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Homotopie des Structures Dendroïdales

Homotopy theory of dendroidal structures

Thèse de Doctorat présentée par

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Résumé

Cette thèse s'inscrit dans le domaine de la topologie algébrique, à l'intersection de la théorie de l'homotopie, de la théorie des opérades et de la théorie des catégories supérieures — ce que l'on appelle souvent algèbre supérieure. On étudie des structures algébriques homotopiquement cohérentes sur des familles d'espaces, et on le fait en utilisant les ∞-opérades, qui permettent d'exprimer des opérations avec plusieurs entrées et une sortie, où la composition est définie seulement à des homotopies près.

Au cœur de notre travail se trouvent la relation entre les ∞-opérades et leurs analogues discrètes, plus simples, ne faisant intervenir aucune homotopie, ainsi que l'utilisation de différents formalismes équivalents pour les ∞-opérades, en particulier le formalisme dendroidal et celui de Lurie. Dans ce cadre, on étudie selon différents points de vue la théorie de l'homotopie des ∞-opérades, avec l'objectif d'améliorer la compréhension de leurs algebres.

On commence par construire deux adjonctions dites "straightening-unstraightening" opéradique, qui établissent une équivalence entre les algèbres opéradiques et les fibrations à gauche opéradiques – certains morphismes d'opérades qui modelisent des ∞-opérades cofibrées en espaces. La première équivalence vaut pour toute ∞-opérade et est définie comme une équivalence d'∞-catégories. La deuxième est définie pour les opérades discrètes seulement et est définie comme une équivalence de Quillen entre catégories de modèles; elle utilise le formalisme dendroidal, où les opérations sont modelisées par des arbres. Le lien entre les équivalences de un/straightening est fort: on montre que la deuxième est une presentation de la première. Pour ce faire, nous démontrons l'indépendance par rapport aux modèles des fibration opéradiques à gauche par rapport au formalisme dendroidal et à celui de Lurie, comblant ainsi une lacune dans la littérature.

Dans un second temps, on démontre que toute ∞-opérade est équivalente à la localisation d'une opérade discrète, généralisant ainsi un résultat analogue établi par Joyal pour les ∞-catégories. Ce résultat fournit un cadre conceptuel qui relie les ∞-opérades et leurs analogues discrètes. À titre d'application, nous exploitons les équivalences opéradiques de un/straightening construites précédemment pour caractériser l'∞-catégories des algèbres sur une ∞-opérade comme celle des algèbres localement constantes sur sa résolution discrète. Nous en déduisons également une version locale du théorème de un/straightening qui vaut pour toutes les ∞-operades.

Mots clefs: Topologie Algébrique, Théorie de l'homotopie, Opérade, Infinie-catégorie

Abstract

This thesis lies in the domain of algebraic topology, at the intersection of homotopy theory, operad theory and higher category theory– what is often called higher algebra. We study homotopy coherent algebraic structures on families of spaces; this is done by means of ∞-operads, which allow to encode operations with multiple inputs and one output, where composition is defined only up to coherent homotopies.

Central to our work are the relationship between ∞-operads and their simpler, discrete counterparts, where no homotopies are involved, as well as the use of different equivalent formalisms for ∞-operads, notably Lurie's and the dendroidal formalisms. Under these lenses, we investigate from different complementary perspectives the homotopy theory of ∞-operads in order to improve the understanding of their algebras.

First, we construct two operadic un/straightening adjunctions, which establish equivalences between algebras over an operad and operadic left fibrations – certain maps of operads expressing ∞-operads cofibred in spaces. The first equivalence holds for any ∞-operad and is defined in Lurie's formalism as an equivalence of ∞-categories. The second is defined for discrete operads only and realized as a Quillen equivalence of model categories by using the dendroidal formalism, where operations are modeled by trees. The connection between the two is strong: we prove that the latter equivalence is in fact a presentation of the former. To do this, we prove model-independence of operadic left fibrations with respect to Lurie's and the dendroidal formalisms, filling a gap in the literature.

Secondly, we employ the dendroidal formalism to demonstrate that every ∞-operad is equivalent to the localization of a discrete operad, extending an analogous result for ∞-categories due to Joyal. This provides a conceptual bridge between ∞-operads and their discrete counterparts. As an application, we use the constructed operadic un/straightening equivalences to characterize the ∞-category of algebras over a dendroidal ∞-operad as that of locally constant algebras over its discrete resolution; we also deduce a local version of the un/straightening theorem for every ∞-operads.

Key words: Algebraic Topology, Homotopy theory, Operad, Infinity-Category

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... Mi sono fatta prendere la mano dal lirismo. Ma quando mi ricapita?

Francesca

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Chapter 1

Introduction - français

Cette introduction est conçue pour être lue indépendamment de la thèse et résume les résultats et motivations principaux de mon travail, tout en esquissant des directions naturelles pour des travaux futurs ; elle fournit également un aperçu détaillé de la structure de la thèse afin d'aider les lecteurs et lectrices à naviguer dans son contenu.

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1.1 Contexte

1.1.1 Opérades en théorie de l'homotopie

Les opérades sont des objets combinatoires modélisant des catégories d'algèbres. Elles ont été introduites pour la première fois par May ([May06]) et Boardman-Vogt ([BV73]) au début des années '70 , afin de décrire les structures algébriques homotopiquement cohérentes sur les espaces de lacets itérés des espaces topologiques, ainsi que les opérations en homologie qui leur sont associées. Depuis lors, elles ont trouvé de nombreuses applications dans divers domaines des mathématiques, notamment en géométrie algébrique, en combinatoire et en physique mathématique.

Pour mieux comprendre le type de structures algébriques concernées, considérons l'espace des lacets pointés $\Omega_x X := \mathsf{Map}(\mathbb{S}^1, X)$ d'un espace topologique pointé (X, x). La concaténation de lacets dote $\Omega_x X$ d'une opération qui n'est pas strictement associative, mais qui l'est seulement *à homotopie près* — plus précisément, à des reparamétrisations des lacets près. De plus, les homotopies entre des concaténations différemment paramétrées satisfont elles-mêmes à des relations analogues d'associativité à homotopie près, et ce de manière cohérente en toute dimension supérieure. Si l'on quotiente $\Omega_x X$ par les équivalences d'homotopie, on obtient la structure de groupe usuelle sur $\pi_1(X)$, le groupe fondamental de X. Cependant, dans ce processus on perd une grande partie de la structure algébrique sur $\Omega_x X$, que la notion d'algèbre associative n'est pas apte à capturer.

Pour décrire cette structure algébrique dans sa totalité, c'est-à-dire, conserver toutes les cohérences homotopiques de la concaténation de lacets, on se sert donc de la notion d'opérade, et on regarde $\Omega_x X$ avec sa structure naturelle d'*algèbre sur l'opérade des petits disques* $\mathbb{E}_1$.

Celui-ci est donné par une collection d'espaces $\mathbb{E}_1 = \{\mathbb{E}_1(k)\}_{k\geqslant 0}$ où, pour tout entier $k \geqslant 0$, l'espace $\mathbb{E}_1(k)$ est défini comme l'espace des configurations de k intervalles disjoints à l'intérieur de l'intervalle unitaire. On pense $\mathbb{E}_1(k)$ comme l'espace des opérations de $\mathbb{E}_1$ d'arité k; la structure de $\mathbb{E}_1$-algèbre sur $\Omega_x X$ provient du fait que chaque configuration dans $\mathbb{E}_1(k)$ détermine une manière de concaténer k lacets basés dans X, ce qui donne une application continue d'espaces topologiques

$$\mathbb{E}_1(k) \longrightarrow \mathsf{Map}((\Omega_x X)^{\times k}, \Omega_x X).$$

Le fait que les configurations puissent être imbriquées les unes dans les autres fournit des lois de composition partielle sur la famille $\mathbb{E}_1 = \mathbb{E}_1(k)_k$, et cela gouverne les lois de composition pour les concaténations de lacets et leurs cohérences homotopiques.

L'opérade $\mathbb{E}_1$ est la première d'une famille d'opérades $\{\mathbb{E}_n\}_{n\geqslant 1}$, où $\mathbb{E}_n$ est appelée *opérade des petits n-disques*, qui gouvernent des structures algébriques homotopiquement associatives et de plus en plus homotopiquement commutatives, lorsque n tend vers l'infini. Pour tout $k \geqslant 0$, l'espace $\mathbb{E}_n(k)$ des opérations d'arité k de $\mathbb{E}_n$ est donné par l'espace des configurations de k petits disques n-dimensionnels disjoints à l'intérieur du disque unitaire de dimension n. Par exemple, pour tout $n \geqslant 1$, l'espace des n-lacets $\Omega^n_x X$ d'un espace topologique pointé (X, x) possède la structure d'une $\mathbb{E}_n$-algèbre — et en fait, à homotopie près, ce sont les seules (group-like) $\mathbb{E}_n$-algèbres dans les espaces topologiques, d'après le théorème de reconnaissance de May ([May06]).

Plus généralement, on autorise ce qu'on appelle une opérade P, et qui est souvent appelée par *opérade colorée* ou *multicategorie*, à posséder un ensemble d'objets $\mathsf{Ob}(P)$ et, pour chaque $k \geqslant 0$, un espace (plus tard spécifié comme un ensemble simplicial) d'opérations $P(x_1, \ldots, x_k; y)$ pour tous objets $x_1, \ldots, x_k, y$ (dans le cas à un seul objet, on note $P(k)$ pour $P(\underbrace{*, \ldots, *}_{k}; *)$). Ces espaces sont munis de lois de composition opéradiques

$$\circ_{x_i} \colon P(x_1, \ldots, x_k; y) \times P(z_1, \ldots, z_m; x_i) \longrightarrow P(x_1, \ldots, x_{i-1}, z_1, \ldots, z_m, x_{i+1}, \ldots, x_k; y)$$

qui sont unitaires et associatives, et équivariantes par rapport à l'action des groupes symétriques qui permutent les variables.

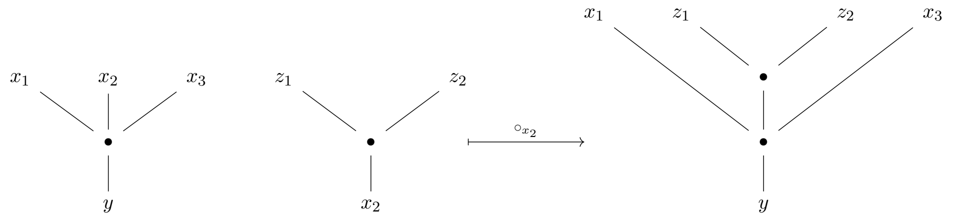

Figure 1.1: Une représentation graphique de la composition opéradique $\circ_{x_2}$.

Une algèbre sur l'opérade P est une famille d'espaces $F = F(x)_{x\in\mathsf{Ob}(P)}$ sur laquelle l'opérade agit via des applications continues

$$P(x_1, \dots, x_k; y) \longrightarrow \mathsf{Map}(\prod_{i=1}^{k} F(x_i), F(y))$$

respectant les compositions, les unités et les symétries de l'opérade P.

En théorie de l'homotopie, les espaces topologiques sont considérés uniquement à équivalence faible d'homotopie près, et on traite la cohérence homotopique via des structures de catégorie de modèles à la Quillen ou des ∞-catégories. À cette fin, les espaces topologiques sont représentés par des ensembles simpliciaux, c'est-à-dire des préfaisceaux sur la catégorie des ordres linéaires finis Δ (par exemple, on peut prendre les chaînes singulières d'un espace topologique), et on étudie leur théorie homotopique via la structure de modèle de Kan-Quillen sur les ensembles simpliciaux. Le fait de considérer les espaces à homotopie près conduit naturellement à s'intéresser aux opérades simpliciales à équivalences faibles, dites *de Dwyer-Kan*, près, qui sont les aux morphismes d'opérades qui sont homotopiquement pleinement fidèles et essentiellement surjectifs. Cela amène aussi à étudier la théorie homotopique des algèbres sur les opérades à homotopie près.

Peut-être le moyen le plus intuitif d'étudier la théorie homotopique des opérades simpliciales est de construire une structure de modèle sur les opérades simpliciales où les équivalences faibles sont précisement les équivalences de Dwyer-Kan ; cela a été fait indépendamment par Cisinski-Moerdijk [CM13b] et Robertson [Rob11]. La structure supplémentaire fournie par une catégorie de modèles à la Quillen permet d'étudier des constructions invariantes par homotopie ; par exemple, en présence d'une structure de modèle, on dispose d'un moyen intrinsèque de définir les algèbres à homotopie près sur une opérade — plus précisément, comme des algèbres sur un remplacement cofibrant de l'opérade.

Bien que facile à comprendre, cette approche est difficile à utiliser lorsqu'il s'agit d'effectuer des constructions catégoriques élémentaires, et l'on rencontre des difficultés analogues à celles que l'on trouve dans le cas des catégories simpliciales. Par exemple, le calcul des espaces de morphismes dérivés est difficile à réaliser : bien que le produit tensoriel de Boardman-Vogt confère à la catégorie des opérades simpliciales un enrichissement simplicial, celui-ci n'est pas compatible avec la structure de modèle. En d'autres termes, la structure de modèle mentionnée ci-dessus sur la catégorie des opérades simpliciales n'est pas une catégorie de modèles monoïdale. [1].

1.1.2 Un formalisme plus souple : les ∞-opérades

Ces difficultés peuvent être contournées en adoptant le formalisme des ∞-*opérades*, dans lequel certains des axiomes définissant les opérades simpliciales sont relâchés, et où la composition n'est définie qu'à homotopie près. Bien que les opérades simpliciales puissent être vues comme des exemples d'∞-opérades (via les constructions du nerf), la notion d'∞-opérade est plus souple et mieux adaptée aux objectifs de la théorie moderne de l'homotopie.

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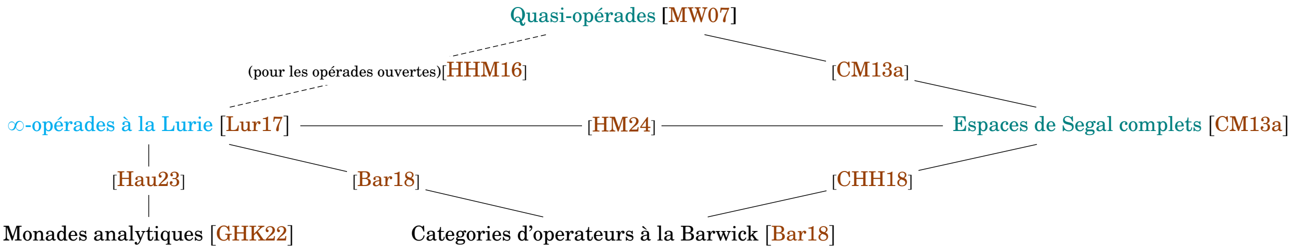
[1] Plus précisément, la structure de modèle sur les opérades simpliciales n'est pas monoïdale au sens de Quillen: en particulier, l'axiome du produit-quotient échoue, comme c'était déjà le cas pour la structure de modèle de Bergner sur les catégories simpliciales ; voir la discussion sur *https://mathoverflow.net/questions/198205/boardman-vogt-tensor-product.*

Il existe plusieurs modèles équivalents pour les ∞-opérades, que l'on peut représenter schématiquement de la manière suivante, ainsi que les équivalences entre eux :



Dans ce travail, nous utilisons deux formalismes : le premier est celui des ∞-opérades de Lurie, et le second est le *formalisme dendroïdal*, qui se décline en deux modèles : les quasi-opérades et les espaces de Segal dendroïdaux complets.

De manière intuitive, l'approche de Lurie repose sur l'idée qu'une opérade peut être vue comme une forme affaiblie de catégorie monoïdale symétrique, où le produit tensoriel $x_1 \otimes \cdots \otimes x_n$ est spécifié uniquement via le foncteur covariant $y \mapsto \mathrm{Hom}(x_1, \ldots, \otimes x_n; y)$, lequel n'a pas besoin d'être représentable. La formalisation de cette notion homotopiquement cohérente d'opérade comme relâchement de celle de catégorie monoïdale symétrique repose de façon fondamentale sur la théorie des fibrations coCartésiennes.

Le formalisme dendroïdal est basé sur une catégorie d'arbres Ω, appelée la *catégorie dendroïdale*, introduite par Moerdijk-Weiss dans [MW07], qui modélise les opérations et la composition opéradique d'une manière qui reflète l'intuition diagrammatique présentée dans la Figure 1.1. Ce formalisme repose sur le fait que les opérades constituent une généralisation des catégories — en effet, les opérades sont parfois appelées *multicatégories symétriques* —, dans le sens que les morphismes $f\colon x \to y$ (à une seule entrée et une seule sortie) sont remplacés par des multimorphismes $(x_1, \ldots, x_n) \to y$ qui autorisent plusieurs entrées (en nombre fini) et une seule sortie. La catégorie Ω est donc conçue pour capturer la structure de tels multimorphismes et leur composition de la même manière que la catégorie simpliciale Δ modélise les morphismes et la composition dans les catégories ordinaires.

1.2 Motivations

Comme on vient d'évoquer, le formalisme des ∞-opérades fournit des outils pour traiter de manière combinatoire les cohérences homotopiques et effectuer des constructions homotopiquement cohérentes. Cependant, il existe une classe importante d'∞-opérades pour lesquelles cette complexité n'est pas nécessaire : il s'agit des opérades *discrètes*, dont les espaces d'opérations ne possèdent aucune structure topologique et sont simplement des ensembles. Cela rend les opérades discrètes beaucoup plus simples à étudier que leurs homologues ∞-catégoriques dans de nombreux cas, car il n'est pas nécessaire de s'occuper des éventuelles cohérences homotopiques des espaces des multimorphismes, car la topologie est triviale. D'un autre côté, leur rigidité impose parfois de les étudier comme des cas particuliers d'∞-opérades. Ce contraste entre la rigidité et simplicité des opérades discrètes et la flexibilité des ∞-opérades donne lieu à une interaction subtile mais féconde, dont l'analyse constitue le fil conducteur de ce travail.

1.2.1 Opérades discrètes et localisations

Les opérades discrètes jouent un rôle fondamental pour obtenir des ∞-opérades non triviales, grace à la notion d'*opérade relative*, c'est-à-dire, une opérade discrète P munie d'un ensemble de morphismes S, appelés équivalences faibles. À partir d'une opérade relative (P, S), on peut inverser formellement les flèches de S dans l'∞-opérade $\mathcal{N}^{\otimes}(P)$ déterminée par le nerf de P, et obtenir une ∞-opérade $\mathcal{N}(P)^{\otimes}[S^{-1}]$, que on appelle la *localisation de P en S*: elle est caractérisée par la propriété universelle usuelle d'être initiale parmi les ∞-opérades sous $\mathcal{N}^{\otimes}(P)$ dans lesquelles les flèches de S deviennent des équivalences.

Une première classe fondamentale d'∞-opérades obtenues de cette manière est celle des ∞-catégories: le théorème de délocalisation de Joyal, énoncé dans [Joy07, §13.6] et démontré par Stevenson dans [Ste17], garantit que toute ∞-catégorie X, que l'on peut représenter sous forme d'un ensemble simplicial (c'est-à-dire une quasicatégorie), est équivalente à une localisation du nerf de sa catégorie des éléments Δ/X. Cette équivalence est réalisée via le *foncteur du dernier sommet* (déjà apparu dans les travaux de Waldhausen [Wal85, §1.6]), une application d'ensembles simpliciaux $\mathcal{N}(\Delta/X) \to X$, du nerf de Δ/X vers X, qui évalue un n-simplexe de X au $n^{\text{ième}}$ sommet. Dans le théorème il est montré que cette application induit un morphisme d'ensemble simpliciaux

$$\mathcal{N}(\Delta/X)[\mathcal{J}^{-1}] \xrightarrow{\sim} X$$

qui présente X comme la localisation ∞-catégorique de $\mathcal{N}(\Delta/X)$ en l'ensemble $\mathcal{J}$ des morphismes donné par les préimages des identités.

Une autre classe importante d'∞-opérades qui ne sont pas simplement des ∞-catégories et qui apparaissent comme des localisations d'opérades relatives est celle des opérades des petits disques $\mathbb{E}_n$ introduites dans les sections précédentes. Rappelons que, pour tout $k \geqslant 0$, l'espace des opérations k-aires $\mathbb{E}_n(k)$ est l'espace des configurations de k disques unitaire n-dimensionnels disjoints à l'intérieur du disque unitaire n-dimensionnel. Un point de $\mathbb{E}_n(k)$ peut être représenté par un diagramme de plongement d'inclusions :

$$\begin{tikzcd}
\bigsqcup_{i=1}^{k} \mathbb{D}^n \arrow[rr, hook] \arrow[dr, hook] & & \mathbb{D}^n \arrow[dl, hook'] \\
& \mathbb{R}^n &
\end{tikzcd} \tag{1.2.1}$$

qui commute à isotopie près ; les chemins dans $\mathbb{E}_n(k)$ sont donnés par des isotopies entre de telles configurations, et ainsi de suite.

Considérons maintenant l'opérade discrète Disks_n, qui possède un ensemble d'objets non trivial donné par l'ensemble des inclusions $\mathbb{D}^n \hookrightarrow \mathbb{R}^n$ de disques ouverts unitaires n-dimensionnels dans $\mathbb{R}^n$, et dans laquelle une opération de k tels plongements vers un autre est un triangle comme dans le diagramme (1.2.1), mais qui commute strictement. Il existe un morphisme naturel d'∞-opérades $\mathcal{N}^{\otimes}\mathsf{Disks}_n \longrightarrow \mathbb{E}_n$ du nerf de Disks_n vers $\mathbb{E}_n$, envoyant chaque inclusion d'une réunion disjointe finie de disques sur la configuration correspondante dans $\mathbb{R}^n$. Chaque flèche dans Disks_n est envoyée sur une équivalence dans $\mathbb{E}_n$, et dans [Lur17, §5.4.5.], Lurie montre que le morphisme induit depuis la localisation

∞-opéradique[2] en la preimage des équivalences

$$\mathcal{N}^{\otimes}(\mathsf{Disks}_n)[\mathcal{J}^{-1}] \xrightarrow{\sim} \mathbb{E}_n$$

est une équivalence d'∞-opérades. La conséquence fondamentale est que le foncteur induit entre les ∞-catégories d'algèbres

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S}) \hookrightarrow \mathsf{Alg}_{\mathsf{Disks}_n}(\mathcal{S})$$

est pleinement fidèle et permet d'identifier les $\mathbb{E}_n$-algèbres avec les *algèbres localement constantes* sur Disks_n, c'est-à-dire les Disks_n-algèbres pour lesquelles les morphismes unaires de Disks_n correspondent à des équivalences. Cette caractérisation des $\mathbb{E}_n$-algèbres permet de démontrer que toute $\mathbb{E}_n$-algèbre peut être étendue de manière cocontinue d'une algèbre sur l'∞-opérade $\mathcal{O}(\mathbb{R}^n)$ de *tous* les sous-espaces ouverts de $\mathbb{R}^n$, par un processus de prolongement de Kan opéradique à gauche ([Lur17, Théorème 5.5.2.5]). En fait, le même résultat vaut pour les ∞-opérades $\mathbb{E}_M$ et Disks_M obtenues en remplaçant $\mathbb{R}^n$ par une variété M n-dimensionnelle équipé d'une trivialisation du fibré tangent (dit *framed manifold*) [3], et constitue le cœur de la définition de l'*homologie de factorisation* de la variété M, un invariant topologique et algébrique relié à l'homologie de Hochschild et aux théories d'homologie ([Lur17], [AF15], [AFT17]).

La localisation s'avère donc être un outil fondamental pour comprendre la théorie des ∞-opérades, et comme nous l'avons vu, exprimer une ∞-opérade comme la localisation d'une opérade discrète peut simplifier certains arguments dans des calculs concrets. À la lumière de cela, et en suivant le principe selon lequel les résultats pour les ∞-catégories devraient admettre des extensions au cadre opéradique (que l'on peut appeler aussi des extensions *multiplicatives*) il est naturel de poser la question suivante :

Question 1. Est-ce que *toute* ∞-opérade peut s'écrire comme la localisation d'une opérade discrète ? Peut-on décrire explicitement une telle opérade discrète ? Peut-on y parvenir en étendant le théorème de délocalisation de Joyal ?

Je répondrai par l'affirmative à toute cette question multiple, comme on le verra dans la Section 1.3, et cela nous permettra de décrire l'∞-catégorie des algèbres sur une ∞-opérade en les caractérisant comme des algèbres localement constantes sur leur résolution discrète. Cette perspective ouvre la voie à divers outils pour étudier la théorie homotopique des algèbres opéradiques, notamment à l'aide de la classe cruciale de méthodes connues sous le nom d'*équivalences opéradiques de un/straightening*, que nous allons à présent explorer.

1.2.2 Équivalences opéradiques de un/straightening

De manière générale, une *équivalence de un/straightening* (à lire 'équivalence de straightening-unstraightening', et qu'en français on pourrait peut-être traduire par 'équivalence de rectification-dérectification') consiste à décrire des diagrammes dans l'∞-catégorie des espaces $\mathcal{S}$ (et même dans celle des ∞-catégories) à l'aide de certaines fibrations plus faciles à comprendre. En topologie algébrique classique, on peut penser

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[2] Pour être précis, pour traiter la localisation des ∞-opérades, Lurie utilise la théorie des *approximations faibles* (weak approximations).

[3] En réalité, même une variété stratifiée, voir les travaux de Ayala-Francis-Tanaka [AFT17].

à la correspondance entre les revêtements $E \to X$ d'un espace topologique X et les ensembles munis d'une action du groupe fondamental $\pi_1(X)$. Dans le langage moderne, l'équivalence de un/straightening a été formulée et démontrée pour la première fois par Lurie dans le cadre du formalisme des ∞-catégories ([Lur17]): cela consiste, pour toute ∞-catégorie $\mathcal{C}$, en une équivalence d∞-catégories

$$\mathsf{St}^{\mathcal{C}}\colon \mathsf{Left}_{\mathcal{C}} \underset{\perp}{\rightleftarrows} \mathsf{Fun}(\mathcal{C},\mathcal{S}) :\mathsf{Unst}^{\mathcal{C}},$$

où $\mathsf{Left}_{\mathcal{C}}$ est l'∞-catégorie des *fibrations à gauche* au-dessus de $\mathcal{C}$, une généralisation de la notion de catégories co-fibrées en groupoïdes, et $\mathsf{Fun}(\mathcal{C},\mathcal{S})$ celle des foncteur ∞-catégoriques de $\mathcal{C}$ à $\mathcal{S}$.

Encore une fois, un principe directeur de la théorie des ∞-opérades est qu'un résultat valable pour les ∞-catégories devrait admettre une extension multiplicative au cadre des ∞-opérades. Dans cette généralisation opéradique, les foncteurs deviennent des algèbres, et les objets fibrés deviennent des objets multiplicativement fibrés : une équivalence de un/straightening opéradique consiste donc, pour toute ∞-opérade $\mathcal{P}$, en une équivalence d'∞-catégories

$$\mathsf{St}^{\mathcal{C}}\colon \mathsf{Left}_{\mathcal{C}} \underset{\perp}{\rightleftarrows} \mathsf{Alg}_{\mathcal{P}}(\mathcal{S}) :\mathsf{Unst}^{\mathcal{C}},$$

où $\mathsf{Left}(\mathsf{Op}_{\infty})_{\mathcal{P}}$ est une ∞-catégorie de *fibrations opéradiques à gauche* au-dessus de $\mathcal{P}$, une généralisation ∞-catégorique des opérades co-fibrées en groupoïdes, et $\mathsf{Alg}_{\mathcal{P}}(\mathcal{S})$ celle des $\mathcal{P}$-algèbres à valeurs dans $\mathcal{S}$.

La définition des ∞-catégories intervenant dans l'équivalence opéradique de un/straightening, ainsi que la construction de cette équivalence elle-même, dépendent du modèle d'∞-opérade choisi; dans les cadres de Lurie et dans l'approche dendroïdale, les ∞-catégories $\mathsf{Left}_{\mathcal{P}}(\mathsf{Op}_{\infty})$ et $\mathsf{Alg}_{\mathcal{P}}(\mathcal{S})$ sont bien définies, et certaines équivalences opéradiques de un/straightening ont été établies. Voici un bref résumé de ces résultats.

Dans les modèles dendroidaux, la notion de fibration opéradique à gauche étend au contexte combinatoire donné par les arbres l'analogue existant pour les ∞-catégories. La première définition a été formulée dans le modèle des ensembles dendroidaux **dSets** et remonte à Heuts [Heu11]. Elle repose sur la notion de *leaf horn* (litteralement *cornet de feuille*) d'un arbre, un objet combinatoire qui permet de capturer l'existence et l'unicité, à homotopie supérieure près, des relèvements des opérations. L'∞-catégorie des fibrations opéradiques à gauche au-dessus d'une quasiopérade X est modélisée par la *structure de modèle covariante* sur la catégorie **dSets**$/X$ des ensembles dendroidaux au-dessus de X. Dans [Heu11], Heuts démontre une équivalence opéradique de un/straightening pour un ensemble dendroïdal *normal*[4] X: il construit une équivalence de Quillen entre la structure de modèle covariante sur **dSets**$/X$ et la structure de modèle projective sur la catégorie $\mathsf{Alg}_{W_!(X)}$(sSets) des algèbres simpliciales sur l'opérade simpliciale $W_!(X)$ (une sorte de résolution homotopiquement cohérente à la Boardman-Vogt de X).

Dans le modèle des espaces de Segal dendroidaux complets, une équivalence analogue a été construite par Boavida-Moerdijk dans [BdBM20] pour une notion de fibration opéradique à gauche équivalente à celle donnée dans le cas des énsembles dendroidaux. Si dans ce dernier modèle une fibration opéradique à gauche est définie par une propriété de relèvement à droite par rapport aux inclusions des leaf horns, pour les espaces

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[4] Il s'agit d'une condition de cofibrance, correspondant à une action libre du groupe symétrique sur une opérade.

dendroidaux elle est définie comme un objet local par rapport aux *inclusions de feuilles* d'arbres. Dans les deux cas d'équivalences dendroidaux de un/straightening, l'adjoint à gauche, qui associe à une fibration opéradique à gauche sur une ∞-opérade dendroidale une algèbre sur celle-ci, agit sur les fibrations représentables comme un foncteur d'« algèbre libre ».

Dans le formalisme de Lurie, une équivalence de un/straightening pour une ∞-opérade $\mathcal{P}$ peut être déduite de la construction de Grothendieck monoïdale, plus générale, introduite par Ramzi pour les *∞-catégories monoïdales $\mathcal{P}$* ([Ram22]). Pour toute ∞-catégorie monoïdale $\mathcal{P}$ donnée, que l'on peut voir comme une ∞-opérade cofibrée au-dessus de $\mathcal{P}$, la construction de Ramzi établit une équivalence d'∞-catégories entre l'∞-catégorie des *fibrations à gauche $\mathcal{P}$-monoïdales* au-dessus de $\mathcal{C}$ et celle des foncteurs $\mathcal{P}$-lax monoïdaux $\mathcal{C} \to \mathcal{S}$[5]. Dans ce cadre formel, on retrouve l'∞-catégorie des algèbres sur $\mathcal{P}$ comme celle des foncteurs $\mathcal{P}$-monoïdaux $\mathcal{P} \to \mathcal{S}$, et les fibrations opéradiques à gauche au-dessus de $\mathcal{P}$ comme étant les fibrations à gauche $\mathcal{P}$-monoïdales $\mathcal{D} \to \mathcal{P}$. La même équivalence a également été démontrée par Kern dans [Ker23], en utilisant le formalisme des motifs algébriques. Bien que cette approche se prête bien à des généralisations plus larges, l'introduction des motifs algébriques dépasse le cadre de cette thèse, et nous n'en dirons donc pas davantage.

Dans la section précédente, nous avons observé le rôle central de l'interaction entre opérades discrètes et ∞-opérades via la localisation. Cela nous mène à une motivation essentielle :

Motivation. Étudier l'interaction entre la localisation opéradique et les fibrations opéradiques à gauche. Construire des équivalences opéradiques de un/straightening adaptées à la manipulation des opérades discrètes.

En outre, l'analyse de ces équivalences opéradiques de un/straightening dans les différents modèles d'∞-opérades soulève également des questions d'équivalence des modèles, comme on explique dans la section suivante.

1.2.3 Indépendance vis-à-vis du modèle

L'équivalence entre les différents formalismes pour les ∞-opérades n'implique pas de façon automatique une équivalence entre les différents modèles pour les fibrations opéradiques à gauche — bien qu'une telle équivalence soit, à juste titre, une caractéristique nécessaire de toute définition robuste. En effet, pour définir les fibrations opéradiques à gauche, le principal point technique à traiter est de donner un sens aux relèvements coCartésiens opéradiques homotopiquement cohérents des opérations, ce qui est réalisé à l'aide de techniques qui ne sont pas manifestement indépendantes du modèle choisi: comme on l'a vu, les solutions adoptées dans le cas dendroïdal passent soit par des objets combinatoires, comme les leaf horns ou les inclusions de feuilles d'arbres, alors que, dans le cas de Lurie, on utilise la notion déjà existante de fibration à gauche des ∞-catégories.

Une première question à poser est donc la suivante : l'∞-catégorie des fibrations opéradiques à gauche au-dessus d'une ∞-opérade est-elle indépendante du modèle choisi?

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[5] En fait, il est démontré que ces deux ∞-catégories s'identifient à des ∞-catégories d'algèbres sur $\mathcal{P}$ dans des ∞-catégories monoïdales $\mathcal{P}$, et que l'équivalence provient d'une équivalence $\mathcal{P}$-monoïdale entre ces dernières.

Pour les modèles dendroidaux, la réponse est positive : dans [BdBM20], Boavida-Moerdijk démontrent l'invariance par changement de modèle des fibrations opéradiques à gauche dans les deux modèles dendroidaux, à savoir les espaces de Segal dendroidaux complets et les ensembles dendroidaux, en s'appuyant sur une équivalence directe entre ces deux modèles d'∞-opérades.

À ce jour, aucune réponse n'a été apportée concernant les cadres de Lurie et des modèles dendroidaux, un des défis principaux étant que la première équivalence entre les deux formalismes était indirecte, et faisait intervenir un troisième modèle d'∞-opérades, celui des catégories d'opérateurs de Barwick (voir le diagramme de la Section 1.1.2). Cependant, dans un travail récent, Hinich et Moerdijk [HM24] ont construit une équivalence directe entre les ∞-opérades de Lurie et les modèles dendroidaux, formulée à l'aide des espaces de Segal dendroidaux complets, établissant ainsi une équivalence adjointe d'∞-catégories :

$$\delta\colon \ell\mathsf{Op}_\infty \underset{\leftarrow}{\overset{\perp}{\rightleftarrows}} \mathsf{DOp}_\infty : \lambda.$$

Cela nous permet d'affiner notre question précédente et de formuler la suivante :

Question 2. L'équivalence de Hinich-Moerdijk induit-elle une équivalence entre les ∞-catégories de fibrations opéradiques à gauche dans les modèles de Lurie et dendroidaux?

Dans la Section 1.3.3, nous verrons que c'est effectivement le cas.

1.3 Contributions

Dans cette section, nous présentons les contributions originales de cette thèse. Avant d'entrer dans une exposition plus détaillée, nous proposons un bref récit résumant les résultats principaux.

L'un des points de départ de notre approche consiste à montrer que toute ∞-opérade est équivalente à la localisation d'une opérade discrète, au moyen du *foncteur racine* (ou *root functor*). Cela nous conduit à étudier les constructions de un/straightening opéradique spécifiquement pour les opérades discrètes. Comme le foncteur racine est formulé dans le formalisme dendroïdal et que les opérades discrètes sont fondamentalement 1-catégoriques, nous travaillons avec le modèle des ensembles dendroidaux. Nous construisons, pour toute opérade discrète et Σ-libre, une équivalence de un/straightening opéradique sous forme d'une équivalence de Quillen entre les structures de modèles covariante (pour les fibrations à gauche) et projective (pour les algèbres). La définition de l'adjoint à gauche, qu'on appelle *foncteur de rectification*, est strictement liée à l'enveloppe monoïdale symétrique associée aux opérades discrètes. La démonstration que l'adjonction établi une équivalence de Quillen est faite en deux parties. Dans un premier temps, on montre que l'adjonction est de Quillen. Pour cela, on considère des algèbres définies par des chaînes d'inclusions de sous-arbres. Pour prouver qu'il s'agit d'une équivalence de Quillen, nous considérons les foncteurs induits entre les ∞-catégories sous-jacentes, et nous montrons qu'ils forment une équivalence adjointe. L'étape clé est de construire une nouvelle équivalence de un/straightening opéradique, qui est en fait définie pour *toute* ∞-opérade dans le formalisme de Lurie, puis nous montrons que notre adjonction de Quillen présente cette équivalence lorsque l'opérade est discrète. Cela nécessite de démontrer que les foncteurs de comparaison de Hinich-Moerdijk induisent une équivalence d'∞-catégories entre les catégories de fibrations opéradiques à gauche dans les modèles de Lurie et le modèle dendroïdal, résolvant ainsi la question de l'indépendance vis-à-vis du modèle. Comme application combinée du foncteur racine et des théorèmes de un/straightening, nous montrons que les algèbres sur des ∞-opérades sont équivalentes aux *algèbres localement constantes* sur leur résolution discrète.

Bien que ces résultats soient interconnectés, chacun possède une importance indépendante, et pour cela on va maintenant les présenter de manière systématique.

1.3.1 Le foncteur racine

Dans la Question 1, non demandions si toute ∞-opérade peut être réalisée comme la localisation d'une opérade discrète : le formalisme dendroïdal, et plus précisément la théorie homotopique des ensembles dendroïdaux, nous permet de répondre positivement à cette question.

Les *ensembles dendroidaux* sont la catégorie des préfaisceaux sur la catégorie dendroidale Ω. Munis de la structure de modèle opéradique, ils forment un modèle d'∞-opérades, qui sont représentées par les *quasiopérades*, c'est-à-dire les ensembles dendroidaux satisfaisant une certaine condition de remplissage des cornets internes dendroidaux (appelés, en anglais, *dendroidal inner horns*). Ce modèle étend celui de Joyal pour les quasicatégories, dans le sens qu'une quasicatégorie est exactement une quasiopérade admettant un morphisme vers le point.

Dans ce cadre, nous étendons le foncteur des catégories d'éléments $\Delta/-\colon \mathbf{sSets} \to \mathsf{Cat}$ en un foncteur

$$\mathbf{\Omega}/-\colon \mathbf{dSets} \longrightarrow \mathsf{Op}$$

des ensembles dendroidaux vers les opérades discrètes, qui à un ensemble dendroidal X associe son *opérade des éléments* Ω/X. Les objets sont des paires (T, α), où T est un arbre et α un morphisme de T vers X, et les ensembles d'opérations sont donnés par des morphismes *larges* et *indépendants* entre unions disjointes d'arbres (voir la Definition 6.3.6). Nous montrons alors qu'évaluer un tel α à la racine $\mathscr{r}_T$ de T (la seule arête canonique de T) se prolonge en un morphisme d'ensembles dendroidaux

$$\mathscr{r}_X\colon \mathcal{N}_d(\mathbf{\Omega}/X) \longrightarrow X, \quad (T, \alpha) \mapsto \alpha(\mathscr{r}_T),$$

où $\mathcal{N}_d$ désigne le nerf dendroidal, et que nous appelons le *foncteur racine* (Proposition 6.3.10).

Un élément (T, α) de $\mathbf{\Omega}/X$ peut être interprété comme une décoration de l'arbre T avec des opérations composables de X. Ce qui se passe alors, c'est que les données homotopiquement cohérentes de X sont 'résolues' (ou 'strictifiées') en devenant des objets dans l'opérade discrète Ω/X, puis récupérées en identifiant certains de ces objets — c'est-à-dire en localisant par rapport à un certain ensemble de morphismes. Le foncteur racine est l'application de comparaison qui capture précisément l'information homotopique manquante de X, dans le sens précis suivant :

Theorem (Theorem 6.3.13). *Soit X un ensemble dendroidal normal, et soit $\mathcal{R}$ l'ensemble des morphismes de Ω/X envoyés sur des identités par $\mathscr{r}_X$. Le foncteur racine $\mathscr{r}_X$ induit une équivalence faible opéradique d'ensembles dendroidaux*

$$\overline{\mathscr{r}_X}\colon \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \xrightarrow{\sim} X$$

entre la localisation de $\mathcal{N}_d(\Omega/X)$ en $\mathcal{R}$ et X.

Pour démontrer ce théorème, nous formalisons d'abord la notion de localisation opéradique dans le modèle des ensembles dendroidaux (Definition 6.2.2), ce qui est, à notre connaissance, une nouveauté. Nous montrons ensuite que, lorsque X est un ensemble dendroidal normal, on peut construire un modèle explicite pour la localisation et l'exprimer comme un recollement (pushout). Dans la preuve du théorème, nous procédons par filtration squelettique (skeletal filtration) et nous construisons un inverse homotopique de $\overline{\mathscr{r}_X}$ dans le cas où X est un ensemble dendroidal représentable, c'est-à-dire un arbre.

Un premier corollaire du théorème est que $\mathscr{r}_X$ est une équivalence faible dans la *structure de modèle stable* sur **dSets** (Corollaire 6.3.15). Définie dans [BN14], cette structure étend la structure de modèle de Kan-Quillen sur les ensembles simpliciaux, et sa théorie homotopique est équivalente à celle des espaces de lacets infinis. Autrement dit, tout ensemble dendroidal normal est équivalent (faiblement) à un espace de lacets infini associé à une opérade discrète.

En se restreignant aux ensembles simpliciaux, on obtient également le théorème de délocalisation de Joyal, comme souhaité.

Corollary (Joyal, Stevenson)**.** *Pour tout ensemble simplicial M, il y a un isomorphisme de catégories $\mathbf{\Omega}/M \simeq \Delta/M$, et le foncteur racine coincide avec le foncteur du dernier sommet $\mathcal{N}(\Delta/M) \to M$. En particulier, il induit une équivalence catégorique faible*

$$\mathcal{N}(\Delta/M)[\mathcal{R}^{-1}] \longrightarrow M$$

entre M et la localisation de $\mathcal{N}(\Delta/M)$ à la préimage des identités. En particulier, l'application $\mathcal{N}(\Delta/M) \to M$ est une équivalence faible d'homotopie.

1.3.2 Une équivalence de un/straightening opéradique dans le modèle de Lurie

L'équivalence induite par le foncteur racine met en évidence le rôle central des opérades discrètes, et la nécessité de construire des équivalences de un/straightening opéradique adaptées pour étudier la théorie homotopique des algèbres sur celles-ci.

Pour faire celà, on utilise le formalisme plus souple et general des ∞-operades. Comme dejà remarqué, la construction des structure homotopiquement invariantes beneficie de la transition entre les approches fondées sur les catégories de modèles à la Quillen et celles développées dans le cadre des ∞-catégories. Ainsi, avant de nous concentrer sur les opérades discrètes et la combinatoire dendroidale (le lecteur ou la lectrice intéressé.e pourra aller directement à la Section 1.3.4), nous considérons le modèle de Lurie pour les ∞-opérades, qui fournit un outil jusqu'au present absent du cadre dendroidal : l'enveloppe monoïdale symétrique. Ce functeur, qui étand un functeur pour les opérades discrètes (définie à l'origine sous le nom de PROP dans [Mac65]), relie la théorie des (∞-)opérades à celle des (∞-)catégories monoïdales symétriques.

Le *foncteur d'enveloppe monoïdale symétrique*

$$\mathsf{Env}(-)^{\otimes} \colon \ell\mathsf{Op}_{\infty} \longrightarrow \mathsf{smCat}_{\infty}$$

est l'adjoint à gauche du foncteur d'oubli de l'∞-catégorie des ∞-catégories monoïdales symétriques vers celle des ∞-opérades. Dans [Lur17], Lurie construit un tel adjoint à gauche, généralisant la notion correspondante pour les opérades discrètes. L'enveloppe est caractérisée par la propriété universelle suivante : pour toute ∞-opérade $\mathcal{P}$ et toute ∞-catégorie monoïdale symétrique $\mathcal{V}$, il existe une équivalence

$$\mathsf{Alg}_{\mathcal{P}}(\mathcal{V}) \simeq \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{P}), \mathcal{V})$$

entre l'∞-catégorie des $\mathcal{P}$-algèbres dans $\mathcal{V}$ et celle des foncteurs monoïdaux forts $\mathsf{Env}(\mathcal{P}) \to \mathcal{V}$. Nous adoptons cette perspective — les algèbres sur une opérade comme foncteurs monoïdaux forts depuis son enveloppe — pour construire une équivalence de un/straightening opéradique.

Nous commençons par observer que, bien que l'enveloppe ne soit pas pleinement fidèle en général, il résulte de [HK24, Proposition 2.4.3] (Haugseng-Kock) qu'elle est un adjoint à gauche pleinement fidèle lorsqu'on la restreint à la tranche au-dessus d'une ∞-opérade de Lurie $\mathcal{O}^{\otimes}$.

Comme première étape, nous prouvons donc que ce foncteur tranché se restreint à une équivalence

$$\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \xrightarrow{\mathsf{Env}(-)^{\otimes}} \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})}$$

entre l'∞-catégorie des fibrations opéradiques à gauche au-dessus de $\mathcal{O}^{\otimes}$ (la même que dans la construction de Grothendieck monoïdale de Ramzi) et celle des *fibrations à gauche fortement symétriques monoïdales* au-dessus de $\mathsf{Env}(\mathcal{O})^{\otimes}$, qui sont des fibrations à gauche entre ∞-catégories monoïdales symétriques satisfaisant une certaine condition de Segal (Definition 4.3.7). Nous caractérisons ensuite cette dernière ∞-catégorie comme l'image essentielle du straightening monoïdal (voir la Section 1.2.2) appliqué aux foncteurs monoïdaux *forts* $\mathsf{Env}(\mathcal{O}) \to \mathcal{S}$, cette caractérisation étant en fait valable pour toute ∞-catégorie monoïdale symétrique $\mathcal{C}^{\otimes}$. C'est ainsi que nous démontrons le résultat principal de cette section :

Theorem (Theorem 4.5.1)**.** *Soit $\mathcal{O}^{\otimes}$ une ∞-opérade de Lurie. Il existe une équivalence d'∞-catégories entre les fibrations opéradiques à gauche et les $\mathcal{O}^{\otimes}$-algèbres en espaces, où le foncteur de straightening est donné par la composition de l'enveloppe monoïdale symétrique et du foncteur de straightening monoïdal. Formellement,*

$$\mathsf{St}^{\mathcal{O}} \colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \xrightarrow{\mathsf{Env}(-)^{\otimes}_{/}} \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} \xrightarrow{\mathsf{St}^{\mathsf{Env}(-),\otimes}} \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}), \mathcal{S}) \simeq \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}).$$

Son inverse est donné par la composition du foncteur de unstraightening avec le changement de base le long de l'unité de l'adjonction $(\mathsf{Env}(-)^{\otimes}, U)$.

Le foncteur de straightening opéradique $\mathsf{St}^{\mathcal{O}}$ diffère du foncteur de Ramzi (Section 1.2.2), qui coïncide, au niveau des objets sous-jacents, avec l'équivalence classique de un/straightening catégorique. Cette équivalence a l'avantage d'offrir une formulation en termes du foncteur d'enveloppe, qui se prête à des généralisations dans le cadre des motifs algébriques (ou *algebraic patterns*), comme dans la construction de Grothendieck de Kern [Ker23] ou telle que suggérée par Barkan-Haugseng-Steinebrunner dans [BHS22].

1.3.3 Indépendance vis-à-vis du modèle des fibrations opéradiques à gauche

Nous abordons ici la question de l'indépendance vis-à-vis du modèle des fibrations opéradiques à gauche formulée dans Question 2. En partant de l'équivalence de Hinich-Moerdijk entre les ∞-opérades de Lurie et les espaces de Segal dendroidaux complets, nous démontrons le résultat suivant :

Theorem (Theorem 4.2.15)**.** *Soient $\mathcal{O}^{\otimes}$ une ∞-opérade de Lurie et X une ∞-opérade dendroidale, telles que $\mathcal{O}^{\otimes}$ et X soient équivalentes via l'équivalence de Hinich-Moerdijk, c'est-à-dire $\mathcal{O}^{\otimes} \simeq \lambda(X)$. Alors, les foncteurs de comparaison de Hinich-Moerdijk (δ, λ) induisent une équivalence*

$$\delta_{/} \colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \rightleftarrows \mathsf{DLeft}_X : \lambda_{/} \ ,$$

entre l'∞-catégorie des fibrations opéradiques à gauche au-dessus de $\mathcal{O}^{\otimes}$ et celle des fibrations dendroidales à gauche au-dessus de X.

L'élément essentiel pour démontrer ce théorème est d'exprimer à la fois les fibrations opéradiques à gauche et les fibrations dendroidales à gauche comme objets locaux par rapport à certaines flèches anodynes à gauche dans des ∞-catégories données. Une difficulté technique majeure vient du fait que $\ell\mathsf{Op}_{\infty}$ n'est pas une localisation de la catégorie des préfaisceaux $\mathsf{PSh}(\Phi)$, où Φ est une catégorie de forêts, c'est-à-dire d'unions disjointes finies d'arbres, contrairement à ce que l'on voit pour les ∞-opérades dendroidales.

Dans le modèle de Lurie, les flèches anodynes sont les inclusions des cornets gauches simpliciales, tandis que dans le modèle dendroidal, ce sont les inclusions de cornets gauches dendroidaux. Nous montrons alors que la combinatoire peut être simplifiée, de sorte qu'il suffit de considérer uniquement les inclusions $0 \colon \Delta^0 \to \Delta^n$ et $\ell(T) \to T$ respectivement, où $\ell(T)$ désigne l'ensemble des feuilles de l'arbre T.

1.3.4 Rectification des fibrations dendroidales à gauche

Nous revenons maintenant à la discussion sur le foncteur racine et au Theorem 4.5.1, et procédons à la construction d'une équivalence de un/straightening pour les opérades discrètes dans le formalisme des ensembles dendroidaux, où le foncteur racine est défini.

Notre point de départ est l'équivalence de un/straightening pour les catégories discrètes formulée dans le formalisme des ensembles simpliciaux, telle que développée par Heuts et Moerdijk dans [HM15] ; nous étendons cette construction de la façon suivante. Étant donnée une opérade discrète P, on définit le *foncteur de rectification* $\rho^P_!$, l'adjont à gauche, comme un foncteur

$$\rho^P_! : \mathbf{dSets}/\mathcal{N}_d P \longrightarrow \mathsf{Alg}_P(\mathbf{sSets})$$

de la catégorie des ensembles dendroidaux au-dessus du nerf de P vers celle des P-algèbres simpliciales, caractérisé par la propriété suivante : pour tout arbre T et tout élément (T, α) dans $\mathbf{dSets}/\mathcal{N}_d(P)$, il existe un isomorphisme de P-algèbres simpliciales $\rho^P_!(T, \alpha) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/}$, où $\mathsf{Env}(-)$ désigne le nerf de l'enveloppe monoïdale symétrique des opérades discrètes et où T est considéré comme l'opérade libre qu'il engendre.

Nous démontrons le résultat suivant.

Theorem (Corollary 5.5.3)**.** *Soit P une opérade discrète Σ-libre. Le foncteur de rectification fait partie d'une équivalence de Quillen*

$$\rho^P_! : \mathbf{dSets}/\mathcal{N}_d P \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho^*_P$$

entre la structure de modèle covariante sur les ensembles dendroidaux au-dessus du nerf dendroidal de P et la structure de modèle projective sur les P-algèbres simpliciales.

Lorsque P est une catégorie discrète, on retrouve l'équivalence de Heuts-Moerdijk prouvée dans [HM15]. En fait, nous améliorons ce résultat en montrant ce qui suit.

Proposition (Theorem 5.4.4)**.** *Soit A une catégorie monoïdale symétrique discrète. La paire de Quillen*

$$\rho_!^A \colon \mathbf{sSets}/\mathcal{N}A \underset{\perp}{\rightleftarrows} \mathsf{Fun}(A, \mathbf{sSets}) : \rho_A^*$$

est une équivalence de Quillen monoïdale entre catégories de modèles de Quillen monoïdales, relativement à la structure de modèle projective et la convolution de Day sur $\mathsf{Fun}(A, \mathbf{sSets})$*, et à la structure de modèle covariante et le produit naturel* $\boxtimes$ *sur* $\mathbf{sSets}/\mathcal{N}A$*.*

La preuve du Corollary 5.5.3 se déroule en deux étapes distinctes, qui diffèrent par leurs méthodes.

La première étape consiste à montrer que l'adjonction construite est une paire de Quillen, et plus précisément que le foncteur adjoint à droite ρ_P^* est un foncteur de Quillen à droite. Pour cela, on utilise la combinatoire des arbres afin de mieux comprendre les relèvements des flèches dans l'image de ρ_P^* par rapport aux inclusions de bords dendroïdaux et de cornets feuilles (*leaf horns*)l.

La seconde étape, qui consiste à démontrer que l'adjonction $(\rho_!^P, \rho_P^*)$ définit une équivalence de Quillen, est plus difficile à établir par des méthodes de catégories de modèle, le problème central étant que les colimites dans les catégories d'algèbres ne se calculent pas objet par objet (voir la discussion au début de la Section 5.5).

Les choses deviennent plus simples si l'on adopte les méthodes moins rigides des ∞-catégories, et nous considérons alors l'adjonction entre les ∞-catégories sous-jacentes induite par le foncteur de rectification,

$$(\rho_!^P)_\infty \colon \mathsf{dLeft}_{\mathcal{N}_d P} \underset{\perp}{\rightleftarrows} \mathsf{Alg}_P(\mathcal{S}) : (\rho_P^*)_\infty .$$

Le point crucial est désormais d'utiliser le Theorem 5.4.4 pour montrer que, pour P Σ-libre, il existe un diagramme commutatif d'∞-catégories

$$\begin{tikzcd}
\mathsf{dLeft}_{\mathcal{N}_d P} \arrow[r, "(\rho_!^P)_\infty"] \arrow[d, "\lambda'_/"'] & \mathsf{Alg}_P(\mathcal{S}) \\
\mathsf{Left}^{\mathsf{lax}}_{\mathcal{N}P} \arrow[ur, "\mathsf{St}^P"'] &
\end{tikzcd}$$

où $\lambda'_/$ est induit par l'équivalence de Hinich-Moerdijk 'tranchée' (Section 1.3.3) et St^P est le foncteur de rectification opéradique de le Théorème 4.5.1 pour le nerf de P. Comme les deux sont des équivalences d'∞-catégories, par le Théorème 4.2.15 et le Théorème 4.5.1 respectivement, nous concluons que $(\rho!^P)_\infty$ est également une équivalence.

1.3.5 Structure de modèle covariante et algèbres localement constantes

Nous combinons l'équivalence de un/straightening du Théorème 5.5.3 et l'équivalence $\mathcal{N}(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \simeq X$ donnée par le foncteur racine (Théorème 6.3.13) afin de fournir différentes caractérisations des ∞-catégories d'algèbres sur X.

Dans un premier temps, nous étudions la compatibilité de la localisation dendroidale avec la structure de modèle covariante.

Proposition (Proposition 6.4.4)**.** *Pour tout ensemble dendroidal Y et tout ensemble de 1-morphismes $S \subseteq Y_{C_1}$, la flèche de localisation $Y \to Y[S^{-1}]$ induit une équivalence de Quillen*

$$\lambda_! \colon \mathbf{dSets}^S/Y \underset{\perp}{\rightleftarrows} \mathbf{dSets}/Y[S^{-1}] \colon \lambda^*$$

entre la localisation de Bousfield à gauche de la structure de modèle covariante sur $\mathbf{dSets}/Y$ *en* $S_{/Y}$ *et la structure de modèle covariante sur* $\mathbf{dSets}/Y[S^{-1}]$*. Ici,* $S_{/Y}$ *est l'ensemble des morphismes dans* $\mathbf{dSets}/Y$ *de la forme*

$$\begin{tikzcd}
\Delta^0 \arrow[rr, hook, "\{1\}"] \arrow[dr, "b"'] & & \Delta^1 \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

où $f \colon a \to b$ est un morphisme de S et où la flèche $\{1\} \colon \Delta^0 \to \Delta^1$ est l'inclusion du dernier sommet.

En particulier, les objets fibrants pour la structure de modèle sur $\mathbf{dSets}^S/Y$ sont les *fibrations dendroidales à gauche $S_{/Y}$-locales* au-dessus de Y, c'est-à-dire les fibrations dendroidales à gauche $E \to Y$ telles que, pour tout $f \colon a \to b$ dans S, le morphisme dans la catégorie homotopique des espaces entre les fibres $f_! \colon E_a \to E_b$ est un isomorphisme.

L'analogue algébrique de cette notion s'exprime par celle des algèbres S-localement constantes sur une opérade. Si P est une opérade simpliciale et S une sous-classe de 1-morphismes dans P, une P-algèbre simpliciale *localement constante* sur S est une P-algèbre F telle que l'application d'ensembles simpliciaux $f_* \colon F(a) \to F(b)$ soit une équivalence faible d'homotopie pour tout morphisme $f \colon a \to b$ dans S.

En considérant $P = \mathbf{\Omega}/X$ et $Y = \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}]$, on obtient deux résultats : une équivalence de un/straightening « localement constante » pour les ∞-opérades dendroidales, ainsi qu'une caractérisation des algèbres sur ces dernières comme des algèbres localement constantes sur leur opérade d'éléments.

Corollary (Corollary 6.4.9)**.** *Soit X une ∞-opérade dendroidale normale.*

1. *Le foncteur de rectification pour l'opérade des elements $\mathbf{\Omega}/X$ induit une équivalence de Quillen*

$$\rho_!^{\mathbf{\Omega}/X} \colon \mathbf{dSets}^{\mathcal{R}_X}/\mathcal{N}_d\mathbf{\Omega}/X \underset{\perp}{\rightleftarrows} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}) \colon \rho^*_{\mathbf{\Omega}/X}$$

entre la structure de modèle covariante localisée de la Proposition 6.4.4 et la localisation de Bousfield à gauche de la structure de modèle projective sur les $\mathbf{\Omega}/X$-algèbres simpliciales en $\rho_!^{\mathbf{\Omega}/X}(\mathcal{R})$. Les objets fibrants de cette dernière sont précisément les $\mathcal{R}_X$-algèbres localement constantes sur $\mathbf{\Omega}/X$.

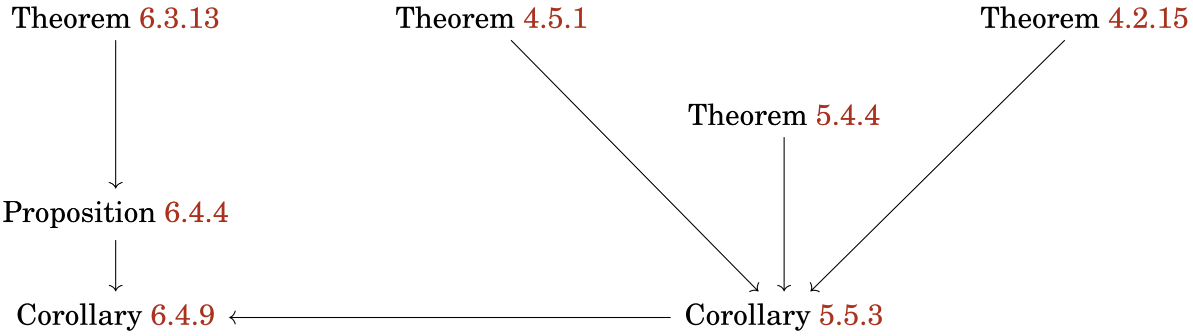
2. *Il existe un zigzag d'équivalences de Quillen*

$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets})$$

entre les structures de modèles projectives pour les algèbres simpliciales $W_!(X)$ et pour les algèbres $\mathcal{R}_X$-localement constantes sur $\mathbf{\Omega}/X$, où $W_!(X)$ est l'opérade simpliciale donnée par la résolution de Boardman-Vogt de X.

1.3.6 Interdépendance des résultats

Le diagramme suivant illustre les dépendances logiques entre les résultats démontrés. Une flèche allant de A vers B signifie que le résultat A est nécessaire pour prouver le résultat B.



1.4 Plan de la thèse

Dans cette introduction, nous avons présenté nos contributions en insistant sur les motivations et leurs interconnexions, dans l'espoir d'offrir une vision cohérente, et aussi avec la volonté de restituer le parcours intellectuel qui nous a mené à nos résultats.

Pour plus de clarté et de rigueur, le contenu est organisé différemment dans le corps principal de la thèse, qui se présente comme suit :

- Le Chapitre 3 contient les bases essentielles sur la théorie des ∞-opérades et les différents formalismes, et établit la notation utilisée tout au long du texte. Il s'agit d'un chapitre explicatif, sans demonstration; l'apport original de la thèse se trouve dans les chapitres 4, 5, 6.
- Le Chapitre 4 contient les résultats des Section 1.3.2 et Section 1.3.3. Mis à part les préliminaires et des ajustements mineurs d'exposition, il correspond à l'article disponible sur arXiv sous le numéro 2501.05263v2, intitulé *A straightening-unstraightening équivalence for ∞-operads* ([Pra25c]).
- Le Chapitre 5 contient les résultats de la Section 1.3.4. Mis à part les préliminaires et des ajustements mineurs d'exposition (principalement dans Section 5.5), il correspond à la version corrigée de l'article disponible sur arXiv sous le numéro 2502.17415v2, intitulé *Rectification of dendroidal left fibrations* ([Pra25a]).
- Le Chapitre 6 contient les résultats de Section 1.3.1 et Section 1.3.5. Mis à part les préliminaires et des ajustements mineurs d'exposition, il correspond à toutes les sections sauf la dernière de l'article disponible sur arXiv sous le numéro 2505.14288, intitulé *The root functor* ([Pra25b]).

Les chapitres sont structurés de façon à pouvoir être lus indépendamment les uns des autres. Au début de chaque chapitre, nous précisons les notions préliminaires nécessaires, avec des références explicites aux sections pertinentes du Chapitre 3. Lorsque des résultats des chapitres précédents sont requis, nous les rappelons dans leur contexte afin que l'exposé reste aussi auto-contenu que possible.

1.4.1 Hypothèses

Nous supposons que la personne qui lit est familière avec les catégories de modèles de Quillen (une bonne référence est [Hir03]), ainsi qu'avec la théorie des ∞-catégories (telle que développée, par exemple, dans [Lur09] ou [Cis19]). Les autres notions de base nécessaires sont rappelées explicitement dans le Chapitre 3, le texte de référence pour la théorie des opérades dendroïdales étant [HM22].

1.5 Perspectives

Dans cette thèse, nous verrons comment plusieurs de nos résultats sont liés à des constructions existantes, suggérant des comparaisons, des équivalences ou des généralisations dont le développement nous semble naturel. Ces points sont discutés et mis en évidence au fil des chapitres, en particulier dans le Chapter 4 (Section 4.1.2, Remarque 4.2.19), Chapitre 5 (Remarque 5.5.5).

Nous nous concentrons ici sur une autre perspective encore, concernant les opérades relatives, les ∞-opérades, et la localisation. Considérons le foncteur de localisation

$$\mathcal{L}\colon \mathsf{RelOp} \longrightarrow \mathsf{Op}_\infty$$

de la 1-catégorie des opérades relatives vers l'∞-catégorie des ∞-opérades, qui envoie une opérade relative sur la localisation ∞-opéradique de son nerf[6]. Le foncteur racine et le Théorème 6.3.13 montrent que $\mathcal{L}$ est essentiellement surjectif, et fournit de plus une section

$$\mathbf{\Omega}/-\colon \mathsf{Op}_\infty \longrightarrow \mathsf{RelOp},$$

où $\mathbf{\Omega}/X$ est regardée comme une opérade relative en considérant comme équivalences faibles les morphismes envoyés par le foncteur racine sur des identités.

Quand on se restreint aux ∞-catégories, le fait que $\mathcal{L}$ est essentiellement surjectif est le contenu du théorème de délocalisation de Joyal. Dans ce cas, on a même un énoncé plus fort : dans [BK12b], Barwick-Kan montrent que la théorie homotopique des catégories relatives est équivalente à celle des ∞-catégories. Plus précisément, ils prouvent que le *diagramme classifiant de Rezk* (prouvé équivalent au foncteur de localisation par Mazel-Gee dans [MG19]) induit une équivalence d'∞-catégories

$$\mathcal{L}\colon \mathsf{RelCat}[\mathsf{DK}^{-1}] \longrightarrow \mathsf{Cat}_\infty,$$

où DK désigne les *équivalences faibles de Dwyer-Kan*, c'est-à-dire les foncteurs induisant une équivalence après localisation. Le résultat de Barwick-Kan est formulé dans le langage des espaces simpliciaux ; récemment, Arakawa a démontré une version quasi-catégorique, montrant que le foncteur du dernier sommet, vu comme un foncteur entre catégories relatives

$$\Delta/-\colon (\mathbf{sSets}, \mathcal{W}_{\mathrm{Joy}}) \longrightarrow (\mathsf{RelCat}, \mathsf{DK})$$

---

[6] Un modèle concret de $\mathcal{L}(P)$ est donné par la localisation dérivée (voir Definition 6.2.2 et Proposition 6.2.4) d'un remplacement cofibrant du nerf dendroïdal de P.

est un inverse homotopique du foncteur de localisation [Ara25, Theorem A.2].

Le foncteur racine suggère qu'un résultat similaire devrait également valoir pour les ∞-opérades ; en fait, en combinant Theorem 6.3.13 avec les techniques de [Ara25], il devrait être possible de prouver que le foncteur de localisation ∞-opéradique induit une équivalence d'∞-catégories

$$\mathsf{RelOp}[\mathsf{DK}^{-1}] \longrightarrow \mathsf{Op}_\infty$$

dont l'inverse homotopique est donné par le foncteur racine. Ici, les équivalences faibles de Dwyer-Kan DK sont les morphismes d'opérades relatives qui induisent une équivalence après localisation. Cependant, cette notion n'est réellement significative que si l'on parvient à la relier aux équivalences de Dwyer-Kan pour les opérades simpliciales et à leur localisation dérivée.

En effet, dans le cas catégorique, l'∞-catégorie $\mathsf{RelCat}[\mathsf{DK}^{-1}]$ est construite et comprise en deux étapes :

- Dans [BK12b], Barwick-Kan construisent une adjonction

  $$K\colon \mathbf{sSpaces} \overset{\perp}{\rightleftarrows} \mathsf{RelCat} : \mathcal{N}_\xi$$

  et relèvent la structure de modèle de Rezk pour les espaces de Segal complets en une structure de modèle de Quillen équivalente sur RelCat, dont les équivalences faibles sont appelées *équivalences faibles de Rezk*.

- Dans [BK12a], les auteurs considèrent ensuite le *foncteur de localisation hamac* $\mathcal{L}^H\colon \mathsf{RelCat} \longrightarrow \mathsf{sCat}_\infty$,

  qui fournit une description explicite pour la localisation dérivée des catégories simpliciales, et ils prouvent que les équivalences faibles de Rezk dans RelCat sont exactement celles envoyées par $\mathcal{L}^H$ sur les équivalences faibles de Dwyer-Kan dans les catégories simpliciales. Ils démontrent aussi que le même résultat vaut si l'on remplace la localisation hamac par d'autres modèles équivalents au sens de Dwyer-Kan.

Il devrait être possible d'obtenir un diagramme similaire pour les ∞-opérades et de prouver que les équivalences faibles de Dwyer-Kan dans RelOp sont celles envoyées sur des équivalences par le foncteur de localisation dérivée $\mathsf{RelOp} \to \mathsf{sOp}_\infty$.

Néanmoins, à l'heure actuelle, nous ne disposons pas d'outils permettant de calculer explicitement la localisation dérivée des opérades simpliciales, comme la localisation hamac le permet pour les catégories simpliciales. Récemment, Basterra-Bobkova-Ponto-Tillmann-Yeakel, dans [BBP$^+$18], ont proposé une généralisation qu'ils appellent *localisation hamac-arbre* (*tree-hammoc localisation*), conçue pour présenter les localisations dérivées des opérades discrètes et simpliciales. Cependant, plusieurs points centraux restent ouverts : en particulier, il n'a pas été démontré que cette construction calcule bien la localisation dérivée, et les algèbres sur la localisation hamac-arbre n'ont été identifiées que partiellement.

Chapter 2

Introduction - english

Designed to be read independently from the thesis, this introduction summarizes the key results and motivations of our work and outlines some natural directions for future work; it also provides a detailed outline of the thesis structure to assist the reader in navigating its contents.

Contents

2.1 Context

2.1.1 Operads in homotopy theory

Operads are combinatorial objects governing categories of algebras. They were first introduced by May ([May06]) and Boardman-Vogt ([BV73]) in the early ’70s to describe the homotopy coherent algebraic structures on iterated loop spaces of topological spaces and the associated homology operations. Since then they have found numerous applications in various fields of mathematics, including algebraic geometry, combinatorics and mathematical physics.

To gain insight into the type of algebraic structures under consideration, consider the based loop space $\Omega_x X \coloneqq \mathsf{Map}(\mathbb{S}^1, X)$ of a pointed topological space (X, x). Concatenation of loops endows $\Omega_x X$ with an operation which is not strictly associative, but which is associative only *up to homotopies* – specifically, up to reparametrizations of the loops. Moreover the homotopies between differently parametrized concatenations satisfy themselves analogous associativity relations up to homotopy,and coherently so in all higher dimensions. If we quotient $\Omega_x X$ by homotopy equivalences, we obtain the familiar group structure on $\pi_1(X)$, the fundamental group of X. However, this process loses much of the full algebraic structure. Instead, $\Omega_x X$ naturally carries the structure of an *algebra over the little disks operad* $\mathbb{E}_1$, which retains all the homotopy coherences.

The operad $\mathbb{E}_1$ is given For every $k \geqslant 0$, let $\mathbb{E}_1(k)$ be the space of configurations of k

disjoint intervals inside the unit interval. Every such configuration determines a way to concatenate k based loops in X, yielding a continuous map of topological spaces

$$\mathbb{E}_1(k) \longrightarrow \mathsf{Map}((\Omega_x X)^{\times k}, \Omega_x X).$$

The fact that configurations can be plugged one into the other yields partial composition laws on the family $\mathbb{E}_1 = \{\mathbb{E}_1(k)\}_k$, and this governs the composition laws for the concatenations of loops and their homotopy coherences. We call $\mathbb{E}_1$ the *little disks operad*.

The operad $\mathbb{E}_1$ is the first of a family of operads $\{\mathbb{E}_n\}_{n\geqslant 1}$, called the *operads of little n-disks*, which govern homotopy associative and increasingly homotopy commutative algebraic structures, as n goes to infinity. For every $k \geqslant 0$, the space $\mathbb{E}_n(k)$ of k-ary operations of $\mathbb{E}_n$ is given by the space of configurations of k disjoint n-dimensional disks into the unit n-dimensional disks. For instance, for every $n \geqslant 1$ the n-fold loop space $\Omega^n_x X$ of a pointed topological space (X, x) has the structure of a $\mathbb{E}_n$-algebra – and in fact, up to homotopy these are the only (group-like) $\mathbb{E}_n$-algebras (in topological spaces), by May's recognition theorem ([May06]).

More generally, we allow what we call operad P, and what is often called *colored operad* or *multicategory*, to have a set of objects $\mathsf{Ob}(P)$ and, for each $k \geqslant 0$, a space of operations (later specified as a simplicial set) $P(x_1, \ldots, x_k; y)$ for any objects $x_1, \ldots, x_k, y$ (in the one-object case, we write $P(k)$ for $P(\underbrace{*, \ldots, *}_{k}; *)$). These come equipped with unital and associative operadic composition laws operadic composition laws

$$\circ_{x_i} \colon P(x_1, \ldots, x_k; y) \times P(z_1, \ldots, z_m; x_i) \longrightarrow P(x_1, \ldots, x_{i-1}, z_1, \ldots, z_m, x_{i+1}, \ldots, x_k; y),$$

equivariant with respect to the action of the symmetric groups given by 'permuting the variables'.

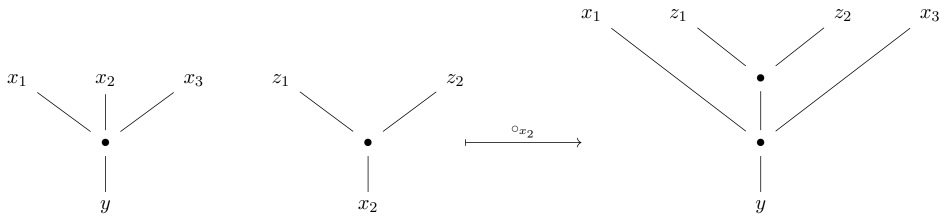


Figure 2.1: A graphical representation of the operadic composition $\circ_{x_2}$.

An algebra over the operad P is a family of spaces $F = \{F(x)\}_{x\in\mathsf{Ob}(P)}$ on which the operad acts via continuous maps, together with action maps

$$P(x_1, \ldots, x_k; y) \longrightarrow \mathsf{Map}(\prod_{i=1}^{k} F(x_i), F(y))$$

respecting composition, units, and symmetries.

In homotopy theory, topological spaces are regarded only up to weak homotopy equivalence, and one deals with homotopy coherence via Quillen model category structures or ∞-categories. To this end, topological spaces are represented by simplicial sets, the presheaves on the category of finite linear orders Δ (for instance, one can take the singular chains on a space), and study their homotopy theory via the Kan-Quillen model structure on simplicial sets. Considering spaces up to weak homotopy equivalences leads to considering simplicial operads up to what are called *Dwyer-Kan weak equivalences*, the homotopically fully faithful and essentially surjective maps of operads, and to study the homotopy theory of algebras over operads up to homotopy.

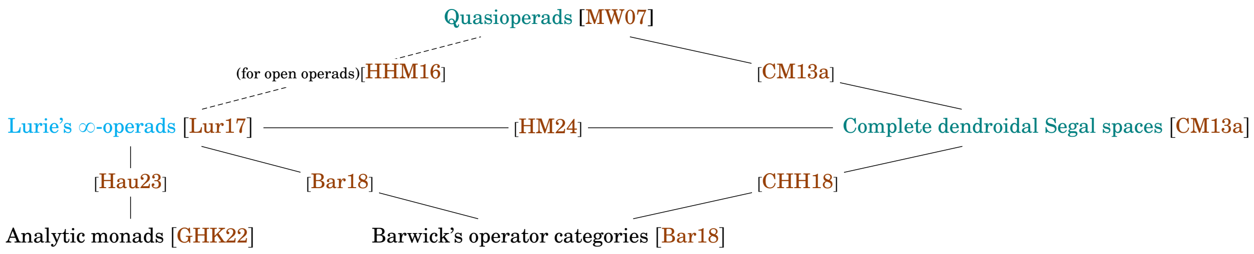
Perhaps the most intuitive way to study the homotopy theory of simplicial operads is by constructing a model structure on simplicial operads where weak equivalences are the Dwyer-Kan weak equivalences; this was done, independently, by Cisinski-Moerdijk [CM13b] and Robertson [Rob11]. The extra structure given by a Quillen model category can help to study homotopy invariant constructions; for instance, in the presence of a model structure, one has an intrinsic way to define homotopy algebras over an operad — specifically, as algebras over a cofibrant replacement of the operad.

Although easy to understand, this approach is hard to use when trying to carry out basic categorical constructions, and one runs into problems similar to those found in the case of simplicial categories. For instance, the calculus of derived mapping spaces is hard to perform: although the Boardman-Vogt tensor product endows the category of simplicial operads with a simplicial enrichment, this is not compatible with the model structure. In other words, the aforementioned model structure on the category of simplicial operads is not a monoidal model category. [1].

2.1.2 A more flexible formalism: ∞-operads

These difficulties can be circumvented by passing to the formalism of *∞-operads*, where some of the defining axioms for simplicial operads are relaxed and composition is only defined up to homotopy. While simplicial operads can be viewed as examples of ∞-operads (via the nerve constructions), the latter notion is more flexible and adequate for the purposes of modern homotopy theory.

There are several, equivalent models for ∞-operads; let us schematically represent some of the existing models and equivalences between them.



In this work, we make use of two formalisms, the first one given by Lurie's ∞-operads and the second one given by the *dendroidal formalism*, which consists in the two models of quasioperads and complete dendroidal Segal spaces.

Intuitively, Lurie's approach to ∞-operads rests on the insight that an operad can be

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[1] More precisely, the model structure on simplicial operads is not model monoidal. In particular, the pushout-product axiom fails, as it was already the case for Bergner's model structure on simplicial categories, see the discussion at *https://mathoverflow.net/questions/198205/boardman-vogt-tensor-product*.

viewed as a weakened form of symmetric monoidal category, where the tensor product $x_1 \otimes \cdots \otimes x_n$ is specified only via the covariant functor $y \mapsto \mathrm{Hom}(x_1, \dots, x_n; y)$, which need not be representable. This leads to a homotopy-coherent notion of operad as a relaxation of that of symmetric monoidal category, whose formalization fundamentally relies on the theory of coCartesian fibrations.

The dendroidal formalism is based on a category of trees $\mathbf{\Omega}$, called the *dendroidal category*, introduced by Moerdijk-Weiss in [MW07], which models operations and operadic composition in a way that reflects the diagrammatic intuition given in Figure 2.1. This formalism relies on the fact that an operads are a generalization of categories – indeed, sometimes operads are referred to as *symmetric multicategories* –, in that morphisms $f \colon x \to y$ (with a single input and output) are replaced by multimorphism $(x_1, \dots, x_n) \to y$ that allow multiple inputs and a single output. The category Ω is then designed to capture the structure of such multimorphisms and their composition, in much the same way that the simplex category Δ models morphisms and composition in ordinary categories.

2.2 Motivations

The formalism of ∞-operads provides tools to deal combinatorially with homotopy coherences and perform homotopy-coherent constructions. However, there is an important class of ∞-operads for which this complexity is not needed: it is given by the *discrete* operads, whose spaces of operations carry no topology structure and are simply sets. In many cases, this makes discrete operads much simpler to study than their ∞-categorical counterpart, as one does not need to keep track of homotopy coherences. On the other hand, their rigidity makes it necessary sometimes to study them as instances of particular ∞-operads. The interplay between these two perspectives is subtle but rich: let us explore how it shapes the motivations of this work.

2.2.1 Discrete operads and localizations

A way to obtain non trivial ∞-operads arises from relative operads. A *relative operad* is a discrete operad P together with a set of unary-operations S called weak equivalences. Starting from a relative operad (P, S), one can formally invert the arrows in S in the ∞-operad $\mathcal{N}^{\otimes}(P)$ determined by the nerve of P and obtain an ∞-operad $\mathcal{N}(P)^{\otimes}[S^{-1}]$, the *localization of P at S*, characterized by the familiar universal property of being initial amongst those ∞-operads under $\mathcal{N}^{\otimes}(P)$ where the arrows in S become equivalences.

A first fundamental class of ∞-operads which arises in this way is given by ∞-categories: Joyal's delocalization theorem, stated in [Joy07, §13.6] and proven by Stevenson in [Ste17], guarantees that every ∞-category X, which can be represented as a simplicial set (that is, a quasicategory), is equivalent to a localization of the nerve of its category of elements Δ/X. This equivalence is realized via the *last vertex functor* (already appearing in the work of Waldhausen [Wal85, §1.6]), a map of simplicial sets $\mathcal{N}(\Delta/X) \to X$, from the nerve of Δ/X to X, which evaluates an n-simplex of X at its n^{th} vertex. In the theorem, it is shown that this map induces a morphism of simplicial sets

$$\mathcal{N}(\Delta/X)[\mathcal{J}^{-1}] \xrightarrow{\sim} X$$

which exhibits X as the ∞-categorical localization of $\mathcal{N}(\Delta/X)$ at the set of morphisms $\mathcal{J}$, given by the preimages of the identity morphisms.

Another important class of ∞-operads which are not simply ∞-categories and arise as the localization of relative operads is that of the little n-disks operads $\mathbb{E}_n$. Recall that, for every $k \geqslant 0$, the space of k-ary operations $\mathbb{E}_n(k)$ is the space of configurations of k disjoint n-dimensional disks into the unit n-dimensional disk. A point in $\mathbb{E}_n(k)$ may be represented as a diagram of embeddings

$$\begin{tikzcd}
\coprod_{i=1}^{k} \mathbb{D}^n \arrow[rr, hook] \arrow[dr, hook] & & \mathbb{D}^n \arrow[dl, hook'] \\
& \mathbb{R}^n &
\end{tikzcd} \tag{2.2.1}$$

which commutes up to a specified isotopy; paths in $\mathbb{E}_n(k)$ are given by isotopies between these, and so on.

Consider now the discrete operad Disks_n, which has a non-trivial set of objects, given by the set of embeddings $\mathbb{D}^n \hookrightarrow \mathbb{R}^n$ of an open n-dimensional disk in $\mathbb{R}^n$, and where an operation from k such embeddings to another is a triangle as in Diagram 5.3.5 but which commutes strictly. There is a natural morphism of ∞-operads $\mathcal{N}^{\otimes}\mathsf{Disk}_n \longrightarrow \mathbb{E}_n$ from the nerve of Disks_n into $\mathbb{E}_n$, sending every inclusion of a finite disjoint union of disks into the corresponding configuration in $\mathbb{R}^n$. Every arrow in Disk_n is sent to an equivalence in $\mathbb{E}_n$, and in [Lur17, §5.4.5] Lurie shows that the induced map from the ∞-operadic localization[2] at the preimage of equivalences

$$\mathcal{N}^{\otimes}(\mathsf{Disks}_n)[\mathcal{J}^{-1}] \xrightarrow{\sim} \mathbb{E}_n$$

is an equivalence of ∞-operads. The fundamental consequence is that the induced functor between the ∞-categories of algebras

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{S}) \hookrightarrow \mathsf{Alg}_{\mathsf{Disk}_n}(\mathcal{S})$$

is fully faithful and allows to identify $\mathbb{E}_n$-algebras with the *locally constant algebras* over Disk_n, that is, those sending unary morphisms into equivalences. This characterization of $\mathbb{E}_n$-algebras allows to prove that any $\mathbb{E}_n$-algebra can be cocontinuously extended to an algebra over the ∞-operad $\mathcal{O}(\mathbb{R}^n)$ of *all* open subspaces of $\mathbb{R}^n$ by a process of operadic left Kan extension ([Lur17, Theorem 5.5.2.5]). In fact, the same result is true for the ∞-operads $\mathbb{E}_M$ and Disks_M, obtained by replacing $\mathbb{R}^n$ by a framed n-manifold M[3], and is at the core of the definition of *factorization homology* of the manifold M, an algebraic and topological invariant connected with Hochschild homology and homology theories ([Lur17], [AF15],[AFT17]).

Localization proves then to be an important tool to understand the theory of ∞-operads, and as we have seen, expressing an ∞-operad as the localization of a discrete one can simplify certain arguments in concrete computations. In light of this, and guided by the principle that results for ∞-categories should admit ”multiplicative” extensions to the operadic setting, it is natural to consider the following question.

Question 1. Does *every* ∞-operad write as the localization of a discrete operad? Can we describe the aforementioned discrete operad explicitly? Can we do this by extending Joyal's delocalization theorem?

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[2] To be precise, to deal with localization of ∞-operads Lurie uses the theory of *weak approximations*.

[3] In fact, even a stratified manifold, see the works of Ayala-Francis-Tanaka [AFT17].

We will answer all these questions in the affirmative, as we will see in Section 2.3. This will allow us to describe the ∞-category of algebras over an ∞-operad through their characterization as locally constant algebras over the discrete resolution. This perspective opens up various tools for studying the homotopy theory of operadic algebras, notably the crucial class of methods known as *operadic un / straightening equivalences*.

2.2.2 Operadic un/straightening equivalences

In a broad sense, a *un / straightening equivalence* (to be read *straightening-unstraightening equivalence*), consists in describing diagrams in the ∞-category of spaces $\mathcal{S}$ (and even in that of ∞-categories) using certain fibrations which are easier to understand. In classical algebraic topology, one can think about the correspondence between covering spaces $E \to X$ of a topological space X and sets with an action of the fundamental group $\pi_1(X)$. In modern language, the un/straightening equivalence for an ∞-category $\mathcal{C}$ was first formulated and proven by Lurie ([Lur17]) and consists of an equivalence of ∞-categories

$$\mathsf{St}^{\mathcal{C}} : \mathsf{Left}_{\mathcal{C}} \rightleftarrows \mathsf{Fun}(\mathcal{C}, \mathcal{S}) : \mathsf{Unst}^{\mathcal{C}},$$

where $\mathsf{Left}_{\mathcal{C}}$ is the ∞-category of *left fibrations* over $\mathcal{C}$, a generalization of the notion of categories cofibred in groupoids, and $\mathsf{Fun}(\mathcal{C}, \mathcal{S})$ is that of ∞-categorical functors from $\mathcal{C}$ to $\mathcal{S}$.

Again, a guiding principle in the theory of ∞-operads is that a result valid for ∞-categories should admit a multiplicative extension to ∞-operads. In the operadic generalization, functors become algebras and fibred objects become multiplicatively fibred objects: an *operadic un / straightening equivalence* for an ∞-operad $\mathcal{P}$ consists of an equivalence of ∞-categories

$$\mathsf{St}^{\mathcal{P}} : \mathsf{Left}\,(\mathsf{Op}_\infty)_{\mathcal{P}} \rightleftarrows \mathsf{Alg}_{\mathcal{P}}(\mathcal{S}) : \mathsf{Unst}^{\mathcal{P}},$$

where $\mathsf{Left}(\mathsf{Op}_\infty)_{\mathcal{P}}$ is an ∞-category of *operadic left fibrations* over $\mathcal{P}$, the higher categorical generalization of operads cofibred in groupoids, and $\mathsf{Alg}_{\mathcal{P}}(\mathcal{S})$ that of $\mathcal{P}$-algebras valued in the ∞-category of spaces $\mathcal{S}$.

The definition of the ∞-categories involved in the operadic un/straightening equivalence, and the construction of the equivalence itself, depend on the chosen model of ∞-operad; in both the frameworks of Lurie and the dendroidal approach, the ∞-categories $\mathsf{Left}(\mathsf{Op}_\infty)\mathcal{P}$ and $\mathsf{Alg}\mathcal{P}(\mathcal{S})$ are well-defined, and certain operadic equivalences of un/straightening have been established. Here is a brief summary of these results.

In Lurie's and the dendroidal settings the ∞-categories $\mathsf{Left}(\mathsf{Op}_\infty)_{\mathcal{P}}$ and $\mathsf{Alg}_{\mathcal{P}}(\mathcal{S})$ are well-defined, and some operadic un/straightening results have been established. A brief summary of these results follows.

In the dendroidal models, the notion of operadic left fibration extends the analogous one for ∞-categories. The first definition was formulated in the model of dendroidal sets and goes back to Heuts [Heu11]. It relies on the notion of *leaf horn* of a tree, a combinatorial object which helps to capture existence and uniqueness up to higher homotopies of lifts of operations. The ∞-category of operadic left fibrations over a quasioperad X is modeled by the *covariant model structure* on the category $\mathbf{dSets}/X$ of dendroidal sets over X. In [Heu11], Heuts proves an operadic un/straightening equivalence for a normal[4]

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[4] It is a cofibrancy condition, it corresponds to the action of the symmetric group on an operad being free.

dendroidal set X, constructing a direct Quillen equivalence between the covariant model structure on $\mathbf{dSets}/X$ and the projective model structure on the category $\mathsf{Alg}_{W_!(X)}(\mathbf{sSets})$ of simplicial algebras over the simplicial operad $W_!(X)$ (a sort of homotopy-coherent Boardman-Vogt resolution of X). In the model of complete dendroidal Segal spaces, an analogous equivalence was constructed by Boavida-Moerdijk in [BdBM20], for a notion of operadic left fibration which results to be equivalent to the one given in the dendroidal case. We observe that, while in the model of dendroidal sets an operadic left fibration is defined by means of a right lifting property (against leaf horn inclusions), for dendroidal spaces this is rather defined as a local object with respect to *leaf inclusions* of trees. In both these dendroidal un/straightening equivalences, the left adjoint, which assigns to an operadic left fibration over a dendroidal ∞-operad an algebra over it, acts on representable dendroidal left fibrations as a 'free algebra' functor.

In Lurie's formalism, a un/straightening equivalence for an ∞-operad $\mathcal{P}$ can be deduced from Ramzi's more general *monoidal Grothendieck construction* for *$\mathcal{P}$-monoidal ∞-categories* ([Ram22]). For any such $\mathcal{P}$-monoidal ∞-category $\mathcal{C}$, which one can regard as an ∞-operad cofibred over $\mathcal{P}$, Ramzi's construction establishes an equivalence of ∞-categories between the ∞-category of *$\mathcal{P}$-monoidal left fibrations* over $\mathcal{C}$ and that of $\mathcal{P}$-lax monoidal functor $\mathcal{C} \to \mathcal{S}$[5]. With this formalism, one recovers the ∞-category of $\mathcal{P}$-algebras as that of $\mathcal{P}$-monoidal functors $\mathcal{P} \to \mathcal{S}$, and operadic left fibrations over $\mathcal{P}$ are the $\mathcal{P}$-monoidal left fibrations $\mathcal{D} \to \mathcal{P}$. The same equivalence was also established by Kern in [Ker23] using the formalism of algebraic patterns. While this approach is well suited for broader generalizations, introducing algebraic patterns lies beyond the scope of this thesis, and we will therefore not elaborate further.

In the previous section, we observed the centrality of the interplay of discrete and ∞-operads through localization. The following becomes for us an important

Motivation. Study the interaction between operadic localization and operadic left fibrations. Construct operadic un/straightening equivalences suited to manipulate discrete operads.

Analyzing the un/straightening equivalences also raises foundational questions, as we shall now see.

2.2.3 Model-independence

The equivalence between different formalisms for ∞-operads does not, in general, directly imply model-independence of operadic left fibrations—although such independence is arguably a necessary feature of any robust definition. Indeed, in order to define operadic left fibrations the main technical point to address is to make sense of operadic homotopy-coherent coCartesian lifts of operations, and this is done by using techniques not evidently independent from the chosen model.

A first question to ask is hence whether the ∞-category of operadic left fibrations over an ∞-operad independent from the chosen model.

For the dendroidal models, the answer is positive: in [BdBM20], Boavida-Moerdijk prove model invariance of operadic left fibrations in the two dendroidal models of complete

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[5] In fact, it is proven that both these ∞-categories arise as the ∞-categories of $\mathcal{P}$-algebras in some $\mathcal{P}$-monoidal ∞-categories, and that the equivalence is induced by a $\mathcal{P}$-monoidal equivalence between the latter.

dendroidal Segal spaces and dendroidal sets, relying on a direct equivalence between the two models for ∞-operads.

To date, no answer has been provided within Lurie's and the dendroidal frameworks, a principal challenge given by the fact that the first equivalence between the two formalism was indirect and featured a third model for ∞-operads, that of Barwick's operator categories (see the diagram in Section 2.1.2). However, in recent work, Hinich and Moerdijk [HM24] constructed a direct equivalence between Lurie and dendroidal ∞-operads, expressed through complete dendroidal Segal spaces, thereby proving an adjoint equivalence of ∞-categories

$$\delta\colon \ell\mathsf{Op}_\infty \underset{\perp}{\rightleftarrows} \mathsf{DOp}_\infty \colon\lambda.$$

This allows us to sharpen our previous question and formulate this second

Question 2. Does Hinich-Moerdijk equivalence induce an equivalence between the ∞-categories of operadic left fibrations in Lurie's and the dendroidal model?

In Section 2.3.3, we will see that this is indeed the case.

2.3 Contributions

In this section, we present the original contributions of this thesis. Before turning to a more detailed exposition, we offer a brief narrative outlining the core results.

We start by showing that every ∞-operad is equivalent to the localization of a discrete operad, via what we call the *root functor*. This leads us to study un/straightening constructions specifically for discrete operads. As the root functor is formulated in the dendroidal formalism and discrete operads are fundamentally 1-categorical, we work with the model of dendroidal sets; we construct, for any discrete Σ-free operad P, a un/straightening equivalence, in the form of a Quillen equivalence between the covariant and the projective model structures. In the construction of the adjunction, we use the symmetric monoidal envelope associated with discrete operads to define the left adjoint, the *rectification functor*. Showing that the adjunction is a Quillen pair involves considering algebras given by chains of inclusions of subtrees. To show that it is a Quillen equivalence, we consider the functors between the underlying ∞-categories, and prove that they form an adjoint equivalence. The key step is to construct a new un/straightening equivalence, which is in fact defined for *any* ∞-operad in Lurie's formalism, which we do in, and then show that our Quillen adjunction presents this equivalence when the operad is discrete. This required proving that Hinich-Moerdijk comparison functors induce an equivalence of ∞-categories between the ∞-category of operadic left fibrations for Lurie's and the dendroidal model. As an application of the root functor and the straightening theorems, we show that algebras over ∞-operads are equivalent to *locally constant algebras* over its discrete resolution.

While interconnected, these results each hold independent significance, and we now present them in a systematic manner.

2.3.1 The root functor

The dendroidal formalism, and more precisely the homotopy theory of dendroidal sets allows us to give a positive answer to Question 1. *Dendroidal sets* are the category of presheaves on the dendroidal category $\boldsymbol{\Omega}$. By means of the operadic model structure, they form a model for ∞-operads, which are represented as quasioperads, dendroidal sets satisfying a certain dendroidal inner horn filling condition. This model extends Joyal's model of quasicategories, in the sense that a quasicategory is precisely a quasioperad which admits a morphism into the point.

In this setting, we extend the category of elements functor $\Delta/-\colon \mathbf{sSets} \to \mathsf{Cat}$ to a functor

$$\mathbf{\Omega}/-\colon \mathbf{dSets} \longrightarrow \mathsf{Op}$$

from dendroidal sets to discrete operads, which assigns to a dendroidal set X its *operad of elements* $\mathbf{\Omega}/X$. The objects are pairs (T,α), with T is a tree and α a morphism from T to X, and the sets of operations are given by *wide* and *independent* maps of disjoint unions of trees (Definition 6.3.6). We then show that evaluating any such α at the root of T (the only 'canonical' edge of T) extends to a morphism of dendroidal sets

$$\mathfrak{r}_X\colon \mathcal{N}_d(\mathbf{\Omega}/X) \longrightarrow X$$

from the dendroidal nerve of $\mathbf{\Omega}/X$ into X, which we call the *root functor* (Proposition 6.3.10).

An element (T,α) can be interpreted as a decoration of the tree T with composable operations of X. What is happening then is that the homotopy-coherences of X are 'resolved' (or 'strictified') by becoming objects in the discrete operad $\mathbf{\Omega}/X$, and then recovered by identifying some of these objects – that is, by localizing at some set of morphisms. The root functor is the comparison map which captures precisely the missing homotopical information of X, in the following precise sense.

Theorem (Theorem 6.3.13)**.** *Let X be a normal dendroidal set, and let $\mathcal{R}$ be the set of morphisms of $\mathbf{\Omega}/X$ sent to identities by $\mathfrak{r}_X$. The root functor $\mathfrak{r}_X$ induces an operadic weak equivalence of dendroidal sets*

$$\overline{\mathfrak{r}_X}\colon \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \xrightarrow{\sim} X$$

between the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}$ and X.

To prove the theorem, we first formalize the notion of localization in the model of dendroidal sets (Definition 6.2.2), which is, to our knowledge, novel. We then show that when X is a normal dendroidal set we can construct an explicit description of the localization and express it as a pushout. In the proof of the theorem, we show that we can proceed by skeletal filtration, and we then construct an homotopy inverse of $\overline{\mathfrak{r}_X}$ in the case when X is a representable dendroidal set, that is, a tree.

A first corollary of Theorem 6.3.13 is that $\mathfrak{r}_X$ is a weak equivalence in the *stable model structure* on $\mathbf{dSets}$ (Corollary 6.3.15). The latter model structure, defined in [BN14], extends the Kan-Quillen model structure on simplicial sets and its homotopy theory is equivalent to that of infinite loop spaces. In other words, every normal dendroidal set is weakly equivalent to the infinite loop space of a discrete operad.

By restricting to simplicial sets, we also obtain Joyal's delocalization theorem, as desired.

Corollary (Joyal, Stevenson)**.** *For any simplicial set M, there is an isomorphism of categories $\mathbf{\Omega}/X \simeq \Delta/X$, and the root functor coincides with the the last vertex functor $\mathcal{N}(\Delta/M) \to M$. In particular, this latter induces a weak categorical equivalence*

$$\mathcal{N}(\Delta/M)[\mathcal{R}^{-1}] \longrightarrow M$$

between M and the localization of $\mathcal{N}(\Delta/M)$ at the preimage of the identities. In particular, the map $\mathcal{N}(\Delta/M) \to M$ is a weak homotopy equivalence.

2.3.2 A un/straightening equivalence in Lurie's model

The equivalence induced by the root functor gives centrality to discrete operads, and to the need of constructing convenient un/straightening equivalences to study the homotopy theory of algebras over them.

Before that, however, we adopt the broader framework of ∞-operads. As noted, ensuring homotopy-invariant constructions benefits from moving between model categorical and ∞-categorical approaches. Therefore, before focusing on discrete operads and dendroidal combinatorics (the impatient reader may go directly to Section 2.3.4), we consider Lurie's model of ∞-operads, which provides a tool available for discrete operads but absent in the dendroidal context: the symmetric monoidal envelope, which relates the theory of (∞-)operads to that of symmetric monoidal (∞-)categories.

The *symmetric monoidal envelope* functor

$$\mathsf{Env}(-)^{\otimes} \colon \ell\mathsf{Op}_{\infty} \longrightarrow \mathsf{smCat}_{\infty}$$

is the left adjoint to the forgetful functor from the ∞-category of symmetric monoidal ∞-categories to that ∞-operads. In [Lur17], Lurie constructs such a left adjoint, in a way which extends the analogous notion for discrete operads (originally defined under the name of PROP in [Mac65]). The envelope is characterized by the following universal property: for any ∞-operad $\mathcal{P}$ and any symmetric monoidal ∞-category $\mathcal{V}$, there is an equivalence

$$\mathsf{Alg}_{\mathcal{P}}(\mathcal{V}) \simeq \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{P}), \mathcal{V})$$

between the ∞-category of $\mathcal{P}$-algebras in $\mathcal{V}$ and that of strong monoidal functors $\mathsf{Env}(\mathcal{P}) \to \mathcal{V}$. We adopt this perspective – algebras as strong monoidal functors – to construct a un/straightening equivalence

We start by observing that although the envelope is not fully faithful, it follows from Haugseng-Kock's [HK24, Proposition 2.4.3] that it is a fully faithful left adjoint when sliced over a Lurie's ∞-operad$\mathcal{O}^{\otimes}$. As a first step, we prove that this sliced functor restricts to an equivalence

$$\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \xrightarrow{\mathsf{Env}(-)^{\otimes}} \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})}$$

between the ∞-category of operadic left fibrations over $\mathcal{O}^{\otimes}$ (the same as in Ramzi's monoidal Grothendieck construction) and that of *strong sm (symmetric monoidal) left fibrations* over $\mathsf{Env}(\mathcal{O})^{\otimes}$, which are left fibrations of symmetric monoidal ∞-categories satisfying a certain Segal condition (Definition 4.3.7). We then characterize this latter ∞-category as the essential image of the monoidal unstraightening (see Section 2.2.2) of *strong* monoidal functors $\mathsf{Env}(\mathcal{O}) \to \mathcal{S}$ (in fact, we obtain such characterization by replacing $\mathsf{Env}(\mathcal{O})^{\otimes}$ any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$). This is how we prove the main result of this section.

Theorem (Theorem 4.5.1)**.** *Let $\mathcal{O}^{\otimes}$ be a Lurie ∞-operad. There exists an equivalence of ∞-categories between operadic left fibrations and $\mathcal{O}^{\otimes}$-algebras in spaces, where the* *straightening functor is given by the composition of the symmetric monoidal envelope and the monoidal straightening functor. In symbols,*

$$\mathsf{St}^{\mathcal{O}}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \xrightarrow{\mathsf{Env}(-)^{\otimes}_{/}} \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} \xrightarrow{\mathsf{St}^{\mathsf{Env}(-),\otimes}} \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}),\mathcal{S}) \simeq \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}).$$

Its inverse is given by the composition of the unstraightening functor with the base-change along the unit of the adjunction $(\mathsf{Env}(-)^{\otimes}, U)$.

The operadic straightening functor $\mathsf{St}^{\mathcal{O}}$ differs from Ramzi's functor (Section 2.2.2), which on the underlying objects coincides with the classical un/straightening equivalence. If any, this un/straightening equivalence has the advantage of giving a formulation in terms of the envelope functor, amenable to generalizations to the context of algebraic patterns, as in Kern's Grothendieck construction [Ker23] or as suggested in Barkan-Haugseng-Steinebrunner [BHS22].

2.3.3 Model independence of operadic left fibrations

We address the question of model-independence of operadic left fibrations raised with Question 2. Starting from Hinich-Moerdijk equivalence between Lurie ∞-operads and complete dendroidal Segal spaces, we prove the following result.

Theorem (Theorem 4.2.15)**.** *Let $\mathcal{O}^{\otimes}$ be a Lurie ∞-operad and X a dendroidal ∞-operad, with $\mathcal{O}^{\otimes}$ and X equivalent under the Hinich-Moerdijk equivalence, that is, $\mathcal{O}^{\otimes} \simeq \lambda(X)$. Then the Hinich-Moerdijk comparison functors (δ, λ) induce an equivalence*

$$\begin{tikzcd} \delta_{/}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{DLeft}_{X} \colon \lambda_{/} \arrow[l, shift left] \end{tikzcd},$$

between the ∞-category of operadic left fibrations over $\mathcal{O}^{\otimes}$ and that of dendroidal left fibrations over X.

The essential point to prove this theorem is expressing both operadic and dendroidal left fibrations as local objects with respect to some left anodyne maps in some ∞-categories, a technical difficulty arising from the fact that $\ell\mathsf{Op}_{\infty}$ is not a localization of the presheaf category $\mathsf{PSh}(\Phi)$, where Φ is a category of forests, that is finite disjoint unions of trees. In Lurie's model the anodynes are left horn inclusions, and in the dendroidal model dendroidal left horn inclusions. We then prove that the combinatorics can be simplified so that one has only to consider the inclusions $\{0\}\colon \Delta^0 \to \Delta^n$ and $\ell(T) \to T$ respectively, where $\ell(T)$ is the set of leaves of T.

2.3.4 Rectification of dendroidal left fibrations

We now return to discussing the root functor and Theorem 4.5.1, and proceed to construct a un/straightening equivalence for discrete operads in the formalism of dendroidal sets, where the root functor is defined.

Our starting point is the un/straightening equivalence for discrete categories in the formalism of simplicial sets, as developed by Heuts and Moerdijk in [HM15], which we extend as follows. Given a discrete operad P, we define the *rectification functor* $\rho^P_!$ as a functor

$$\rho^P_! : \mathbf{dSets}/\mathcal{N}_d P \longrightarrow \mathsf{Alg}_P(\mathbf{sSets})$$

from the category of dendroidal sets over the nerve of P to that of simplicial P-algebras, characterized by the following property: for any tree T and element (T,α) in $\mathbf{dSets}/\mathcal{N}_d(P)$, there is an isomorphism of simplicial P-algebras $\rho_!^P(T,\alpha) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/}$, where $\mathsf{Env}(-)$ is the nerve of the symmetric monoidal envelope of discrete operads and T is regarded as the free operad it generates.

We prove the following result.

Theorem (Corollary 5.5.3)**.** *Let P be a Σ-free discrete operad. The rectification functor is part of a Quillen equivalence*

$$\rho_!^P \colon \mathbf{dSets}/\mathcal{N}_d P \underset{\perp}{\rightleftarrows} \mathsf{Alg}_P(\mathbf{sSets}) : \rho_P^*$$

between the covariant model structure on dendroidal sets over the dendroidal nerve of P and the projective model structure on simplicial P-algebras.

When P is a discrete category, we recover Heuts-Moerdijk equivalence. In fact, we improve on this, by showing the following.

Proposition (Theorem 5.4.4)**.** *Let A be a discrete symmetric monoidal category. The Quillen pair*

$$\rho_!^A \colon \mathbf{sSets}/\mathcal{N}A \underset{\perp}{\rightleftarrows} \mathsf{Fun}(A, \mathbf{sSets}) : \rho_A^*$$

is a monoidal Quillen equivalence of Quillen monoidal model categories, with respect to the projective model structure and Day convolution on $\mathsf{Fun}(A, \mathbf{sSets})$ *and the covariant model structure and the natural $\boxtimes$-product on* $\mathbf{sSets}/\mathcal{N}A$.

The proof of Corollary 5.5.3 is carried on in two distinct steps, which differ in methods.

The first step consists in proving that the constructed adjunction is a Quillen pair, and more precisely that the right adjoint ρ_P^* is right Quillen. To this end, we employ combinatorics of trees in order to better understand lifts of maps in the image of ρ_P^* against dendroidal boundary and leaf horn inclusions.

The second step, showing that the adjunction $(\rho_!^P, \rho_P^*)$ defines a Quillen equivalence, is difficult to establish by applying model-categorical methods, the central issue being that colimits in categories of algebras are not computed objectwise (see the discussion at the beginning of Section 5.5).

Things become easier if we adopt the less rigid methods of ∞-categories, and so we look at the adjunction between the underlying ∞-categories induced by the rectification functor,

$$(\rho_!^P)_\infty \colon \mathsf{dLeft}_{\mathcal{N}_d P} \underset{\perp}{\rightleftarrows} \mathsf{Alg}_P(\mathcal{S}) : (\rho_P^*)_\infty.$$

The crucial point is now to use Theorem 5.4.4 to show that, for P Σ-free, there is a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathsf{dLeft}_{\mathcal{N}_d P} \arrow[r, "(\rho_!^P)_\infty"] \arrow[d, "\lambda'"'] & \mathsf{Alg}_P(\mathcal{S}) \\
\mathsf{Left}^{\mathsf{lax}}_{\mathcal{N}P} \arrow[ur, "\mathsf{St}^P"'] &
\end{tikzcd}$$

where $\lambda'_{/}$ is induced by the sliced Hinich-Moerdijk equivalence (Section 2.3.3) and St^P is the operadic straightening functor of Theorem 4.5.1 for the nerve of P . As both are equivalences of ∞-categories, by Theorem 4.2.15 and Theorem 4.5.1 respectively, we conclude that $(\rho_!^P)_\infty$ is an equivalence as well.

2.3.5 Covariant model structure and locally constant algebras

We combine the un/straightening equivalence of Corollary 5.5.3 and the equivalence $\mathcal{N}(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \simeq X$ given by the root functor (Theorem 6.3.13) to give different characterizations of the ∞-categories of algebras over X.

In the first place, we study the compatibility of dendroidal localization with the covariant model structure.

Proposition (Proposition 6.4.4)**.** *For any dendroidal set Y and set of 1-morphisms $S \subseteq Y_{C_1}$, the localization map $Y \to Y[S^{-1}]$ induces a Quillen equivalence*

$$\lambda_! \colon \mathbf{dSets}^S/Y \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathbf{dSets}/Y[S^{-1}] \colon \lambda^*$$

between the left Bousfield localization of the covariant model structure on $\mathbf{dSets}/Y$ at $S_{/Y}$ and the covariant model structure on $\mathbf{dSets}/Y[S^{-1}]$. Here $S_{/Y}$ is the set of morphisms in $\mathbf{dSets}/Y$ of the form

$$\begin{tikzcd}
\Delta^0 \arrow[rr, hook, "\{1\}"] \arrow[dr, "b"'] & & \Delta^1 \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

where $f \colon a \to b$ is a morphism in S and the arrow $\{1\} \colon \Delta^0 \to \Delta^1$ is the inclusion of the last vertex.

In particular, the fibrant objects for the model structure on $\mathbf{dSets}^S/Y$ are the *$S_{/Y}$-local dendroidal left fibrations* over Y, that is, the dendroidal left fibrations $E \to Y$ for which for every $f \colon a \to b$ in S the morphism in the homotopy category of spaces between the fibres $f_! \colon E_a \to E_b$ is an isomorphism.

The algebraic analogue of this notion is expressed by that of S-locally constant algebras over an operad. If P be a simplicial operad and S a subclass of 1-morphisms in P, a *locally constant* simplicial P-algebra is a P-algebra F for which the map of simplicial sets $f_* \colon F(a) \longrightarrow F(b)$ is a weak homotopy equivalence for any morphism $f \colon a \to b$ in S.

Considering $P = \mathbf{\Omega}/X$ and $Y = \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}]$, we obtain two results: a 'locally constant' un/straightening equivalence for dendroidal ∞-operads, and the characterization of algebras over the latter as locally constant algebras over their operad of elements.

Corollary (Corollary 6.4.9)**.** *Let X be a normal dendroidal ∞-operad.*

1. *The rectification functor for the operad of elements $\mathbf{\Omega}/X$ induces a Quillen equivalence*

   $$\rho_!^{\mathbf{\Omega}/X} \colon \mathbf{dSets}^{\mathcal{R}_X}/\mathcal{N}_d\mathbf{\Omega}/X \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}) \colon \rho^*_{\mathbf{\Omega}/X}$$

   between the localized covariant model structure of Proposition 6.4.4 and the left Bousfield localization of the projective model structure on simpicial $\mathbf{\Omega}/X$-algebras *at* $\rho_!^{\mathbf{\Omega}/X}(\mathcal{R})$*. The fibrant objects of this latter are precisely* $\mathcal{R}_X$*-locally constant* $\mathbf{\Omega}/X$*-algebras.*

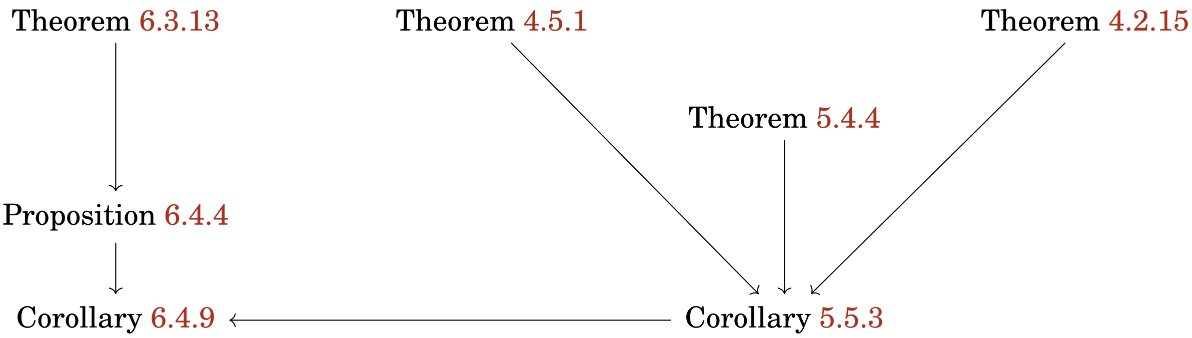
2. *There is a zig-zag of Quillen equivalences*

$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets})$$

between the projective model structures for simplicial $W_!(X)$*-algebras and for* $\mathcal{R}_X$*-locally constant algebras on* $\mathbf{\Omega}/X$*, where* $W_!(X)$ *is the simplicial operad given by the Boardman-Vogt resolution of* X*.*

2.3.6 Interdependence of the results

The following diagram shows the logical dependencies between the proven results. An arrow from A to B means that result A is needed to prove result B.



2.4 Outline of the thesis

In this introduction, we have presented our contributions by emphasizing the underlying motivations and their interconnections, in the hope of offering a coherent perspective, and also with the intention of conveying the intellectual journey that led us to our results.

For the sake of clarity and rigor, we chose a different exposition of the material, and the main body of the thesis is organized as follows:

- Chapter 3 contains the core background on the theory of ∞-operads and the different formalisms, and sets the notation used throughout. It is expository in nature and does not include any proofs.

- Chapter 4 contains the results of Section 2.3.2 and Section 2.3.3. Up to preliminaries and minor expository changes, it consists of the article available on arXiv with the identifier 2501.05263v2, under the name *A straightening-unstraightening equivalence for* ∞*-operads* ([Pra25c]).

- Chapter 5 contains the results of Section 2.3.4. Up to preliminaries and minor expository changes (mostly Section 5.5), it consists of the corrected version of the article available on arXiv with the identifier 2502.17415v2, under the name *Rectification of dendroidal left fibrations* ([Pra25a]).

- Chapter 6 contains the results of Section 2.3.1 and Section 2.3.5. Up to preliminaries and minor expository changes, it consists of all the sections but the last one of the article available on arXiv with the identifier 2505.14288, under the name *The root functor* ([Pra25b]).

The chapters are structured so that each can be read independently of the others. At the beginning of each chapter, we specify the preliminary notions required, with explicit references to the relevant sections of Chapter 3. When results from previous chapters are needed, we recall them in context to ensure that the exposition remains as self-contained as possible.

2.4.1 Assumptions

We assume the reader is familiar with Quillen model categories (a good reference is [Hir03]) as well as the theory of ∞-categories (as developed, for instance, in [Lur09] or [Cis19]). The remaining background material is explicitly reviewed in Chapter 3, with [HM22] serving as the reference text for dendroidal ∞-operads.

2.5 Perspectives

Within this thesis, we will see how several of our results are connected to existing constructions, suggesting possible comparisons, equivalences, or generalizations whose development appears natural to us. These points are discussed and appropriately highlighted throughout the chapters, specifically in Chapter 4 (Section 4.1.2, Remark 4.2.19) Chapter 5 (Remark 5.5.5).

Let us here focus on one more perspective, concerning relative operads, ∞-operads and localization. Consider the localization functor

$$\mathcal{L} \colon \mathsf{RelOp} \longrightarrow \mathsf{Op}_\infty$$

sending a relative operad to the ∞-operadic localization of its nerve[6]. The root functor and Theorem 6.3.13 show that $\mathcal{L}$ is essentially surjective, providing a section

$$\mathbf{\Omega}/- \colon \mathsf{Op}_\infty \longrightarrow \mathsf{RelOp},$$

where the weak equivalences of $\mathbf{\Omega}/X$ are the root preserving morphisms.

For ∞-categories, this is the content of Joyal's delocalization theorem. In this case, more is true: in [BK12b] Barwick-Kan show that the homotopy theory of relative categories is equivalent to that of ∞-categories. More precisely, they prove that *Rezk's classifying diagram* (proven to be equivalent to the localization functor by Mazel-Gee in [MG19]) induces an equivalence of ∞-categories

$$\mathcal{L} \colon \mathsf{RelCat}[\mathsf{DK}^{-1}] \longrightarrow \mathsf{Cat}_\infty,$$

where DK are the *Dwyer-Kan weak equivalences*, the functors inducing an equivalence after localization. Barwick-Kan's result is formulated in the language of simplicial spaces;

---

[6] A concrete model for $\mathcal{L}(P)$ is given by the derived localization (see Definition 6.2.2 and Proposition 6.2.4) of a cofibrant replacement of the dendroidal nerve of P.

recently, Arakawa has proven a quasicategorical version, showing that the last vertex functor, regarded as a functor of relative categories

$$\Delta/- \colon (\mathbf{sSets}, \mathcal{W}_{\mathbf{Joy}}) \longrightarrow (\mathsf{RelCat}, \mathsf{DK})$$

is homotopy inverse to the localization functor [Ara25, Theorem A.2].

The root functor suggests that a similar result should also hold for ∞-operads; in fact, by combining Theorem 6.3.13 with the techniques in [Ara25] it should be possible to prove that the ∞-operadic localization functor induces an equivalence of ∞-categories

$$\mathsf{RelOp}[\mathsf{DK}^{-1}] \longrightarrow \mathsf{Op}_\infty$$

whose homotopy inverse is the root functor. Here, the Dwyer-Kan weak equivalences DK are the maps of relative operads which induce an equivalence after localization. However, observe that this notion becomes meaningful only if we manage to relate it with Dwyer-Kan equivalences of simplicial operads and derived localization of simplicial operads.

Indeed, in the categorical case the ∞-category $\mathsf{RelCat}[\mathsf{DK}^{-1}]$ is constructed and understood in two steps:

- In [BK12b], Barwick-Kan construct an adjunction

  $$K \colon \mathbf{sSpaces} \underset{\perp}{\rightleftarrows} \mathsf{RelCat} \colon \mathcal{N}_\xi$$

  and lift Rezk's model structure for complete Segal spaces on a Quillen equivalent model category structure on RelCat, whose weak equivalences are thus called *Rezk's weak equivalences*.

- In [BK12a], the authors consider then the *hammock localization functor* $\mathcal{L}^H \colon \mathsf{RelCat} \longrightarrow \mathsf{sCat}_\infty$, which yields an explicit model for the derived localization of simplicial categories, and prove that Rezk's weak equivalences in RelCat are precisely those sent by $\mathcal{L}^H$ to Dwyer-Kan weak equivalences of simplicial categories. They also proved that the same is true by replacing the hammock localization with other Dwyer-Kan equivalent models.

It should be possible to obtain a similar diagram for ∞-operads and prove that Dwyer-Kan weak equivalences in RelOp are the ones sent to equivalences by the derived localization functor $\mathsf{RelOp} \to \mathsf{sOp}_\infty$. Nevertheless, at the present moment we do not have tools to compute derived localization of simplicial operads as the Hammock localization did for simplicial categories. Recently, Basterra-Bobkova-Ponto-Tillmann-Yeakel in [BBP+18] have proposed a generalization that they call a *tree-hammock localization*, conceived to present derived localizations of discrete and simplicial operads. However, various central points were left open; namely, it is not proven that the construction does compute derived localization, and algebras over the tree-hammock localization were only partially identified.

Chapter 3

Core Background

This chapter provides the background and notation needed throughout the thesis, with the purpose of making it self-contained up to the proofs of these results. It contains no original results and serves as a reference for standard definitions and results, mainly from [HM22] and [Lur17]. More precisely:

In Section 3.1 we recall the main results allowing to interpret model-categorical results as results of ∞-categories. It is intended to be a dictionary which makes rigorous the flexible adoption that we make of the two languages.

In Section 3.2 we set up notation for ∞-categories.

In Section 3.3 we recall the basic theory of Lurie's ∞-operads.

In Section 3.4 we introduce the dendroidal category Ω and its combinatorics.

In Section 3.5 we recall the theory of complete dendroidal Segal spaces in ∞-categorical terms.

In Section 3.6 we convey the necessary background on the theory of dendroidal sets and the operadic and covariant model structures.

In Section 3.7 we set up notation relative to dendroidal sets.

In Section 3.8 we gather the main results concerning the equivalence between the two dendroidal models for ∞-operads, formulated in ∞-categorical terms.

Contents

3.1 Dictionary

While in the formalism of ∞-categories homotopy-invariance is intrinsic, Quillen model categories offer a more rigid setting amenable to explicit computation[1]. The transition between these two languages is mediated by rectification tools, which we now recall; we follow the approach of [MOR22, §1.1].

To any model category $\mathcal{M}$ one can associate its *underlying ∞-category* (in the sense of quasicategory) $\mathcal{M}_\infty$. Explicitly, one can obtain $\mathcal{M}_\infty$ as:

- the homotopy coherent nerve of a fibrant replacement of the *hammock localization* of $\mathcal{M}$, in which case $\mathcal{M}$ and $\mathcal{M}_\infty$ have the same set of objects (see [DK80]), or
- the derived localization of the nerve of $\mathcal{M}$ at $\mathcal{W}$ (in the sense of Definition 6.2.2), or
- if $\mathcal{M}$ is a simplicial model category, the homotopy coherent nerve of the subcategory $\mathcal{M}^{cf}$ of fibrant-cofibrant objects.

An interesting discussion about the equivalence between the last two models can be read at *https://mathoverflow.net/questions/390076/modern-proofs-for-simplicial-localizations*.

We say that an ∞-category X is *presented by* a model category $\mathcal{M}$ if there exists an equivalence $X \simeq \mathcal{M}_\infty$.

Remark 3.1.1. Not every ∞-category is presented by a model category. This is true only for *presentable* ∞-categories (which, in particular, are complete and cocomplete) ([Dug01], [Lur09, §A.3]).

Any left (resp. right) Quillen functor $F\colon \mathcal{M} \to \mathcal{M}'$ induces a left (resp. right) adjoint functor of ∞-categories $F_\infty\colon \mathcal{M}_\infty \to \mathcal{M}'_\infty$, and $ho(F_\infty) \simeq \mathbb{L}(F)$ (resp. $ho(F_\infty) \simeq \mathbb{R}(F)$) ([Maz16, Theorem 2.1]). On objects, it can be computed as

$$F_\infty(X) = \mathbb{L}F(X) = F(X^{\mathrm{cof}}), \qquad \text{resp.} \qquad F_\infty(X) = \mathbb{R}F(X) \simeq F(X^{\mathrm{fib}})),$$

where $\mathbb{L}F$, resp. $\mathbb{R}F$, denotes the left, resp. right, derived functor of F, and X^{cof}, resp. X^{fib}, is a cofibrant, resp. fibrant, replacement of X in $\mathcal{M}$. Quillen embeddings correspond to fully faithful left adjoints, and any Quillen equivalence induces an equivalence of ∞-categories, as follows from [Maz16, §A.2] and [Lur17, §1.3.4].

Let $\mathcal{G}$ be a functor of ∞-categories, and let F be a Quillen functor between model categories. We say that F *presents* $\mathcal{G}$ if there is an equivalence $\mathcal{G} \simeq F_\infty$.

Any monoidal Quillen model category $\mathcal{M}$ yields a *symmetric monoidal ∞-category* $\mathcal{M}_\infty^\otimes$ (see Definition 3.3.4) whose underlying ∞-category is $\mathcal{M}_\infty$ ([Hin16]).

A functor of ∞-categories $L\colon \mathcal{C} \to \mathcal{D}$ is a *(left Bousfield) localization* if it has a fully-faithful right adjoint. In particular, any localization functor between presentable ∞-categories is cocontinuous and essentially surjective.

A functor of ∞-categories $L\colon \mathcal{C} \to \mathcal{D}$ is a localization if it has a fully faithful right adjoint $\mathcal{D} \hookrightarrow \mathcal{C}$. A full sub ∞-category $\mathcal{D}$ of $\mathcal{C}$ is called a localization of $\mathcal{C}$ if the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint. If $L\colon \mathcal{M} \to \mathcal{N}$ is a left Bousfield localization of Quillen model categories, the induced functor of ∞-categories $L_\infty\colon \mathcal{M}_\infty \to \mathcal{N}_\infty$ is a localization.

---

[1] These two perspectives need not be orthogonal; see for instance [Cis19].

3.2 Notation I

We aim to work with ∞-categories model-independently. When we need to use an explicit model, we adopt that of Joyal's quasicategories. We adopt the following conventions for ∞-categories.

- We write $\mathcal{S}$ for the ∞-category of spaces, presented by the Kan-Quillen model structure on simplicial sets.
- Given an ∞-category $\mathcal{C}$ and objects x, y of $\mathcal{C}$, we write $\mathsf{Map}_{\mathcal{C}}(x, y)$, or sometimes just $\mathsf{Map}(x, y)$, for the mapping space of arrows in $\mathcal{C}$ from x to y.
- Given two ∞-categories $\mathcal{C}$, $\mathcal{D}$, we write $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ for the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$.
- Given an ∞-category $\mathcal{C}$, we write $\mathsf{PSh}(\mathcal{C})$ for the ∞-category of presheaves on $\mathcal{C}$, that is, $\mathsf{PSh}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. It is presented by the projective model structure on $\mathsf{Fun}(\mathfrak{C}(\mathcal{C}), \mathrm{sSets})$, where $\mathfrak{C}\colon \mathrm{sSets} \to \mathsf{sCat}$ is the left adjoint to the homotopy coherent nerve functor.
- We write Cat_∞ for the ∞-category of small ∞-categories, the full sub ∞-category of simplicial presheaves $\mathsf{Psh}(\Delta)$ spanned by complete Segal spaces.

3.3 Lurie's ∞-operads

We recall the theory of Lurie's ∞-operads, algebras over them and symmetric monoidal ∞-categories. Unless stated otherwise, the material in this section follows [Lur17].

Notation 3.3.1. Denote by Fin_* the skeleton of the category of finite pointed sets. Its objects are the finite sets $\underline{n}_+ = \{*, 1, \dots, n\}$, for $n \geqslant 0$, where $*$ is the base point, and a map $f\colon \underline{n}_+ \to \underline{m}_+$ is a map of sets preserving the base point. The map f is called *active* if $f^{-1}(*) = \{*\}$, while it is called *inert* if $\# f^{-1}(i) = 1$ for all $i \in \underline{m}_+ \backslash \{*\}$. We will still denote by Fin_* the simplicial set given by its nerve. For any $n \geqslant 0$, we denote by $\beta\colon \underline{n}_+ \to \underline{1}_+$ the unique active map in Fin_* from $\underline{n}_+$ to $\underline{1}_+$. For any $n \geqslant 0$ and $i \in \{1, \dots, n\}$, we denote by $\rho^i\colon \underline{n}_+ \to \underline{1}_+$ the inert morphism sending $j \neq i$ to the base point $*$ and the element i to 1. We call ρ_i the *projection* on the i-th coordinate. In the degenerate case, when $n = 0$, there is a unique morphism $\rho\colon \underline{0}_+ \to \underline{1}_+$.

Definition 3.3.2. A *Lurie ∞-operad* is an object $(\mathcal{O}^\otimes, p)$ of $\mathsf{Cat}_{\infty/\mathsf{Fin}_*}$ such that the morphism $p\colon \mathcal{O}^\otimes \to \mathsf{Fin}_*$ has the following properties:

1. any inert morphism α of Fin_* admits a coCartesian lift $\alpha_!$ in $\mathcal{O}^\otimes$. As a result, for any inert $\underline{n}_+ \to \underline{m}_+$ there is an induced functor of fibres $\mathcal{O}^\otimes_{\underline{n}_+} \to \mathcal{O}^\otimes_{\underline{m}_+}$;
2. for every $n \geqslant 0$, the functor $\mathcal{O}^\otimes_{\underline{n}_+} \to \prod_{i=1}^n \mathcal{O}^\otimes_{\underline{1}_+}$ induced by the coCartesian lifts $\{\rho^i_!\}_{i=1}^n$ of the projections is an equivalence of spaces;
3. for any arrow $f\colon \underline{n}_+ \to \underline{m}_+$ and objects x of $\mathcal{O}^\otimes_{\underline{n}_+}$ and y of $\mathcal{O}^\otimes_{\underline{m}_+}$, denote by $\mathsf{Map}^f(x, y)$ the subspace of the mapping space $\mathsf{Map}_{\mathcal{O}^\otimes}(x, y)$ spanned by those maps lying over f.

> Given f and y as above, for any x in $\mathcal{O}^{\otimes}_{\underline{n}_+}$, the map
>
> $$\mathsf{Map}^f(x,y) \longrightarrow \prod_{i=1}^{m} \mathsf{Map}^{f\circ\rho^i}(x,y_i)$$
>
> induced by the projections is an equivalence, where $y \to y_i$ is the coCartesian lift of ρ^i at y.

Given such an ∞-operad, we call *inert* the arrows of $\mathcal{O}^{\otimes}$ which are (equivalent to) coCartesian lifts of inert arrows of Fin_*.

We denote a Lurie ∞-operad $(\mathcal{O}^{\otimes}, p\colon \mathcal{O}^{\otimes} \to \mathsf{Fin}_*)$ by $\mathcal{O}^{\otimes}$, leaving the map p implicit.

Observe that condition (1) in Definition 3.3.2 implies that $\mathcal{O}^{\otimes}$ is an ∞-category; as a consequence, the fibres $\mathcal{O}^{\otimes}_{\underline{n}_+}$ as well. We call $\mathcal{O} := \mathcal{O}^{\otimes}_{\underline{1}_+}$ the *underlying category* of $\mathcal{O}^{\otimes}$.

By condition (3) in Definition 3.3.2, any object $x \in \mathcal{O}^{\otimes}_{\underline{n}_+}$ is equivalent to a n-tuple $(x_1,\dots,x_n) \in \mathcal{O}^{\times n}$, and for this reason we write $x \simeq x_1 \oplus \cdots \oplus x_n$. For objects $x_1,\dots,x_n,y$ of $\mathcal{O}$, the space of operations $(x_1,\dots,x_n) \to y$ is spanned by the morphisms $x_1 \oplus \cdots \oplus x_n \to x$ over β, which we denote by

$$\mathsf{Map}^{\beta}_{\mathcal{O}^{\otimes}}((x_1,\dots,x_n);y).$$

Definition 3.3.3. The ∞-category of Lurie ∞-operads $\ell\mathsf{Op}_\infty$ is the non full subcategory of $\mathsf{Cat}_{\infty/\mathsf{Fin}_*}$ where the objects are Lurie ∞-operads $\mathcal{O}^{\otimes}$, and where a morphism $f\colon \mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ in $\mathsf{Cat}_{\infty/\mathsf{Fin}_*}$ is a morphism of ∞-operads if f preserves inert arrows.

The ∞-categoryt $\ell\mathsf{Op}_\infty$ is presented by a combinatorial simplicial model category structure on the category **POp** of *preoperads*, defined as the category of marked simplicial sets over Fin_*, where marking is given by the inert morphisms ([Lur17, Proposition 2.1.4.6.]).

One can define symmetric monoidal ∞-categories as the ∞-operads with *representable* operations. We follow [Lur17, Definition 2.0.0.7].

Definition 3.3.4. A *symmetric monoidal ∞-category* $\mathcal{C}^{\otimes}$ is a Lurie ∞-operad $p\colon \mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ where p is furthermore a coCartesian fibration (that is, all morphisms in Fin_* have coCartesian lifts).

Given such $\mathcal{C}^{\otimes}$, the n-fold tensor product $\mathcal{C}^{\times n} \to \mathcal{C}$ is the functor induced by the coCartesian lifts of the active map $\beta\colon \underline{n}_+ \to \underline{1}_+$.

After [Lur17, Proposition 2.4.1.5.], any ∞-category $\mathcal{C}$ that admits finite products has a canonical symmetric monoidal structure whose tensor product is the cartesian product. In particular, this applies to the ∞-category of spaces $\mathcal{S}$; we denote the corresponding symmetric monoidal ∞-category by $\mathcal{S}^{\times}$.

Definition 3.3.5. The ∞-category of symmetric monoidal ∞-categories smCat_∞ is the non full subcategory of $\mathsf{Cat}_{\infty/\mathsf{Fin}_*}$ where objects are symmetric monoidal ∞-categories, and where a morphism $\mathcal{C}^{\otimes} \to \mathcal{V}^{\otimes}$ over Fin_* is a morphism in smCat_∞ if it preserves all the coCartesian lifts.

Observe that there is a forgetful functor $\mathsf{smCat}_\infty \to \ell\mathsf{Op}_\infty$. It is not full: morphisms of symmetric monoidal ∞-categories correspond to strong monoidal functors, while those of ∞-operads are rather lax monoidal functors. This functor has a left adjoint, the *symmetric monoidal envelope*; we provide a detailed definition in Section 4.4.1.

Definition 3.3.6. Let $\mathcal{O}^{\otimes}$ be a ∞-operad and $\mathcal{V}^{\otimes}$ be a symmetric monoidal ∞-category. A *$\mathcal{O}$-algebra in $\mathcal{V}$* is a morphism of ∞-operads $\mathcal{O}^{\otimes} \to \mathcal{V}^{\otimes}$.

The ∞-category of $\mathcal{O}$-algebras in $\mathcal{V}^{\otimes}$ is defined as the full subcategory of $\mathsf{Fun}_{\mathsf{Cat}_{\infty}/\mathsf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{V}^{\otimes})$ spanned by morphisms of Lurie ∞-operads. We write $\mathsf{Alg}_{\mathcal{O}}(\mathcal{V})$ for such ∞-category.

In Lurie's model there exists a nerve functor which allows to represent simplicial operads as ∞-operads. Let us present this following the presentation in [HHM16].

Let sOp_{∞} be the ∞-category of simplicial operads with Dwyer-Kan weak equivalences. There is a nerve construction

$$\mathcal{N}(-)^{\otimes} \colon \mathsf{sOp}_{\infty} \to \ell\mathsf{Op}_{\infty}$$

which allows to realize every simplicial operad as a Lurie's ∞-operads. It is induced by the functor of relative categories

$$\nu \colon (\mathbf{sOp}, W) \longrightarrow (\mathbf{POp}, W'), \quad P \mapsto \mathcal{N}^h(P^{\otimes})$$

which assigns to any simplicial operad P the homotopy-coherent nerve of its simplicial category of operators $P^{\otimes}$. If P is enriched in Kan-complexes, it follows from [Lur17, Proposition 2.1.1.27.] that $\nu(P)$ is a Lurie's ∞-operad, and the nerve functor $\mathcal{N}^{\otimes}$ is obtained as the functor between the associated ∞-categories. For instance, we have that $\mathsf{Fin}_* \simeq \mathcal{N}^{\otimes}(\mathsf{Comm})$. Given a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, we denote by $\mathsf{CAlg}(\mathcal{C}^{\otimes})$ the ∞-category of commutative monoids in $\mathcal{C}$, that is

$$\mathsf{CAlg}(\mathcal{C}^{\otimes}) \simeq \mathsf{Alg}_{\mathsf{Comm}}(\mathcal{S}).$$

Remark 3.3.7. This nerve construction provides, *a priori*, two distinct approaches to defining the ∞-category of algebras over a (Σ-free) simplicial operad P. One may either view it as the ∞-category presented by the projective model structure on the category $\mathsf{Alg}_P(\mathbf{sSets})$ of simplicial P-algebras, or as the ∞-category of algebras over the ∞-operad $\mathcal{N}^{\otimes}P$ in the sense of Lurie. In [PS14, Theorem 7.11], Pavlov and Scholbach demonstrate that these two models coincide: letting $\mathsf{Alg}_P(\mathbf{sSets})^{cf}$ denote the full subcategory of bifibrant simplicial P-algebras, they show that the canonical map

$$\mathcal{N}(\mathsf{Alg}_P(\mathbf{sSets})^{cf})[\mathcal{W}^{-1}] \longrightarrow \mathsf{Alg}_{\mathcal{N}^{\otimes}(P)}(\mathcal{S})$$

from the ∞-categorical localization of the nerve of $\mathsf{Alg}_P(\mathbf{sSets})^{cf}$ at pointwise weak equivalences to the ∞-category of algebras over $\mathcal{N}^{\otimes}P$ is an equivalence of ∞-categories. Since the left-hand side constitutes a model for $\mathsf{Alg}_P(\mathbf{sSets})_{\infty}$, we may conclude that the ∞-category of algebras over a simplicial operad is independent of the specific choice between Lurie's framework and the model-categorical approach.

3.4 The dendroidal category Ω

The dendroidal category was introduced by Moerdijk-Weiss in [MW07] and led to the development of *dendroidal homotopy theory*, which uses the category $\mathbf{\Omega}$ to model operadic generalization of categorical notions.

The objects of $\mathbf{\Omega}$ are non-planar trees T with finite vertex set $V(T)$ and edge set $E(T)$, together with a specified edge, denoted by r_T, which is attached to a single vertex. We call this edge the *root* of T and this vertex the *root vertex*.

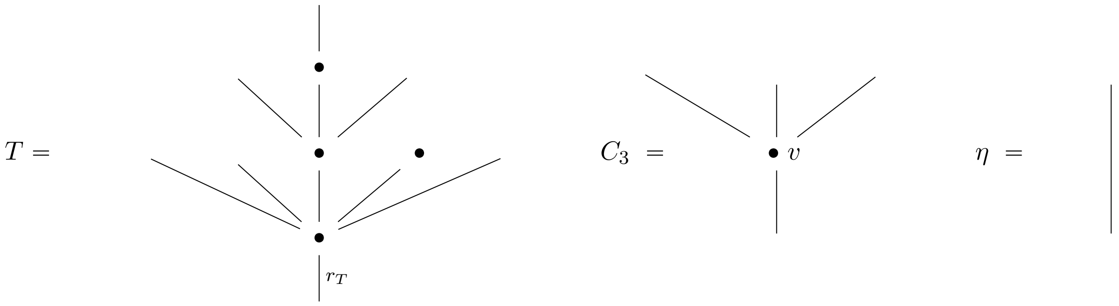


Figure 3.1: Some typical trees in $\mathbf{\Omega}$.

Given a tree T, every vertex v has a single output edge and n input edges, with $n \geqslant 0$. The *leaves* of T are the edges which are not the output edge of a vertex, and a vertex whose set of input edges is non-empty and contained in the set of leaves of T is called a *leaf vertex*. An edge e of T is an *inner edge* if it is both the input edge of two (necessarily distinct) vertices.

We say that S is a subtree of T if it can be obtained from T by successively pruning away external vertices and the outer edges attached to them from T.

Any tree T yields an operad $\Omega(T)$, whose set of objects is the set of edges of T and where, for any choice of edges $e_1, \dots, e_n, e$, one has $\Omega(T)(e_1, \dots, e_n; e) = \{*\}$ if and only if there exists a subtree of T with leaves $\{e_1, \dots, e_n\}$ and root e (necessarily unique), and $\Omega(T)(e_1, \dots, e_n; e) = \varnothing$ otherwise. The operadic composition corresponds to *grafting* of subtrees, which means successive identifications of the root of a subtree with a leaf of another. Given two trees T and S, a morphism $f \colon S \to T$ in $\mathbf{\Omega}$ is by definition a morphism of operads $f \colon \Omega(S) \to \Omega(T)$, determining a fully faithful functor $\Omega \colon \mathbf{\Omega} \hookrightarrow \mathsf{Op}$.

Performing the grafting of trees corresponds then to a pushout in $\mathbf{\Omega}$: given two trees R and S, the tree obtained as the grafting of R onto a leaf ℓ of S corresponds to the pushout in $\mathbf{\Omega}$ of the cospan $S \xleftarrow{\ell} \eta \xrightarrow{r_R} R$.

Observe that any tree can be obtained as the grafting of smaller trees, and the corollas are the indecomposable objects for the grafting, in the sense that, if a corolla C_n is written as the grafting of R along S, then either R or S are trivial (i.e. $\simeq \eta$).

There is a fully faithful inclusion $i \colon \Delta \longrightarrow \mathbf{\Omega}$, realized by sending the finite linear order $[n]$ as the linear tree with $n + 1$ edges and n vertices:

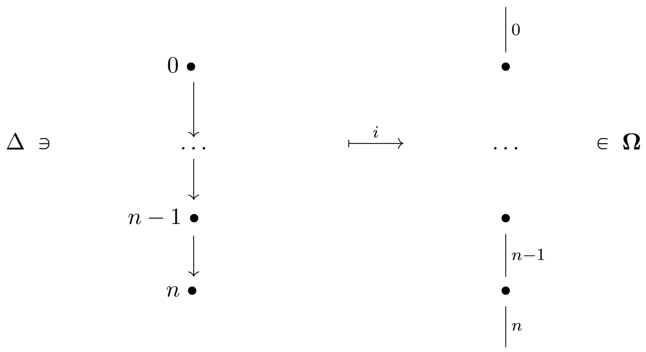


Figure 3.2: The embedding $\Delta \to \mathbf{\Omega}$

so that the following diagram of fully faithful functors commutes

$$\begin{tikzcd}
\Delta \arrow[r, hook, "i"] \arrow[d, hook] & \mathbf{\Omega} \arrow[d, hook] \\
\mathsf{Cat} \arrow[r, hook] & \mathsf{Op}
\end{tikzcd} .$$

Given an operad P, its set of objects inherits a partial order induced by the operations of P: given two objects c, d in P, one says that $c \leqslant d$ if and only if P has an operation whose set of input elements contains c and has target d; any morphism of operads $f\colon P \to Q$ induces a morphism of posets $f\colon \mathsf{Ob}(P) \to \mathsf{Ob}(Q)$.

In particular, given a tree T the partial order on its set of edges corresponds to saying that, given two edges e and f,

$$e \leqslant f \text{ if and only if the (unique) path from } e \text{ to the root of } T \text{ contains } f.$$

In particular, the root r_T is the unique maximal element, while the minimal elements are the leaves of T and the output edges of stumps. Again, any map of trees $f\colon S \to T$ induces a map of posets $f\colon E(S) \to E(T)$ between the sets of edges of S and T. The map assigning to any vertex its output edge also endows the set $V(T)$ of vertices of T with a partial order, where one has $v \leqslant w$ if and only if $\mathsf{out}(v) \leqslant \mathsf{out}(w)$. In particular, any map of trees $f\colon S \to T$ induces maps of posets $E(S) \to E(T)$, $V(S) \to V(T)$; however, unless the trees are linear, a map of posets does not necessarily determine a map of trees, see [HM22, Proposition 3.5].

As it happens for Δ, morphisms in $\mathbf{\Omega}$ can be described in a combinatorial way. We can distinguish four classes of morphisms in $\mathbf{\Omega}$:

- there are isomorphisms $T \xrightarrow{\sim} T'$. Observe that, contrarily to Δ, these can be non-trivial;
- for every inner edge e of T, we call *elementary inner face* the map $\partial_e T \to T$ which comes from contracting e in T and identifying its extremal vertices. If $R \to T$ is obtained by contracting more than one inner edge, we call it *inner face*.
- for every subtree S of T, there is the *external face* consisting in the inclusion $S \hookrightarrow T$;

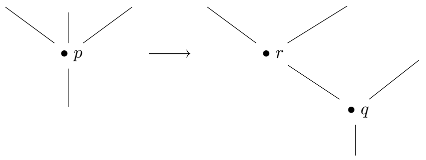
- for every edge e of T, there is a *degeneracy* $\sigma_e T \to T$ which adds a unary vertex in the middle of e.

We call *face map* the composition of inner and external faces and a *degeneracy* the composition of degeneracies. We say that an external face is *elementary face map* if it adds precisely one vertex, and more generally we call *elementary faces* those face maps which add or erase precisely one vertex. Restricted to $\Delta \subseteq \mathbf{\Omega}$, one recovers the usual face and degeneracy morphisms.

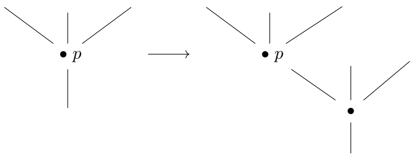
Face maps, degeneracies and isomorphisms generate the morphisms in $\mathbf{\Omega}$; these maps satisfy some relations called *dendroidal identities* (see [HM22, Sec 3.3.4]).

Example 3.4.1. *Intuitively, we can think about the generating morphisms of the category $\mathbf{\Omega}$ in the following informal terms*

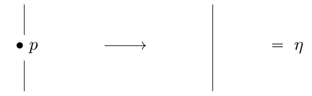
- *an* inner face *corresponds to sending an operation p to an operadic composition $q \circ_i r$, where q and r are unknown.*



- *an* external face *is an inclusion of operads.*



- *a* degeneracy *sends a unary morphism to the identity.*



3.5 Complete dendroidal Segal spaces

Complete dendroidal Segal spaces are a direct extension of Rezk's complete Segal spaces. They were first defined by Cisinski-Moerdijk in [CM13a] in model-categorical terms by means of the category dSets of *dendroidal spaces*, that is, dSpaces $:= \mathsf{Fun}(\mathbf{\Omega}^{\mathrm{op}}, \mathrm{sSets})$. A full account can be found in [HM22]. We give here a presentation in model-categorical terms, following [HM24].

Definition 3.5.1. The ∞-category DOp_∞ of *dendroidal ∞-operads* is the full sub ∞-category of the ∞-category $\mathsf{PSh}(\mathbf{\Omega})$ spanned by the presheaves satisfying the Segal and completeness properties:

1. For an inner edge e of T, calling T_e, resp. T^e, the upper part, resp. the lower part, of T, so that T is given by the grafting $T \simeq T^e \cup_e T_e$, the natural morphism

$$X(T) \longrightarrow X(T_e) \times_{X(e)} X(T^e)$$

is an equivalence.

2. Completeness: $i^*X \in \mathsf{PSh}(\Delta)$ is a complete Segal space.

The ∞-category DOp_∞ can be realized as the one underlying the projective model structure on **dSpaces** localized at the Segal and the completeness condition, or equivalently as the ∞-category underlying the generalized Reedy model structure on **dSpaces** localized at the same classes of morphisms (see [HM22, Remark 12.26]). In particular, this implies that the ∞-category DOp_∞ is a localization of $\mathsf{PSh}(\boldsymbol{\Omega})$, and in fact, that is obtained via two successive localizations

$$\mathsf{PSh}(\boldsymbol{\Omega}) \longrightarrow \mathsf{Seg}(\boldsymbol{\Omega}) \longrightarrow \mathsf{DOp}_\infty,$$

where $\mathsf{Seg}(\boldsymbol{\Omega})$ is the ∞-category of dendroidal Segal spaces, not necessarily complete.

Any simplicial operad is in particular a simplicial object in discrete operads: by applying the dendroidal nerve pointwise (and recalling that the category of dendroidal spaces is equivalent to that of simplicial objects in dendroidal sets), one obtains a nerve functor

$$N \colon \mathbf{sOp} \longrightarrow \mathbf{dSpaces}, P \mapsto ([n] \to \mathcal{N}_d(P_n)),$$

where **sOp** denotes the 1-category of simplicial operads. This functor has a left adjoint, which we denote by τ; the pair (τ, N) is homotopical and induces an equivalence of homotopy categories ([HM22, Theorem 14.21] [2]. The induced equivalence between underlying ∞-categories is easier to understand: after [HM22, Lemma 14.29], it consists of a functor

$$\mathcal{N}_D \colon \mathsf{sOp}_\infty \longrightarrow \mathsf{DOp}_\infty$$

which sends a simplicial operad P in sOp_∞ to the completion of the dendroidal Segal space $T \mapsto \mathsf{Map}_{\mathsf{sOp}_\infty}(\Omega(T), P)$, where $\mathsf{Map}_{\mathsf{sOp}_\infty}(-,-)$ is the mapping space of the ∞-category of simplicial operads.

In particular, one can use the functor of ∞-categories

$$(\tau)_\infty \colon \mathsf{DOp}_\infty \longrightarrow \mathsf{sOp}_\infty$$

to define the ∞-category $\mathsf{Alg}_X(\mathcal{S})$ of algebras over a complete dendroidal Segal space X as the ∞-category underlying the projective model structure on simplicial $\tau_\infty(X)$-algebras. In particular, when $X \simeq \mathcal{N}_D P$, one has

$$\mathsf{Alg}_X(\mathcal{S}) \simeq \mathsf{Alg}_P(\mathbf{sSets})_\infty.$$

3.6 Homotopy theories on dendroidal sets

Dendroidal sets were first introduced by Moerdijk and Weiss in [MW07], where the notion of quasi-operads was also defined. In this section, we recall the fundamentals of dendroidal sets and their homotopy theory, a comprehensive account of which can be found in [HM22], which we refer to for complete demonstrations and proofs.

More specifically, in Section 3.6.1, we define dendroidal sets and recall the tensor product of dendroidal sets.

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[2] This is proven by using the *sparse model structure* on **dSpaces**, a third model structure which is Quillen equivalent to the one(s) for complete dendroidal Segal spaces ([HM22, Theorem 14.17]).

In Section 3.6.2, we review the operadic model structure on dendroidal sets, established by Cisinski and Moerdijk in [CM11].

In Section 3.6.3, we recall the notion of dendroidal left fibrations and the covariant model structure, both developed by Heuts in [Heu11].

3.6.1 Tensor product of dendroidal sets

The category of dendroidal sets is the category of presheaves on $\mathbf{\Omega}$; we denote it by **dSets**. The inclusion $\Omega\colon \mathbf{\Omega} \to \mathsf{Op}$ which associates to a tree the free operad on its edges $\mathbf{\Omega}(T)$ induces the *dendroidal nerve functor*, a fully faithful functor

$$\mathcal{N}_d\colon \mathsf{Op} \longrightarrow \mathbf{dSets}, \quad P \mapsto \mathcal{N}_d(P) := \mathsf{Op}(\Omega(-), P),$$

that fits into the commutative diagram

$$\begin{tikzcd}
\mathsf{Cat} \arrow[r, "\mathcal{N}"] \arrow[d, "j_!"'] & \mathbf{sSets} \arrow[d, "i_!"] \\
\mathsf{Op} \arrow[r, "\mathcal{N}_d"'] & \mathbf{dSets}\,,
\end{tikzcd}$$

where $i_!$ is the left Kan extension of $i\colon \Delta \to \mathbf{\Omega}$ along the Yoneda embedding.

For any tree T, we denote by $\Omega[T]$ the corresponding representable dendroidal set. Observe that there is a natural isomorphism of dendroidal sets

$$\Omega[T] \simeq \mathcal{N}_d(\Omega(T)).$$

The isomorphism $\Delta \simeq \mathbf{\Omega}/\eta$ yields the isomorphism $\mathbf{sSets} \simeq \mathbf{dSets}/\Omega[\eta]$, under which the inclusion $i_!$ is the forgetful functor $\mathbf{dSets}/\Omega[\eta] \to \mathbf{dSets}$. More generally, for any simplicial set M, one has $\mathbf{dSets}/i_!M \simeq \mathbf{sSets}/M$.

For an operad P, the category of simplicial P-algebras (that is in simplicial sets) has a symmetric monoidal structure given by the cartesian product, and one can talk about Q-algebras in $\mathsf{Alg}_P(\mathbf{sSets})$ for another operad Q. The *Boardman-Vogt tensor product* of P and Q ([BV73]) is the operad $P \otimes Q$ with the property that there are equivalences of categories

$$\mathsf{Alg}_Q(\mathsf{Alg}_P(\mathbf{sSets})) \simeq \mathsf{Alg}_{P\otimes Q}(\mathbf{sSets}) \simeq \mathsf{Alg}_P(\mathsf{Alg}_Q(\mathbf{sSets})).$$

It is defined in terms of generators and relations, as follows.

Definition 3.6.1. Given two operads P, Q, its *Boardman-Vogt tensor product* is the operad $P \otimes Q$ defined as follows:

- its set of objects is the product of the set of objects of P and Q;
- for all $p \in P(c_1, \dots, c_n; d)$ and object y of Q, there is an operation

  $$p \otimes y \in P \otimes Q((c_1, y), \dots, (c_n, y); (d, y)),$$

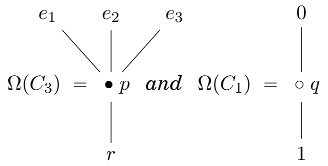
  and for all $q \in Q(y_1, \dots, y_m; z)$ and object c of P, there is an operation

  $$c \otimes q \in P \otimes Q((c, y_1), \dots, (c, y_m); (c, z)),$$

The generators satisfy the following relations:

1. $(p \otimes y) \circ_{(c_i,y)} (p' \otimes y) = (p \circ_{c_i} p') \otimes y$;
2. $(c \otimes q) \circ_{(c,y_j)} (c \otimes q') = c \otimes (q \circ_{y_j} q')$;
3. for any $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$, where Σ_k denotes the symmetric group on k elements, one has
$$\sigma^*(p \otimes y) = \sigma^*(p) \otimes y \quad \text{and} \quad \tau^*(c \otimes q) = c \otimes \tau^*(q),$$
where $(-)^*$ denotes the action of the symmetric group;
4. the *interchange relation*:
$$(p \otimes z) \circ (c_1 \otimes q, \dots, c_n \otimes q) = \sigma^*_{n,m}(d \otimes q) \circ (p \otimes y_1, \dots, p \otimes y_m),$$
where q is an operation of the form $q\colon (y_1, \dots, y_n) \to z$, $\circ$ denotes the total operadic composition and the permutation $\sigma_{n,m}$ is the unique element of Σ_{nm} making sense of the above formula.

Example 3.6.2. *Consider the operads*



The Boardman-Vogt interchange relation for $\Omega(C_3) \otimes \Omega(C_1)$ *yields the identification:*

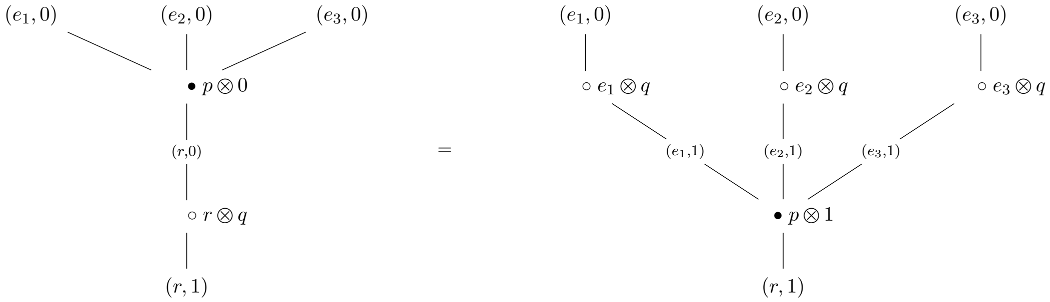


Figure 3.3: The Boardman-Vogt interchange relation for $\Omega(C_3) \otimes \Omega(C_1)$.

Consider the category **dSets** of dendroidal sets. The Boardman-Vogt tensor product extends to a bifunctor of dendroidal sets

$$\otimes\colon \mathbf{dSets} \times \mathbf{dSets} \longrightarrow \mathbf{dSets}$$

characterized by the fact that, for any two trees S and T, one has

$$\Omega[T] \otimes \Omega[S] = \mathcal{N}_d(\Omega(T) \otimes \Omega(S)),$$

and that for dendroidal sets X, Y, with $X = \mathsf{colim}_i X_i$ and $Y = \mathsf{colim}_j Y_j$, one has

$$X \otimes Y \simeq \mathsf{colim}_i(X_i \otimes Y) \simeq \mathsf{colim}_j(X \otimes Y_j).$$

The tensor product of dendroidal sets is symmetric, and when applied to two simplicial sets coincide with the cartesian product ([HM22, Proposition 4.2]).

Cocontinuity in both variables ensures existence of an inner-hom object bifunctor

$$\mathrm{Hom}\colon \mathbf{dSets} \times \mathbf{dSets} \longrightarrow \mathbf{dSets},$$

right adjoint to $\otimes$ once fixed a variable. This means that for any choice of dendroidal sets X, Y, Z, there is a natural isomorphism of sets

$$\mathbf{dSets}(X \otimes Y, Z) \simeq \mathbf{dSets}(X, \mathrm{Hom}(Y, Z)),$$

i.e. $\mathrm{Hom}(Y, Z)$ is the dendroidal sets given by

$$T \mapsto \mathrm{Hom}(Y, Z)_T = \mathbf{dSets}(\Omega[T] \otimes Y, Z).$$

We denote by $\mathrm{hom}(-,-)$ the underlying simplicial set, that is

$$\mathrm{hom}(X, Y) = i^* \,\mathrm{Hom}(X, Y), \quad \mathrm{hom}(X, Y)_n \simeq \mathbf{dSets}(i_!(\Delta^n) \otimes X, Y).$$

Remark 3.6.3. Contrarily to the Boardman-Vogt tensor product, tensor product of dendroidal sets is not associative up to coherent isomorphisms, so $(\mathbf{dSets}, \otimes)$ is not a symmetric monoidal category, and the simplicial object $\mathrm{hom}(-,-)$ does not endow $\mathbf{dSets}$ with a simplicial enrichment. However, the tensor product is associative up to coherent homotopies (cf. the discussion in [HM22, §4.4]), which suffices for many homotopical purposes, as we will see in the next sections.

3.6.2 The operadic model structure on dendroidal sets

Some of the notions of the homotopy theory of simplicial sets can be extended to the context of dendroidal sets. In particular, we recall the following

Definition 3.6.4. Let T be a tree.

- Its *boundary inclusion* $\partial T \to \Omega[T]$ is the morphism of dendroidal sets given by the union of all the faces of T.
- For every inner edge e of T, the *inner horn inclusion* $\Lambda^e T \to \Omega[T]$ is given by the union of all elementary faces except the inner $\partial_e T$.
- A vertex v of T is a *leaf vertex* if its input edges are leaves of T. For any such v, there is an induced *leaf horn incusion* $\Lambda^v T \to \Omega[T]$, defined as the union of all elementary faces except the external $\partial_v T$. If $T \simeq [n]$, its leaf horn inclusion coincides with the left horn $\Lambda^n_0 \hookrightarrow \Delta^n$.

  If $T \simeq C_n$ is a corolla with a unique vertex v, we interpret $\Lambda^v C_n =: \ell(C_n)$ as the disjoint union of the leaves of C_n, and write $\ell(C_n) \to \Omega[C_n]$ for the corresponding leaf horn inclusion.

Quasioperads model operads where composition is only defined up-to-homotopy, and are defined as the dendroidal sets satisfying an inner horn filling condition shaped on trees.

Definition 3.6.5. A dendroidal set X is a *quasioperad* if, for any tree T, any inner horn inclusion $\Lambda^e T \to \Omega[T]$ and any map $\Lambda^e T \to X$, the solid diagram below admits a dotted lift:

$$\begin{tikzcd}
\Lambda^e T \arrow[r] \arrow[d] & X \\
\Omega[T] \arrow[ur, dashed] &
\end{tikzcd}$$

Remark 3.6.6. If $M \in \mathbf{sSets} \overset{i_!}{\hookrightarrow} \mathbf{dSets}$, then $i_!(M)$ is a quasioperad if and only if M is a quasicategory.

Definition 3.6.7. A dendroidal set X is *normal* if, for any tree T, the action on X_T of the group of automorphism of T is free. More generally, a monomorphism of dendroidal sets $Y \to X$ is *normal* if, for any tree T, its group of automorphisms acts freely on $X_T \backslash Y_T$.

Remark 3.6.8. If $X \simeq \mathcal{N}_d(P)$ for a discrete operad P, then X is normal if and only if P is Σ-free.

Theorem 3.6.9. *There exists a left proper and cofibrantly generated model structure on the category* **dSets** *of dendroidal sets, called the* operadic model structure*, where cofibrations are the normal monomorphisms, fibrant objects are the quasioperads and a map between normal dendroidal sets is a weak equivalence if and only if for every quasioperad X, the map*

$$\hom(B, X) \longrightarrow \hom(A, X)$$

is a categorical equivalence of ∞-categories. Moreover, under the isomorphism $\mathbf{dSets}/\Omega[\eta] \simeq \mathbf{sSets}$ *it coincides with Joyal's model structure on simplicial sets.*

If it wasn't for the lack of associativity of the tensor product of dendroidal set, $\hom(-,-)$ would make dendroidal sets with the operadic model structure enriched in the Joyal model structure [HM22, Proposition 9.28]. However, associativity up to homotopy of the tensor product endows the homotopy category $\mathsf{Ho}(\mathbf{dSets})$ with a symmetric monoidal structure. In [HM24], it is proven that this structure can be lifted to the structure of a symmetric monoidal ∞-category on $(\mathbf{dSets})_\infty$. Moreover, we can still use the hom-object to compute the mapping space of **dSets** with respect to the operadic model structure: given a normal dendroidal set A and a quasioperad X, one may choose

$$\mathsf{Map}(A, X) = \hom(A, X)^{\simeq}$$

the maximal sub-Kan complex of the simplicial mapping object $\hom(A, X)$.

The operadic model structure is equivalent to the model structure on simplicial operads for Dwyer-Kan weak equivalences: there is a Quillen equivalence

$$W_! \colon \mathbf{dSets} \underset{\perp}{\rightleftarrows} \mathbf{sOp} : W^*,$$

where the left adjoint $W_!$ is the essentially unique colimit-preserving functor which sends a tree T to the Boardman-Vogt resolution of the discrete operad $\Omega(T)$ generated by T. The right adjoint is called the *homotopy coherent dendroidal nerve functor*, and in this thesis we call $W_!$ the *Boardman-Vogt resolution* of a dendroidal ∞-operad. The Quillen equivalence extends the one between simplicial sets with Joyal's model structure and Bergner's model structure on simplicial categories.

The left Quillen equivalence $W_!$ can be used to define homotopy algebras over a dendroidal set as given by the projective model structure on simplicial $W_!(X)$-algebras. We denote by $\mathsf{Alg}_X(\mathcal{S})$ the underlying ∞-category. This notion is meaningful when X is normal: indeed, if $f\colon X \to Y$ is a weak operadic equivalence of normal dendroidal sets, the base change $f_!\colon \mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \to \mathsf{Alg}_{W_!(Y)}(\mathbf{sSets})$ is a left Quillen equivalence between the projective model structures, so that one has an equivalence of ∞-categories $f_!\colon \mathsf{Alg}_X(\mathcal{S}) \xrightarrow{\sim} \mathsf{Alg}_Y(\mathcal{S})$. In particular, when $X \simeq W^*(P)$ for a simplicial operad P (that we can suppose Σ-free and Kan-enriched), the counit of the adjunction $(W_!, W^*)$ yields an equivalence of ∞-categories

$$\mathsf{Alg}_{W^*(P)}(\mathcal{S}) \simeq \mathsf{Alg}_P(\mathbf{sSets})_\infty =: \mathsf{Alg}_P(\mathcal{S}).$$

3.6.3 The covariant model structure on dendroidal sets

The notion of left fibration between simplicial sets, used to model ∞-categories cofibred in groupoids, generalizes to dendroidal sets, encoding ∞-operads cofibred in groupoids. For a map of dendroidal sets $E \to X$, the existence of a lift of an operation in X with chosen inputs from E is equivalent to the existence of a dotted lift in the following solid diagram

$$\begin{tikzcd}
\ell(C_n) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\Omega[C_n] \arrow[r, "f"] & X
\end{tikzcd}$$

where we have considered the leaf horn inclusion of the leaves of a n-corolla $\ell(C_n) \to \Omega[C_n]$. Indeed, the map $\ell(C_n) \to E$ selects objects $y_1, \dots, y_n$ of E, while f corresponds to an n-ary operation $f \in X(p(y_1), \dots, p(y_n); x)$ for some $x \in X_\eta$; a lift in the diagram is the existence of a object $y \in E_\eta$ and $\overline{f} \in E(y_1, \dots, y_n; y)$ with $p(\overline{f}) = f$. Homotopy-coherent coCartesianity of the lift is then obtained by requiring the existence of lifts against leaf horn inclusions of more general trees. It yields the following definition.

Definition 3.6.10. A morphism of dendroidal sets $p\colon Y \to X$ is a *dendroidal left fibration* if it has the right lifting properties against inner and leaf horn inclusions of trees.

Dendroidal left fibrations over X are in particular objects of the overcategory $\mathbf{dSets}/X$. We write (Y, p) for the morphism $p\colon Y \to X$ when we think of it as an object in the former overcategory. Given an object x of X, we denote by $(Y, p)_x$ the fibre over x, that is,

$$(Y, p)_x \simeq p^{-1}(x).$$

If p is a dendroidal inner fibration, $(Y, p)_x$ is an ∞-category, that is a quasicategory, while if p is dendroidal left fibration, $(Y, p)_x$ is a Kan complex ([HM22, Remark 9.60]). Given a morphism of dendroidal sets $f\colon X \to Y$, there is an induced adjunction

$$f_!\colon \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/Y : f^*,$$

where $f_!$ is the functor that composes a morphism $A \to X$ with f and f^* is the functor that takes the pullback along f; we call it the *base-change adjunction*.

The homotopy theory of dendroidal left fibrations is governed by the covariant model structure, which is also compatible with base-change. Given two elements $(E, f), (B, g)$ in $\mathbf{dSets}/X$, the simplicial set $\hom_X(E, B)$ is defined as the pullback:

$$\begin{tikzcd}
\hom_X(E,B) \arrow[r] \arrow[d] & \hom(E,B) \arrow[d, "g_*"] \\
\Delta^0 \arrow[r, "\{f\}"] & \hom(E,X)
\end{tikzcd}$$

Theorem 3.6.11. *Let X be a dendroidal set. The category $\mathbf{dSets}/X$ carries a left proper cofibrantly generated model structure, called the* covariant model structure*, where the cofibrations are the normal monomorphisms over X, the fibrant objects are the dendroidal left fibrations and a map $(A, u) \to (B, v)$ between normal objects over X is a weak equivalence if and only if for any dendroidal left fibration (Y, p), the map*

$$\hom_X(B, Y) \longrightarrow \hom_X(A, Y)$$

is a weak homotopy equivalence of Kan complexes.

Moreover, under the isomorphism $\mathbf{dSets}/\Omega[\eta] \simeq \mathbf{sSets}$, it yields the covariant model structurefor left fibrations of simplicial sets.

If it wasn't for the lack of associativity of the tensor product of dendroidal sets, the underlying simplicial set $\hom(-,-)$ of the inner-hom object of dendroidal sets would make the covariant model structure on $\mathbf{dSets}/X$ enriched in the Kan-Quillen model structure ([HM22, Proposition 9.66]). We can still use it to compute the mapping spaces for the covariant model structure: given a normal dendroidal set (A, α) over X and a dendroidal left fibration (Y, β) over X, we can choose

$$\mathsf{Map}((A, \alpha), (Y, \beta)) = \hom_X(A, Y) := \hom(A, Y) \times_{\hom(A,X)} \{\alpha\}.$$

In the covariant model structure, weak equivalences between fibrant objects can be characterized fibre-wise. We will often use this important feature, which we record in the following

Theorem 3.6.12. *Let $F\colon (Y, p) \to (Z, q)$ be a morphism between dendroidal left fibrations. Then F is a weak equivalence in the covariant model structure if and only if, for any object x of X, the induced map $(Y, p)_x \to (Z, q)_x$ between the fibres over x is a weak homotopy equivalence of simplicial sets.*

Finally, let us state this important invariance property of the covariant model structure with respect to base-change. Recall that, for a morphism of dendroidal sets $f\colon X \to Y$, the *base-change* adjunction consists in the adjunction between over categories

$$f_!\colon \mathbf{dSets}/X \underset{\perp}{\rightleftarrows} \mathbf{dSets}/Y : f^*$$

where the functor $f_!$ acts by post-composing with f and f^* consists in taking the fibre product of a morphism $Z \to Y$ along f.

Theorem 3.6.13. *Let $f\colon X \to Y$ be a map of dendroidal sets. The base-change adjunction*

$$f_!\colon \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/Y \colon f^*$$

is a Quillen adjunction with respect to the covariant model structure, which is a Quillen equivalence when f is an operadic weak equivalence.

Theorem 3.6.13, as stated above, is proven in [Heu11, Proposition 2.4] and does not appear in [HM22].

3.7 Notation II

We convey here the notations relative to dendroidal theory that we will use throughout the rest of the thesis.

- As $i\colon \Delta \to \mathbf{\Omega}$ is fully faithful, we identify Δ with a full subcategory of $\mathbf{\Omega}$ and drop the i.
- Similarly, as $i_!\colon \mathbf{sSets} \to \mathbf{dSets}$ is fully faithful, we identify $\mathbf{sSets}$ with a full subcategory of $\mathbf{dSets}$, dropping the $i_!$.
- We identify $\mathbf{\Omega}$ both with a full subcategory of Op and of $\mathbf{dSets}$. In particular, given a tree T, we still write T for the operad $\Omega(T)$ and for the dendroidal set $\Omega[T]$. Observe that there are isomorphisms of trees $\eta \simeq [0]$, resp. $C_1 \simeq [1]$, translating in isomorphism of dendroidal sets which in our simplified notation write as $\eta \simeq \Delta^0$ and $C_1 \simeq \Delta^1$.
- We write $E(T)$ for the set of edges of a tree T. It will be often considered with its poset structure.
- Given a model category M, we write $\mathsf{Map}_M(-,-)$ for its mapping space, sometimes just $\mathsf{Map}(-,-)$ when clear from the context.

3.8 Equivalence of the dendroidal formalisms

In Section 3.1 we recalled how to translate model categorical statements into statements of ∞-categories. In particular, let dOp_∞ be the ∞-category underlying the operadic model structure on dendroidal sets. In Section 3.6.2 we have introduced dendroidal sets with the operadic model structure as a model for ∞-operads; as one would expect,and luckily enough, this model is equivalent to the one given by complete dendroidal Segal spaces, in the sense that there is an equivalence of ∞-categories $\mathsf{dOp}_\infty \simeq \mathsf{DOp}_\infty$.

One way to realize such equivalence is model categorical. Indeed, the functor $\mathsf{disc}\colon \mathbf{Sets} \to \mathbf{sSets}$ which realizes every set as a discrete simplicial set induces an adjunction

$$\mathsf{disc}_!\colon \mathbf{dSets} \rightleftarrows \mathbf{dSpaces} \colon \mathsf{disc}^*,$$

and after [HM22, Theorem 12.22], the adjunction is a Quillen equivalence with respect to the operadic model structure on dendroidal sets and the localized Reedy model structure on dSpaces presenting the ∞-category DOp_∞. In particular, this yields an adjoint equivalence of ∞-categories

$$d_!\colon \mathsf{dOp}_\infty \simeq \mathsf{DOp}_\infty\colon d^*,$$

where $d_! = (\mathsf{disc}_!)_\infty$ and $d^* = (\mathsf{disc}^*)_\infty$.

This equivalence is compatible with the nerve constructions: after [HM22, Corollary 14.30], for any simplicial operad P there is a weak equivalence

$$\mathcal{N}_D(P) \simeq d_!(W^*_\infty(P)).$$

In Section 4.2.1 we will introduced the ∞-category DLeft_X of dendroidal left fibrations over a complete dendroidal Segal space X. We record here its equivalence with the analogous notion for dendroidal sets, in the following sense.

Theorem 3.8.1 ([BdBM20]). *Let X be a complete dendroidal Segal space and let Z be the equivalent quasioperad, $Z \simeq d^*(X)$. Let dLeft_Z be the ∞-category underlying the covariant model structure on $\mathbf{dSets}/Z$. There is a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathsf{dOp}_{\infty/Z} \arrow[r, "d_!"] \arrow[d] & \mathsf{DOp}_{\infty/X} \arrow[d] \\
\mathsf{dLeft}_Z \arrow[r, "d_!"] & \mathsf{DLeft}_X
\end{tikzcd}$$

where the vertical arrows are localization. The same holds for X not necessarily complete.

Chapter 4

A straightening-unstraightening equivalence for ∞-operads

In this chapter, we study the ∞-category of operadic left fibrations in the dendroidal and Lurie's formalism and construct an operadic un/straightening equivalences for ∞-operads in Lurie's model.

In a first part, we address model-independency, and prove that the Hinich-Moerdijk comparison functors induce an equivalence between the ∞-categories of operadic left fibrations in the two formalisms. In a second part, and independently, we construct a straightening-unstraightening equivalence for ∞-operads in Lurie's formalism by means of the symmetric monoidal envelope functor.

Contents

4.1 Introduction

As algebras over an operad are a generalization of functors out of a category, to manipulate the ∞-category of space-valued algebras over an ∞-operad we look at the ∞-category of functors $\mathsf{Fun}(\mathcal{C}, \mathcal{S})$ for an ∞-category $\mathcal{C}$. In this case, we can use the straightening-unstraightening equivalence, also known as Lurie-Grothendieck equivalence, present in [Lur09]. It establishes an equivalence

$$\mathsf{St}^{\mathcal{C}} \colon \mathsf{Left}_{\mathcal{C}} \underset{\perp}{\rightleftarrows} \mathsf{Fun}(\mathcal{C}, \mathcal{S}) \colon \mathsf{Unst}^{\mathcal{C}}$$

between the ∞-category of functors $\mathcal{C} \to \mathcal{S}$, on the right, and, and on the left, the ∞-category of left fibrations $\mathcal{D} \to \mathcal{C}$, the higher categorical generalisation of categories fibred in groupoids in ordinary category theory. The un/straightening theorem generalizes Grothendieck's equivalence between categories cofibred in groupoids and set-valued pseudofunctors, as well as the correspondence between covering spaces and sets with an action of the fundamental group of the base, and is a crucial tool in the theory of symmetric monoidal ∞-categories and the theory of ∞-operads. When $\mathcal{C}$ is the underlying ∞-category of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the un/straightening equivalence allows us to similarly work with *lax monoidal* functors $\mathcal{C}^{\otimes} \to \mathcal{S}^{\times}$. This was first proven by Hinich in [Hin15], who proved that the un/straightening equivalence is the underlying functor of an equivalence

$$\mathsf{St}^{\mathcal{C},\otimes} \colon \mathsf{smLeft}_{\mathcal{C}} \underset{\perp}{\rightleftarrows} \mathsf{Fun}^{\mathrm{lax}}(\mathcal{C},\mathcal{S}) : \mathsf{Unst}^{\mathcal{C},\otimes}, \tag{4.1.1}$$

where $\mathsf{smLeft}_{\mathcal{C}^{\otimes}}$ is the ∞-categories of *symmetric monoidal* left fibrations over $\mathcal{C}^{\otimes}$.

This justifies looking for a *straightening-unstraightening equivalence for ∞-operads*, that is, an equivalence

$$\mathsf{St}^{\mathcal{P}} \colon \mathsf{Left}\,(\mathsf{Op}_{\infty})_{\mathcal{P}} \underset{\perp}{\rightleftarrows} \mathsf{Alg}_{\mathcal{P}}(\mathcal{S}^{\times}) : \mathsf{Unst}^{\mathcal{P}}$$

between an ∞-category of *operadic* left fibrations over $\mathcal{P}$, and the ∞-category of $\mathcal{S}$-valued $\mathcal{P}$ algebras. This equivalence has been established both in Lurie's and in the dendroidal model. In Lurie's model, an operadic un/straightening equivalence has been proven by Ramzi in [Ram22], under the name of *$\mathcal{O}$-monoidal Grothendieck construction*: by instantiating [Corollary C] therein with $\mathcal{C} = \mathcal{O}$, one obtains an equivalence

$$\mathsf{Left}_{\mathcal{O}}^{\mathcal{O}-lax} \simeq \mathsf{Fun}^{\mathcal{O}-lax}(\mathcal{O},\mathcal{S}), \tag{4.1.2}$$

between an ∞-category of *$\mathcal{O}$-monoidal left fibrations over $\mathcal{O}^{\otimes}$* and that of $\mathcal{O}$-lax monoidal functors $\mathcal{O}^{\otimes} \to \mathcal{S}^{\otimes}$, equivalent to the ∞-category of $\mathcal{O}$-algebras in spaces. In the dendroidal formalism, a un/straightening equivalence was first established by Heuts for dendroidal sets [Heu11], and later on by Boavida-Moerdijk [BdBM20] for Cisinski-Moerdijk dendroidal model. The equivalences are realized as (zig-zags of) Quillen equivalences between the *covariant model structure* for dendroidal left fibrations and the projective model structure on simplicial algebras over (the Boardman-Vogt resolution of) a dendroidal ∞-operad. Observe that, although the ∞-categories of Lurie's and dendroidal ∞-operads are equivalent, a priori this equivalence depends on the chosen model.

In this chapter, we explore the operadic un/straightening equivalence under two different points of view: first, we prove model-independency of the notion of operadic left fibration in Lurie's and in the dendroidal formalism; second, we construct an alternative version of the equivalence in Lurie's formalism by means of the symmetric monoidal envelope functor.

4.1.1 Strategy and main results

We start with Section 4.2 by addressing model-invariance of the notion of operadic left fibration. The key tool is what we call the *Hinich-Moerdijk comparison functors*, a pair of adjoint functors defined in [HM24] and therein proven to establish a direct equivalence

$$\delta \colon \ell\mathsf{Op}_{\infty} \underset{\perp}{\rightleftarrows} \mathsf{DOp}_{\infty} : \lambda$$

between the ∞-category of Lurie ∞-operads, on the left, and that of dendroidal ∞-operads, on the right.

First, we observe that, given a Lurie ∞-operad $\mathcal{O}^{\otimes}$, the ∞-category of $\mathcal{O}$-monoidal left fibrations over $\mathcal{O}^{\otimes}$ used in [Ram22] is just the full sub ∞-category of $\ell\mathsf{Op}_{\infty/\mathcal{O}^{\otimes}}$ spanned by the objects $(\mathcal{X}^{\otimes}, p)$ for which the map of ∞-operads $p\colon \mathcal{X}^{\otimes} \to \mathcal{O}^{\otimes}$ is also a left fibration of ∞-categories. We call this simply $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$. Given a dendroidal ∞-operad X, we write DLeft_X for the ∞-category of dendroidal left fibrations over X as defined in [BdBM20]. In particular, it sits as a full sub ∞-category of the over ∞-category $\mathsf{DOp}_{\infty/X}$. We then prove the following

Theorem (Theorem 4.2.15)**.** *For any Lurie ∞-operad $\mathcal{O}^{\otimes}$, the Hinich-Moerdijk comparison functors induce an equivalence of ∞-categories*

$$\delta\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \underset{\perp}{\rightleftarrows} \mathsf{DLeft}_X : \lambda,$$

where $X \simeq \delta(\mathcal{O}^{\otimes})$ is the dendroidal model for $\mathcal{O}^{\otimes}$.

We then dedicate the rest of the chapter to proving an operadic un/straightening equivalence in Lurie's model via the *symmetric monoidal envelope functor*. This is a functor

$$\mathsf{Env}(-)^{\otimes}\colon \ell\mathsf{Op}_{\infty} \longrightarrow \mathsf{smCat}_{\infty}$$

from the ∞-category of ∞-operads and that of symmetric monoidal ∞-categories and strong monoidal functors, left adjoint to the forgetful functor. In [Lur17], Lurie gives an explicit construction of it. The universal property of the envelope prescribes an equivalence of ∞-categories

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}), \mathcal{V})$$

for any Lurie ∞-operad $\mathcal{O}^{\otimes}$ and symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$.

Motivated by this property, we construct a un/straightening equivalence where the left adjoint is given by the composition of the symmetric monoidal envelope restricted to operadic left fibrations and a refinement of the monoidal un/straightening of Equation (4.1.1) for strong monoidal functors. The ∞-category standing between operadic left fibrations over $\mathcal{O}^{\otimes}$ and $\mathcal{O}^{\otimes}$-algebras is that of *strong sm-left fibrations* over $\mathsf{Env}(\mathcal{O})^{\otimes}$, which we denote by $\mathsf{Left}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})}$. This is the content of our second main

Theorem (Theorem 4.5.1)**.** *For any Lurie ∞-operad $\mathcal{O}^{\otimes}$, there is an equivalence of ∞-categories*

$$\mathsf{St}^{\mathcal{O}}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \underset{\perp}{\rightleftarrows} \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}) : \mathsf{Unst}^{\mathcal{O}},$$

where the left adjoint is given by the composition

$$\mathsf{St}^{\mathcal{O}}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \xrightarrow{\mathsf{Env}(-)^{\otimes}_{/}} \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} \xrightarrow{\mathsf{St}^{\mathsf{Env}(\mathcal{O}),\otimes}} \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}), \mathcal{S}) \simeq \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}),$$

where the ∞-category $\mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})}$ is the one of Definition 4.3.7.

The proof consists in showing that each functor separately is an equivalence of ∞-categories. For $\mathsf{St}^{\mathsf{Env}(\mathcal{O}),\otimes}$, we provide a characterization of the sm-left fibrations equivalent to the unstraightening *strong* monoidal functors (Proposition 4.3.5). This yields the ∞-category of strong sm-left fibrations $\mathsf{smLeft}^{\mathsf{str}}_{\mathcal{C}}$. We then proceed to prove that slicing the envelope over $\mathcal{O}^{\otimes}$ and restricting it to operadic left fibrations yields an equivalence of these latter with strong sm-left fibrations. The key observation here is due to Haugseng-Kock: although the envelope is not fully faithful, it becomes so when sliced over the terminal ∞-operad ([HK24, Proposition 2.4.3]). In particular, for any Lurie ∞-operad $\mathcal{O}^{\otimes}$ slicing the envelope defines a fully faithful left adjoint

$$\mathsf{Env}(-)^{\otimes}_{/} \colon \ell\mathsf{Op}_{\infty/\mathcal{O}^{\otimes}} \underset{\perp}{\rightleftarrows} \mathsf{smCat}_{\infty/\mathsf{Env}(\mathcal{O}^{\otimes})} : G',$$

where the right adjoint G' is the base-change along the unit composed with the forgetful functor. In Proposition 4.4.7 we prove that this restricts to an adjunction between operadic left fibrations and strong sm-left fibrations, and that this is an equivalence.

4.1.2 Relation with other works

To our knowledge, the equivalence proven in Theorem 4.2.15 is new and might be useful when comparing various models or to construct presentations of the un/straightening equivalence as a Quillen equivalence of model categories.

The use of the envelope functor in the operadic un/straightening equivalence of Theorem 4.5.1 allows to reduce the discussion to symmetric monoidal ∞-categories. In this regard, the spirit of the second part of this article is very much connected with Haugseng-Kock [HK24], where the ∞-category of symmetric monoidal ∞-operad is realised as a full sub ∞-category of symmetric monoidal ∞-categories over finite pointed sets.

Additionally, the reduction to symmetric monoidal ∞-categories allows us to use the un/straightening equivalence for lax monoidal functors already proven by Hinich in [Hin15], without using the full power of Ramzi's construction of [Ram22]. We contribute with Proposition 4.3.5, describing the unstraightening of *strong* monoidal functors out of symmetric monoidal ∞-categories.

Kern [Ker23] defines the symmetric monoidal envelope for dendroidal ∞-operads and proves a straightening/unstraightening equivalence. The condition in Proposition 4.3.5 for $\mathsf{smLeft}^{\mathsf{str}}_{cC^{\otimes}}$ likely corresponds to his notion of equifibredness, suggesting that the Hinich–Moerdijk functors induce an equivalence with Lurie's envelope.

It would be interesting to compare our un/straightening of Theorem 4.5.1 with [BHS22, Theorem E] and with [BHS22, Theorem 2.3.5.]. In this respect, it could also be interesting to consider the work by Haugseng, resp. Haugseng and Kock, in [Hau22], resp. [HK24].

4.1.3 Outline

Each section of this chapter consists in an expository part, where we present the objects we use, followed by the part were we state and prove our contributions. More precisely:

In Section 4.2 we give the definitions of the ∞-categories of operadic left fibrations in Lurie's and the dendroidal models and recall Hinich-Moerdijk equivalence. We then prove Theorem 4.2.15.

In Section 4.3 we present the monoidal Grothendieck construction for symmetric monoidal ∞-categories. We prove Proposition 4.3.5 and define strong sm-left fibrations.

In Section 4.4, we recall the definition of the symmetric monoidal envelope functor, then prove that it establishes an equivalence between operadic left fibrations and strong sm-left fibrations (Proposition 4.4.7).

In Section 4.5, we define the operadic un/straightening equivalence and obtain Theorem 4.5.1.

4.1.4 Preliminaries and notation

The background needed for this chapter is contained in Section 3.3 and Section 3.5. We use the notation therein as well as that in Section 3.2.

4.2 Model-independence of operadic left fibrations

We will work with slice ∞-categories of the form $\mathcal{C}_{/x}$, or occasionally also $\mathcal{C}_{x/}$, with x an object of $\mathcal{C}$. The objects of the *over-∞-category* $\mathcal{C}_{/x}$ are pairs (y, α), where y is an object of $\mathcal{C}$ and $\alpha \colon y \to x$ a morphism in $\mathcal{C}$, a morphism $(y, \alpha) \to (z, \beta)$ in $\mathcal{C}_{/x}$ is a 2-morphism in $\mathcal{C}$ of the form

$$\begin{tikzcd}
y \arrow[rr] \arrow[dr, "\alpha"'] & & z \arrow[dl, "\beta"] \\
& x &
\end{tikzcd}$$

and more generally its n-morphisms are diagrams

$$\Delta^{n+1} \simeq \Delta^n \star \Delta^0 \longrightarrow \mathcal{C}$$

taking the cocone point into x. One can define dually the simplices of the under-∞-category $\mathcal{C}_{x/}$.

By replacing the object $x \colon \Delta^0 \to \mathcal{C}$ by a more general diagram $f \colon \mathcal{D} \to \mathcal{C}$, as for example the one singled out by a n-simplex of $\mathcal{C}$, $f \colon \Delta^n \to \mathcal{C}$, one can similarly define the under-∞-category $\mathcal{C}_{f/}$.

We state the following important

Proposition 4.2.1 ([HM24, Lemma 2.3.4.]). *Let $\mathcal{C}$ be an ∞-category and x an object of $\mathcal{C}$. The Yoneda embedding induces an equivalence of ∞-categories*

$$\mathsf{PSh}(\mathcal{C}_{/x}) \xrightarrow{\simeq} \mathsf{PSh}(\mathcal{C})_{/x}.$$

To present operadic left fibrations for Lurie's ∞-operads and complete dendroidal Segal spaces, we also recall the following

Definition 4.2.2. Let $\mathcal{C}$ be an ∞-category and x an object of $\mathcal{C}$. Given a set S of arrows of $\mathcal{C}$, one says that x is *S-local* if, for any arrow $f \colon a \to b$ in S, the morphism

$$\mathsf{Map}_{\mathcal{C}}(b, x) \xrightarrow{f*} \mathsf{Map}_{\mathcal{C}}(a, x)$$

is an equivalence of spaces.

Given an object (y, f) in the slice $\mathcal{C}_{/x}$, we say that (y, f) is *S-local* if it is $S_{/x}$-local, where $S_{/x}$ consists of those arrows in $\mathcal{C}_{/x}$ of the form

$$\begin{tikzcd}
a \arrow[rr, "s"] \arrow[dr, "\alpha"'] & & b \arrow[dl, "\beta"] \\
& x, &
\end{tikzcd}$$

where s ranges in S and β in the arrows of $\mathcal{C}$.

4.2.1 Dendroidal left fibrations

We work with the model of complete dendroidal Segal spaces, introduced in Section 3.5. For our purposes, it is more convenient to represent complete dendroidal Segal spaces as presheaves on a category of *forests* Φ, obtained from the tree category $\mathbf{\Omega}$ by formally adjoining finite coproducts. Explicitly, the category Φ can be described as the full subcategory of the category of discrete operads spanned by $\Omega(F)$, where $F = \bigsqcup_{i=1}^{n} T_i$ is a finite disjoint union of trees and $\Omega(F)$ is the disjoint union of the operads $\Omega(T_i)$. For the purposes of this work, it will be more convenient to identify DOp_∞

We can use the category Φ to reformulate Definition 3.5.1, and define DOp_∞ as the full sub ∞-category of $\mathsf{PSh}(\Phi)$ spanned by the presheaves satisfying the Segal and completeness properties, as well as the extra (also Segal-type) property:

3. The natural map $X(F) \to \prod_{i=1}^{n} X(T_i)$ for a forest F consisting of the trees T_i is an equivalence. In particular, $X(\varnothing)$ is contractible.

The Yoneda embedding induces fully faithful inclusions

$$\mathbf{\Omega} \hookrightarrow \Phi \hookrightarrow \mathsf{DOp}_\infty,$$

and the inclusion $\mathsf{DOp}_\infty \hookrightarrow \mathsf{PSh}(\Phi)$ still has a left adjoint, that is DOp_∞ is a localization of the ∞-category of presheaves on Φ.

We consider dendroidal left fibrations as first defined in [BdBM20], where they are called *covariant fibrations*. The key idea is defining dendroidal left fibrations as local objects with respect to the inclusions of leaves in a tree – or rather, its analogue with forests.

Definition 4.2.3. For a tree $T \in \mathbf{\Omega}$, let $\ell(T)$ be the disjoint union of its leaves, and denote by $\ell(T) \hookrightarrow T$ the inclusion of these into T. For a forest F in Φ, $F = T_1 \sqcup \cdots \sqcup T_n$, we write $\ell(F)$ for the disjoint union of the leaves of each tree in F, that is, $\ell(F) = \ell(T_1) \sqcup \cdots \sqcup \ell(T_n)$, and $\ell(F) \hookrightarrow F$ for the inclusions of these into F.

Let $\mathcal{L}$ denote the set of morphisms in $\mathsf{PSh}(\Phi)$ given by the inclusions of leaves of a forest,

$$\mathcal{L} := \{\ell(F) \hookrightarrow F\}_{F \in \Phi}$$

Definition 4.2.4. A morphism $\varphi\colon Y \to X$ is a *dendroidal left fibration* if φ is $\mathcal{L}$-local in $\mathsf{DOp}_{\infty/X}$. We write DLeft_X for the full sub ∞-category of $\mathsf{DOp}_{\infty/X}$ spanned by dendroidal left fibrations.

The following will be essential in the proof of Theorem 4.2.15.

Remark 4.2.5. The ∞-category DLeft_X is a localization of the over-category $\mathsf{DOp}_{\infty/X}$, as proven in [HM22, Theorem 13.6]. Moreover, it is a consequence of [HM22, Lemma 13.5] that, when $X = \mathcal{C}$ is an ∞-category, there is a canonical equivalence

$$\mathsf{DLeft}_{\mathcal{C}} \simeq \mathsf{Left}_{\mathcal{C}}.$$

Under the embedding $\Delta \hookrightarrow \Phi$, the leaf inclusion $\ell([n]) \hookrightarrow [n]$ corresponds to the map $\{0\}\colon [0] \to [n]$ selecting the first vertex. In particular, a map of ∞-categories $\mathcal{D} \to \mathcal{C}$ is a left fibration if and only if it is $\{\Delta^0 \xhookrightarrow{\{0\}} \Delta^n\}_{n \geqslant 0}$-local.

In the ∞-category of dendroidal left fibrations, equivalences are detected fibrewise.

Proposition 4.2.6 ([HM22, Theorem 13.6 and Proposition 13.8]). *Let X be a dendroidal ∞-operad, and consider a morphism of dendroidal left fibrations*

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & . \end{tikzcd}$$

The morphism f is an equivalence in DLeft_X if and only if for any object x in X_η, the map of fibres $(Y,p)_x \to (Z,q)_x$ is an equivalence of spaces.

4.2.2 Operadic left fibrations

We now define the notion of left fibration of ∞-operads in Lurie formalism.

Denote by $U\colon \ell\mathsf{Op}_\infty \to \mathsf{Cat}_\infty$ the forgetful functor which, given an ∞-operad $\mathcal{O}^\otimes \to \mathsf{Fin}_*$, simply returns the ∞-category $\mathcal{O}^\otimes$, forgetting the morphism into Fin_*.

Definition 4.2.7. An *operadic left fibration* is a morphism of Lurie ∞-operads $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ such that Uf is a left fibration between ∞-categories.

We denote by $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$ the full subcategory of $\ell\mathsf{Op}_{\infty/\mathcal{O}^\otimes}$ spanned by operadic left fibrations.

Remark 4.2.8. Observe that, after Remark 4.2.5, an element $(\mathcal{D}^\otimes, f)$ in $\ell\mathsf{Op}_{\infty/\mathcal{O}^\otimes}$ is an operadic left fibration if f is $\{\Delta^0 \xrightarrow{\{0\}} \Delta^n\}_{n\geqslant 0}$-local once applied the forgetful functor $\ell\mathsf{Op}_\infty \to \mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$.

We report a useful criterion which allows to understand when a left fibration between ∞-operads is an operadic left fibration.

Proposition 4.2.9 ([Lur17, Proposition 2.1.2.12]). *Let $\mathcal{O}^\otimes$, $\mathcal{D}^\otimes$ be ∞-operads, and consider a morphism $p\colon \mathcal{D}^\otimes \to \mathcal{O}^\otimes$ in $\mathsf{Cat}_{\infty/\mathsf{Fin}_*}$. Suppose that p is a left fibration. The following properties are equivalent:*

- *p is an operadic left fibration;*
- *for any $x \in \mathcal{O}^\otimes_{\underline{n}_+}$, $x \simeq x_1 \oplus \cdots \oplus x_n$, the inert maps $x \to x_i$ induce an equivalence of ∞-categories*

$$\mathcal{D}^\otimes_x \longrightarrow \prod_{i=1}^n \mathcal{D}_{x_i}.$$

We conclude by observing that the ∞-category of operadic left fibrations enjoys the property that equivalences can be detected fibrewise.

Proposition 4.2.10. *Let $\mathcal{O}^\otimes$ be a Lurie ∞-operad and consider a morphism of operadic left fibrations*

$$\begin{tikzcd} \mathcal{T}^\otimes \arrow[rr, "f"] \arrow[dr, "\alpha^\otimes"'] & & \mathcal{Q}^\otimes \arrow[dl, "\beta^\otimes"] \\ & \mathcal{O}^\otimes & \end{tikzcd}$$

Then f is an equivalence in $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$ if and only if, for any object c of $\mathcal{O}$, the map between fibres

$$f_c\colon (\mathcal{T},\alpha)_c \longrightarrow (\mathcal{Q},\beta)_c$$

is an equivalence of spaces.

Proof. As $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$ is a full subcategory of $\ell\mathsf{Op}_{\infty/\mathcal{O}^{\otimes}}$ and f is by assumption a morphism of ∞-operad, we have that f is an equivalence if and only if its underlying map of ∞-categories is an equivalence. But a morphism between (non-operadic) left fibrations is an equivalence if and only if it restricts to an equivalence of fibres. As both $\alpha^{\otimes}$ and $\beta^{\otimes}$ satisfy the Segal condition in Proposition 4.2.9, it suffices to check on fibres over objects in $\mathcal{O}$, so we have the thesis. ☐

Remark 4.2.11. The notion of operadic left fibration over $\mathcal{O}^{\otimes}$, as defined in Definition 4.2.7, is equivalent to that of *$\mathcal{O}$-monoidal left fibration* over $\mathcal{O}^{\otimes}$ as defined in [Ram22, Definition 1.11] On the other hand, it differs from [KK24, Definition 1.8], in that instead of the condition of being a left (rather right) fibration being imposed on $\mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$, it is imposed on the induced map of ∞-categories $\mathsf{Env}(\mathcal{O}) \to \mathsf{Env}(\mathcal{D})$.

4.2.3 Hinich-Moerdijk comparison functors

In [HM24] Hinich-Moerdijk constructed a direct adjoint equivalence

$$\delta\colon \ell\mathsf{Op}_{\infty} \overset{\longrightarrow}{\underset{\perp}{\longleftarrow}} \mathsf{DOp}_{\infty} :\lambda$$

between the ∞-category of Lurie's ∞-operads and that of complete dendroidal Segal spaces. The two functors have explicit descriptions, which we now recall. Crucial to define the functor λ is the following

Remark 4.2.12. The ∞-category $\ell\mathsf{Op}_{\infty}$ is a (*non-full*) sub ∞-category of $\mathsf{PSh}(\mathbb{F})$, where $\mathbb{F} := \Delta/\mathsf{Fin}_*$ is the category of elements of Fin_* (as a simplicial set). Indeed, there is forgetful functor $\ell\mathsf{Op}_{\infty} \to \mathsf{Cat}_{\infty/\mathsf{Fin}_*}$, and the latter ∞-category is a localization of $\mathsf{PSh}(\Delta)_{/\mathsf{Fin}_*}$, which is in turn equivalent to $\mathsf{PSh}(\mathbb{F})$ thanks to Proposition 4.2.1.

Complete dendroidal Segal spaces are a sub ∞-category of that of presheaves on Φ, and there exists a functor from the category of simplices of Fin_* into finite forests,

$$w\colon \mathbb{F} \longrightarrow \Phi,$$

which assigns to every n-simplex of Fin_* a n-leveled forest, hence a forest by forgetting the levels. By left Kan extension, the functor w induces an adjunction

$$w_!\colon \mathsf{PSh}(\mathbb{F}) \overset{\longrightarrow}{\underset{\perp}{\longleftarrow}} \mathsf{PSh}(\Phi) :w^*,$$

and one defines $\lambda'\colon \mathsf{DOp}_{\infty} \to \mathsf{PSh}(\mathbb{F})$ as the restriction of w^* to DOp_{∞}.

Let us give some intuition on the functor w.

- A 0-simplex of Fin_*, that is, an element of Fin_*, is given by the pointed set $\underline{s}_+$ for some $s \geqslant 0$, and we can identify it with the forest given by the disjoint union of s copies of the trivial tree η.

- A 1-simplex $\alpha\colon \Delta^1 \to \mathsf{Fin}_*$ corresponds to a disjoint union of corollas, whose input edges are given by the set of edges given by $\alpha(0)$ and the output edges are given by $\alpha(1)$. More precisely, let $\alpha\colon \underline{m}_+ \to \underline{n}_+$ be the morphism in Fin_* determined by the 1-simplex. The morphism α admits a essentially unique factorization as $\alpha = \alpha' \circ \alpha''$, where $\alpha'\colon \underline{k}_+ \to \underline{n}_+$ is active and $\alpha''\colon \underline{m}_+ \to \underline{k}_+$ is inert, and without loss of generality, we can assume that $\alpha''(i) = i$ if $i \leqslant k$, and $\alpha''(i) = *$ when $i \geqslant k+1$, and that $\alpha'(i) = \alpha(i)$ for all $i \leqslant k$. This means that the shape of the forest determined by α consists in k corollas and $m-k$ copies of η (i.e. the edge without any vertex).

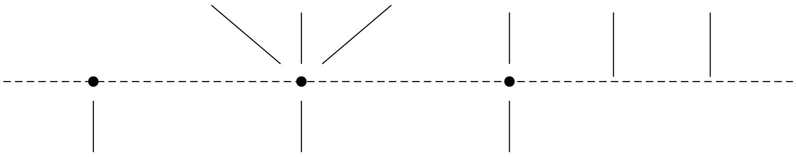


Figure 4.1: The forest of corollas determined by the 1-simplex $\alpha\colon \underline{6}_+ \to \underline{3}_+$ with $\alpha(1) = \alpha(2) = \alpha(3) = 2$, $\alpha(4) = 3$, $\alpha(5) = \alpha(6) = *$

- More generally, a n-simplex $\Delta^n \to \mathsf{Fin}_*$ consists in *a n-leveled forest*, where edges are decorated by objects of $\mathcal{P}$ and vertices by operations in $\mathcal{P}^{\otimes}$, and where being n-leveled means that there is at least a tree in the forest which has a maximal branch of length $n+1$ (equivalently, a maximal chain of vertices of length n).

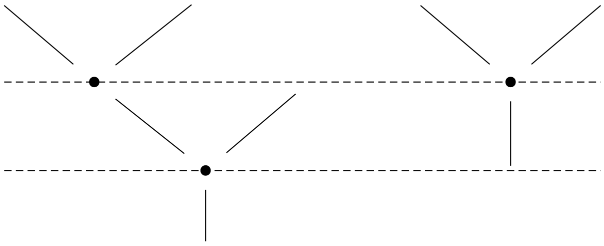


Figure 4.2: The forest determined by the 2-simplex $\underline{4}_+ \xrightarrow{\alpha_1} \underline{3}_+ \xrightarrow{\alpha_2} \underline{1}_+$ with $\alpha_1(1) = 1 = \alpha_1(2)$, $\alpha_1(3) = 3 = \alpha_1(4)$, $\alpha_2(1) = 1 = \alpha_2(2)$, $\alpha_2(3) = *$.

In order to define the functor $\delta\colon \ell\mathsf{Op}_\infty \to \mathsf{DOp}_\infty$, let $i\colon \Phi \hookrightarrow \ell\mathsf{Op}_\infty$ be the inclusion given by identifying Φ with a full subcategory of discrete operads and then applying the nerve functor $\mathcal{N}(-)^{\otimes}$, and let $\delta'\colon \ell\mathsf{Op}_\infty \to \mathsf{PSh}(\Phi)$ be the functor corresponding, under adjunction, to the functor $\mathsf{Map}_{\ell\mathsf{Op}_\infty}(-, i(-))\colon \ell\mathsf{Op}_\infty \times \Phi^{\mathrm{op}} \to \mathcal{S}$.

Theorem 4.2.13. *[HM24, Theorem 3.1.4] The functors λ' and δ' restrict to an adjoint equivalence of ∞-categories*

$$\delta\colon \ell\mathsf{Op}_\infty \underset{\perp}{\rightleftarrows} \mathsf{DOp}_\infty : \lambda.$$

Moreover, if P is a discrete operad, there is an equivalence of Lurie ∞-operads

$$\lambda(\mathcal{N}_D P) \simeq \mathcal{N}(P)^{\otimes},$$

where $\mathcal{N}_D P$ is the nerve of P.

Let us conclude this section with the following

Remark 4.2.14. A variant of the functor w introduced at the beginning of Section 4.2.3 already appears in [HHM16], where Heuts-Hinich-Moerdijk use it to construct an equivalence between *open* (i.e. without nullary operations) Lurie and dendroidal ∞-operads. In that context, the equivalence is relized as a zig-zag of Quillen equivalences of model categories. For this purpose, the category Φ is replaced by a non-full and non-wide subcategory ϕ, where objects are non-empty forests and morphisms are *independent* maps of forests[1], and one constructs a functor of 1-categories

$$\tilde{w} \colon \mathbb{F}_o \longrightarrow \mathbf{Fun}(\phi^{\mathrm{op}}, \mathbf{Sets})_o$$

from the subcategory of open simplices of Fin_* to that of open presheaves on ϕ. The action of $\tilde{w}$ on the simplices of Fin_* differs from w only for the constant simplices of Fin_*. Indeed, given a simplex A of Fin_*, if A is non-constant then $\tilde{w}(A)$ is representable and given by the forest $w(A)$; however, if A is constant, then $\tilde{w}(A)$ is no longer representable and is instead declared to be the empty presheaf. Then the authors consider the adjunction $(\tilde{w}_!, \tilde{w}^*)$ induced by the (1-categorical) left Kan extension of $\tilde{w}$, showing that the right adjoint yields a functor $\overline{w}^* \colon \mathbf{Fun}(\phi^{\mathrm{op}}, \mathbf{Sets})_o^+ \to \underline{0}_+/\mathbf{POp}$ from marked open presheaves on ϕ into open preoperads, which they prove to be a left Quillen equivalences (for appropriate model structures). The procedure is analogous to the one for defining λ in the ∞-categorical case. The need for the category ϕ instead of Φ and the asymmetry in the definition of $\tilde{w}$ is already discussed by the authors in [HHM16, Remark 5.1.1.]: they are needed in order to make $\overline{w}^*$ a left Quillen functor. Theorem 4.2.13 shows that such rigidity is no-longer necessary when working with ∞-categories, and one can hence work with the more uniform definition of w to show the equivalence (with the subtlety of switching from the model of dendroidal sets to the model of dendroidal spaces).

4.2.4 Equivalence between dendroidal and operadic left fibrations

We use the direct equivalence just illustrate to prove model-independence of operadic left fibrations. More precisely, we prove the following.

Theorem 4.2.15. *Let X be a dendroidal ∞-operad, let $\mathcal{O}^{\otimes}$ be a Lurie ∞-operad, with $\lambda X \simeq \mathcal{O}^{\otimes}$. The Hinich-Moerdijk comparison functor induces an adjunction*

$$\delta_{/} \colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \underset{\perp}{\rightleftarrows} \mathsf{DLeft}_X : \lambda_{/}$$

which is an equivalence of ∞-categories.

In order to prove this theorem, we need some standard facts about induced adjunctions between slice ∞-categories, which we now recall. Given a pair of adjoint functors $F \colon \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : G$ and an object y of $\mathcal{D}$, there is an induced pair of adjoint functors

$$\tilde{F} \colon \mathcal{C}_{/G(y)} \underset{\perp}{\rightleftarrows} \mathcal{D}_{/y} : G,$$

where the right adjoint consists in applying G, while the left adjoint consists in applying F and then postcomposing with the counit, that is, it is the composite

$$\mathcal{C}_{/G(y)} \longrightarrow \mathcal{D}_{/FG(y)} \xrightarrow{\epsilon_{y*}} \mathcal{D}_{/y}.$$

---

[1] This category will pop up in Chapter 6, see in particular Definition 6.3.1

If $\mathcal{C}$ has pullbacks, after [Lur09, Proposition 5.2.5.1] for any object x of $\mathcal{C}$ there is another pair of adjoint functors

$$F\colon \mathcal{C}_{/x} \underset{\perp}{\rightleftarrows} \mathcal{D}_{/F(x)} \colon G',$$

where this time the left adjoint consists in applying F, while the right adjoint is the base-change along the unit after having applied G, that is, it is the composite

$$\mathcal{D}_{/F(x)} \longrightarrow \mathcal{C}_{/GF(x)} \xrightarrow{\eta_x^*} \mathcal{C}_{/x}.$$

Proof. The equivalence (δ, λ) induces an equivalence on slices

$$\delta_/\colon \ell\mathsf{Op}_{\infty/\mathcal{O}^\otimes} \underset{\perp}{\rightleftarrows} \mathsf{DOp}_{\infty/X} \colon \lambda_/.$$

Recall that we denoted by $\mathcal{L}$ the set of morphisms in $\mathsf{PSh}(\Phi)$ given by

$$\mathcal{L} := \{\ell(F) \hookrightarrow F\}_{F\in\Phi},$$

and write $\mathcal{I}$ for the set of morphisms in $\mathsf{PSh}(\Delta)$ given by

$$\mathcal{I} := \{\Delta^0 \overset{\{0\}}{\hookrightarrow} \Delta^n\}_{n\geqslant 0}.$$

As dendroidal left fibrations over X are $\mathcal{L}$-local objects in $\mathsf{DOp}_{\infty/X}$ and operadic left fibrations over $\mathcal{O}^\otimes$ are objects in $\ell\mathsf{Op}_{\infty/\mathcal{O}^\otimes}$ which become $\mathcal{I}$-local after post-composing with the forgetful functor $\ell\mathsf{Op}_{\infty/\mathcal{O}^\otimes} \to \mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$, we only need to prove that, for any object $(Y, f) \in \mathsf{DOp}_{\infty/X}$ with $(\mathcal{D}^\otimes, \alpha^\otimes) = \lambda(Y, f)$, one has that

$$(Y, f) \text{ is } \mathcal{L}\text{-local if and only if } (\mathcal{D}^\otimes, \alpha^\otimes) \text{ is } \mathcal{I}\text{-local in } \mathsf{Cat}_{\infty/\mathcal{O}^\otimes}.$$

We go back to the adjunction between categories of presheaves given by $(w_!, w^*)$, where $w_!$ is the left Kan extension of w. Slicing over $\mathcal{O}^\otimes$, we obtain an adjunction

$$\tilde{w}_!\colon \mathsf{PSh}(\mathbb{F})_{/\mathcal{O}^\otimes} \underset{\perp}{\rightleftarrows} \mathsf{PSh}(\Phi)_{/X} \colon w^*,$$

where the left adjoint $\tilde{w}_!$ consists in applying $w_!$ and postcomposing with the counit, while the right adjoint consists in applying w^*.

Let $n \geqslant 0$ and fix $(n, p\colon \Delta^n \to \mathcal{O}^\otimes)$ in $\mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$. Because $\mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$ is a full subcategory of $\mathsf{PSh}(\mathbb{F})_{/\mathcal{O}^\otimes}$, one has

$$\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}^\otimes}}((\Delta^n, p), (\mathcal{D}^\otimes, \alpha^\otimes)) \simeq \mathsf{Map}_{\mathsf{PSh}(\mathbb{F})_{/\mathcal{O}^\otimes}}((\Delta^n, p), (\mathcal{D}^\otimes, \alpha)).$$

As $(\mathcal{D}^\otimes, \alpha^\otimes) \simeq \lambda(Y, f) \simeq w^*(Y, f)$, by adjunction we obtain

$$\mathsf{Map}_{\mathsf{PSh}(\mathbb{F})_{/\mathcal{O}^\otimes}}((\Delta^n, p), (\mathcal{D}^\otimes, \alpha)) \simeq \mathsf{Map}_{\mathsf{PSh}(\Phi)_{/X}}(\tilde{w}_!(\Delta^n, p), (Y, f)).$$

The object $\tilde{w}_!(\Delta^n, p)$ is of the form

$$(w(\overline{p}), q\colon w(\overline{p}) \to X),$$

where $\overline{p}$ is the n-simplex of Fin_* obtained as the composition $\Delta^n \xrightarrow{p} \mathcal{O}^\otimes \to \mathsf{Fin}_*$. In particular, $w(\overline{p})$ belongs to Φ, and hence to DOp_∞ as a representable complete dendroidal

Segal spaces; as X is a dendroidal ∞-operad and DOp_∞ is full in $\mathsf{PSh}(\Phi)$, one has that $\tilde{w}_!(\Delta^n, p)$ belongs to $\mathsf{DOp}_{\infty/X}$. In particular, one has the equivalence

$$\mathsf{Map}_{\mathsf{PSh}(\Phi)_{/X}}((w(\overline{p}), q), (Y, f)) \simeq \mathsf{Map}_{\mathsf{DOp}_{\infty/X}}((w(\overline{p}), q), (Y, f)).$$

Reasoning in the same way for the inclusion $i\colon \Delta^0 \xrightarrow{\{0\}} \Delta^n$ in $\mathcal{I}$, we obtain the commutative diagram

$$\begin{tikzcd}
\mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}^\otimes}}((\Delta^n, p), (\mathcal{D}^\otimes, \alpha^\otimes)) \arrow[r] \arrow[d, "\wr"'] & \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{O}^\otimes}}((\Delta^0, p \circ i, (\mathcal{D}^\otimes, \alpha^\otimes)) \arrow[d, "\wr"] \\
\mathsf{Map}_{\mathsf{DOp}_{\infty/X}}((w(\overline{p}), q), (Y, f)) \arrow[r] & \mathsf{Map}_{\mathsf{DOp}_{\infty/X}}((\ell(w(\overline{p})), q \circ i), (Y, f))
\end{tikzcd}$$

where the vertical arrows are equivalences. This shows that, if (Y, f) is $\mathcal{L}$-local, then $(\mathcal{D}^\otimes, \alpha^\otimes))$ is $\mathcal{I}$-local. After [HM24, Lemma 3.1.2], we know that any forest is a retract of a forest of the form $w(\overline{p})$, which shows that if $(\mathcal{D}^\otimes, \alpha^\otimes)$ is $\mathcal{I}$-local, then (Y, f) is $\mathcal{L}$-local.

We have hence shown that $(Y, f) \in \mathsf{DOp}_{\infty/X}$ is a dendroidal left fibration if and only if $(\mathcal{D}^\otimes, \alpha^\otimes) = \lambda(Y, f)$ is a operadic left fibration, and this concludes the proof. ☐

In particular, one can transfer properties invariant under equivalences from one ∞-category to the other. We hence deduce for instance another proof of Proposition 4.2.10.

Corollary 4.2.16 (Proposition 4.2.10)**.** *Let $\mathcal{O}^\otimes$ be a Lurie ∞-operad and consider a morphism of operadic left fibrations*

$$\begin{tikzcd}
\mathcal{T}^\otimes \arrow[rr, "f"] \arrow[dr, "\alpha^\otimes"'] & & \mathcal{Q}^\otimes \arrow[dl, "\beta^\otimes"] \\
& \mathcal{O}^\otimes &
\end{tikzcd}$$

Then f is an equivalence in $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$ if and only if, for any object c of $\mathcal{O}$, the map between fibres

$$f_c\colon (\mathcal{T}, \alpha)_c \longrightarrow (\mathcal{Q}, \beta)_c$$

is an equivalence of spaces.

Proof. The condition in the statement is invariant under equivalence of ∞-categories, and after Proposition 4.2.6 the condition holds for DLeft_X for any dendroidal ∞-operad X, hence in particular for $X \simeq \delta(\mathcal{O}^\otimes)$. The Segal condition for $\mathcal{O}^\otimes$ ensures that we can look at objects c of $\mathcal{O}$ insteaf of looking at the 'tuples' in $\mathcal{O}^\otimes_{\underline{n}_+}$, and this concludes. ☐

Also, by combining Theorem 4.2.15 with the already existing operadic un/straightening equivalences in Lurie's and the dendroidal model, we obtain the following invariance result on the ∞-categories of algebras over an ∞-operad.

Corollary 4.2.17. *Let $\mathcal{O}^\otimes$ be a Lurie's ∞-operad and X a dendroidal ∞-operad with $\mathcal{O}^\otimes \simeq \lambda X$. There is an equivalence of ∞-categories*

$$\mathsf{Alg}_{\mathcal{O}^\otimes}(\mathcal{S}) \simeq \mathsf{Alg}_X(\mathcal{S}).$$

Remark 4.2.18. As observed in Remark 3.3.7, a similar comparison result was proven by Pavlov-Scholbach ([PS14]) for those ∞-operads equivalent to the nerve of a simplicial operad.

Let us conclude with the following

Remark 4.2.19. Algebras over an operad are the main relevant examples of *left modules* over an operad. One may want to adopt a fibrational approach to study the homotopy theory of *right modules* over an operad, which naturally appear in the domain of embedding calculus[2]. Such approach has been adopted by Barata in [Bar24a], who proved that, for a closed Σ-free simplicial operad P, the homotopy theory of simplicial P-modules is equivalent to the one given by the *contravariant* model structure on forest sets over the nerve of P. In particular, this defines an ∞-category of *dendroidal right fibrations* for dendroidal ∞-operads. One may similarly define *operadic right fibrations* for Lurie's ∞-operads, analogously to Definition 4.2.7, prompting the question of whether the Hinich–Moerdijk equivalence also induce an equivalence between the ∞-categories of operadic right fibrations in the two formalisms. It is important to note that this question is not addressed by Theorem 4.2.15. In the case of ∞-categories, the existence of an order-reversing involution on Δ ensures a duality between left and right fibrations. However, no such involution exists for the dendroidal category Ω, rendering the theories of dendroidal left and right fibrations structurally distinct.

4.3 Unstraightening of strong monoidal functors

Our contribution for this section consists in Proposition 4.3.5, where we characterize the unstraightening of strong monoidal functors on a symmetric monoidal ∞-category. It crucially builds on the following result, which we refer to as the *monoidal un/straightening*.

Theorem 4.3.1 ([Hin15, Corollary A.2.2.]). *For any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$. The un/straightening adjunction for $\mathcal{C}^{\otimes}$ is an equivalence of ∞-categories*

$$\mathsf{St}^{\mathcal{C},\otimes}\colon \mathsf{smLeft}_{\mathcal{C}^{\otimes}} \underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\perp}} \mathsf{Fun}^{\mathsf{lax}}(\mathcal{C}^{\otimes}, \mathcal{S}^{\times}) : \mathsf{Unst}^{\mathcal{C},\otimes},$$

where the ∞-category $\mathsf{smLeft}_{\mathcal{C}^{\otimes}}$ is the ∞-category of Definition 4.3.2 and the functor $\mathsf{St}^{\mathcal{C},\otimes}$, resp. $\mathsf{Unst}^{\mathcal{C},\otimes}$, acts as the straightening, resp. unstraightening, functor of $\mathcal{C}$ on the underlying objects in both ∞-categories.

Definition 4.3.2. A *sm-left fibration* is a morphism $f\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ in smCat_{∞} such that Uf is a left fibration between ∞-categories, where $U\colon \mathsf{smCat}_{\infty} \to \mathsf{Cat}_{\infty}$ is the forgetful functor. We denote by $\mathsf{smLeft}_{\mathcal{C}}$ the full subcategory of $\mathsf{smCat}_{\infty/\mathcal{C}^{\otimes}}$ spanned by the sm-left fibrations over $\mathcal{C}^{\otimes}$.

In a recent work ([Ram22]), Ramzi proves that for any ∞-operad $\mathcal{O}$ there exists what he calls a *$\mathcal{O}$-monoidal Grothendieck construction* for $\mathcal{O}$-monoidal ∞-categories. In particular, Theorem 4.3.1 is just the shadow of a monoidal equivalence between symmetric monoidal ∞-categories, and we devote Section 4.3.1 to an explanation of this latter. It is a purely expository subsection, and while not strictly necessary to prove our contribution (for which the reader may directly jump to Section 4.3.2), it serves to contextualize the constructions used later.

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[2] In [BdBW13], Boavida-Weiss introduced an operadic approach via homotopy sheaves for the construction of the tower of approximation of embedding calculus. For this, one needs to compute the derived mapping spaces for operadic right modules.

4.3.1 Monoidal un/straightening, in context

Given an ∞-operad $\mathcal{O}^{\otimes}$, following [Lur17, Definition 2.1.2.13] one defines a *$\mathcal{O}^{\otimes}$-monoidal ∞-category* as an ∞-operads $\mathcal{P}^{\otimes}$ together with a map of ∞-operads $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ which is also a coCartesian fibration of ∞-categories. For instance, symmetric monoidal ∞-categories are precisely the Fin_*-monoidal ∞-categories.

Theorem 4.3.3 ([Ram22, Corollary C])**.** *Let $\mathcal{C}^{\otimes}$ be a $\mathcal{O}^{\otimes}$-monoidal ∞-category and $\mathcal{C}$ its underlying ∞-category. The ∞-categories $\mathsf{Left}_{\mathcal{C}}$ and $\mathsf{Fun}(\mathcal{C},\mathcal{S})$ inherit a $\mathcal{O}^{\otimes}$-monoidal structure, and that the un / straightening equivalence $\mathsf{St}^{\mathcal{C}}\colon \mathsf{Left}_{\mathcal{C}} \rightleftarrows \mathsf{Fun}(\mathcal{C},\mathcal{S}) : \mathsf{Unst}^{\mathcal{C}}$ can be enhanced to a $\mathcal{O}^{\otimes}$-monoidal equivalence of $\mathcal{O}^{\otimes}$-monoidal ∞-categories. In particular, there is an induced equivalence between the ∞-categories of $\mathcal{O}^{\otimes}$-algebras therein, which we write as follows*

$$\mathsf{Left}_{\mathcal{C}}^{\mathcal{O}-lax} \simeq \mathsf{Fun}^{\mathcal{O}-lax}(\mathcal{C},\mathcal{S}).$$

Rephrasing the above result for $\mathcal{O}^{\otimes} \simeq \mathsf{Fin}_*$, one gets that for any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the ∞-categories $\mathsf{Left}_{\mathcal{C}}$ and $\mathsf{Fun}(\mathcal{C},\mathcal{S})$ have symmetric monoidal structures, which on the latter is given by Day convolution ([Lur17, §2.2.6.]), and that the un/straightening equivalence for $\mathcal{C}$ can be enhanced to a monoidal equivalence of symmetric monoidal ∞-categories

$$(\mathsf{Left}_{\mathcal{C}})^{\otimes} \simeq \mathsf{Fun}(\mathcal{C},\mathcal{S})^{\otimes_{\mathrm{Day}}}. \tag{4.3.1}$$

One recovers Hinich's Theorem 4.3.1 by looking at the induced equivalence between the ∞-categories of commutative monoids; let us see how.

First of all, let us describe the symmetric monoidal ∞-category $(\mathsf{Left}_{\mathcal{C}})^{\otimes}$. Observe that, as $\mathcal{C}$ is a commutative monoid in Cat_{∞}, the ∞-category $\mathsf{Cat}_{\infty/\mathcal{C}}$ has also a symmetric monoidal structure, denoted by $(\mathsf{Cat}_{\infty/\mathcal{C}})^{\otimes}$; given objects $(\mathcal{D}_1,\gamma_1\colon \mathcal{D}_1 \to \mathcal{C}),\dots,(\mathcal{D}_n,\gamma_n),(\mathcal{D}_{\infty},\gamma_{\infty})$, there is an equivalence

$$\mathsf{Map}^{\beta}_{(\mathsf{Cat}_{\infty/\mathcal{C}})^{\otimes}}((\mathcal{D}_1,\gamma_1),\dots,(\mathcal{D}_n,\gamma_n);(\mathcal{D}_{\infty},\gamma_{\infty})) \simeq \mathsf{Map}_{\mathsf{Cat}_{\infty/\mathcal{C}^{\times n}}}\left((\prod_{i=1}^{n}\mathcal{D}_i,\prod_{i=1}^{n}\gamma_i),(\mathcal{C}^{\times n}\times_{\mathcal{C}}\mathcal{D}_{\infty},\Gamma)\right),$$

where Γ is the map in the cartesian diagram

$$\begin{tikzcd}
\mathcal{C}^{\times n}\times_{\mathcal{C}} D_{\infty} \arrow[rr] \arrow[d, "\Gamma"'] & & D_{\infty} \arrow[d, "\gamma"] \\
\mathcal{C}^{\times n} & \mathcal{C}^{\otimes}_{\underline{n}_+} \arrow[l, "\simeq"'] \arrow[r, "\beta_!"] & \mathcal{C}
\end{tikzcd}$$

After [Ram22, Corollary 4.3], the symmetric monoidal ∞-category $(\mathsf{Left}_{\mathcal{C}})^{\otimes}$ can be described as a non-full sub symmetric monoidal ∞-category of $(\mathsf{Cat}_{\infty/\mathcal{C}})^{\otimes}$. Indeed, given *left fibrations* (of ∞-categories) $(\mathcal{D}_1,\gamma_1),\dots,(\mathcal{D}_n,\gamma_n),(\mathcal{D}_{\infty},\gamma_{\infty})$, the fortgetful functor $(\mathsf{Left}_{\mathcal{C}})^{\otimes} \to (\mathsf{Cat}_{\infty/\mathcal{C}})^{\otimes}$ induces a morphism of spaces

$$\mathsf{Map}^{\beta}_{(\mathsf{Left}_{\mathcal{C}})^{\otimes}}((D_1,\gamma_1),\dots,(D_n,\gamma_n);(D_{\infty},\gamma)) \longrightarrow \mathsf{Map}^{\beta}_{\mathsf{Cat}_{\infty/\mathcal{C}^{\times n}}}\left((\prod_{i=1}^{n}\mathcal{D}_i,\prod_{i=1}^{n}\gamma_i),(\mathcal{C}^{\times n}\times_{\mathcal{C}}\mathcal{D}_{\infty},\Gamma)\right)$$

which identifies the former with the subspace of the latter spanned by the components correspoinding to functors $\prod_{i=1}^{n}\mathcal{D}_i \to \mathcal{D}_\infty$ lying over $\beta_! \colon \mathcal{C}^{\times n} \to \mathcal{C}$.

The equivalence in Equation (4.3.1) induces an equivalence between the ∞-categories of commutative monoids,

$$\mathsf{CAlg}((\mathsf{Left}_{\mathcal{C}})^{\otimes}) \simeq \mathsf{CAlg}(\mathsf{Fun}(\mathcal{C},\mathcal{S})^{\otimes}).$$

It is well known that commutative monoids for the Day convolution are *lax* monoidal functors: there is an equivalence of ∞-categories $\mathsf{CAlg}(\mathsf{Fun}(\mathcal{C},\mathcal{S})^{\otimes}) \simeq \mathsf{Fun}^{\mathrm{lax}}(\mathcal{C},\mathcal{S})$, under which the fortgetful functor consists in taking the functor between the underlying ∞-categories,

$$\mathsf{Fun}^{\mathrm{lax}}(\mathcal{C},\mathcal{S}) \longrightarrow \mathsf{Fun}(\mathcal{C},\mathcal{S}), \quad (F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{S}^{\times}) \mapsto (F\colon \mathcal{C} \to \mathcal{S}).$$

The description of the ∞-category of commutative monoids in $\mathsf{Left}_{\mathcal{C}}$ was already present in Hinich's work: it is given by the sm left fibrations over $\mathcal{C}^{\otimes}$.

Proposition 4.3.4 ([Hin15, Lemma A.2.3.]). *For any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, there is an equivalence of ∞-categories*

$$\mathsf{CAlg}((\mathsf{Left}_{\mathcal{C}})^{\otimes}) \simeq \mathsf{smLeft}_{\mathcal{C}}.$$

Under this equivalence, the forgetful functor sends a sm-left fibration to the left fibration between the underlying categories,

$$\mathsf{CAlg}((\mathsf{Left}_{\mathcal{C}})^{\otimes}) \simeq \mathsf{smLeft}_{\mathcal{C}} \to \mathsf{Left}_{\mathcal{C}}, \quad (\mathcal{O}^{\otimes},\alpha^{\otimes}\colon \mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}) \mapsto (\mathcal{O},\alpha\colon \mathcal{O} \to \mathcal{C}).$$

This is how one gets Theorem 4.3.1. We conclude this section by pointing out that cartesianity of the monoidal structure on $\mathcal{S}$ is crucial for Ramzi's monoidal Grothendieck construction. The reason can be seen already at the level of symmetric monoidal ∞-categories and essentially consists in the fact, proven in [Lur17, Proposition 2.4.1.7], that a lax monoidal functor $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{S}^{\otimes}$ can be described as a functor of ∞-categories $F\colon \mathcal{C}^{\otimes} \to \mathcal{S}$ in Cat_∞ satisfying the following Segal condition: for any $c \in \mathcal{C}^{\otimes}_{\underline{n}_+}$, $c \simeq c_1 \oplus \cdots \oplus c_n$, the inert maps ${\rho^i}_! \colon c \to c_i$ induce a weak homotopy equivalence

$$F(c_1 \oplus \cdots \oplus c_n) \xrightarrow{\sim} \prod_{i=1}^{n} F(c_i).$$

4.3.2 Unstraightening of strong monoidal functors

We now characterize, for any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the full subcategory of $\mathsf{smLeft}_{\mathcal{C}}$ whose monoidal straightening corresponds to *strong* monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{S}^{\times}$. Throughout this section, we denote an object of $\mathsf{Fun}^{\mathrm{lax}}(\mathcal{C},\mathcal{S})$ by $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{S}^{\times}$, while we write $F\colon \mathcal{C} \to \mathcal{S}$ for the induced functor $F^{\otimes}_{\underline{1}_+}$ between the underlying ∞-categories.

Proposition 4.3.5. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and let $(\mathcal{D}^{\otimes},\alpha^{\otimes})$ be an object of* $\mathsf{smLeft}_{\mathcal{C}}$. *The following conditions are equivalent:*

1. $(\mathcal{D}^{\otimes},\alpha^{\otimes}) \simeq \mathsf{Unst}^{\mathcal{C},\otimes}(F^{\otimes})$ *for a* strong *monoidal functor $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{S}^{\times}$;*

2. *for any $x \in \mathcal{C}^{\otimes}_{\underline{n}_+}$, $x \simeq x_1 \oplus \dots \oplus x_n$, the induced map between the fibres*

$$(\mathcal{D}^{\otimes}, \alpha^{\otimes})_{x_1 \oplus \dots \oplus x_n} \longrightarrow (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{x_1 \otimes \dots \otimes x_n}$$

induced by $\beta_! \colon x \to x_1 \otimes \dots \otimes x_n$ in $\mathcal{C}^{\otimes}$ is an equivalence.

Remark 4.3.6. By Proposition 4.2.9, the condition of Proposition 4.3.5 is equivalent to asking that the morphism

$$\prod_{i=1}^{n} (\mathcal{D}, \alpha)_{x_i} \xleftarrow{\simeq} (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{x_1 \oplus \dots \oplus x_n} \longrightarrow (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{x_1 \otimes \dots \otimes x_n}$$

is an equivalence.

Proof. Consider an element $(\mathcal{D}^{\otimes}, \alpha^{\otimes})$ in $\mathsf{smLeft}_{\mathcal{C}}$, and let $F^{\otimes} \in \mathsf{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{S})$ be such that $(\mathcal{D}^{\otimes}, \alpha^{\otimes}) \simeq \mathsf{Unst}^{\mathcal{C},\otimes}(F^{\otimes})$. The lax monoidal functor $F^{\otimes} \colon \mathcal{C}^{\otimes} \to \mathcal{S}^{\times}$ is *strong* if it preserves all coCartesian lifts. Since inert-active morphisms in Fin_* form a factorization system and $F^{\otimes}$ is lax monoidal, it is sufficient to check that $F^{\otimes}$ preserves coCartesian lifts of active morphisms, and it is actually enough to see that it preserves coCartesian lifts of $\beta \colon \underline{n}_+ \to \underline{1}_+$.

Let $x_1, \dots, x_n$ be objects of $\mathcal{C}$, let $\beta_! \colon x_1 \oplus \dots \oplus x_n \to x_1 \otimes \dots \otimes x_n$ be the coCartesian lift of β over $x_1 \oplus \dots \oplus x_n$, and consider $F^{\otimes}(\beta_!) \colon F^{\otimes}(x_1 \oplus \dots \oplus x_n) \to F(x_1 \otimes \dots \otimes x_n)$, where $F \colon \mathcal{C} \to \mathcal{S}$ is the functor between the underlying ∞-categories. Since $F^{\otimes}$ is lax, the projections induce an equivalence $F^{\otimes}(x_1 \oplus \dots \oplus x_n) \xrightarrow{\simeq} \prod_{i=1}^{n} F(x_i)$. Then $F^{\otimes}(\beta_!)$ is equivalent to the coCartesian lift in $\mathcal{S}^{\times}$ of β over $F(x_1) \times \dots \times F(x_n)$, if and only if the morphism

$$F(x_1) \times \dots \times F(x_n) \xleftarrow{\simeq} F^{\otimes}_{\underline{n}_+}(x_1 \oplus \dots \oplus x_n) \longrightarrow F(x_1 \otimes \dots \otimes x_n), \tag{4.3.2}$$

is an equivalence.

On the other hand, by taking the fibre over $x_1 \otimes \dots \otimes x_n$ of the n-fold algebra map of $(\mathcal{O}^{\otimes}, \alpha^{\otimes})$ we obtain the following map

$$(\mathcal{D}, \alpha)_{x_1} \times \dots \times (\mathcal{D}, \alpha)_{x_n} \xleftarrow{\simeq} (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{x_1 \oplus \dots \oplus x_n} \xrightarrow{\beta_!} (\mathcal{D}, \alpha)_{x_1 \otimes \dots \otimes x_n}. \tag{4.3.3}$$

Since $(\mathcal{D}^{\otimes}, \alpha^{\otimes}) \simeq \mathsf{Unst}^{\mathcal{C},\otimes}(F^{\otimes})$ and, for any object x of $\mathcal{C}$, there is an equivalence $(\mathcal{D}, \alpha)_x \simeq F(x)$, one can see that the morphisms in Equation (4.3.3) and Equation (4.3.2) are equivalent. This means that $F^{\otimes}$ is strong monoidal if and only if $(\mathcal{D}^{\otimes}, \alpha^{\otimes})$ satisfies the condition in (2), as wanted.

□

Definition 4.3.7. For a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, we write $\mathsf{smLeft}^{\mathsf{str}}_{\mathcal{C}}$ for the full ∞-subcategory of $\mathsf{smLeft}_{\mathcal{C}}$ spanned by those sm-left fibrations satisfying the condition of Proposition 4.3.5.

We now characterize equivalences in the category $\mathsf{smLeft}^{\mathsf{str}}_{\mathcal{C}}$ as those maps which are fibrewise equivalences.

Proposition 4.3.8. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and consider a morphism of sm-left fibrations*

$$\begin{tikzcd}
\mathcal{D}^{\otimes} \arrow[rr, "f^{\otimes}"] \arrow[dr, "\gamma^{\otimes}"'] & & \mathcal{E}^{\otimes} \arrow[dl, "\vartheta^{\otimes}"] \\
& \mathcal{C}^{\otimes} &
\end{tikzcd}$$

Then $f^{\otimes}$ is an equivalence in the ∞-category $\mathsf{smLeft}^{\mathsf{str}}_{\mathcal{C}^{\otimes}}$ if and only if, for any object x of $\mathcal{C}$, the map between fibres

$$f_x \colon (\mathcal{D}, \gamma)_x \longrightarrow (\mathcal{E}, \vartheta)_x$$

is an equivalence of spaces.

Proof. The ∞-category $\mathsf{smLeft}^{\mathsf{str}}_{\mathcal{C}}$ is a full subcategory of $\mathsf{smLeft}_{\mathcal{C}}$, hence it suffices to prove that f is an equivalence in this latter. As $\mathsf{smLeft}_{\mathcal{C}} \simeq \mathsf{CAlg}(\mathsf{Left}^{\otimes}_{\mathcal{C}})$, the equivalences in $\mathsf{smLeft}_{\mathcal{C}}$ are detected by the forgetful functor

$$\mathsf{CAlg}(\mathsf{Left}^{\otimes}_{\mathcal{C}}) \to \mathsf{Left}_{\mathcal{C}}, \quad (\mathcal{F}^{\otimes}, \eta^{\otimes}) \mapsto (\mathcal{F}, \eta),$$

and this implies that $f^{\otimes}$ is an equivalence if and only if $f \colon (\mathcal{D}, \gamma) \to (\mathcal{E}, \vartheta)$ is an equivalence in $\mathsf{Left}_{\mathcal{C}}$, which is true if and only if, for any $x \in \mathcal{C}$, the morphism between fibres

$$f_x \colon (\mathcal{D}, \gamma)_x \to (\mathcal{E}, \vartheta)_x$$

is an equivalence, and this concludes the argument. ☐

Remark 4.3.9. An analogous of Proposition 4.3.8 was pointed out, however without proof, in [BS24, Remark 2.2.12].

4.4 Strong sm-left fibrations

We now work with the symmetric monoidal envelope of an ∞-operad and the induced adjunction between slice categories; we recall the definitions in Section 4.4.1. In Section 4.4.2, we prove that strong sm-left fibrations are equivalent to operadic left fibrations in two steps: first, showing the symmetric monoidal envelope defines a fully faithful left adjoint in Lemma 4.4.6, then proving its essential image coincides with strong sm-left fibrations in Proposition 4.4.7, relying on Theorem 4.2.15.

4.4.1 Symmetric monoidal envelope

Let $\mathsf{Act}(\mathsf{Fin}_*)$ be the nerve of the full subcategory of $\mathsf{Fun}([1], \mathsf{Fin}_*)$ spanned by active morphisms. We recall the definition of Lurie monoidal envelope of an ∞-operad ([Lur17, §2.2.4]).

Definition 4.4.1. Let $\mathcal{O}^{\otimes} \to \mathsf{Fin}_*$ be an ∞-operad. We write $\mathsf{Env}(\mathcal{O})^{\otimes}$ for the fibre product

$$\mathcal{O}^{\otimes} \times_{\mathsf{Fun}(\{0\}, \mathsf{Fin}_*)} \mathsf{Act}(\mathsf{Fin}_*).$$

The target inclusion $\Delta^0 \xrightarrow{\{1\}} \Delta^1$ induces a map $\mathsf{Env}(\mathcal{O})^{\otimes} \to \mathsf{Fin}_*$, so $\mathsf{Env}(-)^{\otimes}$ defines a functor

$$\mathsf{Env}(-)^{\otimes} \colon \mathsf{Op}_{\infty} \longrightarrow \mathsf{Cat}_{\infty/\mathsf{Fin}_*}.$$

Denote by $\mathsf{Env}(\mathcal{O})$ the fibre of $\mathsf{Env}(\mathcal{O})^{\otimes} \to \mathsf{Fin}_*$ over $\underline{1}_+$.

The symmetric monoidal envelope of a ∞-operad can be characterized via the following universal property.

Proposition 4.4.2 ([Lur17, Proposition 2.2.4.9])**.** *The envelope functor* $\mathsf{Env}(-)^{\otimes}$ *takes values in the ∞-category of symmetric monoidal ∞-categories with strong monoidal functors and is left adjoint to the forgetful functors. In symbols,*

$$\mathsf{Env}(-)^{\otimes}\colon \ell\mathsf{Op}_{\infty} \underset{\perp}{\rightleftarrows} \mathsf{smCat}_{\infty} : U.$$

Equivalently, we can reformulate the above result by saying that, for any ∞-operad $\mathcal{O}^{\otimes}$ and any symmetric monoidal ∞-category $\mathcal{V}^{\otimes}$, there is a natural equivalence of ∞-categories

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{V}) \simeq \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}), \mathcal{V}).$$

Remark 4.4.3. Let $\mathsf{Env}(\mathcal{O})$ be the underlying ∞-category of $\mathsf{Env}(\mathcal{O})^{\otimes}$. There is an equivalence of ∞-categories

$$\mathsf{Env}(\mathcal{O}) \simeq \mathcal{O}^{\otimes, act},$$

where $\mathcal{O}^{\otimes, act}$ is the wide subcategory of $\mathcal{O}^{\otimes}$ spanned by active morphisms; in particular, an object of $\mathsf{Env}(\mathcal{O})$ corresponds to an element $\underline{c} \in \mathcal{O}^{\otimes}_{\underline{m}_+}$, that is, a list of objects of $\mathcal{O}$.

More generally, given $n \geqslant 0$, an object x in the fibre $\mathsf{Env}(\mathcal{O})^{\otimes}_{\underline{n}_+}$ writes as $x \simeq (\underline{c}, \alpha)$, with $\underline{c}$ in $\mathcal{O}^{\otimes}_{\underline{m}_+}$ for some $m \geqslant 0$ and α an active morphism $\alpha\colon \underline{m}_+ \to \underline{n}_+$. The Segal condition can be expressed in the following way: given elements $\underline{d}^1, \dots, \underline{d}^n$ in $\mathsf{Env}(\mathcal{O})$, $\underline{d}^i \in \mathcal{O}^{\otimes}_{\underline{m_i}_+}$ for some $\underline{m_i}'s$, the object $\underline{d}^1 \oplus \cdots \oplus \underline{d}^n$ in $\mathsf{Env}(\mathcal{O})^{\otimes}_{\underline{n}_+}$ corresponds to

$$\underline{d}^1 \oplus \cdots \oplus \underline{d}^n \simeq (\underline{d}, \alpha\colon \underline{m}_+ \to \underline{n}_+),$$

where $\underline{d} \simeq \underline{d}^1 \oplus \cdots \oplus \underline{d}^n$ as an element of $\mathcal{O}^{\otimes}_{\underline{m}_+} \simeq \prod_{i=1}^{n} \mathcal{O}^{\otimes}_{\underline{m^i}_+}$ and α appropriately partitions $\underline{m}_+$. If we regard an object of $\mathsf{Env}(\mathcal{O})$ as a list of objects of $\mathcal{O}$, the tensor product in the envelope consists in the concatenation of lists: given $\underline{d}^i$'s as above, we represent the tensor product of the $\underline{d}^i$'s by the morphism

$$\beta_!\colon \underline{d}^1 \oplus \cdots \oplus \underline{d}^n \to \underline{d}^1 \otimes \cdots \otimes \underline{d}^n$$

$$(\underline{d}, \alpha\colon \underline{m}_+ \to \underline{n}_+) \xrightarrow{(\mathrm{id}, \beta)} (\underline{d}, \beta\colon \underline{m}_+ \to \underline{1}_+).$$

Neither the symmetric monoidal envelope nor the forgetful functor of Proposition 4.4.2 are fully faithful. However, the functor $\mathsf{Env}(-)^{\otimes}$ becomes fully faithful when seen as a functor between over-∞-categories. The following proposition is an immediate consequence of [HK24, Proposition 2.4.3].

Proposition 4.4.4. *Let $\mathcal{O}^{\otimes}$ be a ∞-operad and the induced adjunction*

$$\mathsf{Env}(-)^{\otimes}_{/}\colon \ell\mathsf{Op}_{\infty/\mathcal{O}^{\otimes}} \underset{\perp}{\rightleftarrows} \mathsf{smCat}_{\infty/\mathsf{Env}(\mathcal{O})^{\otimes}} : G',$$

where G' is the base-change along the unit after having applied the forgetful functor. Then $\mathsf{Env}(-)^{\otimes}_{/}$ *is fully faithful.*

Remark 4.4.5. In [BHS22] we can find a characterization of the essential image of the sliced symmetric monoidal envelope. However, we so not need this result in the next sections.

4.4.2 Symmetric monoidal envelope of operadic left fibrations

Lemma 4.4.6. *For any ∞-operad $\mathcal{O}^{\otimes}$, the adjunction in Proposition 4.4.4 restricts to an adjunction*

$$\mathsf{Env}(-)^{\otimes}_{/}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \rightleftarrows \mathsf{smLeft}_{\mathsf{Env}(\mathcal{O})} : G.$$

Proof. The categories $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$, resp. $\mathsf{smLeft}_{\mathsf{Env}(\mathcal{O})}$, are full subcategories of $\ell\mathsf{Op}_{\infty/\mathcal{O}^{\otimes}}$, resp. $\mathsf{smCat}_{\infty/\mathsf{Env}(\mathcal{O})^{\otimes}}$, so we only need to see that the restrictions of $\mathsf{Env}(-)^{\otimes}_{/}$ and G' are well defined.

To be completely explicit, recall that, given a sm-left fibration $(\mathcal{T}^{\otimes}, \alpha^{\otimes})$, its image via G' is the left vertical arrow in the following pullback diagram:

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \times_{\mathsf{Env}(\mathcal{O})^{\otimes}} \mathcal{T}^{\otimes} \arrow[r] \arrow[d, "G'(\mathcal{T}^{\otimes}, \alpha^{\otimes})"'] & \mathcal{T}^{\otimes} \arrow[d, "\alpha^{\otimes}"] \\
\mathcal{O}^{\otimes} \arrow[r, "\iota_{\mathcal{O}}"'] & \mathsf{Env}(\mathcal{O})^{\otimes}
\end{tikzcd}$$

Let $(\mathcal{P}^{\otimes}, f)$ be an operadic left fibration. There is a commutative diagram

$$\begin{tikzcd}
\mathsf{Env}(\mathcal{P})^{\otimes} \arrow[r] \arrow[d, "\mathsf{Env}(f)^{\otimes}"'] & \mathcal{P}^{\otimes} \arrow[d, "f"] \\
\mathsf{Env}(\mathcal{O})^{\otimes} \arrow[r] \arrow[d] & \mathcal{O}^{\otimes} \arrow[d] \\
\mathsf{Act}(\mathsf{Fin}_{*}) \arrow[r] & \mathsf{Fin}_{*} ,
\end{tikzcd}$$

where the outer square and the lower square are cartesian. By the pasting lemma, the upper square is cartesian as well, and since left fibrations are stable under pullback, $\mathsf{Env}(f)^{\otimes}$ is also a left fibration. This means that $\mathsf{Env}(\mathcal{P}^{\otimes}, f^{\otimes}) \simeq (\mathsf{Env}(\mathcal{P})^{\otimes}, \mathsf{Env}(f)^{\otimes})$ belongs to $\mathsf{smLeft}_{\mathsf{Env}(\mathcal{O})^{\otimes}}$, as wanted.

Given a sm-left fibration $(\mathcal{T}^{\otimes}, \varphi^{\otimes})$ over $\mathsf{Env}(\mathcal{O})^{\otimes}$, the fact that left fibrations of ∞-categories are stable under pullback ensures that the morphism $G(\mathcal{T}^{\otimes}, \varphi^{\otimes})$ is also a left fibration in Cat_{∞}. As it was already an element in $\ell\mathsf{Op}_{\infty}$, we conclude that $G(\mathcal{T}^{\otimes}, \varphi^{\otimes})$ is an operadic left fibration. This concludes the proof. □

We refine the previous result, and showing that the symmetric monoidal envelope establish an equivalence between operadic left fibrations and *strong* sm-left fibrations.

Proposition 4.4.7. *For any ∞-operad $\mathcal{O}^{\otimes}$, the adjunction in Lemma 4.4.6 restricts to an adjunction of ∞-categories*

$$\mathsf{Env}(-)^{\otimes}_{/}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \rightleftarrows \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} : G.$$

Moreover, this adjunction is an equivalence.

Proof. Consider a operadic left fibration $(\mathcal{D}^{\otimes}, \alpha^{\otimes})$. To see that its image $(\mathsf{Env}(\mathcal{D})^{\otimes}, \mathsf{Env}(\alpha)^{\otimes})$ belongs to $\mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})}$, we apply Proposition 4.3.5 and check that, for any $\underline{c} \in \mathsf{Env}(\mathcal{O})^{\otimes}_{\underline{n}_+}$, $\underline{c} \simeq \underline{c}^1 \oplus \cdots \oplus \underline{c}^n$ for some $\underline{c}^i \in \mathcal{O}^{\otimes}_{\underline{m_i}_+} \subseteq \mathsf{Env}(\mathcal{O})$, the morphism

$$\beta_! \colon (\mathsf{Env}(\mathcal{D})^{\otimes}, \mathsf{Env}(\alpha)^{\otimes})_{\underline{c}^1 \oplus \cdots \oplus \underline{c}^n} \longrightarrow (\mathsf{Env}(\mathcal{D}), \mathsf{Env}(\alpha))_{\underline{c}^1 \otimes \cdots \otimes \underline{c}^n}$$

induced by the coCartesian lifts of $\beta_! \colon \underline{c}^1 \oplus \cdots \oplus \underline{c}^n \to \underline{c}^1 \otimes \cdots \otimes \underline{c}^n$ is an equivalence. Since the fibres of $\alpha^{\otimes} \colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ lie in $\mathcal{D}^{\otimes, act}$ and $\mathsf{Env}(\mathcal{D}) \simeq \mathcal{D}^{\otimes, act}$, for any $z \in \mathcal{O}^{\otimes}_{\underline{m}_+} \subseteq \mathsf{Env}(\mathcal{O})$ there is an equivalence

$$(\mathsf{Env}(\mathcal{D}), \mathsf{Env}(\alpha))_z \simeq (\mathcal{D}^{\otimes, act}, \alpha^{\otimes, act})_z \simeq (\mathcal{D}^{\otimes}, \alpha^{\otimes})_z.$$

By Proposition 4.2.9, the inert maps $\underline{c} \to \underline{c}^i$ in $\mathsf{Env}(\mathcal{O})^{\otimes}$ induce an equivalence

$$(\mathsf{Env}(\mathcal{D})^{\otimes}, \mathsf{Env}(\alpha)^{\otimes})_{\underline{c}^1 \oplus \cdots \oplus \underline{c}^n} \xrightarrow{\sim} \prod_{i=1}^{n} (\mathsf{Env}(\mathcal{D}), \mathsf{Env}(\alpha))_{\underline{c}^i} \simeq \prod_{i=1}^{n} (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{\underline{c}^i}.$$

As a consequence, $\beta_!$ fits into the commutative diagram

$$\begin{tikzcd}
(\mathsf{Env}(\mathcal{D})^{\otimes}, \mathsf{Env}(\alpha)^{\otimes})_{\underline{c}^1 \oplus \cdots \oplus \underline{c}^n} \arrow[rr, "\beta_!"] \arrow[d, "\sim"] & & (\mathsf{Env}(\mathcal{D}), \mathsf{Env}(\alpha))_{\underline{c}^1 \otimes \cdots \otimes \underline{c}^n} \arrow[d, "\simeq"] \\
\prod_{i=1}^{n} (\mathsf{Env}(\mathcal{D}), \mathsf{Env}(\alpha))_{\underline{c}^i} \arrow[r, "\simeq"] & \prod_{i=1}^{n} (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{\underline{c}_i} & (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{\underline{c}^1 \oplus \cdots \oplus \underline{c}^n} \arrow[l, "\psi"']
\end{tikzcd}$$

and $\beta_!$ is an equivalence if and only if ψ is, so let us prove this last fact. For every $i \in \{1, \ldots, n\}$, we can decompose $\underline{c}^i$ as an object in $\mathcal{O}^{\otimes}$ and write $\underline{c}^i \simeq c^i_{j_1} \oplus \cdots \oplus c^i_{j_i}$ for some $c^i_j \in \mathcal{O}$, and by Proposition 4.2.9 the coCartesian lifts of inerts in $\mathcal{O}$ induce an equivalence

$$(\mathcal{D}^{\otimes}, \alpha^{\otimes})_{\underline{c}^1 \oplus \cdots \oplus \underline{c}^n} \xrightarrow{\sim} \prod_{i,j} (\mathcal{D}, \alpha)_{c^i_j}.$$

This equivalence factors through ψ, written as the composition

$$(\mathcal{D}^{\otimes}, \alpha^{\otimes})_{\underline{c}^1 \oplus \cdots \oplus \underline{c}^n} \xrightarrow{\psi} \prod_{i=1}^{n} (\mathcal{D}^{\otimes}, \alpha^{\otimes})_{\underline{c}^i} \xrightarrow{\simeq} \prod_{i,j} (\mathcal{D}, \alpha)_{c^i_j},$$

where the second map is an equivalence again because of Proposition 4.2.9, so we conclude that ψ is an equivalence. This means that $G(\mathcal{D}^{\otimes}, \alpha^{\otimes})$ lies in $\mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})^{\otimes}}$, as wanted.

We have shown that the adjunction in the statement is well defined, so let us now prove that it is an equivalence of ∞-categories.

From [HK24, Proposition 2.4.3.], the symmetric monoidal envelope is a fully faithful functor when regarded as a functor $\ell\mathsf{Op}_{\infty} \simeq \ell\mathsf{Op}_{\infty/\mathsf{Comm}^{\otimes}} \longrightarrow \mathsf{smCat}_{\infty/\mathsf{Env}(\mathsf{Comm})^{\otimes}}$. In particular, its restriction $\mathsf{Env}(-)^{\otimes}_{/} \colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \longrightarrow \mathsf{smLeft}_{\mathsf{Env}(\mathcal{O})}$ is fully faithful as well.

By adjunction, to show that $\mathsf{Env}(-)^{\otimes}_{/}$ is also essentially surjective, it is enough to prove that G is conservative, that is, it reflects weak equivalences.

Consider a morphism of sm-left fibrations $f\colon X \to Y$ such that $G(f)\colon G(X) \to G(Y)$ is an equivalence. By Corollary 4.2.16, this is equivalent to asking that, for any $c \in \mathcal{O}^{\otimes}_{\underline{n}_+}$, for any $n \geqslant 0$, the map between the fibres $(G(f))_c\colon G(X)_c \to G(Y)_c$ is an equivalence of spaces. On the other hand, by Proposition 4.3.8, $f\colon X \to Y$ is an equivalence if and only if, for any object d of $\mathsf{Env}(\mathcal{O})$, the map of fibres $f_d\colon X_d \to Y_d$ is an equivalence of spaces. We conclude by observing that, for any $c \in \mathcal{O}^{\otimes}_{\underline{n}_+} \subseteq \mathsf{Env}(\mathcal{O})$, one has the equivalence $G(X)_c \simeq X_c$, and that any $d \in \mathsf{Env}(\mathcal{O})$ is of the form $d = c \in \mathcal{O}^{\otimes}_{\underline{n}_+}$ for some n. □

4.5 The un/straightening equivalence for ∞-operads

We have gathered everything we need to state and prove our straightening-unstraightening theorem for ∞-operads. Let us summarize the ∞-categories of left fibrations we have introduced so far:

- For a Lurie ∞-operad $\mathcal{O}^{\otimes}$, the ∞-category $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$ of operadic left fibrations over $\mathcal{O}^{\otimes}$ is the full sub ∞-category of $\ell\mathsf{Op}_{\infty/\mathcal{O}^{\otimes}}$ spanned by the elements $(\mathcal{D}^{\otimes}, p)$ such that $p\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ is a left fibration of ∞-categories.

  An object $(\mathcal{D}^{\otimes}, p)$ in $\mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ belongs to $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}}$ if and only if p satisfies the lax monoidality condition of Proposition 4.2.9.

- For a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the ∞-category $\mathsf{smLeft}_{\mathcal{C}}$ of sm left fibrations over $\mathcal{C}^{\otimes}$ is the full sub ∞-category of $\mathsf{smCat}_{\infty/\mathcal{C}}$ spanned by the elements $(\mathcal{D}^{\otimes}, p)$ for which p is a left fibration of ∞-categories. It is equivalent to the ∞-category of commutative monoids in $(\mathsf{Left}_{\mathcal{C}})^{\otimes}$.

- The ∞-category $\mathsf{smLeft}^{\mathsf{str}}_{\mathcal{C}}$ of strong sm left fibrations is the full sub ∞-category of $\mathsf{smLeft}_{\mathcal{C}}$ spanned by the sm left fibrations $(\mathcal{D}^{\otimes}, p)$ satisfying the condition in Proposition 4.3.8.

Also, recall that, after [Cis19], the unstraightening functor for an ∞-category $\mathcal{C}$ sends a copresheaf $F\colon \mathcal{C} \to \mathcal{S}$ to the left fibration

$$\mathsf{Unst}^{\mathcal{C}}(F) \simeq (\mathcal{S}_{\bullet/} \times_{\mathcal{S}} \mathcal{C}, \mathcal{S}_{\bullet/} \times_{\mathcal{S}} \mathcal{C} \to \mathcal{C} \xrightarrow{F} \mathcal{S}),$$

where $\mathcal{S}_{\bullet/} \to \mathcal{S}$ is the forgetful functor on pointed spaces, also called the *universal left fibration*. It enjoys the property that for any object c of $\mathcal{C}$, there is an equivalence of spaces $F(c) \simeq \mathsf{Unst}(F)_c$.

Theorem 4.5.1. *For any Lurie ∞-operad $\mathcal{O}^{\otimes}$, there is an equivalence of ∞-categories*

$$\begin{tikzcd} \mathsf{St}^{\mathcal{O}}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}) : \mathsf{Unst}^{\mathcal{O}}, \arrow[l, shift left] \end{tikzcd}$$

where the left adjoint is given by the composition

$$\mathsf{St}^{\mathcal{O}}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \xrightarrow{\mathsf{Env}(-)^{\otimes}_{/}} \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} \xrightarrow{\mathsf{St}^{\mathsf{Env}(-),\otimes}} \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}), \mathcal{S}) \simeq \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}).$$

Given a $\mathcal{O}^{\otimes}$-algebra F, i.e. a strong monoidal functor $F^{\otimes}\colon \mathsf{Env}(\mathcal{O})^{\otimes} \to \mathcal{S}^{\times}$, the operadic left fibration $\mathsf{Unst}^{\mathcal{O}}(F^{\otimes})$ is the pullback

$$\begin{tikzcd}
\mathcal{S}^{\times}_{\bullet/} \times_{\mathcal{S}^{\times}} \mathcal{O}^{\otimes} \arrow[r] \arrow[d, "\mathsf{Unst}^{\mathcal{O}}(F^{\otimes})"'] & \mathcal{S}^{\times}_{\bullet/} \arrow[d] \\
\mathcal{O}^{\otimes} \arrow[r] & \mathcal{S}^{\times}
\end{tikzcd}$$

with $\mathcal{S}^{\times}_{\bullet/} \to \mathcal{S}^{\times}$ the universal left fibration.

Proof. Consider first the adjunction given by the restriction to operadic left fibrations of the symmetric monoidal envelope: this yields an adjunction

$$\mathsf{Env}(-)^{\otimes}_{/}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \rightleftarrows \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} :G,$$

which we have shown in Proposition 4.4.7 to be an equivalence of ∞-categories. Since $\mathsf{Env}(\mathcal{O})^{\otimes}$ is a symmetric monoidal category, the straightening-unstraightening equivalence is monoidal, and in particular it induces an equivalence between commutative algebras, which acts as the straightening-unstraightening functor on the underlying objects. We consider the restriction of this latter to sm-left fibrations over the envelope of $\mathcal{O}^{\otimes}$ and strong monoidal functors out of $\mathsf{Env}(\mathcal{O})^{\otimes}$; we obtain another adjunction

$$\mathsf{St}^{\mathsf{Env}(\mathcal{O}),\otimes}\colon \mathsf{smLeft}^{\mathsf{str}}_{\mathsf{Env}(\mathcal{O})} \rightleftarrows \mathsf{Fun}^{\mathsf{str}}(\mathsf{Env}(\mathcal{O}),\mathcal{S}) \simeq \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}) :\mathsf{Unst}^{\mathsf{Env}(\mathcal{O}),\otimes},$$

which, thanks to Proposition 4.3.5, is an equivalence as well.

The straightening functor is the composition of the two left adjoints just mentioned, and the unstraightening functor is the composition of the right adjoints. This yields an adjunction

$$\mathsf{St}^{\mathcal{O}}\colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{O}} \rightleftarrows \mathsf{Alg}_{\mathcal{O}}(\mathcal{S}) :\mathsf{Unst}^{\mathcal{O}},$$

which is an equivalence, as wanted. $\square$

Chapter 5

Rectification of dendroidal left fibrations

In this chapter, we consider a discrete colored operad P, we construct an adjunction between the category of dendroidal sets over the nerve of P and the category of simplicial P-algebras. We prove that when P is Σ-free it establishes a Quillen equivalence with respect to the covariant model structure on the former category and the projective model structure on the latter. When $P = A$ is a discrete category, this recovers a Quillen equivalence previously established by Heuts-Moerdijk, of which we provide an independent proof.

To prove the constructed adjunction is a Quillen equivalence, we show that it presents the operadic un/straightening equivalence of Theorem 4.5.1. This involves proving that, for a discrete symmetric monoidal category, the Heuts-Moerdijk equivalence is a monoidal equivalence of monoidal model categories.

Contents

5.1 Introduction

In this chapter, we construct a un/straightening equivalence specifically for discrete operads – the operads whose spaces of operations are discrete, and hence carry no homotopical data.

Our main result is the following

Theorem (Corollary 5.5.3)**.** *Let P be a Σ-free discrete operad. There is a natural Quillen equivalence*

$$\rho_!^P : \mathbf{dSets}/\mathcal{N}_d P \underset{\perp}{\rightleftarrows} \mathsf{Alg}_P(\mathbf{sSets}) : \rho_P^*$$

between the covariant model structure on dendroidal sets over the dendroidal nerve of P and the projective model structure on simplicial P-algebras.

We call the left adjoint $\rho_!^P$ the *(operadic) rectification functor* for P.

The Quillen equivalence of Corollary 5.5.3, which, in the context of this thesis, will find its main application in Section 6.4, extends an analogous one constructed for discrete categories by Heuts-Moerdijk in [HM15]. In fact, we improve their result by showing that, when one considers a discrete symmetric monoidal structure, the above equivalence is a monoidal equivalence of monoidal model categories. More precisely:

Theorem (of Theorem 5.4.4)**.** *Let A be a discrete symmetric monoidal category. The Quillen pair*

$$\rho_!^A : \mathbf{sSets}/\mathcal{N}A \rightleftarrows \mathsf{Fun}(A, \mathbf{sSets}) : \rho_A^*$$

is a monoidal Quillen equivalence of Quillen monoidal model categories, with respect to the projective model structure and Day convolution on $\mathsf{Fun}(A, \mathbf{sSets})$ *and the covariant model structure and the $\boxtimes$-product (5.4.2) on* $\mathbf{sSets}/\mathcal{N}A$*.*

In particular, Theorem 5.4.4 gives a presentation of the monoidal Grothendieck construction of [Ram22, Corollary C] for *discrete* symmetric monoidal ∞-categories.

5.1.1 Strategy

The proof of the main result, Corollary 5.5.3 is carried on in two distinct steps.

The first step consists in constructing the adjunction, which we do by defining the left adjoint. Then we prove that the pair obtained is a Quillen adjunction. Consider a discrete operad P, non necessarily Σ-free.

Proposition (Theorem 5.3.16)**.** *There exists a pair of adjoint functors*

$$\rho_!^P : \mathbf{dSets}/\mathcal{N}_d P \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho_P^*,$$

and it is a Quillen adjunction between the covariant model structure on dendroidal sets over the nerve of P and the projective model structure on simplicial P-algebras.

Let us make the following remarks of the proof of Theorem 5.3.16:

- To construct $\rho_!^P$, we proceed via left Kan extension and impose naturality with respect to base change, so that we reduce to defining a simplicial T-algebra $\mathcal{A}^T = \rho^T(T, \mathrm{id}_T)$ for any tree T in $\mathbf{\Omega}$, regarded as the operad it generates. Given an edge e of T, the simplicial set $\mathcal{A}^T(e)$ is defined as the nerve of the poset of subtrees of T with root e, and the operadic action corresponds to grafting of trees.
- We deduce the description of the P-algebra $\rho_!^P(X, \alpha)$ when $X \simeq T$ is a tree; with this in hand, we are able to prove that ρ_P^* is right Quillen, which makes the pair $(\rho_!^P, \rho_P^*)$ a Quillen adjunction.

Showing that $(\rho_!^P, \rho_P^*)$ is a Quillen equivalence while still using the language of model categories suggests one needs to describe the simplicial P-algebra $\rho_!^P(X, \alpha)$ for any dendroidal set X. We start with the following observation.

Proposition. *Consider a tree T and an element (T, α) in $\mathbf{dSets}/\mathcal{N}_d P$. On an object c of P, the value of the simplicial P-algebra $\rho_!^P(T, \alpha)$ can be written as*

$$\rho_!^P(T, \alpha)(c) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/c},$$

where $\mathsf{Env}(-)$ *is the nerve of the symmetric monoidal envelope of a discrete operad.*

As any dendroidal set can be written as the colimit of the trees mapping into it, and as the rectification functor $\rho_!^P$ is a left adjoint, one has

$$\rho_!^P(X,\alpha) \simeq \mathsf{colim}_{T\to X}\, \rho_!^P(T,\alpha').$$

However, the above colimit of algebras is hard to compute, the main obstacle being the fact that (non sifted) colimits in the category $\mathsf{Alg}_P(\mathbf{sSets})$ are not computed objectwise -a phenomenon we do not observe in functor categories.

Instead, we directly study the left derived functor $\mathbb{L}\rho_!^P$ of $\rho_!^P$, or rather the functor $(\rho_!^P)_\infty$ between the underlying ∞-categories

$$(\rho_!^P)_\infty \colon \mathsf{dLeft}_{\mathcal{N}_d P} \longrightarrow \mathsf{Alg}_P(\mathcal{S})$$

induced by the rectification functor $\rho_!^P$, and prove that this is an equivalence of ∞-categories.

The key step consists in using the model-independence of operadic left fibrations with respect to Lurie's and the dendroidal models, proven in Theorem 4.2.15. Recall that this is an equivalence of ∞-categories

$$\lambda_/ \colon \mathsf{DLeft}_X \rightleftarrows \mathsf{Left}^{opd}_{\lambda(X)} : \delta_/$$

between the ∞-categories of operadic left fibrations in the formalism of complete dendroidal Segal spaces, on the left, and in Lurie's formalism, on the right. For an elment Z in dOp_∞, we can identify $\mathsf{DLeft}_{d_!(Z)}$ with dLeft_Z, and we write $\lambda'_/ \colon \mathsf{dLeft}_Z \to \mathsf{Left}^{opd}_{\lambda'(Z)}$ for the corresponding equivalence of ∞-categories. Observe that $\lambda(d_!(\mathcal{N}_d P)) \simeq \lambda(\mathcal{N}_D P) \simeq \mathcal{N}^{\otimes}(P)$, where this latter is Lurie's nerve of P. We then prove the following

Theorem (Theorem 5.5.2)**.** *Let P be a discrete* Σ-free *operad. There is a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathsf{dLeft}_{\mathcal{N}_d P} \arrow[r, "(\rho_!^P)_\infty"] \arrow[d, "\lambda'_/"'] & \mathsf{Alg}_P(\mathcal{S}) \\
\mathsf{Left}^{\mathsf{opd}}_{\mathcal{N}^{\otimes} P} \arrow[ur, "\mathsf{St}^P"'] &
\end{tikzcd},$$

where St^P is the operadic straightening functor for the Lurie's ∞-operad $\mathcal{N}^{\otimes} P$ appearing in Theorem 4.5.1.

In order to prove Theorem 5.5.2, we make use of Theorem 5.4.4 in the case the discrete category A is the symmetric monoidal envelope of P, which allows to describe the action of St^P as a fibre product (see Corollary 5.4.5). As in [Pra25c] the functor St^P is proved to be an equivalence of ∞-categories, we deduce that the left derived functor $\mathbb{L}\rho_!^P$ is an equivalence between the homotopy categories, which means that $(\rho_!^P, \rho_P^*)$ is a Quillen equivalence.

Besides, we also deduce an explicit description of the left derived functor $\mathbb{L}\rho_!^P$ on *any* dendroidal left fibration over $\mathcal{N}_d P$ (thing we could not do at the point-set level!). We record it in the following final

Corollary. *Let (X, α) be a dendroidal left fibration in $\mathbf{dSets}/\mathcal{N}_d P$. On an object c of P, the value of the simplicial P-algebra $\mathbb{L}\rho_!^P(X, \alpha)$ is weakly homotopy equivalent to the fibre product*

$$\mathbb{L}\rho_!^P(X, \alpha)(c) \simeq \mathsf{Env}(X) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/c},$$

where $\mathsf{Env}(X)$ denotes the underlying ∞-category of Lurie's symmetric monoidal envelope of (the Lurie ∞-operad equivalent to) X.

5.1.2 Outline

1. In Section 5.2, we construct the rectification functor $\rho_!^P$ (Section 5.2.1) and then give an explicit description of its right adjoint ρ_P^* (Section 5.2.2).

2. In Section 5.3, we prove that the pair $(\rho_!^P, \rho_P^*)$ forms a Quillen adjunction by showing that ρ_P^* is right Quillen (Theorem 5.3.16). More precisely, in Section 5.3.1 we relate root preserving dendroidal faces of a tree with chains in the linear order of posets of the tree with a fixed root. We employ this construction in the next two sections, where we give necessary conditions to produce lifts of morphisms of the form $\rho_P^*(F) \to \rho_P^*(G)$ against dendroidal boundary inclusions (Proposition 5.3.11) and inner and left horn inclusions (Proposition 5.3.15).

3. In Section 5.4, we focus on the case where $P = A$ is a discrete category. First, we prove that in this case we have a Quillen equivalence (Theorem 5.4.1), giving an alternative proof of Heuts-Moerdijk's equivalence. After that, we show that when A is symmetric monoidal, both model categories in Theorem 5.3.16 are *monoidal* model categories, and we prove that the Quillen equivalence is a monoidal equivalence of Quillen model categories (Theorem 5.4.4).

4. In Section 5.5 we come back to the case where P is a general discrete operad, proving that when P is moreover Σ-free, the rectification functor presents the straightening functor (Theorem 5.5.2), deducing the Quillen equivalence of Corollary 5.5.3.

5.1.3 Preliminaries and notation

The necessary background for Sections 5.2 through 5.4 can be found in Section 3.6, and we follow the notation for dendroidal sets introduced in Section 3.7.

For Section 5.5, we work within the framework of ∞-categories, relying on the material presented in Section 3.1 and Section 3.8. We also adopt the notation for ∞-categories established in Section 3.2.

In addition to that, we also introduce the following notation.

- Given a tree S with set of leaves given by $\{l_1, \ldots l_n\}$, and $T_1, \ldots, T_n$ are trees, we write $S \underset{l_1,\ldots,l_m}{\circ} (T_1, \ldots, T_n)$, or sometimes, by an abuse of notation, just $S \circ (T_1, \ldots, T_n)$ the tree obtained by grafting each T_i onto the leaf l_i of S.

- Given an edge e of T, write $T_e^\uparrow$ for the biggest subtree of T having e as root. Observe that its set of leaves is contained in that of T.

- For edges $\underline{e} = (e_1, \dots, e_n)$, we write $T^{e}_{\underline{e}}$ for the subtree of $T^{\uparrow}_{e}$ whose root is e and whose leaves are precisely $\underline{e}$, if it exists. The subtree $T^{e}_{\underline{e}}$ does not depend on the ordering of the tuple $\underline{e}$. When it exists, we identify $T^{e}_{\underline{e}}$ with the corresponding n-ary operation of the operad T. In particular, given a T-algebra F, we will write

  $$(T^{e}_{\underline{e}})_* \colon F(e_1) \times \cdots \times F(e_n) \to F(e)$$

  for its action on F.

- In particular, a tree T can denote:
  - the element T in $\mathbf{\Omega}$;
  - the discrete operad $\Omega(T)$;
  - the representable dendroidal set $\Omega[T]$;
  - an operation in a tree R, with $T \subseteq R$.

5.2 Construction of the adjunction

Fix a discrete colored operad P. In this section, we will define an adjunction, natural in P, of the form

$$\rho^P_! \colon \mathbf{dSets}/\mathcal{N}_d P \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) \colon \rho^*_P,$$

and we give explicit descriptions of both the left and right adjoint.

5.2.1 The definition

Any slice category of a presheaf category is itself a presheaf category: in particular, for any dendroidal set X there is an isomorphism

$$\mathbf{dSets}/X \simeq \mathsf{Fun}((\mathbf{\Omega}/X)^{\mathrm{op}}, \mathsf{Set}).$$

The category $\mathbf{\Omega}/X$ is the category of elements of X: its objects are of the form (T, α), where $T \in \mathbf{\Omega}$ and $\alpha \colon T \to X$ is a morphism of dendroidal sets, and a morphism $f \colon (T, \alpha) \to (S, \beta)$ is a map of trees $f \colon T \to S$ making the obvious triangle commute.

When $X \simeq \mathcal{N}_d P$, fully faithfulness of the nerve ensures that a morphism of dendroidal sets $T \to \mathcal{N}_d P$ is just a morphism of operads $T \to P$, so let us write simply $\mathbf{\Omega}/P$ for $\mathbf{\Omega}/\mathcal{N}_d P$, and refer to the objects in $\mathbf{\Omega}/P \overset{\mathcal{Y}}{\hookrightarrow} \mathbf{dSets}/\mathcal{N}_d P$ as the *representables* of $\mathbf{dSets}/\mathcal{N}_d P$.

Strategy. We define $\rho^P_!$ as the left Kan extension of a functor $\rho^P \colon (\mathbf{\Omega}/\mathcal{N}_d P)^{\mathrm{op}} \to \mathsf{Alg}_P(\mathbf{sSets})$, where $\mathbf{\Omega}/\mathcal{N}_d P$ is the category of elements of the presheaf $\mathcal{N}_d P$. To define the functor ρ^P, we require that the family of functors ${\rho^P_!}_P$ is natural with respect to base change of operads, which means that, for any map of operads $\varphi \colon P \to Q$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{dSets}/\mathcal{N}_d P \arrow[r, "\rho^P_!"] \arrow[d, "(\mathcal{N}_d f)_!"'] & \mathsf{Alg}_P(\mathbf{sSets}) \arrow[d, "f_!"] \\
\mathbf{dSets}/\mathcal{N}_d Q \arrow[r, "\rho^Q_!"] & \mathsf{Alg}_Q(\mathbf{sSets}) .
\end{tikzcd} \tag{5.2.1}$$

These two requirements reduce the definition of $\rho^P_!$ to the construction of simplicial T-algebras $\mathcal{A}^T$ for any tree T and functorially in T, and it goes as follows.

Construction 5.2.1. Fix a tree T. For any edge e of T, let $P(T_e^{\uparrow})$ be the set of subtrees of $T_e^{\uparrow}$ having e as root. The elements of $P(T_e^{\uparrow})$ are of the form $T_e^{\underline{e}}$ for edges $\underline{e} = (e_1, \dots, e_n)$ for which $T(e_1, \dots, e_n; e) \neq \varnothing$. We endow $P(T_e^{\uparrow})$ with the partial order given by reversed inclusion of subtrees: there is an arrow $R \to S$ in $P(T_e^{\uparrow})$ if and only if $S \subseteq R$.

Let e be an edge of T, and define $\mathcal{A}^T(e)$ as the nerve of the poset $P(T_e^{\uparrow})$, i.e.

$$\mathcal{A}^T(e) := \mathcal{N}(P(T_e^{\uparrow})).$$

For any choice of edges $e_1, \dots, e_n, e$ of T for which $T(e_1, \dots, e_n; e) \neq \varnothing$, we define the morphism

$$\mathcal{A}^T(e_1) \times \cdots \times \mathcal{A}^T(e_n) \longrightarrow \mathcal{A}^T(e)$$

as the nerve of the map of posets

$$P(T_{e_1}^{\uparrow}) \times \cdots \times P(T_{e_n}^{\uparrow}) \longrightarrow P(T_e^{\uparrow}) : (R_1, \dots, R_n) \mapsto T_e^{\underline{e}} \circ_{\underline{e}} (R_1, \dots, R_n),$$

which sends a n-tuple of subtrees $(R_1, \dots, R_n)$, with respective roots $\underline{e} = (e_1, \dots, e_n)$, to the tree obtained as the grafting of $(R_1, \dots, R_n)$ onto $T_e^{\underline{e}}$. Since operadic composition in T is precisely the grafting of trees and a map of trees sends subtrees to subtrees, respecting the inclusion relation, this endows the family $\mathcal{A}^T := \{\mathcal{A}^T(e)\}_e$ with a simplicial T-algebra structure.

The construction is natural in T in the following sense. Given a map of trees $f\colon S \to T$ and an edge e of S, there is a morphism of simplicial sets $f_e^{\mathcal{A}}\colon \mathcal{A}^S(e) \to \mathcal{A}^T(f(e)) \simeq f^*(\mathcal{A}^T)(e)$ which sends a subtree to its image via f. These maps are compatible with the operadic composition, as one has the equality

$$f_e^{\mathcal{A}}\big(S_e^{\underline{e}} \circ_{\underline{e}} (R_1, \dots, R_n)\big) = S_{f(e)}^{f(\underline{e})} \circ_{f(\underline{e})} \big(f_{e_1}^{\mathcal{A}}(R_1), \dots, f_{e_n}^{\mathcal{A}}(R_n)\big)$$

for any choice of edges $\underline{e} = (e_1, \dots, e_n), e$ of S for which $S(e_1, \dots, e_n; e) \neq \varnothing$. It follows that the family $f^{\mathcal{A}} := \{f_e^{\mathcal{A}}\}_e$ assembles into a morphism of S-algebras $f^{\mathcal{A}}\colon \mathcal{A}^S \to f^*(\mathcal{A}^T)$. Under the adjunction $(f_!, f^*)$, the transpose of $f^{\mathcal{A}}$ yields a morphism of T-algebras

$$f_*^{\mathcal{A}}\colon f_!(\mathcal{A}^S) \longrightarrow \mathcal{A}^T.$$

We are ready to define the functor whose left Kan extension will determine $\rho_!^P$.

Definition 5.2.2. Let P be a discrete operad. We define a functor

$$\rho^P : \mathbf{\Omega}/P \longrightarrow \mathsf{Alg}_P(\mathbf{sSets})$$

as follows: for an object (T, α) in $\mathbf{\Omega}/P$, we set

$$\rho^P(T, \alpha) := \alpha_!(\mathcal{A}^T).$$

Given another object (S, β) and a morphism $h\colon (T, \alpha) \to (S, \beta)$ in $\mathbf{\Omega}/P$, the map $\rho^P(h) = h_*$ is defined via base change along β, that is

$$h_* := \beta_!(h_*^{\mathcal{A}})\colon \alpha_!(\mathcal{A}^T) \longrightarrow \beta_!(\mathcal{A}^S).$$

Observe that any morphism of trees $f\colon S \to T$ can be regarded as a morphism $f\colon (S, f) \to (T, \mathrm{id}_T)$ in $\mathbf{\Omega}/T$, in which case we have $f_* = f_*^{\mathcal{A}}$.

Definition 5.2.3. Let P be a discrete colored operad. The *rectification functor* of P is the functor

$$\rho_!^P \colon \mathbf{dSets}/\mathcal{N}_d P \to \mathsf{Alg}_P(\mathbf{sSets})$$

obtained as the left Kan extension of $\rho^P \colon \mathbf{\Omega}/P \to \mathsf{Alg}_P(\mathbf{sSets})$ along the Yoneda embedding $\mathbf{\Omega}/P \to \mathbf{dSets}/\mathcal{N}_d P$.

The adjoint functor theorem ensures the existence of an adjunction

$$\rho_!^P \colon \mathbf{dSets}/\mathcal{N}_d P \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho_P^*. \tag{5.2.2}$$

Definition 5.2.4. The *relative dendroidal nerve functor* of a discrete operad P is the functor $\rho_P^* \colon \mathsf{Alg}_P(\mathbf{sSets}) \to \mathbf{dSets}/\mathcal{N}_d P$ right adjoint of the rectification functor.

Observe that by the definition of $\rho_!^P$, it is easy to describe the algebras given by the rectification of the representables in $\mathbf{dSets}/\mathcal{N}_d P$.

First of all, by construction, for any tree T the rectification of the identity morphism is the simplicial T-algebra $\mathcal{A}^T$,

$$\rho_!^T(T, \mathrm{id}_T) = \mathcal{A}^T.$$

More generally, we can describe the image of $\rho_!^P$ on a representable (T, α) of $\mathbf{dSets}/\mathcal{N}_d P$ more explicitly, as by inspection we observe that there is a canonical isomorphism

$$\rho_!^P(T, \alpha) \simeq \mathcal{N}(\alpha/c),$$

where $\mathcal{N}(\alpha/c)$ is the nerve of what can be seen as an 'operadic slice', defined as follows.

Definition 5.2.5. Given a morphism $\alpha \colon T \to P$ and an object c of P, we define $\mathcal{N}(\alpha/c)$ as the nerve of the poset α/c, whose objects are pairs $(\underline{e}, z)$, with $\underline{e}$ a tuple of edges of T and z an operation in $P(\alpha(\underline{e}); c)$, and where there is an arrow

$$((\underline{e}^1, \ldots, \underline{e}^n), z') \to (\underline{e}, z)$$

if and only if

$$z' = z \circ (\alpha(T_{e_1}^{\underline{e}^1}), \ldots, \alpha(T_{e_n}^{\underline{e}^n})).$$

Remark 5.2.6. The simplicial set $\mathcal{N}(\alpha/c)$ has the homotopy type of the union of the discrete multi-hom sets $P(\alpha(\ell_1), \ldots, \alpha(\ell_m); c)$, where $(\ell_1, \ldots, \ell_m)$ ranges over tuples of leaves of T. Indeed, for any edge e of T there exist unique leaves $\underline{\ell} = (\ell_1, \ldots, \ell_m)$ such that $T(\ell_1, \ldots, \ell_m; e) \neq \varnothing$. In particular, any tuple of edges $\underline{e} = (e_1, \ldots, e_n)$ determines a tuple of leaves $(\underline{\ell}^1, \ldots, \underline{\ell}^n)$, with $T(\underline{\ell_i}; e_i) = \{T_{e_i}^{\underline{\ell}_i}\} \neq \varnothing$ for every $i \in \{1, \ldots, n\}$. It follows that, for every element $(\underline{e}, z)$ of $\mathcal{N}(\alpha/c)$ one has

$$(\underline{\ell}^1, \ldots, \underline{\ell}^n, z') \to (\underline{e}, z),$$

with $z' := z \circ (\alpha(T_{e_1}^{\underline{\ell}^1}), \ldots, (\alpha(T_{e_n}^{\underline{\ell}^n})) \in P(\alpha(\ell^1), \ldots, \alpha(\ell^n); c)$.

Remark 5.2.7. If T is linear, that is, $T \simeq [n] \in \Delta$, for any object $i \in [n]$ there is an isomorphism $\mathcal{A}^{[n]}(i) \simeq \Delta^i$, natural in i. In particular, given a discrete category A, the rectification functor $\rho_!^A$ of Definition 5.2.2 coincides with Heuts-Moerdijk rectification functor $r_!^A \colon \mathbf{sSets}/\mathcal{N}A \to \mathsf{Fun}(A, \mathbf{sSets})$ defined in [HM15, §4]. By essential uniqueness of right adjoints, we can identify the functor ρ_A^* with Lurie's relative nerve functor ρ_A^* ([Lur09, §3.2.5]).

The poset α/c admits a nice description in terms of a fibred product of the symmetric monoidal envelopes of the discrete operads T and P. Recall the definition of this latter.

Definition 5.2.8. The *(discrete) symmetric monoidal envelope* of a discrete operad P is the symmetric monoidal category $(\mathsf{env}(P), \boxplus, \varnothing)$ defined as follows:

- the objects are given by the strings $(c_1, \dots, c_n)$ of objects of P with $n \geqslant 0$, where the empty string is represented by $\varnothing$;
- a morphism $(c_1, \dots, c_n) \to (d_1, \dots, d_m)$ is given by a pair $(f, \{p_j\}_{j=1}^m)$, where $f\colon \{1, \dots, n\} \to \{1, \dots, m\}$ is a function of finite sets and p_j is an operation in $P(\{c_i\}_{f(i)=j}; d_j)$. Composition of morphisms is induced by the operadic composition of operations of P.
- The tensor product $\boxtimes$ is the concatenation of sequences, and the unit is the empty sequence.

We denote by $\mathsf{Env}(P)$ the nerve of the underlying category of $\mathsf{env}(P)$.

The following is a simple observation that will be useful in Section 5.5.

Lemma 5.2.9. *Let (T, α) be a representable dendroidal set over $\mathcal{N}_d P$. There is an isomorphism of simplicial P-algebras*

$$c \mapsto \rho_!^P(T, \alpha)(c) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/c},$$

where the P-algebra structure on $\mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/-}$ is induced by operadic composition of P and the symmetric monoidal structure of $\mathsf{env}(T)$.

5.2.2 The relative dendroidal nerve functor

Consider a simplicial P-algebra F. The object $\rho_P^*(F)$ is a presheaf on $\mathbf{\Omega}/P$, henceforth to describe it it suffices to describe its values on the elements (T, α) in $\mathbf{\Omega}/P$. Fix one of such. By definition, one has

$$\rho_P^*(F)_{(T,\alpha)} := \mathrm{Hom}_{\mathbf{dSets}/\mathcal{N}_d P}((T, \alpha), \rho_P^*(F)) \simeq \mathrm{Hom}_{\mathsf{Alg}_P}(\rho_!^P(T, \alpha), F).$$

Unwinding the definition, one has

$$\rho_P^*(F)_{(T,\alpha)} \simeq \mathsf{Hom}_{\mathsf{Alg}_P(\mathbf{sSets})}(\alpha_!(\mathcal{A}^S), F) \simeq \mathsf{Hom}_{\mathsf{Alg}_S(\mathbf{sSets})}(\mathcal{A}^S, \alpha^* F).$$

The elements in $\rho_P^*(F)_{(T,\alpha)}$ can be described as follows.

Proposition 5.2.10. *The following data are equivalent:*

1. *An element ${}^t\chi$ in $(\rho^* F)_{(T,\alpha)}$.*
2. *A collection*

$$(\gamma_u\colon \Delta^k \to F(\alpha(e)))_{u,e,k}$$

where e ranges over the edges of T, u over the non-degenerate k-simplices $u\colon \Delta^k \to \mathcal{A}^T(e)$, k ranges over natural numbers and the collection has to satisfy the following compatibility condition*: whenever $T(e_1, \dots, e_n; e) \neq \varnothing$, for any face map $v\colon \Delta^k \to \Delta^{k'}$ fitting into a commutative square as the one on the left, the induced diagram on the right also commutes:*

$$\begin{tikzcd}
\Delta^k \arrow[r, "(u_1,\dots,u_n)"] \arrow[d, "v"'] & \mathcal{A}^T(e_1) \times \cdots \times \mathcal{A}^T(e_n) \arrow[d, "(T^{e}_{\underline{e}})_*"] \\
\Delta^{k'} \arrow[r, "u"] & \mathcal{A}^T(e)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^k \arrow[r, "(\gamma_{u_1},\dots,\gamma_{u_n})"] \arrow[d, "v"'] & F(\alpha(e_1)) \times \cdots \times F(\alpha(e_n)) \arrow[d, "(T^{e}_{\underline{e}})_*"] \\
\Delta^{k'} \arrow[r, "\gamma_u"] & F(\alpha(e))
\end{tikzcd}$$

Proof. Suppose to have the data in (1). By adjunction, the element ${}^t\chi$ corresponds to a map of P-algebras $\chi\colon \rho_!^P(T,\alpha) \to F$. Consider the morphism of T-algebras

$$\eta\colon \mathcal{A}^T \to \alpha^*\alpha_!\mathcal{A}^T$$

given by the unit of the adjunction $(\alpha_!, \alpha^*)$ evaluated at $\mathcal{A}^T$. It consists of maps of simplicial sets $\eta_e\colon \mathcal{A}^T(e) \to \mathcal{N}(\alpha/\alpha(e))$ for every edge e of T which are natural with respect to multimorphisms in P. Given an edge e of T, a natural number $k \geqslant 0$ and a non-degenerate k-simplex $u\colon \Delta^k \to \mathcal{A}^T(e)$, we define γ_u as the composition

$$\Delta^k \xrightarrow{u} \mathcal{A}^T(e) \xrightarrow{\eta_e} \mathcal{N}(\alpha/\alpha(e)) \xrightarrow{\chi_{\alpha(e)}} F(\alpha(e)).$$

It is straightforward to check that the compatibility condition holds, since η and $\alpha^*(\chi) = \chi_{\alpha(-)}$ are maps of algebras.

Let us now prove that (2) implies (1), and suppose to be given a collection $\{\gamma_u\}_{u,e,k}$ as in point (2). We construct the morphism of simplicial P-algebras $\chi\colon \rho_!(T,\alpha) \to F$ by constructing the maps of simplicial sets $\{\chi_c\colon \rho_!(\alpha)_c \to F_c\}_{c \in C(P)}$ by induction on the simplices, checking, at the ℓ-th step, that we have constructed a map of P-algebras in ℓ-truncated simplicial sets, that is, that for any choice of objects $c_1, \dots, c_m; c$ and operation $w \in P(c_1, \dots, c_m; c)$, the following diagram in $\mathbf{sSets}_{\leqslant \ell}$ commutes:

$$\begin{tikzcd}
\mathcal{N}(\alpha/c_1)_{\leqslant\ell} \times \cdots \times \mathcal{N}(\alpha/c_m)_{\leqslant\ell} \arrow[r, "\chi_{c_1}\times\cdots\times\chi_{c_m}"] \arrow[d, "w_*"'] & F(c_1)_{\leqslant\ell} \times \cdots \times F(c_m)_{\leqslant\ell} \arrow[d, "w_*"] \\
\mathcal{N}(\alpha/c)_{\leqslant\ell} \arrow[r, "\chi_c"'] & F(c)_{\leqslant\ell}
\end{tikzcd} \tag{5.2.3}$$

Let $\ell = 0$, let c be a object of P and consider an element $(\underline{e}, z)$ in $\mathcal{N}(\alpha/c)_0$, with $\underline{e} = (e_1, \dots, e_n)$ and $z \in P(\alpha(e_1), \dots, \alpha(e_n); c)$. For every $i \in \{1, \dots, n\}$, let $e_i\colon \Delta^0 \to \mathcal{A}^T(e_i)$ be the morphism selecting $\eta = e_i$. Since F is a P-algebra, we have a map $z_*\colon F(\alpha(e_1)) \times \cdots \times F(\alpha(e_n)) \to F(c)$, and set

$$\chi_c(\underline{e}, z) = z_* \circ (\gamma_{e_1}, \dots, \gamma_{e_n})\colon \Delta^0 \longrightarrow F(c).$$

This defines a map of sets

$$(\chi_c)_0\colon \mathcal{N}(\alpha/c)_0 \to (F_c)_0.$$

To check commutativity of Equation (5.2.3), consider $w \in P(c_1, \dots, c_m; c)$, and elements $z_i \in P(\alpha(e^i_1), \dots, \alpha(e^i_{k_i}); c_i)$, $i = 1, \dots, m$; commutativity is proven by the following computation

$$w_*(\chi_{c_1}(\underline{e}^1, z_1), \dots, \chi_{c_m}(\underline{e}^m, z_m)) = w_* \circ ((z_1)_*(\gamma_{e^1_1}, \dots, \gamma_{e^1_{k_1}}), \dots, (z_m)_*(\gamma_{e^m_1}, \dots, \gamma_{e^m_{k_m}})) =$$

$$= (w \circ (z_1, \dots, z_m))_*(\gamma_{e^1_1}, \dots, \gamma_{e^m_{k_m}}) = \chi_c((\underline{e}^1, \dots, \underline{e}^m, w_*(z_1, \dots, z_m))),$$

where the last equality holds because of the compatibility condition.

Let $\ell = 1$ and consider a non degenerate 1-simplex $p\colon (\underline{e}', z') \to (\underline{e}, z) \in \mathcal{N}(\alpha/c)_1$, with $z' = z \circ (\alpha(T^{\underline{e}^1}_{e_1}), \dots, \alpha(T^{\underline{e}^n}_{e_n}))$ for subtrees $T^{\underline{e}^i}_{e_i}$, $i = 1, \dots, n$, with $\underline{e}' = (\underline{e}^1, \dots, \underline{e}^n)$. The subtree inclusion $\eta = e_i \subseteq T^{\underline{e}^i}_{e_i}$ corresponds to a 1-simplex $u_i\colon \Delta^1 \to \mathcal{A}^T(e_i)$, and we define

$$\chi_c(p) = z_* \circ (\gamma_{u_1}, \dots, \gamma_{u_m}),$$

where, if u_i is degenerate we set $\gamma_{u_i} := \gamma_{e_i}\sigma$, where σ is the degeneracy.

If p is degenerate, that is $p = z\sigma$, we set $\chi_c(p) := \chi_c(\underline{e}, z)\sigma$; compatibility with face maps is given by the compatibility condition with respect to the diagrams

$$\begin{tikzcd}
\Delta^0 \arrow[r, "(e^i_1, \dots, e^i_{n_i})"] \arrow[d, "\partial^1"'] & \mathcal{A}^T(e^i_1) \times \cdots \times \mathcal{A}^T(e^i_{n_i}) \arrow[d, "(T^{\underline{e}^i}_{e_i})_*"] \\
\Delta^1 \arrow[r, "u_i"] & \mathcal{A}^T(e_i) \\
\Delta^0 \arrow[r, "e_i"] \arrow[u, "\partial^0"] & \mathcal{A}^T(e_i) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

for $i \in \{1, \dots, n\}$, and it still follows from the compatibility condition and the structure of P-algebra of F that the diagram Equation (5.2.3) commutes also at the level of 1-simplices.

Let $\ell \geqslant 2$ and t be a ℓ-simplex of $\mathcal{N}(\alpha/c)$, $t\colon (\underline{e}^{(l)}, z^{(\ell)}) \to \cdots \to (\underline{e}^{(1)}, z^{(1)}) \to (\underline{e}^{(0)}, z^{(0)})$. The ℓ-simplex $\chi_c(t)$ of $F(c)$ is defined as follows. For any $i \in \{1, \dots, n\}$, let $u_i\colon \Delta^\ell \to \mathcal{A}^T(e_i)$ be a ℓ-simplex corresponding to the chain of subtrees with root e_i whose subsequent composition yields $z^{(i)}$. If each of the u_i's is non-degenerate, then by hypothesis we have maps γ_{u_i} and we set

$$\chi_c(t) := z_*(\chi_{u_1}, \dots, \chi_{u_n}).$$

If some u_i is degenerate, we define $\gamma_{u_i} := \gamma_{\overline{u}_i} \circ \sigma$ for $\overline{u}_i$ the unique non-degenerate simplex and σ a degeneracy such that $u_i = \overline{u}_i \circ \sigma$, and then define $\chi_c(t)$ in the same way.

It is a lengthy but straightforward computation to check that the simplicial relations involving face maps are satisfied thanks to the compatibility condition enjoyed by the family $\{\gamma_u\}_{e,u,k}$ and that those involving degeneracies are satisfied by construction, and just as readily, one checks that the diagram in Equation (5.2.3) commutes at the level of ℓ-simplices.

We have hence defined a map of simplicial P-algebras $\chi\colon \rho_!(\alpha) \to F$. By adjunction, it yields an element in $\rho^* F_{(T,\alpha)}$, and this concludes the proof. ☐

When the domain of α is a linear tree, i.e. an element of Δ, the above description can be further simplified.

Corollary 5.2.11. *Consider a representable of* $\mathbf{dSets}/\mathcal{N}_d P$ *of the form* $([n], \alpha)$, $n \geqslant 0$, *and let* F *be a simplicial* P*-algebra. An element* ${}^t\chi \in \rho^*(F)_{([n],\alpha)}$ *is determined by a family of morphisms*

$$\{\gamma_u\colon \Delta^k \to F(\alpha(u(k)))\}_u$$

as u ranges over the maps $u\colon \Delta^k \to \Delta^n$, which satisfies the following property: given maps u, u' and a face map $v\colon \Delta^k \to \Delta^{k'}$ making the diagram below on the left commute, the induced diagram on the right commutes as well,

$$\begin{tikzcd}
\Delta^k \arrow[d, "v"'] \arrow[dr, "u"] & \\
\Delta^{k'} \arrow[r, "u'"'] & \Delta^n
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Delta^k \arrow[r, "\gamma_u"] \arrow[d, "v"'] & F(\alpha(u(k))) \arrow[d, "f_v"] \\
\Delta^{k'} \arrow[r, "\gamma_{u'}"] & F(\alpha(u'(k')))
\end{tikzcd}$$

where f_v denotes the composition of the morphisms from $\alpha(u(k))$ to $\alpha(u'(k'))$.

Proof. This follows from Proposition 5.2.10 and the following considerations. First, as observed in Remark 5.2.7, there is a natural isomorphism $\mathcal{A}^{[n]}(j) \simeq \Delta^j$ for any $j \in [n]$. Second, any map of simplicial sets $w\colon \Delta^k \to \Delta^n$ uniquely factors through the inclusion $i\colon \Delta^{w(k)} \hookrightarrow \Delta^n$, $i(x) = x$, and writes as the composition $\Delta^k \xrightarrow{\overline{w}} \Delta^{w(k)} \simeq \mathcal{A}^{[n]}(w(k)) \xhookrightarrow{i} \Delta^n$. Given a face map $v\colon \Delta^k \to \Delta^{k'}$, the commutative triangle $u = u' \circ v$ corresponds to the commutative square $z \circ \overline{w} = \overline{w'} \circ v$, where $z\colon \mathcal{A}^{[n]}(w(k)) \to \mathcal{A}^{[n]}(w'(k'))$ is the image of the arrow $w(k') \leqslant w'(k')$ via the functor $\mathcal{A}^{[n]}\colon [n] \to$ sSets, and commutativity of the square on the right is just the compatibility condition of Proposition 5.2.10 applied to $z \circ \overline{w} = \overline{w'} \circ v$,.

Vice versa, if we have a map $u\colon \Delta^n \to \mathcal{A}^{\Delta^n}(j) \simeq \Delta^j$, then we produce a morphism $\tilde{u}\colon \Delta^n \to \Delta^k$ by postcomposing with the same natural inclusion $\Delta^j \hookrightarrow \Delta^k$. ☐

We deduce the immediate simplification of the above criterion.

Corollary 5.2.12. *In the situation of Corollary 5.2.11, an n-simplex $\chi \in (\rho^* F)_\alpha$ is completely determined by a sequence of simplices $\chi_i\colon \Delta^i \to F_{\alpha(i)}$, for $i \in \{0, 1, \ldots, n\}$, such that $f_i(\chi_{i-1}) = d_i \chi_i$.*

Let us give an

Example 5.2.13. *When F is a P-algebra in sets, i.e. $F(c)$ is a set for every color c of P, we can give a simple description of $\rho^*(F)$: given a representable (T, α), there is a natural bijection of sets*

$$\rho^*(F)_{(T,\alpha)} = \prod_{\ell \in \mathsf{Leaves}(T)} F(\alpha(\ell)) \ ,$$

and if we write $\rho^*(F)$ as a morphism of dendroidal sets $q\colon E \to \mathcal{N}_d P$, we see that for every tree T one has

$$E_T = \bigcup_{\alpha\colon T \to \mathcal{N}_d P} \ \prod_{\ell \in \mathsf{Leaves}(T)} F(\alpha(\ell)) \ ,$$

and that the map q_T sends each tuple to its component, namely

$$q_T\colon E_T \to \mathcal{N}_d P_T \quad (\alpha, \underline{x}) \mapsto \alpha.$$

To see this, we observe that by Proposition 5.2.10 an element $\gamma \in \rho^*(F)_{(T,\alpha)}$ consists of a family of objects

$$(\gamma_u \in F(\alpha(e)))_{k,e,u}$$

satisfying the compatibility condition, where k ranges over natural numbers, e over the edges of T and u over the non-degenerate k-simplices of $\mathcal{A}^T(e)$. The compatibility condition for $T^{\underline{e}}_{e} = id$ impose the relation $\gamma_u = \gamma_{u'} \circ d$ for every two simplices u, u' such that $u = u' \circ d$ for a face map d. As $F(\alpha(e))$ is discrete, this means that $\gamma_u = \gamma_{u'}$, and since the simplicial set $\mathcal{A}^T(e)$ is connected, we get that for every non-degenerate k-simplex u of $\mathcal{A}^T(e)$ we have

$$\gamma_u = \gamma_e,$$

where $e\colon \Delta^0 \to \mathcal{A}^T(e)$ is the 0-simplex selecting the trivial tree $e = \eta$ in $\mathcal{A}^T(e)$. If we now instantiate the compatibility condition with $k = k' = 0$ and $v = id$, we obtain that, whenever $T(e_1, \ldots, e_n; e) = \{T^{\underline{e}}_{e}\} \neq \varnothing$, there is the relation

$$\gamma_e = \alpha(T^{\underline{e}}_{e})_*(\gamma_{e_1}, \ldots, \gamma_{e_n}).$$

Since for any edge e of T there exist unique leaves $\ell_1, \ldots, \ell_n$ (possibly with $n = 0$) such that $T(\ell_1, \ldots, \ell_n; e) \neq \varnothing$, this proves that to define an element of $\rho^*(F)_{(T,\alpha)}$ it is sufficient to choose an element $\gamma_\ell \in F(\alpha(\ell))$ for every leaf ℓ of T. Since viceversa any such choice determines an element of $\rho^*(F)_{(T,\alpha)}$, this gives the wanted description of $\rho^*(F)_{(T,\alpha)}$, and the rest follows from the chain of equalities

$$E_T = \{p\colon T \to E\} = \bigcup_{\alpha\colon T \to \mathcal{N}_d P} \{p\colon T \to E \mid qp = \alpha\} = \bigcup_{\alpha\colon T \to \mathcal{N}_d P} \rho^*(F)_{(T,\alpha)}.$$

Observe that, for every color c of P, there is an isomorphism $\rho^*(F)_c \simeq F(c)$, where $\rho^*(F)_c$ denotes the fibre of q over c.

The isomorphism $\rho^*(F)_c \simeq F(c)$ observed in Example 5.2.13 holds more generally, as we prove in the following

Lemma 5.2.14. *Let F be a simplicial P-algebra, c a object of P, and denote by $(\rho^* F)_c$ the fibre of $\rho^* F$ over c. There is an isomorphism of simplicial sets*

$$(\rho^* F)_c \simeq F(c)$$

natural in P-algebras, meaning that, given a map of P-algebras $\varphi\colon F \to G$, the induced map between the fibres $\rho^*\phi_c\colon \rho^* F_c \to \rho^* G_c$ is the map $\varphi_c\colon F(c) \to G(c)$.

Proof. Given a morphism of dendroidal sets $E \to X$ and an object c of X, the fibre of $E \to X$ over c is the simplicial set E_c obtained as the pullback

$$\begin{tikzcd}
E_c \arrow[r] \arrow[d] & E \arrow[d] \\
\eta \arrow[r, "\{c\}"] & X
\end{tikzcd}$$

where $\{c\}\colon \eta \to X$ is the map selecting c as object. If $E \to X$ is a dendroidal left fibration, the map $E_c \to \eta$ is a left fibration of simplicial sets as well; as η is a Kan complex, the map $E_c \to \eta$ is also a right fibration, which means that E_c is a Kan complex.

By definition, for any element (E, p) in $\mathbf{dSets}/X$, there is a canonical isomorphism of simplicial sets

$$E_c \simeq \hom_X(\{c\}, E),$$

where $\{c\}\colon \eta = \Delta^0 \to X$ is the map selecting c. In particular, by taking $X = \mathcal{N}_d P$ and $(E, p) = \rho^*(F)$, we see that the fibre of $\rho^* F$ over c can be described as the pullback

$$\rho^* F_c = \hom_{\mathcal{N}_d P}(\{c\}, \rho^* F);$$

an n-simplex of $\hom_{\mathcal{N}_d P}(\{c\}, \rho^* F)$ is a morphism $\Delta^n \to \rho^* F$ over $\mathcal{N}_d P$, where the map $\Delta^n \to \mathcal{N}_d P$ is given by the composition $\Delta^n \to \Delta^0 = \eta \xrightarrow{c} \mathcal{N}_d P$. By Corollary 5.2.11 and Corollary 5.2.12, any such simplex is determined by a single element $x_n\colon \Delta^n \to F(c)$, and one can check that the bijection is natural in Δ. This concludes the proof. □

5.3 The Quillen adjunction

Fix a discrete operad P and write $\rho_!$, resp. ρ^*, for the functor $\rho_!^P$, resp. ρ_P^*. The main goal of this section is to prove that the pair $(\rho_!, \rho^*)$ is a Quillen adjunction with respect to the covariant model structure for dendroidal left fibrations and the projective model structure on simplicial P-algebras. We will prove this in Theorem 5.3.16, and our strategy consists in showing that ρ^* is a right Quillen functor. To this end, we need to better understand lifts of maps of the form $\rho^*(F) \to \rho^*(G)$ against dendroidal boundary and leaf horn inclusions. Let us first begin with some combinatorial construction.

5.3.1 Root preserving faces and non-degenerate maximal chains for a tree

Let us start by giving some definitions.

Definition 5.3.1. We say that a face map $\partial\colon S \to T$ is *root preserving* if $\partial(r_S) = r_T$; for a face of linear trees $\partial\colon [i] \to [n]$, this means that $\partial(i) = n$.

Let T be a tree. A n-simplex u of $\mathcal{A}^T(r_T)$ can be written as a sequence of reversed subtree inclusions of the form

$$u\colon\; T_0 \to \cdots \to T_{n-1} \to T_n,$$

and u is non-degenerate if and only if in T_i there is at least one more vertex than T_{i+1}.

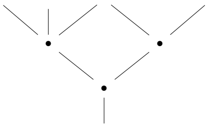
Definition 5.3.2. Let T be a tree. A *non-degenerate maximal chain* for T is a simplex

$$u\colon \Delta^n \to \mathcal{A}^T(r_T), \qquad u\colon\; T_0 \to \cdots \to T_{n-1} \to T_n$$

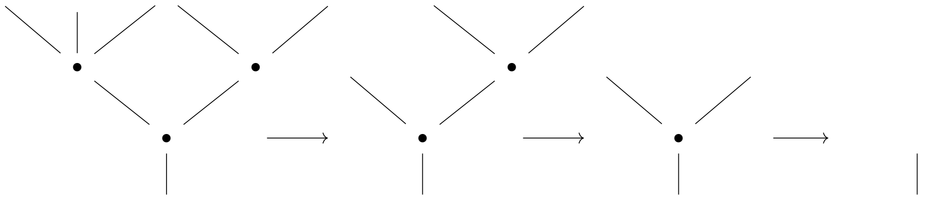
such that $T_0 = T$, $T_n = \eta = r_T$ and each subtree T_i is obtained from T_{i+1} by adding exactly one vertex (and all its input edges).

Put differently, a non-degenerate maximal chain for T is sequence of root preserving subtree inclusions $T_n \subseteq \cdots \subseteq T_1 \subseteq T_0$ in Ω with $T_n = r_T$, $T_0 = T$ and where each T_i is obtained from T_{i+1} by adding exactly one vertex. In particular, observe that in general a non-degenerate maximal chain may not be unique, but the number n is, as $n = \#V(T)$.

Example 5.3.3. *For a tree T of the form*



there are two non-degenerate maximal chains, given by



and

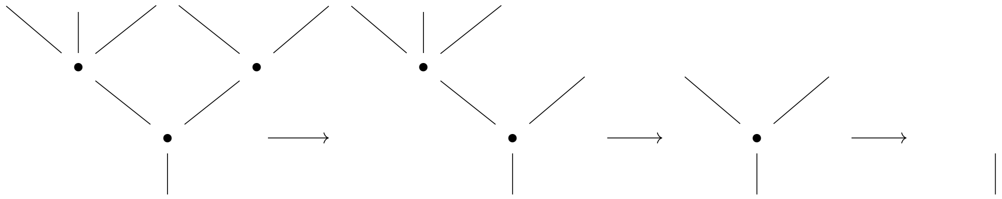


Figure 5.1: The two non-degenerate maximal chains for T

Remark 5.3.4. When defining the partial order on $P(T_e^{\uparrow})$ (Construction 5.2.1), we could have chosen the opposite order, that is, the natural inclusion of subtrees, which would make the definition above more natural. However, by choosing *reverse* inclusion we obtain that, for a leaf vertex x of T, the *leaf* horn $\Lambda^x\mathcal{A}^T(r_T)$ of $\mathcal{A}^T(r_T)$, which we define in Definition 5.3.12, is a *left*, and not *right*, anodyne morphism (see Remark 5.3.13 and Lemma 5.3.18).

Consider a root preserving face map $\partial\colon S \to T$. We still denote by ∂ the map of simplicial sets

$$\partial\colon \mathcal{A}^S(r_S) \longrightarrow \mathcal{A}^T(r_T)$$

induced by the map of posets sending a subtree in $P(S^{\uparrow}_{r_S}) \to P(T^{\uparrow}_{r_T})$ which sends a subtree in S to its image in T via ∂.

The next result relates root preserving face maps and non-degenerate maximal chains of trees. It will be fundamental in the proofs of Proposition 5.3.11 and Proposition 5.3.15, as well as in the previous constructions Construction 5.3.10 and Construction 5.3.14.

Lemma 5.3.5. *Let $\overline{u}\colon \Delta^n \to \mathcal{A}^T(r_T)$ be a non-degenerate maximal chain and $d\colon \Delta^i \to \Delta^n$ a root preserving face.*

(i) There exist a tree $T(d,\overline{u})$, a root preserving dendroidal face $\partial\colon T(d,\overline{u}) \to T$ and a non-degenerate simplex $u\colon \Delta^i \to \mathcal{A}^{T(d,\overline{u})}(r_{T(d,\overline{u})})$ such that the following diagram commutes

$$\begin{tikzcd} \Delta^i \arrow[r, "d"] \arrow[d, "u"'] & \Delta^n \arrow[d, "\overline{u}"] \\ \mathcal{A}^{T(d,\overline{u})}(r_{T(d,\overline{u})}) \arrow[r, "\partial"] & \mathcal{A}^T(r_T) \end{tikzcd} \tag{5.3.1}$$

and such that the triple $(T(d,\overline{u}),\partial,u)$ is initial amongst the triples satisfying the properties of above. In particular, such $(T(d,\overline{u}),\partial,u)$ is essentially unique.

(ii) If $d(i-1) < n-1$ or $i = 1 < n$, the face map $\partial\colon T(d,\overline{u}) \to T$ is not an isomorphism of trees. In other words, the tree $T(d,\overline{u})$ and the face map ∂ contribute to the colimit defining the dendroidal set ∂T.

Proof. Let us first prove point (i). We represent the non-degenerate maximal chain $\overline{u}$ as a chain of root preserving subtree inclusions $\overline{u}_n \subseteq \overline{u}_{n-1} \subseteq \cdots \subseteq \overline{u}_0$, with $\overline{u}_n = r_T$, $\overline{u}_0 = T$ and such that at every step we add exactly one vertex of the tree.

Observe that the diagram in Equation (5.3.1) is in fact the nerve of the diagram of categories, and more precisely of posets, given by the following

$$\begin{tikzcd}
{[i]} \arrow[r, "d"] \arrow[d, "u"'] & {[n]} \arrow[d, "\overline{u}"] \\
P(T(d,\overline{u})^{\uparrow}_{r_{T(d,\overline{u})}}) \arrow[r, "\partial"] & P(T^{\uparrow}_{r_T}) \, .
\end{tikzcd} \tag{5.3.2}$$

We construct the triple $(T(d,\overline{u}), \partial, u)$ inductively. More precisely, we start by inductively constructing a chain of root-preserving inclusions of trees $u_i \subseteq u_{i-1} \subseteq \cdots \subseteq u_0$, the base case being $k = i$, and dendroidal face maps $\partial_k \colon u_k \to \overline{u}_{d(k)}$ for any $k \in [i]$, such that:

1. for every $k \in [i]$, the face ∂_k is root preserving and $\partial(u_k) = \overline{u}_{d(k)}$, that is, the diagram in Equation (5.3.2) commutes at the level of objects,

2. the square

$$\begin{tikzcd}
u_k \arrow[r, hook] \arrow[d, "\partial_k"'] & u_{k-1} \arrow[d, "\partial_{k-1}"] \\
\overline{u}_{d(k)} \arrow[r, hook] & \overline{u}_{d(k-1)}
\end{tikzcd} \tag{5.3.3}$$

   commutes for any $k > 0$, that is, the diagram in Equation (5.3.2) commutes at the level of morphisms,

3. at the k^{th} step, the triple $(u_k, \overline{\partial}_k, u_k \to u_{k-1} \to \cdots \to u_{i-1} \to u_i)$, where $\overline{\partial}_k$ is the dendroidal face given by the composition $u_k \xrightarrow{\partial_k} \overline{u}_{d(k)} \subseteq T(d,\overline{u})$, is initial with respect to the tree T, the non-degenerate maximal chain $\overline{u}$ and the root preserving face map given by the composition $\Delta^{i-k} \hookrightarrow \Delta^i \xrightarrow{d} \Delta^n$, where the inclusion $\Delta^{i-k} \hookrightarrow \Delta^i$ is the face map defined by the assignment $[i-k] \ni j \mapsto j + k \in [i]$.

Observe that condition (1) implies that ∂_k is an inner face map, so in particular it also induces a bijection between the sets of leaves of the tree in the domain and in the codomain.

Let $k = i$. As d is root preserving, we have $\overline{u}_{d(i)} = \overline{u}_n = \eta$, and we define $u_i := \eta$ and $\partial_i = \mathrm{id} \colon u_i \to \overline{u}_n$. This is clearly an initial inner face map (condition (2) at this stage is empty).

Consider $k = i-1$. As $\overline{u}$ is non-degenerate, the tree $\overline{u}_{d(i-1)}$ has at least one vertex. Let ℓ be the number of leaves of $\overline{u}_{d(i-1)}$ and C_ℓ the corolla with ℓ leaves. There exists an essentially unique inner face map from $C_\ell \to \overline{u}_{d(i-1)}$, which is the identity if $d(i-1) = n-1$, and which is clearly initial amongst the inner face maps $S \to \overline{u}_{d(i-1)}$. We set $u_{i-1} := C_\ell$, that is, the corolla obtained by contracting all the inner edges of $\overline{u}_{d(i-1)}$, and we define $\partial_{i-1} \colon u_{i-1} \to \overline{u}_{d(i-1)}$ as the unique inner face map just mentioned. By construction, the conditions $(1), (2)$ and (3) are satisfied.

Let $0 < k \leqslant i-1$, and suppose we have constructed a chain of root preserving inclusions $u_i \subseteq \cdots \subseteq u_k$ satisfying the requirements; we construct $(u_{k-1}, \partial_{k-1})$ as follows. Write $\overline{u}_{d(k-1)}$ as the grafting

$$\overline{u}_{d(k-1)} = \overline{u}_{d(k)} \circ (\overline{v}_1, \dots, \overline{v}_{m_k}),$$

for some uniquely determined subtrees $\overline{v}_j$. Since $\overline{u}$ is non-degenerate, at least one $\overline{v}_j$ is different from η. For any $j = 1, \dots, m_k$, if $\overline{v}_j \neq \eta$, we denote by v_j the corolla with the same number of leaves as $\overline{v}_j$ and write χ_j for the unique inner face map $v_j \to \overline{v}_j$; we set $v_j = \eta$ and $\chi_j = \mathrm{id}$ otherwise. We define

$$u_{k-1} := u_k \circ (v_1, \dots, v_{m_k}),$$

and the face map ∂_{k-1} as the grafting

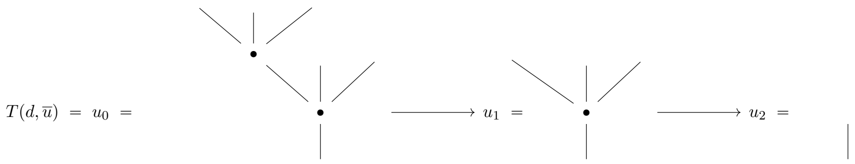
$$\partial_{k-1} := \partial_k \circ (\chi_1, \dots, \chi_{m_k}).$$

By construction, $\partial_{k-1}(u_{k-1}) = \overline{u}_{d(k-1)}$ and the diagram in Equation (5.3.2) commutes. Moreover, as each of the χ_j is initial and by inductive hypothesis ∂_k is initial as well, we have that ∂_{k-1} initial as well, as wanted.

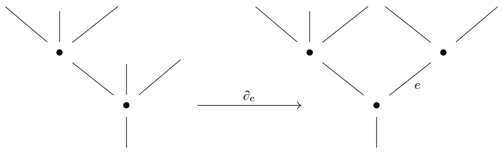
In conclusion, we have shown that the triple given by $(T(d,\overline{u}), \partial, u)$ is initial, where $T(d,\overline{u}) = u_0$, $\partial = \overline{\partial}_0 \colon T(d,\overline{u}) = u_0 \xrightarrow{\partial_0} \overline{u}_{d(0)} \subseteq \overline{u}_n = T$ and u is the non-degenerate simplex defined by $u_0 \to \dots u_{i-1} \to u_i$. It is straightforward to check that initiality implies essential uniqueness, so this concludes the proof of (i).

Let us now address point (ii). If $d(i-1) < n-1$, maximality of $\overline{u}$ ensures that in $\overline{u}_{d(i-1)}$ there are at least two vertices, which is equivalent to say that $\overline{u}_{d(i-1)}$ has at least one inner edge. By construction, $\partial_{i-1} \colon u_{i-1} \to \overline{u}_{d(i-1)}$ is the face obtained by contracting all the inner edges of $\overline{u}_{d(i-1)}$, it is not an isomorphism. As $\partial_{i-1} = \partial_{|_{u_{i-1}}}$, the dendroidal face ∂ cannot be an isomorphism of trees either, as claimed. On the other hand, when $i = 1$ the tree $T(d,\overline{u}) = u_0$ is always a corolla, but if $n > 1$, then T has at least two vertices, which means that the face map $\partial \colon T(d,\overline{u}) \to T$ cannot be an isomorphism of trees. $\square$

Example 5.3.6. *Consider the tree T of Example 5.3.3 and let $\overline{u}$ be the first non-degenerate maximal chain of Figure 5.1. Let $d \colon \Delta^2 \to \Delta^3$ be the face map defined by $d(0) = 0, d(1) = 1, d(2) = 3$. Then we have that $(T(d,\overline{u}), \partial, u)$ is determined by the non-degenerate simplex u (in this case also maximal) given by*



where the face map ∂ is given by the inner face map



obtained by contracting the inner edge e.

Consider now the face map d defined by $d(0) = 0, d(1) = 2, d(2) = 3$, which does not fall in the hypothesis of point (ii) of Lemma 5.3.5. In this case we have $u_2 = \eta$, $u_1 = C_2$ and $u_0 = T$ and $\partial = id$.

We conclude this part with the following

Definition 5.3.7. Given a tree T and a non-degenerate simplex $w\colon \Delta^k \to \mathcal{A}^T(r_T)$, a *maximal extension* of w is a non-degenerate maximal chain $\overline{w}\colon \Delta^n \to \mathcal{A}^T(r_T)$ together with a face map $d\colon \Delta^k \to \Delta^n$ such that $w = \overline{w} \circ d$.

Observe that, even if there are multiple maximal extensions of w, the face map $d\colon \Delta^k \to \Delta^n$ is unique.

5.3.2 Lifts against dendroidal boundaries

Given a tree T, we write $\iota\colon \partial T \to T$ for the dendroidal boundary inclusion. If T is the domain of a representable (T, α) in $\mathbf{dSets}/\mathcal{N}_d P$, by precomposing with ι one obtains another element $(\partial T, \alpha\iota)$, no longer representable.

We need to introduce the following object.

Definition 5.3.8. Let T be a tree. The *boundary of* $\mathcal{A}^T(r_T)$ is the subsimplicial set $\partial\mathcal{A}^T(r_T)$ of $\mathcal{A}^T(r_T)$ given by the chains $u_0 \to \cdots \to u_k$ such that at least one of the following conditions hold:

1. $u_k \neq \eta$;
2. $u_0 \neq T$;
3. there exists at least one $i \in \{0, \dots, k-1\}$ such that the tree u_i is obtained from u_{i+1} by adding at least two vertices v, w with $v < w$.

Condition (3) in Definition 5.3.8 may be informally stated by saying that u_i is obtained from u_{i+1} by adding at least two vertices which are one on top of the other. We will say that two such vertices are *dependent*.

Remark 5.3.9. As noticed in Remark 5.2.7, when $T = [n]$ one has the isomorphism $\mathcal{A}^T(r_T) \simeq \Delta^n$. In this case, there is a natural isomorphism $\partial\mathcal{A}^T(r_T) \simeq \partial\Delta^n$.

Given a simplicial P-algebra F and a representable (T, α), in Construction 5.3.10 we explain how to associate to a morphism $\chi\colon (\partial T, \alpha\iota) \to \rho^*(F)$ a map of simplicial sets $\partial\chi\colon \partial\mathcal{A}^T(r_T) \to F(\alpha(r_T))$.

Construction 5.3.10. Let F and χ be as above; recall the characterization of the dendrices of $\rho^*(F)$ detailed in Proposition 5.2.10. We define $\partial\chi\colon \partial\mathcal{A}^T(r_T) \to F(\alpha(r_T))$ on non-degenerate simplices only, and then extend it to all simplices by imposing the naturality conditions for degeneracies. Consider a non-degenerate k-simplex u of $\partial\mathcal{A}^T(r_T)$, that is, $u\colon \Delta^k \to \partial\mathcal{A}^T(r_T)$.

If $u_k \neq \eta$, then u_k has at least one vertex. Let $\{e_1, \dots, e_m\}$ be the set of leaves of u_k. If it is non-empty, for every $i \in \{1, \dots, m\}$ there exists a unique chain $v^i\colon \Delta^k \to \mathcal{A}^T(e_i)$, possibly degenerate, such that $u = (u_k)_* \circ (v^1, \dots, v^m)$. For every $i \in \{1, \dots, m\}$, the restriction of $\alpha\iota$ to the face $T^{\uparrow}_{e_i} \hookrightarrow T$ yields a map $\chi_{v^i}\colon \Delta^k \to F(\alpha(e_i))$, and we set

$$(\partial\chi)(u) := \alpha(u_k)_* \circ (\chi_{v^1}, \dots, \chi_{v^m}).$$

If the set of leaves of u_k is empty, the fact that u is non-degenerate means that necessarily $k = 0$ (and moreover, although we don't need this here, that $u_k = T$). As F is a P-algebra, u_k determines a map $\alpha(u_k)_* \colon \Delta^0 \to F(\alpha(r_T))$, and we set

$$(\partial\chi)(u) := \alpha(u_k)_*.$$

If $u_k = \eta$ and $u_0 \neq T$, chose a maximal extension $\overline{u} \colon \Delta^n \to \mathcal{A}^T(r_T)$ of u, with $d \colon \Delta^k \to \Delta^n$ the root preserving face map such that $u = \overline{u} \circ d$. By Lemma 5.3.5, there exists a root-preserving face map $\partial \colon T(d, \overline{u}) \to T$ and a non-degenerate simplex $u' \colon \Delta^k \to T(d, \overline{u})$ such that $\overline{u} \circ d = \partial \circ u'$, and we define

$$(\partial\chi)(u) := \chi_{u'}.$$

This definition does not depend on the chosen maximal extension $\overline{u}$: if we write S for the tree u_0, by assumption the simplex u factors through the face $S \to T$, defining a simplex $w \colon \Delta^k \to \mathcal{A}^S(r_S)$. Initiality of the triple $(T(d,\overline{u}), \partial, u')$ garantees that it exists a face map $\delta \colon T(d, \overline{u}) \to S$ through which ∂ factors and which yields a commutative diagram and that the morphism induces a commutative diagram

$$\begin{tikzcd}
\Delta^k \arrow[r, "w"] \arrow[d, "u'"'] & \mathcal{A}^S(r_S) \arrow[d] \\
\mathcal{A}^{T(d,\overline{u})}(r_{T(d,\overline{u})}) \arrow[ur, "\delta"] \arrow[r] & \mathcal{A}^T(r_T)
\end{tikzcd}$$

Coherence of χ on nested faces then ensures that

$$\chi_{u'} = \chi_w.$$

Lastly, suppose that $u_k = \eta, u_0 = T$ and that u satisfies condition (3) of Definition 5.3.8. Similarly to above, choose a maximal extension $\overline{u}$ of u and use Lemma 5.3.5 to obtain a root preserving face map $T(d, \overline{u}) \to T$ and a non-degenrate simplex $u' \colon \Delta^k \to \mathcal{A}^T$, and define

$$(\partial\chi)(u) = \chi_{u'}.$$

Again, the choice does not depend on the chosen maximal extension: let S be the tree obtained from T by contracting all the inner edges between any two dependend vertices added in a jump $u_i \supseteq u_{i+1}$. It yields a face map $S \to T$ through which the simplex u factors, defining a unique k-simplex $w \colon \Delta^k \to \mathcal{A}^S(r_S)$. Initiality of $(T(d,\overline{u}), \partial, u')$ together with coherence of χ on nested subfaces ensure that one has

$$\chi_{u'} = \chi_w.$$

We use this construction for the following

Proposition 5.3.11. *Let $f \colon F \to G$ be a morphism of P-algebras and (T, α) a representable in* $\mathbf{dSets}/\mathcal{N}_d P$*. To have a dotted lift in the solid diagram*

$$\begin{tikzcd}
(\partial T, \alpha \circ \iota) \arrow[r, "\chi"] \arrow[d] & \rho^*(F) \arrow[d] \\
(T, \alpha) \arrow[r, "\xi"'] \arrow[ur, dashed, "\Lambda"] & \rho^*(G)
\end{tikzcd}$$

it suffices to have a lift in the diagram of simplicial sets

$$\begin{tikzcd}
\partial\mathcal{A}^T(r_T) \arrow[r, "\partial\chi"] \arrow[d] & F(\alpha(r_T)) \arrow[d, "f_{\alpha(r_T)}"] \\
\mathcal{A}^T(r_T) \arrow[r, "\bar{\xi}"'] \arrow[ur, dashed, "\lambda"] & G(\alpha(r_T))
\end{tikzcd}$$

where $\partial\chi$ *is the map given by Construction 5.3.10 and* $\bar{\xi}$ *is defined as the evaluation at the root* r_T *of the transpose of* $\xi^t\colon \rho_!(T,\alpha) = \alpha_!(\mathcal{A}^T) \to G$ *under the adjunction* $(\alpha_!, \alpha^*)$.

Proof. For any non-degenerate simplex $w\colon \Delta^k \to \mathcal{A}^T(e)$, we construct a map of simplicial sets $\Lambda_w\colon \Delta^k \to F(\alpha(e))$ such that:

1. the family $\{\Lambda_w\}_w$ satisfies the compatibility condition of Proposition 5.2.10, defining a map $\Lambda\colon (T,\alpha) \to \rho^*(F)$ in $\mathbf{dSets}/\mathcal{N}_d P$;
2. for any face $\partial\colon S \to T$ such that w factors as
$$\Delta^k \xrightarrow{w'} \mathcal{A}^S(e') \to \mathcal{A}^T(\partial(e')) = \mathcal{A}^T(e)$$
for some w', one has $\Lambda_w = \chi_{w'}$. This means that $\Lambda\iota = \chi$;
3. for any edge e of T and w non-degenerate, $f_{\alpha(e)}\Lambda_w = \xi_w$, which means that $\rho^*(f)\Lambda = \xi$.

Consider hence an edge e of T, and a non-degenerate simplex $w\colon \Delta^k \to \mathcal{A}^T(e)$.

If $e \neq r_T$, the tree inclusion $\partial\colon T_e^{\uparrow} \hookrightarrow T$ contributes to the colimit defining ∂ and yields the equality $\mathcal{A}^T(e) = \mathcal{A}^{T_e^{\uparrow}}(r_{T_e^{\uparrow}})$. The restriction of χ to ∂ yields a map $\chi_{|\partial}\colon (T_e^{\uparrow}, \alpha\partial) \to \rho^*(F)$. After Proposition 5.2.10, this gives a simplex $\chi_w\colon \Delta^k \to F(\alpha(\partial(r_S))) = F(\alpha(e))$, and we define $\Lambda_w := \chi_w$.

If $e = r_T$, we define $\Lambda_w := \lambda \circ w$.

Let us now prove that the family just defined satisfies the three conditions above.

1. Consider edges $e_1, \ldots, e_n, e$ of T such that $T(e_1, \ldots, e_n; e) = \{p\} \neq \varnothing$, a face map $v\colon \Delta^k \to \Delta^{k'}$ and a commutative diagram
$$\begin{tikzcd}
\Delta^k \arrow[r, "(u_1,\ldots,u_n)"'] \arrow[d, "v"'] & \mathcal{A}^T(e_1) \times \cdots \times \mathcal{A}^T(e_n) \arrow[d, "p_*"] \\
\Delta^{k'} \arrow[r, "u"] & \mathcal{A}^T(e)
\end{tikzcd} \tag{5.3.4}$$
where $u_1, \ldots, u_n, u$ are non-degenerate. We need to check that the following diagram commutes:
$$\begin{tikzcd}
\Delta^k \arrow[r, "(\Lambda_{u_1},\ldots,\Lambda_{u_n})"'] \arrow[d, "v"'] & F(\alpha(e_1)) \times \cdots \times F(\alpha(e_n)) \arrow[d, "\alpha(p)_*"] \\
\Delta^{k'} \arrow[r, "\Lambda_u"] & F(\alpha(e))
\end{tikzcd} \tag{5.3.5}$$

Suppose $e \neq r_T$, and write S, resp. S_i, for $T_e^{\uparrow}$, resp. $T_{e_i}^{\uparrow}$, $i = 1, \dots, n$. In this case, we can write $u \colon \Delta^{k\prime} \to \mathcal{A}^S(r_S)$ and $u_i \colon \Delta^k \to \mathcal{A}^{S_i}(r_{S_i})$, and by definition we have $\Lambda_u = \chi_u$ and $\Lambda_{u_i} = \chi_{u_i}$. Moreover, since χ is coherent on nested subtrees of T and each u_i can be written as $u_i = u_i^S \colon \Delta^k \to \mathcal{A}^S(e_i)$, we have that $\chi_{u_i} = \chi_{u_i^S}$. As a consequence, if we denote by $i \colon S \to T$ the subtree inclusion, the diagram (5.3.5) is a compatibility diagram for the restriction of χ to i, that is, $\chi_{|_i} \colon (S, \alpha i) \to \rho^*(F)$; in particular, this means that the diagram commutes, as wanted.

If $e = r_T$, by definition we have $\Lambda_u = \lambda \circ u$. If $p \neq \mathrm{id}$, then the k-simplex $uv \colon \Delta^k \to \mathcal{A}^T(r_T)$ is such that $(uv)_k \neq \eta$, hence it belongs to $\partial\mathcal{A}^T(r_T)$. By hypothesis, we have

$$\lambda \circ (uv) = (\partial\chi)(uv),$$

which implies that

$$\Lambda_u \circ v = \alpha((uv)_k)_* \circ (\chi_{v^1}, \dots, \chi_{v^m}), \tag{5.3.6}$$

where $\{l_1, \dots, l_m\}$ is the set of leaves of the tree $(uv)_k$ and for every $i \in \{1, \dots, m\}$, $v^i \colon \Delta^k \to \mathcal{A}^T(l_i)$ is the unique chain such that $uv = ((uv)_k)_* \circ (v^1, \dots, v^m)$. Moreover, we have that

$$(uv)_k = p_* \circ ((u_1)_k, \dots, (u_n)_k),$$

and uniqueness of the v^i's imply that the following diagram is commutative (up to appropriately permuting the factors in the product, which we implicitly do)

$$\begin{tikzcd}
\Delta^k \arrow[r, "(v^1, \dots, v^m)"] \arrow[d, "(u_1, \dots, u_n)"'] & \mathcal{A}^T(l_1) \times \cdots \times \mathcal{A}^T(l_m) \arrow[d, "((uv)_k)_*"] \arrow[dl, "(((u_1)_k)_*, \dots, ((u_n)_k)_*)"'] \\
\mathcal{A}^T(e_1) \times \cdots \times \mathcal{A}^T(e_n) \arrow[r, "p_*"'] & \mathcal{A}^T(r_T)
\end{tikzcd} \tag{5.3.7}$$

As $\mathcal{A}^T(e_i) = \mathcal{A}^{T_{e_i}^{\uparrow}}(r_{T_{e_i}^{\uparrow}})$, we see that Equation (5.3.7) is a product of compatibility diagrams for each $T_{e_i}^{\uparrow}$, which, together with coherence of χ on nested subfaces, implies that we have the equaity

$$(\chi_{u_1}, \dots, \chi_{u_n}) = (\alpha((u_1)_k)_*, \dots, \alpha((u_n)_k)_*) \circ (\chi_{v^1}, \dots, \chi_{v^m}).$$

In particular, referring back to the equality in Equation (5.3.6), we can continue the chain of equalities and get

$$\Lambda_u \circ v = \alpha(p)_* \circ (\alpha((u_1)_k)_*, \dots, \alpha((u_n)_k)_*) \circ (\chi_{v^1}, \dots, \chi_{v^m}) = \alpha(p)_* \circ (\chi_{u_1}, \dots, \chi_{u_n}) = \alpha(p)_*(\Lambda_{u_1}, \dots, \Lambda_{u_n})$$

as wanted.

If $p = \mathrm{id}$, diagram in Equation (5.3.5) commutes by construction, as we have

$$\Lambda_{u_1} = \lambda \circ u_1 = \lambda \circ u \circ v = \Lambda_u \circ v.$$

2. Consider ∂, w and w' as specified in (2); we need to prove that $\Lambda_w = \chi_{w'}$, and for this it suffices to consider the case where w' is non-degenerate as well.

   If $e \neq r_T$, we can write $w \colon \Delta^k \to \mathcal{A}^{T_e^{\uparrow}}(r_{T_e^{\uparrow}}) = \mathcal{A}^T(e)$ and we have by definition that $\Lambda_w = \chi_w$. The restriction of ∂ to the subtree $S_{e'}^{\uparrow}$ yields the face map $S_{e'}^{\uparrow} \to T_e^{\uparrow}$ and induces a factorization of w of the form $\Delta^k \xrightarrow{w'} \mathcal{A}^{S^{\uparrow}_{e'}}(r_{S^{\uparrow}_{e'}}) = \mathcal{A}^S(e') \to \mathcal{A}^T(e)$, so $\chi_w = \chi_{w'}$ and the thesis is proven.

If $e = r_T$, then $\Lambda_w = \lambda \circ w$. Moreover, one necessarily has that $\partial \colon S \to T$ is a root preserving face, either external or internal. In both cases, this ensures that w belongs to $\partial\mathcal{A}^T(r_T)$, which means that $\Lambda_w = (\partial\chi)(w)$.

Let us first suppose that $w_k \neq \eta$: in this case, reasoning as in Construction 5.3.10, we can write $w = (w_k)_* \circ (v^1, \dots, v^m)$ for unique chains v^j's, and by construction we have

$$\Lambda_w = (\partial\chi)(w) = \alpha(w_k)_* \circ (\chi_{v^1}, \dots, \chi_{v^m}).$$

As ∂ is root preserving, necessarily one has $w'_k \neq \eta$ and

$$w' = (w'_k)_* \circ (\underline{v}^1, \dots, \underline{v}^{m'})$$

for unique chains $\underline{v}^j$'s. As ∂ is a morphism of trees, it respects operadic composition, i.e. grafting of trees, and since $\partial(w') = w$, from the uniqueness of the v^j's we deduce that $m' = m$ and that for every $j \in \{1, \dots, m\}$ we have $\partial(\underline{v}^j) = v^j$. The compatibility condition for the restriction of χ to ∂ ensures that we can write

$$\chi_{w'} = \alpha(w'_k)_* \circ (\chi_{\underline{v}^1}, \dots, \chi_{\underline{v^m}})$$

and coherence of χ on nested subfaces yields $\chi_{\underline{v}^j} = \chi_{v^j}$. As $\alpha \colon \partial T \to \mathcal{N}_d P$ is also coherent on nested subfaces, we have $\alpha(w'_k) = \alpha(w_k)$, and we deduce that $\chi_{w'} = (\partial\chi)(w) = \Lambda_w$, as wanted.

If now $w_k = \eta$, then w must fall into cases (2) or (3) of Definition 5.3.8. After Construction 5.3.10, this means that we have $(\partial\chi)(w) = \chi_W$ for some maximal extension $\overline{w}$ of w yielding the tree $T(d, \overline{w})$, a face map $\delta \colon T(d, \overline{w}) \to T$ and the simplex $W \colon \Delta^k \to \mathcal{A}^{T(d,\overline{w})}(r_{T(d,\overline{w})})$. Initiality of the triple $(T(d, \overline{u}), \delta, W)$ and coherence of χ on nested subfaces then guarantee $\chi_W = \chi_{w'}$, as wanted.

3. If $e \neq r_T$, then $\Lambda_w = \chi_w$ and by hypothesis $f_{\alpha(e)} \circ \chi_w = \xi_w$. If $e = r_T$, then $f_{\alpha(r_T)} \circ \Lambda_w = f_{\alpha(r_T)} \circ \lambda \circ w = \overline{\xi} \circ w = \xi_w$, where the last equality holds by construction.

We have checked all the required conditions, so this concludes the proof. □

5.3.3 Lifts against leaf and dendroidal inner horns

We now provide a similar construction of lifts of morphisms of the form $\rho^*(f) \colon \rho^*(F) \to \rho^*(G)$ against dendroidal leaf and inner horn inclusions. Most of constructions and proofs will be similar to the ones of the previous section, so we will only stress the points where the constructions diverge.

First of all, we need to introduce the following objects.

Definition 5.3.12. Let T be a tree and x an inner edge of T. The *horn of* $\mathcal{A}^T(r_T)$ *relative to* x is the subsimplicial set $\Lambda^x\mathcal{A}^T(r_T)$ of $\mathcal{A}^T(r_T)$ given by the chains $u_0 \to \dots \to u_k$ such that at least one of the following conditions hold:

1. $u_k \neq \eta$;

2. $u_0 \neq T$;
3. there exists at least one $i \in \{0, \dots, k-1\}$ such that the tree u_i is obtained from u_{i+1} by adding at least two vertices v, w with $v < w$ and $\{v, w\} \neq \{I(x), O(x)\}$, where $I(x)$, resp. $O(x)$ is the vertex above, resp. below, x.

Let x be a leaf vertex of T. The *horn of* $\mathcal{A}^T(r_T)$ *relative to* x is the subsimplicial set $\Lambda^x\mathcal{A}^T(r_T)$ of $\mathcal{A}^T(r_T)$ given by the chains $u_0 \to \cdots \to u_k$ such that at least one of the following conditions hold:

1. $u_k \neq \eta$;
2. $u_0 \neq \partial_x T$;
3. there exists at least one $i \in \{0, \dots, k-1\}$ such that the tree u_i is obtained from u_{i+1} by adding at least two vertices v, w with $v < w$.

Remark 5.3.13. If $T = [n]$ and x is an inner edge, that is $x \in \{1, \dots, n-1\}$, there is a natural isomorphism of simplicial sets $\Lambda^x\mathcal{A}^T(r_T) \simeq \Lambda^n_x$. If x is the unique leaf vertex of $T = [n]$, that is, the vertex between the edge labelled with 0 and that labelled with 1, then we have $\Lambda^x\mathcal{A}^T(r_T) \simeq \Lambda^n_0$. The correspondence leaf vertex- *left* anodyne depends on the choice of the reverse order for the posets $P(T_e^{\uparrow})$, as already anticipated in Remark 5.3.4.

Let F be a simplicial P-algebra and (T, α) a representable of $\mathbf{dSets}/\mathcal{N}_d P$. Let x be an inner edge or a leaf vertex of T, and let $j\colon \Lambda^x T \to T$ be the corresponding dendroidal horn inclusion. In Construction 5.3.14 we show how to associate to a morphism $\chi\colon (\Lambda^x T, \alpha j) \to \rho^*(F)$ a map of simplicial sets $\Lambda^x\chi\colon \Lambda^x\mathcal{A}^T(r_T) \to F(\alpha(r_T))$.

Construction 5.3.14. Let T, x, F and χ as above, and recall the characterization of the dendrices of $\rho^*(F)$ detailed in Proposition 5.2.10. Also recall that, given a tree T and a leaf vertex or an inner edge x of T, the horn $\Lambda^x T$ is defined as the colimit on all the faces of T except for $\partial_x\colon \partial_x T \to T$. We define $\Lambda^x\chi\colon \Lambda^x\mathcal{A}^T(r_T) \to F(\alpha(r_T))$ on non-degenerate simplices only, and then extend it to all simplices by imposing the naturality conditions for degeneracies. We proceed as in Construction 5.3.10, checking at each step that whenever we consider the restriction of χ to a face $\partial\colon S \to T$, this is well defined, that is, we have $\partial \neq \partial_x\colon \partial_x T \to T$.

Consider hence a non-degenerate k-simplex u of $\Lambda^x\mathcal{A}^T(r_T)$, that is, $u\colon \Delta^k \to \Lambda^x\mathcal{A}^T(r_T)$. If $u_k \neq \eta$, we can proceed as in the corresponding case of Construction 5.3.10, because for x inner edge or leaf vertex, and e any edge of T, a tree of the form $T_e^{\uparrow}$ cannot be isomorphic to a tree of the form $\partial_x T$.

Suppose now that $u_k = \eta$ and that it satisfies conditions (2) or (3) of Definition 5.3.12. We can still define $\Lambda^x\chi$ as in the corresponding cases of Construction 5.3.10, since the conditions in Definition 5.3.12 ensure that $T(d, \overline{u}) \neq \partial_x T$ for every maximal extension $\overline{u}$ of u.

Proposition 5.3.15. *Let $f\colon F \to G$ be a morphism of P-algebras and (T, α) a representable in* $\mathbf{dSets}/\mathcal{N}_d P$. *Let x be an inner edge or a leaf vertex of T. To have a lift Γ in the solid* *diagram*

$$\begin{tikzcd}
(\Lambda^x T, \alpha \circ j) \arrow[r, "\chi"] \arrow[d] & \rho^* F \arrow[d, "\rho^* f"] \\
(T, \alpha) \arrow[r, "\xi"] \arrow[ur, dotted, "\Gamma"] & \rho^* G
\end{tikzcd}$$

it is sufficient to have a lift in the diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^x \mathcal{A}^T(r_T) \arrow[r, "\Lambda^x \chi"] \arrow[d] & F_{\alpha(r_T)} \arrow[d, "f_{\alpha(r_T)}"] \\
\mathcal{A}^T(r_T) \arrow[r, "\bar{\xi}"'] \arrow[ur, dashed, "\gamma"] & G_{\alpha(r_T)}
\end{tikzcd}$$

where $\Lambda^x \chi$ *is the map constructed as in Construction 5.3.14 and* $\bar{\xi}$ *is defined as the evaluation at the root* r_T *of the transpose of* $\xi^t \colon \rho_!(T, \alpha) = \alpha_!(\mathcal{A}^T) \to G$ *under the adjunction* $(\alpha_!, \alpha^*)$.

Proof. The proof of Proposition 5.3.11 can be adapted to this context with minor changes; let us explain how. Assume we are given a lift γ as in the statement. We construct, for any edge e of T and any non-degenerate chain $w \colon \Delta^k \to \mathcal{A}^T(e)$, a map of simplicial sets $\Gamma_w \colon \Delta^k \to F_{\alpha(e)}$ such that:

1. the family $\{\Gamma_w\}_w$ defines a morphism $\Gamma \colon (\Lambda^x T, \alpha j) \to \rho^*(F)$;
2. $\Gamma j = \chi$, which means that if $\partial \colon S \to T$ is a dendroidal face map with $\partial \neq \partial_x$, and $w \colon \Delta^k \to \mathcal{A}^T(e)$ factors as $\Delta^k \xrightarrow{w'} \mathcal{A}^S(e') \to \mathcal{A}^T(\partial(e'))$ for some w' then $\Gamma_w = \chi_{w'}$;
3. $\rho^*(f) \circ \Gamma = \xi$.

Fix such T, e and w. If $e \neq r_T$, the subtree inclusion $\partial \colon T_e^{\uparrow} \to T$ is necessarily different from ∂_x, hence the restriction of χ to ∂ is well-defined and we can define

$$\Gamma_w := \chi_w.$$

If $e = r_T$, we define

$$\Gamma_w := \gamma \circ w.$$

To see that the family $\{\Gamma_w\}_w$ satisfies the three required conditions, let us go over the arguments used in Proposition 5.3.11 and see why they can be adapted to this case as well:

1. Consider the diagrams as in point (1) of the proof of Proposition 5.3.11. Whether $e \neq r_T$ or $e = r_T$, that the compatibility condition holds can be proven by using the same arguments as in Proposition 5.3.11, since they do not involve restricting χ to the face ∂_x.

2. Consider a dendroidal face $\partial\colon S \to T$, $\partial \neq \partial_x$, and a simplex $w\colon \Delta^i \to \mathcal{A}^T(e)$ which factors as $\Delta^i \xrightarrow{w'} \mathcal{A}^S(e') \xrightarrow{\partial} \mathcal{A}^T(\partial(e'))$ for some $e' \in E(S)$ with $\partial(e') = e$. If $e \neq r_T$, as the tree $T_e^{\uparrow}$ cannot be isomorphic to the face $\partial_x T$, we can reason as in the proof of Proposition 5.3.11 and prove that $\Gamma_w = \gamma_{w'}$. If $e = r_T$, we can still apply the same arguments, because ∂ being root preserving and the assumption $\partial \neq \partial_x$ ensure that w belongs to $\Lambda^x\mathcal{A}^T(r_T)$, hence we get $\Gamma_w = \gamma_{w'}$ in this case as well.

3. This point can be proven in a completely analogous way as point (3) in the proof of Proposition 5.3.11.

This concludes the proof. □

5.3.4 The Quillen adjunction

We are ready to prove the main result of this section.

Theorem 5.3.16. *The straightening-unstraightening adjunction*

$$\rho_! : \mathbf{dSets}/\mathcal{N}_d P \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho^*$$

is a Quillen adjunction between the covariant model structure on $\mathbf{dSets}/\mathcal{N}_d P$ *and the projective model structure on* $\mathsf{Alg}_P(\mathbf{sSets})$.

In order to prove the theorem, we first show in Lemma 5.3.18 that the inclusion $\Lambda^x\mathcal{A}^T(r_T) \to \mathcal{A}^T(r_T)$ is anodyne. Recall that a map of simplicial sets is called *anodyne* if it has the left lifting property with respect to Kan fibrations, and *left*, resp. *inner anodyne* if it has the left lifting property with respect to left fibrations, resp. inner Kan fibration. Anodyne morphisms, as well as left and inner anodyne morphisms, form a saturated class of morphisms, which means that they are closed under transfinite composition, retracts and pushouts. We will need the following definition.

Definition 5.3.17. Let E be a non-empty subset of $\{0, 1, \ldots, n\}$. The *generalized horn* Λ^n_E is defined as the subsimplicial set of Δ^n obtained as the union of the faces $\partial_i \Delta^n$ with $i \notin E$. This determines an inclusion $\Lambda^n_E \hookrightarrow \Delta^n$, which we call *generalized horn inclusion*.

After [HM22, Lemma 5.23], a generalized horn inclusion $\Lambda^n_E \to \Delta^n$ is anodyne, and it is left anodyne if $E \subseteq \{0, 1, \ldots, n-1\}$, inner anodyne if $E \subseteq \{1, \ldots, n-1\}$.

We can now prove the next fundamental lemma.

Lemma 5.3.18. *Let T be a tree, and let x be either an inner edge or a leaf vertex. The inclusion*

$$\Lambda^x\mathcal{A}^T(r_T) \hookrightarrow \mathcal{A}^T(r_T)$$

is an anodyne morphism of simplicial sets. In particular, if x is an inner edge it is inner anodyne, and if x is a leaf vertex it is left anodyne.

Proof. We use the critherion of [HM22, Lemma 5.20] and prove the thesis by showing that the inclusion $\Lambda^x\mathcal{A}^T(r_T) \to \mathcal{A}^T(r_T)$ can be obtained as the transfinite (in fact, finite) composition of pushouts of generalized horn inclusions.

Let us start with the case x is an inner edge of T; the case of x leaf vertex will be similar. First of all, observe that any non-degenerate simplex in $\mathcal{A}^T(r_T)$ can be obtained as the face of a maximal chain in $\mathcal{A}^T(r_T)$. In particular, any maximal chain u has the property of having index j such that u_{j-1} is obtained from u_j by adding the output vertex of x, for which we write $u_j = u_{j+1} \cup \{O(x)\}$. We introduce the following terminology: a non-degenerate, not necessarily maximal, k-simplex u of $\mathcal{A}^T(r_T)$ has ***height*** j, for $j \in \{0, 1, \dots, k-1\}$, if $u_{k-j} = u_{k-j+1} \cup \{O(x)\}$. In other words, having height j means that x appears for the first time in the j^{th} smallest tree of the chain of inclusions determined by u, and that x first appears as a leaf. Observe that there are no chains of height 0, and in general the maximal possible height cannot be greater than the number $\#V(T)$ of vertices of T.

Consider the sequence of inclusions

$$\Lambda^x \mathcal{A}^T(r_T) = B^{(0)} \subseteq B^{(1)} \subseteq \cdots \subseteq B^{(k)} \subseteq \dots,$$

where $B^{(k)}$ is the subsimplicial set of $\mathcal{A}^T(r_T)$ generated by $\Lambda^x \mathcal{A}^T(r_T)$ and the non-degenerate simplices of $\mathcal{A}^T(r_T)$ of height at most k. We have that $B^{(k)} = B^{(k+1)}$ for $k \geqslant \#V(T)$, and by the previous discussion

$$\mathcal{A}^T(r_T) = \bigcup_k B^{(k)}.$$

It follows that it suffices to show that each inclusion $B^{(k)} \hookrightarrow B^{(k+1)}$ is inner anodyne. To prove this, we apply the same method and observe that we can write a chain of inclusions

$$B^{(k)} = B_0^{(k+1)} \subseteq B_1^{(k+1)} \subseteq \cdots \subseteq B_h^{(k+1)} \subseteq \dots,$$

where $B_h^{(k+1)}$ is the subsimplicial set of $B^{(k+1)}$ generated by $B^{(k)}$ and the non-degenerate simplices of dimension at most h and of height precisely $k+1$. Observe that we have

$$B_h^{(k+1)} = B^{(k)} \text{ for } h \leqslant k, \text{ and } B_h^{(k+1)} = B_{h+1}^{(k+1)} \text{ for } h \geqslant \#V(T) + 1.$$

Moreover, we have

$$B^{(k+1)} = \bigcup_h B_h^{(k+1)},$$

so we need just to show that for $k \leqslant h \leqslant \#V(T)$, the inclusion $B_h^{(k+1)} \hookrightarrow B_{h+1}^{(k+1)}$ is inner anodyne. Consider then a $(h+1)$-simplex u of height $k+1$ which does not belong to $\Lambda^x \mathcal{A}^T(r_T)$, and let $d_\ell(u)$ be its ℓ^{th} face, with $\ell \in \{0, 1, \dots, h+1\}$. If $0 \leqslant \ell < k-h$, $d_\ell(u)$ is a non-degenerate h-simplex of height $k+1$, hence it belongs to $B_h^{(k+1)}$. If $k-h+1 < \ell \leqslant h+1$, then $d_\ell(u)$ is a non-degenerate h-simplex of height k, hence it belongs to $B^{(k)}$. For $\ell \in \{h-k, h-k+1\}$, if u does not belong to $\Lambda^x \mathcal{A}^T(r_T)$, then $d_\ell(u)$ cannot belong to it either, and $d_\ell(u)$ cannot belong to $B_h^{(k+1)}$ either: indeed, if $\ell = h-k$, than $d_\ell(u)$ is a chain where x firstly appears as an inner edge, while for $\ell = h-k+1$, in the chain $d_\ell(u)$ the subtree where x first appears (as a leaf) is obtained from the previous one by adding at least two vertices. Moreover, one can check that in both cases $d_\ell(u)$ cannot be obtained as the face of a smaller simplex of height $k+1$ nor of a simplex of height k. The same reasoning applies to the intersection of these particular kinds of inner faces $d_\ell(u)$, showing that they are not contained in $B_h^{(k+1)}$, and by applying [HM22, Lemma 5.24], which yields a critherion on how to identify generalized horns, we obtain a pushout diagram of the following form

$$\begin{tikzcd}
\coprod_u \Lambda_E^{h+1} \arrow[r] \arrow[d] & B_h^{(k+1)} \arrow[d] \\
\coprod_u \Delta^{h+1} \arrow[r] & B_{h+1}^{(k+1)}
\end{tikzcd} \tag{5.3.8}$$

where u ranges over all the non-degenerate $h+1$-simplices of height $k+1$, and $E = \{h-k, h-k+1\}$. Since x is an inner edge, one necessarily has $E \subseteq \{1, \ldots, h\}$, and this shows that $B_h^{(k+1)} \to B_{h+1}^{(k+1)}$ is inner anodyne ([HM22, Lemma 5.23]). It follows that the inclusion $B^{(k)} \to B^{(k+1)}$ is inner anodyne, and so is the morphism $\Lambda^x \mathcal{A}^T(r_T) \to \mathcal{A}^T(r_T)$, as wanted.

Suppose now that x is a leaf vertex of T. To show that $\Lambda^x \mathcal{A}^T(r_T) \to \mathcal{A}^T(r_T)$ is anodyne we proceed similarly as above but adjusting the filtrations to the case of x being a leaf vertex. More precisely, we go as follow. First, we say that a non-degenerate k-simplex u of $\mathcal{A}^T(r_T)$ has height j if u_{k-j} can be obtained from u_{k-j+1} just by adding the vertex x, for which we write $u_{k-j} = u_{k-j+1} \cup \{x\}$. We then define $B^{(k)}$ as the subsimplicial set of $\mathcal{A}^T(r_T)$ generated by $\Lambda^x \mathcal{A}^T(r_T)$ and the non-degenerate simplices of height at most k. We have a chain of inclusions

$$\Lambda^x \mathcal{A}^T(r_T) = B^{(0)} \subseteq B^{(1)} \subseteq \cdots \subseteq B^{(k)} \subseteq \ldots$$

and

$$\mathcal{A}^T(r_T) = \bigcup_k B^{(k)},$$

therefore it suffices to show that $B^{(k)} \hookrightarrow B^{(k+1)}$ is left anodyne. We then write

$$B^{(k)} = B_0^{(k+1)} \subseteq B_1^{(k+1)} \subseteq \ldots,$$

with $B_h^{(k+1)}$ the subsimplicial set of $B^{(k+1)}$ generated by $B^{(k)}$ and the non-degenerate simplices of height $k+1$ and dimension at most h. We have

$$B^{(k+1)} = \bigcup_h B_h^{(k+1)},$$

and again we reduce to showing that $B_h^{(k+1)} \hookrightarrow B_{h+1}^{(k+1)}$ is left anodyne. This is true because if we consider a non-degenerate $h+1$-simplex u of height $k+1$ which does not belong to $\Lambda^x T$ and $\ell \in \{0, 1, \ldots, h+1\}$, we see that $d_\ell(u)$ belongs to $B_h^{(k+1)}$ for $0 \leqslant \ell < h-k$, to $B^{(k)}$ for $h-k+1 < \ell \leqslant h+1$ and that it cannot belong to $B_h^{(k+1)}$ for $\ell \in \{h-k, h-k+1\}$; in this latter case, the same holds for the intersections of these faces. We therefore obtain a pushout diagram as that in Equation (5.3.8), where $E = \{h-k, h-k+1\}$ is now necessarily contained in $\{0, 1, \ldots, h\}$, as x is a leaf vertex. This shows $B_h^{(k+1)} \to B_{h+1}^{(k+1)}$ is left anodyne, therefore also $B^{(k)} \hookrightarrow B^{(k+1)}$ is so, which means that the inclusion $\Lambda^x \mathcal{A}^T(r_T) \hookrightarrow \mathcal{A}^T(r_T)$ is left anodyne, as wanted.

□

Proof (of Theorem 5.3.16). Recall that a morphism $(X, f) \to (Y, g)$ in $\mathbf{dSets}/\mathcal{N}_d P$ is a left fibration, resp. trivial fibration, if it has the right lifting property against inner and leaf horn inclusions $(\Lambda^x T, \alpha \circ j) \to (T, \alpha)$, resp. boundary inclusions $(\partial T, \alpha \circ \iota) \to (T, \alpha)$. A morphism of P-algebras $F \to G$ is a projective fibration, resp. projective trivial fibration, if for any object c of P, the map of simplicial sets $F(c) \to G(c)$ has the right lifting property with respect to anodyne morphisms, resp. inclusions of simplicial sets.

After Proposition 5.3.11, the functor ρ^* sends covariant trivial fibrations to projective trivial fibrations, and combining Proposition 5.3.15 with Lemma 5.3.18 we have that ρ^* sends projectively fibrant objects to left fibrations. As covariant fibrations between fibrant objects are left fibrations, this means that ρ^* preserves fibrations between fibrant objects. It is a standard model-categorical fact that this is enough to ensure it preserves all fibrations.

We have shown that ρ^*_P is a right Quillen functor, and this concludes the proof. □

5.4 Monoidal rectification of left fibrations

Consider $P = A$ a discrete category. The rectification functor and the relative dendroidal nerve for A yield an adjunction

$$\rho^A_! : \mathbf{sSets}/\mathcal{N}A \rightleftarrows \mathsf{Fun}(A, \mathbf{sSets}) : \rho^*_A$$

which is a Quillen adjunction with respect to the covariant model structure on the left hand side and the projective model structure on the right hand side. As observed in Remark 5.2.7, this adjunction recovers the one defined in [HM15], where it is proven that it is a Quillen equivalence.

In this section, we

1. give a new independent proof that $(\rho^A_!, \rho^*_A)$ is a Quillen equivalence for a discrete category A. Contrarily to the one in [HM15], it does not rely on comparing $\rho^A_!$ with the homotopy colimit functor;
2. improve the above result by showing that, when A is a symmetric monoidal category, the Quillen equivalence $(\rho^A_!, \rho^*_A)$ is in fact a monoidal Quillen equivalence of monoidal model categories.

In Section 5.5 we will actually prove that $(\rho^P_!, \rho^*_P)$ is a Quillen equivalence for *any* Σ-free discrete operad P, but that will require to use ∞-categories, while in this section we are still able to prove our results with model categorical and point-set level methods.

5.4.1 The Quillen equivalence: discrete categories

Fix a discrete category A and write $\rho_!$, resp. ρ^*, for $\rho^A_!$, resp. ρ^*_A. Consider a simplicial set X and an element (X, f) in $\mathbf{sSets}/\mathcal{N}A$. The element (X, f) is the colimit of representables,

$$(X, f) \simeq \operatorname*{colim}_{([n],\alpha)\to(X,f)}([n], \alpha).$$

As colimits of functors can be computed objectwise, for every object a of A one has

$$\rho_!(X, f) \simeq \operatorname*{colim}_{([n],\alpha)\to(X,f)} \mathcal{N}(\alpha/a).$$

Now, the nerve of the poset α/a can be written as the pullback

$$\mathcal{N}(\alpha/a) \simeq \Delta^n \times_{\mathcal{N}A} \mathcal{N}(A/a),$$

hence

$$\rho_!(A,\alpha) \simeq X \times_{\mathcal{N}A} \mathcal{N}(A/a). \tag{5.4.1}$$

With this description, we can prove that the Quillen adjunction $(\rho_!, \rho^*)$ is in fact a Quillen equivalence. This was already shown in [HM15] by comparing the right derived functor of ρ^* with the left derived one of the homotopy colimit functor. Here, we present a different proof based entirely on simplicial sets and anodyne maps; our strategy is essentially due to Gijs Heuts.

Theorem 5.4.1. *For any discrete category A, the Quillen adjunction*

$$\rho_! \colon \mathbf{sSets}/\mathcal{N}A \rightleftarrows \mathsf{Fun}(A, \mathbf{sSets}) : \rho^*$$

is a Quillen equivalence between the covariant and the projective model structure.

Proof. It is a standard result in model category theory that it is enough to prove that ρ^* reflects weak equivalences between fibrant objects and that the derived unit evaluated at a bifibrant object is a weak equivalence. Let us prove these two facts.

Let $\varphi\colon F \to G$ be a morphism between Kan-enriched functors. Since ρ^* is right Quillen, both $\rho^*(F)$ and $\rho^*(G)$ are left fibrations, which means that $\rho^*(\varphi)$ is a covariant weak equivalence if and only if it is a fibrewise weak homotopy equivalence of spaces. Given an object a of A, after Lemma 5.2.14 there is a natural equivalence between the map of fibres

$$\rho^*(\varphi)_a \colon \rho^*(F)_a \longrightarrow \rho^*(G)_a$$

and the map between the evaluations

$$\varphi_a \colon F(a) \longrightarrow G(a).$$

Therefore $\rho^*(\varphi)$ is a weak equivalence if and only if φ is, so in particular ρ^* reflects weak equivalences.

For the second condition, observe that every object in $\mathbf{sSets}/\mathcal{N}A$ is cofibrant, so let (X, f) be a left fibration and $\mathbb{L}\eta_{(X,\alpha)}$ be the derived unit evaluated at (X, f). This can be written as the composition

$$\mathbb{L}\eta_{(X,f)} \colon (X, f) \xrightarrow{\eta_{(X,f)}} \rho^*\rho_!(X, f) \xrightarrow{\rho^*(\mathcal{R})} \rho^*(F),$$

where $\mathcal{R}\colon \rho_!(X, f) \xrightarrow{\sim} F$ is a fibrant replacement for $\rho_!(X, \alpha)$. As both domain and codomain are left fibrations and weak equivalences between left fibrations are fibrewise weak homotopy equivalences, the derived unit is a weak equivalence if and only if it is fibrewise so. Consider an object a of A; as $\rho^*(\mathcal{R})_a \simeq \mathcal{R}_a$ and $\mathcal{R}$ is a projective weak equivalence, it is sufficient to show that the map of fibres $(X, f)_a \to \rho^*(\rho_!(X, f))_a \simeq \rho_!(X, f)(a)$ given by the non-derived unit is a weak homotopy equivalence of simplicial sets. This map appears as the top horizontal arrow in the pullback

$$\begin{tikzcd}
(X,f)_a \arrow[r] \arrow[d] & X \times_{\mathcal{N}A} \mathcal{N}(A/a) \arrow[d] \\
\Delta^0 \arrow[r, "\{\mathrm{id}_a\}"] & \mathcal{N}(A/a) .
\end{tikzcd}$$

The map $\Delta^0 \xrightarrow{\{\mathrm{id}_a\}} \mathcal{N}(A/a)$ is *right anodyne*, which means it belongs to the closure under pushouts, retracts and transfinite compositions of the set of right horn inclusions $\{\Lambda^n_k \to \Delta^n\}_{n,0<k\leqslant n}$. As left fibrations are stable under pullbacks, also the map $X \times_{\mathcal{N}A} \mathcal{N}(A/a) \to \mathcal{N}(A/a)$ is a left fibration. As proven in [HHR21, Proposition 2.10], the pullback of a right anodyne map along a left fibration is again right anodyne. As right anodyne maps are in particular weak homotopy equivalences of simplicial sets, this concludes the proof. ☐

5.4.2 The monoidal Quillen equivalence

Consider now a symmetric monoidal category $A = (A, \otimes, 1_A)$. As simplicial sets are monoidal with respect to cartesian product, the category $\mathsf{Fun}(A, \mathbf{sSets})$ has a symmetric monoidal structure given by *Day convolution*: given functors $F, G\colon A \to \mathbf{sSets}$, their tensor product $F \underset{Day}{\otimes} G\colon A \to \mathbf{sSets}$ is the two-variables left Kan extension of the product of F and G

$$A \times A \longrightarrow \mathbf{sSets}\ , \ (a,b) \mapsto F(a) \times G(b)$$

along the tensor product of A, so that

$$\left(F \underset{Day}{\otimes} G\right)(c) \simeq \underset{a\otimes b \to c}{\mathsf{colim}}\ F(a) \times G(b).$$

The unit is given by the Yoneda embedding of the unit of A, seen as a discrete simplicial presheaf on A.

The over-category $\mathbf{sSets}/\mathcal{N}A$ also has a symmetric monoidal structure $\boxtimes$, defined as

$$\boxtimes\colon \mathbf{sSets}/\mathcal{N}A \times \mathbf{sSets}/\mathcal{N}A \xrightarrow{-\times-} \mathbf{sSets}/(\mathcal{N}A \times \mathcal{N}A) \xrightarrow{m_!} \mathbf{sSets}/\mathcal{N}A,$$

where $m_!$ is the post-composition with the nerve of the tensor product of A. The unit for the monoidal structure on $\mathbf{sSets}/\mathcal{N}A$ is given by the map $(*, \mathbb{1}_A\colon * \xrightarrow{\{1_A\}} \mathcal{N}A)$ selecting the tensor unit of A.

A *monoidal Quillen model category* is a symmetric monoidal category $(M, \otimes, 1_M)$ endowed with a model category structure and whose tensor product which is appropriately compatible with the model category structure, ensuring, for instance, that the homotopy category is also symmetric monoidal. In particular, in a monoidal model category the tensor product has to be a *left Quillen bifunctor*: given two cofibrations $i\colon X \to Y$, $j\colon U \to V$, their *pushout product*

$$i \square j\colon X \otimes V \bigcup_{X\otimes U} Y \otimes U \longrightarrow Y \otimes V$$

is a cofibration, which is also a weak equivalence if i or j is.

Proposition 5.4.2. *Let $(A, \otimes, 1_A)$ be a symmetric monoidal category. The covariant model structure on the symmetric monoidal category $(\mathbf{sSets}/\mathcal{N}A, \boxtimes, \mathbb{1}_A)$ is model monoidal.*

Proof. The functor $m_!$ is a left Quillen functor ([HM22]), and as the unit of the tensor product is cofibrant, the statement reduces to showing that $- \boxtimes -$ is a left Quillen bifunctor. It is clearly cocontinuous in both variables, and it is also easy to see that the pushout product of two cofibrations is a cofibration as well. Consider a trivial cofibration

$i\colon (X,f) \to (Y,g)$; its pushout-product $i \mathbin{\square} j$ with $j\colon (U,h) \to (V,k)$ is the morphism appearing in the following diagram

$$\begin{tikzcd}
(X,f) \times (U,h) \arrow[r] \arrow[d, "i \times \mathrm{id}"'] & (X,f) \times (V,k) \arrow[d] \arrow[rdd, bend left, "i \times \mathrm{id}"] & \\
(Y,g) \times (U,h) \arrow[r] \arrow[rrd, bend right] & \bullet \arrow[rd, "i \mathbin{\square} j"] & \\
 & & (Y,g) \times (V,k)
\end{tikzcd}$$

By [NS17, Lemma 3.3], for any two simplicial sets S, T the functor

$$- \times S\colon \mathbf{sSets}/T \longrightarrow \mathbf{sSets}/T \times S, \quad (Z,\beta) \mapsto (Z \times S, \beta \times \mathrm{id})$$

is left Quillen with respect to the *contravariant* model structure, which is obtained from the covariant model structure by applying the duality $\Delta \to \Delta,\ [n] \mapsto [n]^{\mathrm{op}}$. It follows that $- \times S$ is left Quillen also with respect to the covariant model structure.

In particular, $i \times \mathrm{id}$ is a weak equivalence; since the covariant model structure is also left proper, the morphism $(X,f) \times (V,k) \to \bullet$ is a weak equivalence as well, and by the 2-out-of-3 property $i \mathbin{\square} j$ is a weak equivalence as well. This concludes the argument. $\square$

Remark 5.4.3. In particular, Proposition 5.4.2 gives a presentation of the symmetric monoidal ∞-category $(\mathsf{Left}_{\mathcal{N}(A)})^{\otimes}$ introduced in Section 4.3.1. That this was abstractly possible – to present a presentable symmetric monoidal ∞-category by a simplicial, combinatorial and left proper symmetric monoidal model category – follows from Nikolaus-Sagave's [NS17, Theorem 1.1].

After [Isa09, Proposition 2.2.15], for any symmetric monoidal category $(A, \otimes, 1_A)$, the projective model structure on $\mathsf{Fun}(A, \mathbf{sSets})$ is model monoidal with respect to Day convolution. We now prove that the rectification functor respects the monoidal structure.

Theorem 5.4.4. *If A is a symmetric monoidal category, the Quillen adjunction*

$$\rho_!\colon \mathbf{sSets}/\mathcal{N}A \underset{\perp}{\rightleftarrows} \mathsf{Fun}(A, \mathbf{sSets}) : \rho^*$$

is a monoidal Quillen equivalence of monoidal model categories for the above monoidal model structures.

Proof. After Theorem 5.4.1, the adjunction $(\rho_!, \rho^*)$ is a Quillen equivalence, so we only need to prove $\rho_!$ is monoidal. Write $X := \mathcal{N}A$, and fix two elements (U,h) and (V,k) in $\mathbf{sSets}/X$. We prove that $\rho_!$ is monoidal in three steps.

1. First, we prove that $\rho_!$ is lax monoidal. We construct the laxity map

$$\rho_!(U,h) \otimes_{\mathrm{Day}} \rho_!(V,k) \longrightarrow \rho_!((U,h) \boxtimes (V,k))$$

as follows. Fix an object a of A and a map $\phi\colon b \otimes c \to a$ for objects b and c. There is a commutative diagram

$$\begin{tikzcd}
(U \times V) \underset{\mathcal{N}A\times\mathcal{N}A}{\times} (\mathcal{N}(A/a) \times \mathcal{N}(A/b)) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & U \times V \arrow[rr, "\beta\times\gamma"] & & \mathcal{N}A \times \mathcal{N}A \arrow[d, "\otimes"] \\
\mathcal{N}(A/a) \times \mathcal{N}(A/b) \arrow[r, "m_!"] & \mathcal{N}(A/a \otimes b) \arrow[r, "\phi_*"] & \mathcal{N}(A/c) \arrow[r] & \mathcal{N}A
\end{tikzcd}$$

and they induce the map

$$\left(\rho_!(U,h) \underset{\mathrm{Day}}{\otimes} \rho_!(V,k)\right)(c) \simeq \underset{a\otimes b\to c}{\mathrm{colim}}\,(U\times V) \underset{\mathcal{N}A\times\mathcal{N}A}{\times} (\mathcal{N}(A/a) \times \mathcal{N}(A/b)) \longrightarrow (U\times V)\times_{\mathcal{N}A}\mathcal{N}(A/c),$$

as wanted.

2. The functor $\rho_!$ is also colax monoidal, and the colaxity map

$$\rho_!((U,h) \boxtimes (V,k)) \longrightarrow \rho_!(U,h) \underset{\mathrm{Day}}{\otimes} \rho_!(V,k)$$

is constructed as follows. Let c be an object in A and let $n \geqslant 0$. A n-simplex X of $\rho_!((U,h) \boxtimes (V,k))(c)$ corresponds to a tuple $((a,b),x)$, where $a \in U_n$, $b \in V_n$, $x \in \mathcal{N}(A/c)_n$, satisfying the property that $(h \boxtimes k)_n(a,b) = \mathcal{U}(x)$, where $\mathcal{U}\colon \mathcal{N}(A/c) \to \mathcal{N}(A)$ is the nerve of the forgetful functor. In other words, if we write

$$h_n(a) = a_0 \to \cdots \to a_n, \quad k_n(b) = b_0 \to \dots b_n, \text{ and } x = c_0 \to c_1 \to \cdots \to c_n \to c \quad ,$$

the property that $((a,b),x)$ satisfies is that

$$c_i = a_i \otimes b_i \text{ and } c_i \to c_{i+1} = (a_i \to a_{i+1}) \otimes (b_i \to b_{i+1})$$

for all i for which it makes sense.

In particular, the couple

$$((a, a_0 \to \cdots \to a_n \xrightarrow{id} a_n), (b, b_0 \to \cdots \to b_n \xrightarrow{id} b_n))$$

determines a n-simplex of

$$\left(U \underset{\mathcal{N}A}{\times} \mathcal{N}(A/a_n)\right) \times \left(V \underset{\mathcal{N}A}{\times} \mathcal{N}(A/b_n)\right)$$

which determines a n-simplex of the colimit $(\rho_!(h) \underset{\mathrm{Day}}{\otimes} \rho_!(k))(c)$ relative to the component determined by the morphism $(c_n = a_n \otimes b_n) \to c$.

3. $\rho_!$ is strong monoidal, because the laxity and colaxity maps of, resp., points (1) and (2), are one the inverse of the other. That the composite

$$\rho_!((U,h) \boxtimes (V,k)) \longrightarrow \rho_!(U,h) \underset{\mathrm{Day}}{\otimes} \rho_!(V,k) \longrightarrow \rho_!(h \boxtimes k)$$

is the identity is immediate to check; to see that the composite

$$\rho_!(U,h) \underset{\mathrm{Day}}{\otimes} \rho_!(V,k) \longrightarrow \rho_!((U,h) \boxtimes (V,k)) \longrightarrow \rho_!(U,h) \underset{\mathrm{Day}}{\otimes} \rho_!(V,k)$$

is the identity as well, it suffices to notice that, given a morphism $\varphi\colon d_1 \otimes d_2 \to c$ (using the same notations as in the previous point), the n-simplices

$$((a, a_0 \to \cdots \to a_n \to d_1), (b, b_0 \to \cdots \to b_n \to d_2)) \text{ in } \rho_!(U,h)(d_1) \times \rho_!(V,k)(d_2)$$

and

$$((a, a_0 \to \cdots \to a_n \xrightarrow{\mathrm{id}} a_n), (b, b_0 \to \cdots \to b_n \xrightarrow{\mathrm{id}} b_n)) \text{ in } \rho_!(U,h)(a_n) \times \rho_!(V,k)(b_n)$$

are identified under Day convolution.

□

We can reformulate this result in the language of ∞-categories (see Section 3.1), obtaining the following

Corollary 5.4.5 (Theorem 5.4.4)**.** *Let A be a discrete symmetric monoidal category. There is an equivalence of symmetric monoidal ∞-categories*

$$(\rho_!^A)_\infty^\otimes \colon (\mathsf{Left}_{\mathcal{N}A})^\otimes \overset{\perp}{\rightleftarrows} \mathsf{Fun}(A, \mathcal{S})^\otimes : (\rho_A^*)_\infty^\otimes,$$

which on the underlying categories coincides with the equivalence $((\rho_!^A)_\infty, (\rho_A^*)_\infty)$. In particular, given a left fibration (X, α) over $\mathcal{N}A$, there is an equivalence of functors

$$(\rho_!^A)_\infty(A, \alpha) \simeq X \times_{\mathcal{N}A} \mathcal{N}A_{/-}.$$

Remark 5.4.6. In other words, Corollary 5.4.5 states that $(\rho_!^A, \rho_A^*)$ presents Ramzi's monoidal Grothendieck construction for a discrete symmetric monoidal ∞-category A. Observe that the fact that it was abstractly possible to present the monoidal Grothendieck construction as a Quillen equivalence between monoidal model category was ensured by Nikolaus-Sagave's presentability results of [NS17].

5.5 The Quillen equivalence

After Section 5.4, we know that for a discrete category A and an element (M, f) of sSets$/\mathcal{N}A$, for every object a of A one has

$$\rho_!^A(M,f)(a) \simeq X \times_{\mathcal{N}A} \mathcal{N}(A/a).$$

Consider now a discrete operad P. As observed in Lemma 5.2.9, if (T, α) is a representable dendroidal set over $\mathcal{N}_d P$, we have an equivalence of simplicial P-algebras

$$\rho_!^P(T, \alpha) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/-}.$$

However, explicitely describing the simplicial P-algebra $\rho_!^P(X, f)$ for (X, f) not representable in **dSets**$/\mathcal{N}_d P$ is a hard procedure, essentially because although we can express (X, f) as colimit of representables, (non-filtered) colimits in $\mathsf{Alg}_P(\mathrm{sSets})$ cannot be computed object-wise when P has also non-unary multimorphisms.

The solution we adopt is to move to the language of ∞-categories. Recalling the dictionary in Section 3.1, we fist observe that the Quillen adjunction proven with Theorem 5.3.16 yields the following

Corollary 5.5.1 (of Theorem 5.3.16)**.** *Let P be a discrete operad. The Quillen adjunction $(\rho_!^P, \rho_P^*)$ induces an adjunction of ∞-categories*

$$(\rho_!^P)_\infty \colon \mathsf{dLeft}_{\mathcal{N}_d P} \rightleftarrows \mathsf{Alg}_P(\mathcal{S}) : (\rho_P^*)_\infty,$$

where $\mathsf{dLeft}_{\mathcal{N}_d P}$ is the ∞-category underlying the covariant model structure on $\mathbf{dSets}/\mathcal{N}_d P$ and $\mathsf{Alg}_P(\mathcal{S})$ the one underlying the projective model structure on simplicial P-algebras.

We now relate $(\rho_!^P)_\infty$ with the operadic straightening functor St^P for the Lurie ∞-operad $\mathcal{N}^\otimes P$ constructed in Theorem 4.5.1. Recall that it consists of an equivalence of ∞-categories

$$\mathsf{St}^P \colon \mathsf{Left}^{\mathsf{lax}}_{\mathcal{N}^\otimes P} \longrightarrow \mathsf{Alg}_P(\mathcal{S})$$

from the ∞-category of operadic left fibrations over $\mathcal{N}^\otimes(P)$ (Definition 4.2.7) to that of P-algebras, obtained as the composition of the symmetric monoidal envelope functor followed by the categorical straightening functor for this latter; in symbols, and somehow imprecisely, we write

$$\mathsf{St}^P(\mathcal{O}, \alpha) := \mathsf{St}^{\mathsf{Env}(P)}(\mathsf{Env}(\mathcal{O}), \mathsf{Env}(\alpha))$$

for the P-algebra associated to an operadic left fibration $(\mathcal{O}, \alpha)$ in $\mathsf{Left}^{\mathsf{lax}}_{\mathcal{N}^\otimes P}$.

To transition between Lurie's and the dendroidal model for operadic left fibrations, we exploit the fact that Hinich-Moerdijk comparison functors induce an equivalence between dendroidal and operadic left fibrations, which we showed in Theorem 4.2.15. By using compatibility between the two dendroidal models of complete dendroidal Segal spaces and quasioperads (see Section 3.8), we obtain an equivalence

$$\lambda'_/ \colon \mathsf{dLeft}_{\mathcal{N}_d P} \xrightarrow{d_!} \mathsf{DLeft}_{d_! \mathcal{N}_d P} \xrightarrow{\lambda_/} \mathsf{Left}^{\mathsf{opd}}_{\mathcal{N}^\otimes P}.$$

We now have all the tools to prove the following result.

Theorem 5.5.2. *Let P be a discrete Σ-*free *operad. There is a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathsf{dLeft}_{\mathcal{N}_d P} \arrow[r, "(\rho_!^P)_\infty"] \arrow[d, "\lambda'_/"'] & \mathsf{Alg}_P(\mathcal{S}) \\
\mathsf{Left}^{\mathsf{opd}}_{\mathcal{N}^\otimes P} \arrow[ur, "\mathsf{St}^P"'] &
\end{tikzcd}$$

In other words, the Quillen adjunction $(\rho_!^P, \rho_P^*)$ gives a presentation of the equivalence of ∞-categories given by the operadic un/straightening equivalence $(\mathsf{St}^P, \mathsf{Unst}^P)$. In particular, $(\rho_!^P)_\infty$ is an equivalence of ∞-categories. From this we deduce our final wanted result.

Corollary 5.5.3. *Let P be a Σ-free discrete operad. The adjoint pair*

$$\rho_!^P \colon \mathbf{dSets}/\mathcal{N}_d P \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho_P^*$$

is a Quillen equivalence between the covariant and the projective model structure.

Let us now give the proof of Theorem 5.5.2.

Proof. [*of Theorem 5.5.2*] First of all, let us represent both $(\rho_!^P)_\infty$ and St^P as cocontinuous functors of ∞-categories

$$\mathsf{DLeft}_X \longrightarrow \mathsf{Alg}_P(\mathcal{S}),$$

where $X \simeq d_!(\mathcal{N}_d P)$. Since DLeft_X is a localization of $\mathsf{DOp}_{\infty/X}$, precomposition with the localization functor yields two cocontinuous functors

$$\mathsf{DOp}_{\infty/X} \longrightarrow \mathsf{Alg}_P(\mathcal{S}),$$

which we denote, respectively, by ϱ and Ξ, and it is enough to prove an equivalence $\varrho \simeq \Xi$. Recall now that $\mathsf{DOp}_{\infty/X}$ is a localization of the presheaf ∞-category

$$\mathsf{PSh}(\mathbf{\Omega})_{/X} \simeq \mathsf{PSh}(\mathbf{\Omega}/X),$$

and by [Lur09, Corollary 5.1.5.8.] this latter is generated under colimits by elements of the form (T, α), with T a tree and $\alpha\colon T \to P$ a map of operads. In particular, also $\mathsf{DOp}_{\infty/X}$ is (non-freely) generated by the elements of the form (T, α), and by cocontinuity of both ϱ and Ξ it is enough to prove that for any such object one has

$$\varrho(T, \alpha) \simeq \Xi(T, \alpha).$$

By Lemma 5.2.9, the P-algebra $\varrho(T, \alpha)$ can be written as

$$\varrho(T, \alpha) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/-}.$$

To compute $\Xi(T, \alpha)$, consider the following commutative diagram of ∞-categories:

$$\begin{tikzcd}
\mathsf{DOp}_\infty \arrow[r, "\lambda"] & \ell\mathsf{Op}_\infty \arrow[r, "\mathsf{Env}^\otimes"] & \mathsf{smCat}_\infty \\
& \mathsf{Op} \arrow[ul, "d_!\mathcal{N}_d"] \arrow[u, "\mathcal{N}^\otimes"] \arrow[r, "\mathsf{env}"] & \mathsf{smCat} \arrow[u, "\mathcal{N}^\otimes"']
\end{tikzcd}$$

where the vertical arrows are the nerve constructions. Combining it with the fact that the rectification functor of a discrete symmetric monoidal category presents the monoidal Grothendieck construction (Corollary 5.4.5), we also get an equivalence of P-algebras

$$\Xi(T, \alpha) \simeq \mathsf{Env}(T) \times_{\mathsf{Env}(P)} \mathsf{Env}(P)_{/-}.$$

This means that Ξ and ϱ are equivalent functors, and this concludes the argument. ☐

Remark 5.5.4. When P is not Σ-free, it may happen that the Quillen adjunction $(\rho_!^P, \rho_P^*)$ is *not* a Quillen equivalence.

Indeed, consider the case of $P = \mathsf{Com}$ the commutative operad, defined by $\mathsf{Com}(n) = *$ for all $n \geqslant 0$, with obvious composition law and symmetric group action. The dendroidal nerve of Com is the terminal object in the category of dendroidal sets, so that

$$\mathbf{dSets}/\mathcal{N}\mathsf{Com} \simeq \mathbf{dSets},$$

and the covariant model structure on the over category coincides with the so-called *absolute covariant model structure* on **dSets**, which models $\mathbb{E}_\infty$-algebras ([HM22, Corollary

13.41]). On the other hand, as detailed in [Whi17, Example 4.4], in the projective model structure on $\mathsf{Alg}_P(\mathbf{sSets})$ every path connected commutative algebra is equivalent to a generalized Eilenberg-Mac Lane space, i.e. product of Eilenberg-Mac Lane spaces. The existence of spaces like $QS = \Omega^\infty\Sigma^\infty\mathbb{S}^0$, which has an $\mathbb{E}_\infty$-algebra structure but is not a generalized Eilenberg-Mac Lane space, demonstrates that the Quillen adjunction cannot be a Quillen equivalence.

Let us conclude with the following

Remark 5.5.5. In [Bar24b], Barata defines the symmetric monoidal envelope of a dendroidal ∞-operad as a functor

$$\mathsf{dEnv}(-)^{\otimes} \colon \mathbf{dSets} \to \mathbf{sSets}/\mathsf{Fin}_*$$

from the category of dendroidal sets to that of simplicial sets over Fin_*. In ([Bar24b, Proposition 4.11]), it is shown that if X is a dendroidal ∞-operad, then $\mathsf{dEnv}(X)^{\otimes}$ is a symmetric monoidal ∞-category (à la Lurie), and in [Bar24b, Proposition 4.12.], it is shown moreover that, when $X \simeq \mathcal{N}_d P$ is the nerve of a discrete operad, the dendroidal envelope agrees with the usual symmetric monoidal envelope of operads. At present, the homotopical behavior of this functor remains insufficiently understood—for example, it is not yet known whether it preserves weak equivalences. Nevertheless, we believe this question merits further investigation, with the long-term aim of obtaining a dendroidal presentation of Lurie's symmetric monoidal envelope (as recalled in Section 4.4.1) in the form of a Quillen adjunction between model categories.

Chapter 6

The root functor

In this chapter we show that any ∞-operad is equivalent to the localization of a discrete Σ-free operad, working in the formalism of dendroidal sets. The key point is defining the *root functor* of a dendroidal set X, a functor from the dendroidal nerve of a discrete operad $\mathbf{\Omega}/X$ into X, which we show to be an operadic weak equivalence after localizing $\mathbf{\Omega}/X$. This extends an analogous result for ∞-categories due to Joyal: when X is a simplicial set, $\mathbf{\Omega}/X$ is its category of elements, and the root functor is the last vertex map. Combining the root functor with the rectification functor constructed in Chapter 5, we deduce that the ∞-category of algebras over an ∞-operad is equivalent to that of locally constant algebras over its discrete resolution.

Contents

6.1 Introduction

Given an ∞-operad $\mathcal{P}$, one may want to invert a class of unary morphisms and identify objects up to some set of maps $\mathcal{W}$: the *localization* of $\mathcal{P}$ at $\mathcal{W}$ is an ∞-opewrad $\mathcal{L}_{\mathcal{W}}\mathcal{P}$ with a morphism $\lambda\colon \mathcal{P} \to \mathcal{L}_{\mathcal{W}}\mathcal{P}$ sending $\mathcal{W}$ to equivalences, initial with this property.

When $\mathcal{Q} \simeq \mathcal{L}_{\mathcal{W}}\mathcal{P}$, one can study $\mathcal{P}$ to deduce properties of $\mathcal{Q}$. The ideal situation is when $\mathcal{P}$ is discrete, that is has discrete spaces of operations, so one does not need to deal with homotopy coherences.

A well-known example of this phenomenom is the little disks operad $\mathbb{E}_n$, the topological operad with a single object whose space of k-ary operations is given by the space of framed embeddings

$$\mathbb{E}_n(k) \simeq \mathsf{Emb}^{\mathsf{fr}}(\bigsqcup_{i=1}^{k} \mathbb{D}^n, \mathbb{D}^n)$$

of the disjoint union of k n-dimentional disks into another one. The data of the homotopy coherences can be resolved by constructing a discrete operad Disk_n whose set of objects is given by all the embeddings $\mathbb{D}^n \hookrightarrow \mathbb{R}^n$, and the set of operations from k such embeddings to another one is given by embeddings $\coprod_{i=1}^{k} \mathbb{D}^n \hookrightarrow \mathbb{D}^n$ strictly commuting over $\mathbb{R}^n$. This operad captures most of the homotopical information about $\mathbb{E}_n$: in [Lur17, Theorem 5.4.5.9], Lurie shows that there is a forgetful functor $\mathcal{N}^{\otimes}\mathsf{Disk}_n \to \mathbb{E}_n$ and that it induces an equivalence between the ∞-category of $\mathbb{E}_n$-algebras and the full sub ∞-category of *locally constant* Disk_n-algebras, that is, those sending an embedding of one disk into another into equivalences. In fact, the discrete operad Disk_n can be further simplified: it is equivalent to the discrete operad Disj_n whose objects are the open subsets of $\mathbb{R}^n$ abstractly homeomorphic to a n-dimensional disk, and where the sets of operations $\mathsf{Disj}_n(U_1, \dots, U_k; V)$ is a singleton if the U_i's are pairwise disjoint and contained in V, and empty otherwise ([Lur17, Remark 5.4.5.7.]). In particular, $\mathbb{E}_n$-algebras in a symmetric monoidal ∞-category can be characterized as functors from the poset of objects of Disj_n sending disjoint union to tensor product and inclusions $U \subseteq V$ which are isotopy equivalences to weak equivalences. In other words, although formulated with the language of weak approximations, this result states that the map of ∞-operads $\mathcal{N}^{\otimes}\mathsf{Disj}_n \to \mathbb{E}_n$ is an equivalence of ∞-operads *up to localizing* $\mathcal{N}\mathsf{Disk}_n^{\otimes}$ at isotopy equivalences. The language of localization of ∞-operads in Lurie's formalism is first used to prove the same thing for generalizations of the little disks operad in the domain of factorization homology by Ayala-Francis-Tanaka ([AFT17]). In both cases, the comparison plays a fundamental role in recognizing certain colimit expressions in this theory, and hence for instance proving the existence of operadic Kan extensions.

Exhibiting an ∞-operad as a localization, and even better as the localization of a discrete operad, can simplify the description of the ∞-category of algebras over it. In this perspective, aside from the abstract information that localization exists, it is important to be able to construct such localizations in the models at our disposal. So we ask the following

Question. Can we express *any* ∞-operad as the localization of a *discrete* operad?

If we restrict to ∞-categories, the answer is positive: Joyal's delocalization theorem, stated in [Joy07, §13.6] and proven by Stevenson in [Ste17], ensures that every ∞-category X, which one can represent as a simplicial set in the form of a quasicategory, is equivalent to the localization of the nerve of its category of elements Δ/X. The equivalence is realized via the *last vertex functor* (already appeared in Waldhausen's work [Wal85, §1.6]), a map of simplicial sets $\mathcal{N}(\Delta/X) \to X$ from the nerve of Δ/X to X which evaluates a n-simplex of X at the n^{th} vertex, which is shown to exhibit X as the ∞-categorical localization of $\mathcal{N}(\Delta/X)$ at the preimages of the identities.

In this chapter we give a positive answer to Section 6.1 and provide a model for such discrete operad and the localizing morphisms in the formalism of dendroidal sets. We will construct the *root functor*, a map $\mathcal{N}_d\mathbf{\Omega}/X \to X$ from the dendroidal nerve of a certain discrete operad $\mathbf{\Omega}/X$ into X which extends the last vertex functor, and prove that it exhibits any normal dendroidal set X as the operadic localization of $\mathbf{\Omega}/X$. This allows in particular to deduce a characterization of the ∞-category of algebras over an ∞-operad as that of locally constant algebras over its discrete resolution.

6.1.1 Main results and outline

We start by formalizing the notion of derived localization of dendroidal sets, derived with respect to the operadic model structure, which we define in Section 6.2, providing an explicit model for it.

Section 6.3 is dedicated to the construction of the root functor and the proof of the equivalence after localization. We start by studying the class of *wide* and *independent* maps between disjoint unions of trees, i.e. forests (Section 6.3.1), and in Definition 6.3.6 we define, for any dendroidal set X, its *operad of elements* $\mathbf{\Omega}/X$. It is a discrete operad whose objects are the elements of the presheaf X, that is pairs (T,α), with T a tree and $\alpha\colon T \to X$ a morphism from the representable dendroidal set given by T into X, and where operations are described in terms of wide independent morphisms $T_1 \sqcup \cdots \sqcup T_n \to S$ over X. For an object (T,α) of $\mathbf{\Omega}/X$, the evaluation of α at the root of T yields an object of X. We prove that the endofunctor

$$\mathbf{dSets} \to \mathbf{dSets} \quad X \mapsto \mathcal{N}_d(\mathbf{\Omega}/X)$$

is cocontinuous, and we define the *root functor* of a dendroidal set X, a morphism

$$\mathscr{r}_X \colon \mathcal{N}_d(\mathbf{\Omega}/X) \to X,$$

from the dendroidal nerve of the operad $\mathbf{\Omega}/X$ into X, natural in X.

The root functor captures most of the homotopical information of X up to inverting some class of 1-morphisms. This is the content of our first main

Theorem (6.3.13)**.** *Let X be a dendroidal set, and let $\mathcal{R}$ be the set of morphisms of $\mathbf{\Omega}/X$ sent to identities by $\mathscr{r}_X$. The root functor $\mathscr{r}_X$ induces an operadic weak equivalence of dendroidal sets*

$$\overline{\mathscr{r}_X} \colon \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \xrightarrow{\sim} X$$

between the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}$ and X.

The proof of the theorem relies on the construction of a homotopy inverse of $\overline{\mathscr{r}_X}$ in the case when X is a representable dendroidal set, that is, a tree.

After [BN14], the operadic model structure on dendroidal sets admits a left Bousfield localization, called the *stable model structure*, which, after [BdBM20], is Quillen equivalent to that of group-like $\mathbb{E}_\infty$-spaces. The latter is equivalent to the homotopy theory of infinite loop spaces. When sliced over the point, one gets back the Kan-Quillen model structure on simplicial sets. A first consequence of Theorem 6.3.13 can be phrased in the following

Corollary (Corollary 6.3.15)**.** *The root functor $\mathscr{r}_X \colon \mathcal{N}_d(\mathbf{\Omega}/X) \to X$ is an equivalence in the stable model structure.*

In other words, every dendroidal set X is weakly equivalent, to the infinite loop space given by the nerve of the operad of elements $\mathbf{\Omega}/X$.

When $X = M$ is a simplicial set, the root functor coincides with the last vertex functor, an we recover Joyal's delocalization theorem as a corollary of Theorem 6.3.13.

Corollary (Joyal, Stevenson)**.** *For any simplicial set M, the last vertex functor $\mathcal{N}(\Delta/M) \to M$ induces a weak categorical equivalence*

$$\mathcal{N}(\Delta/M)[\mathcal{R}^{-1}] \longrightarrow M$$

between M and the localization of $\mathcal{N}(\Delta/M)$ at the preimage of the identities.

In particular, the last vertex functor is a weak homotopy equivalence of simplicial sets.

We dedicate the subsequent Section 6.4 to formalizing how the equivalence given by the root functor allows to describe the ∞-category of algebras over a normal dendroidal ∞-operad as that of locally-constant algebras over its operad of elements $\mathbf{\Omega}/X$.

First of all, in Section 6.4.1 we study compatibility of localization of a dendroidal set with the covariant model structure for dendroidal left fibrations. It turns out that dendroidal left fibrations over the localization of X at a set S of morphisms correspond to $S_{/X}$*-local* dendroidal left fibrations over X (Construction 6.4.1), the fibrant objects of a left Bousfield localization of the covariant model structure on $\mathbf{dSets}/X$, which we denote by $\mathbf{dSets}^S/X$ and refer to as the covariant model structure for $S_{/X}$-local dendroidal left fibrations. More precisely, we show the following

Proposition (Proposition 6.4.4)**.** *For any dendroidal set X and set of 1-morphisms $S \subseteq X_{C_1}$, the localization map $X \to X[S^{-1}]$ induces a Quillen equivalence*

$$\lambda_! \colon \mathbf{dSets}^S/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*$$

between the left Bousfield localization of the covariant model structure on $\mathbf{dSets}/X$ at $S_{/X}$ and the covariant model structure on $\mathbf{dSets}/X[S^{-1}]$.

Afterwards, we consider the operadic un/straightening equivalence for Σ-free discrete operads constructed in Chapter 5, which relates the homotopy category of simplicial P-algebras with that of dendroidal left fibrations $X \to \mathcal{N}_d P$ over the nerve of P, establishing a Quillen equivalence between the covariant model structure for the former and the projective model structure for the latter. In Proposition 6.4.8 we show that the transfer of the localization of Proposition 6.4.4 along this Quillen equivalence yields a localization of the projective model structure on $\mathsf{Alg}_P(\mathbf{sSets})$ where the fibrant objects are S*-locally constant algebras*, that is, those P-algebras sending the morphisms in S to equivalences. Combining this with the straightening equivalence for dendroidal ∞-operads established by Heuts in [Heu11], we get the following description of algebras over a dendroidal ∞-operad and a 'local' version of the un/straightening equivalence for X.

Corollary (Corollary 6.4.9)**.** *Let X be a normal dendroidal ∞-operad.*

1. *The rectification functor for the operad of dendrices $\mathbf{\Omega}/X$ induces a Quillen equivalence*
$$\rho_!^{\mathbf{\Omega}/X} : \mathbf{dSets}^{\mathcal{R}_X}/\mathcal{N}_d(\mathbf{\Omega}/X) \rightleftarrows \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}) : \rho^*_{\mathbf{\Omega}/X}$$
between the covariant model structure for $\mathcal{R}_X$-local dendroidal left fibrations over $\mathcal{N}_d\mathbf{\Omega}/X$ and the projective model structure for $\mathcal{R}_X$-locally constant $\mathbf{\Omega}/X$-algebras.

2. *There is a zig-zag of Quillen equivalences*
$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets})$$
between the projective model structures for simplicial $W_!(X)$-algebras and for $\mathcal{R}_X$-locally constant algebras on $\mathbf{\Omega}/X$.

6.1.2 Preliminaries and notation

The background needed for this chapter is contained in Section 3.6. We use the notation therein as well as that in Section 3.7.

6.2 Localization of dendroidal ∞-operads

We now define derived localization of dendroidal sets, in a way that extends that of quasi-categories in the sense of Joyal and Lurie ([Lur17]). We then construct a model for normal dendroidal sets in Proposition 6.2.4. For compatibility of localization of dendroidal sets with the covariant model structure we direct the reader to Section 6.4.1.

6.2.1 The definition of localization

Let us start by some preliminary

Definition 6.2.1. A *normalization* of a dendroidal set X is a trivial fibration $X' \xrightarrow{\sim} X$ in the operadic model structure, with X' normal.

Given a morphism of dendroidal sets $f\colon X \to Y$, a *normalization* of f is a commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\wr"'] & Y' \arrow[d, "\wr"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where both vertical arrows are normalizations.

Normalizations exist and are unique up to operadic weak equivalence. As explained in [HM22, Remark 9.21], an explicit construction of a normalizarion with contractible fibres is given by the projection $X \times W^*\mathcal{E} \to X$, where $\mathcal{E}$ is the simplicial Barrat-Eccles operad and W^* is the operadic homotopy coherent nerve functor, right adjoint to $W_!$.

Definition 6.2.2. Let $\lambda\colon X \to Y$ be a morphism between normal dendroidal sets and let $S \subseteq X_{C_1}$ be a subset of 1-morphisms. The map λ is a *localization of X at the set of morphisms S* if, for any dendroidal ∞-operad Z, the morphism between the simplicial hom objects

$$\hom(Y, Z) \longrightarrow \hom(X, Z)$$

is fully faithful, with essential image given by those maps $X \to Z$ sending S to equivalences.

If $\lambda\colon X \to Y$ is a morphism between non-necessarily normal dendroidal sets, we say that λ is a localization if any, or equivalently one, normalization λ' of λ is.

Localization is unique up to operadic weak equivalence, and we denote by $X[S^{-1}]$ 'the' localization.

Observe that, if we denote by $\hom_S(X, Z)$ the full sub-simplicial set of $\hom(X, Z)$ spanned by those maps $X \to Z$ sending S to equivalences in Z, the universal property of the localization allows to identify $\hom(X[S^{-1}], Z)$ with $\hom_S(X, Z)$.

Remark 6.2.3. It is immediate to see that the definition applied to a morphism of dendroidal sets recovers the localization of quasi-categories in the sense of [Lur17, Definition 1.3.4.1].

We can construct an explicit model for the localization of a normal dendroidal set.

Proposition 6.2.4. *Denote by J the nerve of the connected groupoid on two objects. Given a normal dendroidal set X and a subset $S \subseteq X_{C_1}$ of 1-morphisms in X, the localization of X at S is realized by the map $\lambda\colon X \to \mathcal{L}(X, S)$ defined by the pushout diagram*

$$\begin{tikzcd}
\coprod_{s\in S} C_1 \arrow[r] \arrow[d] & X \arrow[d, "\lambda"] \\
\coprod_{s\in S} J \arrow[r] & \mathcal{L}(X,S) .
\end{tikzcd} \tag{6.2.1}$$

Proof. Observe that, by left properness of the operadic model structure, the pushout in Equation (6.2.1) is a homotopy pushout. Let Z be a dendroidal ∞-operad and consider the map

$$\lambda^*\colon \hom(\mathcal{L}(X,S),Z) \longrightarrow \hom(X,Z).$$

The essential image of λ^* consists of all functors sending S to equivalences, so we just need to show λ is also fully faithful. To this end, it is sufficient to show that the diagram

$$\begin{tikzcd}
\hom(\mathcal{L}(X,S)\otimes C_1, Z) \arrow[r] \arrow[d] & \hom(X\otimes C_1, Z) \arrow[d] \\
\hom(\mathcal{L}(X,S)\otimes \partial C_1, Z) \arrow[r] & \hom(X\otimes \partial C_1, Z)
\end{tikzcd}$$

is a homotopy pullback for the Joyal model structure, which happens if the diagram

$$\begin{tikzcd}
X\otimes \partial C_1 \arrow[r] \arrow[d] & X\otimes C_1 \arrow[d] \\
\mathcal{L}(X,S)\otimes \partial C_1 \arrow[r] & \mathcal{L}(X,S)\otimes C_1
\end{tikzcd} \tag{6.2.2}$$

is a homotopy pushout for the operadic model structure. Since the diagram in (6.2.1) is a transfinite composition of homotopy pushouts along the morphisms $\{C_1 \xrightarrow{s} J\}_{s\in S}$, it is sufficient to show that it is a homotopy pushout just in the case when $S=\{s\}$, where it appears as the front face of the commutative cube

$$\begin{tikzcd}
C_1\otimes\partial C_1 \arrow[rr] \arrow[dr, "s\otimes\mathrm{id}"] \arrow[dd] & & C_1\otimes C_1 \arrow[dr, "s\otimes\mathrm{id}"] \arrow[dd] & \\
& X\otimes\partial C_1 \arrow[rr] \arrow[dd] & & X\otimes C_1 \arrow[dd] \\
J\otimes\partial C_1 \arrow[rr] \arrow[dr] & & J\otimes C_1 \arrow[dr] & \\
& \mathcal{L}(X,S)\otimes\partial C_1 \arrow[rr] & & \mathcal{L}(X,S)\otimes C_1
\end{tikzcd}$$

All but the front face of the cube are homotopy pushouts, hence we conclude that the front face is one as well. This concludes the proof. ☐

The above construction essentially means that to localize a dendroidal set at a set of 1-morphisms, we can first localize its underlying simplicial set and then glue it to the original dendroidal set X, as we explain in the next

Remark 6.2.5. Let X be a normal dendroidal set, and write $M := i^*X$ for its underlying simplicial set. Of course, we have $S \subseteq X_{C_1} = M_1$, so we can localize M at S. Let $\overline{\lambda}\colon M \to \mathcal{L}(M,S)$ be the localization map. The localization of X at S can be realized as the following homotopy pushout

$$\begin{tikzcd}
M = i^*X \arrow[r, "\epsilon_X"] \arrow[d, "\overline{\lambda}"] & X \arrow[d, "\lambda"] \\
\mathcal{L}(M,S) \arrow[r] & \mathcal{L}(X,S)
\end{tikzcd}$$

where the top horizontal map is the counit of the adjunction $(i_!, i^*)$.

Given two endofunctors of X which preserve S, there is an easy way to check when they are homotopy equivalent as endofunctors of the localization $X[S^{-1}]$. Recall the following

Definition 6.2.6. Let $f, g\colon X \to Y$ be two maps of dendroidal sets. An *homotopy* between f and g is a morphism $h\colon X \otimes C_1 \to Y$ in **dSets** with the property that $h_0 = f$ and $h_1 = g$, where h_i is the restriction of h along the leaf, resp. root, inclusions $\eta \xrightarrow{\{i\}} C_1$, $i = 0$, resp. $i = 1$.

For $x \in X_\eta$, the arrow $h_x\colon f(x) \to g(x)$ in Y_{C_1} is called a *component* of h.

Lemma 6.2.7. *Let X be normal, $S \subseteq X_{C_1}$ a set of 1-morphisms. Let $f, g\colon X \to X$ be two maps sending S to itself. If f and g are homotopic via a homotopy whose components are in S, then f and g are homotopy equivalent as maps $f, g\colon X[S^{-1}] \to X[S^{-1}]$.*

Proof. Let $h\colon X \otimes C_1 \to X$ be the homotopy between f and g whose components lie in S. The transpose of $h^*\colon \hom(X,Z) \to \hom(X \otimes C_1, Z) \simeq \hom(C_1, \hom(X,Z))$ induces a morphism of simplicial sets $\hom_S(X,Z) \times \Delta^1 \to \hom(X,Z)$. By the hypotheses on f and g and fullness of $\hom_S(X,Z)$ in $\hom(X,Z)$, there is an induced homotopy $\mathfrak{h}\colon \hom_S(X,Z) \times \Delta^1 \to \hom_S(X,Z)$. To prove that $\mathfrak{h}$ is a natural equivalence, it suffices to see that, for any object ϕ of $\hom_S(X,Z)$, the arrow $\mathfrak{h}_\phi\colon f^*(\phi) \to g^*(\phi)$ is an equivalence in $\hom_S(X,Z)$, that is in $\hom(X,Z)$. So we only need to check that for any $x \in X_\eta$, the 1-morphism $(\mathfrak{h}_\phi)_x\colon \phi(f(x)) \to \phi(g(x))$ is an equivalence in Z. As $(\mathfrak{h}_\phi)_x = \phi(h_x)$, and since by hypothesis $\phi(h_x)$ is a weak equivalence as h_x is an arrow in S, we have that $\mathfrak{h}$ establishes an equivalence between f^* and g^*. This concludes the proof. □

The homotopy theory of algebras over a dendroidal ∞-operad X is governed by the covariant model structure on the over-category **dSets**$/X$, so it is a natural question to investigate compatibility of the localization of dendroidal ∞-operads with the covariant model structure for dendroidal left fibrations. We study this in Section 6.4.1 (Proposition 6.4.4).

6.3 The root functor

We start Section 6.3.1 by introducing the formalism of forests and wide independent maps between them. We define the operad of elements and the root functor of a dendroidal set in Section 6.3.3, and prove the localization result, namely Theorem 6.3.13, in Section 6.3.4.

6.3.1 The category of forests

A *forest* is a finite disjoint union of trees. We denote by Φ be the *category of forests* obtained from $\mathbf{\Omega}$ by formally adjoining finite coproducts. By extending the inclusion $\mathbf{\Omega} \hookrightarrow \mathsf{Op}$ to forests under finite-coproduct, one has $\Omega\colon \Phi \hookrightarrow \mathsf{Op}$, where, given a forest $F = \bigsqcup_{i=1}^{n} T_i$, the operad $\Omega(F)$ is defined as $\Omega(F) := \bigsqcup_{i=1}^{n} \Omega(T_i)$. For any such F, every tree T_i is called a *constituent* of F, and $\mathbf{\Omega}$ embeds into Φ by seeing a tree to a forest with one constituent. Let ϕ be the subcategory of Φ whose objects are non-empty forests and where morphisms are the *independent* maps of forests, where

Definition 6.3.1. A morphism of forests $f\colon F \longrightarrow G$ is called *independent* if F and G are non-empty and, writing $F = \sqcup_{i=1}^{n} T_i$ and $G = \bigsqcup_{j=1}^{m} S_j$, f is specified by a map of sets $\alpha\colon \{1, \dots, n\} \to \{1, \dots, m\}$ and morphisms of trees $f_i\colon T_i \to S_{\alpha(i)}$, for $i = 1, \dots, n$, such that, whenever $i \neq i'$ and $\alpha(i) = \alpha(i') = j$, for all edges $e \in E(T_i)$, $e' \in E(T_{i'})$ the edges $f_i(e)$ and $f_{i'}(e')$ are incomparable in the poset $E(S_j)$.

By the independency condition, it suffices to check that the edges $f(r_{T_i})$ and $f(r_{T_{i'}})$ are independent.

The restriction of the disjoint union to ϕ endows it with a non-unital symmetric monoidal structure; we write $\oplus := \bigsqcup_{|\phi}$ and call it direct sum. In the category ϕ, every morphism can be written as a direct sum of maps whose target has only one constituent. Observe also that ϕ is no longer the coproduct in ϕ, as a candidate for the codiagonal $F \oplus F \to F$ would not satisfy the independence property.

Remark 6.3.2. The two categories of forests presented here are the ones appearing in various comparisons of the dendroidal and Lurie's model for ∞-operads: the category Φ is the one considered in [HM24], while the category ϕ is the one appearing in [HHM16].

6.3.2 Wide maps of forests

These were first introduced in [HM22].

Definition 6.3.3. A map of forests $f\colon F \to G$ is called *wide* if, for any constituent tree S of the forest G, any maximal monotonic path in the poset $E(S)$, that is from a minimal element to the root r_S, contains one element of the form $f(r_T)$ for some T constituent tree of F.

Observe that if f is also independent, if there exists such a constituent T, then it is unique.

When the target of a map of forest has only one constituent, we can reformulate the wideness condition in the following useful way.

Lemma 6.3.4. *Let $T_1, \dots, T_n, S$ be trees and $f\colon T_1 \oplus \dots \oplus T_n \to S$ a map of forests. If f is independent, then f is wide if and only if $S(f(r_{T_1}), \dots, f(r_{T_n}); r_S) \neq \varnothing$.*

Proof. It suffices to observe that, given edges $e_1, \dots, e_n$ of S, the only obstruction to the existence of a subtree with leaves $e_1, \dots, e_n$ and root r_S is the existence of a maximal monotonic path $\mathfrak{p} = \{l < a_1 < \dots < a_m < r_S\}$ in the poset of edges of S such that $\{e_1, \dots, e_n\} \cap \mathfrak{p} = \varnothing$. ☐

Every wide morphism is the direct sum of morphisms of the above type, and with Lemma 6.3.4 it is immediate to see that wide maps are closed under composition. We introduce some classes of wide maps in the following

Example 6.3.5.

1. *All maps of linear trees* $[n] \to [m]$ *are wide and independent.*
2. *All maps of trees* $T \to S$ *are independent maps of forests.*
3. *A morphism of trees* $f\colon T \to S$ *is* root preserving *if* $f(r_T) = r_S$*. Any root preserving map is wide, and the converse is true whenever the root vertex of* S *is not unary.*
4. *Given a forest* F*, one can construct the tree* $\overline{F}$ *as follows. Write* $F = \oplus_{i=1}^{n} T_i$ *and let* $\overline{F}$ *be the the tree obtained as the grafting of the trees* $(T_1, \dots, T_n)$ *on the leaves* $\{l_1, \dots, l_n\}$ *of an* n*-corolla* C_n*,*
$$\overline{F} = C_n \circ (T_1, \dots, T_n).$$
The forest root face *of* $\overline{F}$ *is the natural inclusion*
$$F = T_1 \oplus \cdots \oplus T_n \longrightarrow C_n \circ (T_1, \dots, T_n) = \overline{F}.$$

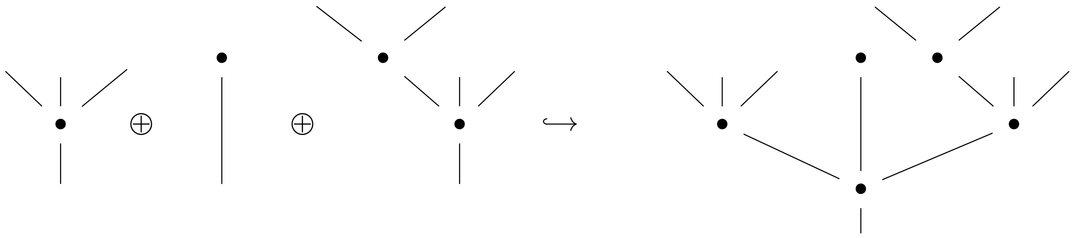


Figure 6.1: A forest root face.

In fact, it is easy to see that wide independent maps of forests are generated, under composition and direct sum, by forest root face inclusions and root preserving morphisms.

6.3.3 The operad of elements and the root functor

We have collected all the tools to be able to give the following

Definition 6.3.6. Let X be a dendroidal set. Its *operad of elements* $\mathbf{\Omega}/X$ is the suboperad of the one given by the symmetric monoidal category $(\mathbf{dSets}_{/X}, \sqcup)$ specified by the following data:

- the set of objects of $\mathbf{\Omega}/X$ is the set of elements of X as a presheaf, that is,
$$\mathsf{Ob}(\mathbf{\Omega}/X) = \{(T, \alpha) \mid T \in \mathbf{\Omega} \text{ and } \alpha\colon T \to X\};$$
- for objects $(S_1, \alpha_1), \dots, (S_n, \alpha_n)$, (R, β), the set of operations
$$\mathbf{\Omega}/X((S_1, \alpha_1), \dots, (S_n, \alpha_n); (R, \beta))$$
is given by the wide and independent maps of forests $f\colon S_1 \oplus \cdots \oplus S_n \to R$ such that
$$\beta \circ f = (\alpha_1, \dots, \alpha_n)\colon S_1 \bigsqcup \dots \bigsqcup S_n \to X.$$

To be explicit, the partial composition of f as above with $g \in (\mathbf{\Omega}/X)((Q_1, \beta_1), \dots, (Q_m, \beta_m); (S_i, \alpha_i))$ along (S_i, α_i) is given by the wide map of forests

$$(S_1 \oplus \cdots \oplus S_{i-1}) \oplus (\oplus_{j=1}^{m} Q_j) \oplus S_{i+1} \cdots \oplus S_n \xrightarrow{(\mathrm{id}^{\oplus i-1}, g, \mathrm{id}^{\oplus n-i})} \oplus_{h=1}^{n} S_h \xrightarrow{f} R.$$

Remark 6.3.7. By the independency condition on forest morphisms, the operad of elements $\mathbf{\Omega}/X$ is Σ-free.

Example 6.3.8.

- *For a simplicial set M, the operad of elements $\mathbf{\Omega}/M$ coincides with the category of elements of M, usually denoted by Δ/M. Its objects are the pairs $([n], f)$, with $n \geqslant 0$ and $f\colon \Delta^n \to M$; a morphism $F\colon ([n], f) \to ([m], g)$ is just a map of representable simplicial sets $F\colon \Delta^n \to \Delta^m$ over M.*
- *The dendroidal nerve functor $\mathcal{N}_d$ is a right adjoint: if $X \simeq \mathcal{N}_d P$ for a discrete operad P, an object of $\mathbf{\Omega}/X$ is a pair (T, α) with α an element in $\mathrm{Hom}_{\mathsf{Op}}(T, P)$. Similarly, if X is given by the homotopy coherent nerve of a simplicial operad $\mathbb{P}$, that is, $X \simeq W^*\mathbb{P}$, an object of $\mathbf{\Omega}/X$ is given by a pair (S, β), with S a tree and β an element in $\mathrm{Hom}_{\mathsf{sOp}}(W(S), \mathbb{P})$, where $W(S)$ is the simplicial operad given by the Boardman-Vogt resolution of the discrete operad generated by S.*

The construction $X \mapsto \mathbf{\Omega}/X$ is functorial in X via postcomposition, and we get an endofunctor

$$\mathcal{N}_d(\mathbf{\Omega}/-)\colon \mathbf{dSets} \to \mathbf{dSets}.$$

We shall now prove that this latter is equivalent to the left Kan extension of its restriction to representables, and moreover in a homotopy-coherent way.

Proposition 6.3.9. *The functor $\mathcal{N}_d(\mathbf{\Omega}/-)$ is cocontinuous. Moreover, it preserves normal monomorphisms of dendroidal sets.*

Proof. We denote by θ the restriction of $\mathcal{N}_d(\mathbf{\Omega}/-)$ to the representables, that is $\theta = \mathcal{N}_d(\mathbf{\Omega}/-)_{|\mathbf{\Omega}}$, and we write $\widehat{\theta}$ for its left Kan extension along the Yoneda embedding.

Given a dendroidal set X and a tree T, the set $\widehat{\theta}(X)_T$ may be described as

$$\widehat{\theta}(X)_T = \{((Q, x), (T, u)) \mid Q \in \mathbf{\Omega}, x\colon Q \to X, u\colon T \to \theta(Q)\}/\sim,$$

where, for any $\alpha\colon S \to Q$, one identifies

$$((Q, x), (R, \alpha_* u)) \sim ((S, x\alpha), (R, u)),$$

where $\alpha_* = \mathcal{N}_d(\mathbf{\Omega}/\alpha)$. Functoriality in T is given by letting faces and degeneracies act on the second component, while functoriality in X is obtained via the first component.

We construct a natural equivalence

$$\psi_X\colon \widehat{\theta}(X) \to \mathcal{N}_d(\mathbf{\Omega}/X)$$

by defining its components $(\psi_X)_T$ by induction on the number of vertices of the tree T. When $T \simeq \eta$, we set

$$\psi_X((Q,x),(\eta,u)) := x_* \circ u \colon \eta \to \mathcal{N}_d(\mathbf{\Omega}/X).$$

The assignment respects the equivalence relation, as (with the same notations as above) one has

$$\psi_X((Q,x),(\eta,\alpha_* u)) = x_*(\alpha_* u) = (x\alpha)_* u = \psi_X((S,x\alpha),(\eta,u)),$$

and it is straightforward to see that it induces a bijection.

Similarly, given a n-corolla C_n, $n \geqslant 0$, and an element $((S,x),(C_n,u))$, we set

$$\psi_X((S,x),(C_n,u)) := x_* u \colon C_n \to \mathcal{N}_d(\mathbf{\Omega}/X),$$

and it is the same calculation which shows that it descends to the quotient and s a bijection.

For the inductive step, let us consider a tree T with at least two vertices and decompose it as the grafting $T = R \cup_a S$, with $R, S \neq \eta$. For any dendroidal set Y, there is a natural isomorphism

$$\mathcal{N}_d(\mathbf{\Omega}/Y)_T \simeq \mathcal{N}_d(\mathbf{\Omega}/Y)_R \underset{\mathcal{N}_d(\mathbf{\Omega}/Y)_\eta}{\times} \mathcal{N}_d(\mathbf{\Omega}/Y)_S.$$

The isomorphism is compatible with the equivalence relation defining $\widehat{\theta}(X)$, which means that $\widehat{\theta}(X)$ satisfies the same strict Segal condition, that is, there is a natural isomorphism

$$\widehat{\theta}(X)_T \simeq \widehat{\theta}(X)_R \underset{\widehat{\theta}(X)_\eta}{\times} \widehat{\theta}(X)_S.$$

One defines the map $(\psi_X)_T$ as

$$(\psi_X)_T := (\psi_X)_R \underset{(\psi_X)_\eta}{\times} (\psi_X)_S.$$

It respects the equivalence relation and is a bijection, so we only need to check that ψ_X is well defined. This follows from the fact that any tree T decomposes as the grafting of corollas, and the decomposition is unique up to isomorphism and operadic associativity relations, and the Segal isomorphism is compatible with these latter, which shows that the definition of $(\psi_X)_T$ does not depend on the decomposition of T, as wanted. □

Now, let T be a tree and (S,α) an object of $\mathbf{\Omega}/T$. Evaluation of α at the root of S yields an assignment

$$\mathsf{Ob}(\mathbf{\Omega}/T) \ni (S,\alpha) \mapsto \alpha(r_S) \in \mathsf{Ob}(T) = E(T). \tag{6.3.1}$$

Proposition 6.3.10. *The assignment in Equation* (6.3.1) *extends to a map of operads*

$$\mathfrak{r}_T \colon \mathbf{\Omega}/T \longrightarrow T$$

that we call the root functor *for* T.

Proof. As the set of operations of T is either empty or a singleton, we just need to check that, if there exists a wide independent map of forests $f \colon (S_1,\alpha_1),\dots,(S_n,\alpha_n) \to (R,\beta)$, then $T(\alpha_1(r_{S_1}),\dots,\alpha_n(r_{S_n});\beta(r_R)) \neq \varnothing$. By Lemma 6.3.4, $R(f(r_{S_1}),\dots,f(r_{S_n});r_R) \neq \varnothing$, and since β is a map of operads and $\beta \circ f = (\alpha_1,\dots,\alpha_n)$ we have the thesis, as wanted. □

Because of cocontinuity of $\mathcal{N}_d(\mathbf{\Omega}/-)$ (Proposition 6.3.9), we can extend the root functor to every dendroidal set.

Definition 6.3.11. Let X be a dendroidal set. The *root functor* of X is the morphism of dendroidal sets

$$\mathfrak{r}_X \colon \mathcal{N}_d(\mathbf{\Omega}/X) \longrightarrow X$$

defined as the colimit

$$\mathfrak{r}_X := \operatorname*{colim}_{T \to X} \mathfrak{r}_T \colon \mathcal{N}_d(\mathbf{\Omega}/T) \longrightarrow T.$$

In particular, given an object (S, α) of $\mathcal{N}_d(\mathbf{\Omega}/X)$, its image via the root functor consists in the evaluation of α at the root of S,

$$\mathfrak{r}_X(S, \alpha) = \alpha(r_S) \ \in X_\eta.$$

Observe that, for any tree T and morphism $\alpha \colon T \to X$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{\Omega}/T \arrow[r, "\mathbf{\Omega}/\alpha"] \arrow[d, "\mathfrak{r}_T"'] & \mathbf{\Omega}/X \arrow[d, "\mathfrak{r}_X"] \\
T \arrow[r, "\alpha"] & X
\end{tikzcd}$$

Remark 6.3.12. If M is a simplicial set, the root functor coincides with the *last vertex functor*

$$\mathfrak{r}_M \colon \Delta/M \to M, \quad \mathfrak{r}_M([n], f) = f(n).$$

6.3.4 The localization result

In Example 6.3.5 we introduced root preserving morphisms of trees, which are in particular wide independent maps of forests. Given a dendroidal set X, we say that a morphism $f \colon (S, \alpha) \to (R, \beta)$ is *root preserving* if $f \colon S \to R$ is a root preserving map of trees. Denote by $\mathcal{R}_X$ the set of root preserving morphisms of $\mathbf{\Omega}/X$.

If $f \colon (S, \alpha) \to (R, \beta)$ is root preserving, then $\mathfrak{r}(f) = \mathrm{id}_{f(r_S)}$, so the root functor factors via the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}_X$, as

$$\begin{tikzcd}
\mathcal{N}_d(\mathbf{\Omega}/X) \arrow[rr, "\mathfrak{r}_X"] \arrow[dr, "\lambda"'] & & X \\
& \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}_X^{-1}] \arrow[ur, "\overline{\mathfrak{r}_X}"'] &
\end{tikzcd} .$$

Theorem 6.3.13. *For any normal dendroidal set X, the root functor induces an operadic weak equivalence of dendroidal sets*

$$\overline{\mathfrak{r}_X} \colon \mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}_X^{-1}] \xrightarrow{\sim} X.$$

Proof. Let us use Proposition 6.2.4 and choose $\mathcal{L}(\mathcal{N}_d(\mathbf{\Omega}/X), \mathcal{R}_X)$ for the model of the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at $\mathcal{R}_X$. By Proposition 6.3.9, the functor $\mathcal{L}(-, \mathcal{R}_-) \colon \mathbf{dSets} \to \mathbf{dSets}$ is cocontinuous and preserves normal monomorphisms, so we proceed by skeletal filtration, reducing the proof to the case of $X \simeq T$ a tree, that is a representable dendroidal set. The root functor is the nerve of the map of operads $\mathscr{r}\colon \mathbf{\Omega}/T \to T$, and we construct a section $\mathfrak{l}_T$ of $\mathscr{r}_T$,

$$\mathfrak{l}_T\colon T \to \mathbf{\Omega}/T$$

and a homotopy

$$h\colon \mathcal{N}_d(\mathbf{\Omega}/T) \otimes C_1 \to \mathcal{N}_d(\mathbf{\Omega}/T)$$

between $\mathfrak{l}_T \circ \mathscr{r}_T$ and $\mathrm{id}_{\mathbf{\Omega}/T}$, such that its components are root preserving morphisms. After Lemma 6.2.7, this implies that h becomes an equivalence between $\mathfrak{l}_T \circ \mathscr{r}_T$ and id_T after localizing at $\mathcal{R}_T$, which means that $\mathfrak{l}_T$ and $\mathscr{r}_T$ are homotopy inverses of the other once localized.

To construct h, consider an edge e of T, and let $T_e^{\uparrow}$ be the biggest subtree of T having e as root. We denote by $\iota_e\colon T_e^{\uparrow} \hookrightarrow T$ the associated subtree inclusion. Observe that, if $e \leqslant f$, then T_e is a subtree of T_f.

We define the morphism $\mathfrak{l}_T$ on an edge e of T as

$$\mathfrak{l}_T(e) \coloneqq (T_e^{\uparrow}, \iota_e) \in \mathsf{Ob}(\mathbf{\Omega}/T).$$

Given edges $e_1, \dots, e_n, e$ such that $T(e_1, \dots, e_n; e) = \{*\}$, the map

$$\mathfrak{l}_T\colon \{*\} \longrightarrow \mathbf{\Omega}/T((T_{e_1}^{\uparrow}, \iota_{e_1}), \dots, (T_{e_n}^{\uparrow}, \iota_{e_n}); (T_e^{\uparrow}, \iota_e))$$

selects the operation of $\mathbf{\Omega}/T$ given by the forest inclusion

$$\oplus_{i=1}^n T_{e_i}^{\uparrow} \hookrightarrow T_e^{\uparrow},$$

defined by the composition of the forest root face $\oplus_{i=1}^n T_{e_i}^{\uparrow} \to \overline{\oplus_{i=1}^n T_{e_i}^{\uparrow}}$ and the inner face $\partial\colon \overline{\oplus_{i=1}^n T_{e_i}^{\uparrow}} \to T_e^{\uparrow}$ sending each $T_{e_i}^{\uparrow}$ to itself and the new root vertex to the subtree $T_e^{\underline{e}}$.

As the operadic composition for T is the grafting of subtrees, we can check that it is a well defined map of operads $\mathfrak{l}_T\colon T \to \mathbf{\Omega}/T$. One checks that $\mathscr{r}_T \circ \mathfrak{l}_T = \mathrm{id}_T$; for the other composition $\mathfrak{l}_T \circ \mathscr{r}_T$, consider an object (S, α) in $\mathbf{\Omega}/T$. We have that

$$\mathfrak{l}_T \circ \mathscr{r}_T(S, \alpha) = \mathfrak{l}_T(\alpha(r_S))) = (T_{\alpha(r_S)}^{\uparrow}, \iota_{\alpha(r_S)}) \,.$$

Since the image of α is contained in the subtree $T_{\alpha(r_S)}^{\uparrow}$, we can always write α as a composition

$$\begin{tikzcd}
S \arrow[rr, "\alpha"] \arrow[dr, "h_{(S,\alpha)}"'] & & T \\
& T_{\alpha(r_S)}^{\uparrow} \arrow[ur, hook, "\iota_{\alpha(r_S)}"'] &
\end{tikzcd}$$

for an unique root preserving morphism $h_{(S,\alpha)}$. In particular, $h_{(S,\alpha)}$ is a morphism in $\mathbf{\Omega}/T$ and we obtain the collection of morphisms

$$h \coloneqq \{h_{(S,\alpha)}\colon (S, \alpha) \to (T_{\alpha(r_S)}, \iota_{\alpha(r_S)})\}_{(S,\alpha) \in \mathsf{Ob}(\mathbf{\Omega}/T)}.$$

Let us show that h defines a homotopy between $\mathrm{id}_{\mathbf{\Omega}/T}$ and $\mathfrak{l}_T \circ \mathscr{r}_T$. For this purpose, we need to check the Boardman-Vogt interchange relation, so consider objects

$(S_1, \beta_1), \dots, (S_n, \beta_n), (R, \gamma)$ and an operation $f \in \mathbf{\Omega}/T((S_1, \beta_1), \dots, (S_n, \beta_n); (R, \gamma))$. Since $\gamma \circ f = (\beta_1, \dots, \beta_n)$, we have that

$$\gamma \circ f = (\iota_{\mathfrak{r}_T(\beta_1)}, \dots, \iota_{\mathfrak{r}_T(\beta_n)}) \circ (h_{(S_1,\beta_1)}, \dots, h_{(S_n,\beta_n)}),$$

which is precisely the wanted relation. Consider the functor of dendroidal set given by the dendroidal nerve of h; precomposing $\mathcal{N}_d(h)$ with the natural map $\mathcal{N}_d(\mathbf{\Omega}/T) \otimes C_1 \to \mathcal{N}_d(\mathbf{\Omega}/T \otimes C_1)$, we obtain a homotopy between between the identity of $\mathbf{\Omega}/T$ and the composition $\iota_T \circ \mathfrak{r}_T$. As the components of this homotopy are root preserving morphisms, it becomes an equivalence after localizing at $\mathcal{R}_T$, and this concludes the argument. □

Remark 6.3.14. In general, there is no reason for the localization of a discrete dendroidal ∞-operad to be discrete again. Theorem 6.3.13 tells us in particular that this is the case for the localization of $\mathcal{N}_d(\mathbf{\Omega}/X)$ at root preserving morphisms whenever X is equivalent to the dendroidal nerve of a discrete Σ-free operad.

The operadic model structure on dSets admits a left Bousfield localization whose homotopy category is equivalent to that of group-like $\mathbb{E}_\infty$-algebras, in turn equivalent to infinite loop spaces. It is called the *stable model structure*, and was first introduced by Bašić-Nikolaus in [BN14]. It has the property that the induced model structure on the overcategory sSets $\simeq$ dSets$/\eta$ coincides with the Kan-Quillen model structure. In particular, one localizes also by the arrow $C_1 \to J$, which becomes an equivalence, so from Theorem 6.3.13 we deduce the following

Corollary 6.3.15. *For any normal dendroidal set X, the root functor $\mathfrak{r}_X \colon \mathcal{N}(\mathbf{\Omega}/X) \to X$ is a weak equivalence in the stable model structure for* dSets.

Specializing the above results to simplicial sets, we hence obtain the following well-known results.

Corollary 6.3.16 (Joyal [Joy07], Stevenson [Ste17])**.** *For any simplicial set M, the last vertex functor induces a categorical weak equivalence*

$$\Delta/M[\mathcal{R}_M^{-1}] \xrightarrow{\sim} M,$$

where $\mathcal{R}_M$ is the class of morphisms $f \colon ([n], \alpha) \to ([m], \beta)$ such that $f(n) = m$.

In particular, the last vertex functor $\Delta/M \to M$ is a weak homotopy equivalence.

6.4 An application: locally constant algebras

In this section, we describe the homotopy theory of algebras over a dendroidal ∞-operad in terms of locally constant algebras over its operad of elements. To this end, we study the compatibility of dendroidal localization with the covariant model structure in Section 6.4.1, and exploit the operadic weak equivalence given by the root functor (Theorem 6.3.13), combined with two operadic un/straightening equivalences: one for Σ-free discrete operads, constructed in Chapter 5, and another for dendroidal ∞-operads, proven by Heuts in [Heu11], which we recall in due course.

6.4.1 Localization and the covariant model structure

For a dendroidal set X and any subset of morphisms $S \subseteq X_{C_1}$, the localization map $\lambda \colon X \to X[S^{-1}]$ induces an adjunction of over-categories

$$\lambda_! \colon \mathbf{dSets}/X \underset{\perp}{\rightleftarrows} \mathbf{dSets}/X[S^{-1}] : \lambda^*,$$

which is a Quillen adjunction with respect to the covariant model structure for dendroidal left fibrations. It is natural to expect dendroidal left fibrations over the localization to be weakly equivalent to those left fibrations over X for which all the maps of fibres $f_! \colon Y_a \to Y_b$ induced by the morphisms $f \colon a \to b$ in S are weak homotopy equivalences of spaces. This is indeed the case, as we show in Proposition 6.4.4. Let us introduce some constructions first.

Construction 6.4.1. Let X be a dendroidal set and S a subset of X_{C_1}. For any $f \colon a \to b$ in S, one can construct the morphism

$$r_f \colon (\eta, \{b\}) \longrightarrow (C_1, f)$$

in $\mathbf{dSets}/X$ defines as follows:

$$\begin{tikzcd}
\eta \arrow[rr, hook, "r"] \arrow[dr, "b"'] & & C_1 \arrow[dl, "f"] \\
& X &
\end{tikzcd}$$

where the arrow $r \colon \eta \to C_1$ is the inclusion of the edge η into the root of C_1, or more familiary the map $\{1\} \colon [0] \to [1]$.

We write $S_{/X}$ for the set of morphisms of the form r_f, for f ranging in S.

Given a model category $\mathcal{M}$ and some set $\mathcal{S}$ of morphisms in it, we can talk about *$\mathcal{S}$-local* objects in $\mathcal{M}$: there are the fibrant objects M for which, for any morphism $s \colon A \to B$ in $\mathcal{S}$, the morphism of mapping spaces

$$s_* \colon \mathsf{Map}_{\mathcal{M}}(B, M) \longrightarrow \mathsf{Map}_{\mathcal{M}}(A, M)$$

is a weak homotopy equivalence of spaces. For $\mathcal{M}$ given by the covariant model structure for dendroidal left fibrations over a dendroidal set X, we have an explicit way of computing mapping spaces between fibrant-cofibrant objects:

Remark 6.4.2. If (A, u) is a normal dendroidal set over X and (E, p) a dendroidal left fibration over X, we have an equivalence of spaces

$$\mathsf{Map}_{\mathbf{dSets}/X}((A, u), (E, p)) \simeq \hom_X(A, E) = \hom(A, E) \times_{\hom(A,X)} \{u\}.$$

As maps in $S_{/X}$ have cofibrant domains and codomains, we can rephrase $S_{/X}$-locality as follows.

Definition 6.4.3. A (E, p) dendroidal left fibration over X is $S_{/X}$-local if, for any f in S, the morphism

$$(r_f)_* \colon \hom(C_1, E) \times_{\hom(C_1,X)} \{f\} \longrightarrow \hom(\eta, E) \times_{\hom(\eta,X)} \{b\} \simeq E_b = p^{-1}(b),$$

induced by precomposition with the root inclusion $\eta \hookrightarrow C_1$, is a weak homotopy equivalence of Kan complexes.

We write $\mathbf{dSets}^S/X$ for the left Bousfield localization of the covariant model structure on $\mathbf{dSets}/X$ at the set of arrows $S_{/X}$, which, recall, is left proper and cofibrantly generated. In this model structure, the fibrant objects are the $S_{/X}$-local dendroidal left fibrations, while the cofibrations are the normal monomorphisms over X. In particular, a morphism between $S_{/X}$-local left fibrations $\varphi\colon (E,p) \to (E',p')$ is a weak equivalence in the localization if and only if it is a covariant weak equivalence, hence a fibrewise weak homotopy equivalence. We call the model structure

$$\mathbf{dSets}^S/X$$

the *$S_{/X}$-local covariant model structure*. It is precisely this localization that encodes the compatibility of localization with the covariant model structure:

Proposition 6.4.4. *Let X be a normal dendroidal set and $S \subseteq X_{C_1}$ a subset of morphisms. The localization map $\lambda\colon X \to X[S^{-1}]$ induces a Quillen adjunction*

$$\lambda_!\colon \mathbf{dSets}^S/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*$$

between the $S_{/X}$-local covariant model structure on $\mathbf{dSets}/X$ and the covariant model structure on $\mathbf{dSets}/X[S^{-1}]$. Moreover, the adjunction is a Quillen equivalence.

Proof. We know that the adjunction

$$\lambda_!\colon \mathbf{dSets}/X \rightleftarrows \mathbf{dSets}/X[S^{-1}] : \lambda^*$$

is a Quillen adjunction between covariant model structures ([HM22, Proposition 9.62]). To prove that it is still a Quillen adjunction after localizing on the left, it suffices to show that, given any dendroidal left fibration (E,p) over X, the element (E,p) is $S_{/X}$-local if and only if there exists a dendroidal left fibration (Y,p') over $X[S^{-1}]$ and a covariant weak equivalence $(E,p) \xrightarrow{\sim} \lambda^*(Y,p)$ in $\mathbf{dSets}/X$.

Fix any such dendroidal left fibration (E,p). Write $S \times C_1$, resp. $S \times J$, for the simplicial set given by the union $S \times C_1 = \bigsqcup_{s\in S} C_1$, resp. $S \times J = \bigsqcup_{s \in S} J$, where J is the nerve of the connected groupoid on two objects.

The pullback

$$E' := (S \times C_1) \times_X E \longrightarrow S \times C_1$$

is a left fibration of simplicial sets, and after [Lur09, Proposition 2.1.3.1] it is a Kan fibration. As such, it can be factored as the composition $E' \to E'' \to S \times C_1$, where $E' \to E''$ is a trivial Kan fibration and $E'' \to S \times C_1$ is a minimal Kan fibration. As explained in [GZ67, §5.4], there is an isomorphism $E'' \simeq S \times C_1 \times M$ for some minimal Kan complex M, and the map $E'' \to S \times C_1$ is the projection. In particular, the map $S \times J \times M \to S \times J$ is a minimal Kan fibration, fitting into the pullback diagram

$$\begin{tikzcd}
E'' \arrow[r, "\phi"] \arrow[d] & S \times J \times M \arrow[d] \\
S \times C_1 \arrow[r] & S \times J
\end{tikzcd}$$

An argument due to Joyal (see the proof of [KL18, Lemma 2.2.5]) shows that we can find a trivial Kan fibration $Z \to S \times J \times M$ and an isomorphism $E' \simeq \phi^*(Z)$ over E''. Thus we obtain a commutative diagram

$$\begin{tikzcd}
E \arrow[d] & E' \arrow[l] \arrow[r] \arrow[d] & Z \arrow[d] \\
X & S \times C_1 \arrow[l] \arrow[r] & S \times J
\end{tikzcd}$$

where the right hand square is a pullback and the map $Z \to S \times J$ is a Kan fibration. Define

$$Y := E \bigcup_{E'} Z$$

there is a canonical map $p' \colon Y \to X[S^{-1}]$, and its pullback along the surjection $(S \times J) \sqcup X \to X[S^{-1}]$ consists in the dendroidal left fibration $Z \sqcup E \to (S \times J) \sqcup X$. In particular, p' is a dendroidal left fibration as well. Moreover, for any object x of X, there is a weak homotopy equivalence of fibres $(E,p)_x \to \lambda^*(Y,p')_x$, hence a covariant weak equivalence $(E,p) \to \lambda^*(Y,p')$ over X, as wanted. This concludes the proof that there is an induced Quillen adjunction. It is straightforward to check that $\mathbb{R}\lambda^*$ is fully faithful, hence we conclude that the induced adjunction is a Quillen equivalence, as wanted. □

Remark 6.4.5. The above proof essentially extends to the dendroidal context Stevenson's proof of [Ste17, Proposition 5.11], stating an analogous result for simplicial sets.

6.4.2 Locally constant algebras

Whenever we are given an operad P with a choice of weak equivalences S, one can study the homotopy theory of *S-locally constant P-algebras*, where

Definition 6.4.6. A simplicial P-algebra A is *S-locally constant* if A sends S to equivalences, that is for any $f \colon a \to b$, the map $A(f) \colon A(a) \to A(b)$ is a weak homotopy equivalence of simplicial sets.

As a model category structure on a category $\mathcal{M}$ is determined, if it exists, by the cofibrations and the fibrant objects, we can safely give the following

Definition 6.4.7. Let P be a Σ-free operad and S a subset of morphisms of P. The *projective model structure for S-locally constant algebras* is the model structure on simplicial P-algebras whose fibrant objects are S-locally constant and projectively fibrant P-algebras, and where the cofibrations are the projective cofibrations. We denote it by $\mathsf{Alg}^S_{\mathbf{P}}(\mathbf{sSets})$.

Let us recall the operadic un/straightening equivalence constructed in Chapter 5: it consists, for any Σ-free operad P, of a Quillen equivalence

$$\rho^P_! : \mathbf{dSets}/\mathcal{N}_d(P) \rightleftarrows \mathsf{Alg}_P(\mathbf{sSets}) : \rho^*_P \tag{6.4.1}$$

between the covariant model structure on the left and the projective model structure on the right (Corollary 5.5.3). The left adjoint $\rho^P_!$, which we call the *rectification functor*, is defined as the essentially unique cocontinuous functor characterized by the fact that, for any tree T and morphism $\alpha \colon T \to \mathcal{N}_d(P)$, the P-algebra $\rho^P_!(T,\alpha)$ is defined as

$$\mathsf{Ob}(P) \ni c \mapsto \rho^P_!(T,\alpha)(c) \simeq \mathsf{Env}(T) \underset{\mathsf{Env}(P)}{\times} \mathsf{Env}(P)_{/c},$$

where Env denotes is the nerve of the symmetric monoidal envelope functor for discrete operads (Definition 5.2.8).

We can now prove the following

Proposition 6.4.8. *Let P be a discrete Σ-free operad. The operadic un/straightening adjunction $(\rho_!^P, \rho_P^*)$ induces a Quillen equivalence*

$$\rho_!^P : \mathbf{dSets}^S/\mathcal{N}_d(P) \rightleftarrows \mathsf{Alg}_P^S(\mathbf{sSets}) : \rho_P^*$$

between the covariant model structure for S-local left fibrations and the projective model structure for S-locally constant P-algebras.

Proof. In general, given a Quillen adjunction of model categories $F\colon \mathcal{M} \rightleftarrows \mathcal{N} : G$ and a choice of (cofibrant) arrows $\mathcal{S}$ in $\mathcal{M}$, one can consider the left Bousfield localizations $\mathcal{L}_{\mathcal{S}}\mathcal{M}$ on $\mathcal{M}$ and the transfered localization on $\mathcal{N}$, resp. $\mathcal{L}_{\mathbb{L}F(\mathcal{S})}\mathcal{N}$, of $\mathcal{M}$ at $\mathcal{S}$, resp. of $\mathcal{N}$ at $\mathbb{L}F(\mathcal{S})$. It is a standard result of model categories ([Hir03]) that, if (F, G) is a Quillen equivalence, then there is an induced Quillen equivalence $F\colon \mathcal{L}_{\mathcal{S}}\mathcal{M} \rightleftarrows \mathcal{L}_{\mathbb{L}F(\mathcal{S})}\mathcal{N} : G$, where $\mathcal{L}_{\mathbb{L}F(\mathcal{S})}\mathcal{N}$ is the left Bousfield localization of $\mathcal{N}$ at $\mathbb{L}F(\mathcal{S})$. This implies that we only need to characterize the fibrant objects of $\mathcal{L}_{\rho_!^P(S_{/\mathcal{N}_d(P)})}\mathsf{Alg}_P(\mathbf{sSets})$ as the locally constant fibrant P-algebras.

An object A in $\mathcal{L}_{\rho_!^P(\mathcal{S})}(\mathsf{Alg}_P(\mathbf{sSets}))$ is fibrant if and only if it is projectively fibrant and $\rho_!^P(\mathcal{S}_{/\mathcal{N}_d P})$-local. As the pair $(\rho_!^P, \rho_P^*)$ is a Quillen adjunction, this is equivalent to asking $\rho_P^*(A)$ to be $S_{/\mathcal{N}_d P}$-local, that is, that the map

$$\mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((C_1, f), \rho_P^*(A)) \longrightarrow \mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{b\}), \rho_P^*(A))$$

is a weak homotopy equivalence for every $f\colon a \to b$ in S. As A is projectively fibrant, $\rho_P^*(A)$ is a dendroidal left fibration, so in particular local with respect to the map $(\eta, \{a\}) \to (C_1, f)$ induced by the leaf inclusion $\ell\colon \eta \hookrightarrow C_1$, so the above is an equivalence if and only if the morphism in the homotopy category of spaces

$$\mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{a\}), \rho_P^*(A)) \longrightarrow \mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{b\}), \rho_P^*(A))$$

is an isomorphism. After Remark 6.4.2, there is an equivalence of spaces

$$\mathsf{Map}_{\mathbf{dSets}/\mathcal{N}_d(P)}((\eta, \{x\}), \rho_P^*(A)) \simeq \rho_P^*(A)_x,$$

and we check that the above map corresponds to the morphism of fibres

$$f_!\colon \rho_P^*(A)_a \longrightarrow \rho_P^*(A)_b.$$

After Lemma 5.2.14, there is an equivalence

$$\rho_P^*(A)_x \simeq A(x),$$

natural in x, therefore we conclude that $\rho_P^*(A)$ is $S_{/\mathcal{N}_d P}$-local if and only if the map

$$A(f)\colon A(a) \longrightarrow A(b)$$

is a weak homotopy equivalence of spaces, that is, if A is S-locally constant. This concludes the proof. ☐

6.4.3 Algebras over dendroidal ∞-operads

We can now state our main results for this section.

Corollary 6.4.9. *Let X be a normal dendroidal ∞-operad.*

1. *The rectification functor for the operad of dendrices $\mathbf{\Omega}/X$ induces a Quillen equivalence*
$$\rho_!^{\mathbf{\Omega}/X} : \mathbf{dSets}^{\mathcal{R}_X}/\mathcal{N}_d(\mathbf{\Omega}/X) \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}) : \rho^*_{\mathbf{\Omega}/X}$$
between the covariant model structure for $\mathcal{R}_X$-local dendroidal left fibrations over $\mathcal{N}_d\mathbf{\Omega}/X$ and the projective model structure for $\mathcal{R}_X$-locally constant $\mathbf{\Omega}/X$-algebras.

2. *There is a zig-zag of Quillen equivalences*
$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets})$$
between the projective model structures for simplicial $W_!(X)$-algebras and for $\mathcal{R}_X$-locally constant algebras on $\mathbf{\Omega}/X$.

Proof. The first point is just Proposition 6.4.8 instantiated with $P = \mathbf{\Omega}/X$, which we know to be Σ-free, so let us focus on point (2). By Theorem 3.6.13 and Theorem 6.3.13, the root functor yields a Quillen equivalence of covariant model structures

$$\overline{\mathfrak{r}_X}_! : \mathbf{dSets}/\mathcal{N}_d(\mathbf{\Omega}/X)[\mathcal{R}^{-1}] \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathbf{dSets}/X : \overline{\mathfrak{r}_X}^*$$

Applying Proposition 6.4.4, we obtain that the root functor yields a Quillen equivalence

$$(\mathfrak{r}_X)_! : \mathbf{dSets}^{\mathcal{R}_X}/\mathbf{\Omega}/X \mathrel{\substack{\longrightarrow \\ \perp \\ \longleftarrow}} \mathbf{dSets}/X : \mathfrak{r}_X^*$$

between the localization of the covariant model structure for S-local dendroidal left fibrations over $\mathbf{\Omega}/X$ and the covariant model structure on $\mathbf{dSets}/X$. Following [Heu11, Theorem 2.7], this latter is Quillen equivalent to the projective model structure on $\mathsf{Alg}_{W_!(X)}(\mathbf{sSets})$, and combining this with point (1) we get the wanted zig-zag of Quillen equivalences
$$\mathsf{Alg}_{W_!(X)}(\mathbf{sSets}) \xleftarrow{\sim} \bullet \xrightarrow{\sim} \mathsf{Alg}^{\mathcal{R}_X}_{\mathbf{\Omega}/X}(\mathbf{sSets}).$$

□

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Resumé

Cette thèse porte sur la théorie des infinies-opérades, une théorie générale des structures algébriques dans des contextes homotopiques et organisées de façon fonctorielle et cohérente. Notamment, étant donnée une opérade P on étudie l'infinie-catégorie des P-algèbres dans l'infinie-catégorie des espaces topologiques à homotopie près, aussi dits infinies-groupoïdes. Premièrement, on travaille avec la théorie des fibrations à gauche sur une infinie-opérade, en considérant deux problèmes: celui de comparer la notion de fibration à gauche dans le cadre des enveloppes monoïdales symétriques suivant Lurie, et celui des espaces de Segal dendroïdaux. Dans le premier cas, on montre que les approches dendroïdale et de Lurie sont compatibles, à travers une équivalence explicite de Hinich et Moerdijk. Pour la deuxième question, on prouve une version de l'équivalence de rectification, traduisant les fibrations à gauche en termes d'algèbres, via une fibration à gauche opéradique universelle. Dans une suite logique de cet approche, on construit une deuxième équivalence de rectification en termes de catégories de modèles de Quillen. Dans un deuxième temps, on étudie les infinies-opérades à travers le processus de localisation, et on étend au cas opéradique un résultat de rectification non triviale. Notre idée est de généraliser la construction de la categorie des éléments d'un préfaisceu simpliciale, et definir l'opérade des éléments d'un préfaisceu dendroïdale. On montre donc que, comme toute infinie-catégorie est équivalente à la localisation d'une 1-catégorie, toute infinie-opérade est équivalente à la localisation d'une 1-opérade. On obtient ainsi une autre description de l'infinie-catégorie des algèbres sur une infinie-opérade.

Summary

This thesis concerns the theory of infinity-operads, a general theory of algebraic structures in homotopical contexts, organized in a functorial and coherent way. In particular, given an infinity-operad P, we study the infinity-category of P-algebras in topological spaces up to weak homotopy equivalence, also called infinity-groupoids. Firstly, we work with the theory of left fibrations over an infinity-operad in the context of the asymmetric monoidal envelope à la Lurie, and that of dendroidal Segal spaces. For the former, we show that the dendroidal and Lurie's approaches are compatible, by means of an equivalence defined by Hinich and Moerdijk. For the latter, we prove a version of the straightening equivalence, which translates left fibrations into operadic algebras, by means of a universal operadic left fibration. As a logical continuation of this approach, we construct a second straightening equivalence in terms of Quillen model categories. At a later stage, we study infinity-operads by means of localization, and extend to infinity-operads a non trivial rectification result for infinity-categories. Our idea is to generalize the category of elements construction for simplicial presheaves and define the operad of elements of a dendroidal presheaf, With this, we show that, as every infinity-category is equivalent to the localization of a 1-category, every infinity-operad is equivalent to the localization of a 1-operad. This allows to obtain a new description of the infinity-category of algebras over an infinity-operad.