_i$ can be written as a φ_q-stable direct summand of some *finite free* φ_q-module over A. We may thus reduce to the case $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both finite free.

Pick an A-basis $e_1, \ldots, e_{d_1}$ of $\mathfrak{M}_1$, and let $A_1 \in M_{d_1}(A[1/d])$ be the matrix giving the action of $\varphi_{\mathfrak{M}_1}$ on this basis, i.e. $\varphi_{\mathfrak{M}_1}(e_1, \ldots, e_{d_1}) = (e_1, \ldots, e_{d_1})A_1$; similarly let $A_2 \in M_{d_2}(A[1/d])$ be the matrix giving the action of $\varphi_{\mathfrak{M}_2}$ on some fixed basis of $\mathfrak{M}_2$. As α is φ-equivariant, we see that if $Y \in M_{d_1 d_2}(A\langle d/\pi\rangle[1/\pi])$ denotes the matrix of α relative to the chosen bases, then

$$YA_1 = A_2\varphi_q(Y).$$

We need to show that Y in fact has entries in $A[1/\pi]$. As A_1 is invertible, we can write the above equation as $d^h Y = B\varphi_q(Y)C$ for some $h \geq 0$, and matrices B, C with entries in A. Then by Proposition 3.4.13 below, Y has entries in $A[1/\pi]$, as wanted. □

[11]For instance, take $K = \mathbf{Q}_p$ and $\pi_K = -p$. Then $E(u) = u + p$ and the coefficient of $u^{1+p+\ldots+p^{n-1}}$ in $\lambda = \prod_{n\geq 0}(1 + u^{p^n}/p)$ is $1/p^n$ and of course one can make n arbitrarily large.

The following lemma was used above.

Lemma 3.4.16. *Let $(A,(d))$ be an $\mathcal{O}_E$-prism. Any object in $\mathrm{Vect}^{\varphi_q}(A)$ can be realized as a φ_q-stable direct summand of some finite free object.*

Proof. The proof is similar to that of [EG23, Lemma 4.3.1]. Fix $\mathfrak{M} \in \mathrm{Vect}^{\varphi_q}(A)$. Pick an A-module $\mathfrak{N}$ so that $\mathfrak{F} := \mathfrak{M} \oplus \mathfrak{N}$ is finite free. Then $(\mathfrak{F} \oplus \mathfrak{N}) \oplus \mathfrak{M} = \mathfrak{F} \oplus \mathfrak{F}$ is finite free, and by fixing an isomorphism $\varphi_F : \varphi_q^*\mathfrak{F} \simeq \mathfrak{F}$, one can endow $\mathfrak{F} \oplus \mathfrak{N}$ with a φ_q-structure via the composition

$$\begin{aligned}\varphi_q^*(\mathfrak{F} \oplus \mathfrak{N})[1/d] &\overset{\varphi_{\mathfrak{F}}}{\simeq} \mathfrak{F}[1/d] \oplus \varphi_q^*\mathfrak{N}[1/d] = \mathfrak{M}[1/d] \oplus \mathfrak{N}[1/d] \oplus \varphi_q^*\mathfrak{N}[1/d] \\ &\overset{\varphi_{\mathfrak{M}}^{-1}}{\simeq} \varphi_q^*\mathfrak{M}[1/d] \oplus \mathfrak{N}[1/d] \oplus \varphi_q^*\mathfrak{N}[1/d] = \varphi_q^*\mathfrak{F} \oplus \mathfrak{N}[1/d] \\ &\overset{\varphi_{\mathfrak{F}}}{\simeq} (\mathfrak{F} \oplus \mathfrak{N})[1/d].\end{aligned}$$

□

Proof of essential surjectivity in Theorem 3.4.7. Fix $T \in \mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K)$. Let $D \in \mathrm{MF}^{\varphi_q}(K)$ be the weakly admissible filtered φ_q-module over K corresponding to $T[1/\pi]$ under the equivalence in Theorem 3.2.4. Fix a Breuil–Kisin prism $(\mathfrak{S}_E, I)$ in $X_{\mathbb{\Delta}}$. We note firstly that the functor $\mathcal{M} : \mathrm{MF}^{\varphi_q}(K) \to \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E\langle I/\pi\rangle[1/\pi])$ from §3.3.2 in fact lifts to a functor

$$\mathcal{M} : \mathrm{MF}^{\varphi_q}(K) \to \mathrm{Vect}^{\varphi_q}(X_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}\langle \mathcal{I}_{\mathbb{\Delta}}/\pi\rangle[1/\pi]) \tag{3.4.16.1}$$

by applying exactly the same construction for each object in $X_{\mathbb{\Delta}}$ (cf. [BS23, Construction 6.5][12]). Moreover, we have seen in §3.3.13 that the weak admissibility of D implies that $\mathcal{M}(D)$ extends to an object $\mathfrak{M} \in \mathrm{Vect}^{\varphi_q}(\mathfrak{S}_E)$, which therefore comes equipped with a descent datum

$$\alpha : \mathfrak{M} \otimes_{\mathfrak{S}_E, p_1} \mathfrak{S}_E^{(1)}\langle I/\pi\rangle[1/\pi] \simeq \mathfrak{M} \otimes_{\mathfrak{S}_E, p_2} \mathfrak{S}_E^{(1)}\langle I/\pi\rangle[1/\pi]$$

by the last sentence. Proposition 3.4.15 then shows that α extends uniquely to a descent datum

$$\alpha : \mathfrak{M} \otimes_{\mathfrak{S}_E, p_1} \mathfrak{S}_E^{(1)}[1/\pi] \simeq \mathfrak{M} \otimes_{\mathfrak{S}_E, p_2} \mathfrak{S}_E^{(1)}[1/\pi];$$

in other words, $\mathfrak{M}[1/\pi]$ lifts naturally to an object in $\mathrm{Vect}^{\varphi_q}(X_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}[1/\pi])$. The rest of the arguments in [BS23, §6.4] now carry over to our setting (applied to the object $\mathcal{M}' := \mathfrak{M} \otimes_{\mathfrak{S}} \mathbb{\Delta}_{\mathcal{O}_C} \in \mathrm{Vect}^{\varphi_q}(\mathbb{\Delta}_{\mathcal{O}_C})$), finishing the proof[13]. □

[12] Keep in mind that in defining the constant F-crystal $D \otimes_{W_{\mathcal{O}_E}(k)} \mathcal{O}_{\mathbb{\Delta}}\langle \mathcal{I}_{\mathbb{\Delta}}/\pi\rangle[1/\pi]$, we use the canonical $W_{\mathcal{O}_E}(k)$-algebra structure on objects of $(\mathcal{O}_K)_{\mathbb{\Delta},\mathcal{O}_E}$; see Remark 3.4.3.

[13] We can also argue slightly differently as follows. Namely, we first check that $T(\mathfrak{M})[1/\pi] \simeq T[1/\pi]|_{G_{K_\infty}}$, whence by Lemma 3.3.14, we may pick $\mathfrak{M}$ uniquely so that $T(\mathfrak{M}) \simeq T|_{G_{K_\infty}}$; here K_∞ as usual denotes the Kummer extension $K(\pi_K^{1/q^\infty})$ of K. But as $\mathfrak{M}[1/\pi]$ lifts to an F-crystal on $\mathcal{O}_K$, its "étale realization" $T(\mathfrak{M})[1/\pi] \in \mathrm{Vect}(E)$ carries a G_K-action (extending the natural G_{K_∞}-action), which is in fact E-crystalline by exactly the same argument as in the proof of Theorem 3.4.7. By full faithfulness of the restriction functor $\mathrm{Rep}^{\mathrm{cris}}_E(G_K) \to \mathrm{Rep}_E(G_{K_\infty})$, we deduce that $T(\mathfrak{M})[1/\pi] \simeq T[1/\pi]$ G_K-equivariantly. Thus, we have arranged so that $\mathfrak{M} \otimes \mathbb{\Delta}_{\mathcal{O}_C}[1/I]^\wedge_\pi$ is stable under the G_K-action on $\mathfrak{M} \otimes \mathbb{\Delta}_{\mathcal{O}_C}[1/I]^\wedge_\pi[1/\pi]$ (coming from the descent datum α). Now we can simply follow the arguments in the fourth paragraph of [BS23, §6.4] to conclude. The difference is that we do not need to run another modification as in *loc. cit.*: the resulting prismatic F-crystal has étale realization T on the nose.

3.4.5 Relation with Kisin–Ren's theory [KR09]

In this subsection, we show that Theorem 3.4.7 encodes the classification of Galois stable lattices in E-crystalline representations in [KR09] upon specializing to a suitable prism in $(\mathcal{O}_K)_{\Delta}$, in the same way that it encodes the theory of Breuil–Kisin theory in [Kis06] by specializing to a Breuil–Kisin prism.

Let $\mathcal{G}$ be the Lubin–Tate formal $\mathcal{O}_E$-module over $\mathcal{O}_E$ corresponding to a uniformizer $\pi \in E$. Pick a coordinate X for $\mathcal{G}$, i.e. an isomorphism $G \simeq \mathrm{Spf}(\mathcal{O}_E[[X]])$. For $a \in \mathcal{O}_E$, denote by $[a] \in \mathcal{O}_E[[X]]$ the power series giving the action of a on $\mathcal{G}$. Let $K_\infty \subseteq \overline{K}$ be the subfield generated by the π-power torsion points of $\mathcal{G}$ and write $\Gamma := \mathrm{Gal}(K_\infty/K)$. The Tate module $T_p\mathcal{G}$ is a free $\mathcal{O}_E$-module of rank one, and the action of G_K on $T_p\mathcal{G}$ is given by the Lubin–Tate character $\chi : G_K \to \mathcal{O}_E^\times$.

Fix a generator $v = (v_n)_{n\geq 0}$ of $T_p\mathcal{G}$. As in [KR09], we will assume in what follows that $K \subseteq K_{0,E}K_\infty$. Fix $m \geq 1$ so that $K \subseteq K_{0,L}(v_m)$. Let $Q(u) := [\pi^m](u)/[\pi^{m-1}](u)$. As before, write $\mathfrak{S}_E := W_{\mathcal{O}_E}(k)[[u]]$; however we now equip $\mathfrak{S}_E$ with a φ_q-action and a Γ-action given respectively by $\varphi_q(u) := [\pi](u)$ and $\gamma(u) = [\chi(\gamma)](u)$ for $\gamma \in \Gamma$.

It is easy to check that the preceding φ_q-action makes the pair $(\mathfrak{S}_E, (Q(u)))$ into an $\mathcal{O}_E$-prism. Moreover, as the map $\mathfrak{S}_E \twoheadrightarrow \mathcal{O}_{K_{0,E}(v_m)}, u \mapsto v_m$ is surjective with kernel $(Q(u))$, our assumption on m gives a map $\mathcal{O}_K \to \mathfrak{S}_E/(Q(u))$, making $(\mathfrak{S}_E, (Q(u))$ into an Γ-equivariant object of $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$, which we will denote by $\mathfrak{S}'_E$ to distinguish with the Breuil–Kisin prism $\mathfrak{S}_E$ introduced earlier. Note also that mapping $u \mapsto \varphi_q^{-(m-1)}([v]_{\mathcal{G}})$ defines a G_K-equivariant map $\mathfrak{S}'_E \to \Delta_{\mathcal{O}_C}$ in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$. In particular, $\mathfrak{S}'_E$ again gives a cover of the final object in the associated topos.

Definition 3.4.17. Let $\mathrm{Vect}^{\varphi_q,\Gamma}(\mathfrak{S}'_E)$ denote the category of $\mathfrak{M} \in \mathrm{Vect}^{\varphi_q}(\mathfrak{S}'_E)$ equipped with a semilinear action of Γ which commutes with φ_q, and such that Γ acts trivially on $\mathfrak{M}/u\mathfrak{M}$. Inside this, we have a full subcategory $\mathrm{Vect}^{\varphi_q,\Gamma,\mathrm{an}}(\mathfrak{S}'_E)$ consisting of objects for which the Γ-action is analytic in a suitable sense; see [KR09, §(2.1.3)] and [KR09, §(2.4.3)].

Proposition 3.4.18. *Consider the functor*

$$D_{\mathfrak{S}'_E} : \mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K) \to \mathrm{Vect}^{\varphi_q,\Gamma}(\mathfrak{S}'_E)$$

defined by composing the inverse of the equivalence in Theorem 3.4.7 with the evaluation at $(\mathfrak{S}'_E, (Q(u))) \in (\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$. Then $D_{\mathfrak{S}'_E}$ is fully faithful with essential image $\mathrm{Vect}^{\varphi_q,\Gamma,\mathrm{an}}(\mathfrak{S}'_E)$.

Proof. We first check that $D_{\mathfrak{S}'_E}$ is well-defined, i.e. given any object $\mathcal{E} \in \mathrm{Vect}^{\varphi_q}((\mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$, the value $\mathcal{E}(\mathfrak{S}'_E)$ is naturally an object in $\mathrm{Vect}^{\varphi_q,\Gamma}(\mathfrak{S}'_E)$. As $\mathfrak{S}'_E$ is a Γ-equivariant object in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$, $\mathfrak{M} := \mathcal{E}(\mathfrak{S}'_E)$ carries a natural Γ-action. Now the map $\mathfrak{S}'_E \to \Delta_{\mathcal{O}_C}$ induces by reducing modulo u a G_K-equivariant map

$$(W_{\mathcal{O}_E}(k), (\pi)) \to (W_{\mathcal{O}_E}(\overline{k}), (\pi))$$

in $(\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$; as $W_{\mathcal{O}_E}(k)$ is fixed by the natural G_K-action on $W_{\mathcal{O}_E}(\overline{k})$, the crystal property of $\mathcal{E}$ again implies that Γ acts trivially on $\mathcal{E}(W_{\mathcal{O}_E}(k)) \simeq \mathfrak{M}/u\mathfrak{M}$, as wanted.

Next, we show that $D_{\mathfrak{S}'_E}$ lands in the subcategory of analytic objects. By the very definition of the latter ([KR09, §(2.4.3)]), it suffices to prove the analogous statement for the composition

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K)[1/\pi] \overset{D_{\mathfrak{S}'_E}[1/\pi]}{\to} \mathrm{Vect}^{\varphi_q,\Gamma}(\mathfrak{S}'_E)[1/\pi] \to \mathrm{Vect}^{\varphi_q,\Gamma}(\mathfrak{S}'_E\langle I/\pi\rangle[1/\pi]) \simeq \mathrm{Vect}^{\varphi_q,\Gamma}(\mathcal{O}'),$$

where $\mathcal{O}'$ again denotes the ring of functions on the rigid open unit disk over $K_{0,E}$, but now equipped with the Frobenius $u \mapsto [\pi](u)$. By unwinding definitions, this coincides (upon identifying $\mathrm{Rep}^{\mathrm{cris}}_E(G_K) \simeq \mathrm{MF}^{\varphi_q,w.a}(K)$) with the functor

$$\mathcal{M}'(\cdot) : \mathrm{MF}^{\varphi_q,w.a.}(K) \to \mathrm{Vect}^{\varphi_q,\Gamma}(\mathcal{O}')$$

from [KR09, §2.2] (cf. Remark 3.3.6); in particular, we know from Lemma (2.2.1) of *loc. cit.* that it indeed factors through the subcategory of analytic objects.

Consider now the composition

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K) \overset{D_{\mathfrak{S}'_E}}{\to} \mathrm{Vect}^{\varphi_q,\Gamma,\mathrm{an}}(\mathfrak{S}'_E) \overset{T}{\to} \mathrm{Rep}_{\mathcal{O}_E}(G_K),$$

where T again denotes the étale realization functor $\mathfrak{M} \mapsto (\mathfrak{M} \otimes_{\mathfrak{S}'_E} W(\mathbf{C}^\flat))^{\varphi_q=1}$. Unwinding again the construction of $D_{\mathfrak{S}'_E}$, we see that this is nothing but the forgetful functor. Moreover, by [KR09, Corollary (3.3.8)], T defines an equivalence onto the subcategory $\mathrm{Rep}^{\mathrm{cris}}_{\mathcal{O}_E}(G_K)$. It follows that $D_{\mathfrak{S}'_E}$ is also an equivalence, as wanted. □

3.4.6 Relation with π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$

In this subsection, we combine Theorem 3.4.7 with a key result on *minuscule* prismatic F-crystals from [AL23] to deduce a classification result for π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ (Theorem 3.4.24).

Definition 3.4.19 (Minuscule Breuil–Kisin modules)**.** Let (A, I) be an $\mathcal{O}_E$-prism. An object $M \in \mathrm{Vect}^{\varphi_q}(A, I)$ is called minuscule (or effective of height 1) if φ_M is induced by a map $\varphi_q^* M \to M$ with cokernel killed by I. An object $\mathcal{E} \in \mathrm{Vect}^{\varphi_q}(X_{\Delta}, \mathcal{O}_{\Delta})$ is called minusucle if for all $(A, I) \in X_{\Delta}$, the value $\mathcal{E}(A)$ is minuscule. Following [AL23], we denote the resulting categories by $\mathrm{BK}_{\min}(A, I)$ and $\mathrm{DM}(X)$, respectively.

Proposition 3.4.20 ([AL23],[Ito23])**.** *Fix a Breuil–Kisin prism $(\mathfrak{S}_E, I) \in (\mathcal{O}_K)_{\Delta,\mathcal{O}_E}$. Then evaluation at $\mathfrak{S}_E$ defines an equivalence*

$$\mathrm{DM}(\mathcal{O}_K) \simeq \mathrm{BK}_{\min}(\mathfrak{S}_E).$$

Proof. This is proved in [AL23, Theorem 5.12] in case $E = \mathbf{Q}_p$ (and for a more general class of rings in place of $\mathcal{O}_K$). The case of general E is then proved in [Ito23, Proposition 7.1.1]. □

Theorem 3.4.21 (Fontaine, Kisin, Raynaud, Tate)**.** *Sending a p-divisible group to its p-adic Tate module defines an equivalence*

$$\mathrm{BT}(\mathcal{O}_K) \simeq \mathrm{Rep}^{\mathrm{cris},\{0,1\}}_{\mathbf{Z}_p}(G_K)$$
$$G \mapsto T_p(G);$$

here the source denotes the category of p-divisible groups over $\mathcal{O}_K$, and the target denotes the category of lattices in crystalline G_K-representations with Hodge–Tate weights in $\{0, 1\}$.

Proof. This is well-known; see e.g. [Liu13, Theorem 2.2.1]. □

Definition 3.4.22 (cf. [Fal02])**.** Let R be an $\mathcal{O}_E$-algebra. A π-divisible $\mathcal{O}_E$-module over R is a p-divisible group G over R together with an action $\mathcal{O}_E \to \mathrm{End}(G)$, which is strict in the sense that the induced action of $\mathcal{O}_E$ on $\mathrm{Lie}(G)$ agrees with the action through the structure map $\mathcal{O}_E \to R$.

Lemma 3.4.23. *Let G be a p-divisible group over $\mathcal{O}_K$. Then an action $\mathcal{O}_E \to \mathrm{End}(G)$ makes G into a π-divisible $\mathcal{O}_E$-module over $\mathcal{O}_K$ if and only if the E-representation $V_p(G)$ is E-crystalline in the sense of Definition 3.2.1.*

Proof. This follows from the Hodge–Tate decomposition for G:

$$V_p(G) \otimes_{\mathbf{Q}_p} C \simeq (\mathrm{Lie}(G^*)^* \otimes_{\mathcal{O}_K} C) \oplus (\mathrm{Lie}(G) \otimes_{\mathcal{O}_K} C(1)). \tag{3.4.23.1}$$

More precisely, as $E \otimes_{\mathbf{Q}_p} C \simeq \prod_{\tau : E \hookrightarrow C} C$, an $(E \otimes_{\mathbf{Q}_p} C)$-module V always decomposes uniquely as $V = \oplus_\tau V_\tau$. As 3.4.23.1 is functorial in G, it is $(E \otimes_{\mathbf{Q}_p} \mathbf{C})$-linear, and hence must respect the corresponding decompositions of both sides. In particular,

$$\bigoplus_{\tau \neq \tau_0} (V_p(G) \otimes_{E,\tau} C) \simeq \bigoplus_{\tau \neq \tau_0} (\mathrm{Lie}(G^*)^* \otimes_{\mathcal{O}_K} C)_\tau \oplus \bigoplus_{\tau \neq \tau_0} (\mathrm{Lie}(G) \otimes_{\mathcal{O}_K} C(1))_\tau.$$

As the first summand on the right is clearly trivial as a C-representation, it remains to observe that the $\mathcal{O}_E$-action on $\mathrm{Lie}(G)$ is strict precisely when the second summand vanishes. □

Combining Theorem 3.4.7, Proposition 3.4.20, Theorem 3.4.21, and Lemma 3.4.23, we obtain the following classification of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ (including the case $p = 2$).

Theorem 3.4.24 (cf. [Che18, Theorem 1.1], [CL16, Theorem 1.0.3])**.** *There is a natural equivalence between the category of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ and the category of minuscule Breuil–Kisin modules over $\mathfrak{S}_E$.*

Remark 3.4.25. In [CL16], the authors have obtained (by a different approach) a similar equivalence even for a large class of Frobenius lifts. In their result, the relevant category of p-divisible groups over $\mathcal{O}_K$ is formed by those which come equipped with an action of $\mathcal{O}_E$ *for which* the rational Tate module is an E-crystalline representation. However, as far as we understand, the fact that this is in fact identified with the category of π-divisible $\mathcal{O}_E$-modules over $\mathcal{O}_K$ (Lemma 3.4.23) was not observed by them.

Appendices

3.A E-crystalline representations and filtered isocrystals

In this appendix, we prove Theorem 3.2.4, thereby giving equivalent characterizations for the category of E-crystalline Galois representations. We will follow closely [FF18, Chapitre 10], which treats the case $E = \mathbf{Q}_p$.

3.A.1 Recap on the Fargues–Fontaine curve

Let X_E be the Fargues–Fontaine curve associated to E and the perfectoid $\mathbf{F}_q$-algebra $F := C^\flat$, where the $\mathbf{F}_q$-algebra structure on F is defined using the fixed inclusion $\tau_0 : E \hookrightarrow K \subseteq C$. As G_K acts naturally on C (hence on $F = C^\flat$), we obtain an induced E-linear action of G_K on X_E. Recall that we also have a canonical identification $X_E = X_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} E$ (cf. [FF18, Théorème 6.5.2 (2)]). Under this identification, $g \in G_K$ acts on X_E as $g \otimes \mathrm{id}_E$.

Let $\pi : X_E \to X_{\mathbf{Q}_p}$ be the projection; this is a G_K-equivariant finite étale covering of degree $[E : \mathbf{Q}_p]$. Let $\infty \in |X_{\mathbf{Q}_p}|$ be the distinguished closed point corresponding to the tautological untilt $\mathbf{Q}_p \hookrightarrow C$ of F. Recall that ∞ is fixed by G_K, and in fact the unique closed point of $X_{\mathbf{Q}_p}$ whose G_K-orbit is finite (see [FF18, Proposition 10.1.1]). For each $\tau : E \hookrightarrow C$, let ∞_τ be the closed point in X_E corresponding to the (Frobenius isomorphism class of the) untilt $\tau : E \hookrightarrow C$. Concretely, ∞_τ is given by the (image of the) closed immersion $\mathrm{Spec}(C) \xrightarrow{(\infty,\tau)} X_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} E = X_E$.

Lemma 3.A.1. *The assignment $\tau \mapsto \infty_\tau$ defines a G_K-equivariant bijection $\mathrm{Hom}_{\mathbf{Q}_p}(E, C) \simeq \pi^{-1}(\infty)$*

Proof. From the previous description of ∞_τ, we see easily that the map is G_K-equivariant. As π is finite étale of degree $[E : \mathbf{Q}_p]$ and $\infty \in X_{\mathbf{Q}_p}$ is a closed point with algebraically closed residue field (namely C), $\pi^{-1}(\infty) \subseteq |X_E|$ is a finite set of $[E : \mathbf{Q}_p]$ closed points. In particular, it suffices to show that the map is surjective. Indeed, a point $x \in \pi^{-1}(\infty)$ necessarily has residue field $k(x) = C$, and it is clear from the construction that $x = \infty_\tau$, where $\tau : E \hookrightarrow C$ is given by the composition $\mathrm{Spec}(k(x)) \hookrightarrow X_E \to \mathrm{Spec}(E)$. □

In particular, we see that the point ∞_{τ_0} (given by the fixed embedding τ_0) is fixed by G_K.

Recall that we have a canonical identification $\widehat{(\mathcal{O}_{X_{\mathbf{Q}_p}})}_\infty \xrightarrow{\sim} B_{\mathrm{dR}}^+$, where B_{dR}^+ is the usual Fontaine's period ring constructed using C. Also, the inclusion $\overline{K} \hookrightarrow C$ lifts uniquely to a (necessarily G_K-equivariant) section $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$. Now we have a canonical identification

$$B_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} E \xrightarrow{\sim} \prod_{\tau : E \hookrightarrow \overline{K}} \widehat{(\mathcal{O}_{X_E})}_{\infty_\tau}$$

(see e.g. [Sta23, Tag 07N9]). Using the section $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ above, we can rewrite the left side as

$$B_{\mathrm{dR}}^+ \otimes_{\overline{K}} (\overline{K} \otimes_{\mathbf{Q}_p} E) = \prod_{\tau : E \hookrightarrow \overline{K}} B_{\mathrm{dR}}^+.$$

Thus, for each τ, there is an E-algebra isomorphism $\widehat{(\mathcal{O}_{X_E})}_{\infty_\tau} \xrightarrow{\sim} B_{\mathrm{dR}}^+$, where the right side is regarded as an E-algebra via the composition $\tau : E \hookrightarrow \overline{K} \hookrightarrow B_{\mathrm{dR}}^+$.

Recall that for each compact interval $I \subseteq]0,1[$, $B_{E,I}$ denotes the completion of

$$B_E^b := \{ \sum_{n \gg -\infty} [x_n]\pi^n \in W_{\mathcal{O}_E}(C^\flat) \mid (x_n) \text{ bounded} \}$$

with respect to the family of norms $|\cdot|_\rho, \rho \in I$. We then let $B_E := \varprojlim_I B_{E,I}$. Similarly using the ring $B_E^{b,+} := W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})[1/\pi]$, we can define B_E^+ and $B_{E,I}^+$. See [FF18, Chapitre I] for a more detailed discussion.

Lemma 3.A.2. *For each compact interval $I \subseteq]0,1[$,*

$$(\mathrm{Frac}(B_{E,I}))^{G_K} = K_{0,E},$$

where $K_{0,E} := K_0 \otimes_{E_0} E$. In particular, as $B_E \subseteq B_{E,I}$, we have $(\mathrm{Frac}(B_E))^{G_K} = K_{0,E}$.

Proof. For $E = \mathbf{Q}_p$, this is [FF18, Proposition 10.2.7] for $E = \mathbf{Q}_p$, but the same argument works also for general E. One can also deduce the general case from the case $E = \mathbf{Q}_p$ as follows. By Proposition 1.6.9 of *loc.cit.* (and scaling I), it suffices to show that $(\mathrm{Frac}(B_{\mathbf{Q}_p,I} \otimes_{E_0} E))^{G_K} = K_{0,E}$. But by Proposition 10.2.7 of *loc.cit.*, $\mathrm{Frac}(B_{\mathbf{Q}_p,I}) \otimes_{E_0} E$ is already a field, so we have $\mathrm{Frac}(B_{\mathbf{Q}_p,I} \otimes_{E_0} E) = \mathrm{Frac}(B_{\mathbf{Q}_p,I}) \otimes_{E_0} E$, and hence $(\mathrm{Frac}(B_{\mathbf{Q}_p,I} \otimes_{E_0} E))^{G_K} = (\mathrm{Frac}(B_{\mathbf{Q}_p,I}))^{G_K} \otimes_{E_0} E = K_0 \otimes_{E_0} E = K_{0,E}$, as wanted. □

3.A.2 Relation with filtered isocrystals

As usual, we denote by φ_q the E-linear q-Frobenius on $K_{0,E} = W_{\mathcal{O}_E}(k)[1/\pi]$.

Definition 3.A.3. Let $\mathrm{Vect}^{\varphi_q}(K_{0,E})$ be the category of φ_q-modules (or isocrystals) (D, φ_q) over $K_{0,E}$, i.e. finite dimensional $K_{0,E}$-vector spaces D equipped with a linear isomorphism $\varphi_q^* D \xrightarrow{\sim} D$.

Let $\mathrm{MF}^{\varphi_q}(K)$ be the category of filtered φ_q-modules over K, i.e. triples $(D, \varphi_q, \mathrm{Fil}^\bullet D_K)$, where $(D, \varphi_q) \in \mathrm{Vect}^{\varphi_q}(K_{0,E})$, and $\mathrm{Fil}^\bullet D_K$ is a decreasing filtration on $D_K := D \otimes_{K_{0,E}} K$.

Fix $t_E \in (B_E^+)^{\varphi_q = \pi}$ such that $V^+(t_E) = \{\infty_{\tau_0}\}$ (t_E is uniquely determined up to multiplication in $E^\times$). Let $B_{e,E} := \Gamma(X_E \setminus \{\infty_{\tau_0}\}, \mathcal{O}_{X_E}) = (B_E^+[1/t_E])^{\varphi_q = 1}$; this is a PID equipped with an action of G_K. As in [FF18, Définition 10.1.2], we let $\mathrm{Rep}_{B_{e,E}}(G_K)$ denote the category of finite free $B_{e,E}$-modules equipped with a continuous semilinear action of G_K.

Definition 3.A.4. Define functors

$$\begin{aligned} D_{\mathrm{cris},E} : \mathrm{Rep}_{B_{e,E}}(G_K) &\to \mathrm{Vect}^{\varphi_q}(K_{0,E}) \\ M &\mapsto (M \otimes_{B_{e,E}} B_E^+[1/t_E])^{G_K} \end{aligned}$$

and

$$\begin{aligned} V_{\mathrm{cris},E} : \mathrm{Vect}^{\varphi_q}(K_{0,E}) &\to \mathrm{Rep}_{B_{e,E}}(G_K) \\ (D, \varphi_q) &\mapsto (D \otimes_{K_{0,E}} B_E^+[1/t_E])^{\varphi_q=1}. \end{aligned}$$

Proposition 3.A.5 ([FF18, Proposition 10.2.12]). (1) *The functors $D_{\mathrm{cris},E}$ and $V_{\mathrm{cris},E}$ are well-defined, and form an adjoint pair with $V_{\mathrm{cris},E}$ being the left adjoint.*

(2) *$V_{\mathrm{cris},E}$ is fully faithful, i.e. the unit*

$$\mathrm{id} \xrightarrow{\sim} D_{\mathrm{cris},E} \circ V_{\mathrm{cris},E}$$

is an isomorphism.

(3) *For each M in $\mathrm{Rep}_{B_{e,E}}(G_K)$, the counit*

$$V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M)) \hookrightarrow M$$

is an injection.

Proof. The case $E = \mathbf{Q}_p$ is treated in [FF18, Proposition 10.2.12]. We begin by constructing a natural isomorphism $D_{\mathrm{cris},E} \circ V_{\mathrm{cris},E} \xrightarrow{\sim} \mathrm{id}$. Let $(D, \varphi_q) \in \mathrm{Vect}^{\varphi_q}(K_{0,E})$. We claim that the natural map

$$((D \otimes_{K_{0,E}} B_E^+[1/t_E])^{\varphi_q=1} \otimes_{B_{e,E}} B_E^+[1/t_E])^{G_K} \to D$$

is an isomorphism (which is clearly φ_q-equivariant). As $(B_E^+[1/t_E])^{G_K} = K_{0,E}$ by Lemma 3.A.2, it suffices to show that the natural map

$$(D \otimes_{K_{0,E}} B_E^+[1/t_E])^{\varphi_q=1} \otimes_{B_{e,E}} B_E^+[1/t_E] \to D \otimes_{K_{0,E}} B_E^+[1/t_E] \tag{3.A.5.1}$$

is an isomorphism. By replacing D with $D \otimes_{K_{0,E}} W_{\mathcal{O}_E}(\overline{k})[1/\pi]$, we may assume k is algebraically closed. Then by the Dieudonné–Manin theorem, we may reduce to the case $D^\vee = \mathcal{O}(d/h)$ is isoclinic (for some $(d, h) \in \mathbf{Z} \times \mathbf{Z}_{\geq 1}$ with $(d, h) = 1$). Recall that by definition $\mathcal{O}(d/h)$ admits a $K_{0,E}$-basic $x_0, \dots, x_{h-1}$ for each $\varphi_q(x_i) = x_{i+1}$ for $0 < i < h-1$, and $\varphi_q(x_{h-1}) = \pi^d x_0$. Thus,

$$(D \otimes_{K_{0,E}} B_E^+[1/t_E])^{\varphi_q=1} = \mathrm{Hom}_{\varphi_q}(D^\vee, B_E^+[1/t_E]) = (B_E^+[1/t_E])^{\varphi_q^h=\pi^d},$$

and so we are reduced to showing that the map

$$\begin{aligned} (B_E^+[1/t_E])^{\varphi_q^h=\pi^d} \otimes_{B_{e,E}} B_E^+[1/t_E] &\to (B_E^+[1/t_E])^{\oplus h} \\ x \otimes a &\mapsto (ax, a\varphi_q(x), ..., a\varphi_q^{h-1}(x)) \end{aligned}$$

is bijective. As usual let $E_h \subseteq C$ denotes the unique unramified extension of E of degree h; in particular we have $B^+_{E_h} = B^+_E$ canonically (cf. the proof of [FF18, Proposition 1.6.9]). Now let $t_{E_h} \in (B^+_E)^{\varphi_q^h=\pi}$ be such that $\{\infty_{t_{E_h}}\}$ maps into $\{\infty_t\}$ under the projection map $\pi : X_{E_h} \to X_E$ where $V^+(t_{E_h}) = \{\infty_{t_{E_h}}\}$. Then $(B^+_E[1/t_E])^{\varphi_q^h=\pi^d}$ is a free of rank one over $E_h \otimes_E B_{e,E}$ with basis $t^d_{E_h}$. As t_{E_h} is a unit in $B^+_E[1/t_E]$[14], it suffices to show that the map

$$\begin{aligned} E_h \otimes_E B^+_E[1/t_E] &\to (B^+_E[1/t_E])^{\oplus h} \\ x \otimes a &\mapsto (\varphi_q^i(a)x)_{0\le i\le h-1} \end{aligned}$$

is an isomorphism. This follows immediately from the analogue decomposition $E_h \otimes_E E_h \xrightarrow{\sim} \prod_{0\le i\le h-1} E_h$. This gives the isomorphism in part (2). Note that the argument also shows that $(D \otimes_{K_{0,E}} B^+_E[1/t_E])^{\varphi_q=1}$ is finite free over $B_{e,E}$ of rank $\dim_{K_{0,E}}(D)$; in other words, the functor $D \mapsto V_{\mathrm{cris},E}(D)$ is rank-preserving and indeed lands in $\mathrm{Rep}_{B_{e,E}}(G_K)$.

For part (3), it suffices to show that for each M in $\mathrm{Rep}_{B_{e,E}}(G_K)$, the natural map

$$V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M)) = ((M \otimes_{B_{e,E}} B^+_E[1/t_E])^{G_K} \otimes_{K_{0,E}} B^+_E[1/t_E])^{\varphi_q=1} \to M$$

is injective. As $(B^+_E[1/t_E])^{\varphi_q=1} = B_{e,E}$, we are reduced to show that the map

$$(M \otimes_{B_{e,E}} B^+_E[1/t_E])^{G_K} \otimes_{K_{0,E}} B^+_E[1/t_E] \to M \otimes_{B_{e,E}} B^+_E[1/t_E] \tag{3.A.5.2}$$

is injective. Upon replacing $B^+_E[1/t_E]$ by $\mathrm{Frac}(B^+_E[1/t_E])$, the result follows readily from the equality $(\mathrm{Frac}(B^+_E[1/t_E]))^{G_K} = K_{0,E}$ in Lemma 3.A.2. Note that the isomorphism 3.A.5.2 also shows that $\dim_{K_{0,E}} D_{\mathrm{cris,E}}(M) \le \mathrm{rank}_{B_{e,E}} M < \infty$ and that the linearization $\varphi_q^* D_{\mathrm{cris},E}(M) \to D_{\mathrm{cris},E}(M)$ is an isomorphism. Indeed, as the source and target have the same (finite) $K_{0,E}$-dimension, it suffices to show that the map is injective, which in turn can be checked after the faithfully flat extension $K_{0,E} \to B^+_E[1/t_E]$. Thus, we see that the functor $M \mapsto D_{\mathrm{cris},E}(M)$ indeed lands in $\mathrm{Vect}^{\varphi_q}(K_{0,E})$, and moreover satisfies $\dim_{K_{0,E}} D_{\mathrm{cris},E}(M) \le \mathrm{rank}_{B_{e,E}} M$. This finishes the proof of (3).

Finally, (1) follows by combining (2) and (3). □

Definition 3.A.6 ([FF18, Définition 10.2.13]). (1) A representation $M \in \mathrm{Rep}_{B_{e,E}}(G_K)$ is called crystalline if $M \cong V_{\mathrm{cris},E}(D)$ for some $(D, \varphi_q) \in \mathrm{Vect}^{\varphi_q}(K_{0,E})$. We denote by $\mathrm{Rep}^{\mathrm{cris}}_{B_{e,E}}(G_K)$ the full subcategory of crystalline objects in $\mathrm{Rep}_{B_{e,E}}(G_K)$.

(2) A G_K-equivariant vector bundle $\mathcal{E}$ on $X_E\backslash\{\infty\}$ is called crystalline if the $B_{e,E}$-representation $H^0(X_E \setminus \{\infty\}, \mathcal{O}_{X_E})$ is crystalline.

Lemma 3.A.7. *M is crystalline if and only if the $B^+_E[1/t_E]$-representation $M \otimes_{B_{e,E}} B^+_E[1/t_E]$ is trivial.*

[14] We claim that t_{E_h} divides t_E in B^+_E. As $\varphi_q^h(t) = \pi^h t$, we can use [FF18, Théorème 6.2.1] to write $t = t_1 \dots t_h$ where $t_i \in (B^+_E)^{\varphi_q^h=\pi}$ for each i. Then $\infty_{t_{E_h}} \in \pi^{-1}(\infty_t) = V^+(t_1) \cup \dots \cup V^+(t_h)$, so we must have $t_{E_h} \in E_h^\times t_i$ for some i (cf. Théorème 6.5.2 of *loc.cit.*); in particular, we have $t_{E_h} | t$, as claimed.

Proof. This is essentially contained in the proof of Proposition 3.A.5. If $M \otimes_{B_{e,E}} B_E^+[1/t_E]$ is trivial, then necessarily

$$(M \otimes_{B_{e,E}} B_E^+[1/t_E])^{G_K} \otimes_{K_{0,E}} B_E^+[1/t_E] \xrightarrow{\sim} M \otimes_{B_{e,E}} B_E^+[1/t_E]$$

whence $V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M)) \xrightarrow{\sim} M$ by taking φ_q-invariants. Conversely, if $M \cong V_{\mathrm{cris},E}(D)$, then by 3.A.5.1, we have

$$\begin{aligned} M \otimes_{B_{e,E}} B_E^+[1/t_E] &\cong V_{\mathrm{cris},E}(D,\varphi_q) \otimes_{B_{e,E}} B_E^+[1/t_E] \\ &\xleftarrow{\sim} D \otimes_{K_{0,E}} B_E^+[1/t_E] \end{aligned}$$

is indeed trivial. □

Remark 3.A.8. Recall that $B_{\mathrm{cris},E}$ denote Fontaine's crystalline period ring defined using E and $\tau_0 : E \hookrightarrow K \subseteq C$. As in the case $E = \mathbf{Q}_p$ ([FF18, Proposition 1.10.12]), one can check that $B_E^+ = \cap_{n\geq 0}\varphi_q^n(B_{\mathrm{cris},E}^+)$ (resp. $B_E^+[1/t_E] = \cap_{n\geq 0}\varphi_q^n(B_{\mathrm{cris},E})$) is the maximal subring of $B_{\mathrm{cris},E}^+$ (resp. $B_{\mathrm{cris},E}$) over which Frobenius is an automorphism. It follows that $D_{\mathrm{cris},E}(M)$ can also be computed as $(M \otimes_{B_{e,E}} B_{\mathrm{cris},E})^{G_K}$ and that M is crystalline if and only if $M \otimes_{B_{e,E}} B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-representation. Indeed, as φ_q is an automorphism on $D := (M \otimes_{B_{e,E}} B_{\mathrm{cris},E})^{G_K}$, we have $D \subseteq \cap_{n\geq 0}\varphi_q^n(M \otimes_{B_{e,E}} B_{\mathrm{cris},E}) = M \otimes_{B_{e,E}} \cap_{n\geq 0}\varphi_q^n(B_{\mathrm{cris},E}) = M \otimes_{B_{e,E}} B_E^+[1/t_E]$, whence $D = D_{\mathrm{cris},E}(M)$.

Remark 3.A.9. We have seen that M is crystalline precisely when the natural injective map $V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M)) \hookrightarrow M$ is an isomorphism. In the case $E = \mathbf{Q}_p$, it in fact suffices to require that the source and target have the same $B_{e,E}$-rank, i.e. $\dim_{K_{0,E}} D_{\mathrm{cris},E}(M) = \mathrm{rank}_{B_{e,E}} M$. Indeed, in this case, ∞ is the unique closed point in $X_{\mathbf{Q}_p}$ with finite G_K-orbit, so any G_K-equivariant coherent sheaf on $X_{\mathbf{Q}_p} \setminus \{\infty\}$ must be torsion-free (as its torsion part must have empty support), and hence a vector bundle. Thus, if $\dim_{K_{0,E}} D_{\mathrm{cris},E}(M) = \mathrm{rank}_{B_{e,E}} M$, then the torsion $B_{e,E}$-module $\mathrm{coker}(V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M)) \hookrightarrow M)$ must be zero, as claimed. On the other hand, this does not seem to be enough if $E \neq \mathbf{Q}_p$ since in general there can be more closed points in X_E with finite G_K-orbit (for instance, if K contains the Galois closure of E in $\overline{K}$, then G_K fixes ∞_τ for all $\tau : E \hookrightarrow \overline{K}$). (See, however, the proof of Proposition 3.A.13 below.) This is also related to the fact that the ring $B_E^+[1/t_E]$ is not (E, G_K)-regular (as opposed to the case $E = \mathbf{Q}_p$, cf. [FF18, Corollaire 10.2.8]): the line Et is G_K-stable yet $t \notin (B_E^+[1/t_E])^\times$.

Lemma 3.A.10. (1) *The functor* $D_{\mathrm{cris},E}$ *defines an equivalence* $\mathrm{Rep}^{\mathrm{cris}}_{B_{e,E}}(G_K) \xrightarrow{\sim} \mathrm{Vect}^{\varphi_q}(K_{0,E})$ *with quasi-inverse* $V_{\mathrm{cris},E}$.

(2) *Both* $D_{\mathrm{cris},E}$ *and* $V_{\mathrm{cris},E}$ *are exact.*

(3) *The category* $\mathrm{Rep}^{\mathrm{cris}}_{B_{e,E}}(G_K)$ *is stable under direct summand subquotients, tensor products, and duals. Moreover,* $D_{\mathrm{cris},E}$ *naturally respects these operations.*

Proof. (1) follows immediately from definition and Proposition 3.A.5.

(2) We will show that $V_{\mathrm{cris},E}$ is exact (the argument for $D_{\mathrm{cris},E}$ being analogous). Let $0 \to D_1 \to D_2 \to D_3 \to 0$ be an exact sequence in of isocrsyals over $K_{0,E}$. As the inclusion

$B_{e,E} \hookrightarrow B_E^+[1/t_E]$ is faithfully flat[15], it suffices to show that the induced sequence after applying $V_{\mathrm{cris},E}(\cdot) \otimes_{B_{e,E}} B_E^+[1/t_E]$ is exact. We are now done because this functor is naturally identified with $(\cdot) \otimes_{K_{0,E}} B_E^+[1/t_E]$.

(3) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence in $\mathrm{Rep}_{B_{e,E}}(G_K)$ with M_2 crystalline. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M_1)) & \longrightarrow & V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M_2)) & \longrightarrow & V_{\mathrm{cris},E}(D_{\mathrm{cris},E}(M_3)) & & \\ & & \downarrow & & \downarrow \simeq & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

As the middle vertical arrow is an isomorphism, a simple diagram chasing shows that the two outer maps are also isomorphisms, i.e. M_1 and M_2 are crystalline, as wanted.

The other claims can be proved e.g. using Lemma 3.A.7. □

Recall that we have a natural functor $(D, \varphi_q) \mapsto \mathcal{E}(D, \varphi_q)$ from $\mathrm{Vect}^{\varphi_q}(K_{0,E})$ to the category of G_K-equivariant vector bundles on X_E, where $\mathcal{E}(D, \varphi_q)$ is the $\mathcal{O}_{X_E}$-module associated to the graded module

$$\bigoplus_{n \geq 0} (D \otimes_{K_{0,E}} B_E^+)^{\varphi_q = \pi^n}.$$

By Beauville–Laszlo glueing (applied to the locus $\infty_{\tau_0} \hookrightarrow X_E$), the datum of a G_K-equivariant vector bundle on X_E is equivalent to the data of a triple $(M_e, M_{\mathrm{dR}}^+, u)$ where $M_e \in \mathrm{Rep}_{B_{e,E}}(G_K)$, $M_{\mathrm{dR}}^+ \in \mathrm{Rep}_{B_{\mathrm{dR}}^+}(G_K)$, and u is a G_K-equivariant isomorphism $M_e \otimes B_{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}}^+[1/t_E]$. In terms of this description, $\mathcal{E}(D, \varphi_q)$ corresponds to the triple $(V_{\mathrm{cris},E}(D), D_K \otimes_K B_{\mathrm{dR}}^+, \iota)$ (with $D_K := D \otimes_{K_{0,E}} K$, and ι being the natural isomorphism). In particular, by definition, a G_K-equivariant vector bundle $\mathcal{E}$ on X_E is crystalline if and only if there exists (D, φ_q) so that there is a G_K-equivariant isomorphism $\mathcal{E}|_{X_E \setminus \{\infty_{\tau_0}\}} \cong \mathcal{E}(D, \varphi_q)|_{X_E \setminus \{\infty_{\tau_0}\}}$.

Lemma 3.A.11. *Let V be a continuous semilinear representation of G_K on a finite free B_{dR}^+-module. Then V is trivial if and only if $V \otimes_{B_{\mathrm{dR}}^+} C$ is trivial as a C-semilinear representation of G_K.*

Proof. See [Du19, Proposition 2.18 (1)]. (The proof of *loc. cit.* uses the usual B_{dR}^+ (and the cyclotomic period t), but we have seen that the natural map $\widehat{(\mathcal{O}_{X_{\mathbf{Q}}})}_\infty \to \widehat{(\mathcal{O}_{X_E})}_{\tau_0}$ is a G_K-equivariant isomorphism.) □

Lemma 3.A.12. *Let $V \in \mathrm{Rep}_E(G_K)$ be an E-representation of G_K. Then V is crystalline if and only if $\dim_{K_{0,E}}(V \otimes_{E_0} B_{\mathrm{cris}})^{G_K} = \dim_E V$.*

[15] As $B_{e,E}$ is a PID and $B_E^+[1/t_E]$ is a domain, the map is flat. It remains to show that $\mathfrak{m}B_E^+[1/t_E] \neq (1)$ for each $\mathfrak{m} \in \mathrm{Max}(B_{e,E})$. By [FF18, Théorème 6.5.2], such $\mathfrak{m}$ is generated by t'/t_E for some $t' \in (B_E^+)^{\varphi_q=\pi} \setminus Et_E$. Now t'/t is a not a unit in $B_E^+[1/t_E]$ as otherwise it would be already a unit in $(B_E^+[1/t_E])^{\varphi_q=1} = B_{e,E}$.

Proof. We have

$$D_{\text{cris}}(V) = (V \otimes_{\mathbf{Q}_p} B_{\text{cris}})^{G_K} = (V \otimes_{E_0} E_0 \otimes_{\mathbf{Q}_p} B_{\text{cris}})^{G_K}$$
$$= \bigoplus_{0 \le i \le f-1} (V \otimes_{E_0, \varphi_p^i} B_{\text{cris}})^{G_K},$$

For each i, we have $\dim_{E \otimes_{E_0, \varphi_p^i} K_0} (V \otimes_{E_0, \sigma_p^i} B_{\text{cris}})^{G_K} \le \dim_E V$ with equality if and only if the $E \otimes_{E_0, \varphi_p^i} B_{\text{cris}}$-representation $V \otimes_{E_0, \varphi_p^i} B_{\text{cris}}$ is trivial[16]. As the latter can be obtained from $V \otimes_{E_0} B_{\text{cris}}$ by extending scalars along the (G_K-equivariant) map $\varphi_p^i : B_{\text{cris}} \to B_{\text{cris}}$, it in fact suffices to require that $V \otimes_{E_0} B_{\text{cris}}$ is trivial. The lemma now follows by counting dimensions. □

We can now give a geometric interpretation of the notion of E-crystalline representations of Kisin–Ren in terms of vector bundles on the Fargues–Fontaine curve.

Proposition 3.A.13. *Let $V \in \mathrm{Rep}_E(G_K)$. Then the following are equivalent:*

(1) *The G_K-equivariant vector bundle $V \otimes_E \mathcal{O}_X$ is crystalline in the sense of Definition 3.A.6.*

(2) *V is E-crystalline.*

Example 3.A.14. Let $V = E(1)$ denotes the E-representation of G_K given by the Lubin–Tate character $\chi_{\text{LT}} : G_K \to \mathcal{O}_E^\times$ associated to the uniformizer π (and the embedding $\tau_0 : E \hookrightarrow K$). Then V satisfies condition (1) of Proposition 3.A.13. Indeed, $V \otimes_E B_E^+[1/t_E]$ has a G_K-invariant basis given by $v \otimes t_E^{-1}$ where v is an E-basis in V. More generally, this holds for the p-adic Tate module of any π-divisible $\mathcal{O}_E$-module over $\mathcal{O}_K$; see Lemma 3.4.23 (the previous example being the case of the Lubin–Tate formal $\mathcal{O}_E$-module associated to π).

Proof of Proposition 3.A.13. Assume (1). By Lemma 3.A.7, $V \otimes_E B_E^+[1/t_E]$ is trivial as a $B_E^+[1/t_E]$-representation of G_K. As t_E is invertible in $\widehat{(X_E)}_{\infty_\tau}$ for each $\tau \neq \tau_0$ (as $V^+(t_E) = \infty_{\tau_0}$), by extending scalars, $\bigoplus_{\tau \neq \tau_0} V \otimes_{E,\tau} B_{\text{dR}}^+$ is also trivial (as a B_{dR}^+-representation), whence the same is true for $\oplus_{\tau \neq \tau_0} V \otimes_{E,\tau} C$.

Moreover, by extending scalars along $B_E^+[1/t_E] \hookrightarrow B_{\text{cris}} \otimes_{E_0} E$[17], $V \otimes_{E_0} B_{\text{cris}}$ is also trivial as a $B_{\text{cris}} \otimes_{E_0} E$-representation, and hence taking G_K-invariants yields $\dim_{K_{0,E}} (V \otimes_{E_0} B_{\text{cris}})^{G_K} = \dim_E V$. By Lemma 3.A.12, V is crystalline, as wanted.

Conversely, assume (2) holds. Let $D := D_{\text{cris},E}(V) = (V \otimes_E B_E^+[1/t_E])^{G_K}$. We need to show that the natural inclusion

$$V_{\text{cris},E}(D) = (D \otimes_{K_{0,E}} B_E^+[1/t_E])^{\varphi_q = 1} \hookrightarrow V \otimes_E B_{e,E} \tag{3.A.14.1}$$

is an isomorphism. We first show that the source and target have the same rank, i.e. $\dim_E V = \dim D$. As $B_E^+[1/t_E] \subseteq B_{\text{cris}} \otimes_{E_0} E$, we always have $D \subseteq (V \otimes_{E_0} B_{\text{cris}})^{G_K}$, and the latter has

[16] This follows from the usual property of admissible representations, and the fact that $E \otimes_{E_0, \varphi_p^i} B_{\text{cris}}$ is (E, G_K)-regular, which in turn can be proved in exactly the same way as in the case $E = \mathbf{Q}_p$.

[17] This inclusion is defined as follows. By [Col02, Lem. 9.17], t_E divides t in $B_{\max}^+ \otimes_{E_0} E$, and so we have $B_E^+[1/t_E] = (B_{\mathbf{Q}_p}^+ \otimes_{E_0} E)[1/t_E] \subseteq B_{\max}^+[1/t] \otimes_{E_0} E = B_{\max} \otimes_{E_0} E$. As $\varphi_p(B_{\max}) \subseteq B_{\text{cris}}$ and φ_q is an automorphism on $B_E^+[1/t_E]$, we also have $B_E^+[1/t_E] \subseteq B_{\text{cris}} \otimes_{E_0} E$, as wanted.

dimension $\dim_E V$ by Lemma 3.A.12. We will show that $D = (V \otimes_{E_0} B_{\mathrm{cris}})^{G_K}$. As $\mathrm{Rep}^{\mathrm{cris}}_{B_{e,E}}(G_K)$ is stable under tensor products by Lemma 3.A.10, by replacing V with $V \otimes_E E(n)$ for $n \ll 0$, we may assume that the Hodge–Tate weights of V are all non-negative. In particular, we have

$$(V \otimes_{E_0} B_{\mathrm{cris}})^{G_K} = (V \otimes_{E_0} B^+_{\mathrm{cris}})^{G_K}.$$

Moreover, as φ_q is an automorphism on $(V \otimes_{E_0} B^+_{\mathrm{cris}})^{G_K}$, we deduce that

$$\begin{aligned}(V \otimes_{E_0} B^+_{\mathrm{cris}})^{G_K} &= (V \otimes_{E_0} \cap_{n \geq 0} \varphi_q^n(B^+_{\mathrm{cris}}))^{G_K} \\ &= (V \otimes_{E_0} B^+_{\mathbf{Q}_p})^{G_K} \\ &= (V \otimes_E (E \otimes_{E_0} B^+_{\mathbf{Q}_p}))^{G_K} \\ &= (V \otimes_E B^+_E)^{G_K} \subseteq D,\end{aligned}$$

as wanted. (For the second equality, see e.g. [FF18, §1.10].)

Next, we claim that the induced map

$$\bigoplus_{\tau \neq \tau_0} D \otimes_{K_{0,E},\tau} B^+_{\mathrm{dR}} \hookrightarrow \bigoplus_{\tau \neq \tau_0} V \otimes_{E,\tau} B^+_{\mathrm{dR}} \qquad (3.A.14.2)$$

on completed stalks is an isomorphism. As V is E-crystalline, both the source and target are trivial as a B^+_{dR}-representation by Lemma 3.A.11. As any such representation W satisfies $W = (W[1/t])^{G_K} \otimes_K B^+_{\mathrm{dR}}$, it suffices to observe that 3.A.14.2 becomes an isomorphism after taking $\otimes B_{\mathrm{dR}}$ (being an injection between B_{dR}-vector spaces of the same (finite) dimension).

Thus, the cokernel of 3.A.14.1 is a torsion equivariant coherent sheaf $\mathcal{F}$ on $X_E \setminus \{\infty_{\tau_0}\}$ satisfying $\widehat{\mathcal{F}}_{\infty_\tau} = 0$ for all $\tau \neq \tau_0$. As $\{\infty_\tau\}_\tau = \pi^{-1}(\infty)$ is precisely the set of closed points in X_E with finite G_K-orbit, $\mathcal{F}$ must be supported on the set $\{\infty_\tau\}_{\tau \neq \tau_0}$, and hence must be zero, as claimed. □

Remark 3.A.15. Combining with Remark 3.A.8, we see that an object $V \in \mathrm{Rep}_E(G_K)$ is E-crystalline if and only if $V \otimes_E B_{\mathrm{cris},E}$ is trivial as a $B_{\mathrm{cris},E}$-representation. Thus, the notion of E-crystalline representations is in some sense indeed a natural extension of the usual notion for $\mathbf{Q}_p$-representations.

Lemma 3.A.16. *Let V be a finite dimensional K-vector space. Then the association $\mathrm{Fil}^\bullet V \mapsto \mathrm{Fil}^0(V \otimes_K B_{\mathrm{dR}})$ gives a bijection between the set of (finite, separated, exhausted) descreasing filtrations on V, and the set of G_K-equivariant B^+_{dR}-lattices in $V \otimes_K B_{\mathrm{dR}}$. The inverse bijection is given by $W \mapsto (t_E^\bullet W)^{G_K}$.*

Proof. See [FF18, Proposition 10.4.3]. □

Combining the above lemma with Beauville–Lazlo's glueing theorem, we deduce the following result.

Lemma 3.A.17. *The functor*

$$\begin{aligned} \mathrm{MF}^{\varphi_q}(K) &\xrightarrow{\sim} \mathrm{Fib}^{G_K,\mathrm{cris}}(X_E) \\ (D, \varphi_q, \mathrm{Fil}^\bullet D_K) &\mapsto \mathcal{E}(D, \varphi_q, \mathrm{Fil}^\bullet D_K) \end{aligned}$$

defines an equivalence onto the category of crystalline G_K-equivariant vector bundles on X_E. Here $\mathcal{E}(D, \varphi_q, \mathrm{Fil}^\bullet D_K)$ is the modification of $\mathcal{E}(D, \varphi_q)$ at ∞_{τ_0}, defined using the G_K-stable B_{dR}^+-lattice $\mathrm{Fil}^\bullet(D_K \otimes_K B_{\mathrm{dR}})$.

3.A.3 "Weakly admissible implies admissible"

In this subsection, we finish the proof that E-crystalline representations are equivalent to weakly admissible filtered isocrystals over K (Theorem 3.A.19).

We begin by recalling the notion of weak admissibility for filtered φ_q-modules. Namely, for a 1-dimensional object D in $\mathrm{MF}^{\varphi_q}(K)$, we pick a basis vector $v \in D$ and let $t_N(D) := v_\pi(\alpha)$ where $\alpha \in (K_{0,E})^\times$ is such that $v_q(v) = \alpha v$. We let $t_H(D_K)$ (or more precisely, $t_H(\mathrm{Fil}^\bullet D_K)$) be the unique integer $i \in \mathbf{Z}$ such that $\mathrm{Fil}^i D_K = D_K$ and $\mathrm{Fil}^{i+1} D_K = 0$. For a general D, we define $t_H(D_K) := t_H(\det(D_K))$ and $t_N(D) := t_N(\det(D))$. We say that an object D in $\mathrm{MF}^{\varphi_q}(K)$ is weakly admissible if $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all subobjects $D' \subseteq D$. As in the case $E = \mathbf{Q}_p$, the degree function and the rank function

$$\begin{aligned} \deg : (D, \varphi_q, \mathrm{Fil}^\bullet D_K) &\mapsto t_H(D_K) - t_N(D, \varphi_q), \\ \mathrm{rank} : (D, \varphi_q, \mathrm{Fil}^\bullet D_K) &\mapsto \mathrm{rank}(D, \varphi_q) \end{aligned}$$

make $\mathrm{MF}^{\varphi_q}(K)$ into a slope category with slope function $\mu := \deg/\mathrm{rank}$. In particular, each object in $\mathrm{MF}^{\varphi_q}(K)$ admits a unique Harder–Narasimhan filtration, and the resulting abelian subcategory of semistable objects of slope 0 is precisely formed by those weakly admissible objects in the preceding sense.

For ease of notation, in what follows we will simply write D for a filtered φ_q-module over K, and $\mathcal{E}(D)$ for $\mathcal{E}(D, \varphi_q, \mathrm{Fil}^\bullet)$.

Proposition 3.A.18 ([FF18, Proposition 10.5.6])**.** *Let D be a filtered φ_q-module over K.*

(1) *We have* $\mathrm{rank}(D) = \mathrm{rank}(\mathcal{E}(D)), \deg(D) = \deg(\mathcal{E}(D))$*, and* $\mu(D) = \mu(\mathcal{E}(D))$.

(2) *If $0 = D_0 \subsetneq \ldots \subsetneq D_r = D$ is the Harder–Narasimhan filtration of D, then that of $\mathcal{E}(D, \varphi_q, \mathrm{Fil}^\bullet D_K)$ is given by*

$$0 = \mathcal{E}(D_0) \subsetneq \ldots \subsetneq \mathcal{E}(D_r) = \mathcal{E}(D).$$

In particular, D is weakly admissible if and only if $\mathcal{E}(D)$ is semistable of slope 0.

Proof. (1) We have seen in the proof of Proposition 3.A.5 that the functor $(D, \varphi_q) \mapsto V_{\mathrm{cris}}(D, \varphi_q)$ is rank-preserving. Thus $\mathrm{rank}(\mathcal{E}(D)) = \mathrm{rank}(\mathcal{E}(D, \varphi_q)) = \mathrm{rank}(D)$. It remains to show $\det(D) =$

$\deg(\mathcal{E}(D))$. As $\mathcal{E}(D)$ is defined as the modification of $\mathcal{E}(D,\varphi_q)$ at ∞_{τ_0} using the lattice $\mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})$, we have

$$\deg(\mathcal{E}(D)) = \deg(\mathcal{E}(D,\varphi_q)) - [D_K \otimes_K B_{\mathrm{dR}}^+ : \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})]^{18}$$

Now $\deg(\mathcal{E}(D,\varphi_q)) = -t_N(D,\varphi_q)$ (recall that if $(D,\varphi_q) = \mathcal{O}(\lambda)$, then $\mathcal{E}(D,\varphi_q) = \mathcal{O}(-\lambda)$), while it follows easily by choosing a splitting of the filtration on D_K that

$$[D_K \otimes_K B_{\mathrm{dR}}^+ : \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})] = -t_H(\mathrm{Fil}^\bullet D_K),$$

as desired.

(2) We follow the proof of [FF18, Proposition 10.5.6]. Observe firstly that by uniqueness, the Harder–Narasimhan filtration of $\mathcal{E}(D)$ is G_K-equivariant. Thus, by part (1) and definition of semistability, it suffices to show that if $\mathcal{E}' \subseteq \mathcal{E}(D)$ is a G_K-stable subbundle, then $\mathcal{E}' = \mathcal{E}(D')$ for a (necessarily unique) subobject $D' \subseteq D$ (as a filtered φ_q-module). As $\mathcal{E}'$ is a subbundle, $\mathcal{E}'|_{X_E \setminus \{\infty_{\tau_0}\}}$ is in particular is a Galois stable direct summand of $V_{\mathrm{cris},E}(D)$, and hence crystalline by Lemma 3.A.10. Thus, $\mathcal{E}' = \mathcal{E}(D',\varphi_q)$ on $X_E \setminus \{\infty_{\tau_0}\}$ for some φ_q-module $D' \subseteq D$. The lattice $\widehat{(\mathcal{E}')}_{\infty_{\tau_0}}$ determines a filtration on D'_K, which we claim is simply the one inherited from D_K. Indeed, if $\mathcal{E}''$ denotes the equivariant vector bundle given by this latter filtration, then it follows from the (explicit) bijection in Lemma 3.A.16 that $\mathcal{E}' \subseteq \mathcal{E}''$. As $\mathcal{E}'$ is a subbundle in $\mathcal{E}$ (hence in $\mathcal{E}''$) and $\mathrm{rank}(\mathcal{E}') = \mathrm{rank}(\mathcal{E}'') = \dim D'$, we must have $\mathcal{E}' = \mathcal{E}''$. Thus, we see that $\mathcal{E}' = \mathcal{E}(D')$ for a subobject $D' \subseteq D$, as claimed. ☐

After the classification of vector bundles ([FF18, Théorème 8.2.10]), any vector bundle $\mathcal{E}$ on X_E is of the form

$$\mathcal{E} \simeq \mathcal{O}(\lambda_1) \oplus \ldots \oplus \mathcal{O}(\lambda_n)$$

for a unique tuple $(\lambda_1 \geq \ldots \geq \lambda_n)$ of rational numbers. As

$$\dim_E H^0(X, \mathcal{O}(\lambda)) = \begin{cases} 0 & \text{if } \lambda < 0, \\ 1 & \text{if } \lambda = 0, \\ \infty & \text{if } \lambda > 0, \end{cases}$$

$\mathcal{E}$ is semistable of slope 0 if and only if $\dim_E H^0(X_E, \mathcal{E}) = \mathrm{rank}(\mathcal{E})$. Combining with Proposition 3.A.18, we see that a filtered φ_q-module D over K is weakly admissible if and only if $\dim_{K_{0,E}} D = \dim_E V_E(D)$, where $V_E(D) := H^0(X_E, \mathcal{E}(D)) = (D \otimes_{K_{0,E}} B_E^+[1/t_E])^{\varphi_q = 1} \cap \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})$.

Motivated by the case $E = \mathbf{Q}_p$, we next proceed to show that the functor $D \mapsto V_E(D)$ defines an equivalence between the category of weakly admissible filtered φ_q-modules over K, and the category of E-crystalline representations of G_K.

Let $V \in \mathrm{Rep}_E(G_K)$. Define

$$D_{\mathrm{cris},E}(V) := (V \otimes_E B_E^+[1/t_E])^{G_K}.$$

[18] For an effective modification $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0$ (so that $\mathcal{F}$ is a skyscraper sheaf, supported at ∞_{τ_0}), this follows from additivity of degree: $\deg(\mathcal{E}) = \deg(E') + \mathrm{length}(\mathcal{F}) = \deg(\mathcal{E}) + [\widehat{(\mathcal{E})}_{\infty_{\tau_0}} : \widehat{(\mathcal{E}')}_{\infty_{\tau_0}}]$. In general, we can choose $n \gg 0$ so that $\widehat{(\mathcal{E}')}_{\infty_{\tau_0}} \subseteq t_E^{-n} \widehat{(\mathcal{E})}_{\infty_{\tau_0}}$, and hence reduce to effective case.

Of course, this is nothing but $D_{\mathrm{cris},E}(M)$ where $M := V \otimes_E B_{e,E}$. In particular, we have seen that $D_{\mathrm{cris},E}$ is naturally a φ_q-module over $K_{0,E}$, of dimension $\leq \dim_E V$. Via the natural inclusion $B_E^+[1/t_E] \otimes_{K_{0,E}} K \hookrightarrow B_{\mathrm{dR}}$, we can endow

$$D_{\mathrm{cris},E}(V) \otimes_{K_{0,E}} K \hookrightarrow (V \otimes_E B_{\mathrm{dR}})^{G_K}$$

with the subspace filtration from $V \otimes_E B_{\mathrm{dR}}$. In this way, $D := D_{\mathrm{cris},E}(V)$ is naturally a filtered φ_q-module over K.

Theorem 3.A.19. *The functor*

$$D_{\mathrm{cris},E} : \mathrm{Rep}_E^{\mathrm{cris}}(G_K) \to \mathrm{MF}^{\varphi_q}(K)$$

is fully faithful. Moreover, the essential image is precisely the subcategory of weakly admissible objects.

Proof. Recall that $B_{e,E} \cap B_{\mathrm{dR}}^+ = E$ (as follows from the fundamental exact sequence $0 \to E \to B_{e,E} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$). The first statement follows rather formally from this. More precisely, for each V in the source, we have

$$V = (V \otimes_E B_E^+[1/t_E])^{\varphi_q=1} \cap \mathrm{Fil}^0(V \otimes_E B_{\mathrm{dR}}) = V_E(D_{\mathrm{cris},E}(V)).$$

Taking G_K-invariants yields, $V^{G_K} = \mathrm{Fil}^0(D_{\mathrm{cris},E}(V)^{\varphi_q=1})$. Using a suitable internal Hom, this implies full faithfulness of $D_{\mathrm{cris},E}$. We remark also that V_E is a quasi-inverse on the essential image of $D_{\mathrm{cris},E}$. We next show that $D := D_{\mathrm{cris},E}(V)$ is weakly admissible. This follows from the equality $\dim_E V_E(D) = \dim_E V = \dim_{K_{0,E}} D$, Proposition 3.A.18, and the classification of vector bundles on X_E.

It remains to show that if D is a weakly admissible filtered φ_q-module over K, then $V := V_E(D)$ is E-crystalline, and $D_{\mathrm{cris},E}(V) \simeq D$ as filtered φ_q-modules. We will follow the proof of [CF00, Proposition 4.5]. Let C_E denote the fraction field of $B_E^+[1/t_E]$. As $(C_E)^{G_K} = K_{0,E}$ by Lemma 3.A.2, by [CF00, Lemme 4.6], there exists a (necessarily unique) $K_{0,E}$-vector space $D' \subseteq D$ such that $D' \otimes_{K_{0,E}} C_E$ equals the C_E-subspace of $D \otimes_{K_{0,E}} C_E$ generated by V. As V is fixed by φ_q, D' is φ_q-stable, and hence naturally a subobject of D (as a filtered φ_q-module). Moreover, as $V \subseteq D' \otimes C_E$ and $V \subseteq D \otimes B_E^+[1/t_E]$, we have $V \subseteq D' \otimes B_E^+[1/t_E]$, whence $V = V_E(D')$. Let $d_1, \ldots, d_r$ be a $K_{0,E}$-basis of D'. Choose also $v_1, \ldots, v_r \in V$ which spans $D' \otimes_{K_{0,E}} C_E$ over C_E. For each i, write $v_i = \sum_j b_{ij} d_j$ for some $b_{ij} \in B_E^+[1/t_E]$. Then $b := \det(b_{ij})$ is nonzero, and so

$$w := v_1 \wedge \ldots \wedge v_r = b(d_1 \wedge \ldots \wedge d_r)$$

is a nonzero element in $W := V_E(\wedge^r D') \subseteq B_E^+[1/t_E] \otimes \wedge^r D'$. As $t_H(D') \leq t_N(D')$ (by weak admissibility of D), it follows from Lemma 3.A.20 below that $t_H(D') = t_N(D')$, $W = Ew$ and that b is a unit in $B_E^+[1/t_E]$. Thus the natural map $V \otimes_E B_E^+[1/t_E] \to D' \otimes B_E^+[1/t_E]$ is surjective, and so as the source and target are abstractly isomorphic (as $\dim_E V = \dim_E V_E(D') = \dim_{K_{0,E}} D'$ by weak admissibility of D'), it is in fact an isomorphism. Thus V is E-crystalline and $D_{\mathrm{cris},E}(V) = D' \subseteq D$. Finally as $\dim D' = \dim_E V = \dim D$, we must have $D' = D$. □

Lemma 3.A.20. *Let D be a filtered φ_q-module over K. Assume D is 1-dimensional with a basis d. Then*

$$\dim_E V_E(D) = \begin{cases} 0 & \text{if } t_H(D) < t_N(D), \\ 1 & \text{if } t_H(D) = t_N(D), \\ \infty & \text{if } t_H(D) > t_N(D). \end{cases}$$

Moreover, in case $\dim V_E(D) = 1$, any basis of $V_E(D)$ is of the form bd for some unit $b \in B_E^+[1/t_E]$.

Proof. Write $\varphi_q(d) = \pi^{t_N(D)} ud$ with $u \in W_{\mathcal{O}_E}(k)^\times$. Choose $x \in W_{\mathcal{O}_E}(\overline{k})^\times$ so that $\varphi_q(x) = ux$. One then checks easily that

$$V_E(D) = t_E^{-t_H(D)} x^{-1} \mathrm{Fil}^0(B_E^+[1/t_E])^{\varphi_q = \pi^{t_H(D) - t_N(D)}} d.$$

The lemma now follows from this and the fundamental exact sequence. □

Remark 3.A.21. For a weakly admissible D in $\mathrm{MF}^{\varphi_q}(K)$, let $V_E'(D) := (D \otimes_{K_{0,E}} (B_{\mathrm{cris}} \otimes_{E_0} E))^{\varphi_q = 1} \cap \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})$. As $B_E^+[1/t_E] \subseteq B_{\mathrm{cris}} \otimes_{E_0} E$, $V_E(D) \subseteq V_E'(D)$. Moreover, by [KR09, Proposition (3.3.4)], $\dim_E V_E'(D) \leq \dim D$. As $\dim V_E(D) = \dim H^0(X_E, \mathcal{E}(D)) = \dim D$ (recall that $\mathcal{E}(D)$ is semistable of slope 0 by weak admissibility of D), $V_E'(D) = V_E(D)$. Thus, the definition of $V_E(D)$ here agrees with the one in [KR09, §3] (for weakly admissible D).

Bibliography

[AL23] Johannes Anschütz and Arthur-César Le Bras. "Prismatic Dieudonné theory". In: *Forum Math. Pi* 11 (2023), Paper No. e2, 92.

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. "On the Beilinson fiber square". In: *Duke Math. J.* 171.18 (2022), pp. 3707–3806.

[Ber04] Laurent Berger. "Limites de représentations cristallines". In: *Compos. Math.* 140.6 (2004), pp. 1473–1498.

[Ber08] Laurent Berger. "Équations différentielles p-adiques et (ϕ, N)-modules filtrés". In: 319. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules. 2008, pp. 13–38.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310.

[BS23] Bhargav Bhatt and Peter Scholze. "Prismatic F-crystals and crystalline Galois representations". In: *Camb. J. Math.* 11.2 (2023), pp. 507–562.

[Bre02] Christophe Breuil. "Integral p-adic Hodge theory". In: *Algebraic geometry 2000, Azumino (Hotaka)*. Vol. 36. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2002, pp. 51–80.

[CL16] Bryden Cais and Tong Liu. "On F-crystalline representations". In: *Doc. Math.* 21 (2016), pp. 223–270.

[Che18] Chuangxun Cheng. "Breuil $\mathcal{O}$-windows and π-divisible $\mathcal{O}$-modules". In: *Trans. Amer. Math. Soc.* 370.1 (2018), pp. 695–726.

[Col02] Pierre Colmez. "Espaces de Banach de dimension finie". In: *J. Inst. Math. Jussieu* 1.3 (2002), pp. 331–439.

[CF00] Pierre Colmez and Jean-Marc Fontaine. "Construction des représentations p-adiques semi-stables". In: *Invent. Math.* 140.1 (2000), pp. 1–43.

[Du19] Heng Du. *Arithmetic Breuil-Kisin-Fargues modules and comparison of integral p-adic Hodge theories*. 2019. arXiv: 1910.02939.

[DL22] Heng Du and Tong Liu. *A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals*. 2022. arXiv: 2107.12240.

[EG23] Matthew Emerton and Toby Gee. *Moduli stacks of Étale (ϕ, Γ)-modules and the existence of crystalline lifts*. Vol. 215. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2023, pp. ix+298.

[Fal02] Gerd Faltings. "Group schemes with strict $\mathcal{O}$-action". In: vol. 2. 2. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. 2002, pp. 249–279.

[FF18] Laurent Fargues and Jean-Marc Fontaine. "Courbes et fibrés vectoriels en théorie de Hodge p-adique". In: *Astérisque* 406 (2018). With a preface by Pierre Colmez, pp. xiii+382.

[FL82] Jean-Marc Fontaine and Guy Laffaille. "Construction de représentations p-adiques". In: *Ann. Sci. École Norm. Sup. (4)* 15.4 (1982), 547–608 (1983).

[GL12] Alain Genestier and Vincent Lafforgue. "Structures de Hodge–Pink pour les $\varphi/\mathfrak{S}$-modules de Breuil et Kisin". In: *Compositio Mathematica* 148.3 (2012), pp. 751–789.

[Ito23] Kazuhiro Ito. *Prismatic G-display and descent theory*. 2023. arXiv: 2303.15814.

[Ked04] Kiran S. Kedlaya. "A p-adic local monodromy theorem". In: *Ann. of Math. (2)* 160.1 (2004), pp. 93–184.

[Ked08] Kiran S. Kedlaya. "Slope filtrations for relative Frobenius". In: 319. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules. 2008, pp. 259–301.

[Kim09] Wansu Kim. "Galois deformation theory for norm fields and its arithmetic applications". Available at https://deepblue.lib.umich.edu/handle/2027.42/63878. PhD thesis. University of Michigan, 2009.

[Kis06] Mark Kisin. "Crystalline representations and F-crystals". In: *Algebraic geometry and number theory*. Vol. 253. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 459–496.

[KR09] Mark Kisin and Wei Ren. "Galois representations and Lubin-Tate groups". In: *Doc. Math.* 14 (2009), pp. 441–461.

[Laz62] Michel Lazard. "Les zéros des fonctions analytiques d'une variable sur un corps valué complet". In: *Inst. Hautes Études Sci. Publ. Math.* 14 (1962), pp. 47–75.

[Liu13] Tong Liu. "The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$". In: *J. Théor. Nombres Bordeaux* 25.3 (2013), pp. 661–676.

[Mar23] Samuel Marks. *Prismatic F-crystals and Lubin-Tate (φ_q, Γ)-modules*. 2023. arXiv: 2303.07620.

[Sta23] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.

[Wu21] Zhiyou Wu. "Galois representations, (φ, Γ)-modules and prismatic F-crystals". In: *Doc. Math.* 26 (2021), pp. 1771–1798.