UNIVERSITÉ PARIS XIII – SORBONNE PARIS NORD

École doctorale Sciences, Technologies, Santé Galilée

Transformées de Siegel, équidistribution et comptage des approximations rationnelles sur les variétés de drapeaux

Siegel transforms, equidistribution and counting rational approximations on flag varieties

---

THÈSE DE DOCTORAT
présentée par

René PFITSCHER

Laboratoire Analyse, Géométrie et Applications

pour l'obtention du grade de
DOCTEUR EN MATHEMATIQUES

soutenue le 25.06.2025 devant le jury d'examen constitué de :

BURRIN Claire, University of Zurich, Examinatrice
DE SAXCE Nicolas, Université Sorbonne Paris-Nord, Directeur de thèse
DE THELIN Henry, Université Sorbonne Paris-Nord, Examinateur
PAULIN Frédéric, Université Paris Saclay, Rapporteur
QUEGUINER MATHIEU Anne, Université Sorbonne Paris-Nord, Examinatrice
YU Shucheng, University of Science and Technology of China, Rapporteur

Résumé

Dans une variété de drapeaux, obtenue comme quotient d'un groupe algébrique semi-simple défini sur $\mathbb{Q}$ par un sous-groupe parabolique maximal, nous développons plusieurs techniques nouvelles pour compter les approximations rationnelles d'un point réel choisi aléatoirement selon la mesure uniforme. En particulier, nos résultats s'appliquent aux grassmanniennes et aux hypersurfaces quadratiques projectives. Les contributions de cette thèse se répartissent selon les trois thèmes suivants :

1. *Intégrabilité des transformées de Siegel*
2. *Comptage* à *l'exposant diophantien*
3. *Comptage* en dessous *de l'exposant diophantien*

Plus précisément, nous généralisons la transformée de Siegel aux représentations rationnelles irréductibles de plus haut poids des $\mathbb{Q}$-groupes semi-simples et, dans la perspective de la formule de moyenne de Siegel et de la formule des moments supérieurs de Rogers pour la transformée classique de Siegel, nous étudions l'intégrabilité au sens de L^p de la transformée de Siegel des fonctions mesurables, bornées et à support compact pour $p = 1, 2, \infty$.

Pour compter les approximations rationnelles sur les variétés de drapeaux de rang un à l'exposant diophantien, nous utilisons une approche ergodique et une formule de moyenne pour les transformées de Siegel, également développée dans cette thèse. Nous affinons ensuite ce résultat en passant d'une formule asymptotique à une estimation avec un terme d'erreur explicite. Pour cela, nous démontrons un résultat d'équidistribution effective de certaines orbites translatées d'un sous-groupe compact maximal, ainsi qu'une propriété d'intégrabilité des transformées de Siegel.

Le comptage par rapport aux exposants en dessous de l'exposant diophantien requiert une approche différente. Nous utilisons le mélange expo-

nentiel dans l'espace des réseaux, des outils issus de la géométrie des nombres ainsi qu'une méthode de comptage célèbre due à Eskin-McMullen et Duke-Rudnick-Sarnak. Cette méthode permet d'obtenir des estimations effectives. Enfin, nous en déduisons l'équidistribution effective des points rationnels de hauteur bornée sur les variétés de drapeaux de rang un.

Abstract

On a generalized flag variety of rank one, we present several novel techniques for counting rational approximations to a real point chosen randomly according to the Riemannian volume. In particular, our results apply to Grassmann varieties and projective quadric hypersurfaces. The contributions of this thesis can be categorized into the following three topics:

1. *Integrability of Siegel transforms;*
2. *Counting* at *the Diophantine exponent;*
3. *Counting* below *the Diophantine exponent.*

More precisely, we generalize the Siegel transform to rational irreducible highest-weight representations of semisimple $\mathbb{Q}$-groups and, with a view toward Siegel's mean value formula and Rogers' higher moment formula for the classical Siegel transform, characterize the L^p-integrability of the Siegel transform of measurable bounded compactly supported functions for $p = 1, 2, \infty$.

To count rational approximations on rank-one flag varieties with respect to the Diophantine exponent, we use an ergodic-theoretic approach and a mean value formula for Siegel transforms, also developed in this thesis. We then refine this result from an asymptotic formula to an estimate with an explicit error term. To this end, we prove effective equidistribution of expanding translates of orbits of maximal compact subgroups, along with an integrability property of Siegel transforms.

Counting with respect to exponents below the Diophantine exponent requires a different approach. We use exponential mixing in the space of lattices, tools from the geometry of numbers, and a famous counting method due to Eskin-McMullen and Duke-Rudnick-Sarnak. This method allows for effective estimates. Finally, we deduce effective equidistribution of rational points of bounded height on rank-one flag varieties.

Acknowledgments

First and foremost, I would like to thank my doctoral advisor, Nicolas de Saxcé. His kindness and regular availability for patient discussions about ideas and techniques, as well as his deep insights, were always very encouraging and helpful. I could learn a lot, both on a mathematical and a personal level, from his approach to mathematics and the simplicity and directness of his mathematical style. I am very grateful to have had him as my advisor.

Moreover, I would like to express my deep gratitude to Frédéric Paulin, rapporteur of my thesis defence jury, for having read my thesis so carefully, for pointing out numerous suggestions that have already improved, and will continue to improve, my mathematical writing style, and for the embarrassing number of typos he found. I am truly grateful for the great effort he put into correcting my thesis and for his many helpful suggestions. For instance, thanks to his attentive corrections, a restrictive hypothesis in Theorem 1.2.2 (namely, that $\beta_\chi \leqslant 1$) could be removed.

I would like to thank Shucheng Yu, also rapporteur on my thesis defence committee, for having read my thesis so carefully, for his many helpful suggestions, and for his interest in my work. I am also grateful to him for inviting me on this inspiring visit to his faculty in China and for his continued support throughout my mathematical career.

I am also grateful to the other members of my thesis jury — Claire Burrin, Anne Queguiner Mathieu, and Henry de Thélin — for participating in my thesis defence as jury members and for their interest and effort.

I would like to thank Wooyeon Kim for valuable input on the proof of Lemma 6.4.1, Gabriel Pallier for helping me to prove Lemma 4.2.1, and Manuel Hauke, Yuval Yifrach, Fernando Argentieri, Zouhair Ouaggag, Andrea Aveni and Sundar Narasimhan for their regular and stimulating discussions. I am also very grateful to have had such kind and supportive colleagues at our mathematics department, LAGA, throughout these three years. The many nice discussions, shared lunches, and coffee breaks consistently created a nice working environment.

Finally, the loving support of my family - Sandra & Andreas, Lukas & Lara, Ruth & Florian - and close friends - especially Andreas Bergmeister and Luis Hafner - was invaluable to me throughout my PhD.

Contents

1 Introduction **8**

1.1 Integrability of Siegel transforms 11
1.2 Counting at the Diophantine exponent 14
1.3 Counting below the Diophantine exponent 15
1.4 Some examples 17
1.5 Notation and conventions 19

2 An alternative proof of Schmidt's theorem **21**

2.1 Method of proof 23
2.2 Coordinates and measures 25
2.3 Second moment of the Siegel transform 27
2.4 Diophantine approximation and counting lattice points 32
2.5 Measure estimates 33
2.6 Well-roundedness 35
2.7 From mean square discrepancy to counting 38
2.8 Proof of Theorem 2.0.1 42

3 Integrability of Siegel transforms **43**

3.1 Some preliminaries 44
3.1.1 Some structure theory of parabolic $\mathbb{Q}$-subgroups 44
3.1.2 Relative fundamental weights 47
3.1.3 Strongly rational representations 48
3.1.4 Reduction theory 50
3.2 Generalizing the Siegel transform 51
3.2.1 Incomplete Eisenstein series 52
3.3 L^1-integrability 57
3.4 L^2-integrability 68

3.5 L^∞-integrability 72

4 Diophantine approximation and counting lattice points 78
4.1 Height on $\mathbf{X}(\mathbb{Q})$ 79
4.2 Distance on X 80
4.3 Diophantine exponent 82
4.4 Cone over X, coordinates and measures 82
4.5 Lattice reduction with respect to $\mathbf{M}$ 85
4.6 Exponential mixing 88
4.7 Counting lattice points 88

5 Counting at the Diophantine exponent 90
5.1 Ergodic-theoretic approach: An asymptotic formula without error term 91
5.2 Counting with an error term: The method 96
5.3 Approximation of $\mathcal{E}_{\beta_\chi}(T)$ 98
5.4 A uniform upper bound for the K-average of the Siegel transform 101
5.5 Equidistribution of compact-orbit translates 104
5.6 Non-escape of mass 110
5.7 Approximation by smooth functions 112
5.8 Proof of Theorem 1.2.2 116
5.8.1 Upper bound estimate 118

6 Counting below the Diophantine exponent 126
6.1 Method of proof 127
6.2 Reduction of Theorem 1.3.1 130
6.3 Diagonal action and well roundedness 132
6.4 From exponential mixing to counting 138
6.4.1 Sobolev norms 139
6.4.2 Equidistribution of translated L-orbits 140
6.4.3 A lattice point counting lemma 143
6.4.4 Proof of Proposition 6.2.1 147
6.5 Proof of Theorem 1.3.2 and a corollary 149
6.6 Applications 151
6.6.1 Projective quadrics 152
6.6.2 Grassmann varieties 152

Chapter 1

Introduction

An important branch of number theory is Diophantine approximation, that is, the study of rational approximations to real numbers with prescribed properties. Given a real number $x \in \mathbb{R}$, a typical goal is to approximate x well by rational numbers p/q with small denominators q, ensuring an efficient approximation. A central result in this field, due to Dirichlet, states that, for any $\theta \in \mathbb{R}$ and $Q > 1$, there exist coprime integers p and q with $1 \leqslant q < Q$ such that $|q\theta - p| \leqslant 1/Q$. An immediate consequence is that for *any* irrational $x \in \mathbb{R}$, the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

admits infinitely many coprime solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. On the other hand, the Thue-Siegel-Roth Theorem [Rot55] shows that for any *algebraic* number $x \in \mathbb{R}$ and for any $\varepsilon > 0$, the inequality

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}$$

admits at most finitely many coprime solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$. Indeed, one can show that beyond the algebraic numbers, this holds for almost every real number. In particular, the exponent $\beta = 2$ plays a critical role for approximation of real numbers by rational ones and is referred to as the *Diophantine exponent* of $\mathbb{R}$.

In 1926, Khintchine [Khi26] proved a refinement of Dirichlet's result. More precisely, for any non-increasing function $\psi : \mathbb{N} \to \mathbb{R}_+$, he showed that the inequality

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}$$

admits infinitely (resp. finitely) many solutions $(p, q) \in \mathbb{Z} \times \mathbb{N}$ for almost every $x \in \mathbb{R}$, if the series $\sum \psi(q)$ diverges (resp. converges). In the case where the series is divergent, Schmidt [Sch60a] strengthened Khintchine's Theorem. In fact, for $x \in \mathbb{R}$ and $T \in \mathbb{N}$, he considered the counting function

$$\mathcal{N}_\psi(x, T) = \# \{(p, q) \in \mathbb{Z} \times \mathbb{N} : 0 \leqslant qx - p < \psi(q),\ 1 \leqslant q \leqslant T\} \qquad (1.0.1)$$

and showed that for almost every $x \in \mathbb{R}$, $\mathcal{N}_\psi(x, T)$ is, up to a scalar, asymptotically equal to $\sum_{1 \leqslant q < T} \psi(q)$ as T tends to infinity, with an explicit error term. This does not only apply to $\mathbb{R}$, but to any Euclidean space $\mathbb{R}^n$. We state his result here.

Theorem 1.0.1 (Schmidt, 1960)**.** *Let $n \geqslant 1$ and, for each $1 \leqslant i \leqslant n$, let $\psi_i : \mathbb{N} \to \mathbb{R}_+$ be a non-negative function and assume that $\psi(q) = \prod_i \psi_i(q)$ is non-increasing. For each $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $T \in \mathbb{N}$, define*

$$\mathcal{N}_\psi(\boldsymbol{x}, T) = \# \{(\boldsymbol{p}, q) \in \mathbb{Z}^n \times \mathbb{N} : \forall i,\ 0 \leqslant qx_i - p_i < \psi_i(q),\ 1 \leqslant q \leqslant T\}.$$

Moreover, define

$$\Psi(T) = \sum_{q=1}^{T} \psi(q), \quad \Omega(T) = \sum_{q=1}^{T} \psi(q) q^{-1}$$

Let $\varepsilon > 0$. Then for almost every $\boldsymbol{x} \in \mathbb{R}^n$,

$$\mathcal{N}_\psi(\boldsymbol{x}, T) = \Psi(T) + O_{\boldsymbol{x}} \left(\Psi(T)^{1/2} \Omega(T)^{1/2} \log^{2+\varepsilon} \Psi(T) \right).$$

More generally, given an algebraic variety $\mathbf{X}$ defined over $\mathbb{Q}$ such that the rational points $\mathbf{X}(\mathbb{Q})$ are dense in the set of real points $X = \mathbf{X}(\mathbb{R})$ and a point $x \in X$, it is natural to investigate the quality of approximations of x by points in $\mathbf{X}(\mathbb{Q})$. This area of study, proposed by Lang [Lan65] in 1965 and relatively unexplored until recently, seems to have regained considerable interest. For instance, de Saxcé [Sax20], extending previous work of Fishman, Kleinbock, Merrill, and Simmons [Fis+22], used methods from homogeneous dynamics, particularly those developed by Margulis and his collaborators [Dan85; KM98; KM99], to develop a theory of Diophantine approximation on generalized flag varieties. Simple examples of such varieties include projective spaces, projective quadrics, and Grassmannians. However, many important classical results still lack general analogs in this theory.

The goal of my doctoral thesis is to prove versions of Schmidt's theorem, where the Euclidean space $\mathbb{R}^n$ is replaced by the space of real points $X = \mathbf{X}(\mathbb{R})$ of a generalized flag variety $\mathbf{X}$ defined over $\mathbb{Q}$. Recall that a generalized flag variety $\mathbf{X}$ defined over $\mathbb{Q}$ is a projective algebraic variety defined over $\mathbb{Q}$ that can be expressed as a quotient $\mathbf{X} = \mathbf{G}/\mathbf{P}$, where $\mathbf{G}$ is a connected semisimple algebraic $\mathbb{Q}$-group and $\mathbf{P} < \mathbf{G}$ is a parabolic $\mathbb{Q}$-subgroup. Suppose that the unipotent radical of $\mathbf{P}$ is abelian. This assumption allows us to work with a Riemannian metric on X instead of a sub-Riemannian one and implies that $\mathbf{P}$ is a *maximal* parabolic $\mathbb{Q}$-subgroup, or in other words, that the $\mathbb{Q}$-rank of $\mathbf{X}$ is 1. Let K be a maximal compact subgroup of $G = \mathbf{G}(\mathbb{R})$. We measure the distance between a real point and its rational approximations using a K-invariant Riemannian distance $d(\cdot,\cdot)$ on X, and we equip X with the unique K-invariant probability measure σ_X. Moreover, we equip the rational points $\mathbf{X}(\mathbb{Q})$ with a height H_χ associated with a dominant $\mathbb{Q}$-weight χ of $\mathbf{G}$. The height takes the place of the denominator q of a rational number p/q written in reduced form. Rational points on X form a finite union of orbits under the action of the arithmetic subgroup $\Gamma = \mathbf{G}(\mathbb{Z})$ of G (see [Bor69, Proposition 15.6]). Problems in Diophantine approximation on X can thus be reformulated in terms of certain diagonal orbits in the space of lattices $\Omega = G/\Gamma$.

Given an arbitrary $\tau > 0$ we shall consider the decreasing approximation function $\psi(t) = t^{-\tau}$. In analogy to (1.0.1), for $x \in X$ and $T > 1$, we thus define

$$\mathcal{N}_\tau(x,T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : d(x,v) < H_\chi(v)^{-\tau},\ 1 \leqslant H_\chi(v) < T\}. \tag{1.0.2}$$

It was shown by de Saxcé [Sax20, Théorème 2.4.5] that there exists a rational number $\beta_\chi > 0$ such that the inequality

$$d(x,v) < H_\chi(v)^{-\tau} \tag{1.0.3}$$

has infinitely (resp. finitely) many solutions $v \in \mathbf{X}(\mathbb{Q})$ for σ_X-almost every $x \in X$ if $\tau \leqslant \beta_\chi$ (resp. $\tau > \beta_\chi$). We refer to β_χ as the *Diophantine exponent* of X with respect to χ. We strengthen this statement by providing an asymptotic formula for the number of solutions $\mathcal{N}_\tau(x,T)$ as $T \to +\infty$, when $x \in X$ is chosen randomly according to the measure σ_X on X.

The contributions of this thesis can be subdivided into the following three themes:

1. *Integrability of Siegel transforms;*
2. *Counting* at *the Diophantine exponent;*
3. *Counting* below *the Diophantine exponent.*

We now describe each contribution.

1.1 Integrability of Siegel transforms

The classical Siegel transform, introduced in 1945 by Carl Ludwig Siegel [Sie45], maps a function of sufficient decay on the Euclidean space $\mathbb{R}^n$ to a function on the space of unimodular lattices $\Omega = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, by averaging over the lattice. We recall that, for any bounded compactly supported measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the *primitive* Siegel transform of f is defined by

$$\forall\, g \in \mathrm{SL}_n(\mathbb{R}), \quad Sf(g) = \sum_{\boldsymbol{v} \in \mathbb{Z}^n_{\mathrm{pr}}} f(g\boldsymbol{v}).$$

Here $\mathbb{Z}^n_{\mathrm{pr}}$ denotes the subset of $\mathbb{Z}^n$ consisting of all the vectors that are primitive: a vector $\boldsymbol{v} \in \mathbb{Z}^n$ is called *primitive* if it is non-zero and $\frac{1}{k}\boldsymbol{v} \notin \mathbb{Z}^n$ for all positive integers $k \geqslant 2$. The space of unimodular lattices carries a unique $\mathrm{SL}_n(\mathbb{R})$-invariant probability measure that we fix and denote by μ. The Siegel mean value theorem expresses the average of the Siegel transform of a function f in terms of the average of f:

$$\int_\Omega Sf\, d\mu = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f\, d\lambda.$$

Here ζ denotes the Riemann zeta function and λ denotes the usual Lebesgue measure on $\mathbb{R}^n$. Later, generalizing Siegel's result, Rogers [Rog55] proved a general k-th moment formula for the Siegel transform for k up to $n-1$. For instance, a remarkable application of the second moment formula was given by Schmidt [Sch60a], who proved asymptotic formulas for counting lattice points in an increasing family of sets in Euclidean space, with an explicit error term. As we will see later, one can in fact use this second moment formula to give an alternative proof of Schmidt's theorem in metric Diophantine approximation for the projective space $\mathbb{P}^n(\mathbb{R})$, one of the simplest flag varieties. In particular, one should think of the classical primitive Siegel transform as the one corresponding to the projective space. For any connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ and *any* parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ (here we need no assumptions on the unipotent radical of $\mathbf{P}$), we shall attach a Siegel transform to the generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$. To do so, we generalize the classical primitive Siegel transform to the setting of irreducible representations of $\mathbf{G}$ that are strongly rational over $\mathbb{Q}$. With a view towards Rogers' higher moment formulas, we shall characterize its integrability properties. Some of these results will also be essential for counting rational approximations at and below the Diophantine exponent in the last two chapters.

To state our results, we need to introduce some notation. As before, we let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group. Let $\mathbf{P}_0 = \mathscr{Z}(\mathbf{T})\mathbf{U}_0$ be a minimal parabolic $\mathbb{Q}$-subgroup, where $\mathbf{U}_0 = R_u(\mathbf{P}_0)$ is the unipotent radical of $\mathbf{P}$ and $\mathscr{Z}(\mathbf{T})$ is the centralizer in $\mathbf{G}$ of a maximal $\mathbb{Q}$-split torus $\mathbf{T}$; see [Bor69, § 11.7]. Let Δ be the associated set of simple roots relative to $\mathbf{T}$, for the ordering associated with $\mathbf{P}_0$. An element χ in the $\mathbb{Q}$-character group $X^*(\mathbf{P}_0)_{\mathbb{Q}}$ of $\mathbf{P}_0$ is said to be *dominant* if its restriction to $\mathbf{T}$ satisfies $(\chi, \alpha) \geqslant 0$ for all $\alpha \in \Delta$, where $(\cdot, \cdot)$ is a fixed admissible Euclidean inner product on $X^*(\mathbf{T}) \otimes \mathbb{R}$. Given a dominant $\mathbb{Q}$-weight $\chi \in X^*(\mathbf{P}_0)_{\mathbb{Q}}$, there exists a unique, up to rational equivalence, irreducible finite-dimensional representation $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ which is strongly rational over $\mathbb{Q}$ and a $\mathbf{P}_0$-invariant line $\mathbf{D}_\chi \subset \mathbf{V}_\chi$ that is defined over $\mathbb{Q}$, on which $\mathbf{P}_0$ acts via χ; see Borel and Tits [BT65, §12]. Let $\mathbf{P}$ be the stabilizer in $\mathbf{G}$ of the line $\mathbf{D}_\chi$. Then $\mathbf{P}$ is a parabolic $\mathbb{Q}$-subgroup containing $\mathbf{P}_0$. Fix a rational vector $\boldsymbol{e}_\chi \in \mathbf{D}_\chi \smallsetminus \{\mathbf{0}\}$ (to any such vector we shall refer to as a rational highest weight vector) and let $\mathbf{L}$ be the stabilizer in $\mathbf{G}$ of $\boldsymbol{e}_\chi$. Let $\widetilde{X}_0$ be the space of real points of the cone $(\mathbf{G} \cdot \mathbf{D}_\chi)\backslash\{\mathbf{0}\}$ of the representation π_χ and denote by $\widetilde{X} = \widetilde{X}_0 \sqcup \{\mathbf{0}\}$ its closure inside $V_\chi = \mathbf{V}_\chi(\mathbb{R})$. Fix a rational lattice $\mathbf{V}_\chi(\mathbb{Z})$ of $V_\chi = \mathbf{V}_\chi(\mathbb{R})$; its stabilizer Γ in $G = \mathbf{G}(\mathbb{R})$ is an arithmetic subgroup. Let $\mathcal{L}_\chi$ denote the set of all the primitive vectors in $\mathbf{V}_\chi(\mathbb{Z})$ that are contained in $\widetilde{X}$. Then for any bounded compactly supported measurable function $f : \widetilde{X} \to \mathbb{R}$ we define the *Siegel transform* of f at χ by

$$\forall\, g \in G, \quad S_\chi f(g) = \sum_{\boldsymbol{v} \in \mathcal{L}_\chi} f(g\boldsymbol{v}).$$

Since f is bounded and compactly supported, this is a finite sum and hence converges absolutely. Since $\mathcal{L}_\chi$ is invariant under the action of Γ, the Siegel transform $S_\chi f$ is right Γ-invariant and hence defines a function on the homogeneous space $\Omega = G/\Gamma$. Let μ be the unique G-invariant probability measure on Ω. We prove the following characterizations; the formula (1.1.1) below is essentially a consequence of Weil's integration formula [Wei53].

Theorem 1.1.1 (L^1-integrability)**.** *The following assertions are equivalent.*

(1) The Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^1(\Omega)$.

(2) There exists a unique (up to scaling) G-invariant Radon measure $\lambda_{\widetilde{X}}$ on $\widetilde{X}$ and one has a convergent mean value formula: for all $f \in L^1(\widetilde{X})$,

$$\int_\Omega S_\chi f \, d\mu = \int_{\widetilde{X}} f \, d\lambda_{\widetilde{X}} \tag{1.1.1}$$

In particular, for all $f \in C_c(\widetilde{X})$,

$$\|S_\chi f\|_{L^1(\Omega)} \leqslant \|f\|_{L^1(\widetilde{X})},$$

and hence the Siegel transform S_χ can be extended to a bounded operator $S_\chi : L^1(\widetilde{X}) \to L^1(\Omega)$.

(3) The Lie group $L = \mathbf{L}(\mathbb{R})$ is unimodular and $L \cap \Gamma$ is a lattice in L.

(4) The parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ of $\mathbf{G}$ is maximal.

(5) There exists $\varepsilon > 0$ such that S_χ maps $C_c(\widetilde{X})$ into $L^{1+\varepsilon}(\Omega)$.

We studied the L^2-integrability of $S_\chi f$ in terms of the Dynkin diagram of $\mathbf{G}$. We recall that there is a correspondence between subsets $\theta \subseteq \Delta$ of the set of simple roots Δ and parabolic $\mathbb{Q}$-subgroups $\mathbf{P}_\theta$ containing $\mathbf{P}_0$. In particular, one has $\mathbf{P}_\varnothing = \mathbf{P}_0$ and $\mathbf{P}_\Delta = \mathbf{G}$. Observe that $\mathbf{P}_\theta$ is maximal if and only if $\theta = \Delta\backslash\{\alpha\}$ for some simple root $\alpha \in \Delta$. In this case we shall simply write $\mathbf{P}_{^c\{\alpha\}} = \mathbf{P}_\theta$.

Theorem 1.1.2 (L^2-integrability)**.** *Suppose that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^2(\Omega)$. Then $\mathbf{P} = \mathbf{P}_\alpha$ is a maximal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and the simple root α has at most one neighbor in the associated Dynkin diagram.*

Question: In the case of $\mathbf{G} = \mathrm{SL}_n$ for $n \geqslant 2$ the converse to Theorem 1.1.2 is true in the sense that if $\mathbf{P} = \mathbf{P}_\alpha$ is a maximal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and the simple root α has at most one neighbor in the associated Dynkin diagram then the Siegel transform S_χ (being the classical primitive Siegel transform with $\widetilde{X} = \mathbb{R}^n$) maps $C_c(\widetilde{X})$ into $L^2(\Omega)$. Does this hold, more generally, for any connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ as considered above?

Theorem 1.1.3 (L^∞-integrability)**.** *The following assertions are equivalent.*

(1) The Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^\infty(\Omega)$.

(2) The $\mathbb{Q}$-rank of $\mathbf{G}$ is 1.

(3) The discrete group $L \cap \Gamma$ is a cocompact lattice in L.

We note that the Siegel transform as defined above is a finite sum of incomplete/pseudo Eisenstein series, and the Mellin transform provides a passage to the theory of Eisenstein series. We also used some of the integrability results in order to count rational approximations at and below the Diophantine exponent, as we now describe.

1.2 Counting at the Diophantine exponent

We assume here that $\mathbf{X} = \mathbf{G}/\mathbf{P}$ is a generalized flag variety obtained as the quotient of a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ by a parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$, whose unipotent radical is *abelian*. Let $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ be an irreducible representation defined over $\mathbb{Q}$ for which there exists a $\mathbf{P}$-stable rational line $\mathbf{D}_\chi \subset \mathbf{V}_\chi$, on which $\mathbf{P}$ acts via a dominant $\mathbb{Q}$-weight χ of $\mathbf{G}$. We let $\tau = \beta_\chi$ be the Diophantine exponent with respect to χ and, for all x in $X = \mathbf{X}(\mathbb{R})$ and $T > 1$, we consider the associated counting function

$$\mathcal{N}_{\beta_\chi}(x,T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : d(x,v) < H_\chi(v)^{-\beta_\chi},\ 1 \leqslant H_\chi(v) < T\}.$$

First, one defines a height $H_\chi : \mathbb{P}(\mathbf{V}_\chi)(\mathbb{Q}) \to \mathbb{R}$ on the projective space (we will give the details below) and then obtains a height on $\mathbf{X} = \mathbf{G}/\mathbf{P}$ by embedding it into the projective space $\mathbb{P}(\mathbf{V}_\chi)$ via $g\mathbf{P} \mapsto \pi_\chi(g)[\boldsymbol{e}_\chi]$, where $[\boldsymbol{e}_\chi]$ denotes the point in the projective space corresponding to $\boldsymbol{e}_\chi$, and restricting the height H_χ to X The height so obtained will still be denoted by H_χ. The problem in Diophantine approximation is then translated to a problem in geometry of numbers, specifically counting lattice points in a certain increasing family of sets in the Euclidean space V_χ. We generalized the ergodic-theoretic approach of Alam and Ghosh [AG22], who counted rational approximations on spheres, which in turn builds on ideas from Athreya-Parrish-Tseng [APT16]. Together with the mean value formula for Siegel transforms as developed above, we were able to give the following explicit formula for the asymptotic behavior of the counting function $\mathcal{N}_{\beta_\chi}(x,T)$ as T approaches infinity.

Theorem 1.2.1. *Let* $\mathbf{G}$ *be a connected semisimple algebraic* $\mathbb{Q}$*-group,* $\mathbf{P}$ *a parabolic* $\mathbb{Q}$*-subgroup with abelian unipotent radical and* $\mathbf{X} = \mathbf{G}/\mathbf{P}$*. Then there exists a constant* $\varkappa > 0$ *such that for* σ_X*-almost every* $x \in X$,

$$\mathcal{N}_{\beta_\chi}(x,T) \sim \varkappa \log(T) \quad \text{as } T \to +\infty. \tag{1.2.1}$$

In order to obtain an effective estimate and not merely the asymptotic, we extended a recent effective result of Ouaggag [Oua23], who counted rational approximations on spheres with respect to the Diophantine exponent, to the setting of rank-one flag varieties. The methods are inspired by his arguments and use the effective equidistribution of translates of orbits of a maximal compact subgroup K of G, as well as the integrability results for Siegel transforms established above.

Theorem 1.2.2. *Let* $\mathbf{G}$ *be a connected semisimple algebraic* $\mathbb{Q}$*-group,* $\mathbf{P}$ *a parabolic* $\mathbb{Q}$*-subgroup with abelian unipotent radical and* $\mathbf{X} = \mathbf{G}/\mathbf{P}$*. Then there exist constants* $\varkappa > 0$ *and* $\varepsilon > 0$ *such that for* σ_X*-almost every* $x \in X$,

$$\mathcal{N}_{\beta_\chi}(x,T) = \varkappa \log(T) \left(1 + O_x(\log(T)^{-\varepsilon})\right). \tag{1.2.2}$$

1.3 Counting below the Diophantine exponent

We assume here again that $\mathbf{X} = \mathbf{G}/\mathbf{P}$ is a generalized flag variety obtained as the quotient of a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ by a parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$, whose unipotent radical is abelian. As above, we let H_χ be a height function on $\mathbf{X}(\mathbb{Q})$ associated with a dominant $\mathbb{Q}$-weight χ and we let β_χ be the corresponding Diophantine exponent. In order to count with respect to exponents $0 < \tau < \beta_\chi$ below the Diophantine exponent, a different approach is used. Namely, exponential mixing in the space of lattices, tools from geometry of numbers, and a famous counting method due to Eskin-McMullen [EM93] and Duke-Rudnick-Sarnak [DRP93]. This method allows for effective estimates. Our proof was inspired by the work of Huang and de Saxcé [HS24] on the local distribution of rational points on flag varieties. Let $d = \dim X$ be the dimension of X as a real manifold. For every decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $x \in X$, and $T > 1$, we define

$$\mathcal{N}_\psi(x, T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : d(x, v) < \psi(H_\chi(v)),\, 1 \leqslant H_\chi(v) < T\}.$$

Theorem 1.3.1. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function satisfying that there exist $\tau \in (0, \beta_\chi)$ and $C > 1$ such that $C^{-1}y^{-\tau} \leqslant \psi(y) \leqslant Cy^{-\tau}$ for all sufficiently large y. Define the integral*

$$\forall\, T > 1, \quad \Psi(T) = \int_1^T \psi(y)^d y^{\beta_\chi d} \frac{dy}{y}.$$

Then there exist constants $\varkappa > 0$ and $\varepsilon > 0$ such that for σ_X-almost every $x \in X$, we have

$$\mathcal{N}_\psi(x, T) = \varkappa\, \Psi(T) \left(1 + O_x(\Psi(T)^{-\varepsilon})\right). \tag{1.3.1}$$

We can even treat the case $\tau = 0$, which corresponds to the equidistribution of rational points in X as the height tends to infinity. In [MG14, Theorem 4], Mohammadi and Salehi Golsefidy counted rational points of bounded height on a flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$ with respect to an arbitrary metrized line bundle and arbitrary parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$. Our method yields the following effective equidistribution result in X. More precisely, for $x \in X$ and $r > 0$ denote by $B_r(x)$ the open ball in X with center x and radius r, and define the counting function

$$\mathcal{N}_\chi(x, r, T) = \#\left\{v \in \mathbf{X}(\mathbb{Q}) : v \in B_r(x),\, 1 \leqslant H_\chi(v) < T\right\}. \tag{1.3.2}$$

Our final main result is as follows. We recall that $d = \dim X$.

Theorem 1.3.2. *There exist constants $\varkappa > 0$ and $\varepsilon > 0$ such that for all $r > 0$ and $x \in X$,*

$$\mathcal{N}_\chi(x, r, T) = \varkappa\, T^{\beta_\chi d} \sigma_X(B_r(x)) \left(1 + O_r(T^{-\varepsilon})\right). \tag{1.3.3}$$

This result can be regarded as a corollary of Theorem 1.3.1 and we prove it in Section 6.5. It essentially follows by setting $\tau = 0$ in the proof of the former theorem.

1.4 Some examples

In this section, we illustrate our main results using rational projective quadric hypersurfaces and Grassmann varieties as examples of generalized flag varieties of $\mathbb{Q}$-rank one.

In the special case where X is a sphere or a rational ellipsoid, the problem of counting rational approximations to a real point chosen randomly according to the uniform probability measure has received much attention recently. In particular, Alam and Ghosh [AG22] and Ouaggag [Oua23] counted rational approximations on the sphere of arbitrary dimension n, but only for approximation functions $\psi : \mathbb{N} \to \mathbb{R}_+$ of the form $\psi(q) = cq^{-1}$ with $c > 0$. On the other hand, Kelmer and Yu [KY23], using the spectral theory of Eisenstein series, were able to deal with a general approximation function, but their result does not apply to spheres of dimension $n > 1$ with $n \equiv 1 \mod 8$.

More generally, let now X_Q be the space of real points of an n-dimensional projective rational quadric hypersurface $\mathbf{X}_Q$, given as the set of zeros in $\mathbb{P}(\mathbb{R}^{n+2})$ of a nonsingular rational quadratic form Q in $n+2$ variables:

$$X_Q = [Q^{-1}(0)] = \{x \in \mathbb{P}(\mathbb{R}^{n+2}) : x = [\boldsymbol{x}] \text{ with } Q(\boldsymbol{x}) = 0\}. \qquad (1.4.1)$$

The distance $d(\cdot,\cdot)$ and the height H are obtained by restriction of the usual distance and height on $\mathbb{P}(\mathbb{R}^{n+2})$, respectively. Let K be a maximal compact subgroup of the special orthogonal group $\mathrm{SO}_Q(\mathbb{R})$ associated with Q and let σ_Q be the K-invariant probability measure on X_Q. Furthermore, one assumes that X_Q contains a rational point; by stereographic projection, this implies in fact that $\mathbf{X}_Q(\mathbb{Q})$ is dense in X_Q. Fishman, Kleinbock, Merrill, and Simmons [Fis+22] have obtained the first remarkable results for intrinsic Diophantine approximation in this setting.

As a corollary of our main theorems, we obtain the following statement, which complements the above-mentioned results. Contrarily to the Kelmer-Yu theorem, we need no congruence condition on the dimension n, and we may in fact take X to be any nonsingular quadric hypersurface as considered by Fishman-Kleinbock-Merrill-Simmons. Moreover, let $X_0 \subseteq \mathbb{P}(\mathbb{R}^4)$ be the quadric hypersurface defined by the equation $Q_1(\boldsymbol{x}) = x_0x_3 - x_1x_2$. The $\mathbb{Q}$-rank of X_0 is 2 (see §6.6 for details) and the formulation of the Khintchine-type theorem for quadric hypersurfaces [Fis+22, Theorem 6.3] depends on whether or not X_Q is rationally isomorphic to X_0. Using [Fis+22, Theorem 1.5], the inequality $d(x,v) < H(v)^{-\tau}$ has infinitely (resp. finitely) many solutions $v \in \mathbf{X}_Q(\mathbb{Q})$ for almost every $x \in X_Q$, if $\tau \leqslant 1$ (resp. $\tau > 1$), that is, the Diophantine exponent of X_Q is 1.

Theorem 1.4.1. *Let X_Q be the space of real points of a nonsingular projective rational quadric hypersurface that is not rationally isomorphic to X_0 and contains at least one rational point. Let $\tau \in (0,1]$. Define the integral*

$$\forall\, T > 1, \quad \Psi(T) = \int_1^T y^{(1-\tau)n}\, \tfrac{dy}{y}.$$

Then there exist constants $\varkappa > 0$ and $\varepsilon > 0$ such that for almost every $x \in X_Q$, we have

$$\mathcal{N}_\tau(x,T) = \varkappa\, \Psi(T) \left(1 + O_x(\Psi(T)^{-\varepsilon})\right).$$

Our next result shows that rational points of bounded height are effectively equidistributed in X_Q. For every $x \in X_Q$ and $r > 0$, denote by $B_r(x)$ the open ball in X_Q with center x and radius r, and define the counting function

$$\mathcal{N}_\chi(x,r,T) = \#\left\{v \in \mathbf{X}_Q(\mathbb{Q}) : v \in B_r(x),\, 1 \leqslant H(v) < T\right\}. \tag{1.4.2}$$

Theorem 1.4.2. *Let X_Q be the space of real points of a nonsingular projective rational quadric hypersurface that is not rationally isomorphic to X_0 and contains at least one rational point. Then there exist constants $\varkappa > 0$ and $\varepsilon > 0$ such that for all $r > 0$ and $x \in X_Q$, we have*

$$\mathcal{N}_\chi(x,r,T) = \varkappa\, T^n \sigma_Q(B_r(x)) \left(1 + O_r(T^{-\varepsilon})\right). \tag{1.4.3}$$

Remark 1.4.3. Unfortunately, it seems that our method does not yield a similar result for the exceptional quadric hypersurface X_0. We will provide more details in §6.6.1, after introducing the necessary notation throughout the paper.

For integers $1 \leqslant \ell < n$, our main results also apply to the Grassmann variety $X_\ell = \mathrm{Gr}_{\ell,n}(\mathbb{R})$ of ℓ-dimensional linear subspaces in the Euclidean space $\mathbb{R}^n$, which represents another novelty. As in Schmidt's paper [Sch67], we use the Plücker embedding to define the height $H(v)$ of a rational subspace

v, and study the approximation of a real subspace chosen at random by rational subspaces. The distance used on X_ℓ is the usual Riemannian distance and we equip X_ℓ with the unique probability measure σ_ℓ invariant under rotations. Write d for the dimension $\dim_{\mathbb{R}} X_\ell = \ell(n-\ell)$ and set $\beta_\ell = \frac{n}{\ell(n-\ell)}$. As follows from [Sax22, Théorème 6], the inequality $d(x,v) < H(v)^{-\tau}$ has infinitely (resp. finitely) many solutions $v \in \mathrm{Gr}_{\ell,n}(\mathbb{Q})$ for almost every subspace $x \in X_\ell$, if $\tau \leqslant \beta_\ell$ (resp. $\tau > \beta_\ell$).

Theorem 1.4.4. *For integers $1 \leqslant \ell < n$, let $X_\ell = \mathrm{Gr}_{\ell,n}(\mathbb{R})$ be the Grassmann variety of ℓ-dimensional subspaces in $\mathbb{R}^n$ and let $\tau \in (0, \beta_\ell]$. Define the integral*

$$\forall\, T > 1, \quad \Psi(T) = \int_1^T y^{(\beta_\ell - \tau)d}\, \frac{dy}{y}.$$

Then there exist constants $\varkappa > 0$ and $\varepsilon > 0$ such that for almost every subspace $x \in X_\ell$, we have

$$\mathcal{N}_\tau(x,T) = \varkappa\, \Psi(T) \left(1 + O_x(\Psi(T)^{-\varepsilon})\right).$$

Finally, we show that rational points of bounded height are effectively equidistributed on the Grassmann variety X_ℓ. For every $x \in X_\ell$ and $r > 0$, denote by $B_r(x)$ the open ball in X_ℓ with center x and radius r, and define the counting function

$$\mathcal{N}_\chi(x,r,T) = \#\left\{v \in \mathrm{Gr}_{\ell,n}(\mathbb{Q}) : v \in B_r(x),\ 1 \leqslant H(v) < T\right\}. \tag{1.4.4}$$

Theorem 1.4.5. *For integers $1 \leqslant \ell < n$, let $X_\ell = \mathrm{Gr}_{\ell,n}(\mathbb{R})$ be the Grassmann variety of ℓ-dimensional subspaces in $\mathbb{R}^n$. Then there exist constants $\varkappa > 0$ and $\varepsilon > 0$ such that for all $r > 0$ and $x \in X_\ell$, we have*

$$\mathcal{N}_\chi(x,r,T) = \varkappa\, T^n \sigma_\ell(B_r(x)) \left(1 + O_r(T^{-\varepsilon})\right). \tag{1.4.5}$$

1.5 Notation and conventions

For two positive quantities A and B, we will use the notation $A \lesssim B$ or $A = O(B)$ to mean that there is a constant $C > 0$ such that $A \leqslant CB$, and we will use subscripts to indicate the dependence of the constant on parameters. We will write $A \asymp B$ for $A \lesssim B \lesssim A$. We denote by $\mathbb{N}$ the set of integers that are $\geqslant 0$ and by $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Moreover, we denote by $\mathbb{R}_+$ the set of non-negative real numbers $[0, +\infty)$. We denote algebraic varieties defined over $\mathbb{Q}$

by bold letters and their sets of real points by ordinary letters. Furthermore, following the exposition in Borel [Bor69], we shall, by abuse of notation, identify an algebraic variety defined over $\mathbb{Q}$ with its set of complex points. For instance, we write $X = \mathbf{X}(\mathbb{R})$ and $\mathbf{X} = \mathbf{X}(\mathbb{C})$ to denote the sets of real and complex points of an algebraic variety $\mathbf{X}$ defined over $\mathbb{Q}$, respectively. Given a $\mathbb{Q}$-subgroup $\mathbf{H}$ of an algebraic $\mathbb{Q}$-group $\mathbf{G}$ and an arithmetic subgroup Γ of $\mathbf{G}$, we denote $\Gamma_H = \Gamma \cap H$. Discrete subgroups of G are always equipped with the counting measure. Moreover, we denote by $X^*(\mathbf{G}) = \mathrm{Hom}(\mathbf{G}, \mathrm{GL}_1)$ (resp. $X_*(\mathbf{G}) = \mathrm{Hom}(\mathrm{GL}_1, \mathbf{G})$) the group of characters (resp. cocharacters) of $\mathbf{G}$, and by $X^*(\mathbf{G})_{\mathbb{Q}}$ (resp. $X_*(\mathbf{G})_{\mathbb{Q}}$) the subgroup of $X^*(\mathbf{G})$ (resp. $X_*(\mathbf{G})$) consisting of characters (resp. cocharacters) that are defined over $\mathbb{Q}$.

Chapter 2

An alternative proof of Schmidt's theorem

In this chapter, we give an alternative proof of Schmidt's result [Sch60a] for the projective space $\mathbb{P}(\mathbb{R}^n)$, one of the simplest flag varieties. The approach adapts arguments in a recent work of Kelmer and Yu [KY23] and has the advantage that it allows for an arbitrary decreasing approximation function. Roughly speaking, the method of proof relies on a mean square bound for the discrepancy that one deduces from a second moment formula of the Siegel transform, mapping functions of sufficient decay on $\mathbb{R}^n$ to functions on the space of unimodular lattices $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, by averaging over the lattice.

Before stating the result and describing the method in more detail, we need to introduce some notation. Fix an integer $n \geqslant 2$ and let $\mathbf{X} = \mathbb{P}^{n-1}$ be the projective $(n-1)$-space. Write $X = \mathbb{P}(\mathbb{R}^n)$ for the set of real points. Recall that X parametrizes lines passing through the origin in $\mathbb{R}^n$. More precisely, X is the space $(\mathbb{R}^n \smallsetminus \{\mathbf{0}\})/\sim$, where for $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}\}$ one has $\boldsymbol{v} \sim \boldsymbol{w}$ if and only if there exists $x \in \mathbb{R}^\times$ such that $\boldsymbol{v} = x\boldsymbol{w}$. We shall refer to points in projective space $X = \mathbb{P}(\mathbb{R}^n)$ as lines. We denote by $[\boldsymbol{v}] \in X$ the line corresponding to a nonzero vector $\boldsymbol{v} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}\}$ and we say that $\boldsymbol{v}$ is a representative for $[\boldsymbol{v}]$.

Define a height H on the set of rational points $\mathbf{X}(\mathbb{Q}) = \mathbb{P}(\mathbb{Q}^n)$ as follows. For $v \in \mathbf{X}(\mathbb{Q})$, choose a primitive vector $\boldsymbol{v} \in \mathbb{Z}^n$ that spans the rational line v and put

$$H(v) = \|\boldsymbol{v}\|,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

The Euclidean inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ induces a Euclidean norm on $\bigwedge^2\mathbb{R}^n$, that we also denote by $\|\cdot\|$, by setting, for decomposable elements $\boldsymbol{v}_1\wedge\boldsymbol{v}_2$,

$$\|\boldsymbol{v}_1\wedge\boldsymbol{v}_2\| = \sqrt{\det((\langle\boldsymbol{v}_i,\boldsymbol{v}_j\rangle)_{1\leqslant i,j\leqslant 2})}.$$

To measure the distance between a real line and a rational approximation, we use the following distance function on X. Given two lines $x,x'\in X$, choose non-zero vectors $\boldsymbol{x},\boldsymbol{x}'\in\mathbb{R}^n$ representing x,x', respectively, and set

$$d(x,x') = \frac{\|\boldsymbol{x}\wedge\boldsymbol{x}'\|}{\|\boldsymbol{x}\|\|\boldsymbol{x}'\|}.$$

The projective space $X=\mathbb{P}(\mathbb{R}^n)$ can be viewed as a homogeneous space. More precisely, let $G=\mathbf{G}(\mathbb{R})=\mathrm{SL}_n(\mathbb{R})$ be the group of real points of the special linear group of rank n and note that G acts transitively on X by $g\cdot[\boldsymbol{v}]=[g\boldsymbol{v}]$. The stabilizer in G of the line through the first standard basis vector $\boldsymbol{e}_1=(1,0,\ldots,0)$ is given explicitly by

$$P=\left\{\begin{pmatrix} a & \boldsymbol{u}^t\\ \mathbf{0} & H\end{pmatrix}\in G : a\in\mathbb{R}^\times,\ \boldsymbol{u}\in\mathbb{R}^{n-1},\ H\in\mathrm{GL}_{n-1}(\mathbb{R})\right\}.$$

In particular, P is a parabolic subgroup of G and we may identify X and G/P via the map $gP\mapsto g\cdot[\boldsymbol{e}_1]$. Moreover, note that already the maximal compact subgroup $K=\mathrm{SO}_n(\mathbb{R})$ acts transitively on X; this is a consequence of the well-known Iwasawa decomposition $G=KP$ of G. In particular, if dk denotes the Haar probability measure on K, then its push-forward to X via the orbit map $k\mapsto k\cdot[\boldsymbol{e}_1]$ defines a K-invariant probability measure on X that we fix and denote by σ_X. We also note that the distance $d(\cdot,\cdot)$ is K-invariant.

Let $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be a non-increasing function. Borrowing the terminology of [KM15], we say that a point $x\in X$ is *ψ-approximable* in X, if there are infinitely many rational points $v\in\mathbf{X}(\mathbb{Q})$ with $d(x,v)<\psi(H(v))$. For $x\in X$ and $T>1$ we define the counting function

$$\mathcal{N}_\psi(x,T) = \#\left\{v\in\mathbf{X}(\mathbb{Q}) : d(x,v)<\psi(H(v)),\ 1\leqslant H(v)<T\right\}. \tag{2.0.1}$$

Note that the function $\mathcal{N}_\psi(x,T)$ is an increasing function of T and that it is unbounded if and only if x is ψ-approximable in X.

Let $\zeta(s)$ be the Riemann ζ-function and let $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $\mathrm{Re}(s) > 0$ be the Γ-function. Define $d(P) = n(n-1)$ and the constant

$$\varkappa_n = \frac{\pi^{(n-1)/2}}{\zeta(n)\Gamma(\frac{n+1}{2})}. \tag{2.0.2}$$

The goal of this chapter is to prove the following analogue of Schmidt's theorem [Sch60a, Theorem 1] for the projective space.

Theorem 2.0.1. *Let $n \geqslant 3$ and let $X = \mathbb{P}(\mathbb{R}^n)$ be the projective space. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function satisfying $\lim_{t\to+\infty} \psi(t) = 0$ and define the two integrals*

$$\Psi(T) = \int_1^T t^{n-1}\psi(t)^{n-1}dt \quad \text{and} \quad \Omega(T) = \int_1^T t^{n-1}\psi(t)^{n+1}dt. \tag{2.0.3}$$

Suppose that $\Psi(T)$ diverges as $T \to +\infty$. Then for almost every $x \in X$, we have

$$\mathcal{N}_\psi(x,T) = \varkappa_n\Psi(T) + O_x\left(\Psi(T)^{\frac{2+d(P)}{3+d(P)}}\log(\Psi(T)) + \Omega(T)\right).$$

2.1 Method of proof

Write $\Omega = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ for the homogeneous space of unimodular lattices in $\mathbb{R}^n$ equipped with the unique $\mathrm{SL}_n(\mathbb{R})$-invariant Haar probability measure μ_Ω. A key input to the proof of Theorem 2.0.1 is a second moment formula for the classical primitive Siegel transform introduced in 1945 by Carl Ludwig Siegel [Sie45]. This transform maps a function f of sufficient decay on the Euclidean space $\mathbb{R}^n$ to a function on the space of unimodular lattices Ω, by averaging over the primitive vectors of the lattice. Let $\mathbb{Z}^n_{\mathrm{pr}}$ denote the set of primitive elements in $\mathbb{Z}^n$. More precisely, the primitive Siegel transform of f is defined for every $g \in \mathrm{SL}_n(\mathbb{R})$ by

$$Sf(g) = \sum_{\boldsymbol{v}\in\mathbb{Z}^n_{\mathrm{pr}}} f(g\boldsymbol{v}),$$

provided the sum converges absolutely. The Siegel mean value theorem expresses the average of the Siegel transform of f in terms of the average of f.

More precisely, denoting by $\lambda_{\mathbb{R}^n}$ the standard Lebesgue measure on $\mathbb{R}^n$, and denoting, for every $\boldsymbol{v} \in \mathbb{R}^n$, $f^-(\boldsymbol{v}) = f(-\boldsymbol{v})$, one has

$$\int_\Omega Sf \, d\mu_\Omega = \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} f \, d\lambda_{\mathbb{R}^n}. \tag{2.1.1}$$

Later, Rogers [Rog55] proved a second moment formula for the Siegel transform of a bounded compactly supported measurable function f, expressing the average of $|Sf|^2$ as a quadratic form on f:

$$\int_\Omega |Sf|^2 \, d\mu_\Omega = \frac{1}{\zeta(n)^2} \left(\int_{\mathbb{R}^n} f \, d\lambda_{\mathbb{R}^n} \right)^2 + \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} \left(|f|^2 + f f^- \right) \, d\lambda_{\mathbb{R}^n}. \tag{2.1.2}$$

It is convenient to denote by $d\,\mathrm{vol}$ the normalized Lebesgue measure $\zeta(n)^{-1}\, d\lambda_{\mathbb{R}^n}$ on $\mathbb{R}^n$.

Now, the outline of the proof is as follows. Similarly as in Schmidt's paper [Sch60b], for every $g \in \mathrm{SL}_n(\mathbb{R})$ and bounded Borel subset $B \subset \mathbb{R}^n$, we define the *discrepancy* of the lattice $g\mathbb{Z}^n$ in B by

$$D(g, B) = |\#(g\mathbb{Z}^n_{\mathrm{pr}} \cap B) - \mathrm{vol}(B)|.$$

Applying the second moment formula (2.1.2) to the Siegel transform of the indicator function of B, $S\mathbb{1}_B(g) = \#(g\mathbb{Z}^n_{\mathrm{pr}} \cap B)$, one obtains the following mean square bound on the discrepancy:

$$\int_\Omega |D(g, B)|^2 \, d\mu(g) \lesssim \mathrm{vol}(B). \tag{2.1.3}$$

To make use of this mean square bound, one translates the problem of counting Diophantine approximations of bounded height in the projective space to counting primitive lattice points in a certain family $(\mathcal{E}_\psi(T))_{T>1}$ of growing sets in the Euclidean space $\mathbb{R}^n$. The counting function $\mathcal{N}_\psi(x, T)$ can then be expressed as

$$\mathcal{N}_\psi(x, T) = \# \left(k_x^{-1} \mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T) \right)$$

for a certain $k_x \in \mathrm{SO}_n(\mathbb{R})$ depending on x. We wish to show that the right-hand side is asymptotically equal, up to a scalar, to the volume of $\mathcal{E}_\psi(T)$, as $T \to +\infty$. After showing that the sets $(\mathcal{E}_\psi(T))_{T>1}$ satisfy a certain well-roundedness condition, the desired asymptotic estimate is then deduced from the mean square bound on the discrepancy (2.1.3) and a standard Borel-Cantelli argument.

2.2 Coordinates and measures

Here, we introduce some group-theoretic and measure-theoretic notation. Let $\boldsymbol{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ be the first standard basis vector and set $x_0 = [\boldsymbol{e}_1] \in X$ for the corresponding point in the projective space. We regard x_0 as the base point in X. Let L be the stabilizer in G of the vector $\boldsymbol{e}_1$ and let P be the maximal parabolic subgroup fixing the line spanned by $\boldsymbol{e}_1$. The group G has an Iwasawa decomposition $G = KP$, with $K = \mathrm{SO}_n(\mathbb{R})$ a maximal compact subgroup. Furthermore, P has a Langlands decomposition $P = MAU$ with

$$U = \left\{ u_{\boldsymbol{x}} = \begin{pmatrix} 1 & \boldsymbol{x}^t \\ \boldsymbol{0} & I_{n-1} \end{pmatrix} : \boldsymbol{x} \in \mathbb{R}^{n-1} \right\}$$

the unipotent radical of P (note that $u_{\boldsymbol{x}}$ fixes $\boldsymbol{e}_1$),

$$A = \left\{ a(y) = \mathrm{diag}(y^{-1}, y^{\frac{1}{n-1}}, \ldots, y^{\frac{1}{n-1}}) : y > 0 \right\}$$

the $\mathbb{R}$-split torus with $a(y)$ acting on $\boldsymbol{e}_1$ as $a(y)\boldsymbol{e}_1 = y^{-1}\boldsymbol{e}_1$ and

$$M = \left\{ m = \begin{pmatrix} \varepsilon & \boldsymbol{0}^t \\ \boldsymbol{0} & \tilde{m} \end{pmatrix} : \varepsilon = \pm 1,\, \tilde{m} \in \mathrm{GL}_{n-1}(\mathbb{R}),\, \det \tilde{m} = \varepsilon \right\}.$$

Note that $L = M^\circ U$, where $M^\circ \cong \mathrm{SL}_{n-1}(\mathbb{R})$ is the identity component of M. Any $g \in G$ can be written as $g = ka(y)mu_{\boldsymbol{x}}$ with $k \in K$, $m \in M$, $a(y) \in A$, and $u_{\boldsymbol{x}} \in U \cong \mathbb{R}^{n-1}$. While km, $a(y)$ and $u_{\boldsymbol{x}}$ are uniquely determined by g, the elements k and m are only determined up to multiplication of k from the right and of m from the left by an element of $K \cap M$.

Let $d\boldsymbol{x}$ be the usual Lebesgue measure on U, dm the Haar measure of M normalized so that the induced M-invariant measure on $M/M(\mathbb{Z})$ is a probability measure, and dk the probability Haar measure of K. For every $a = \mathrm{diag}\,(a_1, \ldots, a_n) \in G$, let

$$\rho(a) = \prod_{i<j} \frac{a_i}{a_j}$$

be the sum of the positive roots of G relative to the full diagonal subgroup with multiplicities counted. Then ρ evaluates on $a(y)$ to $\rho(a(y)) = y^{-n}$ for every $y > 0$. Following [Kna02, §8.4 Integration], in these coordinates the Haar measure μ_G of G is given by

$$d\mu_G(g) = \omega_n\, \rho(a(y))\, dk\, dm\, \frac{dy}{y}\, d\boldsymbol{x} = \omega_n\, y^{-(n+1)}\, dk\, dm\, dy\, d\boldsymbol{x}, \tag{2.2.1}$$

where $\omega_n > 0$ is a constant so that the induced measure μ_Ω on G/Γ is a probability measure.

The subgroup L is unimodular with its Haar measure given by

$$d\mu_L(mu_{\boldsymbol{x}}) = dm\, d\boldsymbol{x}. \tag{2.2.2}$$

Since L is the stabilizer of $\boldsymbol{e}_1$ and G acts transitively on $\mathbb{R}^n \smallsetminus \{\boldsymbol{0}\}$, we can identify $\mathbb{R}^n \smallsetminus \{\boldsymbol{0}\}$ with the homogeneous space G/L. Explicitly, setting $K_L = L \cap K \cong \mathrm{SO}_{n-1}(\mathbb{R})$ and further identifying $K/K_L \times A$ with G/L via the map $(kK_L, a(y)) \mapsto ka(y)\boldsymbol{e}_1$ give natural polar coordinates on $\mathbb{R}^n \smallsetminus \{\boldsymbol{0}\}$: Every $\boldsymbol{v} \in \mathbb{R}^n \smallsetminus \{\boldsymbol{0}\}$ can be written uniquely as $\boldsymbol{v} = ka(y)\boldsymbol{e}_1$ for some $k \in K/K_L$ and $a(y) \in A$. Let σ_n be the unique right K-invariant probability measure on $K/K_L \cong \mathbb{S}^{n-1}$. Since the volume of the n-dimensional unit ball is $\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, in these coordinates the measure $d\lambda_{\mathbb{R}^n}$ is given by

$$d\lambda_{\mathbb{R}^n}(ka(y)\boldsymbol{e}_1) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\frac{dy}{y^{n+1}}d\sigma_n(k). \tag{2.2.3}$$

Let $\mu_{L/L(\mathbb{Z})}$ be the unique L-invariant Borel probability measure on the quotient $L/L(\mathbb{Z})$. Let us denote by $\mu_{G/L(\mathbb{Z})}$ the unique G-invariant Borel measures on the quotient $G/L(\mathbb{Z})$ induced from the Haar measure μ_G on G, that is, so that locally the projection $G \to G/L(\mathbb{Z})$ is measure-preserving. In particular, the following integration formula holds: for all $F \in C_c(G/L(\mathbb{Z}))$,

$$\int_{G/L(\mathbb{Z})} F\, d\mu_{G/L(\mathbb{Z})} = \frac{1}{\zeta(n)} \int_{K/K_L} \int_0^{+\infty} \int_{L/L(\mathbb{Z})} F(ka(y)\ell) d\mu_{L/L(\mathbb{Z})}(\ell) \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\frac{dy}{y^{n+1}} d\sigma_n(k). \tag{2.2.4}$$

As already said, it follows from the Iwasawa decomposition $G = KP$ that the fiber bundle map $K \to X$ defined by $k \to k[\boldsymbol{e}_1]$ is onto. In particular, if

$$\begin{aligned} X &\to K \\ x &\mapsto k_x \end{aligned}$$

is a measurable section of this map, we have for all $x \in X$, $x = k_x[\boldsymbol{e}_1]$. The group P is not unimodular and we let Δ_P be the modulus character of P. Fix a right Haar measure μ_P on P. As follows from [Fol15, page 64], the K-invariant measure σ_X on $X = G/P$ is *strongly quasi-invariant*, meaning there exists a continuous function $\nu_P : G \times G/P \to (0, +\infty)$ such that the translated measures $\sigma_{X,g}(E) = \sigma_X(gE)$ (where $g \in G$ and E is a measurable subset of X), satisfy

$$d\sigma_{X,g}(x) = \nu_P(g,x)\sigma_X(x).$$

Moreover, using [Fol15, Theorem 2.61], there exists a constant $c > 0$ such that the continuous function $\delta_P : G \to (0, +\infty)$ defined by $\delta_P(g) = c\,\nu_P(g, x_0)$ satisfies the functional equation:

$$\forall\, g \in G,\ \forall\, p \in P, \quad \delta_P(gp) = \Delta_P(p)\delta_P(g), \tag{2.2.5}$$

and the following integration formula holds:

$$\forall\, f \in C_c(G), \quad \int_G f(g)\delta_P(g)d\mu_G(g) = \int_{G/P}\int_P f(gp)d\mu_P(p)d\sigma_X(gP). \tag{2.2.6}$$

2.3 Second moment of the Siegel transform

Here, we give a new elementary proof of a well-known second moment formula for the Siegel transform on the space of unimodular lattices that is used to prove a mean square bound for the discrepancy. We recall that the primitive Siegel transform $Sf : \Omega = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}) \to \mathbb{R}$ of a measurable bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support is defined, for every $g \in \mathrm{SL}_n(\mathbb{R})$, by

$$Sf(g\mathbb{Z}^n) = \sum_{\boldsymbol{v} \in \mathbb{Z}^n_{\mathrm{pr}}} f(g\boldsymbol{v}).$$

We shall simply write $Sf(g)$ for $Sf(g\mathbb{Z}^n)$ and note that the right Γ-invariance of $g \mapsto Sf(g)$ implies that $Sf(g)$ is well-defined on the space of unimodular lattices $\Omega = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. Using Siegel's mean value formula (2.1.1), for any bounded Borel subset $B \subset \mathbb{R}^n$, one has

$$\int_\Omega S\mathbb{1}_B\, d\mu_\Omega = \frac{1}{\zeta(n)}\int_{\mathbb{R}^n} \mathbb{1}_B\, d\lambda_{\mathbb{R}^n} = \mathrm{vol}(B), \tag{2.3.1}$$

Later on in 1955, Rogers [Rog55, Theorem 5] proved a second moment formula for more general functions on Ω for $n \geqslant 3$. The following is a special case. In fact, in our result there is a constant $c > 0$, that we weren't able to determine yet. However, by [Rog55, Theorem 5], its value is $\zeta(n)^{-2}$.

Theorem 2.3.1 (Rogers)**.** *Let $n \geqslant 3$. There exists a constant $c > 0$ such that for every measurable bounded compactly supported function $f : \mathbb{R}^n \smallsetminus \{\mathbf{0}\} \to \mathbb{R}$, we have*

$$\int_\Omega |Sf|^2 \, d\mu_\Omega \;=\; c \left(\int_{\mathbb{R}^n} f \, d\lambda_{\mathbb{R}^n} \right)^2 + \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} \left(|f|^2 + f f^- \right) d\lambda_{\mathbb{R}^n}. \quad (2.3.2)$$

The case $n = 2$ was treated by Schmidt [Sch60b]. Let ϕ be the standard Euler ϕ-function and, for every $n \in \mathbb{Z}$, let $J_n = \begin{pmatrix} 1 & 1 \\ 0 & n \end{pmatrix}$. Given a 2×2 matrix $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{R})$, we denote by $[h]_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ the first column and by $[h]_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ the second.

Theorem 2.3.2. *Let $f : \mathbb{R}^2 \backslash \{\mathbf{0}\} \to \mathbb{R}$ be a measurable bounded compactly supported function. Then*

$$\int_\Omega |Sf|^2 \, d\mu_\Omega = \sum_{n \in \mathbb{Z} \backslash \{0\}} \frac{\phi(n)}{\zeta(2)^2} \int_{\mathrm{SL}_2(\mathbb{R})} f([gJ_n]_1) f([gJ_n]_2) \, d\mu_\Omega(g) + \frac{1}{\zeta(2)} \int_{\mathbb{R}^2} \left(|f|^2 + f f^- \right) d\lambda_{\mathbb{R}^n}.$$

Inspired by Schmidt's approach, we present a simple proof of Rogers' second moment formula in the case $n \geqslant 3$. We remark that a similar approach was also used by Marklof and Strömbergsson [MS10, §7, Lemma 7.7]. Before presenting the proof, we first state a consequence of Weil's integration formula [Wei53, Section 9], which we will use multiple times in this thesis. In practice, this formula is often referred to as "folding/unfolding".

Theorem 2.3.3. *Let H be a second countable locally compact unimodular group with Haar measure dh and $H_1 < H_2 < H$ closed unimodular subgroups. Then the following are true.*

1. *There exist unique (up to scaling) invariant measures μ_{H/H_1}, μ_{H/H_2}, and μ_{H_2/H_1} on H/H_1, H/H_2, and H_2/H_1, respectively.*

2. *We have $f \in L^1(H/H_1)$ if and only if for almost every $h \in H$, the*

function $h_2H_1 \mapsto f(hh_2H_1)$ is in $L^1(H_2/H_1)$ and the function

$$\begin{aligned} H/H_2 &\to [0,+\infty] \\ hH_2 &\mapsto \int_{H_2/H_1} |f(hh_2H_1)|\, d\mu_{H_2/H_1}(h_2H_1) \end{aligned}$$

is in $L^1(H/H_2)$. In this case, there exists a constant $\omega > 0$ so that the following formula holds:

$$\int_{H/H_1} f\, d\mu_{H/H_1} = \omega \int_{H/H_2} \int_{H_2/H_1} f(hh_2H_1)\, d\mu_{H/H_2}(hH_2)\, d\mu_{H_2/H_1}(h_2H_1). \tag{2.3.3}$$

3. *Assume $f : H/H_1 \to [0,+\infty]$ is measurable. Then the formula (2.3.3) holds.*

Proof of Theorem 2.3.1. Fix a measurable bounded compactly supported function $f : \mathbb{R}^n \smallsetminus \{\mathbf{0}\} \to \mathbb{R}$. Notice that $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ acts transitively on $\mathbb{Z}^n_{\mathrm{pr}}$. In particular, we may write

$$Sf(g) = \sum_{\boldsymbol{v}\in\mathbb{Z}^n_{\mathrm{pr}}} f(g\boldsymbol{v}) = \sum_{\gamma\in\Gamma/L(\mathbb{Z})} f(g\gamma\boldsymbol{e}_1),$$

where $L(\mathbb{Z}) = \Gamma \cap L$ is the stabilizer in Γ of $\boldsymbol{e}_1$. Let $\mu_{L/L(\mathbb{Z})}$ be the probability Haar measure on $L/L(\mathbb{Z})$. Using Weil's integration formula (2.3.3) (applied with $H = G$, $H_1 = \Gamma$, and $H_2 = L(\mathbb{Z})$) and the integration formula (2.2.4) (applied with $F(gL(\mathbb{Z})) = f(g\boldsymbol{e}_1)Sf(g)$), we have

$$\begin{aligned} \int_\Omega |Sf|^2\, d\mu_\Omega &= \int_\Omega \sum_{\gamma\in\Gamma/L(\mathbb{Z})} \left(f(g\gamma\boldsymbol{e}_1)Sf(g)\right)\, d\mu_\Omega(g\Gamma) \\ &= \int_{G/L(\mathbb{Z})} f(g\boldsymbol{e}_1)Sf(g)\, d\mu_{G/L(\mathbb{Z})}(gL(\mathbb{Z})) \\ &= \frac{1}{\zeta(n)} \int_{K/K_L} \int_0^{+\infty} f(g\boldsymbol{e}_1) \int_{L/L(\mathbb{Z})} Sf(g\ell)\, d\mu_{L/L(\mathbb{Z})}(\ell) \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{dy}{y^{n+1}} d\sigma_n(k). \end{aligned}$$

Our goal is to show that there exists a constant $c_n > 0$ such that for every measurable bounded compactly supported function $f : \mathbb{R}^n \smallsetminus \{\mathbf{0}\} \to \mathbb{R}$, we have

$$\int_{L/L(\mathbb{Z})} Sf(g\ell)\, d\mu_{L/L(\mathbb{Z})}(\ell L(\mathbb{Z})) = c_n \int_{\mathbb{R}^n} f\, d\lambda_{\mathbb{R}^n} + f(ka(y)\boldsymbol{e}_1) + f(-ka(y)\boldsymbol{e}_1).$$

We first decompose $\mathbb{Z}^n_{\mathrm{pr}}$ into a disjoint union of $L(\mathbb{Z})$-orbits. The vectors $\pm\boldsymbol{e}_1$ are fixed under the action of $L(\mathbb{Z})$. So let $\boldsymbol{v} \in \mathbb{Z}^n_{\mathrm{pr}}$ be a vector different from $\pm\boldsymbol{e}_1$. Write the vector $\boldsymbol{v} = (w_1, \boldsymbol{w})$ for some $w_1 \in \mathbb{Z}$ and $\boldsymbol{w} \in \mathbb{Z}^{n-1}\backslash\{\boldsymbol{0}\}$. By multiplying with a suitable element $\gamma \in L(\mathbb{Z})$, we can make sure that

$$\gamma\boldsymbol{v} = (l, \gcd(\boldsymbol{w}), 0, \ldots, 0)$$

for some unique $0 \leqslant l < \gcd(\boldsymbol{w})$ with $\gcd(l, \gcd(\boldsymbol{w})) = 1$. In fact, write $\boldsymbol{w} = \gcd(\boldsymbol{w})\boldsymbol{w}'$ for some unique $\boldsymbol{w}' \in \mathbb{Z}^{n-1}_{\mathrm{pr}}$ and let $\gamma = \begin{pmatrix} 1 & \boldsymbol{u}^t \\ \boldsymbol{0} & \gamma' \end{pmatrix} \in L(\mathbb{Z})$ where $\gamma' \in \mathrm{SL}_{n-1}(\mathbb{Z})$ is such that $\gamma'\boldsymbol{w}' = (1, 0, \ldots, 0)$ and $\boldsymbol{u}^t \in \mathbb{Z}^{n-1}$ is such that $l = w_1 + \gcd(\boldsymbol{v})\, \boldsymbol{u}^t \cdot \boldsymbol{w}'$ satisfies $0 \leqslant l < \gcd(\boldsymbol{w})$ and $\gcd(l, \gcd(\boldsymbol{w})) = 1$.

On the other hand, two primitive vectors $(l, q, 0, \ldots, 0)$ and $(l', q', 0, \ldots, 0)$ with $0 \leqslant l < q$ and $0 \leqslant l' < q'$ lie in the same $L(\mathbb{Z})$-orbit if and only if $l = l'$ and $q = q'$. For l, q as above, we define for simplicity $\boldsymbol{v}_{l,q} = (l, q, 0, \ldots, 0)$. Denote the stabilizer of $\boldsymbol{v}_{l,q}$ in L by L_1 and notice that it does not depend on the particular choice of l and $1 \leqslant q$. Consequently, we have

$$\begin{aligned}
&\int_{L/L(\mathbb{Z})} Sf(g\ell)\, d\mu_{L/L(\mathbb{Z})}(\ell L(\mathbb{Z})) \\
&\qquad = \int_{L/L(\mathbb{Z})} \sum_{\substack{\boldsymbol{v}\in\mathbb{Z}^n_{\mathrm{pr}},\\ \boldsymbol{v}\neq\pm\boldsymbol{e}_1}} f(ka(y)\ell\boldsymbol{v})\, d\mu_{L/L(\mathbb{Z})}(\ell L(\mathbb{Z})) + f(ka(y)\boldsymbol{e}_1) + f(-ka(y)\boldsymbol{e}_1) \\
&\qquad = \sum_{q=1}^{\infty} \sum_{\substack{0\leqslant l<q,\\ (l,q)=1}} \int_{L/L(\mathbb{Z})} \Bigg(\sum_{\gamma\in L(\mathbb{Z})/L_1(\mathbb{Z})} f(ka(y)\ell\gamma\boldsymbol{v}_{l,q})\, d\mu_{L/L(\mathbb{Z})}(\ell L(\mathbb{Z})) \\
&\qquad\qquad + f(ka(y)\boldsymbol{e}_1) + f(-ka(y)\boldsymbol{e}_1) \Bigg).
\end{aligned}$$

We equip L_1 with the Haar measure μ_{L_1} normalized so that the induced L_1-invariant measure on $L_1/L_1(\mathbb{Z})$ is a probability measure. Using Weil's integration formula (2.3.3) again and the L_1-invariance of $\boldsymbol{v}_{l,q}$, there exists a constant $\widehat{c} > 0$ such that

$$\sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} \int_{L/L(\mathbb{Z})} \sum_{\gamma \in L(\mathbb{Z})/L_1(\mathbb{Z})} f(ka(y)\ell\gamma \boldsymbol{v}_{l,q})\, d\mu_{L/L(\mathbb{Z})}(\ell L(\mathbb{Z}))$$

$$= \sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} \int_{L/L_1(\mathbb{Z})} f(ka(y)\ell \boldsymbol{v}_{l,q})\, d\mu_{L/L_1(\mathbb{Z})}(\ell L_1(\mathbb{Z}))$$

$$= \sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} \widehat{c} \int_{L/L_1} \int_{L_1/L_1(\mathbb{Z})} f(ka(y)\ell\ell_1 \boldsymbol{v}_{l,q})\, d\mu_{L_1/L_1(\mathbb{Z})}(\ell_1) d\mu_{L/L_1}(\ell)$$

$$= \widehat{c} \sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} \int_{L/L_1} f(ka(y)\ell \boldsymbol{v}_{l,q})\, d\mu_{L/L_1}(\ell).$$

Now, we write

$$\ell \boldsymbol{v}_{l,q} = \ell(l\boldsymbol{e}_1 + q\boldsymbol{e}_2) = l\boldsymbol{e}_1 + q\ell\boldsymbol{e}_2,$$

and identify L/L_1 with $\mathbb{R}^n \smallsetminus \{\boldsymbol{0}\}$ via $L \ni \ell \mapsto \ell\boldsymbol{e}_2$ (and thus equipping L/L_1 with the Lebesgue measure $\lambda_{\mathbb{R}^n}$). Therefore, using the substitution $q\boldsymbol{v} \mapsto \boldsymbol{v}$, the translation invariance of $\lambda_{\mathbb{R}^n}$, and that $ka(y)$ has determinant 1, we have

$$\widehat{c} \sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} \int_{L/L_1} f(ka(y)(l\boldsymbol{e}_1 + q\ell\boldsymbol{e}_2))\, d\mu_{L/L_1}(\ell)$$

$$= \widehat{c} \sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} \int_{\mathbb{R}^n} f(ka(y)(l\boldsymbol{e}_1 + q\boldsymbol{v}))\, d\lambda_{\mathbb{R}^n}(\boldsymbol{v})$$

$$= \widehat{c} \sum_{q=1}^{\infty} \sum_{\substack{0 \leqslant l < q,\\ (l,q)=1}} q^{-n} \int_{\mathbb{R}^n} f(ka(y)\boldsymbol{v})\, d\lambda_{\mathbb{R}^n}(\boldsymbol{v})$$

$$= \widehat{c} \sum_{q=1}^{\infty} \frac{\phi(q)}{q^n} \int_{\mathbb{R}^n} f(ka(y)\boldsymbol{v})\, d\lambda_{\mathbb{R}^n}(\boldsymbol{v})$$

$$= \widehat{c}\, \frac{\zeta(n-1)}{\zeta(n)} \int_{\mathbb{R}^n} f(ka(y)\boldsymbol{v})\, d\lambda_{\mathbb{R}^n}(\boldsymbol{v})$$

$$= \widehat{c}\, \frac{\zeta(n-1)}{\zeta(n)} \int_{\mathbb{R}^n} f\, d\lambda_{\mathbb{R}^n},$$

as required. The constant $c > 0$ in the statement of Theorem 2.3.1 is given by $c = \widehat{c}\,\frac{\zeta(n-1)}{\zeta(n)^2}$.

□

For any $g \in G$ and Borel subset $B \subset \mathbb{R}^n$ of finite measure, recall that the discrepancy of $g\mathbb{Z}^n$ in B is defined as

$$D(g, B) = \left|\#(g\mathbb{Z}^n_{\mathrm{pr}} \cap B) - \mathrm{vol}(B)\right|.$$

We obtain from [Rog55, Theorem 5] the following mean square bound for the discrepancy.

Lemma 2.3.4. *Let B be a bounded Borel subset of $\mathbb{R}^n$. Then*

$$\int_\Omega |D(g, B)|^2 \, d\mu(g) \lesssim \mathrm{vol}(B). \tag{2.3.4}$$

Proof. Observe that $\#(g\mathbb{Z}^n_{\mathrm{pr}} \cap B) = S\mathbb{1}_B(g)$. Using Siegel's mean value formula (2.1.1) applied with $f = \mathbb{1}_B$, we have

$$\begin{aligned}\int_\Omega |D(g, B)|^2 \, d\mu_\Omega(g) &= \int_\Omega |S\mathbb{1}_B|^2 \, d\mu_\Omega(g) - 2\,\mathrm{vol}(B) \int_\Omega S\mathbb{1}_B \, d\mu_\Omega(g) + \mathrm{vol}(B)^2 \\ &= \int_\Omega |S\mathbb{1}_B|^2 \, d\mu_\Omega - \mathrm{vol}(B)^2.\end{aligned}$$

Now, applying [Rog55, Theorem 5] to $f = \mathbb{1}_B$ gives

$$\int_\Omega |S\mathbb{1}_B|^2 \, d\mu_\Omega = \mathrm{vol}(B)^2 + \frac{1}{\zeta(n)} \int_{\mathbb{R}^n} |\mathbb{1}_B(\boldsymbol{v})|^2 + \mathbb{1}_B(\boldsymbol{v})\mathbb{1}_B(-\boldsymbol{v}) \, d\lambda_{\mathbb{R}^n}(\boldsymbol{v}).$$

Thus, altogether, we have $\int_\Omega |D(g, B)|^2 \, d\mu_\Omega(g) \lesssim \mathrm{vol}(B)$, as required. □

2.4 Diophantine approximation and counting lattice points

In this short section, we translate the problem of counting Diophantine approximations of bounded height in the projective space to corresponding estimates for counting primitive lattice points in certain families of growing sets in the Euclidean space.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. For every $T > 1$, define

$$\mathcal{E}_\psi(T) = \left\{ \boldsymbol{v} \in \widetilde{X} : d(x_0, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|), 0 < \|\boldsymbol{v}\| < T \right\}.$$

We have the following simple lemma.

Lemma 2.4.1. *Fix x in X and let $k_x \in K$ be such that $k_x[\boldsymbol{e}_1] = x$. Then for any $T > 1$, we have*

$$\mathcal{N}_\psi(x, T) = \frac{1}{2} \# \left(k_x^{-1} \mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T) \right). \tag{2.4.1}$$

Proof. It suffices to show that $\mathcal{N}_\psi(x, T) = \frac{1}{2} \# \left(\mathbb{Z}^n_{\mathrm{pr}} \cap k_x \mathcal{E}_\psi(T) \right)$. We first note that

$$k_x \mathcal{E}_\psi(T) = \{ \boldsymbol{v} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}\} : d(x, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|),\ 0 < \|\boldsymbol{v}\| < T \}.$$

Now a rational point $v \in X(\mathbb{Q})$ satisfies $d(x, v) < \psi(H(v))$ and $0 < H(v) < T$ if and only if any of the two primitive vectors $\pm \boldsymbol{v} \in \mathbb{Z}^n_{\mathrm{pr}}$ representing v satisfies $d(x, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|)$ and $0 < \|\boldsymbol{v}\| < T$. This finishes the proof of the lemma. $\square$

2.5 Measure estimates

In the last section, we have seen that for every $x \in X$,

$$\mathcal{N}_\psi(x, T) = \#(k_x^{-1} \mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T)).$$

Noting that $\#(g\mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T))$ equals the Siegel transform $S\mathbb{1}_{\mathcal{E}_\psi(T)}(g)$ of the indicator function of $\mathcal{E}_\psi(T)$ and applying the first moment formula (2.3.1), the expected value is given by

$$\int_\Omega \#(g\mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T))\, d\mu(g) = \mathrm{vol}(\mathcal{E}_\psi(T)).$$

Let $d(P) = n(n-1)$. In Theorem 2.7.1 below, we will show that for almost every $x \in X$ and for all $T > 1$ sufficiently large,

$$\#(k_x^{-1} \mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T)) = \mathrm{vol}(\mathcal{E}_\psi(T)) + O\left(\mathrm{vol}(\mathcal{E}_\psi(T))^{\frac{2+d(P)}{3+d(P)}} \log(\mathrm{vol}(\mathcal{E}_\psi(T))) \right).$$

In order to make the link with our main Theorem 2.0.1, we explicitly compute $\mathrm{vol}(\mathcal{E}_\psi(T))$ in this section.

Let $\boldsymbol{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \smallsetminus \{\mathbf{0}\}$ and note that

$$d(x_0, [\boldsymbol{v}]) = \frac{\|\boldsymbol{e}_1 \wedge \boldsymbol{v}\|}{\|\boldsymbol{e}_1\|\|\boldsymbol{v}\|} = \frac{\|\boldsymbol{v}'\|}{\|\boldsymbol{v}\|}, \tag{2.5.1}$$

where by $\boldsymbol{v}'$ we always denote the vector $\boldsymbol{v}' = (v_2, \ldots, v_n) \in \mathbb{R}^{n-1}$. Let $r \in (0,1)$ and set $\mathfrak{D}_r = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{S}^{n-1} : \|\boldsymbol{x}'\| < r, \, x_1 > 0\}$. Since $\boldsymbol{x} \in \mathbb{S}^{n-1}$ and thus $|x_1|^2 + \|\boldsymbol{x}'\|^2 = 1$, we may alternatively write

$$\mathfrak{D}_r = \{\boldsymbol{x} \in \mathbb{S}^{n-1} : x_1 > \sqrt{1 - r^2}\}.$$

Observe that $\mathfrak{D}_r$ is a polar cap centered at the north pole on $\mathbb{S}^{n-1}$.

Lemma 2.5.1. *For every $r > 0$, we have*

$$\sigma_n(\mathfrak{D}_r) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{1}{2}}\Gamma(\frac{n+1}{2})} r^{n-1} + O_n(r^{n+1}). \tag{2.5.2}$$

Proof. By the form of the error term, it suffices to prove the lemma for all small r, say $r \in (0, \frac{1}{2})$. Using [KY23, Lemma 5.2], for all $r' \in (0,1)$, we have

$$\sigma_n\left(\left\{\boldsymbol{x} \in \mathbb{S}^{n-1} : x_1 > 1 - \frac{r'^2}{2}\right\}\right) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{1}{2}}\Gamma(\frac{n+1}{2})} (r')^{n-1} + O_n((r')^{n+1}) \tag{2.5.3}$$

Setting $r' = \sqrt{2(1 - \sqrt{1 - r^2})}$ for small enough $r > 0$, one can check that

$$\mathfrak{D}_r = \{\boldsymbol{x} \in \mathbb{S}^{n-1} : x_1 > 1 - \frac{r'^2}{2}\}.$$

Taylor expansion at $r = 0$ gives

$$\sqrt{2(1 - \sqrt{1 - r^2})} = r + O(r^3),$$

and plugging this into (2.5.3) yields the claim. $\square$

Lemma 2.5.2. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and define the integrals*

$$\Psi(T) = \int_0^T t^{n-1}\psi(t)^{n-1} dt, \quad \Omega(T) = \int_0^T t^{n-1}\psi(t)^{n+1} dt.$$

Then, as T tends to $+\infty$, we have

$$\operatorname{vol}(\mathcal{E}_\psi(T)) = \varkappa_n \Psi(T) + O(\Omega(T)), \tag{2.5.4}$$

where $\varkappa_n$ was defined in (2.0.2).

Proof. Recall that $K_L = L \cap K \cong \mathrm{SO}_{n-1}(\mathbb{R})$ and that the map of $K/K_L \times A$ to G/L given by $(kK_L, a(y)) \mapsto ka(y)\boldsymbol{e}_1$ gives natural polar coordinates on $\mathbb{R}^n \smallsetminus \{\mathbf{0}\}$: Every $\boldsymbol{v} \in \mathbb{R}^n \smallsetminus \{\mathbf{0}\}$ can be written uniquely as $\boldsymbol{v} = ka(y)\boldsymbol{e}_1$ for some $k \in K/K_L$ and $a(y) \in A$. In these coordinates, the set $\mathcal{E}_\psi(T)$ can be described as follows

$$\mathcal{E}_\psi(T) = \left\{\boldsymbol{v} = ka(y)\boldsymbol{e}_1 \in \widetilde{X} : \|(k\boldsymbol{e}_1)'\| < \psi(y^{-1}),\, y > T^{-1}\right\}.$$

Using equation (2.2.3) and Lemma 2.5.1, we have

$$\begin{aligned}
\operatorname{vol}(\mathcal{E}_\psi(T)) &= \frac{2\pi^{n/2}}{\zeta(n)\Gamma(\frac{n}{2})} \int_{T^{-1}}^{+\infty} \sigma_n(\mathfrak{D}_{\psi(y^{-1})}) \frac{dy}{y^{n+1}} \\
&= \frac{2\pi^{n/2}}{\zeta(n)\Gamma(\frac{n}{2})} \int_{T^{-1}}^{+\infty} \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{1}{2}}\Gamma(\frac{n+1}{2})} \psi(y^{-1})^{n-1} + O_n(\psi(y^{-1})^{n+1}) \frac{dy}{y^{n+1}} \\
&= \varkappa_n \int_0^T t^{n-1}\psi(t)^{n-1}\, dt + O_n\left(\int_0^T t^{n-1}\psi(t)^{n+1}\, dt\right)
\end{aligned}$$

This finishes the proof of the lemma. □

2.6 Well-roundedness

The term *well-roundedness* was introduced in [EM93] and [DRP93], and has been frequently used since then (see, for instance, [GN10], [GN12]). It refers to the regularity property of a set to be almost invariant under the action of a small ball centered at the identity in G, and it allows for asymptotic lattice point counts with an error term. Our approach in this section is inspired by the recent work of Kelmer and Yu [KY23].

The goal is to show that the family $(\mathcal{E}_\psi(T))_{T>1}$ is well-rounded with respect to a certain family $\mathcal{O} = (P_\varepsilon)_{0<\varepsilon<1}$ of symmetric identity neighborhoods in the maximal parabolic subgroup P. We recall that P is defined as the stabilizer in $G = \mathrm{SL}_n(\mathbb{R})$ of the line through $\boldsymbol{e}_1$.

We will work with the following definition of well-rounded sets.

Definition 2.6.1 (Well-roundedness)**.** Let $\mathcal{O} = (P_\varepsilon)_{0<\varepsilon<1}$ be an increasing family of symmetric identity neighborhoods of P. We say that a family $\mathcal{B}'$ of finite-measure Borel subsets in $\mathbb{R}^n \backslash \{\mathbf{0}\}$ is *well-rounded* with respect to $\mathcal{O}$ if there exist $C_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for any $B' \in \mathcal{B}'$ and any $\varepsilon \in (0, \varepsilon_0)$, the Borel sets

$$\underline{B}'_\varepsilon = \bigcap_{p \in P_\varepsilon} pB' \subseteq \bigcup_{p \in P_\varepsilon} pB' =: \overline{B}'_\varepsilon \quad \text{satisfy} \quad \operatorname{vol}(\overline{B}'_\varepsilon \backslash \underline{B}'_\varepsilon) \leqslant C_0 \varepsilon \operatorname{vol}(B'). \tag{2.6.1}$$

Let $d_G(\cdot,\cdot)$ be a left-invariant Riemannian distance on $G = \mathrm{SL}_n(\mathbb{R})$. Let $d_P(\cdot,\cdot)$ denote the induced left-invariant Riemannian distance on P by viewing P as an embedded submanifold of the Riemannian manifold G. Given $0 < \varepsilon \leqslant 1$ and $c > 0$ (to be determined), we define $P_\varepsilon = B_P(c\varepsilon)$ to be the metric open ball in P centered at the identity with radius $c\varepsilon$. Since the dimension of P is $n(n-1)$, we have $\mu_P(P_\varepsilon) \asymp \varepsilon^{n(n-1)}$, and we set $d(P) = n(n-1)$. Next, we verify that the family $(\mathcal{E}_\psi(T))_{T>1}$ is well-rounded with respect to the family $\mathcal{O} = (P_\varepsilon)_{0<\varepsilon<1}$. Let $\psi : \mathbb{R}_+ \to [0, 1/2]$ be a non-increasing function and define for every $t \in \mathbb{R}_+$ and $\varepsilon \in (0, \frac{1}{2})$ the functions $\psi_\varepsilon^\pm(t) = (1+\varepsilon)^{\pm 1}\psi((1+\varepsilon)^{\mp 1}t)$.

Lemma 2.6.2. *Let $\psi : \mathbb{R}_+ \to [0, 1/2]$ be a non-increasing function. Then there exists $c > 0$ such that for all $T > 1$, $p \in P_\varepsilon$, and $\varepsilon \in (0, \frac{1}{2})$,*

$$\mathcal{E}_{\psi_\varepsilon^-}((1+\varepsilon)^{-1}T) \subseteq p\mathcal{E}_\psi(T) \subseteq \mathcal{E}_{\psi_\varepsilon^+}((1+\varepsilon)T). \tag{2.6.2}$$

Proof. Fix $\varepsilon \in (0, \frac{1}{2})$, $p = ma(y)u_{\boldsymbol{x}} \in P_\varepsilon$ and $T > 1$. We first prove the relation

$$p\mathcal{E}_\psi(T) \subseteq \mathcal{E}_{\psi_\varepsilon^+}((1+\varepsilon)T).$$

That is, given $\boldsymbol{v} \in \mathcal{E}_\psi(T)$, we need to verify the following two conditions:

(1) $d(x_0, [p\boldsymbol{v}]) < \psi_\varepsilon^+(\|p\boldsymbol{v}\|)$,

(2) $0 < \|p\boldsymbol{v}\| < (1+\varepsilon)T$.

Since each $\boldsymbol{v} \in \mathcal{E}_\psi(T)$ satisfies $0 < \|\boldsymbol{v}\| < T$, in order to prove (2), it suffices to show that

$$0 < \|p\boldsymbol{v}\| < (1+\varepsilon)\|\boldsymbol{v}\|.$$

Applying the triangle inequality, one has $\|p\boldsymbol{v}\| \leqslant \|\boldsymbol{v}\| + \|p\boldsymbol{v} - \boldsymbol{v}\|$, and, since $\|p\boldsymbol{v} - \boldsymbol{v}\| \leqslant \|p - \mathrm{id}\|\|\boldsymbol{v}\|$, the claim follows by choosing $c > 0$ so that $\|p - \mathrm{id}\| < \varepsilon$ for every $p \in B_P(c\varepsilon)$.

Next, we show condition (1). Combining the estimate $\|p\boldsymbol{v}\| < (1+\varepsilon)\|\boldsymbol{v}\|$ with the fact that ψ is non-increasing, we obtain

$$\psi(\|\boldsymbol{v}\|) \leqslant \psi((1+\varepsilon)^{-1}\|p\boldsymbol{v}\|).$$

Therefore it suffices to show that $d(x_0, [p\boldsymbol{v}]) < (1+\varepsilon)\psi(\|\boldsymbol{v}\|)$. By Equation (2.5.1), we have $\|\boldsymbol{e}_1 \wedge p\boldsymbol{v}\| = \|(p\boldsymbol{v})'\|$. By choosing $c > 0$ even smaller if necessary, a calculation in the explicit coordinates of $\boldsymbol{v} = (v_1, \boldsymbol{v}')$ and $p = ma(y)u_{\boldsymbol{x}}$ shows that

$$d(x_0, [p\boldsymbol{v}]) = \frac{\|(p\boldsymbol{v})'\|}{\|p\boldsymbol{v}\|} \leqslant (1+\varepsilon)\frac{\|\boldsymbol{v}'\|}{\|\boldsymbol{v}\|} = (1+\varepsilon)d(x_0, [\boldsymbol{v}]).$$

Using that $d(x_0, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|)$ by assumption, we get $p\boldsymbol{v} \in \mathcal{E}_{\psi_\varepsilon^+}((1+\varepsilon)T)$, as required. This finishes the proof of the relation $p\mathcal{E}_\psi(T) \subseteq \mathcal{E}_{\psi_\varepsilon^+}((1+\varepsilon)T)$. Similarly, using the fact that P_ε is symmetric and applying the first part to $\psi = \psi_\varepsilon^-$, one gets

$$p^{-1}\mathcal{E}_{\psi_\varepsilon^-}((1+\varepsilon)^{-1}T) \subseteq \mathcal{E}_\psi(T).$$

This completes the proof of the lemma. □

Proposition 2.6.3. *Let $\psi : \mathbb{R}_+ \to [0, \frac{1}{2}]$ be a non-increasing function satisfying $\lim_{t\to+\infty}\psi(t) = 0$ and $\int_0^{+\infty} t^{n-1}\psi(t)^{n-1}dt = +\infty$. Then the family $(\mathcal{E}_\psi(T))_{T>1}$ is well-rounded with respect to $\mathcal{O} = (P_\varepsilon)_{0<\varepsilon<1}$.*

Proof. We show that there exists $C_0 > 0$ (depending only on n) such that for any $\mathcal{E}_\psi(T)$ with $T > 1$ and for any $\varepsilon \in (0, 1/2)$, there exist Borel subsets $\overline{B}_\varepsilon, \underline{B}_\varepsilon$ satisfying

$$\underline{B}_\varepsilon \subseteq \bigcap_{p\in P_\varepsilon} p\mathcal{E}_\psi(T) \subseteq \bigcup_{p\in P_\varepsilon} p\mathcal{E}_\psi(T) \subseteq \overline{B}_\varepsilon \quad \text{and} \quad \mathrm{vol}(\overline{B}_\varepsilon \backslash \underline{B}_\varepsilon) \leqslant C_0\varepsilon\,\mathrm{vol}(B). \tag{2.6.3}$$

For this, we take $\underline{B}_\varepsilon = \mathcal{E}_{\psi_\varepsilon^-}((1+\varepsilon)^{-1}T)$ and $\overline{B}_\varepsilon = \mathcal{E}_{\psi_\varepsilon^+}((1+\varepsilon)T)$, where $\psi_\varepsilon^\pm$ are as in Lemma 2.6.2. The above inclusion relations follow from Lemma 2.6.2. For the measure bound, we note that $\overline{B}_\varepsilon \backslash \underline{B}_\varepsilon \subseteq R_1 \cup R_2$ with

$$R_1 = \left\{ka(y)\boldsymbol{e}_1 \in \mathbb{R}^n\backslash\{\boldsymbol{0}\} : \psi_\varepsilon^-(y^{-1}) \leqslant \|(k\boldsymbol{e}_1)'\| \leqslant \psi_\varepsilon^+(y^{-1}),\ 0 < y^{-1} \leqslant (1+\varepsilon)^{-1}T\right\},$$

and

$$R_2 = \left\{ka(y)e_1 \in \mathbb{R}^n\backslash\{\boldsymbol{0}\} : \|(k\boldsymbol{e}_1)'\| \leqslant \psi_\varepsilon^+(y^{-1}),\ (1+\varepsilon)^{-1}T \leqslant y^{-1} \leqslant (1+\varepsilon)T\right\}.$$

By Lemma 2.5.1 (noting also that by our assumptions $\psi_\varepsilon^+(t) < 1$ for all $t \in \mathbb{R}_+$), we have

$$\operatorname{vol}(R_1) \lesssim_n \int_0^{(1+\varepsilon)^{-1}T} t^{n-1} \left(\psi_\varepsilon^+(t)^{n-1} - \psi_\varepsilon^-(t)^{n-1}\right) dt$$

and

$$\operatorname{vol}(R_2) \lesssim_n \int_{(1+\varepsilon)^{-1}T}^{(1+\varepsilon)T} t^{n-1}\psi_\varepsilon^+(t)^{n-1}\, dt.$$

This, together with the measure estimate $\operatorname{vol}(\mathcal{E}_\psi(T)) \asymp_n \Psi(T)$ (cf. Lemma 2.5.2) implies that

$$\begin{aligned}\operatorname{vol}(\overline{B}_\varepsilon \backslash \underline{B}_\varepsilon) &\lesssim_n \int_0^{(1+\varepsilon)T} t^{n-1}\psi_\varepsilon^+(t)^{n-1}\, dt - \int_0^{(1+\varepsilon)^{-1}T} t^{n-1}\psi_\varepsilon^-(t)^{n-1}\, dt \\ &= \left((1+\varepsilon)^n - (1+\varepsilon)^{-n}\right) \int_0^T t^{n-1}\psi(t)^{n-1}\, dt \lesssim_n \varepsilon \operatorname{vol}(\mathcal{E}_\psi(T)).\end{aligned}$$

This finishes the proof. □

2.7 From mean square discrepancy to counting

In this section, we prove effective counting estimates on $\#(k_x^{-1}\mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T))$ as $T \to +\infty$. Our approach is inspired by the arguments given in [KY23, §3] to count rational approximations on spheres. More precisely, we use the well-roundedness of the family $(\mathcal{E}_\psi(T))_{T>1}$ as established in Proposition 2.6.3) and the mean square bound on the discrepancy (2.3.4) to deduce the following counting result, which holds for almost every $x \in X$ as $T \to +\infty$. Throughout, we let $d(P) = n(n-1)$.

Theorem 2.7.1. *Let $\psi : \mathbb{R}_+ \to [0, 1/2]$ be a non-increasing function satisfying $\lim_{t\to+\infty}\psi(t) = 0$ and $\int_0^{+\infty} t^{n-1}\psi(t)^{n-1}dt = +\infty$. Then for almost every $x \in X$ and for all sufficiently large $T > 1$, we have*

$$\#\left(k_x^{-1}\mathbb{Z}^n_{\mathrm{pr}} \cap \mathcal{E}_\psi(T)\right) = \operatorname{vol}(\mathcal{E}_\psi(T)) + O\left(\operatorname{vol}(\mathcal{E}_\psi(T))^{\frac{2+d(P)}{3+d(P)}} \log(\operatorname{vol}(\mathcal{E}_\psi(T)))\right).$$

The following proposition, which is a key ingredient for Theorem 2.7.1, is a special case of [KY23, Proposition 3.6]. Recall that for $\varepsilon \in (0, \frac{1}{2})$ we defined

$P_\varepsilon = B_P(c\varepsilon)$ to be the metric open ball in P centered at the identity with radius $c\varepsilon$ for some sufficiently small and fixed $c > 0$.

Proposition 2.7.2 (Kelmer-Yu)**.** *Let $C_0 > 0$ be as in Equation (2.6.3). Then for all sufficiently large $T > 1$ and for all $0 < Z < C_0 \operatorname{vol}(\mathcal{E}_\psi(T))$, we have*

$$\sigma_X(\{x \in X : D(k_x^{-1}, \mathcal{E}_\psi(T)) > Z\}) \lesssim_{\mathcal{O},C_0,n} \frac{\operatorname{vol}(\mathcal{E}_\psi(T))^{1+d(P)}}{Z^{2+d(P)}}.$$

For the convenience of the reader, we reproduce their argument here.

Proof. For any $T > 1$ and any $Z \in (0, C_0 \operatorname{vol}(\mathcal{E}_\psi(T)))$, we set

$$\mathcal{M}_{T,Z} = \left\{x \in X : D(k_x^{-1}, \mathcal{E}_\psi(T)) > Z\right\}.$$

Let us also denote by

$$\mathcal{M}_{T,Z}^{\pm} = \left\{x \in X : \pm\left(\#(k_x^{-1}\mathbb{Z}_{\mathrm{pr}}^n \cap \mathcal{E}_\psi(T)) - \operatorname{vol}(\mathcal{E}_\psi(T))\right) > Z\right\},$$

so that $\mathcal{M}_{T,Z} = \mathcal{M}_{T,Z}^{+} \cup \mathcal{M}_{T,Z}^{-}$. It thus suffices to show that for all sufficiently large T,

$$\sigma_X(\mathcal{M}_{T,Z}^{\pm}) \lesssim_{\mathcal{O},C_0,n} \frac{\operatorname{vol}(\mathcal{E}_\psi(T))^{1+d(P)}}{Z^{2+d(P)}}.$$

We prove this for $\mathcal{M}_{T,Z}^{+}$, the other case is shown similarly.

So take $\varepsilon = \frac{Z}{2C_0 \operatorname{vol}(\mathcal{E}_\psi(T))}$, so that $Z - C_0\varepsilon \operatorname{vol}(\mathcal{E}_\psi(T)) = \frac{Z}{2}$. Note that since $0 < Z < C_0 \operatorname{vol}(\mathcal{E}_\psi(T))$, we have $\varepsilon \in (0, \frac{1}{2})$. Let $x \in \mathcal{M}_{T,Z}^{+}$. Then for every $p \in P_\varepsilon$, by Proposition 2.6.3 and since P_ε is symmetric, for $\overline{B}_\varepsilon$ as in (2.6.3), we have

$$\begin{aligned}
\#(k_x^{-1}\mathbb{Z}_{\mathrm{pr}}^n \cap p\overline{B}_\varepsilon) &\geqslant \#(k_x^{-1}\mathbb{Z}_{\mathrm{pr}}^n \cap \mathcal{E}_\psi(T)) \\
&> \operatorname{vol}(\mathcal{E}_\psi(T)) + Z \\
&\geqslant \operatorname{vol}(\overline{B}_\varepsilon) - C_0\varepsilon \operatorname{vol}(\mathcal{E}_\psi(T)) + Z \\
&= \operatorname{vol}(\overline{B}_\varepsilon) + \frac{Z}{2},
\end{aligned}$$

implying that

$$D(p^{-1}k_x^{-1}, \overline{B}_\varepsilon) \geqslant \#(p^{-1}k_x^{-1}\mathbb{Z}_{\mathrm{pr}}^n \cap \overline{B}_\varepsilon) - \operatorname{vol}(\overline{B}_\varepsilon) > \frac{Z}{2}.$$

Hence

$$\frac{Z^2}{4}\sigma_X(\mathcal{M}^+_{T,Z})\mu_P(P_\varepsilon) < \int_X \int_{P_\varepsilon} D(p^{-1}k_x^{-1}, \overline{B}_\varepsilon)^2 \, d\mu_P(p) \, d\sigma_X(x).$$

Let us denote by I this double integral.

On the other hand, letting $\mathcal{S}_\varepsilon = \{k_x p : x \in X,\ p \in P_\varepsilon\} \subseteq G$ and using the measure description in equation (2.2.6), since G is unimodular, we have

$$I = \int_{\mathcal{S}_\varepsilon} D(g^{-1}, \overline{B}_\varepsilon)^2 \delta_P(g) \, d\mu_G(g) = \int_{\mathcal{S}_\varepsilon^{-1}} D(g, \overline{B}_\varepsilon)^2 \delta_P(g^{-1}) \, d\mu_G(g).$$

Since $P_\varepsilon \subseteq P_1$ and both $P_1 \subseteq P$ and X are compact, $\mathcal{S}_\varepsilon^{-1}$ can be covered by a finite number, possibly depending on P_1, of fundamental domains for G/Γ. Moreover, since δ_P is continuous and strictly positive on G and since the closure of $\mathcal{S}_\varepsilon$ is compact, we have $\delta_P(g^{-1}) \asymp_{P_1} 1$ for all $g \in \mathcal{S}_\varepsilon^{-1}$. Thus, using the mean square bound on the discrepancy as in Lemma 2.3.4 and that $\mathrm{vol}(\overline{B}_\varepsilon) < (1 + C_0\varepsilon)\,\mathrm{vol}(\mathcal{E}_\psi(T)) \lesssim_{C_0} \mathrm{vol}(\mathcal{E}_\psi(T))$, we have

$$\begin{aligned} I &\lesssim_{P_1} \int_{\mathcal{S}_\varepsilon^{-1}} D(g, \overline{B}_\varepsilon)^2 \, d\mu_G(g) \\ &\lesssim_{P_1} \int_\Omega D(g, \overline{B}_\varepsilon)^2 \, d\mu_\Omega(g\Gamma) \lesssim \mathrm{vol}(\overline{B}_\varepsilon) \lesssim_{C_0} \mathrm{vol}(\mathcal{E}_\psi(T)). \end{aligned}$$

Thus, recalling that $\mu_P(P_\varepsilon) \asymp \varepsilon^{d(P)}$, we have

$$\begin{aligned} \sigma_X(\mathcal{M}^+_{T,Z}) &\lesssim_{C_0,P_1} \tfrac{\mathrm{vol}(\mathcal{E}_\psi(T))}{Z^2}\mu_P(P_\varepsilon)^{-1} \\ &\lesssim \tfrac{\mathrm{vol}(\mathcal{E}_\psi(T))}{Z^2}\left(\tfrac{Z}{2C_0\,\mathrm{vol}(\mathcal{E}_\psi(T))}\right)^{-d(P)} \asymp_{C_0,d(P)} \tfrac{\mathrm{vol}(\mathcal{E}_\psi(T))^{1+d(P)}}{Z^{2+d(P)}}. \end{aligned}$$

Similarly, we also have $\sigma_X(\mathcal{M}^-_{T,Z}) \lesssim_{\mathcal{O},C_0,n} \frac{\mathrm{vol}(\mathcal{E}_\psi(T))^{1+d(P)}}{Z^{2+d(P)}}$, finishing the proof. □

We can now prove Theorem 2.7.1 using a Borel-Cantelli-type argument. The proof follows the arguments in [KY23, Lemma 3.2 and Corollary 3.3].

Proof of Theorem 2.7.1. Let $\psi : \mathbb{R}_+ \to [0, \frac{1}{2}]$ be as in this theorem. By Proposition 2.6.3, the increasing family $(\mathcal{E}_\psi(T))_{T>1}$ is well-rounded with respect to the family of identity neighborhoods $\mathcal{O} = (P_\varepsilon)_{0<\varepsilon<1}$. Thus we can apply Proposition 2.7.2 to get for all $T > V$, where $V > 0$ is some large positive constant, and for all $0 < Z < C_0 \operatorname{vol}(\mathcal{E}_\psi(T))$,

$$\sigma_X(\{x \in X : D(k_x^{-1}, \mathcal{E}_\psi(T)) > Z\}) \lesssim_{\mathcal{O},C_0,n} \frac{\operatorname{vol}(\mathcal{E}_\psi(T))^{1+d(P)}}{Z^{2+d(P)}}. \tag{2.7.1}$$

Now to prove the theorem, it suffices to show that for almost every $x \in X$ and for all sufficiently large $T > 1$,

$$D(k_x^{-1}, \mathcal{E}_\psi(T)) \lesssim \operatorname{vol}(\mathcal{E}_\psi(T))^{\frac{2+d(P)}{3+d(P)}} \log\left(\operatorname{vol}(\mathcal{E}_\psi(T))\right).$$

Set $\nu = \frac{1}{3+d(P)}$ and $\omega = \frac{1}{\nu}$. Let $(\mathcal{E}_\psi(T_k))_{k\in\mathbb{N}^*}$ be the increasing sequence given by $\operatorname{vol}(\mathcal{E}_\psi(T_k)) = \max\{V, k^\omega \log(k)\}$. Since the family $(\mathcal{E}_\psi(T))_{T>1}$ is increasing, any $\mathcal{E}_\psi(T)$ with $\operatorname{vol}(\mathcal{E}_\psi(T)) > V$ satisfies $\mathcal{E}_\psi(T_k) \subseteq \mathcal{E}_\psi(T) \subseteq \mathcal{E}_\psi(T_{k+1})$ for some k and we can estimate for all k sufficiently large

$$\begin{aligned}\operatorname{vol}(\mathcal{E}_\psi(T_{k+1}) \backslash \mathcal{E}_\psi(T_k)) &\lesssim k^{\omega-1}\log(k) \lesssim k^{\omega-1}\log(k)^{2-\nu} \\ &\asymp_n \operatorname{vol}(\mathcal{E}_\psi(T_k))^{1-\nu}\log(\operatorname{vol}(\mathcal{E}_\psi(T_k))).\end{aligned} \tag{2.7.2}$$

We can also estimate

$$\begin{aligned}&\sum_{k=1}^{+\infty} \operatorname{vol}(\mathcal{E}_\psi(T_k))^{\frac{2+d(P)}{3+d(P)}-1}\log(\operatorname{vol}(\mathcal{E}_\psi(T_k)))^{-(2+d(P))} \\ &\qquad\lesssim \sum_{k=1}^{+\infty} k^{-1}\log(k)^{-(2+d(P))} < +\infty.\end{aligned} \tag{2.7.3}$$

Set $\mathcal{M}_k = \{x \in X : D(k_x^{-1}, \mathcal{E}_\psi(T_k)) > \operatorname{vol}(\mathcal{E}_\psi(T_k))^{1-\nu}\log(\operatorname{vol}(\mathcal{E}_\psi(T_k)))\}$. We then need to show that the set

$$\mathcal{M}_\infty = \bigcap_{m\geqslant 1}\bigcup_{k\geqslant m}\mathcal{M}_k$$

has σ_X-measure zero. We can apply equation (2.7.1) to get for all k sufficiently large,

$$\sigma_X(\mathcal{M}_k) \lesssim \operatorname{vol}(\mathcal{E}_\psi(T_k))^{\frac{2+d(P)}{3+d(P)}-1}\log(\operatorname{vol}(B_k))^{-(2+d)}.$$

This estimate and the summability in (2.7.3) then imply that $\sigma_X(\mathcal{M}_\infty) = 0$. Thus we have shown that for σ_X-a.e. $x \in X$ and for all k sufficiently large, we have

$$D(k_x^{-1}, \mathcal{E}_\psi(T_k)) \lesssim \operatorname{vol}(\mathcal{E}_\psi(T_k))^{\frac{2+d(P)}{3+d(P)}} \log\left(\operatorname{vol}(\mathcal{E}_\psi(T_k))\right).$$

For arbitrary $T > 1$ with sufficiently large $\mathrm{vol}(\mathcal{E}_\psi(T))$, the claim follows from the estimate in (2.7.2) as $1 - \nu = \frac{2+d(P)}{3+d(P)}$ and the observation

$$D(k_x^{-1}, \mathcal{E}_\psi(T)) \leqslant \max\left\{D(k_x^{-1}, \mathcal{E}_\psi(T_k)), D(k_x^{-1}, \mathcal{E}_\psi(T_{k+1}))\right\} + \mathrm{vol}(\mathcal{E}_\psi(T_{k+1}) \backslash \mathcal{E}_\psi(T_k)),$$

whenever k is such that $\mathcal{E}_\psi(T_k) \subseteq \mathcal{E}_\psi(T) \subseteq \mathcal{E}_\psi(T_{k+1})$. This finishes the proof. □

2.8 Proof of Theorem 2.0.1

In view of Lemma 2.4.1 and the measure estimates in §2.5, Theorem 2.0.1 now essentially follows from Theorem 2.7.1.

Proof of Theorem 2.0.1. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. We first assume that $\psi(\mathbb{R}_+) \subseteq [0, 1/2]$. We can thus apply Lemma 2.4.1 and Theorem 2.7.1 to get for almost every $x \in X$ and for all sufficiently large T,

$$\mathcal{N}_\psi(x, T) = \mathrm{vol}(\mathcal{E}_\psi(T)) + O\left(\mathrm{vol}(\mathcal{E}_\psi(T))^{\frac{2+d(P)}{3+d(P)}} \log(\mathrm{vol}(\mathcal{E}_\psi(T)))\right).$$

Next, applying the measure estimate from Lemma 2.5.2 for $\mathcal{E}_\psi(T)$, we get

$$\mathcal{N}_\psi(x, T) = \varkappa_n \Psi(T) + O(\Psi(T)^{\frac{2+d(P)}{3+d(P)}} \log(\Psi(T)) + \Omega(T)).$$

Finally, if $\psi(\mathbb{R}_+) \nsubseteq [0, 1/2]$, since $\lim_{t \to +\infty} \psi(t) = 0$, there exists some $t_0 > 0$ such that $\psi(t) < \frac{1}{2}$ for all $t \geqslant t_0$. Define $\psi_0 : \mathbb{R}_+ \to [0, 1/2]$ by $\psi_0(t) = \psi(t_0)$ if $t < t_0$ and $\psi_0(t) = \psi(t)$ if $t \geqslant t_0$. Applying the previous result for ψ_0 we get for σ_X-a.e. $x \in X$ and for all sufficiently large T,

$$\mathcal{N}_{\psi_0}(x, T) = \varkappa_n \Psi_0(T) + O_n(\Psi_0(T)^{\frac{2+d(P)}{3+d(P)}} \log(\Psi_0(T)) + \Omega_0(T)).$$

We can then conclude the proof by noting that $\mathcal{N}_\psi(x, T) = \mathcal{N}_{\psi_0}(x, T) + O_\psi(1)$, $\Psi(T) = \Psi_0(T) + O_\psi(1)$ and $\Omega(T) = \Omega_0(T) + O_\psi(1)$ for any $T > 1$ and $x \in X$. □

Chapter 3

Integrability of Siegel transforms

As we have seen in the previous chapter, we can obtain an alternative proof of Schmidt's classical result [Sch60a] for the projective space $\mathbb{P}(\mathbb{R}^n)$ by adapting the recent approach of Kelmer and Yu [KY23] and using Rogers' second moment formula for the classical primitive Siegel transform (see Theorem 2.3.1).

Let us elaborate briefly on the work of Kelmer and Yu [KY23], as it inspired our generalization of the classical primitive Siegel transform to the setting of connected semisimple algebraic $\mathbb{Q}$-groups. Specifically, they introduced and studied the light-cone Siegel transform that they associated with a rational isotropic quadratic form Q of signature $(n+1, 1)$ for $n \geqslant 1$. They observed that this transform can be expressed as a finite sum of incomplete Eisenstein series. Using the spectral theory of Eisenstein series, they established a second moment formula and showed how it yields effective counting results for rational points on spheres. Notably, they proved a version of Schmidt's theorem in metric Diophantine approximation on the sphere $\mathbb{S}^n$ for $n \not\equiv 1 \pmod 8$, with respect to general non-increasing approximation functions. Their light-cone Siegel transform is a special case of our generalization.

Motivated by their approach and by our alternative proof of Schmidt's theorem (Theorem 2.0.1) for the projective space $\mathbb{P}(\mathbb{R}^n)$, using second moment formulas for Siegel transforms, we now turn to studying the integrability properties of our new transform, that can be attached to any generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$, obtained as a quotient of a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ by a parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$. In particular, our goal is to determine when this transform is in L^2. Unfortunately, it turns out that this transform fails to be square-integrable when the defining parabolic subgroup $\mathbf{P}$ is not maximal. Consequently, the method employed in the previous chapter cannot be extended directly to this more general setting, showing a limitation of this approach.

However, when $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup, our generalized Siegel transform admits a natural extension of Siegel's classical mean value formula (2.1.1) and, moreover, lies in $L^{1+\varepsilon}$ for some small $\varepsilon > 0$. This integrability property, together with effective mixing in the space of lattices and tools from the geometry of numbers, is crucial in establishing a version of Schmidt's theorem for the real points of rank-one generalized flag varieties, as developed in Chapters 5 and 6.

3.1 Some preliminaries

In this section, for the convenience of the reader, we record precise statements with references for certain results that are used in an crucial way in the rest of this thesis.

3.1.1 Some structure theory of parabolic $\mathbb{Q}$-subgroups

In this subsection, we record some fact concerning the structure of parabolic $\mathbb{Q}$-subgroups $\mathbf{P}$ of a connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$, as presented in [Bor69, § 11.7].

We first recall that a $\mathbb{Q}$-group $\mathbf{G}$ is an *almost direct product* of $\mathbb{Q}$-subgroups $\mathbf{G}_1, \ldots, \mathbf{G}_m \leqslant \mathbf{G}$ if these are normal subgroups and the product map

$$\begin{aligned} \mathbf{G}_1 \times \cdots \times \mathbf{G}_m &\to \mathbf{G} \\ (g_1, \ldots, g_m) &\mapsto g_1 \cdots g_m \end{aligned}$$

is a $\mathbb{Q}$-isogeny, that is, a surjective morphism of algebraic $\mathbb{Q}$-groups with finite kernel (this implies in particular that for $i \neq j$ the subgroups $\mathbf{G}_i$ and $\mathbf{G}_j$ centralize each other).

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-subgroup. We recall that a *parabolic subgroup* of $\mathbf{G}$ is a closed subgroup $\mathbf{P}$ that contains a Borel subgroup

$\mathbf{B}$ of $\mathbf{G}$ (that is, a maximal connected solvable subgroup). We say that $\mathbf{P}$ is a *parabolic* $\mathbb{Q}$*-subgroup* if, in addition to being a parabolic subgroup, it is defined over $\mathbb{Q}$.

Fix a minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$ of $\mathbf{G}$ containing a maximal $\mathbb{Q}$-split torus $\mathbf{T}$. Let Δ be the associated set of simple roots of $\mathbf{G}$ relative to $\mathbf{T}$, for the ordering associated with $\mathbf{P}_0$. Then $\mathbf{P}_0 = \mathcal{Z}(\mathbf{T})\mathbf{U}_0$ is the semi-direct product of the centralizer $\mathcal{Z}(\mathbf{T})$ in $\mathbf{G}$ of $\mathbf{T}$ and the unipotent radical $\mathbf{U}_0$ of $\mathbf{P}_0$. This centralizer is a connected reductive $\mathbb{Q}$-subgroup of $\mathbf{G}$. We construct a family of parabolic $\mathbb{Q}$-subgroups that are representatives of the conjugacy classes under $\mathbf{G}(\mathbb{Q})$. To this end, we associate to each subset θ of Δ, a $\mathbb{Q}$-subtorus of $\mathbf{T}$:

$$\mathbf{T}_\theta = \left(\bigcap_{\alpha\in\theta} \ker(\alpha)\right)^\circ,$$

where $(\cdot)^\circ$ means the connected component with respect to the Zariski topology, and a parabolic $\mathbb{Q}$-subgroup that contains $\mathbf{P}_0$:

$$\mathbf{P}_\theta = \mathcal{Z}(\mathbf{T}_\theta)\mathbf{U}_0.$$

In fact, this group is a semi-direct product

$$\mathbf{P}_\theta = \mathcal{Z}(\mathbf{T}_\theta)\mathbf{U}_\theta$$

of $\mathcal{Z}(\mathbf{T}_\theta)$ and its unipotent radical $\mathbf{U}_\theta$. In particular, this is a Levi decomposition of $\mathbf{P}_\theta$ with Levi subgroup $\mathcal{Z}(\mathbf{T}_\theta)$. The characters of $\mathbf{T}$ that arise from the adjoint representation of $\mathbf{T}$ on the Lie algebra $\mathfrak{u}_\theta$ of $\mathbf{U}_\theta(\mathbb{R})$ are the positive roots that contain at least one simple root that is not in θ. As for the roots of $\mathcal{Z}(\mathbf{T}_\theta)$, these are the roots whose simple components are all in θ. Thus, the set of $\mathbb{Q}$-roots of $\mathbf{P}_\theta$ consists the positive roots and of the negative roots that are linear combinations of the simple roots that are in θ. The map $\theta \mapsto \mathbf{P}_\theta$ is increasing and one has:

$$\mathbf{P}_\theta \cap \mathbf{P}_{\theta'} = \mathbf{P}_{\theta\cap\theta'}, \quad \mathbf{P}_\varnothing = \mathbf{P}_0, \quad \mathbf{P}_\Delta = \mathbf{G}.$$

Let $[\theta]$ be the set of $\mathbb{Q}$-roots that are linear combinations of elements of θ. Let $\mathbf{Q}$ be the largest connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T})$. There exist a connected semisimple $\mathbb{Q}$-subgroup $\mathbf{M}_\theta$ of $\mathcal{Z}(\mathbf{T}_\theta)$ and a connected $\mathbb{Q}$-subgroup $\mathbf{Q}_\theta$ of $\mathbf{Q}$ such that:

- the $\mathbb{Q}$-rank of $\mathbf{M}_\theta$ is equal to the number of elements of θ;

- $\mathbf{T}_{\mathbf{M}_\theta} = (\mathbf{M}_\theta \cap \mathbf{T})^\circ$ is a maximal $\mathbb{Q}$-split torus of $\mathbf{M}_\theta$;
- $[\theta]$ is the system of $\mathbb{Q}$-roots of $\mathbf{M}_\theta$;
- $\mathcal{Z}(\mathbf{T}_\theta)$ is the almost direct product of $\mathbf{Q}_\theta$, $\mathbf{M}_\theta$, $\mathbf{T}_\theta$.

The Lie algebra of $\mathbf{M}_\theta$ is the sum of the subspaces $\mathfrak{g}_\alpha + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, where $\mathfrak{g}_\alpha$ is the root subspace corresponding to the root α and α runs through $[\theta]$.

Lemma 3.1.1. *Let $\mathbf{M}$ be the identity component of the intersection of the kernels of the $\mathbb{Q}$-characters of $\mathcal{Z}(\mathbf{T}_\theta)$. Then $\mathbf{M} = \mathbf{Q}_\theta \mathbf{M}_\theta$.*

Proof. By [Bor69, Proposition 10.7], the group $\mathcal{Z}(\mathbf{T}_\theta)$ is the almost direct product of a $\mathbb{Q}$-split $\mathbb{Q}$-torus $\mathbf{S}$ and a connected $\mathbb{Q}$-subgroup $\mathbf{M}$ satisfying that its $\mathbb{Q}$-character group, $X^*(\mathbf{M})_{\mathbb{Q}}$, is trivial. Moreover, this decomposition is unique and $\mathbf{M}$ is explicitly given as the identity component of the intersection of the kernels of the $\mathbb{Q}$-characters of $\mathcal{Z}(\mathbf{T}_\theta)$. Since $\mathcal{Z}(\mathbf{T}_\theta) = (\mathbf{Q}_\theta \mathbf{M}_\theta)\mathbf{T}_\theta$ is another such decomposition, we must have $\mathbf{M} = \mathbf{Q}_\theta \mathbf{M}_\theta$ and $\mathbf{S} = \mathbf{T}_\theta$. $\square$

Theorem 3.1.2 ([Bor69, Theorem 11.8])**.** *Let $\mathbf{G}$ be a connected semisimple $\mathbb{Q}$-group.*

1. *Every parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ is conjugated by an element of $\mathbf{G}(\mathbb{Q})$ to a unique subgroup $\mathbf{P}_\theta$.*
2. *Two parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$ that are conjugated over $\overline{\mathbb{Q}}$ are conjugated over $\mathbb{Q}$.*
3. *The projection map $\mathbf{G}(\mathbb{Q}) \to (\mathbf{G}/\mathbf{P})(\mathbb{Q})$ is surjective. In particular, one has $(\mathbf{G}/\mathbf{P})(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$.*

Let $\mathbf{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. Let θ be the unique subset of Δ such that $\mathbf{P}_\theta$ is conjugated to $\mathbf{P}$ by an element of $\mathbf{G}(\mathbb{Q})$. We say that $\mathbf{P}$ is a *maximal* parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$, if there exists a simple root $\alpha \in \Delta$ such that $\theta = \Delta \smallsetminus \{\alpha\}$. Let us also recall that the $\mathbb{Q}$-*rank of* $\mathbf{P}$, which is denoted by $\operatorname{rank}_{\mathbb{Q}} \mathbf{P}$, is defined to be the cardinality of the set θ. In particular, $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ if and only if $\operatorname{rank}_{\mathbb{Q}} \mathbf{P} = \operatorname{rank}_{\mathbb{Q}} \mathbf{G} - 1$.

3.1.2 Relative fundamental weights

Let us recall here the notion of relative fundamental weights. We follow the exposition given in [MG14, §3] (where $\mathbf{G}$ is assumed to be simply connected) and refer the reader to [BT65, §12] for the details.

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group. We first work over $\overline{\mathbb{Q}}$ and then consider the corresponding "relativized" notions over $\mathbb{Q}$, denoting the former with a tilde.

So let $\mathbf{S}$ be a maximal $\mathbb{Q}$-torus in $\mathbf{G}$ and $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing the torus $\mathbf{S}$. Let $\widetilde{\Phi} = \widetilde{\Phi}(\mathbf{G}, \mathbf{S})$ be the associated absolute roots of $\mathbf{G}$ relative to $\mathbf{S}$ and let $\widetilde{\Delta} \subset \widetilde{\Phi}$ be the set of simple roots with respect to the ordering induced on $\widetilde{\Phi}$ by the Borel subgroup $\mathbf{B}$. We denote by $X^*(\mathbf{S})$ (resp. $X_*(\mathbf{S})$) the groups of characters (resp. cocharacters) of $\mathbf{S}$. Let $\widetilde{\Phi}^\vee \subset X_*(\mathbf{S})$ be the absolute coroots of $\mathbf{G}$ relative to $\mathbf{S}$. The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the absolute Weyl group $\widetilde{W} = (N(\mathbf{S})/\mathbf{S})(\overline{\mathbb{Q}})$ act linearly on $\mathbf{S}(\overline{\mathbb{Q}})$. Let k be a finite Galois extension of $\mathbb{Q}$ such that $\mathbf{S}$ splits over k. Then the absolute Galois group acts on $\mathbf{S}(\mathbb{Q})$ through $H = \mathrm{Gal}(k/\mathbb{Q})$ and $\widetilde{W} = (N(\mathbf{S})/\mathbf{S})(k)$. We fix an inner product $\langle \cdot, \cdot \rangle$ on $E = X^*(\mathbf{S}) \otimes \mathbb{R}$ that is invariant under the action of $H \ltimes \widetilde{W}$. We refer to such an inner product on E as *admissible* and we use it to identify $X_*(\mathbf{S})$ as the dual of $X^*(\mathbf{S})$ in E (in fact, using the natural pairing between $X_*(\mathbf{S})$ and $X^*(\mathbf{S})$, every element $a_* \in X_*(\mathbf{S})$ induces a linear functional Λ_{a_*} on E and hence is of the form $\langle \cdot, a^* \rangle$ for some unique element $a^* \in X^*(\mathbf{S})$ and we identify a_* with a^*).

Let now $\mathbf{T}$ be a maximal $\mathbb{Q}$-split torus contained in $\mathbf{S}$ and $\mathbf{P}_0$ a minimal parabolic $\mathbb{Q}$-subgroup containing the Borel subgroup $\mathbf{B}$. Let $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ be the roots of $\mathbf{G}$ relative to $\mathbf{T}$ and Δ be the set of simple roots corresponding to the ordering coming from the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. Let $j : \mathbf{T} \to \mathbf{S}$ be the injection map. It induces a surjection $j^* : X^*(\mathbf{S}) \to X^*(\mathbf{T})$ and an injection $j_* : X_*(\mathbf{T}) \to X_*(\mathbf{S})$. Via these maps, $X_*(\mathbf{T})$ gets identified with

$$Y_* = \{v \in X_*(\mathbf{S}) : H \cdot v = v\},$$

the extension of j^* to $X^*(\mathbf{S}) \otimes \mathbb{R} \to X^*(\mathbf{T}) \otimes \mathbb{R}$ gets identified with the orthogonal projection π onto

$$V = \{v \in E : H \cdot v = v\},$$

and $X^*(\mathbf{T})$ gets identified with $Y^* = \pi(X^*(\mathbf{S}))$. The root system Φ of $\mathbf{G}$ relative to $\mathbf{T}$ is then $\pi(\widetilde{\Phi}) \setminus \{0\}$ and $\widetilde{\Delta}$ can be chosen such that $\Delta = \pi(\widetilde{\Delta}) \setminus \{0\}$.

For any absolute Galois automorphism h, there exists a unique w_h in the absolute Weyl group such that $w_h(h(\widetilde{\Delta})) = \widetilde{\Delta}$. This induces a new linear action τ of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X^*(\mathbf{S})$. Let $\widetilde{\phi} \in \widetilde{\Phi}$ be such that $\pi(\widetilde{\phi}) = \phi \in \Phi$. Then $\pi^{-1}(\phi) = \tau(H)(\widetilde{\phi})$. We recall that the *anisotropic kernel* of $\mathbf{G}$ is the connected component of the derived subgroup of the centralizer of $\mathbf{T}$ in $\mathbf{G}$. The absolute root system of the anisotropic kernel of $\mathbf{G}$ is given by $\Phi_0 = \ker(\pi) \cap \widetilde{\phi}$ and $\Delta_0 = \widetilde{\Delta} \cap \Phi_0$ is a set of simple roots in Φ_0.

For any $\widetilde{\phi} \in \widetilde{\Phi}$ (resp. $\phi \in \Phi$), under the above identification, we have that $\widetilde{\phi}^\vee = 2\widetilde{\phi}/(\widetilde{\phi}, \widetilde{\phi})$ is a coroot (resp. $\phi^\vee = 2\phi/(\phi, \phi)$ is a relative coroot). Let $\{\widetilde{\omega}_{\widetilde{\alpha}} : \widetilde{\alpha} \in \widetilde{\Delta}\}$ be the absolute fundamental weights of $\mathbf{G}$. This means that for every $\widetilde{\alpha} \in \widetilde{\Delta}$, the element $\widetilde{\omega}_{\widetilde{\alpha}} \in X^*(\mathbf{S})$ is the unique character of $\mathbf{S}$ such that for every $\widetilde{\beta} \in \widetilde{\Delta}$ we have $(\widetilde{\omega}_{\widetilde{\alpha}}, \widetilde{\beta}) = \delta_{\widetilde{\alpha},\widetilde{\beta}}$. For any $\alpha \in \Delta$, put

$$\omega_\alpha = \pi \left(\sum_{\widetilde{\alpha} \in \pi^{-1}(\alpha)} \widetilde{\omega}_{\widetilde{\alpha}} \right).$$

These are called the *relative fundamental* $\mathbb{Q}$*-weights* of $\mathbf{G}$. Moreover, one has

$$\langle \omega_\alpha, \beta \rangle = \frac{\|\widetilde{\alpha}\|}{\|\alpha\|} \delta_{\alpha,\beta},$$

where $\widetilde{\alpha}$ is any element in $\pi^{-1}(\alpha) \cap \widetilde{\Delta}$. A character $\chi \in X^*(\mathbf{T})$ is called a *dominant* $\mathbb{Q}$*-weight* if, for every $\alpha \in \Delta$, $\langle \chi, \alpha \rangle \geqslant 0$. In particular, the relative fundamental $\mathbb{Q}$-weights are dominant $\mathbb{Q}$-weights. Let $\mathbf{P}$ be a parabolic $\mathbb{Q}$-subgroup containing $\mathbf{P}_0$. By a slight abuse of notation, we shall say that $\chi \in X^*(\mathbf{P})_{\mathbb{Q}}$, the $\mathbb{Q}$-character group of $\mathbf{P}$, is a *dominant* $\mathbb{Q}$*-weight* if its restriction to $\mathbf{T}$ is so. Moreover, by a *highest* $\mathbb{Q}$*-weight* $\chi \in X^*(\mathbf{T})$ we shall always mean a dominant $\mathbb{Q}$-weight $\chi \in X^*(\mathbf{T})$ that is the highest $\mathbb{Q}$-weight of a representation $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ which is strongly rational over $\mathbb{Q}$, as defined below.

3.1.3 Strongly rational representations

Let us now recall the notion of a representation which is strongly rational over $\mathbb{Q}$. We refer the reader again to [BT65, §12] for the details.

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ and suppose that $\mathbf{S}$ is a maximal $\mathbb{Q}$-torus contained in $\mathbf{B}$.

Let $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ be an irreducible representation. There exists a unique line $\mathbf{D} \subset \mathbf{V}$ that is invariant under the Borel subgroup $\mathbf{B}$. In particular, $\mathbf{S}$ acts on the line $\mathbf{D}$ through a character $\widetilde{\chi} \in X^*(\mathbf{S})$ called the *highest weight* of π: $\forall\, s \in \mathbf{S}$, $v \in \mathbf{D}$, we have $\pi(s)v = \widetilde{\chi}(s)v$. The orbit under $\mathbf{G}$ of the line $\mathbf{D}$ is called the *cone of the representation* π and we denote it by $\mathcal{C}_\pi$. The stabilizers of the lines in $\mathcal{C}_\pi$ form a class of parabolic subgroups of $\mathbf{G}$, denoted by $\mathcal{P}_\pi$, that are conjugated to each other. The representation π is called *strongly rational over* $\mathbb{Q}$ if it is defined over $\mathbb{Q}$ and if $\mathcal{P}_\pi$ contains a parabolic subgroup $\mathbf{P}$ which is defined over $\mathbb{Q}$.

The character $\widetilde{\chi} \in X^*(\mathbf{S})$ determines the representation π up to equivalence. Moreover, the highest weight $\widetilde{\chi}$ is a linear combination $\widetilde{\chi} = \sum_{\widetilde{\alpha} \in \widetilde{\Delta}} c_{\widetilde{\alpha}}\, \widetilde{\omega}_{\widetilde{\alpha}}$ with $c_{\widetilde{\alpha}} \in \mathbb{N}$ of the absolute fundamental weights $\{\widetilde{\omega}_{\widetilde{\alpha}}\}_{\widetilde{\alpha} \in \widetilde{\Delta}}$ and conversely, if $\mathbf{G}$ is simply connected, every linear combination $\sum_{\widetilde{\alpha} \in \widetilde{\Delta}} c_{\widetilde{\alpha}}\, \widetilde{\omega}_{\widetilde{\alpha}}$ with $c_{\widetilde{\alpha}} \in \mathbb{N}$ is the highest weight of an irreducible representation $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}')$.

We suppose now until the end of this subsection that $\mathbf{T}$ is a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbf{G}$ contained in $\mathbf{S}$ and that $\mathbf{P}_0$ is a minimal parabolic $\mathbb{Q}$-subgroup containing $\mathbf{B}$. Let $X^*(\mathbf{P}_0)_\mathbb{Q}$ be the group of $\mathbb{Q}$-characters of $\mathbf{P}_0$. By [BT65, Corollary 12.11], an irreducible representation $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ is strongly rational over $\mathbb{Q}$ if and only if its highest weight $\widetilde{\chi}$ is the restriction to $\mathbf{S}$ of an element of $X^*(\mathbf{P}_0)_\mathbb{Q}$. Let now $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ be a strongly rational representation. Then the stabilizer $\mathbf{P}$ of the unique $\mathbf{B}$-invariant line $\mathbf{D} \subset \mathbf{V}$ is a parabolic $\mathbb{Q}$-subgroup containing $\mathbf{P}_0$. The *weights of* π are the characters $\mu \in X^*(\mathbf{S})$ for which there exists a non-zero $v \in \mathbf{V}$ such that, $\forall\, s \in \mathbf{S}$, we have $\pi(s)v = \mu(s)v$. The $\mathbb{Q}$-*weights of* π are the restrictions to $\mathbf{T}$ of the weights of π; the *highest* $\mathbb{Q}$-*weight of* π, denoted by χ, is the restriction of $\widetilde{\chi}$ to $\mathbf{T}$. This character determines π up to rational equivalence. Hence we denote from now on π by π_χ and $\mathbf{V}$ by $\mathbf{V}_\chi$. By [BT65, Proposition 12.13], the highest $\mathbb{Q}$-weight χ of an irreducible representation $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ which is strongly rational over $\mathbb{Q}$ is a linear combination $\chi = \sum_{\alpha \in \Delta} c_\alpha\, \omega_\alpha$ with $c_\alpha \in \mathbb{N}$ of the relative fundamental weights $\{\omega_\alpha\}_{\alpha \in \Delta}$ and, for every $\alpha \in \Delta$, there exists an integer $d_\alpha \geqslant 1$ such that for every $c_\alpha \in \mathbb{N}$, $c_\alpha\, d_\alpha\, \omega_\alpha$ is the highest $\mathbb{Q}$-weight of an irreducible representation $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ which is strongly rational over $\mathbb{Q}$. If $\mathbf{G}$ is simply connected, every linear combination $\sum_{\alpha \in \Delta} c_\alpha\, \omega_\alpha$ with $c_\alpha \in \mathbb{N}$ is the highest $\mathbb{Q}$-weight of an irreducible representation $\pi : \mathbf{G} \to \mathrm{GL}(\mathbf{V})$ which is strongly rational over $\mathbb{Q}$. For every $\mathbb{Q}$-weight $\mu \in X^*(\mathbf{T})$ of a representation $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ which is strongly rational over $\mathbb{Q}$, let

$$\mathbf{V}^{\mu} = \{\boldsymbol{v} \in \mathbf{V}_{\chi} : \forall t \in \mathbf{T},\, \pi_{\chi}(t)\boldsymbol{v} = \mu(t)\boldsymbol{v}\}.$$

This is a $\mathbb{Q}$-subspace of $\mathbf{V}_{\chi}$. It is known that $\mathbf{V}_{\chi}$ is the direct sum of the linear subspaces $\mathbf{V}^{\mu}$, that $\mathbf{V}^{\chi}$ is one-dimensional, and that every $\mathbb{Q}$-weight of π_{χ} has the form

$$\mu = \chi - \sum_{\alpha \in \Delta} c_{\alpha}(\mu)\alpha \tag{3.1.1}$$

with $c_{\alpha}(\mu) \in \mathbb{N}$. We fix once and for all a vector $\boldsymbol{e}_{\chi} \in \mathbf{V}^{\chi}(\mathbb{Q}) \smallsetminus \{\mathbf{0}\}$.

3.1.4 Reduction theory

Finally, we shall briefly recall the Bruhat decomposition of $\mathbf{G}$ and some notions from reduction theory, as it is presented in [BT65]. Let W be the Weyl group associated to $\mathbf{G}$ and $\mathbf{T}$, given as the quotient of the normalizer of $\mathbf{T}$ by its centralizer. The Bruhat decomposition [BT65, Théorème 5.15] of $\mathbf{G}(\mathbb{Q})$ allows to express

$$\mathbf{G}(\mathbb{Q}) = \bigsqcup_{w \in W} \mathbf{P}_0(\mathbb{Q}) x_w \mathbf{P}_0(\mathbb{Q}), \tag{3.1.2}$$

where $(x_w)_{w \in W}$ is a system of representatives of W in $N(\mathbf{T})(\mathbb{Q})$. In what follows, by a slight abuse of notation, we shall simply write w for x_w. Moreover, we need the notion of a Siegel set. Let K be a maximal compact subgroup of G, the group of real points of $\mathbf{G}$. Moreover, let $\mathbf{P}_0$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and $\mathbf{T}$ a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbf{G}$ contained in $\mathbf{P}_0$. Let $\mathbf{U}_0 = R_u(\mathbf{P}_0)$ denote the unipotent radical of $\mathbf{P}_0$ and let $\mathbf{M}_0$ be the maximal connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of the centralizer $\mathcal{Z}(\mathbf{T})$ of $\mathbf{T}$ in $\mathbf{G}$. By a slight abuse of notation, we let $\mathfrak{a}$ denote the Lie algebra of $T = \mathbf{T}(\mathbb{R})$. For every $\tau > 0$ let $\mathfrak{a}_{\tau} = \{X \in \mathfrak{a} : \forall \alpha \in \Delta,\, \alpha(X) \leqslant \tau\}$ and set $A_{\tau} = \exp \mathfrak{a}_{\tau}$. Denote by M_0 and U_0 the groups of real points of $\mathbf{M}_0$ and $\mathbf{U}_0$, respectively. Let $\boldsymbol{\omega}$ be compact neighborhood of the identity in $M_0 U_0$. Then $\mathfrak{S} = \mathfrak{S}_{\tau,\boldsymbol{\omega}} = K\, A_{\tau}\, \boldsymbol{\omega}$ is called a *Siegel set (over* $\mathbb{Q}$*)*. Let Γ be an arithmetic subgroup of G contained in $\mathbf{G}(\mathbb{Q})$. Let us also recall that $\mathcal{F} \subset G$ is called a *fundamental set of* Γ (see [Bor69, Definition 9.6]) if the following conditions hold:

1. $K\,\mathcal{F} = \mathcal{F}$,

2. $\mathcal{F}\Gamma = G$,

3. For every $c \in \mathbf{G}(\mathbb{Q})$, the set $\{\gamma \in \Gamma : \mathcal{F}c \cap \mathcal{F}\gamma \neq \varnothing\}$ if finite.

Theorem 3.1.3 ([Bor69, Theorem 15.5])**.** *Let* $\mathbf{G}$ *be a connected semisimple algebraic* $\mathbb{Q}$*-group and* Γ *an arithmetic subgroup of* G *contained in* $\mathbf{G}(\mathbb{Q})$*. There exists a Siegel set* $\mathfrak{S}$ *over* $\mathbb{Q}$ *of* $\mathbf{G}$ *and a finite subset* $C \subset \mathbf{G}(\mathbb{Q})$ *such that* $\mathfrak{S}C$ *is a fundamental set for* Γ *in* G*. This set has finite Haar measure (since* $X^*(\mathbf{G})_{\mathbb{Q}} = \{1\}$*) and is compact if* $\operatorname{rank}_{\mathbb{Q}}(\mathbf{G}) = 0$*.*

Consequently, if we let $\mathfrak{S} = K\,A_\tau\,\boldsymbol{\omega}$ be as above, every $g \in G$ can be written as $g = kanc\gamma$ with $k \in K$, $a \in A_\tau$, $n \in \boldsymbol{\omega}$, $c \in C$, and $\gamma \in \Gamma$. Such a decomposition is in general not unique. However, the a-component is determined up to bounded error (see, for instance, [Sax20, Proposition 4.2.4.]): if $g = k'a'n'c'\gamma'$ is any other such decomposition of g with $k' \in K$, $a' \in A_\tau$, $n' \in \boldsymbol{\omega}$, $c' \in C$, and $\gamma' \in \Gamma$, then $a'a^{-1} = \exp(O(1))$.

3.2 Generalizing the Siegel transform

Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group. Let $\mathbf{P}_0$ be a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbf{G}$ contained in $\mathbf{P}_0$. Let $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ be a representation which is strongly rational over $\mathbb{Q}$ and let $\chi \in X^*(\mathbf{T})$ be the associated highest $\mathbb{Q}$-weight. In particular, there is a unique $\mathbf{P}_0$-invariant line $\mathbf{D} \subset \mathbf{V}_\chi$ that is defined over $\mathbb{Q}$, on which $\mathbf{T}$ acts via χ. The stabilizer $\mathbf{P}$ in $\mathbf{G}$ of the line $\mathbf{D}$ is a parabolic $\mathbb{Q}$-subgroup containing $\mathbf{P}_0$. Fix a non-zero vector $\boldsymbol{e}_\chi \in \mathbf{D}(\mathbb{Q})$ and let $\mathbf{L}$ be the stabilizer in $\mathbf{G}$ of $\boldsymbol{e}_\chi$. Let $V_\chi = \mathbf{V}_\chi(\mathbb{R})$ and $G = \mathbf{G}(\mathbb{R})$. Let $\widetilde{X}_0 = G \cdot \boldsymbol{e}_\chi \subset V_\chi$ be the G-orbit of $\boldsymbol{e}_\chi$ and denote by $\widetilde{X} = \widetilde{X}_0 \sqcup \{\mathbf{0}\}$. Fix a lattice $\mathbf{V}_\chi(\mathbb{Z})$ of V_χ contained in $\mathbf{V}_\chi(\mathbb{Q})$. Its stabilizer Γ in G is an arithmetic subgroup. Let $\mathcal{L}_\chi$ denote the set of all the primitive vectors in $\mathbf{V}_\chi(\mathbb{Z})$ that are contained in $\widetilde{X}$. We assume that $\boldsymbol{e}_\chi \in \mathcal{L}_\chi$.

Definition 3.2.1 (Siegel transform)**.** For every measurable bounded compactly supported function $f : \widetilde{X} \to \mathbb{R}$, we define the *Siegel transform* of f at χ as the map $S_\chi f : G/\Gamma \to \mathbb{R}$ given by

$$\forall\, g \in G, \quad S_\chi f(g\Gamma) = \sum_{\boldsymbol{v} \in \mathcal{L}_\chi} f(g\boldsymbol{v}). \tag{3.2.1}$$

Since f is compactly supported, for each $g \in G$, the sum on the right-hand side is finite, and hence converges absolutely. Since $\mathcal{L}_\chi$ is stable under the action of Γ, the Siegel transform $S_\chi f$ is a well-defined function on $\Omega = G/\Gamma$. By abuse of notation, we shall write $S_\chi f(g)$ for $S_\chi f(g\Gamma)$.

We equip the Lie algebra $\mathfrak{g}$ of G with the Killing form and choose a maximal compact subgroup K of G whose Lie algebra is orthogonal to that of $T = \mathbf{T}(\mathbb{R})$. We endow V_χ with a Euclidean inner product $\langle \cdot, \cdot \rangle$ for which $\pi_\chi(g)$ is unitary (resp. self-adjoint) whenever $g \in K$ (resp. $g \in T$). In particular, the eigenspaces V^μ are mutually orthogonal. We equip the homogeneous space $\Omega = G/\Gamma$ with the unique G-invariant probability measure, that we denote by μ_Ω. In what follows, for $p \in \{1, 2, \infty\}$, we study the L^p-integrability of the Siegel transform. More precisely, we are interested in finding criteria for the Siegel transform S_χ to map $C_c(\widetilde{X})$ into $L^p(\Omega)$. We start with some observation and reductions.

3.2.1 Incomplete Eisenstein series

We begin by showing that the Siegel transform is a finite sum of certain incomplete Eisenstein series.

Observe that Γ acts on the discrete set $\mathcal{L}_\chi$ and that this action is in general not transitive. However, as we will now show, $\mathcal{L}_\chi$ can always be expressed as a finite union of Γ-orbits. Recall that $\mathbf{P}$ is the stabilizer in $\mathbf{G}$ of the line through the highest weight vector $\boldsymbol{e}_\chi$. By a theorem of Borel and Harish-Chandra [Bor69, Proposition 15.6], the set of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ is finite. According to Theorem 3.1.2 that we recorded above, one has $(\mathbf{G}/\mathbf{P})(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$. In particular, the set of rational points $\mathbf{X}(\mathbb{Q})$ of the generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$ is a finite union of Γ-orbits. Let $x_0 = [\boldsymbol{e}_\chi]$ be the projective point in $\mathbb{P}(\mathbf{V}_\chi)$ corresponding to $\boldsymbol{e}_\chi$. The orbit map $\mathbf{G} \to \mathbb{P}(\mathbf{V}_\chi)$ given by $g \mapsto gx_0$ induces an isomorphism $\mathbf{G}/\mathbf{P} \to \mathbf{G} \cdot x_0$ defined over $\mathbb{Q}$. We identify $\mathbf{X} = \mathbf{G}/\mathbf{P}$ with $\mathbf{G} \cdot x_0$ via this isomorphism. Observe that there is a one-to-one correspondence between $\mathbf{X}(\mathbb{Q})$ and lines passing through elements of $\mathcal{L}_\chi$. Hence there exist finitely many representatives $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\kappa \in \mathcal{L}_\chi$ such that

$$\mathcal{L}_\chi = \bigsqcup_{i=1}^{\kappa} \Gamma \cdot \boldsymbol{v}_i. \tag{3.2.2}$$

We pick $\tau_i \in \mathbf{G}(\mathbb{Q})$ and $\lambda_i > 0$ such that $\boldsymbol{v}_i = \lambda_i \tau_i \boldsymbol{e}_\chi$. Recall that L is the stabilizer in G of the vector $\boldsymbol{e}_\chi$ and that $\mathbf{P}$ is the stabilizer in $\mathbf{G}$ of the line through $\boldsymbol{e}_\chi$. Accordingly, $L_i = \tau_i L \tau_i^{-1}$ is the stabilizer in G of $\boldsymbol{v}_i$ and $\mathbf{P}_i = \tau_i \mathbf{P} \tau_i^{-1}$ is the stabilizer in $\mathbf{G}$ of the line through $\boldsymbol{v}_i$. By abuse of notation, we write $L_i(\mathbb{Z}) = \Gamma \cap L_i$. Consequently, for every $f \in C_c(\widetilde{X})$ and $g \in G$, we have

$$S_\chi f(g) = \sum_{i=1}^{\kappa} \sum_{\gamma \in \Gamma / L_i(\mathbb{Z})} f(g\gamma \boldsymbol{v}_i) = \sum_{i=1}^{\kappa} E_i f(g), \tag{3.2.3}$$

where, for each $i = 1, \ldots, \kappa$, the series

$$E_i f(g) = \sum_{\gamma \in \Gamma / L_i(\mathbb{Z})} f(g\gamma \boldsymbol{v}_i)$$

is the *incomplete Eisenstein series* of f associated to the orbit $\Gamma \cdot \boldsymbol{v}_i$. In particular, the Siegel transform is a finite sum of these incomplete Eisenstein series.

Let us show that, for any $p \in [1, +\infty]$, the L^p-integrability of the Siegel transform S_χ does not depend on the choice of the particular lattice $\mathbf{V}_\chi(\mathbb{Z})$ (and hence does not depend on Γ either) nor is it affected if we replace χ by a dominant weight χ' that is a multiple of χ. To this end, it will be convenient to state a slightly more general result.

Fix a dominant weight $\chi \in X^*(\mathbf{P})_\mathbb{Q}$ with kernel $\mathbf{L} = \ker \chi$ in $\mathbf{P}$ and set $L = \mathbf{L}(\mathbb{R})$. For any arithmetic subgroup Γ_1 of $\mathbf{G}$ and $\tau_1 \in \mathbf{G}(\mathbb{Q})$, we define for any $f \in C_c(G/L_\tau)$, where $L_{\tau_1} = \tau_1 L {\tau_1}^{-1}$, the incomplete Eisenstein series

$$\forall\, g \in G, \quad E_{\tau_1, \Gamma_1} f(g) = \sum_{\gamma \in \Gamma_1 / \Gamma_1 \cap L_{\tau_1}} f(g\gamma L_{\tau_1}).$$

If $\tau_1 = 1$, we simply set $E_{\Gamma_1} f(g) = E_{1, \Gamma_1} f(g)$.

Lemma 3.2.2. *Fix $p \in [1, +\infty]$. Let Γ_1 and Γ_2 be arithmetic subgroups of G and τ_1 and τ_2 be elements of $\mathbf{G}(\mathbb{Q})$. Then E_{τ_1, Γ_1} maps $C_c(G/L_{\tau_1})$ into $L^p(G/\Gamma_1)$ if and only if E_{τ_2, Γ_2} maps $C_c(G/L_{\tau_2})$ into $L^p(G/\Gamma_2)$.*

Given an arithmetic subgroup Γ_1 as above, we identify G/Γ_1 with a fundamental domain $\mathcal{F}_1$ in G and the G-invariant Haar measure on G/Γ_1 with the restriction of the Haar measure μ_G on G to $\mathcal{F}_1$. The proof essentially only uses that the arithmetic subgroups Γ_1 and Γ_2 are *commensurable*, that is, that their intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2.

Proof. It suffices to show by symmetry that if E_{τ_1,Γ_1} maps $C_c(G/L_{\tau_1})$ into $L^p(G/\Gamma_1)$ then E_{τ_2,Γ_2} maps $C_c(G/L_{\tau_2})$ into $L^p(G/\Gamma_2)$.

Let $c_{\tau_1} : G/L \to G/L_{\tau_1}$ be the conjugation map $c_{\tau_1}(gL) = \tau_1 g \tau_1^{-1} L_{\tau_1}$. We first claim that for every $f \in C_c(G/L_{\tau_1})$ and $g \in G$, we have

$$E_{\tau_1,\Gamma_1} f(g) = \sum_{\gamma \in \tau_1^{-1}\Gamma_1\tau_1/\tau_1^{-1}\Gamma_1\tau_1 \cap L} (f \circ c_{\tau_1})(\tau_1^{-1} g \tau_1 \gamma L).$$

In fact, using that the map $c_{\tau_1^{-1}} : \Gamma_1/\Gamma_1 \cap L_{\tau_1} \to \tau_1^{-1}\Gamma_1\tau_1/\tau_1^{-1}\Gamma_1\tau_1 \cap L$ given by conjugation $c_{\tau_1^{-1}}(\gamma_1(\Gamma_1 \cap L_{\tau_1})) = \tau_1^{-1}\gamma\tau_1(\tau_1^{-1}\Gamma_1\tau_1 \cap L)$ is a bijection, we have

$$\begin{aligned} E_{\tau_1,\Gamma_1} f(g) &= \sum_{\gamma \in \Gamma_1/\Gamma_1 \cap L_{\tau_1}} f(g\gamma L_{\tau_1}) \\ &= \sum_{\gamma \in \Gamma_1/\Gamma_1 \cap L_{\tau_1}} (f \circ c_{\tau_1})(\tau_1^{-1} g \gamma \tau_1 L) \\ &= \sum_{\gamma \in \Gamma_1/\Gamma_1 \cap L_{\tau_1}} (f \circ c_{\tau_1})(\tau_1^{-1} g \tau_1 \tau_1^{-1} \gamma \tau_1 L) \\ &= \sum_{\gamma \in \tau_1^{-1}\Gamma_1\tau_1/\tau_1^{-1}\Gamma_1\tau_1 \cap L} (f \circ c_{\tau_1})(\tau_1^{-1} g \tau_1 \gamma L), \end{aligned}$$

as required.

Moreover, using the left-invariance of the Haar measure on G/Γ_1 and the substitution $g \mapsto g\tau_1^{-1}$, one has, for $p < +\infty$,

$$\int_{G/\Gamma_1} |E_{\tau_1,\Gamma_1} f|^p \, d\mu_G = \int_{G/\tau_1^{-1}\Gamma_1\tau_1} \Big| \sum_{\gamma \in \tau_1^{-1}\Gamma_1\tau_1/\tau_1^{-1}\Gamma_1\tau_1 \cap L} (f \circ c_{\tau_1})(g\gamma L) \Big|^p \, d\mu_G(g).$$

If $p = +\infty$, then, as $E_{\tau_1,\Gamma_1} f$ is a continuous function,

$$\|E_{\tau_1,\Gamma_1} f\|_\infty = \sup_{g \in G} |E_{\tau_1,\Gamma_1} f(g)| = \sup_{g \in G} \left| \sum_{\gamma \in (\tau_1^{-1}\Gamma_1\tau_1)/(\tau_1^{-1}\Gamma_1\tau_1 \cap L)} (f \circ c_{\tau_1})(g\gamma L) \right|.$$

A similar argument applies to E_{τ_2,Γ_2}. Thus, up to replacing $\tau_1^{-1}\Gamma_1\tau_1$ with Γ_1 and $\tau_2^{-1}\Gamma_2\tau_2$ with Γ_2, we may assume that $\tau_1 = \tau_2 = 1$. Hence it is enough to show that if E_{Γ_1} maps $C_c(G/L)$ into $L^p(G/\Gamma_1)$ then E_{Γ_2} maps $C_c(G/L)$ into

$L^p(G/\Gamma_2)$. So let $f \in C_c(G/L)$. We may assume without loss of generality that $f \geqslant 0$. We first treat the case $p < +\infty$. Observe that, for every $g \in G$, we have

$$\begin{aligned}|E_{\Gamma_2} f(g)|^p &\leqslant \left| \sum_{\gamma \in \Gamma_2/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma L) \right|^p \\ &\leqslant \max_{\gamma_1 \in \Gamma_2/\Gamma_1 \cap \Gamma_2} \left| \#\big(\Gamma_2/(\Gamma_1 \cap \Gamma_2)\big) \sum_{\gamma_2 \in (\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma_1\gamma_2 L) \right|^p .\end{aligned}$$

Let $\mathcal{F}_2$ be a fundamental domain for Γ_2 in G and let $\mathcal{F}_{1,2}$ be the fundamental domain for $\Gamma_1 \cap \Gamma_2$ in G given by $\mathcal{F}_{1,2} = \bigsqcup_{\gamma_1 \in \Gamma_2/(\Gamma_1 \cap \Gamma_2)} \mathcal{F}_2\gamma_1$. Then, we have

$$\begin{aligned}\int_{\mathcal{F}_2} |E_{\Gamma_2} f|^p \, d\mu_G &\lesssim_{\Gamma_1,\Gamma_2} \sum_{\gamma_1 \in \Gamma_2/(\Gamma_1 \cap \Gamma_2)} \int_{\mathcal{F}_2\gamma_1} \left| \sum_{\gamma \in (\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma L) \right|^p d\mu_G(g) \\ &= \int_{\mathcal{F}_{1,2}} \left| \sum_{\gamma \in (\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma L) \right|^p d\mu_G(g).\end{aligned}$$

Now, since $(\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)$ injects into $\Gamma_1/(\Gamma_1 \cap L)$,

$$\left| \sum_{\gamma \in (\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma L) \right|^p \leqslant \left| \sum_{\gamma \in \Gamma_1/(\Gamma_1 \cap L)} f(g\gamma L) \right|^p .$$

Finally, we let $\mathcal{F}_1$ be a fundamental domain for Γ_1 in G satisfying that $\mathcal{F}_{1,2} = \bigsqcup_{\gamma_1 \in \Gamma_1/(\Gamma_1 \cap \Gamma_2)} \mathcal{F}_1\gamma_1$. Then, using the fact that the function sending g to $\sum_{\gamma \in \Gamma_1/\Gamma_1 \cap L} f(g\gamma L)$ is right Γ_1-invariant, we have

$$\int_{\mathcal{F}_{1,2}} \left| \sum_{\gamma \in (\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma L) \right|^p d\mu_G(g) \lesssim_{\Gamma_1,\Gamma_2} \int_{\mathcal{F}_1} \left| \sum_{\gamma \in \Gamma_1/(\Gamma_1 \cap L)} f(g\gamma L) \right|^p d\mu_G(g)$$

and the latter converges, by assumption. This concludes the proof of the case $p < +\infty$.

Now suppose that $p = +\infty$. It follows from the above that

$$\begin{aligned} |E_{\Gamma_2} f(g)| &\leqslant \max_{\gamma_1 \in \Gamma_2/(\Gamma_1 \cap \Gamma_2)} \#\big(\Gamma_2/(\Gamma_1 \cap \Gamma_2)\big) \left| \sum_{\gamma_2 \in (\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)} f(g\gamma_1\gamma_2 L) \right| \\ &\leqslant \max_{\gamma_1 \in \Gamma_2/(\Gamma_1 \cap \Gamma_2)} \#\big(\Gamma_2/(\Gamma_1 \cap \Gamma_2)\big) \left| \sum_{\gamma_2 \in \Gamma_1/(\Gamma_1 \cap L)} f(g\gamma_1\gamma_2 L) \right|, \end{aligned}$$

where for the second inequality we used that $(\Gamma_1 \cap \Gamma_2)/(\Gamma_1 \cap \Gamma_2 \cap L)$ injects into $\Gamma_1/(\Gamma_1 \cap L)$. Taking upper bounds yields the claim. The proof is complete. □

For every fixed $p \in [1, \infty]$, the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^p(\Omega)$ if and only if each E_i does. Indeed, this follows from the following inequalities and the fact that $L^p(\Omega)$ is closed under addition: for every $f \in C_c(\widetilde{X})$ and $i = 1, \ldots, \kappa$, we have

$$E_i f \leqslant S_\chi |f| \quad \text{and} \quad S_\chi f \leqslant \sum_{i=1}^{\kappa} E_i |f|.$$

In view of the above Lemma 3.2.2, it further suffices to show that this is the case for *some* i. Hence, as far as the integrability of the Siegel transform is concerned, we may without loss of generality assume that $\mathcal{L}_\chi$ consists in a single Γ-orbit. In particular, we have

$$\forall\, g \in G, \quad S_\chi f(g) = \sum_{\gamma \in \Gamma/\Gamma_L} f(g\gamma \boldsymbol{e}_\chi).$$

Let $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ be a representation which is strongly rational over $\mathbb{Q}$ and let $\chi \in X^*(\mathbf{P}_0)_\mathbb{Q}$ be the associated highest $\mathbb{Q}$-weight. Let $\mathbf{D} \subset \mathbf{V}_\chi$ be the unique $\mathbf{P}_0$-invariant line and let $\mathbf{P}$ be the stabilizer in $\mathbf{G}$ of this line. Then $\mathbf{P}$ is the standard parabolic $\mathbb{Q}$-subgroup corresponding to the subset $\theta \subset \Delta$ defined by $\theta = \{\alpha \in \Delta : \langle \chi, \alpha \rangle = 0\}$.

Corollary 3.2.3. *Let $\chi_1, \chi_2 \in X^*(\mathbf{P}_0)_\mathbb{Q}$ be two highest $\mathbb{Q}$-weights such that one is a multiple of the other. Let $\mathbf{P} = \mathbf{P}_\theta$ be the associated standard parabolic $\mathbb{Q}$-subgroup corresponding to the subset $\theta = \{\alpha \in \Delta : \langle \chi_1, \alpha \rangle = 0\} \subset \Delta$. Let $\mathbf{L}$ be the kernel of χ_1 in $\mathbf{P}$ and put $L = \mathbf{L}(\mathbb{R})$. For every $i \in \{1, 2\}$, let $\widetilde{X}_i$ be the closure of $G \cdot \boldsymbol{e}_{\chi_i}$ in V_{χ_i} and let $\Omega_i = G/\Gamma_i$, equipped with the unique G-invariant probability measure μ_{Ω_i}. Let $p \in [1, +\infty]$. Then the Siegel transform S_{χ_1} maps $C_c(\widetilde{X}_1)$ into $L^p(\Omega_1)$ if and only if S_{χ_2} maps $C_c(\widetilde{X}_2)$ into $L^p(\Omega_2)$.*

Proof. Since the two highest $\mathbb{Q}$-weights χ_1, χ_2 are multiples of each other, they define the same parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ and have the same kernel $\mathbf{L}$ in $\mathbf{P}$. The orbit maps $\rho_1 : G/L \to \widetilde{X}_1$ and $\rho_2 : G/L \to \widetilde{X}_2$, given by $\rho_1(gL) = g\boldsymbol{e}_{\chi_1}$ and $\rho_2(gL) = g\boldsymbol{e}_{\chi_2}$, respectively, extend to homeomorphisms and give rise to a homeomorphism between $\widetilde{X}_1$ and $\widetilde{X}_2$. The respective Siegel transforms are given by, for every $f_1 \in C_c(\widetilde{X}_1)$, $f_2 \in C_c(\widetilde{X}_2)$, $g_1, g_2 \in G$,

$$S_{\chi_1} f_1(g_1) = \sum_{\gamma_1 \in \Gamma_1/\Gamma_1 \cap L} f_1(g_1\gamma_1\boldsymbol{e}_{\chi_1}) \quad \text{and} \quad S_{\chi_2} f_2(g_2) = \sum_{\gamma_2 \in \Gamma_2/\Gamma_2 \cap L} f_2(g_2\gamma_2\boldsymbol{e}_{\chi_2}).$$

It suffices, by symmetry, to assume that the Siegel transform S_{χ_1} maps $C_c(\widetilde{X}_1)$ into $L^p(\Omega_1)$ and to show that S_{χ_2} maps $C_c(\widetilde{X}_2)$ into $L^p(\Omega_2)$. So let $f_2 \in C_c(\widetilde{X}_2)$. We first claim that E_{Γ_1}, in the notation of Lemma 3.2.2, maps $C_c(G/L)$ into $L^p(\Omega_1)$. Indeed, observe that for every $f_1 \in C_c(G/L)$ and $g \in G$,

$$E_{\Gamma_1} f_1(g) = \sum_{\gamma_1 \in \Gamma_1/\Gamma_1 \cap L} f_1(g\gamma_1 L) = S_{\chi_1}(f_1 \circ \rho_1^{-1})(g)$$

and, since ρ_1 induces a homeomorphism, the claim follows. Using Lemma 3.2.2, it now follows that E_{Γ_2} maps $C_c(G/L)$ into $L^p(\Omega_2)$. Using that ρ_2 induces a homeomorphism, a similar argument now implies that S_{χ_2} maps $C_c(\widetilde{X}_2)$ into $L^p(\Omega_2)$. The proof is complete. □

3.3 L^1-integrability

Let $\mathbf{G}$, $\mathbf{T}$, $\mathbf{P}_0$, Δ, χ, $\mathbf{P}$, $\boldsymbol{e}_\chi$ and $\mathbf{V}_\chi$ be as in the beginning of Section 3.2. Here, we characterize the L^1-integrability of the Siegel transform S_χ. We will need the following lemma. For every $g \in G$, we define

$$\lambda_\chi(g) = \min_{\boldsymbol{v} \in \mathbf{V}_\chi(\mathbb{Z}) \smallsetminus \{\mathbf{0}\}} \|g\boldsymbol{v}\|$$

to be the length of the shortest non-zero vector in $g\mathbf{V}_\chi(\mathbb{Z})$. We wish to bound the Siegel transform of a function $f \in C_c(\widetilde{X})$ in terms of λ_χ. In order to proceed, we need a consequence of Corollary 6.5.1, which is proved in Chapter 6 and assumes that the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ is maximal. To state it, we need to anticipate some notation. Using Lemma 3.2.2, we can assume, without loss of generality, that the lattice $\mathbf{V}_\chi(\mathbb{Z})$ is spanned over $\mathbb{Z}$ by an basis $(\boldsymbol{v}_i)_{i \in I}$ contained in $\mathbf{V}_\chi(\mathbb{Q})$ consisting of weight vectors for the action of $\mathbf{T}(\mathbb{R})$. Fix a K-invariant Euclidean inner product $\langle \cdot, \cdot \rangle$ on V_χ for which this basis is orthonormal and we denote the implied norm on V_χ by $\|\cdot\|$. Let us write $\mathfrak{a}$ for the Lie algebra of $\mathbf{T}(\mathbb{R})$ and let $Y_\alpha \in \mathfrak{a}$ be the unique element satisfying

$$\alpha(Y_\alpha) = -1 \quad \text{and} \quad \beta(Y_\alpha) = 0 \quad \text{for all } \beta \in \Delta \smallsetminus \{\alpha\}. \tag{3.3.1}$$

Moreover, let $d = \dim X$ be the dimension (as a real manifold) of the space of real points of the generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$. Define $\beta_\chi = -\frac{1}{\chi(Y_\alpha)}$. We obtain a height function H_χ on $\mathbb{P}(\mathbf{V}_\chi)(\mathbb{Q})$ by $H([\boldsymbol{v}]) = \|\boldsymbol{v}\|$, where $\boldsymbol{v}$ is a primitive vector in the lattice $\mathbf{V}_\chi(\mathbb{Z})$ representing $[\boldsymbol{v}]$. This height function is well defined, since a primitive vector in $\mathbf{V}_\chi(\mathbb{Z})$ representing $[\boldsymbol{v}]$ is uniquely determined up to multiplication by ± 1. The map $\iota_\chi : \mathbf{X} \to \mathbb{P}(\mathbf{V}_\chi)$ given by $\iota_\chi(g\mathbf{P}) = g[\boldsymbol{e}_\chi]$ is an embedding defined over $\mathbb{Q}$. One then obtains a height function on $\mathbf{X}(\mathbb{Q})$, which we also denote by H_χ, using the embedding ι_χ. For every $T \geqslant 1$, we define the function

$$\mathcal{N}_\chi(T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : H_\chi(v) < T\}$$

counting rational points in X of height bounded by T. By Corollary 6.5.1, there exists a constant $\varkappa > 0$ such that

$$\mathcal{N}_\chi(T) \sim \varkappa T^{\beta_\chi d} \quad \text{as } T \to +\infty.$$

Moreover, since the projection map $\mathcal{L}_\chi \to \mathbf{X}(\mathbb{Q})$ is surjective with finite fibers (of cardinality $\leqslant 2$), for every $T \geqslant 2$, we have

$$\#\{\boldsymbol{v} \in \mathcal{L}_\chi : \|\boldsymbol{v}\| < T\} \asymp T^{\beta_\chi d}. \tag{3.3.2}$$

Lemma 3.3.1. *Suppose that the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ is maximal. Then, for every $f \in C_c(\widetilde{X})$, we have*

$$\forall\, g \in G, \quad |S_\chi f(g)| \lesssim_{\operatorname{supp}(f)} \|f\|_\infty\, \lambda_\chi(g)^{-\beta_\chi d}. \tag{3.3.3}$$

Although this bound is not optimal, it is sufficient for our purposes.

Proof. Let $f \in C_c(\widetilde{X})$ and pick $r = r(\operatorname{supp}(f)) \geqslant 2$ such that $\operatorname{supp}(f)$ is contained in $B_{\widetilde{X}}(r) = \{\boldsymbol{v} \in \widetilde{X} : \|\boldsymbol{v}\| < r\}$. Using reduction theory as presented in §3.1.4, since $G = \mathfrak{S}\, C\, \Gamma$, where $\mathfrak{S} = K A_\tau \boldsymbol{\omega}$ is a Siegel set (we recall that the sets K and $\boldsymbol{\omega}$ are compact) and $C \subset \mathbf{G}(\mathbb{Q})$ a finite subset, we can express $g = kanc\gamma$ with $k \in K$, $a \in A_\tau$, $n \in \boldsymbol{\omega}$, $c \in C$, and $\gamma \in \Gamma$. We assumed that Γ acts transitively on $\mathcal{L}_\chi$. This implies that Γ acts transitively on the set of rational points of $\mathbf{X} = \mathbf{G}/\mathbf{P}$. Hence, by [Bor69, Proposition 15.6], we can take $C = \{1\}$. There exists a radius $r_0 > 0$ such that $\mathfrak{a}_\tau$ is contained in $\mathfrak{a}_0 + B_{\mathfrak{a}}(r_0)$, where

$$\mathfrak{a}_0 = \{Y \in \mathfrak{a} : \forall\, \alpha \in \Delta,\ \alpha(Y) \leqslant 0\}.$$

Let $k \in K$, $n \in \boldsymbol{\omega}$, $a \in A_\tau$, and $\gamma \in \Gamma$. We express $a = a_0 \exp(O(1))$ with $a_0 \in \exp(\mathfrak{a}_0)$. Using that λ_χ is right Γ-invariant, that K is compact and that $\bigcup_{a \in A_\tau} a\boldsymbol{\omega}a^{-1}$ is relatively compact (see [Bor69, Lemma 12.2]), we have

$$\lambda_\chi(kan\gamma) \asymp \lambda_\chi(a_0). \tag{3.3.4}$$

Let $Y_0 \in \mathfrak{a}_0$ and $a_0 = \exp(Y_0)$. By the description of the $\mathbb{Q}$-weights of the representation π_χ in (3.1.1), for every $\mathbb{Q}$-weight μ of π_χ, we have

$$\chi(Y_0) \leqslant \mu(Y_0).$$

In particular, since we assumed $\mathbf{V}_\chi(\mathbb{Z})$ to be spanned over $\mathbb{Z}$ by an orthonormal basis consisting of $\mathbb{Q}$-weight vectors for the action of $\mathbf{T}(\mathbb{R})$, we have $\lambda_\chi(a_0) = \exp(\chi(Y_0))$. By taking inverses, we see that the largest eigenvalue of a_0^{-1} is $\lambda_\chi(a_0)^{-1}$. Hence $a_0^{-1}B_{\widetilde{X}}(r) \subseteq \lambda_\chi(a_0)^{-1}B_{\widetilde{X}}(r)$. Since the ball $B_{\widetilde{X}}(r)$ is K-invariant, the Siegel transform $S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)}$ is so, too. Using that $S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)}$ is right Γ-invariant, that K is compact and that $\bigcup_{a \in A_\tau} a\boldsymbol{\omega}a^{-1}$ is relatively compact, there exists a constant $C_0 \geqslant 1$, independent of f, such that, for every $g \in G$ with Siegel decomposition $g = kan\gamma$ (and writing $a = a_0 \exp(O(1))$ as above), we have

$$\begin{aligned} |S_\chi f(g)| &\leqslant \|f\|_\infty\, S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)}(g) \\ &\leqslant \|f\|_\infty\, S_\chi \mathbb{1}_{a_0^{-1}B_{\widetilde{X}}(C_0 r)}(1) \lesssim \|f\|_\infty\, S_\chi \mathbb{1}_{B_{\widetilde{X}}(C_0\lambda_\chi(a_0)^{-1}r)}(1). \end{aligned}$$

Using (3.3.4) and noting that

$$S_\chi \mathbb{1}_{B_{\widetilde{X}}(C_0\lambda_\chi(a_0)^{-1}r)}(1) = \#\{\boldsymbol{v} \in \mathcal{L}_\chi : \|\boldsymbol{v}\| < C_0\lambda_\chi(a_0)^{-1}r\},$$

it now follows from (3.3.2) that

$$|S_\chi f(g)| \lesssim_{\mathrm{supp}\,(f)} \|f\|_\infty\, \lambda_\chi(g)^{-\beta_\chi d},$$

as desired. $\square$

For the convenience of the reader, we recall the statement of Theorem 1.1.1. The first four equivalences in this theorem are probably standard.

Theorem (L^1-integrability)**.** *The following assertions are equivalent.*

(1) The Siegel transform S_χ *maps* $C_c(\widetilde{X})$ *into* $L^1(\Omega)$*.*

(2) There exists a unique (up to scaling) G*-invariant Radon measure* $\lambda_{\widetilde{X}}$ *on* $\widetilde{X}$ *and one has a convergent mean value formula: for all* $f \in C_c(\widetilde{X})$*,*

$$\int_\Omega S_\chi f \, d\mu_\Omega = \int_{\widetilde{X}} f \, d\lambda_{\widetilde{X}} \tag{3.3.5}$$

In particular, for any $f \in C_c(\widetilde{X})$*,*

$$\|S_\chi f\|_{L^1(\Omega)} \leqslant \|f\|_{L^1(\widetilde{X})},$$

and hence the Siegel transform S_χ *can be extended to a bounded operator* $S_\chi : L^1(\widetilde{X}) \to L^1(\Omega)$*.*

(3) The Lie group $L = \mathbf{L}(\mathbb{R})$ *is unimodular and* Γ_L *is a lattice in* L*.*

(4) The parabolic $\mathbb{Q}$*-subgroup* $\mathbf{P}$ *of* $\mathbf{G}$ *is* maximal.

(5) There exists $\varepsilon > 0$ *such that* S_χ *maps* $C_c(\widetilde{X})$ *into* $L^{1+\varepsilon}(\Omega)$*.*

Proof of Theorem 1.1.1. We first show that (1) $\Rightarrow$ (2). In view of the Riesz-Markov-Kakutani Representation Theorem, since $\Lambda(f) = \int_\Omega S_\chi f \, d\mu$ defines a positive G-invariant linear functional on $C_c(\widetilde{X})$ by assumption, there exists a unique G-invariant Radon measure $\lambda_{\widetilde{X}}$ on $\widetilde{X}$ such that for all $f \in C_c(\widetilde{X})$,

$$\int_\Omega S_\chi f \, d\mu_\Omega = \int_{\widetilde{X}} f \, d\lambda_{\widetilde{X}}.$$

Note that $\widetilde{X}$ is the disjoint union of the two G-orbits $\widetilde{X}_0$ and $\{\mathbf{0}\}$. One may restrict the functional Λ to the space $C_c(\widetilde{X}_0)$ of functions in $C_c(\widetilde{X})$ that are supported away from the origin, and obtain a G-invariant Radon measure λ' on $\widetilde{X}_0$ such that for all $f \in C_c(\widetilde{X}_0)$, $\int_\Omega S_\chi f \, d\mu_\Omega = \int_{\widetilde{X}} f \, d\lambda'$. The measure $\lambda_{\widetilde{X}} = a\lambda' + b\delta_0$ is a linear combination of λ' and the Dirac delta measure δ_0 supported at the origin. Let us show that $a = 1$ and $b = 0$. So let $f_n \in C_c(\widetilde{X})$ be a decreasing sequence of functions satisfying $0 \leqslant f_n \leqslant 1$,

$\operatorname{supp}(f_n) \subseteq B_{\widetilde{X}}(1/n)$, $f_n(0) = 1$ for all $n \geqslant 1$, and $f_n(\boldsymbol{v}) \to 0$ for all non-zero $\boldsymbol{v} \in \widetilde{X}$. Since for every $n \geqslant 1$, $\operatorname{supp}(f_n) \subseteq B_{\widetilde{X}}(1/n)$, we have, for every $g \in G$, that $\lim_{n\to\infty} S_\chi f_n(g) = 0$. Using this together with the fact that $S_\chi f_n \leqslant S_\chi f_1$ and the latter is in $L^1(\Omega)$ by assumption, Lebesgue's dominated convergence theorem now yields the desired claim: one has

$$\lim_{n\to\infty} \int_\Omega S_\chi f_n \, d\mu_\Omega = 0,$$

but also

$$\lim_{n\to\infty} \int_\Omega S_\chi f_n \, d\mu_\Omega = \lim_{n\to\infty} \int_{\widetilde{X}} f_n \, d\lambda_{\widetilde{X}} = \lim_{n\to\infty} (a\lambda'(f_n) + b) = b.$$

To see the implication (2) $\Rightarrow$ (3), one first notes that since $\widetilde{X}_0 = G/L$ carries a positive G-invariant Radon measure $\lambda_{\widetilde{X}}$, L must be a unimodular subgroup of G (see, for instance, [Fol15, Theorem 2.51]). Fix a Haar measure μ_L on L. The discrete group Γ_L is unimodular and we equip it with the counting measure. By assumption, for every non-negative $f \in C_c(\widetilde{X})$, we have

$$\int_{G/\Gamma} \sum_{\gamma\in\Gamma/\Gamma_L} f(g\gamma\boldsymbol{e}_\chi) \, d\mu_\Omega(g) \leqslant \int_\Omega S_\chi f \, d\mu_\Omega < +\infty.$$

Using Weil's integration formula in the form of Theorem 2.3.3, there exist unique (up to scaling) G and L-invariant measures $\mu_{G/L}$ and μ_{L/Γ_L} on G/L and L/Γ_L, respectively, such that for every non-negative $f \in C_c(\widetilde{X})$, we have

$$\int_{G/\Gamma} \sum_{\gamma\in\Gamma/\Gamma_L} f(g\gamma\boldsymbol{e}_\chi) \, d\mu_\Omega(g) = \int_{G/L} \int_{L/\Gamma_L} f(gl\boldsymbol{e}_\chi) \, d\mu_{L/\Gamma_L}(l) d\mu_{G/L}(g).$$

Using that L stabilizes the vector $\boldsymbol{e}_\chi$, we further have

$$\int_{G/L} \int_{L/\Gamma_L} f(gl\boldsymbol{e}_\chi) \, d\mu_{L/\Gamma_L}(l) d\mu_{G/L}(g) = \mu_{L/\Gamma_L}(L/\Gamma_L) \int_{G/L} f(g\boldsymbol{e}_\chi) \, d\mu_{G/L}(g).$$

Choosing $f \in C_c(\widetilde{X})$ non-negative so that $\int_{G/L} f(g\boldsymbol{e}_\chi) \, d\mu_{G/L}(g) > 0$, we have

$$\mu_{L/\Gamma_L}(L/\Gamma_L) \leqslant \frac{\int_\Omega S_\chi f \, d\mu_\Omega}{\int_{G/L} f(g\boldsymbol{e}_\chi) \, d\mu_{G/L}(g)} < +\infty,$$

which shows that Γ_L is a lattice in L.

We now show that (3) $\Leftrightarrow$ (4). Here we use some facts about the structure of the parabolic $\mathbb{Q}$-subgroups, as recorded in §3.1.1. To maintain the flow of reading, we briefly recall the facts used here. The parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ contains the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$ and it is the standard parabolic $\mathbb{Q}$-subgroup corresponding to a unique subset $\theta \subseteq \Delta$. Therefore, we write $\mathbf{P} = \mathbf{P}_\theta$. Define the $\mathbb{Q}$-split torus $\mathbf{T}_\theta = \left(\bigcap_{\beta\in\theta} \ker(\beta)\right)^\circ$, where $(\cdot)^\circ$ means taking the connected component with respect to the Zariski topology. The group $\mathbf{P}$ is the semidirect product $\mathbf{P} = \mathcal{Z}(\mathbf{T}_\theta)\mathbf{U}_\theta$, where $\mathcal{Z}(\mathbf{T}_\theta)$ is the centralizer of $\mathbf{T}_\theta$ in $\mathbf{G}$ and $\mathbf{U}_\theta$ is its unipotent radical. Let $\mathbf{Q}$ be the largest $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T})$. Furthermore, the centralizer $\mathcal{Z}(\mathbf{T}_\theta)$ is an almost direct product of $\mathbf{Q}_\theta$, $\mathbf{M}_\theta$, and $\mathbf{T}_\theta$, where $\mathbf{M}_\theta$ is a connected semisimple $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T}_\theta)$ and $\mathbf{Q}_\theta$ is a connected $\mathbb{Q}$-subgroup of $\mathbf{Q}$. The stabilizer $\mathbf{L}$ in $\mathbf{G}$ of $\boldsymbol{e}_\chi$ then satisfies $\mathbf{L}^\circ = \mathbf{Q}_\theta\mathbf{M}_\theta(\mathbf{T}_\theta \cap \ker(\chi))^\circ\mathbf{U}_\theta$. Now, by a theorem of Borel and Harish-Chandra [BH62, Theorem 9.4], the Lie group L is unimodular and Γ_L is a lattice in L if and only if $\mathbf{L}^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters. The group of $\mathbb{Q}$-characters $X^*(\mathbf{L}^\circ)_\mathbb{Q}$ can be identified, by restriction, with $X^*(\mathbf{Q}_\theta\,\mathbf{M}_\theta\,(\mathbf{T}_\theta \cap \ker(\chi))^\circ)_\mathbb{Q}$. We claim that the latter is trivial if and only if the central $\mathbb{Q}$-split torus $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is trivial. In fact, if $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is trivial, then $\mathbf{Q}_\theta\,\mathbf{M}_\theta$ does not admit any non-trivial $\mathbb{Q}$-characters since it is an almost direct product of a $\mathbb{Q}$-anisotropic and a semisimple $\mathbb{Q}$-subgroup. Conversely, suppose that the $\mathbb{Q}$-split torus $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is non-trivial. Since $(\mathbf{Q}_\theta\,\mathbf{M}_\theta)$ is a normal $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T}_\theta)$, the quotient map $q : \mathcal{Z}(\mathbf{T}_\theta) \to \mathcal{Z}(\mathbf{T}_\theta)/(\mathbf{Q}_\theta\,\mathbf{M}_\theta)$ is a $\mathbb{Q}$-morphism of algebraic $\mathbb{Q}$-groups. Since its restriction to $\mathbf{T}_\theta$ is still surjective, we see that $\mathcal{Z}(\mathbf{T}_\theta)/(\mathbf{Q}_\theta\,\mathbf{M}_\theta)$ is a non-trivial $\mathbb{Q}$-split torus (see, for instance, [Bor69, Corollary10.4]). In particular, the composition of q with any non-trivial $\mathbb{Q}$-character of $\mathcal{Z}(\mathbf{T}_\theta)/(\mathbf{Q}_\theta\,\mathbf{M}_\theta)$ gives a non-trivial $\mathbb{Q}$-character of $\mathcal{Z}(\mathbf{T}_\theta)$, as required. The torus $\mathbf{T}_\theta$ acts non-trivially on the line through $\boldsymbol{e}_\chi$ via the character χ. In particular, the quotient $\mathbf{T}_\theta/(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is one-dimensional. Thus $(\mathbf{T}_\theta \cap \ker(\chi))^\circ$ is trivial if and only if $\mathbf{T}_\theta$ is one-dimensional if and only if $\mathbf{P}$ is maximal, as claimed.

Next we show that (4) $\Rightarrow$ (5). By Lemma 3.2.2, we may assume that the action of Γ on $\mathcal{L}_\chi$ is transitive. The proof is based on reduction theory, as it is presented in Borel [Bor69, §15, 16]. By assumption, $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup. In particular, there exists a unique simple root $\alpha \in \Delta$ such that $\theta = \Delta \smallsetminus \{\alpha\}$. Moreover, as was shown in the previous paragraph,

$\mathbf{P}$ if maximal if and only if the Lie group L is unimodular and Γ_L is a lattice in L. Thus the cone $\widetilde{X}_0 = G/L$ admits a unique (up to scaling) G-invariant Radon measure (see, for instance, [Fol15, Theorem 2.51]), which we fix and denote by $\lambda_{\tilde{X}}$. It will be convenient to parametrize $\widetilde{X}_0$ using polar coordinates as follows. Let A be the connected component of the group of real points of the one-dimensional torus $\mathbf{T}_\theta$. Setting $K_L = K \cap L$ to be the stabilizer in K of the vector $\boldsymbol{e}_\chi$ and further identifying $\widetilde{X}_0 = G/L$ with $K/K_L \times A$ gives natural polar coordinates on $\widetilde{X}_0$. In fact, every $\boldsymbol{v} \in \widetilde{X}_0$ can be written uniquely as $\boldsymbol{v} = ka\boldsymbol{e}_\chi$ for some $k \in K/K_L$ and $a \in A$. The group A can be parametrized concretely as follows. For $y \in \mathbb{R}_+^\times$ define

$$a(y) = \exp(\log(y)Y_\alpha), \tag{3.3.6}$$

where $Y_\alpha \in \mathrm{Lie}(\mathbf{T}(\mathbb{R}))$ is determined by, for every $\beta \in \Delta$,

$$\beta(Y_\alpha) = \begin{cases} 0 & \text{if } \beta \neq \alpha, \\ -1 & \text{if } \beta = \alpha. \end{cases}$$

In particular, we have $a(y)\boldsymbol{e}_\chi = \exp(\log(y)\,\chi(Y_\alpha))\boldsymbol{e}_\chi = y^{\chi(Y_\alpha)}\boldsymbol{e}_\chi$ for all $y > 0$ and $A = \{a(y) : y > 0\}$.

Let $f \in C_c(\widetilde{X})$ and let $\varepsilon > 0$. Let us prove that $S_\chi f \in L^{1+\varepsilon}(\Omega)$ if $\varepsilon' = \varepsilon\beta_\chi d$ is small enough. Without loss of generality, we can assume that f is non-negative and K-invariant. Then, using Weil's integration formula (2.3.3) (applied to the triple $H = G$, $H_1 = \Gamma$ and $H_2 = \Gamma_L$), there exists a unique G-invariant measure μ_{G/Γ_L} on G/Γ_L such that we have

$$\begin{aligned}\int_\Omega |S_\chi f|^{1+\varepsilon}\, d\mu_\Omega &= \int_\Omega \sum_{\gamma\in\Gamma/\Gamma_L} \left(f(g\gamma\boldsymbol{e}_\chi)\,(S_\chi f)(g)^\varepsilon\right)\, d\mu_\Omega(g\Gamma) \\ &= \int_{G/\Gamma_L} f(g\gamma\boldsymbol{e}_\chi)\,(S_\chi f)(g)^\varepsilon\, d\mu_\Omega(g\Gamma).\end{aligned}$$

Now, using Weil's integration formula (2.3.3) again (but applied to the triple $H = G$, $H_1 = L$ and $H_2 = \Gamma_L$), there exists a unique L-invariant measure μ_{L/Γ_L} on L/Γ_L such that we have

$$\begin{aligned}&\int_{G/\Gamma_L} f(g\gamma\boldsymbol{e}_\chi)\,(S_\chi f)(g)^\varepsilon\, d\mu_\Omega(g) \\ &\qquad\qquad = \int_{G/L}\int_{L/\Gamma_L} f(gl\boldsymbol{e}_\chi)\,(S_\chi f)(gl)^\varepsilon\, d\mu_{L/\Gamma_L}(l\Gamma_L)\, d\lambda_{\tilde{X}}(gL)\end{aligned}$$

Further, using the fact that L stabilizes the vector $\boldsymbol{e}_\chi$, the right hand side equals

$$\int_{G/L} f(g\boldsymbol{e}_\chi) \int_{L/\Gamma_L} (S_\chi f)(gl)^\varepsilon \, d\mu_{L/\Gamma_L}(l) \, d\lambda_{\widetilde{X}}(g).$$

Moreover, using the identification $\widetilde{X}_0 = G/L = K/K_L \times A$, this further equals

$$\int_{\widetilde{X}_0} f(ka(y)\boldsymbol{e}_\chi) \int_{L/\Gamma_L} (S_\chi f)(ka(y)l)^\varepsilon \, d\mu_{L/\Gamma_L}(l) \, d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).$$

By Lemma 3.3.1, we can estimate $|S_\chi f(g)| \lesssim_{\mathrm{supp}\,(f)} \|f\|_\infty \lambda_\chi(g)^{-\beta_\chi d}$. Putting everything together, and using that f and λ_χ are K-invariant, we obtain, with $\varepsilon' = \varepsilon\beta_\chi d$,

$$\begin{aligned} &\int_\Omega |S_\chi f|^{1+\varepsilon} \, d\mu_\Omega \\ &\quad \lesssim \|f\|_\infty^\varepsilon \int_{\widetilde{X}_0} f(a(y)\boldsymbol{e}_\chi) \int_{L/\Gamma_L} \lambda_\chi(a(y)l)^{-\varepsilon'} \, d\mu_{L/\Gamma_L}(l) d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi). \end{aligned} \tag{3.3.7}$$

where the implicit constant depends only on the support of f. Recall that we assumed $\mathbf{V}_\chi(\mathbb{Z})$ to be spanned over $\mathbb{Z}$ by an orthonormal basis consisting of $\mathbb{Q}$-weight vectors for the action of $\mathbf{T}(\mathbb{R})$. Hence $\lambda_\chi(a(y))$ is the smallest eigenvalue of $a(y)$ with respect to this basis. Therefore, we have, for every $y > 0$ and $l \in L$,

$$\lambda_\chi(a(y)l) = \min_{\boldsymbol{v} \in \mathbf{V}_\chi(\mathbb{Z}) \smallsetminus \{\mathbf{0}\}} \|a(y)l\boldsymbol{v}\| \geqslant \lambda_\chi(a(y))\lambda_\chi(l)$$

and, by raising to the power $-\varepsilon'$, $\lambda_\chi(a(y)l)^{-\varepsilon'} \leqslant \lambda_\chi(a(y))^{-\varepsilon'}\lambda_\chi(l)^{-\varepsilon'}$. As a result, we can separate the above integral and get

$$\begin{aligned} &\int_\Omega |S_\chi f|^{1+\varepsilon} \, d\mu_\Omega \\ &\qquad \lesssim \int_{\widetilde{X}_0} f(a(y)\boldsymbol{e}_\chi)\lambda_\chi(a(y))^{-\varepsilon'} \, d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi) \int_{L/\Gamma_L} \lambda_\chi(l)^{-\varepsilon'} \, d\mu_{L/\Gamma_L}(l). \end{aligned}$$

Let us first show that, for every sufficiently small $\varepsilon' > 0$, the first integral in the product on the right-hand side converges. In the above coordinates (see, for instance, the measure description in Chapter 4, specifically Equation (4.4.7)), there exists a constant $\omega_1 > 0$ such that the measure $\lambda_{\widetilde{X}}$ is given by

$$d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi) = \omega_1 y^{-(d+1)}\, d\sigma(k)dy, \tag{3.3.8}$$

where σ is the unique K-invariant probability measure on K/K_L and, as before, $d = \dim X$ is the dimension (as a real manifold) of the space of real points of the generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$. Moreover, since f is compactly supported, there exists $y_f > 0$ such that, for every $y < y_f$, we have (recall that $\chi(Y_\alpha) < 0$)

$$a(y)\boldsymbol{e}_\chi = y^{\chi(Y_\alpha)}\boldsymbol{e}_\chi \notin \operatorname{supp}(f).$$

Therefore, $f(a(y)\boldsymbol{e}_\chi)\lambda_\chi(a(y))^{-\varepsilon'}$ vanishes for every $y < y_f$. Let us now consider the case $y \geqslant y_f$. Recall from the proof of Lemma 3.3.1, that we denote by $\mathfrak{a}$ the Lie algebra of $\mathbf{T}(\mathbb{R})$ and we let

$$\mathfrak{a}_0 = \{Y \in \mathfrak{a} : \forall\, \alpha' \in \Delta,\ \alpha'(Y) \leqslant 0\}.$$

Moreover, we observed that elements $Y_0 \in \mathfrak{a}_0$ satisfy, for every $\mathbb{Q}$-weight μ of π_χ,

$$\chi(Y_0) \leqslant \mu(Y_0).$$

Notice that for $a(y) = \exp(\log(y)\, Y_\alpha)$ we have $\log(y)\, Y_\alpha \in \mathfrak{a}_0$ if and only if $\log(y) \geqslant 0$ (hence, if and only if $y \geqslant 1$). In particular, since we assumed $\mathbf{V}_\chi(\mathbb{Z})$ to be spanned over $\mathbb{Z}$ by an orthonormal basis consisting of $\mathbb{Q}$-weight vectors for the action of $\mathbf{T}(\mathbb{R})$, we have, for every $y \geqslant 1$,

$$\lambda_\chi(a(y)) = \exp(\log(y)\, \chi(Y_\alpha)) = y^{\chi(Y_\alpha)}.$$

Since the interval $[y_f, 1]$ is compact, there exists a positive constant $c_f < 1$ such that for every $y \in [y_f, 1]$, $\lambda_\chi(a(y))$ is bounded from below by $c_f y^{\chi(Y_\alpha)}$. Now, using the measure description (3.3.8), we have

$$\begin{aligned}\int_{\widetilde{X}_0} f(a(y)\boldsymbol{e}_\chi)\lambda_\chi(a(y))^{-\varepsilon'}\, d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi) &= \int_{y\geqslant y_f} f(a(y)\boldsymbol{e}_\chi)\lambda_\chi(a(y))^{-\varepsilon'}\, y^{-(d+1)}\, dy \\ &\lesssim \|f\|_\infty\, c_f^{-\varepsilon'} \int_{y\geqslant y_f} y^{-\varepsilon'\chi(Y_\alpha)}\, y^{-(d+1)}\, dy,\end{aligned}$$

which converges, if $\varepsilon' > 0$ is sufficiently small.

Thus it suffices to show that for $\varepsilon' > 0$ sufficiently small, we have

$$\int_{L/\Gamma_L} \lambda_\chi(l)^{-\varepsilon'}\, d\mu_{L/\Gamma_L}(l) < +\infty.$$

Recall that since $\mathbf{P}$ is maximal, we know that $\mathbf{L} = \mathbf{Q}_\theta \mathbf{M}_\theta (\mathbf{T}_\theta \cap \ker \chi)\mathbf{U}_\theta$ is an almost direct product of a connected $\mathbb{Q}$-subgroup $\mathbf{Q}_\theta$ of the largest $\mathbb{Q}$-anisotropic subgroup of $\mathcal{Z}(\mathbf{T})$, a connected semisimple $\mathbb{Q}$-group $\mathbf{M}_\theta$, a finite group $(\mathbf{T}_\theta \cap \ker \chi)$ and the unipotent radical of $\mathbf{P}$. Without loss of generality, we may assume that $(\mathbf{T}_\theta \cap \ker \chi) = \{1\}$. We note that $\widetilde{\mathbf{M}}_\theta = \mathbf{Q}_\theta \mathbf{M}_\theta$ is a connected reductive $\mathbb{Q}$-group, $\mathbf{L}$ is the semi-direct product of $\widetilde{\mathbf{M}}_\theta$ and $\mathbf{U}_\theta$ and $X^*(\widetilde{\mathbf{M}}_\theta)_\mathbb{Q} = \{1\}$. Since by [BH62, Corollary 6.4] the group $\widetilde{M}_\theta(\mathbb{Z})U_\theta(\mathbb{Z})$ has finite index in Γ_L, up to passing to a larger quotient, we may assume that $\Gamma_L = \widetilde{M}_\theta(\mathbb{Z})U_\theta(\mathbb{Z})$.

As follows from [Bor69, Remark 9.9], if Ω_1 is a fundamental set (see [Bor69, Definition 9.6]) for $\widetilde{M}_\theta(\mathbb{Z})$ in $\widetilde{M}_\theta$, $\widetilde{K}$ a maximal compact subgroup of $\widetilde{M}_\theta$ such that $\widetilde{K}\Omega_1 = \Omega_1$, and Ω_2 a compact subset of U_θ such that $U_\theta = \Omega_2 U_\theta(\mathbb{Z})$, then $\Omega_1\Omega_2$ is a fundamental set for Γ_L in L. Let dm_θ and du_θ be Haar measures on $\widetilde{M}_\theta$ and U_θ, respectively. Hence, after choosing appropriate fundamental sets Ω_1 and Ω_2, it will be enough to check that

$$\int_{\Omega_1}\int_{\Omega_2} \lambda_\chi(mu)^{-\varepsilon'}\, du_\theta dm_\theta$$

is finite.

Fix any compact subset Ω_2 of U_θ such that $U_\theta = \Omega_2 U_\theta(\mathbb{Z})$. For Ω_1 we will choose a large enough Siegel set for $\widetilde{M}_\theta(\mathbb{Z})$ in $\widetilde{M}_\theta$. To keep the exposition simple, we use the same notation as in the reduction theory of Γ in G, but with a tilde for distinction. Hence, using [Bor69, Theorem 13.1], there exists a Siegel set $\widetilde{\mathfrak{S}}$ (with respect to the maximal compact subgroup $\widetilde{K} = K \cap \widetilde{M}_\theta$, the minimal parabolic $\mathbb{Q}$-subgroup $\widetilde{\mathbf{P}_0} = \mathbf{P}_0 \cap \widetilde{\mathbf{M}}_\theta$, and the maximal $\mathbb{Q}$-split torus $\widetilde{\mathbf{T}} = \mathbf{T} \cap \widetilde{\mathbf{M}}_\theta$) and a finite subset $\widetilde{C} \subset \widetilde{\mathbf{M}}_\theta(\mathbb{Q})$ such that

$$\widetilde{M}_\theta = \widetilde{\mathfrak{S}}\, \widetilde{C}\, \widetilde{M}_\theta(\mathbb{Z}).$$

Let $\widetilde{\mathbf{U}_0}$ denote the unipotent radical of $\widetilde{\mathbf{P}_0}$ and let $\widetilde{\mathbf{M}}_0$ be the maximal $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of the centralizer $\mathcal{Z}(\widetilde{\mathbf{T}})^\circ$ of $\widetilde{\mathbf{T}}$ in $\widetilde{\mathbf{M}}_\theta$. Since $\mathbf{P}$ is maximal by assumption, the subset $\theta \subseteq \Delta$ consists of all but one simple root α: $\theta = \Delta \smallsetminus \{\alpha\}$ and $\mathbf{P} = \mathbf{P}_\theta$ is the corresponding standard parabolic $\mathbb{Q}$-subgroup. By the structure theory of the parabolic $\mathbb{Q}$-subgroups, as recorded in Section 3.1.1, we have that $\widetilde{\Delta} = \theta = \Delta \smallsetminus \{\alpha\} \subset \Delta$ is the set of simple roots of $\widetilde{\mathbf{M}}_\theta$ relative to $\widetilde{\mathbf{T}}$.

One has $\widetilde{\mathfrak{S}} = \widetilde{K}\,\widetilde{A}_{\tau'}\,\widetilde{\boldsymbol{\omega}}$, where $\widetilde{A}_{\tau'} = \{a \in \widetilde{T}^\circ : \forall \beta \in \widetilde{\Delta}\,,\, \beta(a) \leqslant \tau'\}$ with $\widetilde{\boldsymbol{\omega}}$ a compact neighborhood of 1 in $\widetilde{M}_0\widetilde{U}_0$. In particular, it is enough to show that

$$\int_{\widetilde{\mathfrak{S}}\cdot\widetilde{C}} \int_{\Omega_2} \lambda_\chi(mu)^{-\varepsilon'}\, du_\theta dm_\theta < +\infty.$$

We claim that $\lambda_\chi(mu)^{-1} \lesssim \lambda_\chi(a)^{-1}$ for all $m = kanc$ with $kan \in \widetilde{\mathfrak{S}}$, $c \in \widetilde{C}$ and $u \in \Omega_2$. Indeed, using the fact that $\bigcup_{a\in\widetilde{A}_{\tau'}} a\widetilde{\boldsymbol{\omega}}a^{-1}$ is relatively compact (see [Bor69, Lemma 12.2]) and since $\min_{\boldsymbol{v}\in\mathbf{V}_\chi(\mathbb{Z})\smallsetminus\{0\},\, c\in\widetilde{C},\, u\in\Omega_2} \|cu\boldsymbol{v}\| > 0$ (as $\widetilde{C}$ and Ω_2 are compact and $\mathbf{V}_\chi(\mathbb{Z})$ is discrete), we have

$$\lambda_\chi(mu) = \min_{\boldsymbol{v}\in\mathbf{V}_\chi(\mathbb{Z})\smallsetminus\{0\}} \|kan(a)^{-1}acu\boldsymbol{v}\| \gtrsim \lambda_\chi(a) \min_{\boldsymbol{v}\in\mathbf{V}_\chi(\mathbb{Z})\smallsetminus\{0\},\, c\in\widetilde{C},\, u\in\Omega_2} \|cu\boldsymbol{v}\|,$$

as required.

Let $\widetilde{\mathfrak{a}} = \mathrm{Lie}(\widetilde{T})$ and $\rho : \widetilde{\mathfrak{a}} \to (0, +\infty)$ be the character which is the sum of all the positive roots of $\widetilde{M}_\theta$ relative to $\widetilde{T}$ with multiplicities counted and can therefore be written in the form $\rho = \sum_{\beta\in\widetilde{\Delta}} n_\beta \beta$, with $n_\beta \geqslant 1$. According to [Kna02, Proposition 8.44], since we have a decomposition $\widetilde{M}_\theta = \widetilde{K}\widetilde{B}$ with $\widetilde{B} = \widetilde{M}_0\widetilde{T}^\circ\widetilde{N}$, the left Haar measures dk, dm_0, da, and dn of $\widetilde{K}$, $\widetilde{M}_0$, $\widetilde{T}^\circ$, and $\widetilde{N}$ can be normalized so that

$$dm_\theta = \rho(a)\, dk\, dm_0\, da\, dn.$$

Hence, it suffices to show that

$$\int_{\widetilde{A}_{\tau'}} \lambda_\chi(a)^{-\varepsilon'} \rho(a)\, da < +\infty.$$

For every $\tau' \geqslant 0$, write $\widetilde{\mathfrak{a}}_{\tau'} = \{X \in \widetilde{\mathfrak{a}} : \forall\, \beta \in \widetilde{\Delta}\,,\, \beta(X) \leqslant \log \tau'\}$. The exponential map is an isomorphism from $\widetilde{\mathfrak{a}}$ onto $\widetilde{T}^\circ$ that carries a Lebesgue measure to a Haar measure. Let $\widetilde{A}_{\tau'} = \exp(\widetilde{\mathfrak{a}_{\tau'}})$. Since $X^*(\widetilde{\mathbf{M}}_\theta)_{\mathbb{Q}} = \{1\}$ is trivial by Section 3.1.1, the set of $\beta \in \widetilde{\Delta}$ forms a basis of $X(\widetilde{\mathbf{T}})$. Hence, their differentials, which we also denote by β, form a system of coordinates on $\widetilde{\mathfrak{a}}$. Let I denote the set of $\mathbb{Q}$-weights on $\mathbf{V}_\chi$. Since $\mathbf{V}_\chi(\mathbb{Z})$ is spanned by an orthonormal basis consisting of $\mathbb{Q}$-weight vectors for the action of $\mathbf{T}$, we have, for every $a \in \widetilde{A}_{\tau'}$, $\lambda_\chi(a) = \min_{\widehat{\chi} \in I} \widehat{\chi}(a)$ and $\widehat{\chi}(a) = e^{\sum_{\beta \in \widetilde{\Delta}} c_\beta(\widehat{\chi})\beta}$ with $c_\beta(\widehat{\chi}) \in \mathbb{R}$. Therefore, for every $a \in \widetilde{A}_{\tau'}$,

$$\lambda_\chi(a)^{-\varepsilon'} = \left(\min_{\widehat{\chi} \in I} \widehat{\chi}(a)\right)^{-\varepsilon'} = \max_{\widehat{\chi} \in I} \widehat{\chi}(a)^{-\varepsilon'} \leqslant \sum_{\widehat{\chi} \in I} \widehat{\chi}(a)^{-\varepsilon'}$$

and it is enough to show that for every $\widehat{\chi} \in I$ the integral

$$\begin{aligned}\int_{\widetilde{A}_{\tau'}} \widehat{\chi}(a)^{-\varepsilon'} \rho(a)\, da &= \int_{\widetilde{\mathfrak{a}}_{\tau'}} \widehat{\chi}(\exp(Y))^{-\varepsilon'} \rho(\exp(Y))\, dY \\ &= \int_{\widetilde{\mathfrak{a}}_{\tau'}} e^{\langle \rho + \delta\chi, X \rangle}\, dX \\ &= \prod_{\beta \in \widetilde{\Delta}} \int_{-\infty}^{\log \tau'} e^{(-c_\beta(\widehat{\chi})\, \varepsilon' + n_\beta)\beta}\, d\beta\end{aligned}$$

converges. By choosing $\varepsilon' > 0$ sufficiently small, the implication (4) $\Rightarrow$ (5) follows.

Since (5) $\Rightarrow$ (1) is immediate, this concludes the proof of the Theorem. $\square$

3.4 L^2-integrability

Let $\mathbf{G}$, $\mathbf{T}$, $\mathbf{P}_0$, Δ, χ, $\mathbf{P}$, $\boldsymbol{e}_\chi$, Γ, $\mathbf{V}_\chi$, L, Γ_L and $\mathcal{L}_\chi$ be as in the beginning of Section 3.2. In this section, we study the L^2-integrability of the Siegel transform S_χ.

Let θ be the subset of Δ such that $\mathbf{P} = \mathbf{P}_\theta$ is the standard parabolic $\mathbb{Q}$-subgroup associated to θ (see Section 3.1.1). Moreover, by Lemma 3.2.2, we may without loss of generality assume that the action of Γ on $\mathcal{L}_\chi$ is transitive. Let W be the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$. We note that, for every $w \in W$, the vector $w\boldsymbol{e}_\chi$ is a highest $\mathbb{Q}$-weight vector with associated $\mathbb{Q}$-weight $w \cdot \chi$. By Lemma 3.2.2, we may furthermore assume that the orbit of $\boldsymbol{e}_\chi$ under W is contained in $\mathcal{L}_\chi$ (by letting $\mathbf{V}_\chi(\mathbb{Z})$ be spanned by a basis of $\mathbb{Q}$-weight vectors for the action of $\mathbf{T}$).

Theorem (Theorem 1.1.2)**.** *Suppose that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^2(\Omega)$. Then the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ is maximal and the unique simple root $\alpha \in \Delta$ with $\theta = \Delta \smallsetminus \{\alpha\}$ has at most one neighbor in the Dynkin diagram of $\mathbf{G}$ relative to $\mathbf{T}$ with respect to $\mathbf{P}_0$.*

The following lemma is the first step in the proof of Theorem 1.1.2.

Lemma 3.4.1. *Suppose that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^2(\Omega)$. Then, for every $f \in C_c(\widetilde{X})$, we have*

$$\int_{L/\Gamma_L} |S_\chi f| \, d\mu_{L/\Gamma_L} < +\infty.$$

For every measurable bounded compactly supported function $f : \widetilde{X} \to \mathbb{R}$ and $g \in G$, we let $g \cdot f : \widetilde{X} \to \mathbb{R}$ denote the function defined by

$$\forall \, \boldsymbol{v} \in V_\chi, \quad (g \cdot f)(\boldsymbol{v}) = f(g^{-1}\boldsymbol{v}).$$

Proof. Let $f \in C_c(\widetilde{X})$. Let $\rho \in C_c(\widetilde{X})$ be a non-negative K-invariant function such that $\rho(\boldsymbol{e}_\chi) > 0$ and for all $k \in K$, $y \in [1/2, 3/2]$ and $\boldsymbol{v} \in \widetilde{X}$, we have $|f(\boldsymbol{v})| < ((ka(y))^{-1} \cdot \rho)(\boldsymbol{v})$. By assumption, we have

$$\int_\Omega |S_\chi \rho|^2 \, d\mu_\Omega < +\infty.$$

On the other hand, using the identification $\widetilde{X}_0 = G/L = K/K_L \times A$ and the corresponding measure description of $\lambda_{\widetilde{X}}$ in Equation (3.3.8), by Weil's integration formula (see Theorem 2.3.3), there exists a constant $\omega > 0$ such that we have

$$\begin{aligned}
\int_\Omega |S_\chi \rho|^2 \, d\mu_\Omega &= \int_{G/\Gamma} \sum_{\gamma \in \Gamma/\Gamma_L} \big(\rho(g\gamma \boldsymbol{e}_\chi)(S_\chi \rho)(g)\big) \, d\mu_\Omega(g) \\
&= \int_{G/\Gamma_L} \rho(g\gamma \boldsymbol{e}_\chi)(S_\chi \rho)(g) \, d\mu_{G/\Gamma_L}(g) \\
&= \omega \int_{\widetilde{X}} \rho(ka(y)\boldsymbol{e}_\chi) \left(\int_{L/\Gamma_L} S_\chi \rho(ka(y)l) \, d\mu_{L/\Gamma_L}(l) \right) d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).
\end{aligned}$$

Since $\rho(\boldsymbol{e}_\chi) > 0$ and ρ is K-invariant, by continuity there exists $0 < \varepsilon < 1/2$ such that $\rho(ka(y)\boldsymbol{e}_\chi) > 0$ for every $k \in K$ and $y \in [1-\varepsilon, 1+\varepsilon]$. Since the above integral converges, for $\lambda_{\widetilde{X}}$-almost every $ka(y)\boldsymbol{e}_\chi$ with $k \in K$ and $y \in [1-\varepsilon, 1+\varepsilon]$, we have

$$\int_{L/\Gamma_L} S_\chi \rho(ka(y)l)\, d\mu_{L/\Gamma_L}(l) < +\infty.$$

Fix such an element $ka(y)\boldsymbol{e}_\chi$. By construction of ρ, for every $\boldsymbol{v} \in \widetilde{X}$, we have $|f(\boldsymbol{v})| < ((ka(y))^{-1} \cdot \rho)(\boldsymbol{v})$, and hence

$$\begin{aligned}\int_{L/\Gamma_L} |S_\chi f|\, d\mu_{L/\Gamma_L} &\leqslant \int_{L/\Gamma_L} S_\chi((ka(y))^{-1} \cdot \rho)\, d\mu_{L/\Gamma_L} \\ &= \int_{L/\Gamma_L} (S_\chi \rho)(ka(y)l)\, d\mu_{L/\Gamma_L}(l) < +\infty,\end{aligned}$$

as desired. □

Proof of Theorem 1.1.2. Suppose that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^2(\Omega)$. By Lemma 3.4.1, for every $f \in C_c(\widetilde{X})$, we have

$$\int_{L/\Gamma_L} |S_\chi f|\, d\mu_{L/\Gamma_L} < +\infty.$$

We write $\mathcal{L}_\chi$ as a disjoint union of Γ_L-orbits, that is, we pick I a countable subset of $\mathcal{L}_\chi$ such that $\mathcal{L}_\chi = \bigsqcup_{\boldsymbol{v} \in I} \Gamma_L \cdot \boldsymbol{v}$. Let $L_{\boldsymbol{v}}$ be the stabilizer in L of $\boldsymbol{v} \in I$, and let $\Gamma_{L_{\boldsymbol{v}}} = \Gamma \cap L_{\boldsymbol{v}}$. In particular, if $f \geqslant 0$, the integral above can be expressed as (after interchanging the sum and integral since $f \geqslant 0$),

$$\int_{L/\Gamma_L} S_\chi f\, d\mu_{L/\Gamma_L} = \sum_{\boldsymbol{v} \in I} \int_{L/\Gamma_L} \sum_{\gamma \in \Gamma_L/\Gamma_{L_{\boldsymbol{v}}}} f(l\gamma \boldsymbol{v})\, d\mu_{L/\Gamma_L}(l).$$

Hence, for every $\boldsymbol{v} \in I$, the map $\Lambda_{\boldsymbol{v}} : C_c(\widetilde{X}) \to \mathbb{R}$ defined by

$$\forall\, f \in C_c(\widetilde{X}), \quad \Lambda_{\boldsymbol{v}}(f) = \int_{L/\Gamma_L} \sum_{\gamma \in \Gamma_L/\Gamma_{L_{\boldsymbol{v}}}} f(l\gamma \boldsymbol{v})\, d\mu_{L/\Gamma_L}(l) < +\infty$$

is a positive L-invariant linear functional on $C_c(\widetilde{X})$, inducing a unique L-invariant Radon measure $\lambda_{\boldsymbol{v}}$ on $\widetilde{X}$, whose support is contained in the closure of the orbit $L \cdot \boldsymbol{v}$. Therefore $L \cdot \boldsymbol{v} \cong L/L_{\boldsymbol{v}}$ carries an L-invariant Radon measure and since L is unimodular, this implies that $L_{\boldsymbol{v}}$ is also unimodular.

Hence, applying Weil's integration formula as in Theorem 2.3.3, there exists an L-invariant Radon measure $\mu_{L/L_{\boldsymbol{v}}}$ on $L/L_{\boldsymbol{v}}$ normalized so that

$$\begin{aligned}\int_{L/\Gamma_L} S_\chi f \, d\mu_{L/\Gamma_L} &= \sum_{\boldsymbol{v}\in I} \int_{L/L_{\boldsymbol{v}}} \int_{L_{\boldsymbol{v}}/\Gamma_{L_{\boldsymbol{v}}}} f(ll'\boldsymbol{v}) \, d\mu_{L_{\boldsymbol{v}}/\Gamma_{L_{\boldsymbol{v}}}}(l') d\mu_{L/L_{\boldsymbol{v}}}(l) \\ &= \sum_{\boldsymbol{v}\in I} \mu_{L_{\boldsymbol{v}}/\Gamma_{L_{\boldsymbol{v}}}}(L_{\boldsymbol{v}}/\Gamma_{L_{\boldsymbol{v}}} \int_{L/L_{\boldsymbol{v}}} f(l\boldsymbol{v}) \, d\mu_{L/L_{\boldsymbol{v}}}(l).\end{aligned}$$

It follows that $\Gamma_{L_{\boldsymbol{v}}}$ is a lattice in $L_{\boldsymbol{v}}$ for every $\boldsymbol{v} \in I$. By a theorem of Borel and Harish-Chandra [BH62, Theorem 9.4], $\mathbf{L}_{\boldsymbol{v}}^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters. Using the Bruhat decomposition

$$\mathbf{G}(\mathbb{Q}) = \bigcup_{w\in W} \mathbf{P}(\mathbb{Q}) w \mathbf{P}(\mathbb{Q}),$$

for every $\boldsymbol{v} \in I$, there exist a scalar $c_{\boldsymbol{v}} > 0$, $w_{\boldsymbol{v}} \in W$, and $p_{\boldsymbol{v}} \in \mathbf{P}(\mathbb{Q})$ such that $\boldsymbol{v} = c_{\boldsymbol{v}} \, p_{\boldsymbol{v}} \, w_{\boldsymbol{v}} \, \boldsymbol{e}_\chi$. In particular, since $L_{\boldsymbol{v}} = \mathrm{Stab}_L(p_{\boldsymbol{v}} \, w_{\boldsymbol{v}} \, \boldsymbol{e}_\chi)$ and $L = \mathrm{Stab}_G(\boldsymbol{e}_\chi)$, we have

$$L_{\boldsymbol{v}} = p_{\boldsymbol{v}} w_{\boldsymbol{v}} L w_{\boldsymbol{v}}^{-1} p_{\boldsymbol{v}}^{-1} \cap L.$$

Moreover, as L is the kernel of the character χ of P, and hence is a normal subgroup of P, we may write $L_{\boldsymbol{v}} = p_{\boldsymbol{v}} \, (w_{\boldsymbol{v}} L w_{\boldsymbol{v}}^{-1} \cap L) \, p_{\boldsymbol{v}}^{-1}$. By assumption, the orbit of $\boldsymbol{e}_\chi$ under W is contained in $\mathcal{L}_\chi$. As conjugation by $p_{\boldsymbol{v}}$ defines a rational isomorphism from $w_{\boldsymbol{v}} L w_{\boldsymbol{v}}^{-1} \cap L$ onto $L_{\boldsymbol{v}}$ for all $\boldsymbol{v} \in I$, we must have that $(w\mathbf{L}w^{-1} \cap \mathbf{L})^\circ$ does not admit any non-trivial $\mathbb{Q}$-characters for all $w \in W$.

Recall that we denote by $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ the root system of $\mathbf{G}$ relative to $\mathbf{T}$, and by Δ the set of simple roots corresponding to the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. For a subset $\theta \subseteq \Delta$, we let $\mathbf{P}_\theta$ be the corresponding standard parabolic $\mathbb{Q}$-subgroup containing $\mathbf{P}_0$. Since the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^2(\Omega)$ and $L^2(\Omega)$ is contained in $L^1(\Omega)$, Theorem 1.1.1 implies that $\mathbf{P}$ is a maximal parabolic $\mathbb{Q}$-subgroup. Hence there exists a unique simple root $\alpha \in \Delta$, such that $\mathbf{P} = \mathbf{P}_{\Delta \smallsetminus \{\alpha\}}$. We wish to prove that the root α has at most one neighbor in the Dynkin diagram of Φ relative to $\mathbf{T}$ with respect to $\mathbf{P}_0$. Let us recall that two simple roots $\beta_1, \beta_2 \in \Delta$ with $\beta_1 \neq \beta_2$ are said to be neighbors if $\langle \beta_1, \beta_2 \rangle \neq 0$. Denote the set of neighbors of α by

$$V(\alpha) = \{\beta \in \Delta \smallsetminus \{\alpha\} : \alpha \text{ and } \beta \text{ are neighbors in the Dynkin diagram of } \Phi\},$$

and put $B(\alpha) = V(\alpha) \cup \{\alpha\}$. Let $w = s_\alpha \in W$ be the reflection about the hyperplane defined by the root α. For simplicity, we write $\mathbf{H}_\alpha = (w\mathbf{L}w^{-1} \cap \mathbf{L})^\circ$.

We claim that $\mathbf{H}_\alpha$ is contained in the standard parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_{\Delta \smallsetminus B(\alpha)}$. Observe that $\mathbf{H}_\alpha$ is normalized by $\mathbf{T}$. Therefore the Lie algebra $\mathfrak{h}_\alpha = \mathrm{Lie}(H_\alpha)$ can be written as a direct sum of a maximal toral subalgebra and corresponding root subspaces. Now, $\mathbf{H}_\alpha$ is contained in $\mathbf{P}_{\Delta \smallsetminus B(\alpha)}$ if the negative roots $\lambda = \sum_{\beta \in \Delta} n_\beta \beta$ arising in the adjoint representation of $\mathbf{T}$ on $\mathbf{H}_\alpha$ satisfy $n_\beta = 0$ for all $\beta \in B(\alpha)$. Since $\mathbf{H}_\alpha$ is invariant under $w = s_\alpha$, we know that $s_\alpha(\lambda)$ is still a root of $\mathbf{H}_\alpha$ and hence cannot contain in its support the root $-\alpha$ (since $\mathbf{H}_\alpha \subseteq \mathbf{L} \subseteq \mathbf{P}_{\Delta \smallsetminus \{\alpha\}}$). Applying the reflection $w = s_\alpha$ gives

$$s_\alpha(\lambda) = \sum_{\beta \in \Delta \smallsetminus \{\alpha\}} n_\beta s_\alpha(\beta) = \left(\sum_{\beta \in \Delta \smallsetminus \{\alpha\}} n_\beta \beta \right) - \left(\sum_{\beta \in V(\alpha)} \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} n_\beta \right) \alpha,$$

and hence $n_\beta = 0$ for all $\beta \in V(\alpha)$. Therefore, $\mathbf{H}_\alpha$ is contained in the standard parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_{\Delta \smallsetminus B(\alpha)}$, as required. Now, the $\mathbb{Q}$-rank of $\mathbf{H}_\alpha$ is at least $\mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) - 2$ (because $\mathbf{H}_\alpha$ contains $\mathbf{T} \cap \ker \chi \cap \ker(w \cdot \chi)$) and the $\mathbb{Z}$-rank of $X^*(\mathbf{P}_{\Delta \smallsetminus B(\alpha)})_{\mathbb{Q}}$ is $|B(\alpha)|$. If $|V(\alpha)| > 1$, and hence $|B(\alpha)| > 2$, it would follow that $\mathbf{H}_\alpha$ admits a non-trivial $\mathbb{Q}$-character. Thus we must have $|V(\alpha)| \leqslant 1$. $\square$

3.5 L^∞-integrability

Let $\mathbf{G}$, $\mathbf{T}$, $\mathbf{P}_0$, Δ, χ, $\mathbf{P}$, $\boldsymbol{e}_\chi$, Γ, V_χ and $\mathcal{L}_\chi$ be as in the beginning of Section 3.2. In this section, we study the L^∞-integrability of the Siegel transform S_χ. For the convenience of the reader, we recall the statement of Theorem 1.1.3.

Theorem (L^∞-integrability)**.** *The following assertions are equivalent.*

(1) The Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^\infty(\Omega)$.

(2) The $\mathbb{Q}$-rank of $\mathbf{G}$ is 1.

(3) The subgroup Γ_L is a cocompact *lattice in L.*

Proof of Theorem 1.1.3. We first show that (1) $\Leftrightarrow$ (2). Suppose that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1. In particular, the set of simple roots Δ consists of a single element, say α, and $\mathbf{P}$ is the standard parabolic $\mathbb{Q}$-subgroup associated to the empty set $\Delta \smallsetminus \{\alpha\} = \varnothing$, and hence equals the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. We need to show that S_χ maps $C_c(\widetilde{X})$ into $L^\infty(\Omega)$. Since any continuous compactly supported function f on $\widetilde{X}$ can be bounded by $\|f\|_\infty \mathbb{1}_{B_{\widetilde{X}}(r)}$ for some large radius $r > 0$ depending only on the support of f, it suffices to show that $S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)} \in L^\infty(\Omega)$. We note that in this case $T = \mathbf{T}(\mathbb{R})^\circ$ is one-dimensional and equals the subgroup A given as the identity component of $\mathbf{T}_{\Delta \smallsetminus \{\alpha\}}(\mathbb{R})$ (see Section 3.1.1). By Lemma 3.2.2, we may assume that Γ acts transitively on $\mathcal{L}_\chi$. Using the reduction theory for Γ in G, as detailed in Section 3.1.4, we have $G = \mathfrak{S}\, C\, \Gamma$, where $\mathfrak{S} = K A_\tau \boldsymbol{\omega}$ is a Siegel set and the set A_τ can be described as $A_\tau = \{a(y) \in A : \alpha(a(y)) = y^{-1} \leqslant e^\tau\}$ for some $\tau > 0$. Since Γ acts transitively on $\mathcal{L}_\chi$, it also acts transitively on the set of rational points of $\mathbf{X} = \mathbf{G}/\mathbf{P}$. Hence, by [Bor69, Proposition 15.6], we have $C = \{1\}$. This allows us to express g as $g = ka(y)n\gamma$, where $k \in K$, $a(y) \in A_\tau$, $n \in \boldsymbol{\omega}$, and $\gamma \in \Gamma$. Using the fact that $B_{\widetilde{X}}(r)$ is K-invariant and that $\bigcup_{a(y)\in A_\tau} a(y)\boldsymbol{\omega} a(y)^{-1}$ is relatively compact, there exists $R \geqslant r$, depending on r, the choice of the Siegel set $\mathfrak{S}$ and the finite set C, such that

$$S_\chi \mathbb{1}_{B_{\widetilde{X}}(r)}(g) = \#(B_{\widetilde{X}}(r) \cap g\mathcal{L}_\chi) \leqslant \#(B_{\widetilde{X}}(R) \cap a(y)\mathcal{L}_\chi).$$

Since the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1, the Weyl group $W \cong \mathbb{Z}/2\mathbb{Z}$ consists of 2 elements and we let $w \in N_G(\mathbf{T})(\mathbb{Q})$ be a representative of the non-trivial element in W. According to [BT65, Théorème 5.15], one has the Bruhat decomposition $G = P \sqcup PwP$. Since $w \cdot \chi = -\chi$, the element $a(y)$ acts on $\boldsymbol{e}_{-\chi} = w\boldsymbol{e}_\chi$ by

$$\begin{aligned} a(y)w\boldsymbol{e}_\chi = ww^{-1}a(y)w\boldsymbol{e}_\chi &= w\chi(w^{-1}a(y)w)\boldsymbol{e}_\chi \\ &= w(w\cdot\chi)(a(y))\boldsymbol{e}_\chi = y^{-\chi(Y_\alpha)}\boldsymbol{e}_{-\chi}, \end{aligned}$$

and since $\chi(Y_\alpha) < 0$ this action expands $\boldsymbol{e}_{-\chi}$ as $y \to +\infty$. We may assume that $\boldsymbol{e}_{-\chi}$ is in the basis $\mathcal{B}$ for V_χ. We first show that any $\boldsymbol{v} \in \mathcal{L}_\chi$ with $\boldsymbol{v} \neq \pm\boldsymbol{e}_\chi$ satisfies $\|a(y)\boldsymbol{v}\| \to +\infty$ as $y \to +\infty$. This follows if we can show that for any such $\boldsymbol{v}$ one has $|\langle \boldsymbol{v}, \boldsymbol{e}_{-\chi}\rangle| \geqslant 1$. Using the Bruhat decomposition $G = P \sqcup PwP$, there exist $p_1, p_2 \in P$ such that $\boldsymbol{v} = p_1wp_2\boldsymbol{e}_\chi$. Now, applying $a(y)$, one gets

$$a(y)\boldsymbol{v} = a(y)p_1a(y)^{-1}ww^{-1}a(y)w\chi(p_2)\boldsymbol{e}_\chi = a(y)p_1a(y)^{-1}wy^{-\chi(Y_\alpha)}\chi(p_2)\boldsymbol{e}_\chi.$$

Noting that $a(y)p_1a(y)^{-1}w$ converges to some element $p_\infty w$ with $p_\infty \in P$ as y tends to infinity, we deduce that $\|a(y)\boldsymbol{v}\|$ grows at the highest possible rate: $y^{-\chi(Y_\alpha)}$. Hence using the fact that $\langle \boldsymbol{v}, \boldsymbol{e}_{-\chi}\rangle \in \mathbb{Z}$, we must have $|\langle \boldsymbol{v}, \boldsymbol{e}_{-\chi}\rangle| \geqslant 1$.

Therefore, choosing y so large that $y^{-\chi(Y_\alpha)} > R$, we have for all $\boldsymbol{v} \in \mathcal{L}_\chi \smallsetminus \{\pm \boldsymbol{e}_\chi\}$,

$$\|a(y)\boldsymbol{v}\| \geqslant \|a(y)\boldsymbol{e}_{-\chi}\| = y^{-\chi(Y_\alpha)} > R.$$

Hence for all y large enough, $B_{\widetilde{X}}(R) \cap a(y)\mathcal{L}_\chi = \{\pm \boldsymbol{e}_\chi\}$. Since $f \in C_c(\widetilde{X})$ was arbitrary, this shows that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^\infty(\Omega)$.

Conversely, suppose that we have $\operatorname{rank}_{\mathbb{Q}} \mathbf{G} \geqslant 2$. Let $\mathfrak{a}$ be the Lie algebra of $\mathbf{T}(\mathbb{R})^\circ$. The negative Weyl chamber is defined by

$$\mathfrak{a}^- = \{X \in \mathfrak{a} : \forall \beta \in \Delta,\ \beta(X) < 0\}.$$

Fix a K-invariant non-negative function $\rho \in C_c(\widetilde{X})$ such that $\rho \geqslant \mathbb{1}_{B_{\widetilde{X}}(1)}$. To show that the Siegel transform S_χ does not map $C_c(\widetilde{X})$ into $L^\infty(\Omega)$, it suffices to show that $S_\chi\rho$ is unbounded on Ω. By continuity of ρ (and hence of $S_\chi\rho$), it further suffices to show that $S_\chi\rho$ is unbounded on $A^- = \exp \mathfrak{a}^-$. Roughly speaking, we would like to find $a \in A^-$ such that $a\mathcal{L}_\chi$ has many short vectors. Let $\mathbf{U}_0$ be the unipotent radical of the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$. Since the orbit of $\boldsymbol{e}_\chi$ under W is contained in $\mathcal{L}_\chi$, it suffices to find $w \in W$ and $Y \in \mathfrak{a}^-$ such that $\Gamma_{U_0} w^{-1}\boldsymbol{e}_\chi \subseteq \mathcal{L}_\chi$ is infinite and gets contracted by $\widehat{a}(y) = \exp(\log(y)Y) \in A^-$ as $y \to +\infty$. For any $Y \in \mathfrak{a}^-$, $u \in U_0$, $w \in W$, we have

$$\begin{aligned}\|\widehat{a}(y)uw^{-1}\boldsymbol{e}_\chi\| &= \|\widehat{a}(y)u\widehat{a}(y)^{-1}w^{-1}w\widehat{a}(y)w^{-1}\boldsymbol{e}_\chi\| \\ &\asymp_{u,w} \|\exp(\log(y)\operatorname{Ad}_w Y)\boldsymbol{e}_\chi\| = y^{(w^{-1}\cdot\chi)(Y)}\end{aligned}$$

Hence it is enough to pick $w \in W$ and $Y \in \mathfrak{a}^-$ such that

$$(w^{-1}\cdot\chi)(Y) = \langle \chi, \operatorname{Ad}_w Y\rangle < 0$$

and $\Gamma_{U_0} w^{-1}\boldsymbol{e}_\chi$ is infinite.

Fact: The group $\Gamma_{U_0} w^{-1}\boldsymbol{e}_\chi$ is infinite, unless w is the identity element.

Proof of fact. Indeed, the stabilizer of $w^{-1}\boldsymbol{e}_\chi$ in Γ_{U_0} is given by

$$\Gamma_{U_0} \cap w^{-1}\Gamma_{U_0}w = U_0 \cap w^{-1}U_0w \cap \Gamma \cap w^{-1}\Gamma w.$$

Since Γ and $w^{-1}\Gamma w$ are commensurable (indeed, conjugation by w defines a $\mathbb{Q}$-isomorphism from $\mathbf{G}$ to itself and the image of Γ under a $\mathbb{Q}$-isomorphism is again an arithmetic subgroup of G), the intersection $\Gamma \cap w^{-1}\Gamma w$ has finite index in Γ. Setting $U_{0,w} = U_0 \cap w^{-1}U_0 w$, it suffices to show that $\Gamma_{U_0}/\Gamma_{U_{0,w}}$ is infinite. Note that the Lie algebra of $U_{0,w}$ is the sum of all the root spaces associated to positive roots $\beta > 0$ such that $w \cdot \beta > 0$. Moreover, define $U^-_{w,0} = U_0 \cap w^{-1}U_0^- w$, where U_0^- is the unipotent subgroup opposite to U_0. Then its Lie algebra is the sum of all the root spaces associated to positive roots $\beta > 0$ such that $w\cdot\beta < 0$. Since the Weyl groups acts simply transitively on the set of bases of Φ, for each non-trivial w there exists a positive root $\beta > 0$ such that $w \cdot \beta < 0$. In particular, $U^-_{0,w}$ is a non-trivial subgroup of U_0 and the Lie algebra of U_0 is the direct sum of the Lie algebras of $U_{0,w}$ and $U^-_{0,w}$. In particular, the group $\Gamma_{U^-_{w,0}}$ is infinite (as it is a lattice in $U^-_{w,0}$) and the inclusion map followed by the quotient map

$$\Gamma_{U^-_{w,0}} \to \Gamma_{U_0} \to \Gamma_{U_0}/\Gamma_{U_{0,w}}$$

is injective. Thus for every non-trivial $w \in W$, the orbit $\Gamma_{U_0} w^{-1}\boldsymbol{e}_\chi$ is infinite, as required. $\square$

Let w_0 be the unique element in the Weyl group W that takes the positive Weyl chamber to the negative one. Our goal is to find $w \in W \setminus \{1, w_0\}$ and $Y \in \mathfrak{a}^-$ such that $\langle \chi, \mathrm{Ad}_w Y\rangle < 0$.

We first claim that there exists $\varepsilon > 0$ and Y_0 in the negative Weyl chamber $\mathfrak{a}^-$ such that the ε-ball $B_{\mathfrak{a}}(Y_0, \varepsilon)$ with center Y_0 is still contained in the negative Weyl chamber and for all $Y \in B_{\mathfrak{a}}(Y_0, \varepsilon)$,

$$\langle \chi, \mathrm{Ad}_{w_0} Y\rangle + \langle \chi, Y\rangle \geqslant 0. \tag{3.5.1}$$

To show this claim, we first suppose that w_0 acts on χ by $w_0 \cdot \chi = -\chi$. Then (3.5.1) holds for all $Y \in \mathfrak{a}^-$, and we are done. Now suppose that $w_0 \cdot \chi \neq -\chi$. Let Y_χ be the unique element in $\mathfrak{a}$ such that $\chi(\cdot) = \langle Y_\chi, \cdot\rangle$. Then we have $\|Y_\chi\| = \|\chi\|$ and $\chi(Y_\chi) = \|\chi\|^2$. Set $Y_0 = \mathrm{Ad}_{w_0^{-1}} Y_\chi$. Then

$$\langle \chi, \mathrm{Ad}_{w_0} Y_0\rangle = \langle \chi, Y_\chi\rangle = \|\chi\|^2,$$

and expressing $w_0 \cdot \chi = a\chi + b\chi^\perp$ as an orthogonal sum with $\chi^\perp$ in the orthogonal complement of $\mathbb{R}\chi$ and $a > -1$, we have

$$\langle \chi, Y_0\rangle = \langle w_0 \cdot \chi, Y_\chi\rangle = \langle a\chi + b\chi^\perp, Y_\chi\rangle = a\|\chi\|^2.$$

Hence $\langle \chi, \mathrm{Ad}_{w_0} Y_0 \rangle + \langle \chi, Y_0 \rangle = \|\chi\|^2 + a\|\chi\|^2 > 0$. If Y_0 lies in a wall of the negative Weyl chamber, we may replace Y_0 with a closeby element Y_0' in the interior of $\mathfrak{a}^-$ and still have that $\langle \chi, \mathrm{Ad}_{w_0} Y_0' \rangle + \langle \chi, Y_0' \rangle > 0$. The claim (3.5.1) follows now by continuity of the inner product.

Observe that for all $Y \in \mathfrak{a}$, we have $\sum_{w \in W} \mathrm{Ad}_w Y = 0$, since the only element of $\mathfrak{a}$ invariant under the Weyl group is 0. In particular, for all $Y \in B_{\mathfrak{a}}(Y_0, \varepsilon)$, using (3.5.1), one has

$$\sum_{w \in W \smallsetminus \{1, w_0\}} \langle \chi, \mathrm{Ad}_w Y \rangle \leqslant 0.$$

Since the $\mathbb{Q}$-rank of G is at least 2, the set $W \smallsetminus \{1, w_0\}$ is non-empty. Fix $Y \in B_{\mathfrak{a}}(Y_0, \varepsilon)$. If there is $w \in W \smallsetminus \{1, w_0\}$ with $\langle \chi, \mathrm{Ad}_w Y \rangle < 0$, then we are done. Otherwise, we have for all such w that $\langle \chi, \mathrm{Ad}_w Y \rangle = 0$. Since the ball $B_{\mathfrak{a}}(Y_0, \varepsilon)$ is not contained in a proper linear subspace of $\mathfrak{a}$, we can pick $Y' \in B_{\mathfrak{a}}(Y_0, \varepsilon)$ closeby such that $\langle \chi, \mathrm{Ad}_w Y' \rangle < 0$, as required.

Finally, we show that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1 if and only if Γ_L is a cocompact (or uniform) lattice in L. To this end, we recall the structure of the stabilizer $\mathbf{L}$ in $\mathbf{G}$ of $\boldsymbol{e}_\chi$, as described in the proof of Theorem 1.1.1. Let $\theta \subset \Delta$ be the proper subset of simple roots such that $\mathbf{P}$, the stabilizer in $\mathbf{G}$ of the line through $\boldsymbol{e}_\chi$, is the associated standard parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_\theta$. Then $\mathbf{L}$ is the semi-direct product of the reductive $\mathbb{Q}$-group $\widetilde{\mathbf{M}_\theta} = \mathbf{Q}_\theta \mathbf{M}_\theta (\mathbf{T}_\theta \cap \ker(\chi))$ and $\mathbf{U}_\theta$ (the unipotent radical of $\mathbf{P} = \mathbf{P}_\theta$), where $\mathbf{Q}_\theta$ is a connected $\mathbb{Q}$-subgroup of the largest $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $\mathcal{Z}(\mathbf{T})$, $\mathbf{M}_\theta$ is a connected semisimple $\mathbb{Q}$-subgroup and $\mathbf{T}_\theta = \left(\bigcap_{\beta \in \theta} \ker \beta \right)^\circ$.

Now, if the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1, then $\theta = \varnothing$ is the empty set and $\mathbf{P} = \mathbf{P}_\varnothing$ is the minimal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}_0$ and $\mathbf{U} = \mathbf{U}_\varnothing$, the unipotent radical of $\mathbf{P}$, is the unipotent radical $\mathbf{U}_0$ of $\mathbf{P}_0$. In particular, $\mathbf{T}_\varnothing$ is the maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus $\mathbf{T}$. By [Bor69, §11.7], the centralizer $\mathcal{Z}(\mathbf{T})$ is an almost direct product of $\mathbf{T}$ and the largest connected $\mathbb{Q}$-anisotropic subgroup $\mathbf{M}$ of $\mathcal{Z}(\mathbf{T})$. Since $\mathrm{char}(\mathbb{Q}) = 0$, we have $X^*(\mathbf{M})_{\mathbb{Q}} = \{1\}$ and $\mathbf{M}(\mathbb{Q})$ consists of semisimple elements (see just below the [Bor69, Definition 10.5]). Let $M = \mathbf{M}(\mathbb{R})$. By [Bor69, Theorem 8.4], $\Gamma_M = \Gamma \cap M$ is thus a cocompact lattice in M. Also, $\Gamma_U = \Gamma \cap U$ is a cocompact lattice in U. Let $\widetilde{K} = K \cap M$. Hence, we can find a compact fundamental set Ω_1 for Γ_M in M such that $\widetilde{K}\Omega_1 = \Omega_1$, and a compact fundamental set for Ω_2 for Γ_U in U (see [Bor69, Definition 9.6]). By [Bor69, Remark 9.9], the compact set $\Omega_1 \Omega_2$ is a fundamental set for $\Gamma_M \Gamma_U$

in L. Since by [BH62, Corollary 6.4] the group $\Gamma_M\Gamma_U$ has finite index in Γ_L, we deduce that Γ_L is a uniform lattice in L, as desired.

Conversely, suppose that Γ_L is a uniform lattice in L. Without loss of generality, we may assume that $\Gamma_L = \Gamma_{\widetilde{M}_\theta}\Gamma_{U_\theta}$. As follows from [Bor69, Theorem 8.7], $X^*(\mathbf{L}^\circ)_\mathbb{Q} = \{1\}$ and every unipotent element of $\mathbf{L}(\mathbb{Q})$ is contained in the unipotent radical of $\mathbf{L}$. Since $\mathbf{L} = \widetilde{\mathbf{M}}_\theta\mathbf{U}_\theta$ is the semi-direct product of a reductive $\mathbb{Q}$-subgroup and the connected unipotent $\mathbb{Q}$-subgroup $\mathbf{U}_\theta$, it follows from [Bor69, §7.15] that $R_u(\mathbf{L}) = \mathbf{U}_\theta$. In view of Theorem 1.1.1, since $X^*(\mathbf{L}^\circ)_\mathbb{Q} = \{1\}$, we must have that $\mathbf{P} = \mathbf{P}_\theta$ is maximal, and hence $\Delta \smallsetminus \theta = \{\alpha\}$ consists of a single element. To conclude, it is enough to show that $\theta = \varnothing$ is the empty set. Since $X^*(\mathbf{L}^\circ)_\mathbb{Q} = \{1\}$ and every unipotent element of $\mathbf{L}(\mathbb{Q})$ is contained in the unipotent radical of $\mathbf{L}$, we must have that $X^*(\widetilde{\mathbf{M}}_\theta^\circ)_\mathbb{Q} = \{1\}$ and every unipotent element of $\widetilde{\mathbf{M}}_\theta(\mathbb{Q})$ is trivial. This implies that $\widetilde{\mathbf{M}}_\theta$ does not contain any non-trivial $\mathbb{Q}$-split torus. In particular, the semisimple $\mathbb{Q}$-subgroup $\mathbf{M}_\theta$ of $\widetilde{\mathbf{M}}_\theta$ does not contain any non-trivial $\mathbb{Q}$-split torus. However, by [Bor69, §11.7], the number of elements in θ is equal to the $\mathbb{Q}$-rank of $\mathbf{M}_\theta$, and hence zero. Thus, we have shown that the $\mathbb{Q}$-rank of $\mathbf{G}$ is 1. The proof of Theorem 1.1.3 is complete. $\square$

Chapter 4

Diophantine approximation and counting lattice points

In this chapter and in the remainder of this thesis, we let $\mathbf{X} = \mathbf{G}/\mathbf{P}$ be a generalized flag variety defined over $\mathbb{Q}$, where $\mathbf{G}$ is a connected semisimple algebraic $\mathbb{Q}$-group and $\mathbf{P}$ is a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. For a definition of the $\mathbb{Q}$-rank of $\mathbf{P}$ we refer to §3.1.1. Let us recall that the $\mathbb{Q}$-*rank of* $\mathbf{X}$ is defined by

$$\operatorname{rank}_{\mathbb{Q}} \mathbf{X} = \operatorname{rank}_{\mathbb{Q}} \mathbf{G} - \operatorname{rank}_{\mathbb{Q}} \mathbf{P}.$$

We shall suppose that the unipotent radical of $\mathbf{P}$ is abelian. This implies that the parabolic subgroup $\mathbf{P}$ is maximal, or in other words, the $\mathbb{Q}$-rank of $\mathbf{X}$ is 1. Let χ be a highest $\mathbb{Q}$-weight of $\mathbf{G}$ (see §3.1.3). The purpose of this chapter is to equip the set of rational points of $X = \mathbf{X}(\mathbb{R})$ with a height H_χ associated to χ and to specify the distance $d(\cdot,\cdot)$ and probability measure σ_X that we use on X. We recall that, given $\tau \geqslant 0$, our goal will be to determine the asymptotic behaviour, as $T \to +\infty$ and for almost every $x \in X$ with respect to the Riemannian measure on X, of the counting function

$$\mathcal{N}_\tau(x,T) = \#\,\{v \in \mathbf{X}(\mathbb{Q}) : d(x,v) < H_\chi(v)^{-\tau},\ H_\chi(v) < T\}.$$

We then translate the problem of counting rational approximations of bounded height in X to the problem of counting primitive lattice points in a certain family of growing sets in the Euclidean space.

The notation introduced in this chapter will be used in the next two Chapters 5 and 6.

4.1 Height on $\mathbf{X}(\mathbb{Q})$

Roughly, a height on $\mathbf{X}(\mathbb{Q})$ is obtained by picking a certain rational representation $(\pi, \mathbf{V})$ of $\mathbf{G}$ such that $\mathbf{X}$ can be embedded into the projective space $\mathbb{P}(\mathbf{V})$ and by restricting to $\mathbf{X}$ a height on $\mathbb{P}(\mathbf{V})$.

We let $\mathbf{P}_0$ be a minimal parabolic $\mathbb{Q}$-subgroup contained in $\mathbf{P}$ and we let $\mathbf{T}$ be a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbf{G}$ contained in $\mathbf{P}_0$. Let Φ, Δ and $\{\lambda_\alpha\}_{\alpha\in\Delta}$ be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$, the set of simple roots for the ordering associated to $\mathbf{P}_0$, and the relative fundamental $\mathbb{Q}$-weights, respectively (see §3.1.2). By the maximality of $\mathbf{P}$, there exists a unique simple root $\alpha \in \Delta$ such that $\mathbf{P} = \mathbf{P}_{\Delta\backslash\{\alpha\}}$ is the standard parabolic $\mathbb{Q}$-subgroup associated to the subset $\Delta\backslash\{\alpha\}$ of Δ (see §3.1.1). As we will now show, we can, without loss of generality, assume that $\mathbf{G}$ is a connected simply-connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group. Write $\mathbf{G} = \mathbf{G}_1 \cdots \mathbf{G}_n$ as an almost direct product of connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-groups $\mathbf{G}_j$, for $1 \leqslant j \leqslant n$, and suppose that the root α belongs to the set of roots of $\mathbf{G}_1$ relative to the maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus $(\mathbf{G}_1 \cap \mathbf{T})^\circ$. Thus, we have $\mathbf{G}_j \subset \mathbf{P}$, for $2 \leqslant j \leqslant n$, and we may assume that $\mathbf{G}$ is almost $\mathbb{Q}$-simple. Next, the universal cover $\widetilde{\mathbf{G}}$ of $\mathbf{G}$ is a connected simply-connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group and the covering map $\widetilde{\pi} : \widetilde{\mathbf{G}} \to \mathbf{G}$ is a $\mathbb{Q}$-isogeny. Then $\widetilde{\mathbf{P}} = \pi^{-1}(\mathbf{P})$ is a maximal parabolic $\mathbb{Q}$-subgroup of $\widetilde{\mathbf{G}}$ and the implied morphism $\widetilde{\mathbf{G}}/\widetilde{\mathbf{P}} \to \mathbf{G}/\mathbf{P}$ is an isomorphism defined over $\mathbb{Q}$. Hence, from now, we assume that $\mathbf{G}$ is a connected simply-connected almost $\mathbb{Q}$-simple $\mathbb{Q}$-group.

Let $\chi \in X^*(\mathbf{P})_{\mathbb{Q}}$ be a non-zero element such that its restriction to $\mathbf{T}$ is the highest $\mathbb{Q}$-weight of $\mathbf{G}$ and let $\pi_\chi : \mathbf{G} \to \mathrm{GL}(\mathbf{V}_\chi)$ be the associated representation which is strongly rational over $\mathbb{Q}$ (see §3.1.3). In particular, there exists a unique $\mathbf{P}$-invariant line $\mathbf{D}$ in $\mathbf{V}_\chi$, which is defined over $\mathbb{Q}$ and on which $\mathbf{T}$ acts through the highest $\mathbb{Q}$-weight χ. For all $g \in \mathbf{G}$ and $\boldsymbol{v} \in \mathbf{V}_\chi$, we abbreviate $\pi_\chi(g)\boldsymbol{v}$ as $g\boldsymbol{v}$. Fix a non-zero vector $\boldsymbol{e}_\chi \in \mathbf{D}(\mathbb{Q})$ and write $x_0 = [\boldsymbol{e}_\chi]$ for the corresponding point in the projective space $\mathbb{P}(\mathbf{V}_\chi)$. Then, for every $p \in \mathbf{P}$, we have $p\boldsymbol{e}_\chi = \chi(p)\boldsymbol{e}_\chi$ and

$$\mathbf{P} = \{g \in \mathbf{G} : gx_0 = x_0\}.$$

We identify the orbit $\mathbf{G} \cdot x_0$ with $\mathbf{X} = \mathbf{G}/\mathbf{P}$ via the map $\iota_\chi(g\mathbf{P}) = gx_0$. Let K be a maximal compact subgroup of $G = \mathbf{G}(\mathbb{R})$, equipped with the unique K-invariant Haar probability measure μ_K. By the Iwasawa decomposition $G = KP$, the group K acts transitively on the set of real points $X = \mathbf{X}(\mathbb{R})$

of the flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$ via the map $k \mapsto kx_0$. We equip X with the pushforward of μ_K along this map; this is the unique K-invariant probability measure on X. Fix a rational basis $(\boldsymbol{v}_i)_{i\in I}$ of $V_\chi = \mathbf{V}_\chi(\mathbb{R})$ and a Euclidean inner product $\langle\cdot,\cdot\rangle$ on V_χ for which the action of K is unitary, the one of T is self-adjoint, and such that the basis $(\boldsymbol{v}_i)$ is orthonormal. We denote the implied norm on V_χ by $\|\cdot\|$. We assume that $\boldsymbol{v}_1 = \boldsymbol{e}_\chi$. This gives us a height function H on $\mathbb{P}(\mathbf{V}_\chi)(\mathbb{Q})$ by $H([\boldsymbol{v}]) = \|\boldsymbol{v}\|$, where $\boldsymbol{v}$ is any primitive vector in the lattice $\mathbf{V}_\chi(\mathbb{Z}) := \oplus_{i\in I}\mathbb{Z}\boldsymbol{v_i}$ representing $[\boldsymbol{v}]$. This height function is well defined, since a primitive vector in $\mathbf{V}_\chi(\mathbb{Z})$ representing $[\boldsymbol{v}]$ is uniquely determined up to multiplication by ± 1. Using the embedding ι_χ, one then obtains a height function H_χ on $\mathbf{X}(\mathbb{Q})$, which is given by

$$\forall\, v \in \mathbf{X}(\mathbb{Q}), \quad H_\chi(v) = H(\iota_\chi(v)).$$

4.2 Distance on X

Here we equip the flag variety X with a Riemannian distance that is compatible with the Euclidean structure of V_χ. Let $\mathbb{S} = \{\boldsymbol{x} \in V_\chi : \|\boldsymbol{x}\| = 1\}$ be the unit sphere in V_χ, viewed as a Riemannian submanifold of V_χ. The K-equivariant projection map $\mathbb{S} \to \mathbb{P}(V_\chi)$, $\boldsymbol{v} \mapsto [\boldsymbol{v}]$, being a smooth local diffeomorphism, induces a K-invariant Riemannian metric on $\mathbb{P}(V_\chi)$, and by restriction also on X. We observe that the associated Riemannian measure equals $\mathrm{vol}(X)\,\sigma_X$, where $\mathrm{vol}(X)$ denotes the total Riemannian volume of X. We denote the induced K-invariant Riemannian distance on X by $d(\cdot,\cdot)$.

Let us denote by $\mathfrak{g}$ the Lie algebra of G. For every $\beta \in \Phi$, let $\mathfrak{g}_\beta = \{Y \in \mathfrak{g} : \forall\, t \in T,\ \mathrm{Ad}(t)Y = \beta(t)Y\}$ and let $\mathfrak{z} = \{Y \in \mathfrak{g} : \forall\, t \in T,\ \mathrm{Ad}(t)Y = Y\}$. The Lie algebra of G has a root space decomposition with respect to the action of T:

$$\mathfrak{g} = \mathfrak{z} \oplus \left(\bigoplus_{\beta\in\Phi} \mathfrak{g}_\beta\right).$$

Let Φ^+ be the set of positive roots and, as before, let $\alpha \in \Delta$ be the unique root such that $\mathbf{P} = \mathbf{P}_{\Delta\setminus\{\alpha\}}$ is the standard parabolic $\mathbb{Q}$-subgroup associated with the subset $\Delta\setminus\{\alpha\}$. Let $\langle\Delta\setminus\{\alpha\}\rangle^-$ be the set of all roots that can be expressed as a negative integer linear combination of elements in $\Delta\setminus\{\alpha\}$. Then the Lie algebras of P and of the unipotent subgroup U^- opposite to P

are, respectively,

$$\mathfrak{p} = \mathfrak{z} \oplus \left(\bigoplus_{\beta \in \Phi^+ \cup \langle \Delta \setminus \{\alpha\} \rangle^-} \mathfrak{g}_\beta \right) \quad \text{and} \quad \mathfrak{u}^- = \bigoplus_{\beta \in \Phi,\, -\alpha \prec \beta} \mathfrak{g}_\beta,$$

where $-\alpha \prec \beta$ means that $-\alpha$ occurs in the support of β. Observe that $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{p}$. Let $\phi : G \to X$ be the projection map $\phi(g) = gx_0$ and let $D_1\phi : \mathfrak{g} \to T_{x_0}X$ be its derivative at the identity $1 \in G$. Observe that $\ker D_1\phi = \mathfrak{p}$. In particular, if we denote the restriction of this projection map from G to U^- still by ϕ, then $D_1\phi : \mathfrak{u}^- \to T_{x_0}X$ is a linear isomorphism. We equip $\mathfrak{u}^-$ with a Euclidean structure for which this isomorphism is an isometry and we denote the implied norm on $\mathfrak{u}^-$ by $\| \cdot \|_{\mathfrak{u}^-}$. It will be convenient for us to relate the distance on X to the one on the Lie algebra $\mathfrak{u}^-$.

Lemma 4.2.1. *For every $u \in \mathfrak{u}^-$, we have*

$$d(x_0, \exp(u)x_0) = \|u\|_{\mathfrak{u}^-} + O\big(\|u\|^2_{\mathfrak{u}^-}\big), \tag{4.2.1}$$

where $\exp : \mathfrak{u}^- \to U^-$ *is the exponential map.*

Proof. It suffices to prove (4.2.1) for all $u \in \mathfrak{u}^-$ sufficiently close to the origin. Let $\| \cdot \|_{x_0}$ be the implied norm on $T_{x_0}X$. For all $Y \in T_{x_0}X$ close to the origin, the Riemannian exponential map $\exp_{x_0} : T_{x_0}X \to X$ at x_0 satisfies: $d(x_0, \exp_{x_0}(Y)) = \|Y\|_{x_0}$. Using the triangle inequality and the fact that the derivative $D_1\phi$ is an isometry yields, for every $u \in \mathfrak{u}^-$ close to the origin,

$$d(x_0, \exp(u)x_0) = \|u\|_{\mathfrak{u}^-} + O\left(d(\exp_{x_0}(D_1\phi(u)), \exp(u)x_0)\right).$$

Moreover, since $\exp_{x_0}$ is locally bi-Lipschitz, it suffices to show that the map $\Phi = \exp_{x_0}^{-1} \circ \phi \circ \exp$, defined on a neighborhood of the origin in $\mathfrak{u}^-$, satisfies:

$$\Phi(u) = D_1\phi(u) + O\big(\|u\|^2_{\mathfrak{u}^-}\big).$$

Since $\Phi(0) = 0$, by Taylor's theorem,

$$\Phi(u) = D_0\Phi(u) + O\big(\|u\|^2_{\mathfrak{u}^-}\big).$$

By the chain rule, $D_0\Phi = D_{x_0} \exp_{x_0}^{-1} \circ D_1\phi \circ D_0 \exp = D_1\phi$, as required. $\square$

4.3 Diophantine exponent

As in [Sax20, Définition 2.4.1], we define the *Diophantine exponent* $\beta_\chi(x)$ *relative to* χ of a point $x \in X = \mathbf{X}(\mathbb{R})$ by

$$\beta_\chi(x) = \inf\{\beta > 0 : \exists c > 0,\ \forall v \in \mathbf{X}(\mathbb{Q}),\ d(x,v) > cH_\chi(v)^{-\beta}\}.$$

By [Sax20, Theorem 2.4.5], this Diophantine exponent is constant almost everywhere on X: there exists a positive rational number $\beta_\chi = \beta_\chi(X) > 0$ such that for σ_X-almost every $x \in X$, we have $\beta_\chi(x) = \beta_\chi$. We refer to β_χ as the *Diophantine exponent of* X *relative to* χ. In particular, the inequality

$$d(x,v) \leqslant H(v)^{-\tau}$$

has infinitely (at most finitely) many solutions $v \in \mathbf{X}(\mathbb{Q})$, if $\tau \leqslant \beta_\chi$ ($\tau > \beta_\chi$).

4.4 Cone over X, coordinates and measures

By abuse of notation, we shall refer to $\widetilde{X}$, the closure of the homogeneous space $\widetilde{X}_0 = G \cdot \boldsymbol{e}_\chi$ in V_χ, which is contained in the punctured affine cone over X, as the *cone over* X. Let $\mathcal{L}_\chi$ be the set of primitive elements in $\mathbf{V}_\chi(\mathbb{Z})$ that are contained in $\widetilde{X}$. Let Γ be the stabilizer in G of $\mathbf{V}_\chi(\mathbb{Z})$; this is an arithmetic subgroup of G that stabilizes $\mathcal{L}_\chi$. We normalize the Haar measure μ_G on G so that the induced G-invariant measure on the quotient Ω, which we denote by μ_Ω, is a probability measure.

We briefly recall a few facts from the structure theory of parabolic subgroups and refer the reader to §3.1.1 for details. By the maximality of $\mathbf{P}$, there exists a unique simple root $\alpha \in \Delta$ such that $\mathbf{P} = \mathbf{P}_{\Delta\setminus\{\alpha\}}$ is the standard parabolic $\mathbb{Q}$-subgroup associated with the subset $\Delta\setminus\{\alpha\}$ of Δ. Define the one-dimensional $\mathbb{Q}$-split $\mathbb{Q}$-torus $\mathbf{T}_{\Delta\setminus\{\alpha\}} = \left(\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \ker\beta\right)^\circ$ and let $\mathcal{Z}(\mathbf{T}_{\Delta\setminus\{\alpha\}})$ be its centralizer in $\mathbf{G}$. The parabolic $\mathbf{P}$ is the semidirect product $\mathbf{P} = \mathcal{Z}(\mathbf{T}_{\Delta\setminus\{\alpha\}}) \ltimes \mathbf{U}$, where $\mathbf{U}$ is its unipotent radical. Let $\mathbf{M}$ be the identity component of the intersection of the kernels of the $\mathbb{Q}$-characters of $\mathcal{Z}(\mathbf{T}_{\Delta\setminus\{\alpha\}})$. By [Bor69, Proposition 10.7, (b)], we have $X^*(\mathbf{M})_\mathbb{Q} = \{1\}$ and $\mathcal{Z}(\mathbf{T}_{\Delta\setminus\{\alpha\}})$ is an almost direct product of $\mathbf{M}$ and $\mathbf{T}_{\Delta\setminus\{\alpha\}}$. Hence $\mathbf{P}$ is an almost direct product of $\mathbf{M}$, $\mathbf{T}_{\Delta\setminus\{\alpha\}}$ and $\mathbf{U}$:

$$\mathbf{P} = \mathbf{M}\,\mathbf{T}_{\Delta\setminus\{\alpha\}}\,\mathbf{U}. \tag{4.4.1}$$

Denote by $\mathbf{L}$ the stabilizer in $\mathbf{G}$ of the vector $\boldsymbol{e}_\chi$ and let $L = \mathbf{L}(\mathbb{R})$. We note that $\mathbf{T}_{\Delta\setminus\{\alpha\}} \cap \mathbf{L}$ is a finite group and we define $\widetilde{\mathbf{M}} = \mathbf{M}\,(\mathbf{T}_{\Delta\setminus\{\alpha\}} \cap \mathbf{L})$.

We claim that $\mathbf{L} = \widetilde{\mathbf{M}} \ltimes \mathbf{U}$. Indeed, the inclusion $\supseteq$ holds because every element of $\widetilde{\mathbf{M}} \ltimes \mathbf{U}$ fixes the vector $\boldsymbol{e}_\chi$. On the other hand, using the decomposition (4.4.1) of $\mathbf{P}$, for every $l \in \mathbf{L}$, we have $l = m\,t\,u$ for some (not necessarily unique) $m \in \mathbf{M}$, $t \in \mathbf{T}_{\Delta\setminus\{\alpha\}}$ and $u \in \mathbf{U}$. We need to show that $\chi(t) = 1$. But since χ is a character of $\mathbf{P}$, this yields

$$1 = \chi(l) = \chi(m\,t\,u) = \chi(t).$$

We let $\widetilde{M} = \widetilde{\mathbf{M}}(\mathbb{R})$ and $A = \mathbf{T}_{\Delta\setminus\{\alpha\}}(\mathbb{R})^\circ$. Note that L is the semi-direct product $L = \widetilde{M} \ltimes U$ and P is the direct product $P = \widetilde{M}AU$. We identify the cone $\widetilde{X}$ with the homogeneous space G/L via the map $gL \mapsto g\boldsymbol{e}_\chi$. By Theorem 1.1.1, since $\mathbf{P}$ is assumed to be maximal, the Lie group L is unimodular and $\Gamma_L = \Gamma \cap L$ is a lattice in L. Let μ_L be the Haar measure on L normalized so that the induced L-invariant measure μ_{L/Γ_L} is a probability measure. In particular, by [Fol15, Theorem 2.51], there exists a unique G-invariant Radon measure $\lambda_{\widetilde{X}} = \mu_{G/L}$ on $\widetilde{X} = G/L$ such that,

$$\forall\, f \in C_c(G), \quad \int_G f(g)\, d\mu_G(g) = \int_{G/L}\int_L f(g\boldsymbol{v})\, d\mu_L(l)\, d\mu_{G/L}(gL). \tag{4.4.2}$$

We shall also use the following parametrization of A. Write $\mathfrak{t}$ for the Lie algebra of $\mathbf{T}(\mathbb{R})$ and $Y_\alpha \in \mathfrak{t}$ for the unique element satisfying

$$\alpha(Y_\alpha) = -1 \quad \text{and} \quad \beta(Y_\alpha) = 0 \quad \text{for all } \beta \in \Delta\setminus\{\alpha\}. \tag{4.4.3}$$

We then have $A = \{a(y) = \exp(\log(y)Y_\alpha) : y > 0\}$. We recall that β_χ denotes the Diophantine exponent of X relative to χ (see §4.3). By [Sax20, Théorème 2.4.5], we have the relation

$$\chi(Y_\alpha) = -\frac{1}{\beta_\chi} \tag{4.4.4}$$

Hence, by Equation (3.3.6), the action of $a(y)$ on $\boldsymbol{e}_\chi$ is given by

$$a(y)\boldsymbol{e}_\chi = y^{-\frac{1}{\beta_\chi}}\boldsymbol{e}_\chi. \tag{4.4.5}$$

Let $\rho : \mathfrak{t} \to \mathbb{R}$ be the sum of all the positive roots with multiplicities counted, that is, $\rho = \sum_{\beta \in \Phi^+} \beta$, where Φ^+ is the set of positive roots of $\mathbf{G}$ relative to $\mathbf{T}$ with respect to $\mathbf{P}_0$. Let Φ^+_α be the set of elements in Φ^+ whose support contains the simple root α. Let $d = \dim X$ be the dimension of X as a real manifold. We claim that $\rho(Y_\alpha) = -d$. Indeed, by the definition of Y_α, we have

$$\rho(Y_\alpha) = \sum_{\beta \in \Phi^+_\alpha} \beta(Y_\alpha).$$

Together with equation (4.4.4) applied to $\chi = \chi_{\mathrm{ac}}$, we have

$$\rho(Y_\alpha) = \sum_{\beta \in \Phi^+_\alpha} \beta(Y_\alpha) = \chi_{\mathrm{ac}}(Y_\alpha) = -\frac{1}{\beta_{\chi_{\mathrm{ac}}}} = -d,$$

as required. In particular, we observe that $\rho(a(y)) = y^{-d}$.

The Haar measure we consider on A is the push-forward of the Haar measure $y^{-1}dy$ on $\mathbb{R}^\times_+$ via the morphism $y \mapsto a(y)$. Let $K_L = K \cap L$ and let σ be the pushforward of the Haar probability measure μ_K on K to K_L via the map $k \mapsto k\,K_L$. By [Bor69, §11.19], we have $G = KP = KP^\circ = KM^\circ AU$ and the product map defines a homeomorphism of $(KM^\circ) \times A \times U$ onto G. Therefore, every $g \in G$ can be written as $g = kma(y)u$ with $u \in U$, $y \in \mathbb{R}^\times_+$, $m \in M^\circ$, and $k \in K$. Let $\Gamma_{\widetilde{M}} = \Gamma \cap \widetilde{M}$ and let $d\mu_{\widetilde{M}}$ be a Haar measure on $\widetilde{M}$, normalized so that the induced $\widetilde{M}$-invariant measure on $\widetilde{M}/\Gamma_{\widetilde{M}}$ is a probability measure. Similarly, we let $\Gamma_U = \Gamma \cap U$ and we let $d\mu_U$ be a Haar measure on U, normalized so that the induced U-invariant measure on U/Γ_U is a probability measure. By [Kna02, Proposition], in these coordinates, there exists a normalizing constant $\omega_0 > 0$ such that the Haar measure μ_G of G is given by

$$d\mu_G(g) = \omega_0\, \rho(a(y))\, d\mu_K(k)\, d\mu_{\widetilde{M}}(m)\, \frac{dy}{y}\, d\mu_U(u). \tag{4.4.6}$$

We note that $d\mu_L = d\mu_{\widetilde{M}}\, d\mu_U$ is a Haar measure on L such that the induced L-invariant Radon measure on L/Γ_L is a probability measure. Moreover, since the product map defines a homeomorphism of $(KM^\circ) \times A \times U$ onto G, the map of $(K/K_L) \times A$ to $\widetilde{X} = G/L$ given by $(k\,K_L, a(y)) \mapsto ka(y)\boldsymbol{e}_\chi$ is a homeomorphism. We will now show that, in these coordinates, there exists a constant $\omega_1 > 0$ such that the measure $\lambda_{\widetilde{X}}$ is given by

$$d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi) = \omega_1\, y^{-(d+1)}\, d\sigma(k)\, dy. \tag{4.4.7}$$

By the uniqueness (up to a scalar) of the G-invariant Radon measure on $\widetilde{X} = G/L$ (see, for instance, [Fol15, Theorem 2.51]), it suffices to show the G-invariance of the measure on the right hand side in (4.4.7). So fix $g_0 \in G$ and $f \in C_c(\widetilde{X})$. By [Fol15, Proposition 2.50], there exists $\phi \in C_c(G)$ such that, for every $g \in G$,

$$f(g\boldsymbol{e}_\chi) = \int_L \phi(gl)\, d\mu_L(l).$$

Hence, using the fact that $\rho(a(y)) = y^{-d}$, that σ is the pushforward from K to K/K_L of μ_K, the measure description in (4.4.6) of μ_G and its invariance, we have

$$\begin{aligned}
\int_{\tilde{X}} f(g_0ka(y)\boldsymbol{e}_\chi)\, y^{-(d+1)}\, dy\, d\sigma(k) &= \int_{\tilde{X}} \int_L \phi(g_0ka(y)l)\, d\mu_L(l)\, y^{-(d+1)}\, dy\, d\sigma(k) \\
&= \int_{\tilde{X}} \int_L \phi(g_0ka(y)l)\, d\mu_L(l)\, y^{-(d+1)}\, dy\, d\sigma(k) \\
&= \int_K \int_A \int_{\widetilde{M}} \int_U \phi(g_0ka(y)mu)\, \rho(a(y)) d\mu_U(u)\, d\mu_{\widetilde{M}}(m)\, \frac{dy}{y}\, d\mu_K(k) \\
&= \omega_0^{-1} \int_G \phi(g_0g)\, d\mu_G(g) = \omega_0^{-1} \int_G \phi(g)\, d\mu_G(g).
\end{aligned}$$

By the same argument, we also have

$$\int_{\tilde{X}} f(ka(y)\boldsymbol{e}_\chi)\, y^{-(d+1)}\, dy\, d\sigma(k) = \omega_0^{-1} \int_G \phi(g)\, d\mu_G(g).$$

This shows that

$$\int_{\tilde{X}} f(g_0ka(y)\boldsymbol{e}_\chi)\, y^{-(d+1)}\, dy\, d\sigma(k) = \int_{\tilde{X}} f(ka(y)\boldsymbol{e}_\chi)\, y^{-(d+1)}\, dy\, d\sigma(k)$$

and, since $g_0 \in G$ was arbitrary, the measure on the right-hand side in (4.4.7) is G-invariant, as required.

4.5 Lattice reduction with respect to M

In the proof of Theorem 1.3.1, we will need the effective equidistribution of translates of certain L-orbits. We note that $\mathbf{L}^\circ$ is the semi-direct product of the connected reductive $\mathbb{Q}$-group $\mathbf{M}$ and the unipotent radical $\mathbf{U}$ of $\mathbf{P}$. We will deduce the required equidistribution result (see Lemma 6.4.1) by using the effective equidistribution of expanding translates of (horospherical) U-orbits (see [KM96, Proposition 2.4.8.]) together with lattice reduction with respect to M. The latter will be the content of this section. By [BH62, Theorem 9.4], since $\mathbf{M}$ does not admit any non-trivial $\mathbb{Q}$-characters, we have that $\Gamma_M = \Gamma \cap M$ is a lattice in M. We note that $K_M = K \cap M$ is a maximal compact subgroup of M, $\mathbf{P}_0 \cap \mathbf{M}$ is a minimal parabolic $\mathbb{Q}$-subgroup of $\mathbf{M}$ and $\mathbf{S} = (\mathbf{T} \cap \mathbf{M})^\circ$ is a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus of $\mathbf{M}$ (see [Bor69, Section 11.7]). Then $[\Delta \smallsetminus \{\alpha\}]$, the set of roots that are linear combinations of elements of $\Delta \smallsetminus \{\alpha\}$, is the root system of $\mathbf{M}$ relative to $\mathbf{S}$. Let $\mathbf{U}_0$ denote the unipotent radical of $\mathbf{P}_0 \cap \mathbf{M}$ and let $\mathbf{M}_0$ be the maximal connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of the centralizer $\mathscr{Z}(\mathbf{S})$ of $\mathbf{S}$ in $\mathbf{M}$. We let $\mathfrak{s}$ denote the Lie algebra of $S = \mathbf{S}(\mathbb{R})$. For every $t > 0$, let

$$\mathfrak{s}_t = \{X \in \mathfrak{s} : \forall \beta \in \Delta \smallsetminus \{\alpha\},\, \beta(X) \leqslant t\}$$

The negative Weyl chamber of $\mathfrak{s}$ is given by

$$\mathfrak{s}^- = \{X \in \mathfrak{s} : \forall \beta \in \Delta \smallsetminus \{\alpha\},\, \beta(X) < 0\}.$$

We set $S^- = \exp(\mathfrak{s}^-)$ and note that there exists a compact neighborhood Q of the identity in S such that $\exp(\mathfrak{s}_t) \subset Q\, S^-$. Denote by M_0 and U_0 the groups of real points of $\mathbf{M}_0$ and $\mathbf{U}_0$, respectively. By [Bor69, Theorem 13.1], there exists $t > 0$, a compact neighborhood $\boldsymbol{\omega}$ of the identity in $M_0 U_0$ and a finite set $C \subset \mathbf{M}(\mathbb{Q})$, such that the Siegel set $\mathfrak{S} = \mathfrak{S}_{\tau,\boldsymbol{\omega}} = K_M \exp(\mathfrak{s}_t)\, \boldsymbol{\omega}$ satisfies $M = \mathfrak{S}\, C\, \Gamma_M$. For every $\delta \in (0,1)$, we define

$$\mathfrak{S}(\delta) = \{m \in \mathfrak{S} : \lambda_1(\mathrm{Ad}(m)_\Gamma) < \delta\}, \tag{4.5.1}$$

and we denote by $\mathfrak{S}(\delta)^c$ the complementary subset of $\mathfrak{S}(\delta)$ in $\mathfrak{S}$. Let μ_M be the Haar measure on M. There exists $c_1 > 0$ such that, for every $\delta \in (0,1)$, we have

$$\mu_M(\mathfrak{S}(\delta)) \lesssim \delta^{c_1}. \tag{4.5.2}$$

In fact, by [Bor69, Lemma 12.2], the union $\cup_{s \in \exp(\mathfrak{s}_t)} s \boldsymbol{\omega} s^{-1}$ is relatively compact, and, for all $k \in K_M\, Q$, $s \in S^-$, $n \in \boldsymbol{\omega}$, $c \in C$, and $\gamma \in \Gamma_M$, we have

$$\lambda_1(\mathrm{Ad}(ksnc\gamma)_\Gamma) \asymp \lambda_1(\mathrm{Ad}(s)_\Gamma). \tag{4.5.3}$$

Moreover, since S is contained in the torus T and the roots Φ are the non-trivial $\mathbb{Q}$-weights in the adjoint representation of G,

$$\forall\, s \in S, \quad \lambda_1(\mathrm{Ad}(s)_\Gamma) \asymp \min_{\beta \in \Phi} \beta(s). \tag{4.5.4}$$

Choose a character $\chi' \in X^*(\mathbf{S})$ such that

$$\forall\, s \in S^-, \quad \chi'(s) \leqslant \min_{\beta\in\Phi} \beta(s).$$

Let $\rho' : \mathfrak{s} \to \mathbb{R}_+^\times$ be the sum of all the positive S-roots with multiplicities counted. By [Kna02, Theorem 8.32], we can decompose the measure μ_M on M according to the decomposition $M = K_M\, M_0\, S\, U_0$ and deduce that, for some constant $C' \geqslant 1$,

$$\begin{aligned} \mu_M(\mathfrak{S}(\delta)) &\lesssim \int_{S^-} \mathbb{1}_{\{\chi'(s)\leqslant C'\,\delta\}} \rho'(s)\, \mu_S(s) \\ &= \int_{\mathfrak{s}^-} \mathbb{1}_{\{\chi'(s)\leqslant \ln(C'\,\delta)\}}\, e^{\rho'(s)}\, s. \end{aligned}$$

Note that the $\mathbb{Q}$-rank of $\mathbf{M}$ is $r = \mathrm{rank}_{\mathbb{Q}}(\mathbf{M}) = \mathrm{rank}_{\mathbb{Q}}(\mathbf{G}) - 1$. Let $(Y_i)_{1\leqslant i\leqslant r}$ be the dual basis of the set of simple roots of $\mathbf{M}$ relative to $\mathbf{S}$. This last integral is the integral of an exponential function over a convex polytope in $\mathfrak{s}$, and is therefore comparable to the maximum of the function on this set, multiplied by a factor corresponding to the dimension of the face on which this maximum is attained. This maximum is attained at one of the vertices of the convex set, which are the points $A_i = \frac{\ln(C'\,\delta)}{\chi'(Y_i)} Y_i$ with $i = 1, \dots, r$. Hence, for all $\delta \in (0,1)$ sufficiently small, we have

$$\int_{\mathfrak{s}^-} \mathbb{1}_{\{\chi'(s)\leqslant \ln(C'\,\delta)\}}\, \rho'(s)\, s \asymp (C'\delta)^a |\ln(C'\delta)|^{b-1} \lesssim \delta^{a/2}$$

where

$$a = \min_{1\leqslant i\leqslant r} \frac{\rho'(Y_i)}{\chi'(Y_i)} \quad \text{and} \quad b = \#\{1 \leqslant i \leqslant r : \frac{\rho'(Y_i)}{\chi'(Y_i)} = a\}.$$

This proves the claim in (4.5.2) with $c_1 = a/2$. Moreover, for every $m \in \mathfrak{S}(\delta)^c$, we have

$$\|\operatorname{Ad}(m)\| \lesssim \delta^{-1}. \tag{4.5.5}$$

Indeed, by (4.5.3) and (4.5.4), for all $m = ksnc\gamma$ with $k \in K_M\, Q$, $s \in S^-$, $n \in \boldsymbol{\omega}$, $c \in C$, and $\gamma \in \Gamma_M$, we have

$$\begin{aligned} \|\operatorname{Ad}(m)\| \asymp \|\operatorname{Ad}(s)\| \asymp \max_{\beta\in\Phi} \beta(s) &= \left(\min_{\beta\in\Phi} \beta(s)\right)^{-1} \\ &\asymp \lambda_1(\operatorname{Ad}(s)_\Gamma)^{-1} \asymp \lambda_1(\operatorname{Ad}(m)_\Gamma)^{-1} \leqslant \delta^{-1}. \end{aligned}$$

4.6 Exponential mixing

Finally, we will need the following theorem, which follows from the results of [KM99, Section 3.4] and [MA20, Section 4.1-4.3] applied to $\mathbf{G}$, $\mathbf{T}$ and Γ as above.

Theorem 4.6.1. *Let Y be an element in the Lie algebra of $\mathbf{T}(\mathbb{R})$ such that, for every projection $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ onto a $\mathbb{Q}$-simple factor, $\pi_i(Y) \neq 0$. For every $y \in \mathbb{R}_+^\times$, let $a'_y = \exp(\ln(y)\, Y)$. Then there exist constants $\ell \in \mathbb{N}$ and $C', c' > 0$ such that for all $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and all $0 < y \leqslant 1$,*

$$\left| \int_\Omega \phi_1(a'_y x)\, \phi_2(x)\, \mu_\Omega(x) \;-\; \mu_\Omega(\phi_1)\, \mu_\Omega(\phi_2) \right| \leqslant C'\, y^{c'}\, \|\Upsilon^\ell \phi_1\|_2\, \|\Upsilon^\ell \phi_2\|_2,$$

where Υ denotes the differential operator

$$\Upsilon = 1 - \sum_i Y_i^2,$$

where (Y_i) is an orthonormal basis of the Lie algebra $\mathfrak{k}$ of the maximal compact subgroup K of G.

4.7 Counting lattice points

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function. For every $x \in X$ and $T \geqslant 1$, we define the counting function

$$\mathcal{N}_\psi(x, T) = \#\left\{v \in \mathbf{X}(\mathbb{Q}) : d(x, v) < \psi(H_\chi(v)),\ H_\chi(v) < T\right\}.$$

Moreover, for every $T \geqslant 1$, we associate to ψ the set

$$\mathcal{E}_\psi(T) = \left\{\boldsymbol{v} \in \widetilde{X} : d(x_0, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|), 1 \leqslant \|\boldsymbol{v}\| < T\right\}. \tag{4.7.1}$$

Fix a section $\mathfrak{s} : X \to K$ of the orbital map $K \to X$ sending k to kx_0. Given $x \in X$, we shall write $k_x := \mathfrak{s}(x)$. As the following lemma shows, estimating the counting function $\mathcal{N}_\psi(x, T)$ amounts to counting lattice points in the increasing family $(\mathcal{E}_T)_{T \geqslant 1}$. Let $[K \cap P : K \cap L] \in \{1, 2\}$ be the index of $K \cap L$ in $K \cap P$.

Lemma 4.7.1. *For all $x \in X$ and $T \geqslant 1$,*

$$\mathcal{N}_\psi(x, T) = [K \cap P : K \cap L]^{-1}\, \#\left(k_x^{-1} \mathcal{L}_\chi \cap \mathcal{E}_\psi(T)\right). \tag{4.7.2}$$

Proof. It suffices to show that $\mathcal{N}_\psi(x,T) = \#\left(\mathcal{L}_\chi \cap k_x\mathcal{E}_\psi(T)\right)$. We first note that

$$k_x\mathcal{E}_\psi(T) = \left\{\boldsymbol{v} \in \widetilde{X} : d(x,[\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|), 1 \leqslant \|\boldsymbol{v}\| < T\right\}.$$

Now a rational point $v = g[\boldsymbol{e}_\chi] \in \mathbf{X}(\mathbb{Q})$ satisfies $d(x,v) < \psi(H_\chi(v))$ and $1 \leqslant H_\chi(v) < T$ if and only if any of the primitive vectors $\boldsymbol{v} \in \mathcal{L}_\chi$ representing v satisfies $d(x,[\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|)$ and $1 \leqslant \|\boldsymbol{v}\| < T$. This finishes the proof of the lemma. □

Chapter 5

Counting at the Diophantine exponent

Let us recall that $\mathbf{X} = \mathbf{G}/\mathbf{P}$ denotes a generalized flag variety defined over $\mathbb{Q}$, where $\mathbf{G}$ is a connected semisimple algebraic $\mathbb{Q}$-group and $\mathbf{P}$ is a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. We suppose that the unipotent radical of $\mathbf{P}$ is abelian. In particular, the parabolic subgroup $\mathbf{P}$ is maximal, or in other words, the $\mathbb{Q}$-rank of $\mathbf{X}$ is 1. As was specified in the previous chapter, the choice of a highest $\mathbb{Q}$-weight χ of $\mathbf{G}$ associated to $\mathbf{P}$ defines a height H_χ on $\mathbf{X}(\mathbb{Q})$. The distance $d(\cdot,\cdot)$ and probability measure σ_X on X are the K-invariant Riemannian distance and the corresponding normalized K-invariant Riemannian measure, respectively. Throughout this chapter, we adopt the notation introduced in Chapter 4.

In this chapter, we shall count rational approximations to a real point chosen at random according to the Riemannian volume with respect to the Diophantine exponent $\tau = \beta_\chi$. More precisely, we shall be interested in determining the asymptotic behaviour of the counting function

$$\mathcal{N}_{\beta_\chi}(x,T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : d(x,v) < H_\chi(v)^{-\beta_\chi},\ 1 \leqslant H_\chi(v) < T\}$$

as $T \to +\infty$ for almost every $x \in X$. As shown in Chapter 3, the Siegel transform is, in general, not in L^2, even in the case the parabolic subgroup is maximal. Therefore, the approach from Chapter 2, which relies on a second moment formula for the Siegel transform, does not apply directly. To start with, we use an ergodic-theoretic approach and a mean value formula for Siegel transforms, as developed previously, to describe the almost sure asymptotic behaviour of $\mathcal{N}_{\beta_\chi}(x,T)$. Indeed, we will show that there exists an explicit constant $\varkappa > 0$ such that for almost every $x \in X$,

$$\mathcal{N}_{\beta_\chi}(x,T) \sim \varkappa \ln T \qquad \text{as } T \to +\infty.$$

We then refine this result from an asymptotic formula to an estimate with an explicit error term. To this end, we use Theorem 1.1.1 giving equivalent criteria for the L^1-integrability of Siegel transforms and prove an effective equidistribution result for expanding translates of orbits of the maximal compact subgroup K of G.

5.1 Ergodic-theoretic approach: An asymptotic formula without error term

The method is inspired by the ergodic-theoretic approach in [AG22], where Alam and Ghosh counted rational approximations on spheres with respect to the Diophantine exponent (which equals 1 for any sphere $\mathbb{S}^n$, $n \geqslant 2$). Roughly, it can be described as follows. In view of Lemma 4.7.1, for any $x \in X$ and $T \geqslant 1$, we have $\mathcal{N}_{\beta_\chi}(x,T) = \#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{\beta_\chi}(T)\right)$, where

$$\mathcal{E}_{\beta_\chi}(T) = \{\boldsymbol{v} \in \widetilde{X} : d(x_0,[\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi}, 1 \leqslant \|\boldsymbol{v}\| < T\}.$$

First, one approximates $\mathcal{E}_{\beta_\chi}(T)$ by regions that admit a nice tessellation with respect to the action of the subgroup A. To conclude, one uses Birkhoff's ergodic theorem and an approximation argument.

Let $\pi^+ : V_\chi \to V_\chi$ be the orthogonal projection onto $\mathbb{R}\boldsymbol{e}_\chi$ and we abbreviate $\pi^+(\boldsymbol{v})$ simply by $\boldsymbol{v}^+$. Recall from §4.2 that $\phi : G \to X$ is the orbital map $g \mapsto gx_0$. Denote by $D_1\phi : \mathfrak{g} \to T_{x_0}X$ its derivative at the identity $1 \in G$. It satisfies $\ker D_1\phi = \mathfrak{p}$, where $\mathfrak{p}$ is the Lie algebra of P, and hence defines an isomorphism $D_1\phi : \mathfrak{u}^- \to T_{x_0}X$. We use this isomorphism to identify $\mathfrak{u}^-$ with $T_{x_0}X$. By [Sax22, Lemme 2.4.2], using the definition

$$\forall\, y > 0, \quad a_y = \exp(\ln(y)Y_\alpha)$$

and the fact that $\mathfrak{u}^-$ is abelian, the derivative action of a_y on $\mathfrak{u}^- = T_{x_0}X$ is given by

$$\forall\, u \in \mathfrak{u}^-, \quad \mathrm{Ad}(a_y)u^- = yu^-.$$

For every non-zero $\boldsymbol{v} \in \widetilde{X}$, such that $[\boldsymbol{v}]$ is close to x_0, we denote by $u_{\boldsymbol{v}}^- \in \mathfrak{u}^-$ the unique element such that $[\boldsymbol{v}] = \exp(u_{\boldsymbol{v}}^-)x_0$. Observe that

$$[a_y\boldsymbol{v}] = a_y[\boldsymbol{v}] = a_y \exp(u_{\boldsymbol{v}}^-)a_y^{-1}a_y x_0 = \exp(\mathrm{Ad}(a_y)u_{\boldsymbol{v}}^-)x_0 = \exp(yu_{\boldsymbol{v}}^-)x_0$$

But one also has $[a_y\boldsymbol{v}] = \exp(u_{a_y\boldsymbol{v}}^-)x_0$. By uniqueness, this gives the simple relation

$$u_{a_y\boldsymbol{v}}^- = y\, u_{\boldsymbol{v}}^-. \tag{5.1.1}$$

In view of (4.2.1), there is a constant $C_0 > 0$ such that

$$d(x_0, [\boldsymbol{v}]) \leqslant \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} + C_0\|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-}^2. \tag{5.1.2}$$

We now approximate the region $\mathcal{E}_{\beta_\chi}(T)$ from inside and from outside by regions

$$\mathcal{E}_{T,c}^+ = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c\,\|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \leqslant \|\boldsymbol{v}^+\| < c\,T\}, \tag{5.1.3}$$

where $c > 0$ is a parameter that will approach 1. By enlarging C_0 if necessary, we can assume that $\|\boldsymbol{v}^+\| \geqslant C_0^{-1}\|\boldsymbol{v}\|$ if $[\boldsymbol{v}]$ is sufficiently close to x_0. For every natural number $\ell \geqslant 1$, let

$$Q_\ell = \{\boldsymbol{v} \in \widetilde{X} : \|\boldsymbol{v}\| \leqslant C_0\ell\} \tag{5.1.4}$$

and put

$$c_\ell = \left(1 + C_0\,\ell^{-\beta_\chi/2}\right)^{-2(1+\beta_\chi)} \in (0,1). \tag{5.1.5}$$

In particular, one has $c_\ell \nearrow 1$ as $\ell \to +\infty$. Fix a left-invariant Riemannian distance $d_G(\cdot,\cdot)$ on G. Write $B_P(r)$ for the (symmetric) open ball in P with radius $r > 0$ and center $1 \in P$ with respect to the distance induced by the distance $d_G(\cdot,\cdot)$ on G.

The following lemma tells us, roughly, that the region $\mathcal{E}_{\beta_\chi}(T)$ can be approximated well by regions of the form $\mathcal{E}_{T,c}^+$, that this approximation is even stable under perturbations of elements p close to the identity in P, and that the regions $\mathcal{E}_{T,c}^+$ admit a tessellation with respect to the action of the subgroup A. For every $c > 0$, let

$$\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c\,\|\boldsymbol{v}^+\|^{-\beta_\chi},\ 1 \leqslant \|\boldsymbol{v}^+\| < 2\}. \tag{5.1.6}$$

Lemma 5.1.1. *For all large enough $\ell \geqslant 1$, $p \in \mathcal{O}_\ell = B_P(\ell^{-\beta_\chi/2})$, and $T \geqslant 1$,*

$$\mathcal{E}^+_{T,c_\ell} \smallsetminus Q_{2\ell} \subseteq p\ \left(\mathcal{E}_{\beta_\chi}(T) \smallsetminus Q_\ell\right) \subseteq \mathcal{E}^+_{T,c_\ell^{-1}}. \tag{5.1.7}$$

Moreover, for all $c > 0$ and $T \geqslant 1$ such that $cT = 2^N$ for some $N \in \mathbb{N}$, we have

$$\mathcal{E}^+_{T,c} = \bigsqcup_{i=0}^{N-1} a_{y_j}^{-1}\,\mathcal{F}_c, \qquad \textit{with } y_j = 2^{\beta_\chi j} \textit{ for every } j \in \mathbb{N}. \tag{5.1.8}$$

Proof. Let us show the left inclusion in Equation (5.1.7). Put $\delta_\ell = \ell^{-\beta_\chi/2}$. We need to show that for all large enough $\ell \geqslant 1$, $p \in \mathcal{O}_\ell$, $T \geqslant 1$, and $\boldsymbol{v} \in \mathcal{E}^+_{T,c_\ell} \smallsetminus Q_{2\ell}$, we have

$$d(x_0, p[\boldsymbol{v}]) < \|p\boldsymbol{v}\|^{-\beta_\chi}, \quad \text{and} \quad C_0\ell < \|p\boldsymbol{v}\| < T.$$

Using the triangle inequality, we get $d(x_0, p[\boldsymbol{v}]) \leqslant d(x_0, [\boldsymbol{v}]) + d([\boldsymbol{v}], p[\boldsymbol{v}])$. Next, writing $[\boldsymbol{v}] = \exp(u^-_{\boldsymbol{v}})x_0$ with $u^-_{\boldsymbol{v}} \in \mathfrak{u}^-$, using that p stabilizes the line x_0, and expressing $\exp(\mathrm{Ad}(p)u^-_{\boldsymbol{v}}) = \exp(u')p'$ with $u' \in \mathfrak{u}^-$, $p' \in P$, we get

$$\begin{aligned} d([\boldsymbol{v}], p[\boldsymbol{v}]) &\lesssim d(\exp(u^-_{\boldsymbol{v}})x_0, p\exp(u^-_{\boldsymbol{v}})x_0) \\ &= d(\exp(u^-_{\boldsymbol{v}})x_0, \exp(\mathrm{Ad}(p)u^-_{\boldsymbol{v}})x_0) \\ &= d(\exp(u^-_{\boldsymbol{v}})x_0, \exp(u')x_0) \asymp \|u' - u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} \end{aligned}$$

Note that the map from a bounded small neighborhood of 0 in the Lie algebra $\mathfrak{g}$ of G to $\mathfrak{u}^-$ defined by $X \mapsto X'$ with $X' \in \mathfrak{u}^-$ and $p_X \in P^\circ$ such that $\exp(X) = \exp(X')p_X$ is Lipschitz. Hence

$$\|u' - u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} \lesssim \|\mathrm{Ad}(p)u^-_{\boldsymbol{v}} - u^-_{\boldsymbol{v}}\|_{\mathfrak{g}} \lesssim \|\mathrm{Ad}(p) - \mathrm{Id}\|\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} \lesssim \delta_\ell\, d(x_0, [\boldsymbol{v}]).$$

Therefore, by enlarging C_0 if necessary, one has $d(x_0, p[\boldsymbol{v}]) \leqslant (1+C_0\delta_\ell)d(x_0, [\boldsymbol{v}])$. Together with the estimate (5.1.2) and the fact that $\boldsymbol{v} \in \mathcal{E}^+_{T,c_\ell}$, we thus get

$$\begin{aligned} d(x_0, p[\boldsymbol{v}]) &\leqslant (1 + C_0\delta_\ell)\,\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-}\ \left(1 + C_0\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-}\right) \\ &< \|\boldsymbol{v}^+\|^{-\beta_\chi}c_\ell\,(1 + C_0\delta_\ell)\left(1 + C_0\|\boldsymbol{v}^+\|^{-\beta_\chi}\right). \end{aligned}$$

Let $\boldsymbol{v}^\perp = \boldsymbol{v} - \boldsymbol{v}^+$ be the projection of $\boldsymbol{v}$ onto the orthogonal complement of $\mathbb{R}\boldsymbol{e}_\chi$. Using the fact that $\boldsymbol{v} \in \mathcal{E}^+_{T,c_\ell} \smallsetminus Q_{2\ell}$, we have

$$\frac{\|\boldsymbol{v}^\perp\|}{\|\boldsymbol{v}^+\|} \asymp d(x_0, v) \asymp \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} \lesssim \|\boldsymbol{v}^+\|^{-\beta_\chi}.$$

Hence, by enlarging C_0 if necessary and using that $\|\boldsymbol{v}\|^2 = \|\boldsymbol{v}^+\|^2 + \|\boldsymbol{v}^\perp\|^2$, we have

$$\|\boldsymbol{v}^+\|^{-\beta_X} = \|\boldsymbol{v}\|^{-\beta_X}\left(1+\frac{\|\boldsymbol{v}^\perp\|^2}{\|\boldsymbol{v}^+\|^2}\right)^{\frac{\beta_X}{2}} \leqslant \|\boldsymbol{v}\|^{-\beta_X}\left(1+C_0\|\boldsymbol{v}^+\|^{-2\beta_X}\right)^{\frac{\beta_X}{2}}.$$

By enlarging C_0, we may assume that $\|\boldsymbol{v}^+\| \geqslant C_0^{-1}\|\boldsymbol{v}\|$. Since $\boldsymbol{v} \notin Q_{2\ell}$, we thus have $\|\boldsymbol{v}^+\| \geqslant C_0^{-1}\|\boldsymbol{v}\| \geqslant 2\ell$. Moreover, we may assume that C_0 is such that $\|p\boldsymbol{v}\| \leqslant (1+C_0\delta_\ell)\|\boldsymbol{v}\|$ for all large $\ell \geqslant 1$ and $p \in B_P(\delta_\ell)$. Putting everything together and using the definition of c_ℓ, we have

$$\begin{aligned} d(x_0, p[\boldsymbol{v}]) &< \|p\boldsymbol{v}\|^{-\beta_X}\left(c_\ell(1+C_0\delta_\ell)^{1+\beta_X}(1+C_0\|\boldsymbol{v}^+\|^{-\beta_X})\left(1+C_0\|\boldsymbol{v}^+\|^{-2\beta_X}\right)^{\frac{\beta_X}{2}}\right) \\ &\leqslant \|p\boldsymbol{v}\|^{-\beta_X}\left(c_\ell(1+C_0\delta_\ell)^{1+\beta_X}(1+C_0\ell^{-\beta_X})\left(1+C_0\ell^{-2\beta_X}\right)^{\frac{\beta_X}{2}}\right) \\ &\leqslant \|p\boldsymbol{v}\|^{-\beta_X}, \end{aligned}$$

as desired. Moreover, by the definition of c_ℓ in (5.1.5), we have

$$\|p\boldsymbol{v}\| \leqslant (1+C_0\,\delta_\ell)\|\boldsymbol{v}^+\|\frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}^+\|} \leqslant (1+C_0\,\delta_\ell)c_\ell\frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}^+\|}T < T.$$

Finally, since $\boldsymbol{v} \notin Q_{2\ell}$, we have, when ℓ is large,

$$\|p\boldsymbol{v}\| \geqslant (1+C_0\delta_\ell)^{-1}\|\boldsymbol{v}\| > (1+C_0\delta_\ell)^{-1}C_0(2\ell) \geqslant C_0\ell.$$

This shows the left inclusion in Equation (5.1.7). The right inclusion is proved similarly. To see the last claim, let us recall that a_y acts on $\boldsymbol{v}^+$ by $a_y\boldsymbol{v}^+ = y^{-\frac{1}{\beta_X}}\boldsymbol{v}^+$. Then the claim follows by using (5.1.1) and observing that

$$a_{y_j}^{-1}\cdot\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c\cdot\|\boldsymbol{v}^+\|^{-\beta_X}, 2^j \leqslant \|\boldsymbol{v}^+\| < 2^{j+1}\}.$$

□

We recall that, for every $j \in \mathbb{Z}$, we defined $y_j = 2^{\beta_X j}$. By Moore's ergodicity theorem (see [BM00, Section 3, Theorem 2.1]), the action of the unbounded subgroup $\{a_{y_j} : j \in \mathbb{Z}\}$ of G on the probability space (Ω, μ_Ω) is ergodic. Therefore, by Birkhoff's ergodic theorem (see [BM00, Section 1, Theorem 2.5]), for every $f \in L^1(\Omega)$ and almost every $x \in \Omega$,

$$\frac{1}{N}\sum_{j=0}^{N-1} f(a_{y_j}x) \longrightarrow \int_\Omega f\,d\mu_\Omega \qquad \text{as } N \to +\infty. \tag{5.1.9}$$

A point $x \in \Omega$ satisfying (5.1.9) is called *Birkhoff generic* with respect to f.

We are now ready to give the proof of Theorem 1.2.1. In order not to interrupt the flow of reading, we recall the statement of this theorem:

Theorem (Theorem 1.2.1)**.** *Let* $\mathbf{G}$ *be a connected semisimple* $\mathbb{Q}$*-group,* $\mathbf{P}$ *a parabolic* $\mathbb{Q}$*-subgroup with abelian unipotent radical and* $\mathbf{X} = \mathbf{G}/\mathbf{P}$*. Then there exists a constant* $\varkappa > 0$ *such that for* σ_X*-almost every* $x \in X$*,*

$$\mathcal{N}_{\beta_\chi}(x, T) \sim \varkappa \ln(T) \qquad \text{as } T \to +\infty. \tag{5.1.10}$$

Proof of Theorem 1.2.1. For any $c > 0$, with $\mathcal{F}_c$ as defined in Equation (5.1.6), define the function $F_c : \Omega \to \mathbb{R}$ by $F_c : g\Gamma \mapsto \#(g\mathcal{L}_\chi \cap \mathcal{F}_c)$. By the proof of Lemma 6.4.2 and with $\varkappa_2 > 0$ as in (6.4.9), for every $c > 0$, we have

$$\int_\Omega F_c \, d\mu_\Omega = \varkappa_2 \, \lambda_{\tilde{X}}(\mathcal{F}_c). \tag{5.1.11}$$

By the Iwasawa decomposition, the set $S = \mathcal{O}_1 \cdot K$, where $\mathcal{O}_1 = B_P(1)$ is as above, contains an open neighborhood of the identity in G. Moreover, a Fubini-type argument shows that for almost every $p \in \mathcal{O}_1$ there is a measurable subset $K_p \subset K$ with $\mu_K(K_p) = 1$ such that for every $k \in K_p$ the point $pk\Gamma$ is Birkhoff generic with respect to the function F_c. Consequently, for all $\ell \geqslant 1$, we can find $p_\ell \in \mathcal{O}_\ell$ and a full-measure subset $K_\ell \subset K$ such that for every $k \in K_\ell$ the point $p_\ell k\Gamma$ is Birkhoff generic with respect to the functions F_{c_ℓ} and $F_{c_\ell^{-1}}$. Let $k \in K_\infty = \bigcap_{\ell \geqslant 1} K_\ell$. Since $p_\ell \in \mathcal{O}_\ell$, Lemma 5.1.1 gives for all large ℓ and $T \geqslant 1$, that

$$\mathcal{E}^+_{T,c_\ell} \smallsetminus Q_{2\ell} \subseteq p_\ell \cdot \left(\mathcal{E}_{\beta_\chi}(T) \smallsetminus Q_\ell\right) \subseteq \mathcal{E}^+_{T,c_\ell^{-1}}.$$

Intersecting with $p_\ell k\mathcal{L}_\chi$ and using the fact that the number of lattice points in the set $\mathcal{O}_1 Q_\ell$ is bounded by an absolute constant times $\ell^{\beta_\chi d}$ (see Corollary 6.5.1), up to enlarging C_0 if necessary, we have

$$\#\left(p_\ell k\mathcal{L}_\chi \cap \mathcal{E}^+_{T,c_\ell}\right) - C_0 \ell^{\beta_\chi d} \leqslant \#\left(k\mathcal{L}_\chi \cap \mathcal{E}_T\right) \leqslant \#\left(p_\ell k\mathcal{L}_\chi \cap \mathcal{E}^+_{T,c_\ell^{-1}}\right) + C_0 \ell^{\beta_\chi d}. \tag{5.1.12}$$

Using the decomposition of $\mathcal{E}^+_{T,c}$ in (5.1.8) with ℓ large enough and $T = \frac{1}{c}2^N$ for every integer $N \geqslant 1$, we have

$$\#(p_\ell k\mathcal{L}_\chi \cap \mathcal{E}^+_{T,c}) = \sum_{j=0}^{N-1} F_c(a_{y_j} p_\ell k\Gamma). \tag{5.1.13}$$

For $T \geqslant 1$ and ℓ large, let $N \geqslant 1$ be the unique integer with $\frac{1}{c_\ell}2^N \leqslant T < \frac{1}{c_\ell}2^{N+1}$. Plugging this back into (5.1.12), we get the lower and upper bounds

$$\begin{aligned}\sum_{j=0}^{N-1} F_{c_\ell}(a_{y_j} p_\ell k\Gamma) - C_0 \ell^{\beta_\chi d} &\leqslant \#\left(k\mathcal{L}_\chi \cap \mathcal{E}_T\right) \\ &\leqslant \sum_{j=0}^{N} F_{c_\ell^{-1}}(a_{y_j} p_\ell k\Gamma) + C_0 \ell^{\beta_\chi d}.\end{aligned}$$

Dividing by $\ln(T)$, using that $p_\ell k\Gamma$ is Birkhoff generic with respect to F_{c_ℓ} and $F_{c_\ell^{-1}}$, taking limits $T \to +\infty$ and evaluating them using (5.1.11), we have

$$\frac{\varkappa_2}{\ln(2)}\lambda_{\widetilde{X}}(\mathcal{F}_{c_\ell}) \leqslant \lim_{T\to+\infty} \frac{\#\left(k\mathcal{L}_\chi \cap \mathcal{E}_T\right)}{\ln(T)} \leqslant \frac{\varkappa_2}{\ln(2)}\lambda_{\widetilde{X}}(\mathcal{F}_{c_\ell^{-1}}). \tag{5.1.14}$$

Since the function $c \mapsto \lambda_{\widetilde{X}}(\mathcal{F}_c)$ is continuous, we have $\lambda_{\widetilde{X}}(\mathcal{F}_{c_\ell^{\pm 1}}) \to \lambda_{\widetilde{X}}(\mathcal{F}_1)$ as $\ell \to +\infty$. Let $\varkappa > 0$ be as in (6.2.4). By Lemma 4.7.1 and letting ℓ go to infinity in (5.1.14), for every $k \in K_\infty^{-1}$, we have, as $T \to +\infty$,

$$\mathcal{N}_{\beta_\chi}(kx_0, T) = [K \cap L : K \cap P]\,\#\left(k^{-1}\mathcal{L}_\chi \cap \mathcal{E}_T\right) \sim \varkappa \ln(T),$$

Noting that $X = K/(K \cap P)$ and that, for every $k \in K$ and $p \in K \cap P$, we have $\mathcal{N}_{\beta_\chi}(kpx_0, T) = \mathcal{N}_{\beta_\chi}(kx_0, T)$, implies that, for σ_X-almost every $x \in X$, as $T \to +\infty$,

$$\mathcal{N}_{\beta_\chi}(x, T) \sim \varkappa \ln(T).$$

The proof of Theorem 1.2.1 is complete. □

5.2 Counting with an error term: The method

In the remainder of this chapter, we would like to refine the asymptotic formula to an estimate with an explicit error term. To this end, we use Theorem 1.1.1 giving equivalent criteria for the L^1-integrability of Siegel transforms and we prove an effective equidistribution result for expanding translates of orbits of the maximal compact subgroup K of G. For the convenience of the reader, let us recall the statement of Theorem 1.2.2.

Theorem (Theorem 1.2.2)**.** *Let* $\mathbf{G}$ *be a connected semisimple algebraic* $\mathbb{Q}$*-group,* $\mathbf{P}$ *a parabolic* $\mathbb{Q}$*-subgroup of* $\mathbf{G}$ *with abelian unipotent radical and*

$\mathbf{X} = \mathbf{G}/\mathbf{P}$. *Then there exists an explicit constant* $\varkappa > 0$ *and* $\varepsilon > 0$ *such that for* σ_X*-almost every* $x \in X$*, as* $T \to +\infty$*, we have*

$$\mathcal{N}_{\beta_\chi}(x,T) = \varkappa \ln(T) \left(1 + O_x(\ln(T)^{-\varepsilon})\right).$$

In this section, we sketch the method of proof of Theorem 1.2.2; it is inspired by the arguments of Ouaggag [Oua23] used to count rational approximations on spheres. The detailed proof will be given in Sections 5.3 - 5.8.

In chapter 4, we translated the problem of counting rational approximations to a point $x \in X$ of bounded height to counting primitive lattice points within a growing family $(k_x \mathcal{E}_{\beta_\chi}(T))_{T>1}$ of subsets of the cone $\widetilde{X}$, where $k_x \in K$ is any element such that $x = k_x x_0$. Moreover, we obtained the relationship

$$\forall\, x \in X,\ \forall\, T \geqslant 1, \quad \mathcal{N}_{\beta_\chi}(x,T) = \#(\mathcal{L}_\chi \cap k_x \mathcal{E}_{\beta_\chi}(T)).$$

We will show an asymptotic formula for $\#(\mathcal{L}_\chi \cap k_x \mathcal{E}_{\beta_\chi}(T))$, for almost every $x \in X$, with an explicit error term. Just as in the ergodic-theoretic approach in Section 5.1, we will first approximate $\mathcal{E}_{\beta_\chi}(T)$ with a region $\mathcal{E}_{\beta_\chi}(T)^+$ that admits a tessellation under the one-parameter diagonal subgroup A. For convenience, we shall use a different parametrization of A for the remainder of this chapter. Namely, for $t \in \mathbb{R}$, we put $\widehat{a}(t) = \exp(tY_\alpha)$, where Y_α is as in (4.4.3). Therefore, $\widehat{a}(t)$ will act on $\boldsymbol{e}_\chi$ by $\widehat{a}(t)\boldsymbol{e}_\chi = e^{-\frac{t}{\beta_\chi}}\boldsymbol{e}_\chi$ and on the tangent space at $x_0 \in X$ through scalar multiplication by e^t. In fact, there will be an elementary domain $\mathcal{F} \subset \widetilde{X}$ such that for every $N \in \mathbb{N}$, we have, with $T = e^N$,

$$\mathcal{E}_T^+ = \bigsqcup_{i=0}^{N-1} \widehat{a}(t_i)^{-1} \cdot \mathcal{F}, \qquad \text{with } t_j = \beta_\chi j \text{ for } j \in \mathbb{N}.$$

For the sake of describing the method, we suppose that $\Gamma \cdot \boldsymbol{e}_\chi = \mathcal{L}_\chi$, that is, that $\mathcal{L}_\chi$ is a single Γ-orbit. The definition of the Siegel transform together with the tessellation of $\mathcal{E}_T^+$ will give

$$\#(\mathcal{L}_\chi \cap k_x \mathcal{E}_{\beta_\chi}(T)^+) = \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{k_x \widehat{a}(t_i)^{-1}\mathcal{F}}(\Gamma) = \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}}(\widehat{a}(t_i) k_x^{-1} \Gamma). \quad (5.2.1)$$

Noting that $N = \ln(T)$, Theorem 1.2.2 will follow from the fact that there exists $\varepsilon > 0$ such that for almost every $k \in K$,

$$\sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}}(\widehat{a}(t_i) k \Gamma) = \int_\Omega S_\chi f \, d\mu_\Omega \, N \, \left(1 + O_k(N^{-\varepsilon})\right).$$

This result in turn will be deduced from an L^p-bound, for some $p > 1$, of the form

$$\int_K \left| \sum_{j=a}^{b} \left(S_\chi \mathbb{1}_{\mathcal{F}}(\widehat{a}(t_i)k\Gamma) - \int_\Omega S_\chi f \right) \right|^p dk \lesssim b - a,$$

for all $a, b \in \mathbb{N}$, $a < b$. The latter is shown using the $L^{1+\delta}$-integrability of the Siegel transform as in Theorem 1.1.1 and using the effective double equidistribution of expanding translates of K-orbits. In particular, we will need to work with smooth compactly supported functions that, on translated K-orbits, approximate the Siegel transform $S_\chi \mathbb{1}_{\mathcal{F}}$, which typically is neither smooth nor compactly supported.

5.3 Approximation of $\mathcal{E}_{\beta_\chi}(T)$

So it remains to provide an asymptotic formula for $\#(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{\beta_\chi}(T))$ for almost every $x \in X$. To this end, exactly as in Section 5.1, we now approximate the region $\mathcal{E}_{\beta_\chi}(T)$ from inside and from outside by regions that admit a tessellation with respect to the action of the one-parameter diagonal subgroup A.

Recall that the map $\mathfrak{u}^- \to X$ sending $u \mapsto \exp(u)x_0$ restricts to a diffeomorphism from a neighborhood of $1 \in \mathfrak{u}^-$ to a neighborhood of $x_0 \in X$. In particular, any $\boldsymbol{v} \in \widetilde{X}$, so that $[\boldsymbol{v}]$ is close to x_0, defines an element $u_{\boldsymbol{v}}^-$ in the Lie algebra $\mathfrak{u}^-$ by $[\boldsymbol{v}] = \exp(u_{\boldsymbol{v}}^-)x_0$. The adjoint action of $\widehat{a}(t) \in A$ on $\mathfrak{u}^- = T_{x_0}X$ acts by scalar multiplication $\mathrm{Ad}(\widehat{a}(t))u^- = e^t\, u^-$ by e^t. Observe that

$$[\widehat{a}(t)\boldsymbol{v}] = \widehat{a}(t)[\boldsymbol{v}] = \widehat{a}(t)\exp(u_{\boldsymbol{v}}^-)\widehat{a}(t)\widehat{a}(t)^{-1}x_0 = \exp(\mathrm{Ad}(\widehat{a}(t))u_{\boldsymbol{v}}^-)x_0.$$

But one also has $[\widehat{a}(t)\boldsymbol{v}] = \exp(u_{\widehat{a}(t)\boldsymbol{v}}^-)x_0$. By uniqueness, this gives the simple relation

$$u_{\widehat{a}(t)\boldsymbol{v}}^- = e^t\, u_{\boldsymbol{v}}^-. \tag{5.3.1}$$

Moreover, by the distance estimate (4.2.1), there exists a constant $C_0 > 0$ such that $d(x_0, [\boldsymbol{v}]) \leqslant \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-}(1 + C_0\|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-})$. Let $\pi^+ : V_\chi \to V_\chi$ be the orthogonal projection onto $\mathbb{R}\boldsymbol{e}_\chi$ and we abbreviate $\pi^+(\boldsymbol{v})$ by $\boldsymbol{v}^+$.

For every $T \geqslant 1$ and $c > 0$ close to 1, we will work with exactly the same regions $\mathcal{E}_{T,c}^+$ that were introduced in (5.1.3) and were defined by

$$\mathcal{E}_{T,c}^+ = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c\,\|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \leqslant \|\boldsymbol{v}^+\| < c\,T\}.$$

By enlarging C_0 if necessary, we can assume that $\|\boldsymbol{v}^+\| \geqslant C_0^{-1}\|\boldsymbol{v}\|$ as soon as $d(x_0, [\boldsymbol{v}]) < 1$. For every integer $\ell \geqslant 1$, exactly as in (5.1.4), we let

$$Q_\ell = \{\boldsymbol{v} \in \widetilde{X} : \|\boldsymbol{v}\| \leqslant C_0\,\ell\}$$

and we define

$$\widehat{c}_\ell = \left(1 + C_0\,\ell^{-\beta_\chi}\right)^{-(1+\beta_\chi)} \in (0,1).$$

In particular, one has $\widehat{c}_\ell \nearrow 1$ as $\ell \to +\infty$.

The sets $\mathcal{E}^+_{T,c}$ have the following nice properties. By a slight abuse of notation (see Equation 5.1.6), for every $c > 0$, let

$$\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} < c\|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \leqslant \|\boldsymbol{v}^+\| < e\}. \tag{5.3.2}$$

Lemma 5.3.1. *1. (Approximation) For all large enough $\ell \geqslant 1$ and $T \geqslant 1$, we have*

$$\mathcal{E}^+_{T,\widehat{c}_\ell} \smallsetminus Q_{2\ell} \subseteq \mathcal{E}_{\beta_\chi}(T) \smallsetminus Q_\ell \subseteq \mathcal{E}^+_{T,\widehat{c}_\ell^{-1}}. \tag{5.3.3}$$

2. (Tessellation) For all $c > 0$ and $T \geqslant 1$ such that $cT = e^N$ for some $N \in \mathbb{N}$, we have

$$\mathcal{E}^+_{T,c} = \bigsqcup_{i=0}^{N-1} \widehat{a}(t_i)^{-1}\,\mathcal{F}_c, \qquad \textit{with } t_j = \beta_\chi j \textit{ for } j \in \mathbb{N}. \tag{5.3.4}$$

Proof. Let $\boldsymbol{v} \in \mathcal{E}^+_{T,\widehat{c}_\ell} \smallsetminus Q_{2\ell}$. Let us first prove that for all sufficiently large ℓ and T, we have

$$d(x_0, [\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi},\; C_0\,\ell < \|\boldsymbol{v}\| < T.$$

Let $\boldsymbol{v}^\perp = \boldsymbol{v} - \boldsymbol{v}^+$. Observe that, for large enough ℓ, we have

$$\frac{\|\boldsymbol{v}^\perp\|}{\|\boldsymbol{v}^+\|} \asymp d(x_0, [\boldsymbol{v}]) < \|\boldsymbol{v}\|^{-\beta_\chi}.$$

Therefore, we have

$$\frac{\|\boldsymbol{v}\|^2}{\|\boldsymbol{v}^+\|^2} = 1 + \frac{\|\boldsymbol{v}^\perp\|^2}{\|\boldsymbol{v}^+\|^2} \leqslant 1 + C_0\,\|\boldsymbol{v}^+\|^{-2\beta_\chi},$$

and hence

$$\frac{\|\boldsymbol{v}\|^{\beta_\chi}}{\|\boldsymbol{v}^+\|^{\beta_\chi}} \leqslant \left(1 + C_0\,\|\boldsymbol{v}^+\|^{-2\beta_\chi}\right)^{\beta_\chi/2}.$$

Since $\boldsymbol{v}$ does not lie in $Q_{2\ell}$, we have $\|\boldsymbol{v}^+\| \geqslant C_0^{-1}\|\boldsymbol{v}\| \geqslant 2\ell$. Thus, using the definition of $\widehat{c}_\ell$, we have (by enlarging C_0 where necessary)

$$\begin{aligned} d(x_0, [\boldsymbol{v}]) &\leqslant \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} \left(1 + C_0 \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-}\right) \\ &\leqslant \widehat{c}_\ell \|\boldsymbol{v}^+\|^{-\beta_\chi} \left(1 + C_0 \|\boldsymbol{v}^+\|^{-\beta_\chi}\right) \\ &= \|\boldsymbol{v}\|^{-\beta_\chi} \left(\widehat{c}_\ell \frac{\|\boldsymbol{v}\|^{\beta_\chi}}{\|\boldsymbol{v}^+\|^{\beta_\chi}} (1 + C_0 \|\boldsymbol{v}^+\|^{-\beta_\chi})\right) \\ &\leqslant \|\boldsymbol{v}\|^{-\beta_\chi} \left(\widehat{c}_\ell \left(1 + C_0 \,\ell^{-2\beta_\chi}\right)^{\beta_\chi/2} (1 + C_0 \,\ell^{-\beta_\chi})\right) \\ &\leqslant \|\boldsymbol{v}\|^{-\beta_\chi} \end{aligned}$$

Since $\boldsymbol{v}$ does not lie in $Q_{2\ell}$, it does, in particular, not lie in Q_ℓ. Moreover, we have $\|\boldsymbol{v}\| = \|\boldsymbol{v}^+\| \frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}^+\|} \leqslant \widehat{c}_\ell \frac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}^+\|} T < T$. This shows the left inclusion in Equation (5.3.3). The other inclusion is proved similarly.

To see the last claim in the lemma, let us recall that a_y acts on $\boldsymbol{v}^+$ by $a_y \boldsymbol{v}^+ = y^{-\frac{1}{\beta_\chi}} \boldsymbol{v}^+$. Then, using (5.3.1) and observing that

$$\widehat{a}(t_i)^{-1} \mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c \,\|\boldsymbol{v}^+\|^{-\beta_\chi}, e^j \leqslant \|\boldsymbol{v}^+\| < e^{j+1}\}$$

yields the desired tessellation. $\square$

Intersecting (5.3.3) with the discrete set $k_x^{-1}\mathcal{L}_\chi$ and using the fact that the number of lattice points in the set Q_ℓ is bounded by an absolute constant times $\ell^{\beta_\chi d}$ (see Corollary 6.5.1), we get, by enlarging C_0 if necessary,

$$\# \left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{T,\widehat{c}_\ell}^+\right) - C_0 \ell^{\beta_\chi d} \leqslant \# \left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_T\right) \leqslant \# \left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{T,\widehat{c}_\ell^{-1}}^+\right) + C_0 \ell^{\beta_\chi d}. \tag{5.3.5}$$

Using the tessellation of $\mathcal{E}_{T,c}^+$ as in (5.3.4) with $T = \frac{1}{c}e^N$ for every integer $N \geqslant 1$, we have

$$\begin{aligned} \#(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{T,c}^+) &= \#(\mathcal{L}_\chi \cap k_x \mathcal{E}_{T,c}^+) \\ &= \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{k_x \widehat{a}(t_i)^{-1} \mathcal{F}_c}(\Gamma) = \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}(t_i) k_x^{-1} \Gamma). \end{aligned} \tag{5.3.6}$$

Plugging this back into (5.3.5), we get the lower and upper bounds, for every

$T' \geqslant 1$ and large enough ℓ,

$$\sum_{i=0}^{\lfloor T'+\ln \widehat{c}_\ell \rfloor - 1} S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell}}(\widehat{a}(t_i) k_x^{-1}\Gamma) - C_0\, \ell^{\beta_\chi d} \leqslant \#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{\beta_\chi}(e^{T'})\right)$$
$$\leqslant \sum_{i=0}^{\lceil T'-\ln \widehat{c}_\ell \rceil - 1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_\ell^{-1}}}(\widehat{a}(t_i) k_x^{-1}\Gamma) + C_0\, \ell^{\beta_\chi d}. \tag{5.3.7}$$

The proof of Theorem 1.2.2 consists of effectively estimating the left- and right-hand sides of (5.3.7). To this end, we will now develop the necessary tools and ingredients for these estimates.

5.4 A uniform upper bound for the K-average of the Siegel transform

A crucial estimate used in this approximation is given in the following lemma. Using Theorem 1.1.1, we fix $\varepsilon > 0$ such that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^{1+\varepsilon}(\Omega)$.

Lemma 5.4.1. *Let $s \in (0, 1+\varepsilon)$. For all $c > 0$ and $g \in G$, we have*

$$\sup_{t \geqslant 0} \int_K \left| S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k g)\right|^s d\mu_K(k) < +\infty.$$

The upper bound is uniform as g and c vary in compact sets.

Let $d_G(\cdot,\cdot)$ be a left-invariant Riemannian metric on G; this induces left-invariant metrics on the parabolic subgroup P and on the quotient $\Omega = G/\Gamma$, and we denote them by $d_P(\cdot,\cdot)$ and $d_\Omega(\cdot,\cdot)$, respectively. The key observation used in the proof is that the translated sets $\widehat{a}(t)^{-1}\mathcal{F}_c$ are almost invariant under a small neighborhood of the identity in P.

Proof. We fix $s \in (0, 1+\varepsilon)$. We show that for $c > 0$ and g_0 in a compact subset Q of G, we have

$$\sup_{t \geqslant 0} \int_K \left(S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k g_0)\right)^s d\mu_K(k) \lesssim_{s,Q} \int_{\widetilde{X}} \mathbb{1}_{\mathcal{F}_c}\, d\lambda_{\widetilde{X}}.$$

For every $\delta \in (0, \frac{1}{2})$, we define a δ-neighborhood of $\mathcal{F}_c$ by

$$\mathcal{F}_c(\delta) = \{\boldsymbol{v} \in \widetilde{X} : \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} < (1+\delta)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, (1+\delta)^{-1} \leqslant \|\boldsymbol{v}^+\| < (1+\delta)e\}.$$

One can show that $\lambda(\mathcal{F}_c(\delta) \smallsetminus \mathcal{F}_c) \lesssim_c \delta$. For every $r > 0$, let $B_P(r)$ denote the metric open ball in P with radius $r > 0$ and center $1 \in P$. We claim that there exists some small constant $\widetilde{c} > 0$, independent of δ, such that for all $0 < \delta < \frac{1}{2}$, $p \in B_P(\widetilde{c}\delta)$, and $t \geqslant 0$, we have

$$p\widehat{a}(t)^{-1}\mathcal{F}_c \subseteq \widehat{a}(t)^{-1}\mathcal{F}_c(\delta).$$

First, using the relationship (5.3.1) and the fact that, for every $t \in \mathbb{R}$, $\widehat{a}(t)$ acts on $\boldsymbol{e}_\chi$ via $\widehat{a}(t)\, \boldsymbol{e}_\chi = e^{-\frac{t}{\beta_\chi}}\, \boldsymbol{e}_\chi$, we have

$$\widehat{a}(t)^{-1}\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} < c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, e^{\frac{t}{\beta_\chi}} \leqslant \|\boldsymbol{v}^+\| < e^{\frac{t}{\beta_\chi}+1}\},$$

and likewise

$$\begin{aligned}\widehat{a}(t)^{-1}\mathcal{F}_c(\delta) = \{\boldsymbol{v} \in \widetilde{X} : &\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} < (1+\delta)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, \\ &(1+\delta)^{-1} e^{\frac{t}{\beta_\chi}} \leqslant \|\boldsymbol{v}^+\| < (1+\delta) e^{\frac{t}{\beta_\chi}+1}\}.\end{aligned}$$

So we need to show that for some small $\widetilde{c} > 0$, for all $0 < \delta < \frac{1}{2}$, $p \in B_P(\widetilde{c}\delta)$, $t \geqslant 0$, and $\boldsymbol{v} \in \widehat{a}(t)^{-1}\mathcal{F}_c$, we have

(1) $\|u^-_{p\boldsymbol{v}}\|_{\mathfrak{u}^-} < (1+\delta)^{1+\beta_\chi} c\, \|(p\boldsymbol{v})^+\|^{-\beta_\chi}$, and

(2) $(1+\delta)^{-1} e^{\frac{t}{\beta_\chi}} \leqslant \|(p\boldsymbol{v})^+\| < (1+\delta)\, e^{\frac{t}{\beta_\chi}+1}$.

Let us start with condition (2). Since $\boldsymbol{v}$ satisfies $e^{\frac{t}{\beta_\chi}} \leqslant \|\boldsymbol{v}^+\| < e^{\frac{t}{\beta_\chi}+1}$, it is enough to show that for some small $\widetilde{c} > 0$, for all $0 < \delta < \frac{1}{2}$, $p \in B_P(\widetilde{c}\delta)$, $t \geqslant 0$, and $\boldsymbol{v} \in \widehat{a}(t)^{-1}\mathcal{F}_c$, we have

$$(1+\delta)^{-1}\|\boldsymbol{v}^+\| \leqslant \|(p\boldsymbol{v})^+\| < (1+\delta)\|\boldsymbol{v}^+\|. \tag{5.4.1}$$

However, applying the triangle inequality, we already see that

$$\|(p\boldsymbol{v})^+\| - \|\boldsymbol{v}^+\| \leqslant \|(p\boldsymbol{v})^+ - \boldsymbol{v}^+\| \leqslant \|p\boldsymbol{v} - \boldsymbol{v}\| \lesssim \widetilde{c}\delta\|\boldsymbol{v}\| \lessapprox \widetilde{c}\delta\|\boldsymbol{v}^+\|.$$

By shrinking $\widetilde{c} > 0$ if necessary, this gives $\|(p\boldsymbol{v})^+\| < (1+\varepsilon)\|\boldsymbol{v}^+\|$, as required. The other inequality is shown similarly and we omit the details.

So it remains to show the first condition. Recall that $u_{p\boldsymbol{v}} \in \mathfrak{u}^-$ is defined by the relation $p[\boldsymbol{v}] = \exp(u^-_{p\boldsymbol{v}})x_0$. On the other hand, since x_0 is fixed by p, we have $p[\boldsymbol{v}] = \exp(\mathrm{Ad}(p)u^-_{\boldsymbol{v}})x_0$. Consider the decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{p}$, where $\mathfrak{p}$ denotes the Lie algebra of P. The map $\mathfrak{u}^- \oplus \mathfrak{p} \to G$ sending (u^-, p') to $\exp(u^-)\exp(p')$ is a local diffeomorphism. In particular, there exists a unique $(u^-, p') \in \mathfrak{u}^- \oplus \mathfrak{p}$ such that $\exp(\mathrm{Ad}(p)u^-_{\boldsymbol{v}}) = \exp(u^-)\exp(p')$. When applied to x_0, we have

$$p[\boldsymbol{v}] = \exp(\mathrm{Ad}(p)u^-_{\boldsymbol{v}})x_0 = \exp(u^-)x_0 \quad \text{and} \quad p[\boldsymbol{v}] = \exp(u^-_{p\boldsymbol{v}})x_0.$$

This implies that $u^- = u^-_{p\boldsymbol{v}}$. The induced map $\mathfrak{g} \to \mathfrak{u}^-$, sending $Y \in \mathfrak{g}$ to u^-_Y such that $\exp(Y) = \exp(u^-_Y)\exp(p_Y)$ with $p_Y \in \mathfrak{p}$, is Lipschitz on a bounded neighborhood of the origin in $\mathfrak{g}$. Thus

$$\|u^-_{p\boldsymbol{v}}\|_{\mathfrak{u}^-} = \|u^-\|_{\mathfrak{u}^-} \lesssim \|\mathrm{Ad}(p)u^-_{\boldsymbol{v}}\|_{\mathfrak{g}^-} \leqslant \|\mathrm{Ad}(p)\|\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-},$$

where $\|\mathrm{Ad}(p)\|$ denotes the operator norm as an operator on $\mathfrak{g}$. By shrinking $c > 0$ if necessary, one has $p \in B_P(\widetilde{c}\varepsilon)$ implies that $\|\mathrm{Ad}(p)\| < 1 + \varepsilon$.

Since $\boldsymbol{v} \in \widehat{a}(t)^{-1}\mathcal{F}_c$, we get from the first inequality in (5.4.1) applied to p^{-1} that

$$\begin{aligned}\|u^-_{p\boldsymbol{v}}\|_{\mathfrak{u}^-} \leqslant (1+\varepsilon)\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} &\leqslant (1+\varepsilon)\,c\,\|\boldsymbol{v}^+\|^{-\beta_\chi} = (1+\varepsilon)c\,\|(p^{-1}p\boldsymbol{v})^+\|^{-\beta_\chi} \\ &\leqslant (1+\varepsilon)^{1+\beta_\chi}\,c\,\|(p\boldsymbol{v})^+\|^{-\beta_\chi},\end{aligned}$$

as desired. This completes the proof of the claim.

We can now prove Lemma 5.4.1. Letting $g_0 \in Q$, we see that

$$\begin{aligned}\int_K \left|S_\chi \mathbb{1}_{\mathcal{F}_c}(a_t k g_0)\right|^s d\mu_K(k) &= \int_K \left|S_\chi \mathbb{1}_{\widehat{a}(t)^{-1}\mathcal{F}_c}(kg_0)\right|^s d\mu_K(k) \\ &\leqslant \frac{1}{\mu_P(B_P(\widetilde{c}\delta))}\int_{B_P(\widetilde{c}\delta)}\int_K \left|S_\chi \mathbb{1}_{\widehat{a}(t)^{-1}\mathcal{F}_{c(\varepsilon)}}(pkg_0)\right|^s d\mu_K(k)\,d\mu_P(p) \\ &= \frac{1}{\mu_P(B_P(\widetilde{c}\delta))}\int_{B_P(\widetilde{c}\delta)\,K} \left|S_\chi \mathbb{1}_{\widehat{a}(t)^{-1}\mathcal{F}_{c(\varepsilon)}}(gg_0)\right|^s d\mu_G(g) \\ &= \frac{1}{\mu_P(B_P(\widetilde{c}\delta))}\int_{(B_P(\widetilde{c}\delta)\,K)g_0} \left|S_\chi \mathbb{1}_{\widehat{a}(t)^{-1}\mathcal{F}_{c(\varepsilon)}}(g)\right|^s d\mu_G(g).\end{aligned}$$

Similarly as in the proof of Theorem 1.1.1, let $\mathfrak{S}$ be a Siegel set and $C \subset \mathbf{G}(\mathbb{Q})$ a finite subset such that $G = \mathfrak{S}\,C\,\Gamma$. Fix a fundamental domain $F \subseteq \mathfrak{S}\,C$

with nonempty interior for the right action of Γ on G. There exist a finite subset I of G such that the union $\bigcup_{h\in I} hF$ covers the closure of $B_P(\widetilde{c}\delta)\, K\, Q$ in G, which is compact. Together with the G-invariance of μ_Ω, we have

$$\int_K \left|S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k g)\right|^s d\mu_K(k) \lesssim_\varepsilon \sum_{h\in I} \int_{hF} \left|S_\chi \mathbb{1}_{\widehat{a}(t)^{-1}\mathcal{F}_c(\varepsilon)}(g)\right|^s d\mu_G(g)$$
$$\lesssim_\varepsilon \sum_{h\in I} \int_F \left|S_\chi \mathbb{1}_{\widehat{a}(t)^{-1}\mathcal{F}_c(\varepsilon)}(hg)\right|^s d\mu_G(g) \lesssim_{\varepsilon,F} \int_\Omega \left|S_\chi \mathbb{1}_{\mathcal{F}_c(\varepsilon)}(g)\right|^s d\mu_\Omega(g\Gamma).$$

To conclude, let us recall a few facts. First, applying the inequality (3.3.7) in the proof of Theorem 1.1.1 to $f = \mathbb{1}_{\mathcal{F}_c(\varepsilon)}$ and using that $\|f\|_\infty^\delta = 1$, we have

$$\int_\Omega |S_\chi f|^s \, d\mu_\Omega \lesssim \int_{\widetilde{X}} f(a_y \boldsymbol{e}_\chi) \int_{L/L(\mathbb{Z})} \lambda_\chi(a_y l)^{-(s-1)a_\chi} \, d\mu_{L/L(\mathbb{Z})}(l) d\lambda_{\widetilde{X}}(k a_y \boldsymbol{e}_\chi).$$

where the implicit constant depends only on the support of f. Since this support is compact, and consequently $a_y \in A$ varies in a compact set, for every $l \in L$, we have $\lambda_\chi(\widehat{a}(t)l)^{-(s-1)a_\chi} \lesssim_{\operatorname{supp} f} \lambda_\chi(l)^{-(s-1)a_\chi}$. Hence

$$\int_\Omega |S_\chi f|^s \, d\mu_\Omega \lesssim \int_{\widetilde{X}} f(a_y \boldsymbol{e}_\chi)\, d\lambda_{\widetilde{X}}(k a_y \boldsymbol{e}_\chi) \int_{L/L(\mathbb{Z})} \lambda_\chi(l)^{-(s-1)a_\chi} \, d\mu_{L/L(\mathbb{Z})}(l)$$

and the integral $\int_{L/L(\mathbb{Z})} \lambda_\chi(l)^{-(s-1)a_\chi} \, d\mu_{L/L(\mathbb{Z})}(l)$ converges by the choice of s and the arguments in the proof of Theorem 1.1.1. Putting everything together, we have

$$\int_K \left(S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t k \Delta)\right)^s d\mu_K(k) \lesssim_{\varepsilon,F,c,s} \int_{\widetilde{X}} \mathbb{1}_{\mathcal{F}_c(\varepsilon)}\, d\lambda_{\widetilde{X}} \lesssim_{\varepsilon,F,c,s} \int_{\widetilde{X}} \mathbb{1}_{\mathcal{F}_c}\, d\lambda_{\widetilde{X}},$$

as desired. Since the sets $\mathcal{F}_c(\varepsilon)$, for c and ε varying in fixed compact sets, are all contained in a fixed compact subset of $\widetilde{X}$, one sees that the implicit constant only depends on s and the compact set Q, completing the proof. □

5.5 Equidistribution of compact-orbit translates

In this section, we prove an effective double equidistribution result for expanding translates of K-orbits. The proof is based on the one of Ouaggag's result [Oua23, Proposition 4.1], which is derived from the effective equidistribution of unstable horospherical orbits established in [BG21].

In order to state the result, we shall need to introduce certain Sobolev norms on $C_c^\infty(\Omega)$ and $C^\infty(K)$. Each element Z in the Lie algebra $\mathfrak{g}$ of G defines a first order differential operator $\mathcal{D}_Z$ on $C_c^\infty(\Omega)$ by

$$\forall\, \phi \in C_c^\infty(\Omega),\ \forall\, x \in \Omega, \quad \mathcal{D}_Z\phi(x) = \frac{d}{dt}\Big|_{t=0}\phi(\exp(tZ)x).$$

Let D be the dimension of $\mathfrak{g}$ and let $\mathcal{B} = (Z_i)_{1 \leqslant i \leqslant D}$ be a basis of the real vector space $\mathfrak{g}$. Then each monomial

$$\mathcal{D}_Z = \mathcal{D}_{Z_1}^{j_1} \circ \cdots \circ \mathcal{D}_{Z_D}^{j_D} \tag{5.5.1}$$

with $(j_1, \ldots, j_D) \in \mathbb{N}^D$ defines a differential operator of degree $\deg(\mathcal{D}_Z) = j_1 + \cdots + j_D$. For all $r \geqslant 1$ and $\phi \in C_c^\infty(\Omega)$, we define the *degree r Sobolev norm* of ϕ by

$$\mathcal{S}_r(\phi) = \sum_{\deg(\mathcal{D}) \leqslant r} \|\mathcal{D}\phi\|_\infty\,, \tag{5.5.2}$$

where $\mathcal{D}$ ranges over all monomials of elements in $\mathcal{B}$ of degree $\leqslant r$. Similarly, one can define a degree r Sobolev norm on $C^\infty(K)$, which, by abuse of notation, we also denote by $\mathcal{S}_r$.

For every bounded continuous function $\phi : \Omega \to \mathbb{R}$ and time $t \geqslant 0$, we introduce

$$\Delta_\phi(t) = \left| \int_K \phi\big(\widehat{a}(t)k\big)\, d\mu_K(k) - \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right|. \tag{5.5.3}$$

Moreover, for all $t_2 \geqslant t_1 > 0$ we shall write $m(t_1, t_2) = \min\{t_1, t_2 - t_1\}$ and given bounded continuous functions $\phi_1, \phi_2 : \Omega \to \mathbb{R}$, and times $t_2 \geqslant t_1 \geqslant 0$, we introduce

$$\Delta_{\phi_1,\phi_2}(t_1, t_2) = \left| \int_K \phi_1\big(\widehat{a}(t_1)k\big)\phi_2\big(\widehat{a}(t_2)k\big) d\mu_K(k) - \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right|. \tag{5.5.4}$$

Proposition 5.5.1 (Effective single and double equidistribution)**.** *There exist constants $c > 0$, $C > 0$ and an integer $r \geqslant 1$ such that for all $\phi \in C_c^\infty(\Omega)$ and $t \geqslant 0$, we have*

$$\Delta_\phi(t) \;\leqslant\; C\, e^{-c\,t}\mathcal{S}_r(\phi). \tag{5.5.5}$$

and, for all $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and $t_2 \geqslant t_1 \geqslant 0$, we have

$$\Delta_{\phi_1,\phi_2}(t_1, t_2) \;\leqslant\; C\, e^{-c\,m(t_1,t_2)}\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2). \tag{5.5.6}$$

Observe that U^-, the unipotent subgroup opposite to the unipotent radical U of P, is the expanding horospherical subgroup with respect to the one-parameter diagonal subgroup $A = \{\widehat{a}(t) : t \in \mathbb{R}\}$:

$$U^- = \left\{ g \in G : \lim_{t \to -\infty} \widehat{a}(t)\, g\, \widehat{a}(-t) = 1 \right\}.$$

Let μ_{U^-} be a Haar measure on U^-. The idea of proof of Proposition 5.5.1 is to deduce the effective single and double equidistribution of expanding compact-orbit translates from the corresponding effective single and double equidistribution of expanding horospherical translates, as proved by Björklund and Gorodnik (see [BG21, Theorem 1.2]).

Theorem 5.5.2. *There exists a constants $c' > 0$ and an integer $r \geqslant 1$ such that the following hold. Let Q be a compact subset of G. Then, for all $f \in C_c^\infty(U)$, $\phi \in C_c^\infty(\Omega)$, $g \in Q$, and $t \geqslant 0$, we have*

$$\left| \int_{U^-} f(u)\phi\big(\widehat{a}(t_1)u^- g\big)\, d\mu_{U^-}(u^-) - \mu_{U^-}(f)\mu_\Omega(\phi) \right| \lesssim_Q e^{-c'\,t}\mathcal{S}_r(f)\mathcal{S}_r(\phi).$$

and, for all $f \in C_c^\infty(U)$, $\phi_1, \phi_2 \in C_c^\infty(\Omega)$, $g \in Q$, and $t_2 \geqslant t_1 \geqslant 0$, we have

$$\left| \int_{U^-} f(u)\phi_1\big(\widehat{a}(t_1)u^- g\big)\phi_2\big(\widehat{a}(t_2)u^- g\big)\, d\mu_{U^-}(u^-) - \mu_{U^-}(f)\mu_\Omega(\phi_1)\mu_\Omega(\phi_2) \right|$$
$$\lesssim_Q e^{-c'\,m(t_1,t_2)}\mathcal{S}_r(f)\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2). \quad (5.5.7)$$

Proof of Proposition 5.5.1. Let us prove the effective double equidistribution statement in (5.5.6). The proof of the corresponding effective single equidistribution statement in (5.5.5) follows along the same line of argument and we omit the details.

Let $\mathfrak{k}_P$ be the Lie algebra of the stabilizer $K_P = K \cap P$ in K of $x_0 \in X$. Let us define $\mathfrak{s}$ to be the orthogonal complement inside the Lie algebra $\mathfrak{k}$ of K of the Lie algebra $\mathfrak{k}_P$ with respect to the Killing form $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G. In particular, we have $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{k}_P$. Let V be a neighborhood of the origin in $\mathfrak{g}$ such that the exponential map $\exp : \mathfrak{g} \to G$ restricts to a diffeomorphism $\exp|_V : V \to \exp(V)$. Consider the embedded submanifold $S = \exp(V \cap \mathfrak{s})$ of $\exp(V)$. In order to relate the double equidistribution of translated K-orbits to that of translated horospherical orbits, we first construct a local diffeomorphism from S to U^-.

Note that the product map $S \times K_P \to K$ restricts to a diffeomorphism in a neighborhood of the identity, giving a decomposition $k = s(k)k_P(k)$ with $s(k) \in S$ and $k_P(k) \in K_P$ for every $k \in K$ in this neighborhood. We recall that $\mathfrak{u}^-$ and $\mathfrak{p}$ denote the Lie algebras of U^- and P respectively and that the Lie algebra of G decomposes as the direct sum $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{p}$. In particular, every element $g \in G$ close to the identity $1 \in G$ can be uniquely decomposed as $g = u^-(g)p(g)$ with $u^-(g) \in U^-$ and $p(g) \in P$.

We claim that there exists a neighborhood V_S of the identity $1 \in S$ such that the map $h : V_S \to U^-$ given by sending $s \in S$ to $u^-(s)$ is well-defined and defines a diffeomorphism onto a neighborhood of the identity $1 \in U^-$. Observe first that the dimensions of S and U^- agree:

$$\dim S = \dim K - \dim K_P = \dim X = \dim U^-.$$

To show the claim, by the inverse function theorem, it suffices to show that the derivative $D_1 h$ of the map h at $1 \in S$ is injective. This however follows from the fact that $\ker D_1 h$ is contained in $\mathfrak{s} \cap \mathfrak{p} = \{0\}$.

Let μ_{K_P} be the Haar probability measure on K_P. Let us now equip S with a measure μ_S such that, for all integrable functions $f : K \to \mathbb{R}$ supported in a sufficiently small neighborhood of $1 \in K$, we have

$$\int_K f(k)\, d\mu_K(k) = \int_S \int_{K_P} f(sk_P)\, d\mu_S(s) d\mu_{K_P}(k_P).$$

Let μ_{K/K_P} be the pushforward of μ_K along the projection map $K \to K/K_P$. By [Fol15, Theorem 2.51], for all integrable functions $f : K \to \mathbb{R}$ supported in a sufficiently small neighborhood of $1 \in K$, we have

$$\int_K f(k)\, d\mu_K(k) = \int_{K/K_P} \int_{K_P} f(xk_P)\, d\mu_{K/K_P}(x) d\mu_{K_P}(k_P).$$

Since the kernel of the derivative at $1 \in K$ of the projection map $K \to K/K_P$ is $\mathfrak{k}_P$, the restriction to S of this projection map defines a local diffeomorphism at 1. We let μ_S be the pushforward of the restriction of μ_{K/K_P} to a suitable neighborhood along the inverse of this diffeomorphism.

Let $r \geqslant 1$ be as in Theorem 5.5.2. There exist constants $c_1 > 0$ and $c_2 > 0$ such that the following holds. For every small $r_0 > 0$, there exists $N \in \mathbb{N}$ with $N \lesssim r_0^{-c_1}$, non-negative functions $\kappa_i \in C_c^\infty(K)$ with $1 \leqslant i \leqslant N$, all supported in $B_K(r_0)$ and satisfying $\|\kappa_i\|_r \lesssim r_0^{-c_2}$, and elements $k_i \in K$ with $1 \leqslant i \leqslant N$ such that we have a partition of unity: for every $k \in K$, we have $1 = \sum_{i=1}^N \kappa_i(kk_i^{-1})$.

For all $\phi_1, \phi_2 \in C_c^\infty(\Omega)$ and $t_2 \geqslant t_1 \geqslant 0$, we define

$$I_{\phi_1,\phi_2}(t_1, t_2) = \int_K \phi_1\big(\widehat{a}(t_1)k\big)\phi_2\big(\widehat{a}(t_2)k\big)d\mu_K(k).$$

Let $r_0 > 0$ be small, to be fixed later. Using the direct sum decompositions

$$\mathfrak{k} = \mathfrak{k}_P \oplus \mathfrak{s} \quad \text{and} \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-,$$

for every k close the identity in K, we have

$$k = k_P(k)s(k) = k_P(k)p(s(k))u^-(s(k)).$$

Observe that K_P is the centralizer $\mathscr{Z}_K(A)$ in K of A. Putting everything together and letting, for $j = 1, 2$,

$$g(k, t_j) = k_P(k)\widehat{a}(t_j)p(s(k))\widehat{a}(-t_j),$$

we have

$$\begin{aligned} I_{\phi_1,\phi_2}(t_1, t_2) &= \sum_{i=1}^N \int_K \kappa_i(k)\phi_1(\widehat{a}(t_1)kk_i)\phi_2(\widehat{a}(t_2)kk_i)d\mu_K(k) \\ &= \sum_{i=1}^N \int_K \kappa_i(k)\phi_1(g(k,t_1)\widehat{a}(t_1)u^-(s(k))k_i)\phi_2(g(k,t_2)\widehat{a}(t_2)u^-(s(k))k_i)\, d\mu_K(k). \end{aligned}$$

We recall that P is the semi-direct product of the centralizer $\mathscr{Z}_G(A)$ in G of A and the unipotent radical U of P. This unipotent radical is also the contracting horospherical subgroup with respect to the one-parameter diagonal subgroup $A = \{\widehat{a}(t) : t \in \mathbb{R}\}$:

$$U = \left\{ g \in G : \lim_{t \to +\infty} \widehat{a}(t)\, g\, \widehat{a}(-t) = 1 \right\}.$$

In particular, for every $t \geqslant 0$, elements $p \in P$ do not get expanded by the conjugation action of $\widehat{a}(t)$. By Lipschitz continuity of the coordinate maps $k \mapsto k_P(k)$, $k \mapsto p(s(k))$ on $B_K(r_0)$ with $r_0 > 0$ small enough, there exists a constant $C_1 > 0$, independent of t_1 and t_2, such that for every $k \in B_K(r_0)$, we have

$$k_P(k),\ \widehat{a}(t_j)p(s(k))\widehat{a}(-t_j) \in B_G(C_1 r_0).$$

By the Lipschitz continuity of ϕ_1 and ϕ_2, we have

$$\left| I_{\phi_1,\phi_2}(t_1,t_2) - \sum_{i=1}^{N} \int_K \kappa_i(k)\phi_1(\widehat{a}(t_1)u^-(s(k))k_i)\phi_2(\widehat{a}(t_2)u^-(s(k))k_i)\, d\mu_K(k) \right| \\ \lesssim_r\ r_0\, \mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2).$$

Recall that there exists a neighborhood V_S of the identity $1 \in S$ such that the map $h : V_S \to U^-$ given by sending $s \in S$ to $u^-(s)$ is well-defined and defines a diffeomorphism onto a neighborhood of the identity $1 \in U^-$. Hence, denoting by $u^- \mapsto s(u^-)$ the local inverse of this diffeomorphism, there exists a smooth density ρ_0 defined in a neighborhood of $1 \in U^-$ such that for all sufficiently small $r_0 > 0$ and all $f \in C_c(S)$ supported in $B_S(r_0)$, we have

$$\int_S f(s)\, d\mu_S(s) = \int_{U^-} f(s(u^-))\rho_0(u^-)\, d\mu_{U^-}(u^-).$$

Using the local decomposition of the measure μ_K as a product of the measures μ_{K_P} and μ_S, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_K \kappa_i(k)\phi_1(\widehat{a}(t_1)u^-(s(k))k_i)\phi_2(\widehat{a}(t_2)u^-(s(k))k_i)\, d\mu_K(k) \\
&= \sum_{i=1}^{N} \int_{K_P} \int_S \kappa_i(k_P s)\phi_1(\widehat{a}(t_1)u^-(s)k_i)\phi_2(\widehat{a}(t_2)u^-(s)k_i)\, d\mu_S(s) d\mu_{K_P}(k_P) \\
&= \int_{K_P} \sum_{i=1}^{N} \left(\int_{U^-} \kappa_i(k_P s(u^-))\phi_1(\widehat{a}(t_1)u^- k_i)\phi_2(\widehat{a}(t_2)u^- k_i)\rho_0(u^-)\, d\mu_{U^-}(u^-) \right) d\mu_{K_P}(k_P).
\end{aligned} \tag{5.5.8}$$

By Theorem 5.5.2, applied to ϕ_1, ϕ_2 and the functions $f_{k_P,i}$, for $1 \leqslant i \leqslant N$, defined on a neighborhood of $1 \in U^-$ by

$$u^- \mapsto f_{k_P,i}(u^-) = \kappa_i(k_P s(u^-))\rho_0(u^-),$$

there exist constants $c' > 0$, $C' > 0$, independent of $f_{k_P,i}$, ϕ_1, ϕ_2, such that we have

$$\left|\int_{U^-} f_{k_P,i}(u^-)\phi_1(\widehat{a}(t_1)u^-k_i)\phi_2(\widehat{a}(t_2)u^-k_i)\,d\mu_{U^-}(u^-) - \mu_{U^-}(f_{k_P,i})\mu_\Omega(\phi_1)\mu_\Omega(\phi_2)\right|$$
$$\leqslant C'\,e^{-c'\,m(t_1,t_2)}\mathcal{S}_r(f_{k_P,i})\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2).$$

Observing that $\mathcal{S}_r(f_{k_P,i}) \lesssim \mathcal{S}_r(\kappa_i)\mathcal{S}_r(\rho_0)$, that $N \leqslant r_0^{-c_1}$, that $\mathcal{S}_r(\kappa_i) \lesssim r_0^{-c_2}$, that $\mathcal{S}_r(\rho_0|_{B_{U^-}(r_0)}) \lesssim 1$, and that

$$\int_{K_P}\sum_{i=1}^{N}\mu_{U^-}(f_{k_P,i})d\mu_{K_P}(k_P) = \int_{K_P}\int_S\sum_{i=1}^{N}\kappa_i(k_Ps)\,d\mu_S(s)d\mu_{K_P}(k_P) = 1,$$

the integral (5.5.8) is equal to

$$\int_{K_P}\sum_{i=1}^{N}\left(\mu_{U^-}(f_{k_P,i})\mu_\Omega(\phi_1)\mu_\Omega(\phi_2)+O\left(e^{-c'\,m(t_1,t_2)}\,r_0^{-c_2}\,\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2)\right)\right)d\mu_{K_P}(k_P)$$
$$= \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) + O\left(e^{-c'\,m(t_1,t_2)}\,r_0^{-c_1-c_2}\,\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2)\right)$$

Hence, putting everything together, we have

$$I_{\phi_1,\phi_2}(t_1,t_2) = \mu_\Omega(\phi_1)\mu_\Omega(\phi_2) + O\left((e^{-c'\,m(t_1,t_2)}\,r_0^{-c_1-c_2} + r_0)\,\mathcal{S}_r(\phi_1)\mathcal{S}_r(\phi_2)\right).$$

Setting $c = \frac{c'}{1+c_1+c_2}$ and $r_0 = e^{-c\,m(t_1,t_2)}$ now completes the proof of Proposition 5.5.1. □

5.6 Non-escape of mass

We equip the K-orbit $Y = K\Gamma/\Gamma$ of the identity coset $\Gamma \in \Omega$ with the pushforward measure ν_Y, defined via the orbital map $k \mapsto k\Gamma$, of the Haar probability measure μ_K on K. We will be interested in establishing upper bounds for the Siegel transform of bounded compactly supported functions $f : \widetilde{X} \to \mathbb{R}$ when evaluated at points of the translated K-orbit $\widehat{a}(t)Y$. To this end, similarly as in [BG19, Proposition 4.5], we begin by establishing a non-escape of mass property for $\widehat{a}(t)Y$.

We recall that, for every $g \in G$, we defined

$$\lambda_\chi(g) = \min_{\boldsymbol{v} \in \mathbf{V}_\chi(\mathbb{Z}) \smallsetminus \{\mathbf{0}\}} \|g\boldsymbol{v}\|$$

to be the length of the shortest non-zero vector in $g\mathbf{V}_\chi(\mathbb{Z})$. Let us also recall that d stands for the dimension of X and β_χ is the Diophantine exponent of X with respect to χ. For each $0 < \delta < 1$, we define an open cusp neighborhood in Ω by

$$\Omega_\delta = \{g\Gamma \in \Omega : \lambda_\chi(g) < \delta\}.$$

Lemma 5.6.1. *There exists a constant $\kappa > 0$ such that for all $\delta \in (0,1)$ and $t \in [\kappa \ln \delta^{-1}, +\infty)$, we have*

$$\nu_Y\big(\{y \in Y : \lambda_\chi(\widehat{a}(t)y) < \delta\}\big) \lesssim \delta^{\beta_\chi d}.$$

Proof. Let us denote by $\mathbb{1}_{\Omega_\delta^c}$ the indicator function of the complementary subset $\Omega_\delta^c = \Omega \smallsetminus \Omega_\delta$. By Mahler's compactness criterion, the support of $\mathbb{1}_{\Omega_\delta^c}$ is compact. We fix once and for all a non-negative function $\rho_1 \in C_c^\infty(G)$ with $\int_G \rho_1 \, d\mu_G = 1$ and define $\chi_\delta = \rho_1 * \mathbb{1}_{\Omega_\delta^c} : \Omega \to [0, +\infty)$. Since μ_Ω is G-invariant, we have

$$\int_\Omega \chi_\delta \, d\mu_\Omega = \int_\Omega \mathbb{1}_{\Omega_\delta^c} \, d\mu_\Omega = \mu_\Omega(\{\lambda_\chi \geqslant \delta\}).$$

Moreover, by the G-invariance of μ_Ω again, for any differential operator $\mathcal{D}$, we have $\mathcal{D}\chi_\delta = \mathcal{D}(\rho) * \mathbb{1}_{\Omega_\delta^c}$. In particular, we have $\chi_\delta \in C_c^\infty(\Omega)$ and, letting $r \geqslant 1$ be the integer from Proposition 5.5.1, also $\mathcal{S}_r(\chi_\delta) \lesssim \mathcal{S}_r(\rho_1) \lesssim 1$. Moreover, there exists $\xi = \xi(\rho_1) > 1$ such that, for every $g \in \operatorname{supp}(\rho_1)$ and $x \in \Omega$, we have $\lambda_\chi(gx) \leqslant \xi \lambda_\chi(x)$. Therefore, for every $g \in \operatorname{supp}(\rho_1)$, we have

$$\{x \in \Omega : \lambda_\chi(gx) \geqslant \delta\} \subseteq \{x \in \Omega : \lambda_\chi(x) \geqslant \xi^{-1}\delta\}$$

and hence $\chi_\delta \leqslant \mathbb{1}_{\Omega^c_{\xi^{-1}\delta}}$. Thus, for every $t \geqslant 0$, we have

$$\begin{aligned}
\nu_Y(\{y \in Y : \lambda_\chi(\widehat{a}(t)y) \geqslant \xi^{-1}\delta\}) &= \int_Y \mathbb{1}_{\Omega^c_{\xi^{-1}\delta}}(\widehat{a}(t)y) \, d\nu_Y(y) \\
&\geqslant \int_Y \chi_\delta(\widehat{a}(t)y) \, d\nu_Y(y).
\end{aligned}$$

By Proposition 5.5.1, since ν_Y is the pushforward to Y of the Haar probability measure μ_K on K, there exists $c > 0$ such that, for every $t \geqslant 0$, we have

$$\begin{aligned}\int_Y \chi_\delta(\widehat{a}(t)y)\, d\nu_Y(y) &= \int_K \chi_\delta(\widehat{a}(t)k\Gamma)\, d\mu_K(k) \\ &= \int_\Omega \chi_\delta\, d\mu_\Omega + O\left(e^{-ct}\mathcal{S}_r(\chi_\delta)\right) \\ &= \mu_\Omega(\{\lambda_\chi \geqslant \delta\}) + O(e^{-ct}).\end{aligned}$$

By [Sax22, Proposition 3.1.1], we have the measure estimate

$$\mu_\Omega(\Omega_\delta) \asymp \delta^{\beta_\chi d}. \tag{5.6.1}$$

Putting everything together, for every $t \geqslant 1$, this yields

$$\nu_Y(\{y \in Y : \lambda_\chi(\widehat{a}(t)y) \geqslant \xi^{-1}\delta\}) \geqslant \mu_\Omega(\{\lambda_\chi \geqslant \delta\}) + O\left(e^{-ct}\right) = 1 + O\left(\delta^{\beta_\chi d} + e^{-ct}\right).$$

Therefore, since ρ_1 is fixed and ξ only depends on ρ_1, for every $t \geqslant 1$, we have

$$\nu_Y(\{y \in Y : \lambda_\chi(\widehat{a}(t)y) < \delta\}) \lesssim (\xi\delta)^{\beta_\chi d} + e^{-ct} \lesssim \delta^{\beta_\chi d} + e^{-ct}.$$

Letting $\kappa = \frac{\beta_\chi d}{c}$, the claim holds for all $\delta \in (0,1)$ and $t \in [\kappa \ln \delta^{-1}, +\infty)$. □

5.7 Approximation by smooth functions

The Siegel transform of the indicator function of the set $\mathcal{F}_c$ appearing in the tessellation (5.3.4) is neither smooth nor bounded. In order to apply effective equidistribution results, we approximate $S_\chi \mathbb{1}_{\mathcal{F}_c}$ by smooth compactly supported functions. We again fix the integer $r \geqslant 1$ as in Proposition 5.5.1.

Lemma 5.7.1. *For every $\xi > 1$, there exists a family of functions $(D_\delta)_{\delta \in (0,1)}$ in $C_c^\infty(\Omega)$ satisfying*

$$0 \leqslant D_\delta \leqslant 1, \quad D_\delta = 1 \text{ on } \{\lambda_\chi \geqslant \xi\delta\}, \quad D_\delta = 0 \text{ on } \{\lambda_\chi < \xi^{-1}\delta\}, \quad \mathcal{S}_r(D_\delta) \lesssim 1.$$

The proof is essentially analogous to that of [BG19, Lemma 4.11] and we omit the details. We refer to the family $(D_\delta)_{\delta \in (0,1)}$ as a family of smooth cut-off functions on Ω and, fixing once and for all a $\xi > 1$ in the above lemma, we will omit ξ from the notation. For every $\delta \in (0,1)$ and measurable bounded compactly supported function $f : \widetilde{X} \to \mathbb{R}$, we define the *$\delta$-truncated Siegel transform* $S_\chi^{(\delta)} f : \Omega \to \mathbb{R}$ of f by

$$\forall\, g \in G, \quad S_\chi^{(\delta)} f(g) = D_\delta(g)\, S_\chi f(g). \tag{5.7.1}$$

Next, we approximate $\mathbb{1}_{\mathcal{F}_c}$ for c arbitrarily close to 1 by a family of non-negative smooth compactly supported functions.

For every $\varepsilon \in (0,1)$, we recall the definition of the ε-neighborhood $\mathcal{F}_c(\varepsilon)$ of $\mathcal{F}_c$ given by

$$\mathcal{F}_c(\varepsilon) = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, (1+\varepsilon)^{-1} \leqslant \|\boldsymbol{v}^+\| < (1+\varepsilon)e\}.$$

There exists a family $(f_{\varepsilon,c})_{\varepsilon \in (0,1),\, c \in [1/2,3/2]} \subset C_c^\infty(\widetilde{X})$ and $\operatorname{supp}(f_{\varepsilon,c}) \subset \mathcal{F}_c(\varepsilon)$ satisfying the following properties:

$$\forall\, \varepsilon \in (0,1), \quad \mathbb{1}_{\mathcal{F}_c} \leqslant f_{\varepsilon,c} \leqslant 1, \quad \|f_{\varepsilon,c} - \mathbb{1}_{\mathcal{F}_c}\|_{L^1(\tilde{X})} \lesssim \varepsilon, \quad \mathcal{S}_r(f_{\varepsilon,c}) \lesssim \varepsilon^{-r}, \tag{5.7.2}$$

and the implicit constants are uniform in $c \in [1/2, 3/2]$.

Proposition 5.7.2. *There exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,1)$ and $t \in [-\frac{\beta_\chi}{\varepsilon_1} \ln(\beta_\chi \varepsilon), +\infty)$, we have*

$$\int_Y |S_\chi f_{\varepsilon,c}(\widehat{a}(t)y) - S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}(t)y)|\, d\nu_Y(y) \;\lesssim\; \varepsilon.$$

The implicit constant is uniform in $c \in [1/2, 3/2]$, but depends on β_χ and d.

Proof. Since $\operatorname{supp}(f_{\varepsilon,c}) \subset \mathcal{F}_c(\varepsilon)$, we have $S_\chi \mathbb{1}_{\mathcal{F}_c(\varepsilon)} - S_\chi \mathbb{1}_{\mathcal{F}_c} \geqslant S_\chi f_{\varepsilon,c} - S_\chi \mathbb{1}_{\mathcal{F}_c}$. The difference $\mathbb{1}_{\mathcal{F}_c(\varepsilon)} - \mathbb{1}_{\mathcal{F}_c}$ is bounded by the sum $\mathbb{1}_{\mathcal{R}_1(\varepsilon)} + \mathbb{1}_{\mathcal{R}_2(\varepsilon)} + \mathbb{1}_{\mathcal{R}_3(\varepsilon)}$ of indicator functions of the sets

$$\begin{aligned}
\mathcal{R}_1(\varepsilon) &= \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, (1+\varepsilon)^{-1} \leqslant \|\boldsymbol{v}^+\| < 1\},\\
\mathcal{R}_2(\varepsilon) &= \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, e \leqslant \|\boldsymbol{v}^+\| < (1+\varepsilon)e\},\\
\mathcal{R}_3(\varepsilon) &= \{\boldsymbol{v} \in \widetilde{X} : c\, \|\boldsymbol{v}^+\|^{-\beta_\chi} \leqslant \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < (1+\varepsilon)^{1+\beta_\chi}\, c\, \|\boldsymbol{v}^+\|^{-\beta_\chi}, 1 \leqslant \|\boldsymbol{v}^+\| < e\}.
\end{aligned}$$

In particular, we have $S_\chi f_{\varepsilon,c} - S_\chi \mathbb{1}_{\mathcal{F}_c} \leqslant S_\chi \mathbb{1}_{\mathcal{R}_1(\varepsilon)} + S_\chi \mathbb{1}_{\mathcal{R}_2(\varepsilon)} + S_\chi \mathbb{1}_{\mathcal{R}_3(\varepsilon)}$, and it is enough to show that for all $t \geqslant 0$ sufficiently large in terms of ε, we have, for every $i \in \{1,2,3\}$,

$$J_i(t) = \int_Y S_\chi \mathbb{1}_{\mathcal{R}_i(\varepsilon)}(\widehat{a}(t)y)\, d\nu_Y(y) = \sum_{\boldsymbol{v} \in \mathcal{L}_\chi} \int_K \mathbb{1}_{\mathcal{R}_i(\varepsilon)}(\widehat{a}(t)k\boldsymbol{v})\, d\mu_K(k) \lesssim \varepsilon.$$

We start with $J_1(t)$. Using polar coordinates on $\widetilde{X} \smallsetminus \{\mathbf{0}\}$ (see Section 4.4), for every $\boldsymbol{v} \in \mathcal{L}_\chi$ there exist $k_{\boldsymbol{v}} \in K$ and $t(\boldsymbol{v}) \in \mathbb{R}$ such that $\boldsymbol{v} = k_{\boldsymbol{v}}\widehat{a}(t(\boldsymbol{v}))\boldsymbol{e}_\chi$. Taking norms of both sides and recalling that $\widehat{a}(t(\boldsymbol{v}))$ acts through the character χ on $\boldsymbol{e}_\chi$, we have $\boldsymbol{v} = \|\boldsymbol{v}\|k_{\boldsymbol{v}}\boldsymbol{e}_\chi$. Now, the right K-invariance of μ_K gives, for every $t \in \mathbb{R}$,

$$J_1(t) = \sum_{\boldsymbol{v}\in\mathcal{L}_\chi} \int_K \mathbb{1}_{\mathcal{R}_1(\varepsilon)}(\widehat{a}(t)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)\, d\mu_K(k).$$

Let $\boldsymbol{v} \in \mathcal{L}_\chi$ and let us define the two intervals $I_1(\varepsilon) = [(1+\varepsilon)^{-1}, 1)$ and $I_2(\varepsilon) = [0, (1+\varepsilon)^{1+\beta_\chi}c)$. We observe that, for every $k \in K$, we have $\mathbb{1}_{\mathcal{R}_1(\varepsilon)}(\widehat{a}(t)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi) = 1$ if and only if

$$\mathbb{1}_{I_1(\varepsilon)}\left(\|(\widehat{a}(t)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+\|\right)\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{\widehat{a}(t)k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}\,\|(\widehat{a}(t)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+\|^{\beta_\chi}\right) = 1. \quad (5.7.3)$$

For every $\boldsymbol{v} \in \widetilde{X} \smallsetminus \{\mathbf{0}\}$ and $k \in K$, we have $(\widehat{a}(t)\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+ = e^{-\frac{t}{\beta_\chi}}(\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+$ and $\|u^-_{\widehat{a}(t)k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-} = e^t\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}$ (see Equation (5.3.1)). Moreover, we have $\|(\|\boldsymbol{v}\|k\boldsymbol{e}_\chi)^+\| = \|\boldsymbol{v}\|\,|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|$, since $\boldsymbol{e}_\chi$ is unitary. Hence, (5.7.3) holds if and only if

$$\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\|\,|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right)\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}\,(\|\boldsymbol{v}\|\,|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) = 1.$$

By the definition of $\mathcal{R}_1(\varepsilon)$, there exists an absolute constant $\widehat{C} > 1$ such that for every $\boldsymbol{v} \in \mathcal{R}_1(\varepsilon)$, we have $\|u^-_{\boldsymbol{v}}\| < \widehat{C}$. In particular, if $[\mathcal{R}_1(\varepsilon)]$ denotes the corresponding set in X, then $[\mathcal{R}_1(\varepsilon)] \subset B_{\mathfrak{u}^-}(\widehat{C})\,x_0$. This implies that there exists a small constant $\widehat{c} > 0$ such that the region $\mathcal{R}_1(\varepsilon)$ is contained in the set $\mathcal{C} = \{\boldsymbol{v} \in \widetilde{X} \smallsetminus \{\mathbf{0}\} : \|\boldsymbol{v}^+\| \geqslant \widehat{c}\,\|\boldsymbol{v}\|\}$. For every $t \geqslant 0$, the set $\mathcal{C}$ is stable under the action of $\widehat{a}(t)$ and for every $\boldsymbol{v} \in \mathcal{C}$, written as $\boldsymbol{v} = \|\boldsymbol{v}\|k\boldsymbol{e}_\chi$ as above, we have $|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \geqslant \widehat{c}$. Therefore, letting $K(\widehat{c}) = \{k \in K : |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \geqslant \widehat{c}\}$, we have that $J_1(t)$ is given by

$$\sum_{\boldsymbol{v}\in\mathcal{L}_\chi}\int_{K(\widehat{c})}\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right)\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right)d\mu_K(k).$$

Let us write $J_1(t) = B_1(t) + B_2(t)$, where $B_1(t)$ is given by

$$\sum_{\boldsymbol{v}\in\mathcal{L}_\chi}\int_{K((1+\varepsilon)^{-1})}\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right)\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right)d\mu_K(k).$$

and $B_2(t) = J_1(t) - B_1(t)$. Consequently, for every $k \in K$ satisfying that $|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \geqslant (1+\varepsilon)^{-1}$, if $\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|\right) = 1$, then

$$(1+\varepsilon)^{-1}e^{\frac{t}{\beta_\chi}} \leqslant \|\boldsymbol{v}\| \leqslant (1+\varepsilon)e^{\frac{t}{\beta_\chi}}$$

Hence, if also $\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) = 1$, then

$$\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-} \lesssim e^{-t}.$$

The latter implies that we have $d(x_0, kx_0) \lesssim e^{-t}$. Together this gives,

$$\begin{aligned} B_1(t) &\lesssim \sum_{\boldsymbol{v}\in\mathcal{L}_\chi,\, \|\boldsymbol{v}\| = e^{\frac{t}{\beta_\chi}} + O(\varepsilon e^{\frac{t}{\beta_\chi}})} \int_X \mathbb{1}_{B_X(e^{-t})}(x)\, d\sigma_X(x) \\ &\lesssim \sum_{\boldsymbol{v}\in\mathcal{L}_\chi,\, \|\boldsymbol{v}\| = e^{\frac{t}{\beta_\chi}} + O(\varepsilon e^{\frac{t}{\beta_\chi}})} e^{-dt}. \end{aligned}$$

Using Theorem 1.3.2, there exist constants $\varkappa_1 > 0$ and $\varepsilon_1 > 0$ such that

$$\#\{\boldsymbol{v} \in \mathcal{L}_\chi : \|\boldsymbol{v}\| \leqslant T\} = \varkappa_1\, T^{\beta_\chi d}(1 + O(T^{-\varepsilon_1})),$$

and consequently

$$\#\{\boldsymbol{v} \in \mathcal{L}_\chi : (1+\varepsilon)^{-1}e^{\frac{t}{\beta_\chi}} \leqslant \|\boldsymbol{v}\| \leqslant (1+\varepsilon)e^{\frac{t}{\beta_\chi}}\} = 2\,\beta_\chi\, d\, \varkappa_1\, \varepsilon\, e^{dt} + O(\varepsilon^2 e^{dt} + e^{-\frac{\varepsilon_1 t}{\beta_\chi}}\, e^{dt}).$$

Hence we get $B_1(t) \lesssim \beta_\chi\, \varepsilon$ for all $t \geqslant -\frac{\beta_\chi}{\varepsilon_1}\ln(\beta_\chi\, \varepsilon)$.

Let us now bound the term $B_2(t)$, where we integrate over all $k \in K$ such that $\widehat{c} \leqslant |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \leqslant (1+\varepsilon)^{-1}$. The vector $(k\boldsymbol{e}_\chi)^\perp = k\boldsymbol{e}_\chi - \langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle$ satisfies

$$\|u_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-} \asymp d(kx_0, x_0) \asymp \|(k\boldsymbol{e}_\chi)^\perp\| \geqslant 1 - |\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle| \gtrsim \varepsilon.$$

In particular, there exists $\delta_0 > 0$ such that $\|u_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}|\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|^{\beta_\chi} \geqslant \delta_0\varepsilon$ is bounded away from zero. Thus, if $\mathbb{1}_{I_2(\varepsilon)}\left(\|u^-_{k\boldsymbol{e}_\chi}\|_{\mathfrak{u}^-}(\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi, \boldsymbol{e}_\chi\rangle|)^{\beta_\chi}\right) = 1$, then $\mathbb{1}_{I_2(\varepsilon)}\left(\delta_0\varepsilon\|\boldsymbol{v}\|^{\beta_\chi}\right) = 1$ and hence

$$\|\boldsymbol{v}\|^{\beta_\chi} < (1+\varepsilon)^{1+\beta_\chi}c(\delta_0\varepsilon)^{-1}. \tag{5.7.4}$$

Therefore, for every $t \geqslant 0$, the non-negative term $B_2(t)$ is bounded from above by

$$\sum_{\substack{\boldsymbol{v}\in\mathcal{L}_\chi \\ \|\boldsymbol{v}\|^{\beta_\chi}<(1+\varepsilon)^{1+\beta_\chi}c(\delta_0\varepsilon)^{-1}}} \int_{K((1+\varepsilon)^{-1})\smallsetminus K(\widehat{c})} \mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi,\boldsymbol{e}_\chi\rangle|\right)\,d\mu_K(k). \tag{5.7.5}$$

We claim that, for all $t \geqslant \ln\left((1+\varepsilon)^{1+\beta_\chi}c(\delta_0\varepsilon)^{-1}\right)$, the expression (5.7.5) is zero. In fact, by (5.7.4) and since $\widehat{c} \leqslant |\langle k\boldsymbol{e}_\chi,\boldsymbol{e}_\chi\rangle| \leqslant (1+\varepsilon)^{-1}$, we have

$$e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi,\boldsymbol{e}_\chi\rangle| \leqslant e^{-\frac{t}{\beta_\chi}}\left((1+\varepsilon)^{1+\beta_\chi}c(\delta_0\varepsilon)^{-1}\right)^{\frac{1}{\beta_\chi}}(1+\varepsilon)^{-1}$$

and, for all $t \geqslant \ln\left((1+\varepsilon)^{1+\beta_\chi}c(\delta_0\varepsilon)^{-1}\right)$, the right-hand side is smaller than $(1+\varepsilon)^{-1}$ (and hence $\mathbb{1}_{I_1(\varepsilon)}\left(e^{-\frac{t}{\beta_\chi}}\|\boldsymbol{v}\||\langle k\boldsymbol{e}_\chi,\boldsymbol{e}_\chi\rangle|\right)=0$).

The calculations for $J_2(t)$ and $J_3(t)$ are essentially analogous to that for $J_1(t)$ and we omit the details. □

5.8 Proof of Theorem 1.2.2

We first generalize [Har98, Lemma 1.4], a useful tool in the theory of metric Diophantine approximation for deriving effective counting statements from an L^2-bound, to obtain such effective statements from an L^p-bound for arbitrary $p \in (1,2]$. The idea of the proof goes back to the work of H. Weyl [Wey16] on the equidistribution of numbers modulo one. Moreover, W. Schmidt strengthened [Har98, Lemma 1.4] in [Sch60b], obtaining the optimal error term, though we shall content ourselves with the following weaker version.

Lemma 5.8.1. *Let $(\widehat{Y},\nu_{\widehat{Y}})$ be a probability space and let $(\phi_{i,\ell}:\widehat{Y}\to\mathbb{R})_{i,\ell\in\mathbb{N}^*}$ be a family of non-negative random variables. Let $C_1>1$ be a constant and $(\overline{\phi}_{i,\ell})_{i,\ell\in\mathbb{N}^*}$ and $(\overline{\phi}_i)_{i\in\mathbb{N}^*}$ be families of real numbers satisfying, for all $i,\ell\in\mathbb{N}^*$, $0\leqslant\overline{\phi}_{i,\ell}\leqslant\overline{\phi}_i\leqslant C_1$, and put $Z_{i,\ell}=\phi_{i,\ell}-\overline{\phi}_{i,\ell}$. Assume that $\sum_{i=1}^{\infty}\overline{\phi}_i=+\infty$ and that for some $p\in(1,2]$ and $C_2>0$, we have*

$$\forall\, N\in\mathbb{N}^*, \forall\,\ell\in\mathbb{N}^*, \quad \int_{\widehat{Y}}\Big|\sum_{i=1}^{N}Z_{i,\ell}(y)\Big|^p\,d\nu_{\widehat{Y}}(y)\leqslant C_2\sum_{i=1}^{N}\overline{\phi}_i. \tag{5.8.1}$$

Let $\varepsilon > 0$ and let $(\ell_N)_{N\geqslant 1}$ be a sequence of positive integers. Then there is a constant $C_3 > 0$ such that almost surely

$$\forall\, N \in \mathbb{N}^*, \quad \left|\sum_{i=1}^{N} Z_{i,\ell_N}\right| \leqslant C_3 \left(\sum_{i=1}^{N} \overline{\phi}_i\right)^{\frac{2}{p+1}+\varepsilon} \tag{5.8.2}$$

Proof. Let $\varepsilon > 0$ and let $(\ell_N)_{N\geqslant 1}$ be a sequence of positive integers. For every $N \in \mathbb{N}^*$ and $y \in \widehat{Y}$, we define

$$\begin{aligned}
\Psi(N, y) &= \sum_{i=1}^{N} \phi_{i,\ell_N}(y) \\
\Psi(N) &= \sum_{i=1}^{N} \overline{\phi}_{i,\ell_N} \\
\Phi(N) &= \sum_{i=1}^{N} \overline{\phi}_i \\
E(N, y) &= \Psi(N, y) - \Psi(N) = \sum_{i=1}^{N} Z_{i,\ell_N}(y).
\end{aligned}$$

Let $x > 1$. For every $k \in \mathbb{N}^*$, let $N_k \in \mathbb{N}^*$ be the smallest integer with

$$\Phi(N_k) > k^{px-1}. \tag{5.8.3}$$

We remark that the sequence $(N_k)_{k\in\mathbb{N}^*}$ is increasing. Next, for every $k \in \mathbb{N}^*$, we define the subset A_k of $\widehat{Y}$ by

$$A_k = \{y \in \widehat{Y} : |E(N_k, y)| > k^{x+\varepsilon}\}.$$

By Chebyshev's inequality and our assumption (5.8.1), we have

$$\begin{aligned}
\nu_{\widehat{Y}}(A_k) &\leqslant \frac{1}{k^{px+p\varepsilon}} \int_{\widehat{Y}} |E(N_k, y)|^p \, d\nu_Y(y) \\
&\leqslant \frac{C_2 k^{px-1} + C_1}{k^{px+p\varepsilon}} \\
&\lesssim k^{-1-p\varepsilon}.
\end{aligned}$$

Hence $\sum_{k=1}^{\infty} \nu_{\widehat{Y}}(A_k)$ converges. By the Borel-Cantelli lemma, we have, for almost every $y \in Y$, that there exists an integer $k(y) \in \mathbb{N}^*$ such that for all $k \geqslant k(y)$, we have

$$|E(N_k, y)| \leqslant k^{x+\varepsilon}$$

and also, by (5.8.3),

$$k^{x+\varepsilon} \lesssim_x (k-1)^{x+\varepsilon} \lesssim \Phi(N_{k-1})^{\frac{x+\varepsilon}{px-1}}.$$

Now, for an arbitrary $N \in \mathbb{N}^*$ there exists $k \in \mathbb{N}^*$ such that $N_{k-1} \leqslant N < N_k$ and

$$\Psi(N_{k-1}, y) \leqslant \Psi(N, y) \leqslant \Psi(N_k, y).$$

Thus, for almost every $y \in Y$, there exists an integer $k(y) \in \mathbb{N}^*$ such that for all $k \geqslant k(y)$, we have

$$\Psi(N, y) = \Psi(N) + O\left(\Psi(N_k) - \Psi(N_{k-1}) + \Phi(N_{k-1})^{\frac{x+\varepsilon}{px-1}}\right), \tag{5.8.4}$$

Next, we note that, for all $k \in \mathbb{N}^*$, we have

$$\Psi(N_k) - \Psi(N_{k-1}) \leqslant \Phi(N_k) - \Phi(N_{k-1}) \lesssim k^{px-2} \lesssim \Phi(N_{k-1})^{\frac{px-2}{px-1}}. \tag{5.8.5}$$

We put $x = \frac{2+\varepsilon}{p-1}$, so that the exponents in (5.8.4) and (5.8.5) match. Plugging this back into $\frac{x+\varepsilon}{px-1}$, we find that

$$\frac{\frac{2+\varepsilon}{p-1} + \varepsilon}{p\frac{2+\varepsilon}{p-1} - 1} = \frac{2 + \varepsilon(2-p)}{p + 1 + p\varepsilon} \leqslant \frac{2 + \varepsilon(2-p)}{p+1} \leqslant \frac{2}{p+1} + \varepsilon.$$

The proof of Lemma 5.8.1 is complete. $\square$

5.8.1 Upper bound estimate

We recall that, for all $\ell \in \mathbb{N}^*$, we defined

$$\widehat{c}_\ell = \left(1 + C_0\, \ell^{-\beta_X}\right)^{-(1+\beta_X)} \in (0,1)$$

and we let, for every $c \in [1/2, 3/2]$,

$$\mathcal{F}_c = \{\boldsymbol{v} \in \widetilde{X} : \|u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} < c\|\boldsymbol{v}^+\|^{-\beta_X}, 1 \leqslant \|\boldsymbol{v}^+\| < e\}.$$

We also defined, for every $i \in \mathbb{N}$, the times $t_i = \beta_\chi i$ and recall from Section 5.3 that there exists a constant $C_0 > 0$ such that for every $T' \geqslant 1$ and $x \in X$, we have the following lower and upper bounds on the lattice point counting function $\#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{\beta_\chi}(e^{T'})\right)$: for all large enough $\ell \in \mathbb{N}^*$, we have

$$\sum_{i=0}^{\lfloor T'+\ln \widehat{c}_\ell\rfloor -1} S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell}}(\widehat{a}(t_i)k_x^{-1}\Gamma) - C_0\,\ell^{\beta_\chi d} \leqslant \#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_{\beta_\chi}(e^{T'})\right)$$
$$\leqslant \sum_{i=0}^{\lceil T'-\ln \widehat{c}_\ell\rceil -1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_\ell^{-1}}}(\widehat{a}(t_i)k_x^{-1}\Gamma) + C_0\,\ell^{\beta_\chi d}. \tag{5.8.6}$$

The proof of Theorem 1.2.2 consists of effectively estimating the left- and right-hand sides of (5.8.6) and we start with the latter.

Proposition 5.8.2. *There exist $\delta' \in (0,1)$ and $\varepsilon \in (0,1)$ such that the following holds. Let $\varkappa_1 = \lambda_{\widetilde{X}}(\mathcal{F})$ be the volume of $\mathcal{F} = \mathcal{F}_1$ and, for every $N \in \mathbb{N}^*$, let $\ell_N = \lfloor N^{\delta'}\rfloor$. Then for almost every $x \in X$ and for all large enough $N \in \mathbb{N}^*$, we have*

$$\sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_{\widehat{c}_{\ell_N}^{-1}}}(\widehat{a}(t_i)k_x^{-1}\Gamma) + C_0\,\ell_N^{\beta_\chi d} = \varkappa_1\,N + O_x(N^{1-\varepsilon}). \tag{5.8.7}$$

Using the inequality (5.8.6), by Lemma 4.7.1 and Proposition 5.8.2, this prove the desired upper bound in (1.2.2). The lower bound in (1.2.2) is shown analogously and we omit the details.

Proof of Proposition 5.8.2. We recall that Y denotes the K-orbit of the identity coset $\Gamma \in \Omega$ and ν_Y is the pushforward to Y under the orbit map $k \mapsto k\Gamma$ of the Haar probability measure μ_K on K. We will apply Lemma 5.8.1 with $(\widehat{Y}, \nu_{\widehat{Y}}) = (Y, \nu_Y)$. Let δ' and $(\ell_N)_{N\in\mathbb{N}^*}$ be as in the statement of Proposition 5.8.2. Let us define, for all $i \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, the function $\phi_{i,\ell} : Y \to \mathbb{R}$ and the number $\overline{\phi}_{i,\ell}$ by

$$\forall\, y \in Y, \quad \phi_{i,\ell}(y) = S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}}(\widehat{a}(t_i)y), \quad \overline{\phi}_{i,\ell} = \int_\Omega S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}}\,d\mu_\Omega.$$

Moreover, we define the function $Z_{i,\ell} : Y \to \mathbb{R}$ by $Z_{i,\ell} = \phi_{i,\ell} - \overline{\phi}_{i,\ell}$. Let us also put $\overline{\phi}_i = \overline{\phi}_{i,1}$ (so that, $i \in \mathbb{N}^*$ and $\ell \in \mathbb{N}$, $\overline{\phi}_i \geqslant \overline{\phi}_{i,\ell}$). By Lemma 5.8.1, we need to show that there exists $p \in (1,2]$, such that

$$\forall\, N \in \mathbb{N}^*, \forall\, \ell \in \mathbb{N}^*, \quad \int_Y \left|\sum_{i=1}^N Z_{i,\ell}(y)\right|^p d\nu_Y(y) \lesssim N. \tag{5.8.8}$$

Fix $\ell \in \mathbb{N}^*$ and $\varepsilon_0 \in (0,1]$ such that the Siegel transform S_χ maps $C_c(\widetilde{X})$ into $L^{1+\varepsilon_0}(\Omega)$ (see Theorem 1.1.1). Let us simply write $\mathbb{1}_\ell$ for $\mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}}$.

First, we approximate the Siegel transform $S_\chi \mathbb{1}_\ell$ by the δ-truncated Siegel transform $S_\chi^{(\delta)}\mathbb{1}_\ell = D_\delta\, S_\chi \mathbb{1}_\ell$ (with $(D_\delta)_{\delta\in(0,1)}$ as in Lemma 5.7.1) with $\delta \in (0,1)$ to be determined later. That is, we would like to give an upper bound in terms of the truncation parameter δ for

$$\Big\|\Big(S_\chi \mathbb{1}_\ell \circ \widehat{a}(t) - \int_\Omega S_\chi \mathbb{1}_\ell\Big)\, d\mu_\Omega - \Big(S_\chi^{(\delta)}\mathbb{1}_\ell \circ \widehat{a}(t) - \int_\Omega S_\chi^{(\delta)}\mathbb{1}_\ell \, d\mu_\Omega\Big)\Big\|_{L^p(Y)}.$$

Note that, using Theorem 1.1.1 and Hölder's inequality with $p' = 1+\varepsilon_0 > 1$ and $q' = (1-\frac{1}{p'})^{-1}$, we have

$$\begin{aligned}\int_\Omega \big|S_\chi \mathbb{1}_\ell - S_\chi^{(\delta)}\mathbb{1}_\ell\big|\, d\mu_\Omega &= \int_\Omega \left|(S_\chi \mathbb{1}_\ell)\,(1-D_\delta)\right|\, d\mu_\Omega \\ &\leqslant \Big(\int_\Omega (S_\chi \mathbb{1}_\ell)^{p'}\, d\mu_\Omega\Big)^{\frac{1}{p'}} \mu_\Omega\left(\{\lambda_\chi \leqslant \xi^{-1}\,\delta\}\right)^{\frac{1}{q'}} \\ &\lesssim_{\operatorname{supp}(\mathbb{1}_\ell),\, p'} \;\delta^{\beta_\chi d(1-\frac{1}{1+\varepsilon_0})}.\end{aligned}$$

Using Hölder's inequality again with $p \in (1, 1+\varepsilon_0/2)$, $s = 1+\varepsilon_0/2$, and $q = (\frac{1}{p}-\frac{1}{s})^{-1}$, and Lemma 5.6.1 with $t \geqslant \kappa \ln(\delta^{-1})$, we thus have

$$\begin{aligned}&\Big\|\Big(S_\chi \mathbb{1}_\ell \circ \widehat{a}(t) - \int_\Omega S_\chi \mathbb{1}_\ell\, d\mu_\Omega\Big) - \Big(S_\chi^{(\delta)}\mathbb{1}_\ell \circ \widehat{a}(t) - \int_\Omega S_\chi^{(\delta)}\mathbb{1}_\ell \, d\mu_\Omega\Big)\Big\|_{L^p(Y)} \\ &\qquad \leqslant \big\|S_\chi \mathbb{1}_\ell \circ \widehat{a}(t) - S_\chi^{(\delta)}\mathbb{1}_\ell \circ \widehat{a}(t)\big\|_{L^p(Y)} + \int_\Omega \big|S_\chi \mathbb{1}_\ell - S_\chi^{(\delta)}\mathbb{1}_\ell\big|\, d\mu_\Omega \\ &\qquad \lesssim \|S_\chi \mathbb{1}_\ell \circ \widehat{a}(t)\|_{L_{\nu_Y}^s(Y)}\ \nu_Y\big(\{y \in Y : \lambda_\chi(\widehat{a}(t)y) < \delta\}\big)^{1/q} + \delta^{\beta_\chi d(1-\frac{1}{1+\varepsilon_0})} \\ &\qquad \lesssim \delta^{\beta_\chi d(\frac{1}{p}-\frac{1}{1+\varepsilon_0/2})}. \end{aligned} \tag{5.8.9}$$

Next, we approximate the δ-truncated Siegel transform $S_\chi^{(\delta)}\mathbb{1}_\ell$ by the Siegel transform of the smooth compactly supported approximation function $f_{\varepsilon,\ell} = f_{\varepsilon,c_\ell}$ with $\varepsilon \in (0,1)$, as constructed in (5.7.2). That is, we would like to give an upper bound in terms of the truncation parameter δ and in terms of the approximation parameter ε for

$$\Big\|\Big(S_\chi^{(\delta)}\mathbb{1}_\ell \circ \widehat{a}(t) - \int_\Omega S_\chi^{(\delta)}\mathbb{1}_\ell\Big)\, d\mu_\Omega - \Big(S_\chi^{(\delta)} f_{\varepsilon,\ell} \circ \widehat{a}(t) - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell}\, d\mu_\Omega\Big)\Big\|_{L^p(Y)}.$$

By the mean value formula in Theorem 1.1.1 and by the approximation properties of $f_{\varepsilon,\ell}$ (see Equation (5.7.2)), we have

$$\int_{\Omega} \left|S_\chi^{(\delta)} f_{\varepsilon,\ell} - S_\chi^{(\delta)} \mathbb{1}_\ell\right| \, d\mu_\Omega \leqslant \int_{\Omega} \left|S_\chi f_{\varepsilon,\ell} - S_\chi \mathbb{1}_\ell\right| \, d\mu_\Omega = \int_{\widetilde{X}} (f_{\varepsilon,\ell} - \mathbb{1}_\ell) \, d\lambda_{\widetilde{X}} \lesssim \varepsilon.$$

By the estimate in (3.3.3) applied with $\mathbb{1}_\ell - f_{\varepsilon,\ell}$, we have

$$\forall\, g \in G, \quad |S_\chi(\mathbb{1}_\ell - f_{\varepsilon,\ell})(g)| \lesssim_{\operatorname{supp}(\mathbb{1}_\ell - f_{\varepsilon,\ell})} \lambda_\chi(g\Gamma)^{-\beta_\chi d}.$$

This together with the fact that $\operatorname{supp}(D_\delta) \subseteq \{x \in \Omega : \lambda_\chi(x) \geqslant \xi^{-1}\delta\}$, gives

$$\left\| S_\chi^{(\delta)}(\mathbb{1}_\ell - f_{\varepsilon,\ell}) \circ \widehat{a}(t) \right\|_{L^\infty(Y)}^{\frac{p-1}{p}} \lesssim \delta^{-\beta_\chi d \frac{p-1}{p}}.$$

Putting everything together, by Proposition 5.7.2, there exists $\varepsilon_1 > 0$ such that for all $t \in [-\frac{\beta_\chi}{\varepsilon_1} \ln(\beta_\chi \varepsilon), +\infty)$, we have

$$\begin{aligned}
&\left\| \left(S_\chi^{(\delta)} \mathbb{1}_\ell \circ \widehat{a}(t) - \int_\Omega S_\chi^{(\delta)} \mathbb{1}_\ell \, d\mu_\Omega \right) - \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} \circ \widehat{a}(t) - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \, d\mu_\Omega \right) \right\|_{L^p(Y)} \\
&\leqslant \left\| S_\chi^{(\delta)} \mathbb{1}_\ell \circ \widehat{a}(t) - S_\chi^{(\delta)} f_{\varepsilon,\ell} \circ \widehat{a}(t) \right\|_{L^p(Y)} + \int_\Omega \left| S_\chi^{(\delta)} \mathbb{1}_\ell - S_\chi^{(\delta)} f_{\varepsilon,\ell} \right| \, d\mu_\Omega \\
&\leqslant \left\| (S_\chi^{(\delta)}(\mathbb{1}_\ell - f_{\varepsilon,\ell})) \circ \widehat{a}(t) \right\|_{L^\infty(Y)}^{\frac{p-1}{p}} \left\| (S_\chi^{(\delta)}(\mathbb{1}_\ell - f_{\varepsilon,\ell}) \circ \widehat{a}(t) \right\|_{L^1(Y)}^{\frac{1}{p}} + \varepsilon \\
&\lesssim \delta^{-\beta_\chi d \frac{p-1}{p}} \varepsilon^{\frac{1}{p}} + \varepsilon \, \lesssim \, \delta^{-\beta_\chi d (1 - \frac{1}{p})} \varepsilon^{\frac{1}{p}}.
\end{aligned} \tag{5.8.10}$$

Next, using the effective equidistribution of compact-orbit translates (see Proposition 5.5.1), we would like to give an upper bound in terms of the rate of equidistribution on

$$\int_Y \left| \sum_{i=1}^N \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \, d\mu_\Omega \right) \right|^p d\nu_Y = \left\| \sum_{t=1}^N \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \, d\mu_\Omega \right) \circ \widehat{a}(t) \right\|_{L^p(Y)}^p$$

Note that, for every $f \in C_c^\infty(\widetilde{X})$, we have $S_\chi^{(\delta)} f \in C_c^\infty(\Omega)$. Let us now show that, for every $f \in C_c^\infty(\widetilde{X})$, we have

$$\forall\, r \in \mathbb{N}^*, \quad \mathcal{S}_r(S_\chi^{(\delta)} f) \lesssim_{\operatorname{supp}(f)} \delta^{-\beta_\chi d} \mathcal{S}_r(f). \tag{5.8.11}$$

First, for every $f \in C_c^\infty(\widetilde{X})$ and every differential operator $\mathcal{D}$ as in (6.4.1), we note that $\mathcal{D}(S_\chi f) = S_\chi(\mathcal{D} f)$. Then, by the point-wise upper bound estimate (3.3.3) for the Siegel transform applied with $\mathcal{D} f$, we have

$$\forall\, g \in G, \quad |S_\chi(\mathcal{D} f)(g)| \lesssim_{\operatorname{supp}(f)} \mathcal{S}_r(f) \, \lambda_\chi(g)^{-\beta_\chi d}.$$

Since $\operatorname{supp}(D_\delta) \subseteq \{x \in \Omega : \lambda_\chi(x) \geqslant \xi^{-1}\delta\}$ and $\mathcal{S}_r(D_\delta) \lesssim 1$, we deduce (5.8.11), as desired.

We recall that for every $i \in \mathbb{N}$, we defined $t_i = \beta_\chi i$. For every $i, j \geqslant 0$ we shall write $m(t_i, t_j) = \min\{t_i, t_j, |t_i - t_j|\}$. By Proposition 5.5.1, there are constants $c > 0$ and $C > 0$ such that, using also the estimate for the Sobolev norm (5.8.11) and the inequality $\mathcal{S}_r(f_{\varepsilon,\ell}) \lesssim \varepsilon^{-r}$ (see (5.7.2)), we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{N} \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \leqslant \left\| \sum_{i=1}^{N} \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \right\|_{L_{\nu_Y}^2(Y)} \\
&= \left(\sum_{i,j=1}^{N} \int_Y \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_j)\, d\nu_Y \right)^{1/2} \\
&\lesssim \left(\sum_{i,j=1}^{N} C\, \mathcal{S}_r(S_\chi^{(\delta)} f_{\varepsilon,\ell})^2 \, e^{-c\, m(t_i,t_j)} \right)^{1/2} \\
&\lesssim \left(\sum_{i,j=1}^{N} C\, \delta^{-2\beta_\chi d} \, \mathcal{S}_r(f_{\varepsilon,\ell}))^2 \, e^{-c\, m(t_i,t_j)} \right)^{1/2} \\
&\lesssim \delta^{-\beta_\chi d}\, \varepsilon^{-r} \left(\sum_{i,j=1}^{N} e^{-c\, m(t_i,t_j)} \right)^{1/2}.
\end{aligned}$$

Next, we would like to show that

$$\left(\sum_{i,j=1}^{N} e^{-c\, m(t_i,t_j)} \right)^{1/2} \lesssim N^{1/2}. \tag{5.8.12}$$

We have

$$\sum_{\substack{i,j=1 \\ t_1 \leqslant \min\{t_j, |t_i - t_j|\}}}^{N} e^{-c\, m(t_i,t_j)} \leqslant \sum_{i,j=1}^{N} e^{-c\, t_i} = N \sum_{i=1}^{N} e^{-c\, t_i} \lesssim N.$$

By symmetry, we also have $\sum_{\substack{i,j=1 \\ t_j \leqslant \min\{t_i, |t_i - t_j|\}}}^{N} e^{-c\, m(t_i,t_j)} \lesssim N$, and, by letting $n = i - j$ and noting that $t_i - t_j = t_{i-j}$ if $i, j \geqslant 0$, we have

$$\sum_{\substack{i,j=1 \\ |t_i - t_j| \leqslant \min\{t_i, t_j\}}}^{N} e^{-c\, m(t_i,t_j)} \leqslant \sum_{n=-N}^{N} \sum_{\substack{i=1 \\ i \in [1+n, N+n]}}^{N} e^{-c\, |t_n|} \leqslant \sum_{n=-N}^{N} (N - |n|) e^{-c\, |t_{|n|}|} \lesssim N.$$

as desired. Hence, putting everything together, we have shown that

$$\left\| \sum_{i=1}^{N} \left(S_\chi^{(\delta)} f_{\varepsilon,\ell} - \int_\Omega S_\chi^{(\delta)} f_{\varepsilon,\ell} \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \lesssim \delta^{-\beta_\chi d} \, \varepsilon^{-r} \, N^{1/2}. \tag{5.8.13}$$

Let κ be as in Proposition 5.6.1 and ε_1 as in Proposition 5.7.2. Later we will choose $\lambda_1, \lambda_2 > 0$ and let $\delta = N^{-\lambda_1}$ and $\varepsilon = N^{-\lambda_2}$. Let us note that for every such choices of λ_1, λ_2 and for all large $N \in \mathbb{N}^*$, we have

$$\frac{|\{i \in \{1, \ldots, N\} : t_i \geqslant \max\{\kappa \lambda_1 \ln N, \frac{\beta_\chi}{\varepsilon_1} \gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)\}\}}{N} = O\left(\frac{\ln N}{N}\right),$$

Hence the proportion of indices $i \in \{1, \ldots, N\}$ for which Lemma 5.6.1 and Proposition 5.7.2 do not apply to the time t_i is asymptotically negligible. Since we are free to choose $\lambda_1, \lambda_2 > 0$, let us determine them by a heuristic argument. So suppose for a moment that Propositions 5.6.1 and 5.7.2 hold for all $t \geqslant 0$. Then using Minkowski's inequality and combining (5.8.9), (5.8.10), and (5.8.13), we would have

$$\begin{aligned} &\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \, d\mu_\Omega \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \\ &\qquad \lesssim N(\delta^{\beta_\chi d(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})} + \delta^{-\beta_\chi d(1-\frac{1}{p})} \varepsilon^{\frac{1}{p}}) + N^{1/2} \delta^{-\beta_\chi d} \varepsilon^{-r}. \end{aligned} \tag{5.8.14}$$

By setting

$$\delta = N^{-\frac{1}{2\beta_\chi d(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}} \quad \text{and} \quad \varepsilon = \delta^{\beta_\chi dp(1 - \frac{1}{1+\varepsilon_0/2})} \tag{5.8.15}$$

the exponents of the three terms on the right-hand side in (5.8.14) match. Hence we let

$$\lambda_1 = \frac{1}{2\beta_\chi d(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}$$

and

$$\lambda_2 = \frac{p(1 - \frac{1}{1+\varepsilon_0/2})}{2(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}.$$

Now having fixed λ_1 and λ_2, let us show that (up to a multiplicative constant) (5.8.14) holds true. We recall that, by Lemma 5.4.1, there exists an absolute constant $C \geqslant 1$ such that, for every $c \in [1/2, 2/3]$, we have

$$\sup_{t \geqslant 0} \int_Y \left| S_\chi \mathbb{1}_{\mathcal{F}_c}(\widehat{a}_t y) \right|^p d\nu_Y(y) \leqslant C.$$

Applying this estimate in the case where $t \in \mathbb{R}_+$ satisfies

$$t < \max\{\kappa\lambda_1 \ln N, \tfrac{\beta_\chi}{\varepsilon_1}\gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)\},$$

using Minkowski's inequality and combining (5.8.9), (5.8.10), and (5.8.13), we have

$$\begin{aligned}
&\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \lesssim \left(\sum_{\substack{i=1 \\ t_i \geqslant \kappa\lambda_1 \ln(N)}}^{N} \delta^{\beta_\chi d(1-\frac{1}{p})} \right) + \left(\sum_{\substack{i=1 \\ t_i \leqslant \kappa\lambda_1 \ln(N)}}^{N} C \right) \\
&+ \left(\sum_{\substack{i=1 \\ t_i \geqslant \frac{\beta_\chi}{\varepsilon_1}\gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)}}^{N} \delta^{\beta_\chi d(1-\frac{1}{p})} \right) + \left(\sum_{\substack{i=1 \\ t_i < \frac{\beta_\chi}{\varepsilon_1}\gamma_2 \ln(\beta_\chi^{1/\gamma_2} N)}}^{N} C \right) + \delta^{-\beta_\chi d} \varepsilon^{-r} N^{1/2} \\
&\lesssim N(\delta^{\beta_\chi d(1-\frac{1}{p})} + \delta^{-\beta_\chi d(1-\frac{1}{p})} \varepsilon^{\frac{1}{p}}) + N^{1/2} \delta^{-\beta_\chi d} \varepsilon^{-r}.
\end{aligned}$$

Plugging in the formulas for δ and ε, we have

$$\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)} \lesssim N^{1 - \frac{(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})}{2(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}}. \tag{5.8.16}$$

Consider the function $h : (1, 1 + \varepsilon_0/2) \to \mathbb{R}$ given by

$$h(p) = 1 - \frac{(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2})}{2(\frac{1}{p} - \frac{1}{1+\varepsilon_0/2} + 1 + rp(1 - \frac{1}{1+\varepsilon_0/2}))}.$$

This function satisfies $\lim_{p \to 1} h(p) < 1$. Let $p \in (1, 1 + \varepsilon_0/2)$ be maximal with the property that $h(p) \leqslant \frac{1}{p}$. Then we have

$$\left\| \sum_{i=1}^{N} \left(S_\chi \mathbb{1}_\ell - \int_\Omega S_\chi \mathbb{1}_\ell \right) \circ \widehat{a}(t_i) \right\|_{L^p(Y)}^p \lesssim N.$$

Let $\delta' > 0$ and, for every $N \in \mathbb{N}^*$, let $\ell_N = \lfloor N^{\delta'} \rfloor$. By Lemma 5.8.1, for every $\varepsilon' > 0$ and for almost every $y \in Y$, we have

$$\forall\, N \in \mathbb{N}^*, \quad \sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_{\hat{c}_\ell^{-1}}}(\widehat{a}(t_i) k_x^{-1} \Gamma) = \int_\Omega S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}} \, d\mu_\Omega \, N + O\left(N^{\frac{2}{p+1}+\varepsilon'}\right)$$

By the mean value formula in Theorem 1.1.1, we have

$$\int_\Omega S_\chi \mathbb{1}_{\mathcal{F}_{c_\ell^{-1}}} \, d\mu_\Omega = \lambda_{\tilde{X}} \left(\mathcal{F}_{c_\ell^{-1}}\right).$$

There exists a constant $\kappa' > 0$ such that $\lambda_{\tilde{X}} \left(\mathcal{F}_{c_\ell^{-1}}\right) = \lambda_{\tilde{X}} (\mathcal{F}_1) + O(\ell^{-\kappa'})$. Plugging this into the inequality (5.8.6), we have, for all $N \in \mathbb{N}^*$,

$$\begin{aligned}\sum_{i=0}^{N-1} S_\chi \mathbb{1}_{\mathcal{F}_{\hat{c}_\ell^{-1}}}(\widehat{a}(t_i) k_x^{-1} \Gamma) + C_0 \, \ell_N^{\beta_\chi d} \\ = \lambda_{\tilde{X}} (\mathcal{F}_1) \; N + O(\lfloor N^{\delta'} \rfloor^{-\kappa'}) + O\left(N^{\frac{2}{p+1}+\varepsilon'}\right) + O\left(\lfloor N^{\delta'} \rfloor^{\beta_\chi d}\right).\end{aligned}$$

This completes the proof of Proposition 5.8.2 with

$$\delta' = \frac{1}{\beta_\chi d} \frac{2}{p+1}, \quad \varepsilon = 1 - \frac{2}{p+1}, \quad \text{and} \quad \varkappa_1 = \lambda_{\tilde{X}}(\mathcal{F}).$$

□

Chapter 6

Counting below the Diophantine exponent

Here the setting is the same one as in the previous chapter. In particular, $\mathbf{X} = \mathbf{G}/\mathbf{P}$ denotes a generalized flag variety defined over $\mathbb{Q}$, where $\mathbf{G}$ is a connected semisimple $\mathbb{Q}$-group and $\mathbf{P}$ is a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$. We shall again suppose that the unipotent radical of $\mathbf{P}$ is abelian. Throughout this chapter, we adopt the notation introduced in Chapter 4, in particular, the $\mathbb{Q}$-character $\chi \in X^*(\mathbf{P})_{\mathbb{Q}}$ and the height function $H_\chi : \mathbf{X}(\mathbb{Q}) \to \mathbb{R}$.

In this chapter, we shall count rational approximations to a real point of $\mathbf{X}$ chosen at random according to the Riemannian volume on $X = \mathbf{X}(\mathbb{R})$ with respect to exponents $\tau > 0$ less than the Diophantine exponent β_χ. We recall that, given a non-increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, for every $x \in X$ and $T \geqslant 1$, we defined the counting function

$$\mathcal{N}_\psi(x, T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : d(x, v) < \psi(H_\chi(v)),\, 1 \leqslant H_\chi(v) < T\}.$$

Let $\tau \in (0, \beta_\chi)$. In the case where $\psi_\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is the decreasing function defined by $\psi_\tau(t) = t^{-\tau}$, we will simply write $\mathcal{N}_\tau(x, T)$ for $\mathcal{N}_{\psi_\tau}(x, T)$. We shall be interested in determining, for almost every $x \in X$, the asymptotic behaviour of $\mathcal{N}_\tau(x, T)$ as $T \to +\infty$. Let d denote the dimension of X. In fact, we will find that there exists an explicit constant $\varkappa > 0$ such that for almost every $x \in X$, as $T \to +\infty$, we have

$$\mathcal{N}_\tau(x, T) \sim \varkappa T^{(\beta_\chi - \tau)d}.$$

Counting with respect to exponents below the Diophantine exponent requires a different approach than the one of Chapter 5. We use the exponential mixing of diagonal orbits in the space of lattices, tools from the geometry of numbers, and a famous counting method due to Eskin-McMullen and Duke-Rudnick-Sarnak. This method allows for effective estimates. In fact, we even prove that there exists a positive constant $\varepsilon > 0$ such that for almost every $x \in X$, we have

$$\mathcal{N}_\tau(x,T) = \varkappa T^{(\beta_\chi - \tau)d}\left(1 + O_x(T^{-\varepsilon})\right).$$

Finally, in addition to our counting result of rational approximations, we deduce effective equidistribution of rational points of bounded height on rank-one flag varieties, improving on an earlier result of Mohammadi and Salehi-Golsefidy [MG14] in the setting of rank-one flag varieties.

6.1 Method of proof

We sketch here the method of proof of Theorem 1.3.1, which was inspired by the work of Huang and de Saxcé [HS24] on the local distribution of rational points on flag varieties and uses a celebrated counting method due to Eskin-McMullen [EM93] and Duke-Rudnick-Sarnak [DRP93].

We recall that, given a non-increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, for every $T \geqslant 1$, we defined in Equation (6.5.1) the region

$$\mathcal{E}_\psi(T) = \{\boldsymbol{v} \in \widetilde{X} : d(x_0, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|),\ 1 \leqslant \|\boldsymbol{v}\| < T\}.$$

Let $\tau \in (0, \beta_\chi)$. In the case where $\psi_\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is the decreasing function defined by $\psi_\tau(t) = t^{-\tau}$, we will simply write $\mathcal{E}_\tau(T)$ for $\mathcal{E}_{\psi_\tau}(T)$. Using Lemma 4.7.1 in Chapter 4, we translate the problem of counting rational approximations to $x \in X$ of bounded height to counting primitive lattice points within a growing family $(k_x\mathcal{E}_\tau(T))_{T\geqslant 1}$ in the cone $\widetilde{X}$, where $k_x \in K$ is any element such that $x = k_x P$. We find the relation

$$\mathcal{N}_\psi(x,T) = [K \cap P : K \cap L]\,\#(\mathcal{L}_\chi \cap k_x\mathcal{E}_\tau(T)).$$

We prove an asymptotic formula for $\#(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_\tau(T))$ for almost every $x \in X$. We recall that Γ is the stabilizer of $\mathbf{V}_\chi(\mathbb{Z})$ in G and the diagonal subgroup $A = \{a(y) : y > 0\}$ is defined as in the line after Equation (4.4.3).

The set $\mathcal{E}_\tau(T)$ is typically not well-rounded; this term was introduced in [EM93], it refers to the regularity property of a set to be almost invariant

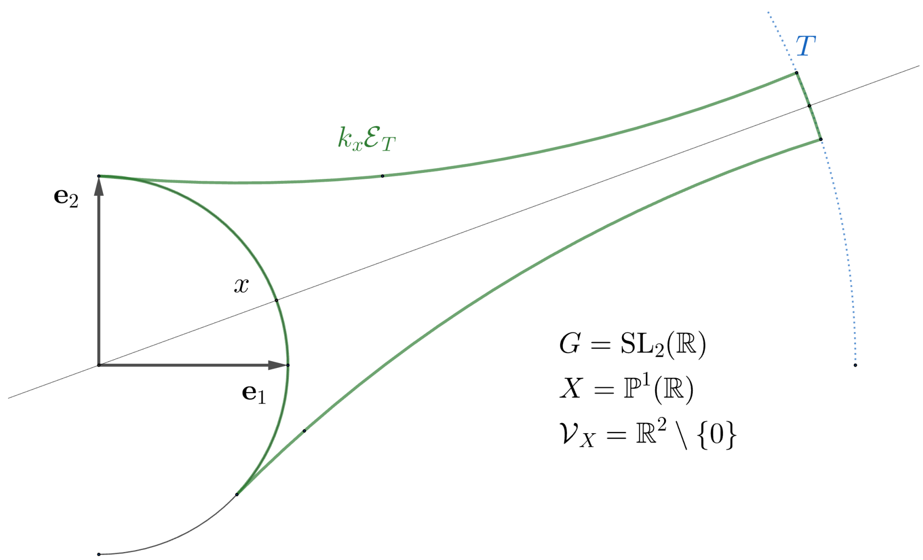

Figure 6.1: The set $k_x\mathcal{E}_\tau(T)$ for the group $G = \mathrm{SL}_2(\mathbb{R})$, $X = \mathbb{P}^1(\mathbb{R})$, $\widetilde{X} = \mathbb{R}^2$, and $\mathcal{L}_\chi = \mathcal{P}(\mathbb{Z}^2)$.

under the action of a small ball centered at the identity in G, and it allows for asymptotic lattice point counts with an error term. The idea is then to tessellate $\mathcal{E}_\tau(T)$ dyadically

$$\mathcal{E}_\tau(T) = \bigsqcup_{j \geqslant 0} \mathcal{F}_{T_j} \quad \text{where } \mathcal{F}_{T_j} = \mathcal{E}_\tau(T/2^j) \backslash \mathcal{E}_\tau(T/2^{j+1}) \text{ for all } j \geqslant 0,$$

and to apply to each $\mathcal{F}_{T_j}$ an element $a(y_j)$ of the diagonal subgroup A in order to obtain a well-rounded set $\mathcal{B}_{T_j} = a(y_j)\mathcal{F}_{T_j}$.

This leads us to the expression

$$\#(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_\tau(T)) = \sum_{j \geqslant 0} \#(a(y_j)k_x^{-1}\mathcal{L}_\chi \cap \mathcal{B}_{T_j}).$$

Now, using the exponential mixing property of diagonal orbits and the fact that $\mathcal{L}_\chi$ is a finite union of Γ-orbits, we show that there exists $\varepsilon > 0$ such that for every $T \geqslant 1$ sufficiently large and every $g \in G$ whose norm is bounded by T^ε, one can estimate $\#\,(g\mathcal{L}_\chi \cap \mathcal{B}_T)$ in terms of the volume of the set $\mathcal{B}_T$:

$$\#\,(g\mathcal{L}_\chi \cap \mathcal{B}_T) = \omega\,\lambda(\mathcal{B}_T)\left(1 + O(T^{-\varepsilon})\right),$$

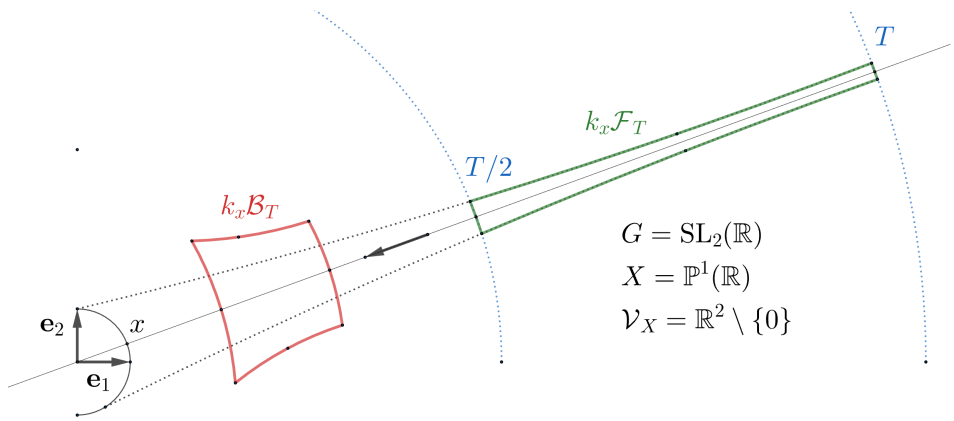

Figure 6.2: The action of $a(y) = \begin{pmatrix} y^{-1/2} & \\ & y^{1/2} \end{pmatrix}$ with $y > 1$ on $\widetilde{X} = \mathbb{R}^2$ contracts the line through $\boldsymbol{e}_\chi = \boldsymbol{e}_1$ and expands the line through $\boldsymbol{e}_2$. We show that for a suitable time y_T, the set $\mathcal{B}_T = a(y_T)\mathcal{F}_T$ is well-rounded and the lattice $a(y_T)k_x^{-1}\mathbb{Z}^2$ is not too distorted for σ_X-almost every $x \in X$. The Gauss circle argument thus suggests that $\#\left(a(y_T)k_x^{-1}\mathcal{L}_\chi \cap \mathcal{B}_T\right)$ should be (up to scalars) approximately given by the volume of $\mathcal{B}_T$.

for some positive constant $\omega > 0$. On the other hand, using the ergodicity of the action of A on the space of lattices and reduction theory, we can show that the first minimum of the lattice $a(y_j)k_x^{-1}\mathbf{V}_\chi(\mathbb{Z})$ is not affected too much. Altogether, we can thus estimate $\#(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_\tau(T))$ in terms of the volume of the set $\mathcal{E}_\tau(T)$. Providing the necessary volume estimates then concludes the proof of Theorem 1.3.1.

6.2 Reduction of Theorem 1.3.1

In this section, using Lemma 4.7.1, we reduce Theorem 1.3.1 to a statement about counting primitive lattice points within a growing family in the cone $\widetilde{X}$. In fact, Theorem 1.3.1 follows from the following result, which provides an asymptotic formula in terms of the measure of the set $\mathcal{E}_\tau(T)$ and whose proof will be given in Sections 6.3 and 6.4. Recall that the measure $\lambda_{\widetilde{X}}$ on $\widetilde{X}$ has been constructed in (4.4.7).

Proposition 6.2.1. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function. Suppose that there exist $\tau \in (0, \beta_\chi)$ and $C > 1$ such that $C^{-1}y^{-\tau} \leqslant \psi(y) \leqslant Cy^{-\tau}$ for all sufficiently large $y \geqslant 1$. Then there exist $\varepsilon' > 0$ and an explicit constant $\varkappa' > 0$ such that for σ_X-almost every $x \in X$, as $T \to +\infty$, we have*

$$\#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}_\tau(T)\right) = \varkappa'\,\lambda_{\widetilde{X}}(\mathcal{E}_\tau(T))\left(1 + O_x(\lambda_{\widetilde{X}}(\mathcal{E}_\tau(T))^{-\varepsilon'})\right).$$

The constant $\varkappa'$ is explicitly given in (6.4.9). Before proving Theorem 1.3.1, we recall its statement:

Theorem (Theorem 1.3.1)**.** *Let $\mathbf{G}$ be a connected semisimple $\mathbb{Q}$-group, $\mathbf{P}$ a parabolic $\mathbb{Q}$-subgroup with abelian unipotent radical and $\mathbf{X} = \mathbf{G}/\mathbf{P}$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function satisfying that there exist $\tau \in (0, \beta_\chi)$ and $C > 1$ such that $C^{-1}y^{-\tau} \leqslant \psi(y) \leqslant Cy^{-\tau}$ for all sufficiently large y. For every $T \geqslant 1$, define the integral*

$$\Psi(T) = \int_1^T \psi(y)^d y^{\beta_\chi d}\frac{dy}{y}.$$

Then there exists an explicit constant $\varkappa > 0$ and $\varepsilon > 0$ such that for σ_X-almost every $x \in X$, we have

$$\mathcal{N}_\psi(x, T) = \varkappa\,\Psi(T)\left(1 + O_x(\Psi(T)^{-\varepsilon})\right). \tag{6.2.1}$$

The constant $\varkappa$ is explicitly given in (6.2.4).

Proof of Theorem 1.3.1. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be as in Theorem 1.3.1. By Lemma 4.7.1 and Proposition 6.2.1, it only remains to provide the desired measure estimate for $\mathcal{E}_\tau(T)$. Let σ be the pushforward of the Haar probability measure μ_K on K to the quotient K/K_L along the quotient map $k \mapsto kK_L$. We recall from Section 4.4 that the map of $(K/K_L) \times A$ to $\widetilde{X}_0 = G/L$ given by $(k\,K_L, a(y)) \mapsto ka(y)\boldsymbol{e}_\chi$ is a homeomorphism and, in these coordinates, there exists a constant $\omega_1 > 0$ such that the measure $\lambda_{\widetilde{X}}$ is given by

$$d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi) = \omega_1\, y^{-(d+1)}\, d\sigma(k)\, dy.$$

By Equation (4.4.5), the action of $a(y)$ on $\boldsymbol{e}_\chi$ is given by

$$a(y)\boldsymbol{e}_\chi = y^{-\frac{1}{\beta_\chi}}\boldsymbol{e}_\chi.$$

Using the fact that the norm $\|\cdot\|$ on V_χ is K-invariant and $\|\boldsymbol{e}_\chi\| = 1$, the set $\mathcal{E}_\tau(T)$ can now be described as

$$\mathcal{E}_\tau(T) = \left\{ka(y)\boldsymbol{e}_\chi \in \widetilde{X}_0 : d(x_0, kx_0) < \psi(y^{-\frac{1}{\beta_\chi}}),\, 1 \leqslant y^{-\frac{1}{\beta_\chi}} < T\right\}.$$

We denote by $B_X(r)$ the ball of radius $r > 0$ centered at $x_0 \in X$. Since σ_X is the pushforward of σ to the quotient $X = K/(K \cap P)$ along the map $k(K \cap L) \mapsto k(K \cap P)$, we have

$$\begin{aligned}
\lambda_{\widetilde{X}}(\mathcal{E}_\tau(T)) &= \int_{T^{-\beta_\chi}}^{1} \int_{K/K\cap L} \mathbb{1}_{B_X(\psi(y^{-\frac{1}{\beta_\chi}}))}(kx_0)\, \omega_1\, d\sigma(k) \frac{dy}{y^{d+1}} \\
&= \int_{T^{-\beta_\chi}}^{1} \sigma_X(B_X(\psi(y^{-\frac{1}{\beta_\chi}})))\, \omega_1\, \frac{dy}{y^{d+1}}.
\end{aligned}$$

Let $\mathrm{vol}_{\mathrm{R}}(X)$ be the total Riemannian volume of X. As is true for any Riemannian manifold [Gra74, Theorem 3.1], for every $r > 0$, we have

$$\sigma_X(B_X(r)) = \mathrm{vol}_{\mathrm{R}}(X)^{-1} \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} r^d + O(r^{d+2}), \tag{6.2.2}$$

where $\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the unit ball in the Euclidean space $\mathbb{R}^d$. Let us define the constant

$$\widehat{\varkappa} = \beta_\chi \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\, \mathrm{vol}_{\mathrm{R}}(X)^{-1}\, \omega_1.$$

Using the change of variable $y \mapsto y^{-\frac{1}{\beta_\chi}}$, the assumption $\psi(y) \asymp y^{-\tau}$, and defining, for every $T \geqslant 1$, the integral $\Psi(T) = \int_1^T \psi(y)^d y^{\beta_\chi d} \frac{dy}{y}$, we have

$$\begin{aligned}\lambda_{\widetilde{X}}(\mathcal{E}_\tau(T)) &= \beta_\chi \int_1^T \sigma_X(B_X(\psi(y))) y^{\beta_\chi d}\, \omega_1 \frac{dy}{y} \\ &= \widehat{\varkappa} \int_1^T \psi(y)^d y^{\beta_\chi d} \frac{dy}{y} + O\left(\int_1^T y^{(\beta_\chi - \tau)d - 2\tau} \frac{dy}{y}\right) \\ &= \widehat{\varkappa}\, \Psi(T) + O\left(T^{(\beta_\chi-\tau)d-2\tau}\right) \\ &= \widehat{\varkappa}\, \Psi(T) \left(1 + O(\Psi(T)^{-\frac{2\tau}{(\beta_\chi-\tau)d}})\right),\end{aligned} \tag{6.2.3}$$

Let $\varepsilon' > 0$ and $\varkappa' > 0$ be as in Proposition 6.2.1. This establishes the proof of Theorem 1.3.1 with

$$\varkappa = \beta_\chi\, [K \cap P : K \cap L]^{-1} \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} \operatorname{vol}_{\mathrm{R}}(X)^{-1}\, \omega_1\, \varkappa'. \tag{6.2.4}$$

and $\varepsilon = \min\{\varepsilon', \frac{2\tau}{(\beta_\chi - \tau)d}\}$. □

6.3 Diagonal action and well roundedness

Let ψ, τ, C be as in Proposition 6.2.1. To establish the lattice point counting in Proposition 6.2.1, we use the exponential mixing property of a certain diagonal flow and the well-roundedness property of certain sets, in the spirit of the counting method developed by Eskin-McMullen [EM93] and Duke-Rudnick-Sarnak [DRP93]. However, the set $\mathcal{E}_\tau(T)$ is typically not well-rounded with respect to the action of G. For every $T \geqslant 1$, we define

$$\mathcal{F}_T = \left\{\boldsymbol{v} \in \widetilde{X} : d(x_0, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|),\ \max\{1, T/2\} \leqslant \|\boldsymbol{v}\| < T\right\}.$$

The idea is to first tessellate $\mathcal{E}_\tau(T)$ dyadically:

$$\mathcal{E}_\tau(T) = \bigsqcup_{j \geqslant 0} \mathcal{F}_{T_j} \quad \text{where } T_j = T/2^j,\ j \geqslant 0.$$

It will turn out that, for every $T \geqslant 1$, if we set $y_T = T^\tau$, then the set

$$\mathcal{B}_T = a(y_T)\mathcal{F}_T$$

is well-rounded as defined below. The purpose of this section is to establish the well-roundedness of the family $\mathcal{B} = (\mathcal{B}_T)_{T\geqslant 1}$.

We shall work with the following definition. Let $d_G(\cdot,\cdot)$ be a left-invariant Riemannian metric on G and $B_G(r)$ the ball with radius $r > 0$ and center $1 \in G$. Our definition is inspired by that of [KY23, Section 3.2], which in turn is inspired by the work of [EM93].

Definition 6.3.1. A family $\mathcal{B}' = (\mathcal{B}'_T)_{T\geqslant 1}$ of Borel subsets of $\widetilde{X}$ with finite measure for $\lambda_{\widetilde{X}}$ is said to be *well-rounded* if there exist constants $C_1 > 0$, $\delta_0 > 0$, and $T_0 > 1$ such that for all $\delta \in (0,\delta_0)$ and $T > T_0$, the sets

$$\overline{\mathcal{B}}'_{T,\delta} := \bigcup_{g\in B_G(\delta)} g\mathcal{B}'_T \quad \text{and} \quad \underline{\mathcal{B}}'_{T,\delta} := \bigcap_{g\in B_G(\delta)} g\mathcal{B}'_T$$

satisfy

$$\lambda_{\widetilde{X}}\left(\overline{\mathcal{B}}'_{T,\delta} \smallsetminus \underline{\mathcal{B}}'_{T,\delta}\right) \leqslant C_1\delta\lambda_{\widetilde{X}}(\mathcal{B}'_T). \tag{6.3.1}$$

Let $(\chi_j)_{j\in J}$ be the finite family of $\mathbb{Q}$-weights of the representation $(\pi_\chi, \mathbf{V}_\chi)$ and $(V_j)_{j\in J}$ the corresponding $\mathbb{Q}$-weight spaces, explicitly given by

$$V_j = \{\boldsymbol{v} \in V_\chi : \forall\, y \in \mathbb{R}_+^\times,\, a(y)\boldsymbol{v} = \chi_j(a(y))\boldsymbol{v}\}.$$

Then the vector space V_χ decomposes as a finite direct sum

$$V_\chi = \bigoplus_{j\in J} V_j. \tag{6.3.2}$$

By our choice of inner product on V_χ, the subspaces V_j are mutually orthogonal (see Section 4.1). We denote by $\pi^+ : V_\chi \to V_\chi$ the orthogonal projection onto $\mathbb{R}\boldsymbol{e}_\chi$ and we simply write $\boldsymbol{v}^+$ for $\pi^+(\boldsymbol{v})$.

For all $\delta \in (0,1)$ and $y > 0$, we define

$$\psi_\delta^-(y) = (1+\delta)^{-1}\psi((1+\delta)y) \quad \text{and} \quad \psi_\delta^+(y) = (1+\delta)\psi((1+\delta)^{-1}y). \tag{6.3.3}$$

Using these functions $\psi_\delta^\pm : \mathbb{R}_+ \to \mathbb{R}_+$, we define the sets

$$\underline{\mathcal{F}}_{T,\delta} = \left\{\boldsymbol{v} \in \widetilde{X}_0 : d(x_0,[\boldsymbol{v}]) < \psi_\delta^-(\|\boldsymbol{v}\|),\, (1+\delta)\tfrac{T}{2} \leqslant \|\boldsymbol{v}\| < (1+\delta)^{-1}T\right\}$$

and

$$\overline{\mathcal{F}}_{T,\delta} = \left\{\boldsymbol{v} \in \widetilde{X}_0 : d(x_0,[\boldsymbol{v}]) < \psi_\delta^+(\|\boldsymbol{v}\|),\, (1+\delta)^{-1}\tfrac{T}{2} \leqslant \|\boldsymbol{v}\| < (1+\delta)T\right\}.$$

Then we put

$$\underline{\mathcal{B}}_{T,\delta} = a(y_T)\underline{\mathcal{F}}_{T,\delta} \quad \text{and} \quad \overline{\mathcal{B}}_{T,\delta} = a(y_T)\overline{\mathcal{F}}_{T,\delta}. \tag{6.3.4}$$

Lemma 6.3.2. *For all $T \geqslant 1$ large enough and $\boldsymbol{w} \in \mathcal{B}_T$, we have $\|\boldsymbol{w}\| \asymp \|\boldsymbol{w}^+\|$.*

Proof. For all $T \geqslant 1$ large enough, every $\boldsymbol{v} \in \mathcal{F}_T$ satisfies that $[\boldsymbol{v}] \in X$ is close to x_0. Since the map $\phi : \mathfrak{u}^- \to X$ given by $\phi : u^- \mapsto \exp(u^-)x_0$ parametrizes a neighborhood of $x_0 \in X$, there exists a unique $u^-_{\boldsymbol{v}} \in \mathfrak{u}^-$ such that $[\boldsymbol{v}] = \exp(u^-_{\boldsymbol{v}})x_0$. Moreover, there exists a unique $y_{\boldsymbol{v}} \in \mathbb{R}^\times_+$ such that $\boldsymbol{v} = \pm\exp(u^-_{\boldsymbol{v}})a_{y_{\boldsymbol{v}}}\boldsymbol{e}_\chi$. Without loss of generality, we may assume that $\boldsymbol{v} = \exp(u^-_{\boldsymbol{v}})a_{y_{\boldsymbol{v}}}\boldsymbol{e}_\chi$. Using the definition of $a_y = \exp(\log(y)Y_\alpha)$ and the fact that $^-$ is abelian, the adjoint action of a_y on $^- = T_{x_0}X$ is given by $\mathrm{Ad}(a_y)u^- = yu^-$ (see [Sax20, Lemme 2.4.2]). Hence the element $\boldsymbol{w} = a_{y_T}\boldsymbol{v} \in \mathcal{B}_T$ can be written as

$$a_{y_T}\boldsymbol{v} = \exp(y_T u^-_{\boldsymbol{v}})a_{y_T}a_{y_{\boldsymbol{v}}}\boldsymbol{e}_\chi.$$

The elements $y_T u^-_{\boldsymbol{v}}$ stay in a fixed compact neighborhood V of the origin in $\mathfrak{u}^-$. In fact, by the estimate (4.2.1), we have $\|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} \asymp d(x_0, [\boldsymbol{v}]) \leqslant \psi(\|\boldsymbol{v}\|) \lesssim T^{-\tau}$, and hence

$$y_T \|u^-_{\boldsymbol{v}}\|_{\mathfrak{u}^-} \lesssim 1.$$

This implies that

$$\|\boldsymbol{w}\| \asymp \|a_{y_T}a_{y_{\boldsymbol{v}}}\boldsymbol{e}_\chi\| = \chi(a_{y_T}a_{y_{\boldsymbol{v}}}). \tag{6.3.5}$$

On the other hand, we observe that for every $u^- \in \mathfrak{u}^-$ and $y \in \mathbb{R}^\times_+$, we have

$$a_y \exp(u^-)\boldsymbol{e}_\chi = \exp(yu^-)\chi(a_y)\boldsymbol{e}_\chi.$$

Thus the vector $\exp(u^-)\boldsymbol{e}_\chi$ gets expanded under the action of a_y at the highest possible rate $\chi(a_y)$ as y tends to zero and we must have $\|(\exp(u^-)\boldsymbol{e}_\chi)^+\| > 0$. By the compactness of V, we thus have $\|(\exp(u^-)\boldsymbol{e}_\chi)^+\| \asymp 1$ for all $u^- \in V$. This yields

$$\|\boldsymbol{w}^+\| = \chi(a_{y_T}a_{y_{\boldsymbol{v}}})\|(\exp(y_T u^-_{\boldsymbol{v}})\boldsymbol{e}_\chi)^+\| \asymp \chi(a_{y_T}a_{y_{\boldsymbol{v}}}). \tag{6.3.6}$$

Putting (6.3.5) and (6.3.6) together completes the proof of Lemma 6.3.2. $\square$

Lemma 6.3.3. *There exists $c_1 > 0$ such that for all $T \geqslant 1$, $\delta \in (0, 1)$, $g \in B_G(c_1\delta)$, we have*

$$\underline{\mathcal{B}}_{T,\delta} \subseteq g\mathcal{B}_T \subseteq \overline{\mathcal{B}}_{T,\delta}. \tag{6.3.7}$$

Proof. We prove the right inclusion in (6.3.7); the proof of the other inclusion $\underline{\mathcal{B}}_{T,\delta} \subseteq g\mathcal{B}_T$ is essentially identical and we omit the details. To simplify the notation, for all $g \in G$ and $T \geqslant 1$, we write $g_T = a(y_T)^{-1} g a(y_T)$ and for $\boldsymbol{v} \in \widetilde{X}_0$ we denote by $v = [\boldsymbol{v}]$ the corresponding projective point.

Fix $\delta \in (0,1)$. First, using the fact that $\mathcal{B}_T = a(y_T)\mathcal{F}_T$ and $\overline{\mathcal{B}}_{T,\delta} = a(y_T)\overline{\mathcal{F}}_{T,\delta}$, we note that the inclusion $g\mathcal{B}_T \subseteq \overline{\mathcal{B}}_{T,\delta}$ holds if and only if $g_T \mathcal{F}_{T,\delta} \subseteq \overline{\mathcal{F}}_{T,\delta}$. Therefore, using the definitions of $\mathcal{F}_{T,\delta}$ and $\overline{\mathcal{F}}_{T,\delta}$, we need to show that there exists a constant $c_1 > 0$ such that for all $T \geqslant 1$, $\boldsymbol{v} \in \mathcal{F}_T$, and $g \in B_G(c_1\delta)$, we have

$$
\begin{aligned}
&(1) \quad (1+\delta)^{-1}\frac{T}{2} \leqslant \|g_T \boldsymbol{v}\| \leqslant (1+\delta)T, \\
&(2) \quad d(x_0, g_T v) < \psi_\delta^+(\|g_T \boldsymbol{v}\|).
\end{aligned}
$$

Since each $\boldsymbol{v} \in \mathcal{F}_T$ satisfies $T/2 \leqslant \|\boldsymbol{v}\| < T$, in order to prove (1), it suffices to show that

$$(1+\delta)^{-1}\|\boldsymbol{v}\| \leqslant \|g_T \boldsymbol{v}\| \leqslant (1+\delta)\|\boldsymbol{v}\|. \tag{6.3.8}$$

We first show the right inequality $\|g_T \boldsymbol{v}\| \leqslant (1+\delta)\|\boldsymbol{v}\|$ in (6.3.8). Applying the triangle inequality, one has $\|g_T \boldsymbol{v}\| \leqslant \|\boldsymbol{v}\| + \|g_T \boldsymbol{v} - \boldsymbol{v}\|$. Thus, we are further reduced to showing that for all $g \in B_G(\delta)$,

$$\|g_T \boldsymbol{v} - \boldsymbol{v}\| \lesssim \delta \|\boldsymbol{v}\|. \tag{6.3.9}$$

Then, by decomposing $\boldsymbol{v} = \sum_{j \in J} \boldsymbol{v}_j$ (recall that we assumed that $\boldsymbol{v}_1 = \boldsymbol{v}^+$), into weight vectors according to (6.3.2), for every $j \in J$, we have

$$\|g_T \boldsymbol{v}_j - \boldsymbol{v}_j\| \leqslant y^{\frac{1}{\beta_X}} \|g a(y_T)\boldsymbol{v}_j - a(y_T)\boldsymbol{v}_j\| \lesssim y_T^{\frac{1}{\beta_X}} \delta \|a(y_T)\boldsymbol{v}_j\| \lesssim \delta \|\boldsymbol{v}^+\|,$$

where for the last inequality we used that $\|a(y_T)\boldsymbol{v}_j\| \lesssim \|a(y_T)\boldsymbol{v}^+\| = y^{\frac{1}{\beta_X}}\|\boldsymbol{v}^+\|$, which follows from Lemma 6.3.2. This implies (6.3.9), which is the right hand side of (6.3.8). The proof of the other inequality in (6.3.8) is very similar and we omit the details.

Let us now show Assertion (2). Using (6.3.8) and the fact that ψ is decreasing, we have

$$\psi_\delta^+(\|g_T \boldsymbol{v}\|) = (1+\delta)\psi((1+\delta)^{-1}\|g_T \boldsymbol{v}\|) \geqslant (1+\delta)\psi(\|\boldsymbol{v}\|).$$

Thus, it is enough to check that

$$d(x_0, g_T v) < (1+\delta)\psi(\|\boldsymbol{v}\|). \tag{6.3.10}$$

Using the triangle inequality and the fact that $\boldsymbol{v} \in \mathcal{F}_T$ satisfies $d(x_0, v) < \psi(\|\boldsymbol{v}\|)$, we have

$$d(x_0, g_T v) \leqslant \psi(\|\boldsymbol{v}\|) + d(v, g_T v).$$

Therefore, in order to prove (2), we only have to prove that, for every $T \geqslant 1$ and $\boldsymbol{v} \in \mathcal{F}_T$, and for all $g \in B_G(\delta)$, we have

$$d(v, g_T v) \lesssim \delta \psi(\|\boldsymbol{v}\|). \tag{6.3.11}$$

We then have

$$d(v, g_T v) = d(\exp(u_{\boldsymbol{v}}^-)x_0, g_T \exp(u_{\boldsymbol{v}}^-)x_0) \lesssim d(x_0, \exp(-u_{\boldsymbol{v}}^-)g_T \exp(u_{\boldsymbol{v}}^-)x_0).$$

Conjugating $\exp(u_{\boldsymbol{v}}^-)$ by $a(y_T)$ yields

$$\exp(-u_{\boldsymbol{v}}^-)g_T \exp(u_{\boldsymbol{v}}^-) = a(y_T)^{-1} \exp(-\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-)g \exp(\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-)a(y_T).$$

Using the decomposition $G = U^- P$, where $U^- = \exp(\mathfrak{u}^-)$ is the unipotent subgroup opposite to P, we may write $\exp(-\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-)g \exp(\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-) = \exp(u')p$ with $u' \in U^-$ and $p \in P$. The distance we want to bound is then

$$d(v, g_T v) \lesssim d(x_0, a(y_T)^{-1} \exp(u')x_0) = d(x_0, \exp(\operatorname{Ad}(a(y_T)^{-1})u')x_0) \lesssim y_T^{-1}\|u'\|_{\mathfrak{u}^-}.$$

Using the definition $y_T = T^\tau$ and the inequality $T^{-\tau} \lesssim \psi(\|\boldsymbol{v}\|)$ gives $y_T^{-1} \lesssim \psi(\|\boldsymbol{v}\|)$. On the other hand, using again that $\|\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-\|_{\mathfrak{u}^-} \lesssim 1$, one has

$$\begin{aligned}\|u'\|_{\mathfrak{u}^-} &\lesssim d_G(1, \exp(u')p) \\ &= d_G(\exp(\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-), g \exp(\operatorname{Ad}(a(y_T))u_{\boldsymbol{v}}^-)) \lesssim d_G(1, g) < \delta,\end{aligned}$$

as desired. □

Proposition 6.3.4. *The family $(\mathcal{B}_T)_{T \geqslant 1}$ is well-rounded.*

Proof. Let $c_1 > 0$ be as in Lemma 6.3.3. We show that there exists $C_1 > 0$ such that for all $T \geqslant 1$ and $\delta \in (0, 1)$, the Borel subsets $\overline{\mathcal{B}}_{T,\delta}$ and $\underline{\mathcal{B}}_{T,\delta}$ as defined in (6.3.4) satisfy

$$\underline{\mathcal{B}}_{T,\delta} \subseteq \bigcap_{g \in B_G(c_1\delta)} g\mathcal{B}_T \subseteq \bigcup_{g \in B_G(c_1\delta)} g\mathcal{B}_T \subseteq \overline{\mathcal{B}}_{T,\delta}.$$

and $\lambda_{\tilde{X}}(\overline{\mathcal{B}}_{T,\delta} \smallsetminus \underline{\mathcal{B}}_{T,\delta}) \leqslant C_1 \delta \lambda_{\tilde{X}}(\mathcal{B}_T)$. The inclusion relations follow from Lemma 6.3.3. Thus, using the G-invariance of the measure $\lambda_{\tilde{X}}$, it suffices to show the measure bound

$$\lambda_{\tilde{X}}(\overline{\mathcal{F}}_{T,\delta} \smallsetminus \underline{\mathcal{F}}_{T,\delta}) \leqslant C_1 \delta \lambda_{\tilde{X}}(\mathcal{F}_T).$$

We have $\overline{\mathcal{F}}_{T,\delta} \smallsetminus \underline{\mathcal{F}}_{T,\delta} \subseteq R_1 \cup R_2 \cup R_3$, where

$$R_1 = \left\{ ka(y)\boldsymbol{e}_\chi \in \widetilde{X} : \begin{array}{r} \psi_\delta^-\left(y^{-\frac{1}{\beta_\chi}}\right) \leqslant d(x_0, kx_0) < \psi_\delta^+\left(y^{-\frac{1}{\beta_\chi}}\right), \\ (1+\delta)\frac{T}{2} \leqslant y^{-\frac{1}{\beta_\chi}} < (1+\delta)^{-1}T \end{array} \right\},$$

$$R_2 = \left\{ ka(y)\boldsymbol{e}_\chi \in \widetilde{X} : \begin{array}{r} d(x_0, kx_0) < \psi_\delta^+\left(y^{-\frac{1}{\beta_\chi}}\right), \\ (1+\delta)^{-1}\frac{T}{2} \leqslant y^{-\frac{1}{\beta_\chi}} < (1+\delta)\frac{T}{2} \end{array} \right\},$$

and

$$R_3 = \left\{ ka(y)\boldsymbol{e}_\chi \in \widetilde{X} : \begin{array}{r} d(x_0, kx_0) < \psi_\delta^+\left(y^{-\frac{1}{\beta_\chi}}\right), \\ (1+\delta)^{-1}T \leqslant y^{-\frac{1}{\beta_\chi}} < (1+\delta)T \end{array} \right\}.$$

Let us start with giving an upper bound on the measure of R_1. By the arguments given in (6.2.3) in the proof of Theorem 1.2.2, using the change of variable $y \mapsto y^{-\frac{1}{\beta_\chi}}$, we have

$$\lambda_{\widetilde{X}}(R_1) \lesssim \int_{(1+\delta)\frac{T}{2}}^{(1+\delta)^{-1}T} \sigma_X\left(B_X(\psi_\delta^+(y)) \smallsetminus B_X(\psi_\delta^-(y))\right) y^{\beta_\chi d}\,\frac{dy}{y}$$

By [Gra74, Theorem 3.1], for all $0 < r_2 < r_1$ small enough, the volume of the difference of two small balls centered at x_0 satisfies

$$\sigma_X(B_X(r_1) \backslash B_X(r_2)) \lesssim r_1^d - r_2^d. \tag{6.3.12}$$

Using this estimate, we have $\lambda_{\widetilde{X}}(R_1) \lesssim I_1 - I_2$, where

$$I_1 = \int_{(1+\delta)\frac{T}{2}}^{(1+\delta)^{-1}T} \psi_\delta^+(y)^d y^{\beta_\chi d}\frac{dy}{y}, \quad I_2 = \int_{(1+\delta)\frac{T}{2}}^{(1+\delta)^{-1}T} \psi_\delta^-(y)^d y^{\beta_\chi d}\frac{dy}{y}.$$

Using the change of variable $y \mapsto (1+\delta)^{-1}y$ in I_1 and rearranging, one gets $I_1 = I_1^{(1)} - I_1^{(2)}$, where

$$I_1^{(1)} = (1+\delta)^{d(1+\beta_\chi)} \int_{\frac{T}{2}}^{T} \psi(y)^d y^{\beta_\chi d}\frac{dy}{y},$$
$$I_1^{(2)} = (1+\delta)^{d(1+\beta_\chi)} \int_{(1+\delta)^{-2}T}^{T} \psi(y)^d y^{\beta_\chi d}\frac{dy}{y}.$$

Similarly, by the change of variable $y \mapsto (1+\delta)y$ in I_2 and rearranging, we have $I_2 = I_2^{(1)} - I_2^{(2)}$, where

$$I_2^{(1)} = (1+\delta)^{-d(1+\beta_X)} \int_{\frac{T}{2}}^{T} \psi(y)^d y^{\beta_X d} \frac{dy}{y},$$
$$I_2^{(2)} = (1+\delta)^{-d(1+\beta_X)} \int_{\frac{T}{2}}^{(1+\delta)^2 \frac{T}{2}} \psi(y)^d y^{\beta_X d} \frac{dy}{y}.$$

Let us show that $I_1^{(1)} - I_2^{(1)}, I_1^{(2)}, I_2^{(2)} \lesssim \delta \lambda_{\widetilde{X}}(\mathcal{F}_T)$. We have

$$\begin{aligned} I_1^{(1)} - I_2^{(1)} &= \left((1+\delta)^{d(1+\beta_X)} - (1+\delta)^{-d(1+\beta_X)}\right) \int_{\frac{T}{2}}^{T} \psi(y)^d y^{\beta_X d} \frac{dy}{y} \\ &\lesssim \delta\, \lambda_{\widetilde{X}}(\mathcal{F}_T). \end{aligned}$$

Next, using that $\psi(y) \asymp y^{-\tau}$ and evaluating the integral, we have

$$\begin{aligned} I_1^{(2)} &= (1+\delta)^{d(1+\beta_X)} \int_{(1+\delta)^{-2}T}^{T} \psi(y)^d y^{\beta_X d} \frac{dy}{y} \\ &\lesssim (1+\delta)^{d(1+\beta_X)} \int_{(1+\delta)^{-2}T}^{T} y^{(\beta_X - \tau)d - 1}\, dy \\ &\lesssim (1+\delta)^{d(1+\beta_X)} \left(T^{(\beta_X-\tau)d} - ((1+\delta)^{-2}T)^{(\beta_X-\tau)d}\right) \\ &= (1+\delta)^{d(1+\beta_X)} \left(1 - ((1+\delta)^{-2})^{(\beta_X-\tau)d}\right) T^{(\beta_X-\tau)d} \\ &\lesssim \delta\, T^{(\beta_X-\tau)d} \asymp \delta\, \lambda_{\widetilde{X}}(\mathcal{F}_T). \end{aligned}$$

By symmetry, we also have $I_2^{(2)} \lesssim \delta \lambda_{\widetilde{X}}(\mathcal{F}_T)$, as required. The measure bounds $\lambda_{\widetilde{X}}(R_2) \lesssim \delta\, \lambda_{\widetilde{X}}(\mathcal{F}_T)$ and $\lambda_{\widetilde{X}}(R_3) \lesssim \delta \lambda_{\widetilde{X}}(\mathcal{F}_T)$ are shown similarly, and we omit the details. □

6.4 From exponential mixing to counting

In this section, we use the exponential mixing property of A, an ingredient from the geometry of numbers, and the well-roundedness of the family $(\mathcal{B}_T)_{T \geqslant 1}$ in order to prove Proposition 6.2.1.

6.4.1 Sobolev norms

Following [EMV09, §3.6, §3.7], we introduce a function $\mathrm{ht} : \Omega \to \mathbb{R}_{>0}$ and a degree $\ell \in \mathbb{N}$ Sobolev norm $\mathcal{S}_\ell$ on the space $C_c^\infty(\Omega)$ of compactly supported smooth functions on Ω. More precisely, we fix an $\mathrm{Ad}(K)$-invariant Euclidean norm $\|\cdot\|_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G such that $\|[\boldsymbol{u}, \boldsymbol{v}]\|_{\mathfrak{g}} \leqslant \|\boldsymbol{u}\|_{\mathfrak{g}} \|\boldsymbol{v}\|_{\mathfrak{g}}$ and choose a rational $\mathrm{Ad}(\Gamma)$-stable lattice $\mathfrak{g}_\Gamma \subset \mathfrak{g}$ satisfying $[\mathfrak{g}_\Gamma, \mathfrak{g}_\Gamma] \subseteq \mathfrak{g}_\Gamma$. For every $x \in \Omega = G/\Gamma$, we then set

$$\mathrm{ht}(x) = \sup \left\{ \| \mathrm{Ad}(g)\boldsymbol{v}\|_{\mathfrak{g}}^{-1} : x = g\Gamma,\, \boldsymbol{v} \in \mathfrak{g}_\Gamma \smallsetminus \{\boldsymbol{0}\} \right\}.$$

Let us show, using lattice reduction theory, that there exist constants $c_3 \in (0,1)$ and $\kappa_0 > 0$ so that the map $G \ni g \mapsto gx \in \Omega$ is injective for $d_G(1,g) < c_3 \mathrm{ht}(x)^{-\kappa_0}$. Let $x \in G/\Gamma$ and suppose that $g_1, g_2 \in G$ satisfy $g_1 x = g_2 x$ and $d_G(e, g_i) < c_3 \mathrm{ht}(x)^{-\kappa_0}$ where the values of the constants $c_3 > 0$, $\kappa_0 > 0$ will still be determined. Choose $g \in G$ with $x = g\Gamma$. Then $g_2^{-1} g_1$ stabilizes the lattice $\mathrm{Ad}(g)\mathfrak{g}_\Gamma$. For every non-zero $\boldsymbol{v} \in \mathrm{Ad}(g)\mathfrak{g}_\Gamma$, we have $\|\boldsymbol{v}\| \geqslant \mathrm{ht}(x)^{-1}$. The covolume of $\mathrm{Ad}(g)\mathfrak{g}_\Gamma$ is independent of g and by lattice reduction theory $\mathrm{Ad}(g)\mathfrak{g}_\Gamma$ admits a basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\dim \mathfrak{g}}$ such that, for every $i \in \{1, \ldots, \dim \mathfrak{g}\}$, we have $\|\boldsymbol{v}_i\| \lesssim \mathrm{ht}(x)^{\dim \mathfrak{g} - 1}$. Thus, if we choose $\kappa_0 = \dim \mathfrak{g}$ and c_3 sufficiently small, we have

$$\|g_2^{-1} g_1 \boldsymbol{v}_i - \boldsymbol{v}_i\| \lesssim d_G(1, g_2^{-1} g_1) \|\boldsymbol{v}_i\| \lesssim \mathrm{ht}(x)^{-1}$$

for every $i \in \{1, \ldots, \dim \mathfrak{g}\}$, and thus $g_2^{-1} g_1$ fixes $\mathrm{Ad}(g)\mathfrak{g}_\Gamma$ pointwise. Thus $g_2^{-1} g_1$ belongs to the finite center of G. By choosing $c_3 > 0$ small enough, the claim follows.

Given a lattice Δ in $\mathfrak{g}$, we denote by $\lambda_1(\Delta)$ the first minimum, that is, the length $\|\boldsymbol{v}\|$ of the shortest non-zero vector $\boldsymbol{v}$ in Δ. In particular, we have $\mathrm{ht}(g\Gamma) = \lambda_1(\mathrm{Ad}(g)\mathfrak{g}_\Gamma)^{-1}$. Each element Z in the Lie algebra $\mathfrak{g}$ of G defines a first order differential operator $\mathcal{D}_Z$ on $C_c^\infty(\Omega)$ by

$$\forall\, \phi \in C_c^\infty(\Omega),\, \forall\, x \in \Omega, \quad \mathcal{D}_Z \phi(x) = \frac{d}{dt}\Big|_{t=0} \phi(\exp(tZ)x).$$

Let D be the dimension of $\mathfrak{g}$ and let $\mathcal{B} = (Z_i)_{1 \leqslant i \leqslant D}$ be an orthonormal basis of the real vector space $\mathfrak{g}$. Then, for all $(j_1, \ldots, j_D) \in \mathbb{N}^D$, the monomial

$$\mathcal{D}_Z = \mathcal{D}_{Z_1}^{j_1} \circ \cdots \circ \mathcal{D}_{Z_D}^{j_D} \tag{6.4.1}$$

defines a differential operator of degree $\deg(\mathcal{D}_Z) = j_1 + \cdots + j_D$. For all $\ell \geqslant 1$ and $\phi \in C_c^\infty(\Omega)$, we define the L^2-*Sobolev norm* $\mathcal{S}_\ell(\phi)$ of ϕ by

$$\mathcal{S}_\ell(\phi)^2 = \sum_{\mathcal{D}} \|\mathrm{ht}(x)^\ell\, \mathcal{D}\phi\|^2_{L^2(\Omega)},$$

where the sum ranges over all monomials $\mathcal{D}$ in the chosen basis of degree $\leqslant \ell$.

6.4.2 Equidistribution of translated L-orbits

The following equidistribution result, derived from the effective equidistribution of translated horospherical orbits (see [KM96, Proposition 2.4.8]), is probably standard. For the sake of completeness, we include a proof. We recall that $\mathbf{L}$ denotes the stabilizer in $\mathbf{G}$ of the vector $\boldsymbol{e}_\chi$ and that Ω denotes the homogeneous space G/Γ. For every $g \in \mathbf{G}(\mathbb{Q})$, we consider the stabilizer $\Gamma_L^g = g\Gamma g^{-1} \cap L$ in L of the rational point $x = g\Gamma$ in Ω and we equip the quotient L/Γ_L^g with the unique finite L-invariant measure μ_{L/Γ_L^g} induced from the Haar measure μ_L on L.

Lemma 6.4.1. *Let Q be a compact subset of G. There exist $c > 0$ and $\ell \in \mathbb{N}^\times$, depending only on G, such that, for every $g \in \mathbf{G}(\mathbb{Q})$, $q \in Q$, $0 < y \leqslant 1$, and $\phi \in C_c^\infty(\Omega)$,*

$$\left| \int_{L/\Gamma_L^g} \phi(q a_y l g)\, d\mu_{L/\Gamma_L^g}(l) - \mu_{L/\Gamma_L^g}(1)\, \mu_\Omega(\phi) \right| \lesssim_{g,G,\Gamma,Q} y^c\, \mathcal{S}_\ell(\phi). \tag{6.4.2}$$

Proof. We recall from Section 4.5 that $\mathbf{L}^\circ$ is the semi-direct product of the reductive $\mathbb{Q}$-subgroup $\mathbf{M}$ of $\mathbf{G}$ and the unipotent radical $\mathbf{U}$ of $\mathbf{P}$. Without loss of generality, we may assume that the total volume of L/Γ_L^g is 1. Note that the group $(\mathbf{L}^\circ)(\mathbb{R})$ is open and closed, normal, and of finite index in $L = \mathbf{L}(\mathbb{R})$, so $(\mathbf{L}^\circ)(\mathbb{R})/(\Gamma^g \cap (\mathbf{L}^\circ)(\mathbb{R}))$ is open and closed in L/Γ_L^g, and the latter is the union of a finite number of translates $h(\mathbf{L}^\circ)(\mathbb{R})/(\Gamma^g \cap (\mathbf{L}^\circ)(\mathbb{R}))$ with $h \in \mathcal{Z}(A)$ (since $L \subset P = \mathcal{Z}(A) \ltimes U$ and U is connected). Hence, we may further assume that $L = (\mathbf{L}^\circ)(\mathbb{R})$. In particular, we have $L = M \ltimes U$. Note that $\Gamma_U^g = \Gamma^g \cap U$ and $\Gamma_M^g = \Gamma^g \cap M$ are lattices in U and M, respectively. Let us denote by μ_{U/Γ_U^g} and μ_{M/Γ_M^g} the unique U and M-invariant probability measures on U/Γ_U^g and M/Γ_M^g, respectively. Let Q' be a compact subset of G. We recall that $\mathbf{G}$ is assumed to be simply connected and almost $\mathbb{Q}$-simple. Hence, by the definition of Y_α in (4.4.3), for every projection $p_i : \mathfrak{g} \to \mathfrak{g}_i$ onto a $\mathbb{Q}$-simple factor $\mathfrak{g}_i$ of $\mathfrak{g}$, we have $p_i(Y_\alpha) \neq 0$. Thus, by Theorem 4.6.1, the flow $a_y = \exp(\ln(y)\, Y_\alpha)$ is exponential mixing on Ω. Therefore, by the proof of [KM96, Proposition 2.4.8], there exists a constant $c' > 0$ and $\ell \in \mathbb{N}^\times$, depending only on G, such that, for every $g \in \mathbf{G}(\mathbb{Q})$, $q \in Q'$, $0 < y \leqslant 1$, and $\phi \in C_c^\infty(G/\Gamma)$, we have

$$\left| \int_{U/\Gamma_U^g} \phi(q a_y u g)\, d\mu_{U/\Gamma_U^g}(u) - \mu_\Omega(\phi) \right| \lesssim_{g,G,\Gamma,Q'} y^{c'}\, \mathcal{S}_\ell(\phi). \tag{6.4.3}$$

By [BH62, Corollary 6.4], the semi-direct product $\Gamma_M^g\, \Gamma_U^g$ has finite index in Γ_L^g, and we may, without loss of generality, assume that $\Gamma_L^g = \Gamma_M^g\, \Gamma_U^g$. Let us now show that, for every $\phi \in C_c^\infty(\Omega)$,

$$\int_{L/\Gamma_L^g} \phi(q a_y l g)\, d\mu_{L/\Gamma_L^g}(l) = \int_{M/\Gamma_M^g} \int_{U/\Gamma_U^g} \phi(q a_y m u g)\, d\mu_{U/\Gamma_U^g}(u) d\mu_{M/\Gamma_M^g}(m). \tag{6.4.4}$$

We first observe that conjugation by $\gamma \in \Gamma_M^g$ defines a map from U/Γ_U^g to itself and sends the U-invariant probability measure μ_{U/Γ_U^g} to a U-invariant probability measure, which, by uniqueness, must be μ_{U/Γ_U^g}. Thus, for every $\phi' \in C_c(L/\Gamma_L^g)$, the map $m \mapsto \int_{U/\Gamma_U^g} \phi'(mu)\, d\mu_{U/\Gamma_U^g}$ on M defines a well-defined function on M/Γ_M^g. Note that

$$\Lambda(\phi') = \int_{M/\Gamma_M^g} \int_{U/\Gamma_U^g} \phi'(mu)\, d\mu_{U/\Gamma_U^g}(u) d\mu_{M/\Gamma_M^g}(m)$$

defines a positive L-invariant linear functional on $C_c(L/\Gamma_L^g)$. Thus, by the Riesz-Markov-Kakutani representation theorem, there exists a unique constant $c_0 > 0$ such that for every $\phi' \in C_c(L/\Gamma_L^g)$, we have

$$\int_{L/\Gamma_L^g} \phi'(l)\, d\mu_{L/\Gamma_L^g}(l) = c_0 \int_{M/\Gamma_M^g} \int_{U/\Gamma_U^g} \phi'(hu)\, d\mu_{U/\Gamma_U^g}(u) d\mu_{M/\Gamma_M^g}(h).$$

By Lebesgue's dominated convergence theorem, this equality still holds for $\phi' = 1$. Hence, we have $c_0 = 1$. Since the restriction of $\phi(q a_y \cdot)$ to $L\Gamma^g/\Gamma^g = L/\Gamma_L^g$ is a continuous compactly supported function, equation (6.4.4) follows. As in Section 4.5, we let $\mathfrak{S}$ be a Siegel set of M with respect to Γ_M^g and we let $C \subset \mathbf{M}(\mathbb{Q})$ be the finite subset such that $M = \mathfrak{S}\, C\, \Gamma_M^g$. Therefore, for every $g \in \mathbf{G}(\mathbb{Q})$, $q \in Q$, $0 < y \leqslant 1$ and $\phi \in C_c^\infty(\Omega)$,

$$\begin{aligned}
&\left| \int_{L/\Gamma_L^g} \phi(qa_y lg)\, d\mu_{L/\Gamma_L^g}(l) - \mu_\Omega(\phi) \right| \\
&\qquad = \left| \int_{M/\Gamma_M^g} \int_{U/\Gamma_U^g} \phi(qma_y ug)\, d\mu_{U/\Gamma_U^g}(u) d\mu_{M/\Gamma_M^g}(m) - \mu_\Omega(\phi) \right| \\
&\qquad \lesssim \int_{\mathfrak{S}} \sum_{c \in C} \left| \int_{U/\Gamma_U^g} \phi(qmca_y ug)\, d\mu_{U/\Gamma_U^g}(u) - \mu_\Omega(\phi) \right| d\mu_M(m).
\end{aligned}$$

We recall from Section 4.5 that, for every $\delta \in (0,1)$, we defined

$$\mathfrak{S}(\delta) = \{m \in \mathfrak{S} : \lambda_1(\mathrm{Ad}(h)\mathfrak{g}_\Gamma) < \delta\}.$$

By (4.5.2), there exists a constant $c_1 > 0$ such that, for all $\delta \in (0,1)$ small enough, we have $\mu_M(\mathfrak{S}(\delta)) \lesssim \delta^{c_1}$. In particular, using that $|\phi| \leqslant \|\phi\|_{L^\infty(\Omega)}$, we have

$$\int_{\mathfrak{S}(\delta)} \sum_{c \in C} \left| \int_{U/\Gamma_U^g} \phi(qmca_y ug)\, d\mu_{U/\Gamma_U^g}(u) - \mu_\Omega(\phi) \right| d\mu_M(m) \lesssim \|\phi\|_{L^\infty(\Omega)}\, \delta^{c_1}. \tag{6.4.5}$$

Denote by $\mathfrak{S}(\delta)^c$ the complementary subset of $\mathfrak{S}(\delta)$ in $\mathfrak{S}$. By (4.5.5), for every $m \in \mathfrak{S}(\delta)^c$, we have $\|\mathrm{Ad}(m)\| \lesssim \delta^{-1}$. Moreover, by [EMV09, Section 3.7], provided ℓ is large enough, there is $c_2 > 0$ such that, for every $g \in G$ and $\phi \in C_c^\infty(\Omega)$, we have $\mathcal{S}_\ell(g \cdot \phi) \lesssim \|\mathrm{Ad}(g)\|^{c_2}\, \mathcal{S}_\ell(\phi)$ and $\|\phi\|_{L^\infty(\Omega)} \lesssim \mathcal{S}_\ell(\phi)$. Applying (6.4.3) with the compact subset $Q' = Q\,\mathfrak{S}(\delta)^c\, C$ of G (but keeping the dependency on $m \in \mathfrak{S}(\delta)^c$ explicit), we have

$$\begin{aligned}
&\int_{\mathfrak{S}(\delta)^c} \sum_{c \in C} \left| \int_{U/\Gamma_U^g} \phi(qmca_y ug)\, d\mu_{U/\Gamma_U^g}(u) - \mu_\Omega(\phi) \right| d\mu_m(m) \\
&\qquad \lesssim_{g,G,\Gamma,Q} \int_{\mathfrak{S}(\delta)^c} \sum_{c \in C} y^{c'}\, \|\mathrm{Ad}(m)\|^{c_2} \mathcal{S}_\ell(\phi)\, d\mu_M(m) \\
&\qquad \lesssim_{g,G,\Gamma,Q} \int_{\mathfrak{S}(\delta)^c} y^{c'}\, \delta^{-c_2} \mathcal{S}_\ell(\phi)\, d\mu_M(m) \\
&\qquad \lesssim_{g,G,\Gamma,Q} y^{c'}\, \delta^{-c_2} \mathcal{S}_\ell(\phi)
\end{aligned} \tag{6.4.6}$$

Therefore, combining (6.4.5) and (6.4.6), we have

$$\left| \int_{L/\Gamma_L^g} \phi(ka_y l)\, d\mu_{L/\Gamma_L^g}(l) - \mu_\Omega(\phi) \right| \lesssim y^{c'}\, \delta^{-c_2} \mathcal{S}_\ell(\phi) + \delta^{c_1} \mathcal{S}_\ell(\phi).$$

Setting $\delta = y^{\frac{c'}{c_1+c_2}}$ proves Lemma 6.4.1 with $c = c'\left(1 - \frac{c_2}{c_1+c_2}\right)$. □

6.4.3 A lattice point counting lemma

The following lemma is one of the key ingredients to Proposition 6.2.1, hence to Theorem 1.3.1.

Lemma 6.4.2. *There exist constants $\varkappa' > 0$, $\varepsilon > 0$, and $T_0 > 1$ such that for all $T > T_0$ and all $g_1 \in G$ with $\mathrm{ht}(g_1\Gamma) < T^\varepsilon$, we have*

$$\#\left(g_1\mathcal{L}_\chi \cap \mathcal{B}_T\right) = \varkappa'\, \lambda(\mathcal{B}_T)\left(1 + O(T^{-\varepsilon})\right).$$

Proof. We analyze the contribution of each Γ-orbit in $\mathcal{L}_\chi$ separately. In fact, by a theorem of Borel and Harish-Chandra [Bor69, Proposition 15.6], the set of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ is finite. Moreover, according to [BT65, Lemma 2.6], one has $(\mathbf{G}/\mathbf{P})(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$. As a consequence, $\mathbf{X}(\mathbb{Q})$ is a finite union of Γ-orbits and, since there is a one-to-one correspondence between $\mathbf{X}(\mathbb{Q})$ and lines passing through elements of $\mathcal{L}_\chi$, there exist finitely many $\boldsymbol{v}_1, \dots, \boldsymbol{v}_\kappa \in \mathcal{L}_\chi$ such that

$$\mathcal{L}_\chi = \bigsqcup_{i=1}^{\kappa} \Gamma\boldsymbol{v}_i, \tag{6.4.7}$$

and we can pick $\tau_i \in \mathbf{G}(\mathbb{Q})$ and $\lambda_i > 0$ such that $\boldsymbol{v}_i = \lambda_i\tau_i\boldsymbol{e}_\chi$. Note that $L_i = \tau_i L \tau_i^{-1}$ is the stabilizer of $\boldsymbol{v}_i$ in G and put $\Gamma_{L_i} = \Gamma \cap L_i$. Then for every $g \in G$, we have

$$\#(g\mathcal{L}_\chi \cap \mathcal{B}_T) = \sum_{i=1}^{\kappa} \#(g\Gamma\boldsymbol{v}_i \cap \mathcal{B}_T) = \sum_{i=1}^{\kappa} \sum_{\gamma\in\Gamma/\Gamma_{L_i}} \mathbb{1}_{\mathcal{B}_T}(g\gamma\boldsymbol{v}_i), \tag{6.4.8}$$

We now fix $1 \leqslant i \leqslant \kappa$ and define the function $F_T^{(i)} : \Omega \to \mathbb{R}$ by

$$F_T^{(i)}(g\Gamma) = \sum_{\gamma\in\Gamma/\Gamma_{L_i}} \mathbb{1}_{\mathcal{B}_T}(g\gamma\boldsymbol{v}_i).$$

Using Weil's integration formula (2.3.3) and the change of variable $g\tau_i \mapsto g$, for every measurable bounded function $\phi : \Omega \to \mathbb{R}$, we have

$$\begin{aligned}
\langle F_T^{(i)}, \phi\rangle_{L^2(\Omega)} &= \int_{G/\Gamma} \sum_{\gamma\in\Gamma/\Gamma_{L_i}} \mathbb{1}_{\mathcal{B}_T}(g\gamma \boldsymbol{v}_i)\phi(g\Gamma)\, d\mu_\Omega(g\Gamma) \\
&= \int_{G/\Gamma_{L_i}} \mathbb{1}_{\mathcal{B}_T}(g\boldsymbol{v}_i)\phi(g\Gamma)\, d\mu_{G/\Gamma_{L_i}}(g\Gamma_{L_i}) \\
&= \int_{G/\tau_i^{-1}\Gamma_{L_i}\tau_i} \mathbb{1}_{\mathcal{B}_T}(\lambda_i g\boldsymbol{e}_\chi)\phi(g\tau_i^{-1}\Gamma)\, d\mu_{G/\tau_i^{-1}\Gamma_{L_i}\tau_i}(g(\tau_i^{-1}\Gamma_{L_i}\tau_i)).
\end{aligned}$$

Let us write $\Gamma_i = \tau_i^{-1}\Gamma\tau_i$ and hence $\tau_i^{-1}\Gamma_{L_i}\tau_i = \Gamma_i \cap L$. Using Weil's integration formula (2.3.3) again, together with the measure decomposition of μ_G and the fact that L fixes $\boldsymbol{e}_\chi$, we have

$$\langle F_T^{(i)}, \phi\rangle_{L^2(\Omega)} = \int_{\widetilde{X}} \int_{L/\Gamma_i\cap L} \mathbb{1}_{\mathcal{B}_T}(\lambda_i k a(y)\boldsymbol{e}_\chi)\phi(ka(y)\ell\tau_i^{-1})\, d\mu_{L/\Gamma_i\cap L}(\ell\Gamma_i\cap L) d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).$$

Let us carry out the change of variable $yy_i \mapsto y$ with $y_i = \lambda_i^{-\beta_\chi}$, so that $a(y_i)\boldsymbol{e}_\chi = y_i^{-\frac{1}{\beta_\chi}}\boldsymbol{e}_\chi = \lambda_i\boldsymbol{e}_\chi$. Then $\langle F_T^{(i)}, \phi\rangle_{L^2(\Omega)}$ is given by

$$\int_{\widetilde{X}} \mathbb{1}_{\mathcal{B}_T}(ka(y)\boldsymbol{e}_\chi) \left(\lambda_i^{-\beta_\chi d} \int_{L/\Gamma_i\cap L} \phi(ka(y_i)^{-1}a(y)\ell\tau_i^{-1})\, d\mu_{L/\Gamma_i\cap L}(\ell)\right) d\lambda_{\widetilde{X}}(ka(y)\boldsymbol{e}_\chi).$$

In particular, setting $\phi = 1$, we have

$$\int_\Omega F_T^{(i)}\, d\mu_\Omega = \nu_i\, \lambda_{\widetilde{X}}(\mathcal{B}_T), \quad \text{with } \ \nu_i = \lambda_i^{-\beta_\chi d} \mu_{L/\tau_i^{-1}\Gamma_{L_i}\tau_i}\left(L/\tau_i^{-1}\Gamma_{L_i}\tau_i\right).$$

Hence we only need to show that there exists $\varepsilon > 0$, independent of i, such that for all sufficiently large T and for all $g_1 \in G$ with $\mathrm{ht}(g_1\Gamma) < T^\varepsilon$, we have

$$F_T^{(i)}(g_1\Gamma) = \#\left(g_1\Gamma\boldsymbol{v}_i \cap \mathcal{B}_T\right) = \nu_i\, \lambda_{\widetilde{X}}(\mathcal{B}_T)\left(1 + O(T^{-\varepsilon})\right).$$

We will obtain Lemma 6.4.2 with

$$\varkappa' = \sum_{i=1}^{\kappa} \nu_i = \sum_{i=1}^{\kappa} \lambda_i^{-\beta_\chi d} \mu_{L/\tau_i^{-1}\Gamma_{L_i}\tau_i}\left(L/\tau_i^{-1}\Gamma_{L_i}\tau_i\right). \tag{6.4.9}$$

Note that

$$\sup\{y \in \mathbb{R}_+^\times : \exists\, k \in K,\ ka(y)\boldsymbol{e}_\chi \in \mathcal{B}_T\} \lesssim T^{-(\beta_\chi-\tau)}.$$

Let $\omega_i = \lambda_i^{\beta_\chi d}\nu_i$. For all $\phi \in C_c^\infty(\Omega)$, by Lemma 6.4.1, we have

$$\begin{aligned}
&\left|\langle F_T^{(i)}, \phi\rangle_{L^2(\Omega)} - \nu_i\,\lambda_{\tilde{X}}(\mathcal{B}_T)\int_\Omega \phi\, d\mu_\Omega\right| \\
&\leqslant \int_{\tilde{X}} \mathbb{1}_{\mathcal{B}_T}(ka(y)\boldsymbol{e}_\chi)\lambda_i^{-\beta_\chi d}\left|\int_{L/\Gamma_i\cap L}\phi(ka(y_i)^{-1}a(y)\ell\tau_i^{-1})\,d\mu_{L/\Gamma_i\cap L}(\ell)\right. \\
&\qquad\qquad\qquad\qquad\left. - \omega_i\int_{G/\Gamma}\phi\,d\mu_\Omega\right| d\lambda_{\tilde{X}}(ka(y)\boldsymbol{e}_\chi) \\
&\lesssim \left(\int_{\tilde{X}} \mathbb{1}_{\mathcal{B}_T}(ka(y)\boldsymbol{e}_\chi)y^c\,d\lambda_{\tilde{X}}(ka(y)\boldsymbol{e}_\chi)\right)\mathcal{S}_\ell(\phi) \\
&\lesssim T^{-c(\beta_\chi-\tau)}\,\lambda_{\tilde{X}}(\mathcal{B}_T)\,\mathcal{S}_\ell(\phi).
\end{aligned}$$

Let $c_1 > 0$ be as in Lemma 6.3.3, and let $\mathcal{O} = (\mathcal{O}_\delta)_{0<\delta<1}$ with

$$\mathcal{O}_\delta = B_G(c_1\delta).$$

Recall that for every $x \in \Omega$, the positive number $c_3\mathrm{ht}(x)^{-\kappa_0}$ is a lower bound for the injectivity radius at x. Using [KM96, Lemma 2.4.7], for each $\varepsilon_1 > 0$ and $g_1 \in G$ with $\mathrm{ht}(g_1\Gamma) < T^{\varepsilon_1}$, we let $\phi \in C_c^\infty(\Omega)$ be a non-negative function so that

(1) $\mu_\Omega(\phi) = 1$,
(2) $\operatorname{Supp}\phi \subseteq \mathcal{O}_\delta.g_1\Gamma$ with $\delta = c_3T^{-\varepsilon_1\kappa_0} \leqslant c_3\mathrm{ht}(g_1\Gamma)^{-\kappa_0}$,
(3) $\mathcal{S}_\ell(\phi) \lesssim T^{\varepsilon_1 D}$, where $D = \kappa_0(\ell + (\dim G)/2) + \ell$.

Putting everything together, we obtain the averaged counting result

$$\left|\langle F_T^{(i)}, \phi\rangle_{L^2(\Omega)} - \nu_i\,\lambda_{\tilde{X}}(\mathcal{B}_T)\right| \lesssim T^{-c(\beta_\chi-\tau)+\varepsilon_1 D}\lambda_{\tilde{X}}(\mathcal{B}_T).$$

Using Proposition 6.3.4, the family $(\mathcal{B}_T)_{T\geqslant 1}$ is well-rounded, and we showed that there exists $C_1 > 0$ such that for all sufficiently large $T \geqslant 1$ and $\delta \in (0,1)$, there are Borel sets $\underline{\mathcal{B}}_{T,\delta}, \overline{\mathcal{B}}_{T,\delta}$ satisfying

$$\underline{\mathcal{B}}_{T,\delta} \subseteq \bigcap_{g\in\mathcal{O}_\delta} g\mathcal{B}_T \subseteq \bigcup_{g\in\mathcal{O}_\delta} g\mathcal{B}_T \subseteq \overline{\mathcal{B}}_{T,\delta}$$

and $\lambda_{\tilde{X}}(\overline{\mathcal{B}}_{T,\delta} \smallsetminus \underline{\mathcal{B}}_{T,\delta}) \leqslant C_1\delta\lambda_{\tilde{X}}(\mathcal{B}_T)$. For $\delta = c_3 T^{-\varepsilon_1\kappa_0} \in (0,1)$, define

$$\underline{F}_{T,\delta}(g) = \sum_{\gamma\in\Gamma/\Gamma_{L_i}} \mathbb{1}_{\underline{\mathcal{B}}_{T,\delta}}(g\gamma\tau_i \boldsymbol{e}_\chi) \quad \text{and} \quad \overline{F}_{T,\delta}(g) = \sum_{\gamma\in\Gamma/\Gamma_{L_i}} \mathbb{1}_{\overline{\mathcal{B}}_{T,\delta}}(g\gamma\tau_i \boldsymbol{e}_\chi).$$

Then for $g\Gamma \in \operatorname{Supp}\phi$, there exists $h \in \mathcal{O}_\delta$ such that $g\Gamma = hg_1\Gamma$. Thus

$$\underline{F}_{T,\delta}(g) = \underline{F}_{T,\delta}(hg_1) = \#\left(g_1\Gamma\tau_i\boldsymbol{e}_\chi \cap h^{-1}\underline{\mathcal{B}}_{T,\delta}\right) \leqslant F_T^{(i)}(g_1).$$

Multiplying by $\phi(g)$ and integrating over the support of ϕ gives

$$\langle \underline{F}_{T,\delta}, \phi\rangle_{G/\Gamma} \leqslant F_T^{(i)}(g_1).$$

Similarly as above, we can estimate

$$\left|\langle \underline{F}_{T,\delta}, \phi\rangle_{G/\Gamma} - \nu_i\lambda_{\tilde{X}}(\underline{\mathcal{B}}_{T,\delta})\right| \lesssim T^{-c(\beta_\chi-\tau)+\varepsilon_1 D}\lambda_{\tilde{X}}(\underline{\mathcal{B}}_{T,\delta}).$$

Moreover, from $\lambda_{\tilde{X}}(\overline{\mathcal{B}}_{T,\delta} \smallsetminus \underline{\mathcal{B}}_{T,\delta}) \leqslant C_1\delta\lambda_{\tilde{X}}(\mathcal{B}_T)$ we get

$$\lambda_{\tilde{X}}(\underline{\mathcal{B}}_{T,\delta}) \geqslant (1 - C_1\delta)\lambda_{\tilde{X}}(\mathcal{B}_T).$$

Therefore, one has

$$\begin{aligned} F_T^{(i)}(g_1) - \nu_i\lambda_{\tilde{X}}(\mathcal{B}_T) &\geqslant \langle \underline{F}_{T,\delta}, \phi\rangle_{G/\Gamma} - \nu_i\lambda_{\tilde{X}}(\underline{\mathcal{B}}_{T,\delta}) + \nu_i\lambda_{\tilde{X}}(\underline{\mathcal{B}}_{T,\delta}) - \nu_i\lambda_{\tilde{X}}(\mathcal{B}_T) \\ &\gtrsim -T^{-c(\beta_\chi-\tau)+\varepsilon_1 D}\lambda_{\tilde{X}}(\underline{\mathcal{B}}_{T,\delta}) - C_1\delta\lambda_{\tilde{X}}(\mathcal{B}_T) \\ &\gtrsim -(T^{-c(\beta_\chi-\tau)+\varepsilon_1 D} + T^{-\varepsilon_1\kappa_0})\lambda_{\tilde{X}}(\mathcal{B}_T). \end{aligned}$$

Similarly, using the subset $\overline{\mathcal{B}}_{T,\delta}$, one can show that

$$F_T^{(i)}(g_1) - \nu_i\lambda_{\tilde{X}}(\mathcal{B}_T) \lesssim (T^{-c(\beta_\chi-\tau)+\varepsilon_1 D} + T^{-\varepsilon_1\kappa_0})\lambda_{\tilde{X}}(\mathcal{B}_T).$$

Together, this gives

$$\left|F_T^{(i)}(g_1) - \nu_i\lambda_{\tilde{X}}(\mathcal{B}_T)\right| \lesssim (T^{-c(\beta_\chi-\tau)+\varepsilon_1 D} + T^{-\varepsilon_1\kappa_0})\lambda_{\tilde{X}}(\mathcal{B}_T).$$

Setting $\varepsilon_1 = \frac{c(\beta_\chi-\tau)}{D+\kappa_0}$ proves Lemma 6.4.2 with $\varepsilon = \varepsilon_1$ and $\varkappa'$ given by Equation (6.4.9). □

6.4.4 Proof of Proposition 6.2.1

Proof of Proposition 6.2.1. By the proof of [Sax20, Theorem 2.4.5], using the ergodicity of the action of A on the probability space (Ω, μ_Ω), for μ_G-almost every $g \in G$, we have

$$\lim_{y\to\infty} \frac{\ln\left(\lambda_1(\mathrm{Ad}(a(y)g)\mathfrak{g}_\Gamma)\right)}{\ln(y)} = 0. \tag{6.4.10}$$

For every $p \in P$, there exists $p_\infty \in P$ such that, as $y \to +\infty$, we have $a(y)pa(y)^{-1} \to p_\infty$. Hence the above limit only depends on the class Pg of g in $P\backslash G$. Therefore, for σ_X-almost every $x = k_xP \in X$, we have

$$\lim_{y\to\infty} \frac{\ln\left(\lambda_1(\mathrm{Ad}(a(y)k_x^{-1})\mathfrak{g}_\Gamma)\right)}{\ln(y)} = 0. \tag{6.4.11}$$

Plugging in the definition $y_T = T^\tau$ of y_T, there exists a full measure subset $X_0 \subseteq X$ such that for every $x \in X_0$, we have, as $T \to +\infty$,

$$\lambda_1(\mathrm{Ad}(a(y_T)k_x^{-1})\mathfrak{g}_\Gamma) = T^{o(1)}. \tag{6.4.12}$$

Fix $x \in X_0$. Since $\mathcal{L}_\chi = \bigsqcup_{1\leqslant i\leqslant\kappa} \Gamma\,\boldsymbol{v}_i$ is a finite union of Γ-orbits (see (6.4.7)), it is sufficient to prove that there exists $\varepsilon \in (0,1)$, independent of i, such that for every sufficiently large $T \geqslant 1$,

$$\#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{E}_\tau(T)\right) = \nu_i\lambda_{\widetilde{X}}(\mathcal{E}_\tau(T))\left(1 + O_x(T^{-\varepsilon})\right). \tag{6.4.13}$$

Recall that for every $T \geqslant 1$, we defined

$$\mathcal{F}_T = \left\{\boldsymbol{v} \in \widetilde{X}_0 : d(x_0, [\boldsymbol{v}]) < \psi(\|\boldsymbol{v}\|),\ \max\{1, T/2\} \leqslant \|\boldsymbol{v}\| < T\right\}.$$

With $T_j = T/2^j$ for every $j \in \mathbb{N}$, this gives the disjoint union

$$\mathcal{E}_\tau(T) = \bigsqcup_{j\geqslant 0} \mathcal{F}_{T_j},$$

and therefore

$$\#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{E}_\tau(T)\right) = \sum_{j\geqslant 0} \#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{F}_{T_j}\right). \tag{6.4.14}$$

For a given $T \geqslant 1$, we now analyze and estimate each term $\#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{F}_T\right)$ individually. In view of (6.4.12) we have $\mathrm{ht}(a(y_T)k_x^{-1}\Gamma) = T^{o(1)}$. Hence

Lemma 6.4.2 and the invariance of the measure $\lambda_{\tilde{X}}$ under A since $\mathcal{B}_T = a(y_T)\mathcal{F}_T$ imply that for all sufficiently large $T > T_0$,

$$\begin{aligned}\#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{F}_T\right) &= \#\left(a(y_T)k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{B}_T\right)\\ &= \nu_i\lambda_{\tilde{X}}(\mathcal{F}_T)\left(1 + O_x(T^{-\varepsilon})\right)\\ &= \nu_i\lambda_{\tilde{X}}(\mathcal{B}_T)\left(1 + O_x(T^{-\varepsilon})\right).\end{aligned}$$

Note that

$$\sum_{T_j \leqslant T_0} \#\left(k_x^{-1}\Gamma\tau_i\boldsymbol{e}_\chi \cap \mathcal{F}_{T_j}\right) - \sum_{T_j \leqslant T_0} \nu_i\lambda_{\tilde{X}}(\mathcal{F}_{T_j}) = O_x\left(\sum_{j \geqslant 0} T_j^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{F}_{T_j})\right).$$

Returning to (6.4.14), we deduce that

$$\begin{aligned}&\#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{E}_\tau(T)\right) = \sum_{j \geqslant 0} \#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{F}_{T_j}\right)\\ &= \sum_{T_j \leqslant T_0} \#\left(k_x^{-1}\Gamma\boldsymbol{v}_i \cap \mathcal{F}_{T_j}\right) + \sum_{T_j > T_0} \nu_i\lambda_{\tilde{X}}(\mathcal{F}_{T_j}) + O_x\left(\sum_{T_j > T_0} T_j^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{F}_{T_j})\right)\\ &= \nu_i\,\lambda_{\tilde{X}}(\mathcal{E}_\tau(T)) + O_x\left(\sum_{j \geqslant 0} T_j^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{F}_{T_j})\right).\end{aligned}$$

Thus to prove the formula (6.4.13), it suffices to show that if $\varepsilon \in (0, 1)$ is small enough, then

$$\sum_{j \geqslant 0} T_j^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{F}_{T_j}) \lesssim T^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{E}_\tau(T)).$$

Using the volume estimate

$$\lambda_{\tilde{X}}(\mathcal{F}_{T_j}) \lesssim \int_{T_{j+1}}^{T_j} y^{(\beta_\chi - \tau)d}\frac{dy}{y} \lesssim T_j^{(\beta_\chi-\tau)d} - T_{j+1}^{(\beta_\chi-\tau)d} = (1 - 2^{-(\beta_\chi-\tau)d})T_j^{(\beta_\chi-\tau)d},$$

and choosing $\varepsilon > 0$ smaller than $(\beta_\chi - \tau)d$ if necessary, we get

$$\begin{aligned}\sum_{j \geqslant 0} T_j^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{F}_{T_j}) &\lesssim \sum_{j \geqslant 0} T_j^{(\beta_\chi-\tau)d-\varepsilon} \lesssim T^{(\beta_\chi-\tau)d-\varepsilon}\sum_{j \geqslant 0}\left(\frac{1}{2^{(\beta_\chi-\tau)d-\varepsilon}}\right)^j\\ &\lesssim T^{-\varepsilon}\lambda_{\tilde{X}}(\mathcal{E}_\tau(T)).\end{aligned}$$

$\square$

6.5 Proof of Theorem 1.3.2 and a corollary

In this section, we briefly explain how Theorem 1.3.2 follows from the proof of Theorem 1.3.1.

For the reader's convenience, we recall its statement. Recall that, for every $x \in X$ and radius $r > 0$, we denote by $B_r(x)$ the open ball in X with center x and radius r, and we define the counting function

$$\mathcal{N}_\chi(x, r, T) = \#\left\{v \in \mathbf{X}(\mathbb{Q}) : v \in B_r(x),\, 1 \leqslant H_\chi(v) < T\right\}.$$

Let $\psi_r : \mathbb{R}_+ \to \mathbb{R}_+$ be the constant function defined by $\psi_r(t) = r$. Then, in the notation of Theorems 1.2.2 and 1.3.1, we remark that, for every $x \in T$ and $T \geqslant 1$, we have $\mathcal{N}_\chi(x, r, T) = \mathcal{N}_{\psi_r}(x, T)$.

Theorem. *Let $X = \mathbf{X}(\mathbb{R})$ be the space of real points of a generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$, obtained as the quotient of a connected semisimple algebraic $\mathbb{Q}$-group by a parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ with abelian unipotent radical. Then, there exists an explicit constant $\varkappa > 0$ and $\varepsilon > 0$ such that for all $r > 0$ and $x \in X$, as $T \to +\infty$, we have*

$$\mathcal{N}_\chi(x, r, T) = \varkappa\, T^{\beta_\chi d} \sigma_X(B_r(x)) \left(1 + O_r(T^{-\varepsilon})\right).$$

Proof of Theorem 1.3.2. Let the constant $\varkappa > 0$ be given as in Equation (6.2.4). Fix $x \in X$ and $r > 0$. For every $T \geqslant 1$, we define the set

$$\mathcal{E}(r, T) = \left\{\boldsymbol{v} \in \widetilde{X}_0 : d(x_0, [\boldsymbol{v}]) < r,\, 1 \leqslant \|\boldsymbol{v}\| < T\right\}. \tag{6.5.1}$$

By Lemma 4.7.1 applied to $\psi = \psi_r$, for every $x \in X$ and $T \geqslant 1$, we have

$$\mathcal{N}_\chi(x, r, T) = [K \cap P : K \cap L]^{-1} \# \left(k_x^{-1} \mathcal{L}_\chi \cap \mathcal{E}(r, T)\right).$$

For $T \geqslant 1$, we define

$$\mathcal{F}(r, T) = \left\{\boldsymbol{v} \in \widetilde{X}_0 : d(x_0, [\boldsymbol{v}]) < r,\, \max\{1, T/2\} \leqslant \|\boldsymbol{v}\| < T\right\}.$$

We tessellate $\mathcal{E}(r, T)$ dyadically:

$$\mathcal{E}(r, T) = \bigsqcup_{j \geqslant 0} \mathcal{F}(r, T_j) \quad \text{where } T_j = T/2^j,\ j \geqslant 0.$$

For any $\delta \in (0,1)$, we define the sets

$$\underline{\mathcal{F}}(r,T,\delta) = \left\{ \boldsymbol{v} \in \widetilde{X}_0 : d(x_0,[\boldsymbol{v}]) < (1+\delta)^{-1}\, r,\ (1+\delta)\tfrac{T}{2} < \|\boldsymbol{v}\| < (1+\delta)^{-1}T \right\}$$

and

$$\overline{\mathcal{F}}(r,T,\delta) = \left\{ \boldsymbol{v} \in \widetilde{X}_0 : d(x_0,[\boldsymbol{v}]) < (1+\delta)\, r,\ (1+\delta)^{-1}\tfrac{T}{2} < \|\boldsymbol{v}\| < (1+\delta)T \right\}.$$

Then, following the notation of Lemma 6.3.3, we simply put (without applying an element of A)

$$\underline{\mathcal{B}}(r,T,\delta) = \underline{\mathcal{F}}(r,T,\delta) \quad \text{and} \quad \overline{\mathcal{B}}(r,T,\delta) = \overline{\mathcal{F}}(r,T,\delta).$$

Then Lemma 6.3.3 and Proposition 6.3.4 still apply, and hence the family $(\mathcal{B}(r,T))_{T\geqslant 1}$ is well-rounded in the sense of Definition 6.3.1. In the proof of Proposition 6.2.1, we used the ergodicity of the action of A on G/Γ to show that for almost every $x \in X$, we have

$$\lambda_1(\mathrm{Ad}(a(y_T)k_x^{-1})\mathfrak{g}_\Gamma) = T^{o(1)} \qquad \text{as } T \to +\infty.$$

Since here $y_T = 1$, and hence $a(y_T) = 1$, we have, for every $x \in X$, that $\lambda_1(\mathrm{Ad}(a(y_T)k_x^{-1})\mathfrak{g}_\Gamma) \asymp 1$ and hence also $\mathrm{ht}(a(y_T)k_x^{-1}\Gamma) \asymp 1$. Hence, by Lemma 6.4.2, there exist constants $\varkappa' > 0$, $\varepsilon' > 0$, and $T_0 > 1$ such that for all $T > T_0$,

$$\#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{B}(r,T)\right) = \varkappa'\, \lambda_{\widetilde{X}}(\mathcal{B}(r,T)) \left(1 + O(T^{-\varepsilon'})\right).$$

The proof now proceeds similarly (with $\tau = 0$ and $\psi = \psi_r$) as the one of Proposition 6.2.1. Thus, by choosing $\varepsilon = \min\{\varepsilon', \beta_\chi d/2\}$, we get

$$\#\left(k_x^{-1}\mathcal{L}_\chi \cap \mathcal{E}(r,T)\right) = \varkappa'\, \lambda_{\widetilde{X}}(\mathcal{E}(r,T)) \left(1 + O(T^{-\varepsilon})\right).$$

By the volume computation in the proof of Theorem 1.3.1, we have

$$\begin{aligned}
\lambda_{\widetilde{X}}(\mathcal{E}(r,T)) &= \int_{T^{-\beta_\chi}}^{1} \int_{K/K\cap L} \mathbb{1}_{B_X(r)}(kx_0)\, \omega_1\, d\sigma(k) \frac{dy}{y^{d+1}} \\
&= \int_{T^{-\beta_\chi}}^{1} \sigma_X(B_X(r))\, \omega_1\, \frac{dy}{y^{d+1}} \\
&= \frac{1}{d}\, \omega_1\, \sigma_X(B_X(r))\, T^{\beta_\chi d} + O(1),
\end{aligned}$$

as required. This proves Theorem 1.3.2 with $\varepsilon = \min\{\varepsilon', \beta_\chi d/2\}$ and

$$\varkappa = \varkappa' \frac{1}{d} [K \cap P : K \cap L]^{-1} \omega_1.$$

$\square$

Let $X = \mathbf{X}(\mathbb{R})$ be the set of real points of a generalized flag variety, obtained as the quotient $\mathbf{X} = \mathbf{G}/\mathbf{P}$ of a connected semisimple algebraic $\mathbb{Q}$-group by a maximal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$. Without the assumption that the unipotent radical of $\mathbf{P}$ is abelian, we can still asymptotically count rational points of bounded height in X. This complements the work of Mohammadi and Salehi Golsefidy [MG14, Theorem 4], where they additionally assume $\mathbf{G}$ to be simply connected. This estimate is used in the proof of Lemma 3.3.1.

We equip X with the Carnot–Carathéodory distance $d(\cdot,\cdot)$, whose construction is detailed in [Sax20, §2.2]. In the case where the unipotent radical of $\mathbf{P}$ is abelian, this distance is a Riemannian distance.

For every $T \geqslant 1$, we define the function

$$\mathcal{N}_\chi(T) = \#\{v \in \mathbf{X}(\mathbb{Q}) : H_\chi(v) < T\}$$

counting rational points in X of height bounded by T. Indeed, observe that the Carnot-Carathéodory distance, for which we were not able to derive estimates similar to (6.2.2) and (6.3.12), is not involved in the definition of $\mathcal{N}_\chi(T)$. Therefore, the steps in the proof of Theorem 1.3.2 above still apply, and we get the following asymptotic estimate.

Corollary 6.5.1. *Let $X = \mathbf{X}(\mathbb{R})$ be the set of real points of a generalized flag variety $\mathbf{X} = \mathbf{G}/\mathbf{P}$, obtained as the quotient of a connected semisimple algebraic $\mathbb{Q}$-group by a maximal parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$. Let H_χ be a height on $\mathbf{X}(\mathbb{Q})$ corresponding to a highest $\mathbb{Q}$-weight χ of $\mathbf{G}$. Then, there exists an explicit constant $\varkappa > 0$ such that, as $T \to +\infty$, we have*

$$\mathcal{N}_\chi(T) \;=\; \varkappa T^{\beta_\chi d}\left(1 + O(T^{-\varepsilon})\right).$$

6.6 Applications

We briefly outline how our applications fit within the framework of Theorems 1.2.2 and 1.3.1.

6.6.1 Projective quadrics

Let $n \geqslant 1$ be a positive integer, and let $Q : \mathbb{R}^{n+2} \to \mathbb{R}$ be a non-degenerate $\mathbb{Q}$-isotropic rational quadratic form in $n+2$ variables. We write $\mathbf{X}_Q = \mathbf{G}/\mathbf{P}$, where $\mathbf{G} = \mathrm{SO}_Q$ is the special orthogonal group associated with the quadratic form Q, and $\mathbf{P} \leqslant \mathbf{G}$ is the parabolic $\mathbb{Q}$-subgroup stabilizing a rational isotropic line in the standard representation. If the quadratic form Q is not conjugated over $\mathbb{Q}$ to the exceptional quadratic form $Q_0(\boldsymbol{x}) = x_1x_4 - x_2x_3$, then the group $\mathbf{G}$ is $\mathbb{Q}$-simple and the parabolic $\mathbb{Q}$-subgroup $\mathbf{P}$ is maximal: $\operatorname{rank}_{\mathbb{Q}} \mathbf{P} = \operatorname{rank}_{\mathbb{Q}} \mathbf{G} - 1$. On the other hand, if Q is conjugated over $\mathbb{Q}$ to Q_0, we have an isomorphism

$$\mathbf{G} \simeq \mathrm{SO}(2,2) \simeq \mathrm{SO}(2,1) \times \mathrm{SO}(2,1),$$

and $\operatorname{rank}_{\mathbb{Q}} \mathbf{P} = \operatorname{rank}_{\mathbb{Q}} \mathbf{G} - 2$. The space of real points of $\mathbf{X}_Q$ is identified with the n-dimensional projective rational quadric hypersurface given as the set of zeros in $\mathbb{P}(\mathbb{R}^{n+2})$ of the quadratic form Q:

$$X_Q = [Q^{-1}(0)] = \big\{x \in \mathbb{P}(\mathbb{R}^{n+2}) : x = [\boldsymbol{x}] \text{ with } Q(\boldsymbol{x}) = 0\big\}. \tag{6.6.1}$$

The distance $d(\cdot,\cdot)$ and the height function H are obtained by restriction of the usual distance and height function on $\mathbb{P}(\mathbb{R}^{n+2})$, respectively. Let K be a maximal compact subgroup of the special orthogonal group $\mathrm{SO}_Q(\mathbb{R})$ associated to Q and let σ_Q be the K-invariant probability measure on X_Q. Furthermore, one assumes that X_Q contains a rational point; by stereographic projection, this implies in fact that $\mathbf{X}_Q(\mathbb{Q})$ is dense in X_Q.

As mentioned in Remark 1.4.3, let us briefly explain why our methods, both for counting below and at the Diophantine exponent, do not apply to the exceptional quadric hypersurface X_0. Indeed, Lemmas 6.4.2 and 6.4.1 rely on the assumption that the parabolic subgroup defining X is maximal, whereas the parabolic subgroup defining X_0 is *not* maximal. This maximality is essential for the existence of a G-invariant measure on the cone $\widetilde{X}_0$, for guaranteeing that $L \cap \Gamma$ is a lattice in L, and for the validity of the mean value formula (1.1.1), which was also used in (5.1.11).

6.6.2 Grassmann varieties

Let $1 \leqslant \ell < n$ be positive integers, and write $\mathbf{X}_\ell = \mathbf{G}/\mathbf{P}$, where $\mathbf{G} = \mathrm{SL}_n$ and $\mathbf{P} \leqslant \mathbf{G}$ is the parabolic $\mathbb{Q}$-subgroup stabilizing a rational line spanned by a pure tensor in the ℓ-th exterior power of the standard representation. Then the parabolic subgroup $\mathbf{P}$ is maximal and X_ℓ, viewed as a subvariety of $\mathbb{P}(\bigwedge^\ell \mathbb{R}^n)$, is the Grassmann manifold $\mathrm{Gr}_{\ell,n}(\mathbb{R})$ of ℓ-dimensional subspaces of $\mathbb{R}^n$. This is in accordance with Schmidt's paper [Sch67], where he used the Plücker embedding to define the height $H(v)$ of a rational subspace v of $\mathbb{R}^n$. The distance used on X_ℓ is the usual Riemannian distance and we equip X_ℓ with the unique probability measure σ_ℓ invariant under $K = \mathrm{SO}_n(\mathbb{R})$. Let $\mathbf{T}$ be the subgroup of $\mathbf{G}$ consisting of all diagonal matrices. Then $\mathbf{T}$ is a maximal $\mathbb{Q}$-split $\mathbb{Q}$-torus. Let $\mathbf{P}_0$ be the Borel subgroup of $\mathbf{G}$ consisting of all upper-triangular matrices. Let $\Phi(\mathbf{G}, \mathbf{T})$ be the associated root system with ordering induced by $\mathbf{P}_0$, Δ the set of simple roots, and $(\lambda_\alpha)_{\alpha\in\Delta}$ the set of fundamental $\mathbb{Q}$-weights. Let $\alpha \in \Delta$ be the simple root such that $\mathbf{P} = \mathbf{P}_{\Delta\smallsetminus\{\alpha\}}$ is the standard parabolic $\mathbb{Q}$-subgroup corresponding to the subset $\Delta \smallsetminus \{\alpha\}$ of simple roots. The representation of $\mathbf{G}$ given by the ℓ-th exterior power is the unique strongly rational representation of $\mathbf{G}$ corresponding to the choice of dominant $\mathbb{Q}$-weight χ given by the fundamental $\mathbb{Q}$-weight λ_α.

We refer to the introduction for the statements of Theorems 1.4.4 and 1.4.5, that are special cases of the Theorems 1.2.2, 1.3.1 and 1.3.2.

Bibliography

[AG22] M. Alam and A. Ghosh. “Quantitative rational approximation on spheres”. In: *Selecta Math. (N.S.), 28(5), Paper No. 86* (2022).

[APT16] J. S. Athreya, A. Parrish, and J. Tseng. “Ergodic theorem and Diophantine approximation for translation surfaces and linear forms”. In: *Nonlinearity, 29(8):2173–2190* (2016).

[BG19] M. Björklund and A. Gorodnik. “Central limit theorems for Diophantine approximants”. In: *Math. Ann. 374, No. 3-4, 1371-1437* (2019).

[BG21] M. Björklund and A. Gorodnik. “Effective multiple equidistribution of translated measures”. In: *doi: 10.48550/ARXIV.2105.05468* (2021).

[BH62] A. Borel and Harish-Chandra. “Arithmetic subgroups of algebraic groups”. In: *Annals of Mathematics, Vol. 75, No. 3, pp. 485-535* (1962).

[Bir31] G. D. Birkhoff. “Proof of the ergodic theorem”. In: *Proc. Nat. Acad. Sci., U.S.A. 17, 656–660* (1931).

[BM00] M. B. Bekka and M. Mayer. “Ergodic theory and topological dynamics of group actions on homogeneous spaces”. In: *Cambridge University Press, London Mathematical Society, Lecture Note Series 269, Cambridge* (2000).

[Bor69] A. Borel. “Introduction aux groupes arithmétiques”. In: *Hermann, Paris* (1969).

[Bou04] N. Bourbaki. “Elements of Mathematics. Integration II, Chapters 7–9”. In: *Springer-Verlag Berlin Heidelberg* (2004).

[BT65] A. Borel and J. Tits. “Groupes réductifs”. In: *Publications mathématiques de l’I.H.É.S., 27, 55-151* (1965).

[Dan85] S. G. Dani. "Divergent trajectories of flows on homogeneous spaces and diophantine approximation". In: *J. Reine Angew. Math., 359 :55-89* (1985).

[DRP93] W. Duke, Z. Rudnick, and P.Sarnak. "Density of integer points on affine homogeneous varieties". In: *Duke Math. J. 71(1): 1n3-179* (1993).

[EM93] A. Eskin and C. McMullen. "Mixing, counting, and equidistribution in Lie groups". In: *Duke Math. J. 71(1): 181-209* (1993).

[EMV09] M. Einsiedler, G. Margulis, and A. Venkatesh. "Effective equidistribution for closed orbits of semisimple groups on homogeneous spaces". In: *Invent. Math., 177, no. 1, 137–212. 2507639* (2009).

[Fis+22] L. Fishman et al. "Intrinsic diophantine approximation on quadric hypersurfaces". In: *J. Eur. Math. Soc. (JEMS), 24, No. 3, 1045-1101* (2022).

[Fol15] G. Folland. "A course in abstract harmonic analysis, second edition". In: *Chapman Hall* (2015).

[GN10] A. Gorodnik and A. Nevo. "The ergodic theory of lattice subgroups". In: *Annals of Mathematics Studies, vol. 172, Princeton University Press,* (2010).

[GN12] A. Gorodnik and A. Nevo. "Counting lattice points". In: *J. reine. angew. Math. 663, 127–176.* (2012).

[Gra74] A. Gray. "The volume of a small geodesic ball of a Riemannian manifold". In: *Michigan Math. J., 20(4): 329-344* (1974).

[Har98] G. Harman. "Metric number theory". In: *LMS Monographs New Series, vol. 18, Clarendon Press* (1998).

[HS24] Z. Huang and N. de Saxcé. "Distribution locale des approximations rationnelles dans les variétés de drapeaux". In: *In preparation* (2024).

[Khi26] A. Khintchine. "Zur metrischen Theorie der diophantischen Approximationen". In: *Mathematische Zeitschrift, 24 :706-714* (1926).

[KM15] D. Y. Kleinbock and K. Merrill. "Rational approximation on spheres". In: *Israel J. Math. 209(1):293–322* (2015).

[KM96] D. Kleinbock and G. Margulis. “Bounded orbits of nonquasiunipotent flows on homogeneous spaces”. In: *Amer. Math. Soc., Ser. 2, vol. 171, pp. 141–172* (1996).

[KM98] D. Y. Kleinbock and G. A. Margulis. “Flows on homogeneous spaces and diophantine approximation on manifolds”. In: *Ann. of Math. (2), 148(1) :339-360* (1998).

[KM99] D. Y. Kleinbock and G. A. Margulis. “Logarithm laws for flows on homogeneous spaces”. In: *Invent. Math., 138(3) :451-494* (1999).

[Kna02] A. W. Knapp. “Lie groups beyond an introduction”. In: *Volume 140 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, second edition* (2002).

[KY23] D. Kelmer and S. Yu. “Second moment of the light-cone Siegel transform and applications”. In: *Adv. Math. 432, 70 p.* (2023).

[Lan65] S. Lang. “Report on Diophantine approximations”. In: *Bull. Soc. Math. Fr., 93 :177-192* (1965).

[MA20] A. Mohammadi M. Einsiedler G. Margulis and A. Venkatesh. “Effective equidistribution and property tau”. In: *J. Am. Math. Soc., 33(1) :223-289* (2020).

[MG14] A. Mohammadi and A. Salehi Golsefidy. “Translate of horospheres and counting problems”. In: *American Journal of Mathematics, vol. 136, no. 5 1301–1346* (2014).

[MS10] J. Marklof and A. Strömbergsson. “The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems”. In: *Ann. Math., (2) 172(3), 1949–2033* (2010).

[Oua23] Z. Ouaggag. “Effective rational approximation on spheres”. In: *J. Number Theory 249, 183-208* (2023).

[Oua24] Z. Ouaggag. “Central limit theorem for Diophantine approximation on spheres”. In: *https://arxiv.org/abs/2409.02970* (2024).

[Pfi24] R. Pfitscher. “Counting rational approximations on rank one flag varieties”. In: *https://arxiv.org/abs/2410.16508* (2024).

[Rog55] C.A. Rogers. “Mean Values over the space of lattices”. In: *Acta Math. 94: 249-287* (1955).

[Rot55] K. F. Roth. “Rational approximations to algebraic numbers and Corrigendum”. In: *Mathematika. Bd. 2* (1955).

[Sax20] N. de Saxcé. “Groupes arithmétiques et approximation diophantienne”. In: *manuscrit disponible à l'adresse https://www.math.univ-paris13.fr/ desaxce/* (2020).

[Sax22] N. de Saxcé. “Approximation diophantienne sur la grassmannienne”. In: *Preprint, https://www.math.univ-paris13.fr/ desaxce/* (2022).

[Sch60a] W. M. Schmidt. “A metrical theorem in diophantine approximation”. In: *Canad. J. Math., 12, 619–631* (1960).

[Sch60b] W. M. Schmidt. “A metrical theorem in geometry of numbers”. In: *Trans. Amer. Math. Soc., 95:516–529* (1960).

[Sch67] W. M. Schmidt. “On heights of algebraic subspaces and diophantine approximations”. In: *Ann. Math. (2), 85 :430-472* (1967).

[Sie45] C. L. Siegel. “A Mean Value Theorem in Geometry of Numbers”. In: *The Annals of Mathematics, 46(2):340* (1945).

[Wei53] A. Weil. “L'intégration dans les groupes topologiques et ses applications, 2nd edn.” In: *Hermann* (1953).

[Wey16] H. Weyl. “Über die Gleichverteilung von Zahlen mod Eins, 2nd edn.” In: *Math. Ann., 77, 313-52* (1916).