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Analyse du Laplacien hypoelliptique de Bismut dans la double asymptotique des grandes frictions et basses températures

THÈSE DE DOCTORAT

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Resumé :

Dans ma thèse, on s'intéresse au bas du spectre du Laplacien hypoelliptique de Bismut. Pour y parvenir, on commence par refaire une démonstration de l'inégalité sous-elliptique pour les opérateurs de Kramers-Fokker-Planck géométriques, une classe d'opérateurs dont fait partie le Laplacien hypoelliptique. Dans un second temps, on établit la convergence spectrale du Laplacien hypoelliptique de Bismut vers le Laplacien de Witten dans la limite de basse température et de grande friction. Cette convergence permet la comparaison entre le spectre de ces deux opérateurs.

Summary :

In my thesis, we focus on the lower part of the spectrum of Bismut's hypoelliptic Laplacian. To achieve this, we begin by revisiting a proof of the sub-elliptic inequality for geometric Kramers-Fokker-Planck operators, a class of operators to which the hypoelliptic Laplacian belongs. In the second step, we establish the spectral convergence of Bismut's hypoelliptic Laplacian to the Witten Laplacian in the low-temperature and high-friction limit. This convergence enables a comparison between the spectra of these two operators.

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Chapitre 1

Introduction

Rappelons d'abord les deux modélisations classiques du mouvement brownien. Historiquement, le premier modèle, lié à l'approche d'Albert Einstein et appelé aussi Langevin suramorti, se formule maintenant à l'aide d'une équation différentielle stochastique, écrite ici dans le cas scalaire euclidien,

$$\left\{\begin{array}{l} dq = -\frac{1}{h}\nabla V(q)dt + dW \\ \mathbb{E}(u_0(q(t))|q(t=0)=q) = u(q,t) \\ \partial_t u = -\frac{1}{h}\nabla V(q)\partial_q u + \frac{1}{2}\Delta_q u \quad , \quad u(q,t=0) = u_0(q) \end{array}\right. ,$$

et qui correspond à une dynamique suivant le gradient du potentiel avec bruit sur la vitesse (dW représente le bruit blanc gaussien). La mesure invariante, solution stationnaire pour l'opérateur adjoint $\partial_q.(\partial_q + \frac{2}{h}\partial_q V)\mu = 0$ est $e^{-\frac{2V}{h}}dq$ et on conjugue par $e^{\frac{V}{h}}$ pour aboutir à l'opérateur

$$\frac{1}{2}(\partial_q - \frac{1}{h}\nabla V).(\partial_q + \frac{1}{h}\nabla V) = \frac{1}{2}\big(\Delta_q - \frac{1}{h^2}|\nabla V|(q)^2 + \frac{1}{h}\Delta V(q)\big) = \frac{-1}{2h^2}\Delta^{(0)}_{V,h}$$

étudié dans l'espace $L^2(\mathbb{R}^d, dq)$. On reconnaît le Laplacien de Witten sur les fonctions $\Delta^{(0)}_{V,h}$ qui est un opérateur auto-adjoint (une fois le domaine précisé) et elliptique. La limite basse température correspond à une limite semiclassique $h \to 0^+$ pour le Laplacien de Witten.
Un modèle plus fin, proposé par Langevin, est donné par la dynamique stochastique

$$\left\{\begin{array}{l} dq = pdt \\ dp = -\frac{1}{m}\nabla V(q)dt - \gamma p dt + \sqrt{\frac{2\gamma}{m\beta}}dW \\ u(q,p,t) = \mathbb{E}(u_0(q(t),p(t)))|q(0)=q, p(0)=p)) \\ \partial_t u = [p.\partial_q - \frac{1}{m}\partial_q V(q).\partial_p]u + \frac{\gamma}{m\beta}(\partial_p - m\beta p).\partial_p u \quad , \quad u(q,p,0) = u_0(q,p), \end{array}\right.$$

où le bruit est sur l'accélération comme un terme de force aléatoire. Les paramètres sont :

- m la masse de la particule ;
- $\gamma > 0$ le coefficient de friction ;
- $\beta = \frac{1}{k_b T}$ l'inverse de la température ;

Un changement d'échelle permet de se ramener à $m = 1$ et à un ensemble réduit de paramètres indépendants

- $h = 2k_B T = \frac{2}{\beta}$ ici
- et $b = \frac{\sqrt{h}}{\gamma} = \frac{\sqrt{2}}{\sqrt{\beta}\gamma}$.

Si $\frac{1}{b}$ n'est pas exactement proportionnel à γ, nous dirons abusivement que la limite $b \to 0^+$ est une limite de grande friction et $h \to 0^+$ une limite de basse température.
L'opérateur associé à la description de Langevin du mouvement brownien est l'opérateur de

Kramers-Fokker-Planck, qui est non auto-adjoint. La mesure invariante est ici, avec $m = 1$ et les paramètres

$$\boxed{(b,h) = (\frac{2\sqrt{2}}{\sqrt{\beta\gamma}}, \frac{2}{\beta})},$$

la maxwellienne $\mu = e^{-2\frac{\frac{|p|^2}{2}+V(q)}{h}}dqdp$ et une conjugaison par $e^{\frac{\frac{|p|^2}{2}+V(q)}{h}}$ conduit à l'étude de l'opérateur

$$[p.\partial_q - \partial_q V(q).\partial_p] + \frac{h^{3/2}}{2b}(-\Delta_p + \frac{|p|^2}{h^2} - \frac{d}{h})$$

ou encore après un changement d'échelle en p

$$\sqrt{h}b\left[\frac{1}{b}(p.\partial_q - \frac{1}{h}\partial_q V.\partial_p) + \frac{1}{b^2}\frac{-\Delta_p + |p|^2 - d}{2}\right]$$

dans $L^2(\mathbb{R}^{2d}, dqdp)$. Cet opérateur n'est plus ni symétrique ni elliptique mais il est hypoelliptique et sa fermeture est maximale accrétive.

Le Laplacien que Witten introduit dans [Wit] est un Laplacien de type Hodge agissant sur les formes différentielles de tout degré qui généralise l'opérateur associé au modèle de Langevin suramorti. Formellement, le Laplacien de Hodge standard domine dans la limite $h \to +\infty$ tandis que la topologie des courbes de niveaux de V et plus précisément la théorie de Morse (si V est une fonction de Morse) domine quand $h \to 0$. C'est ainsi que Witten propose une démonstration analytique des inégalités de Morse, reliant le nombre de points critiques d'indice p avec les nombres de Betti de la variété.
Plus récemment, Bismut dans [Bis041], [Bis042], [Bis05] a introduit un opérateur, appelé maintenant Laplacien hypoelliptique de Bismut. A l'instar du Laplacien de Witten, il agit sur les formes différentielles et généralise la description de Langevin du mouvement brownien avec les deux paramètres $b > 0$ et $h > 0$. Un des objectifs est d'interpoler entre la limite formelle $b \to +\infty$ où le flot géodésique (ou hamiltonien si $V \neq 0$) domine et la limite $b \to 0$, "grande friction" où l'on doit retrouver Langevin suramorti et donc le Laplacien de Witten dans un premier temps. Combiné avec la limite $h \to 0^+$, ce programme vise donc a faire un lien entre propriétés dynamiques de la variété et sa topologie, via la théorie de Morse.
Rappelons brièvement l'histoire récente des travaux sur l'analyse spectrale de ces deux opérateurs dans les asymptotiques $h \to 0^+$ pour le Laplacien de Witten et $(b,h) \to (0^+, 0^+)$ pour le Laplacien hypoelliptique.
Les Figures 1.1 et 1.2 illustrent l'évolution de nos connaissances sur le spectre du Laplacien de Witten.

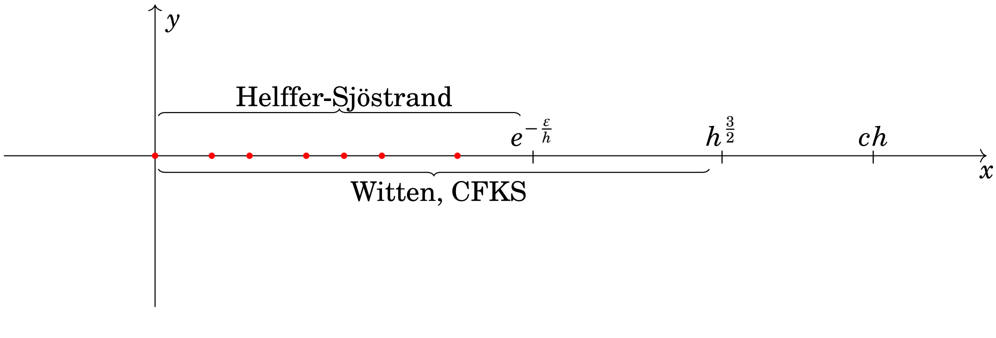

FIGURE 1.1 – Spectre du Laplacien de Witten.

— Dans [Wit] et [CFKS], les auteurs séparent les valeurs propres du Laplacien de Witten avec un potentiel de Morse en deux groupes : celles d'ordre $O(h^{3/2})$ et les autres.
— Ensuite, Helffer–Sjöstrand [HeSj4] améliorent ce résultat : les valeurs propres d'ordre $O(h^{3/2})$ sont exponentiellement petites (c'est-à-dire $O(e^{-\frac{\varepsilon}{h}})$ avec une constante $\varepsilon > 0$), sans donner plus de précision sur leurs tailles respectives.
— Inspirés par la théorie de Freidlin–Wentzell, Bovier–Eckhoff–Gayrard–Klein donnent dans [BEGK], avec des arguments de théorie du potentiel, des valeurs précises des valeurs propres exponentiellement petites dans le cas scalaire, degré $p = 0$. Dans le même temps Helffer–Klein–Nier dans [HKN] obtiennent les mêmes asymptotiques avec une approche purement spectrale et semiclassique.
— Finalement dans [LNV2], Le Peutrec–Nier–Viterbo donnent les échelles exponentielles de toutes les valeurs propres en tout degré pour un potentiel V $\mathscr{C}^\infty$ avec un nombre fini de valeurs critiques. Les constantes $c_n^{(p)}$ de la figure 1.2 sont en fait données par les longueurs des barres de degré p et $p-1$ (avec une extrémité de degré p) dans le code barre du potentiel. Les échelles exponentiellement petites des valeurs propres du Laplacien de Witten en limite semiclassique sont donc liées à l'homologie persistante et sont donc très robustes en tant que quantités topologiques.

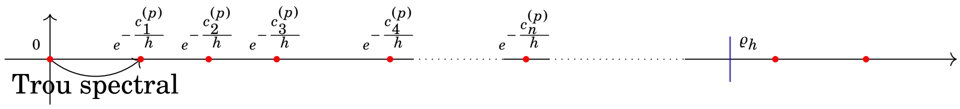

FIGURE 1.2 – Spectre du Laplacien de Witten semiclassique : valeurs propres en • ; trou spectral ϱ_h séparant les valeurs propres exponentiellement petites des autres.

La première valeur propre non nulle du Laplacien de Witten encode le temps moyen nécessaire pour que le mouvement Brownien atteigne l'équilibre. La déterminer de façon précise correspond donc à des estimations dites de trou spectral pour estimer la vitesse de retour à l'équilibre. En fait les valeurs propres exponentiellement petites, et leurs différentes échelles exponentielles, donnent des informations sur la durée de vie d'états métastables On sait depuis Arrhenius que distinguer ces échelles de temps est la base de la cinétique chimique. Par ailleurs ces idées ont dépassé le cadre initial de la chimie et de la physique statistique pour être développées dans le cadre de l'optimisation via des algorithmes de recuit simulé ou de dynamique moléculaire ou de telles informations sont aussi utiles. Enfin le modèle de Langevin, par rapport à Langevin suramorti, offre plus de flexibilité et conduit à des méthodes numériques parfois plus performantes que sa version suramortie. Nous renvoyons au texte de Lelièvre-Stoltz [LeSt] pour plus de détails sur les enjeux de modélisation et de traitement numérique.
Il est donc naturel d'envisager le même programme de détermination précise du bas du spectre pour le Laplacien hypoelliptique de Bismut.
Sur le Laplacien hypoelliptique, Bismut et Lebeau dans [BiLe] étudient essentiellement l'asymptotique $b \to 0^+$ à $h > 0$ fixé, pour relier le spectre du Laplacien hypoelliptique à celui du Laplacien de Witten. Dans [She], Shen étudie directement la double asymptotique $(b,h) \to (0^+,0^+)$ en supposant $b \propto \sqrt{h}$ (γ est borné et $T \to 0^+$) et avec une métrique bien choisie au voisinage des points critiques.
Dans cette thèse nous arrivons a une description plus générale pour la double asymptotique $(b,h) \to (0^+,0^+)$ avec des conditions précisées plus loin. Nous vérifions en particulier que les petites valeurs propres du Laplacien hypoelliptique, sont réelles, ≥ 0 , exponentiellement petites et que avec les bonnes hypothèses de comparaison entre b et h , leurs échelles exponentielles sont données par celles des valeurs du Laplacien de Witten et donc par le code barre de la fonction potentiel.

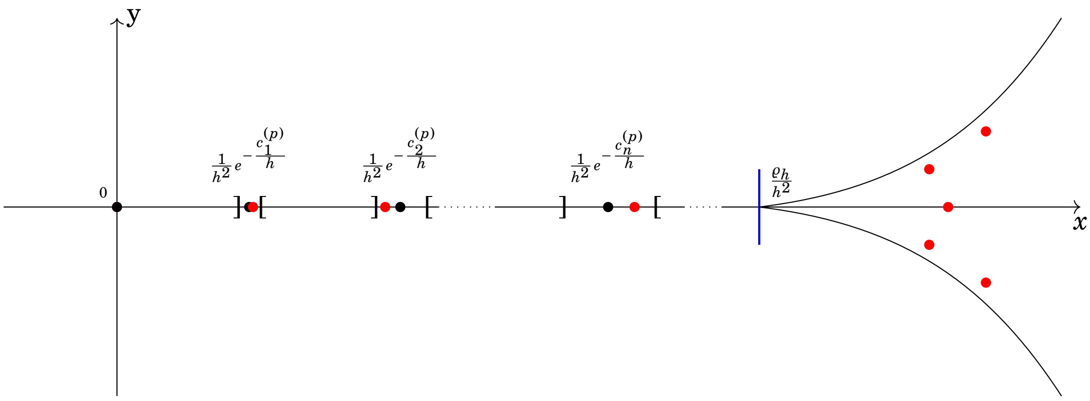

FIGURE 1.3 – Spectre du Laplacien hypoelliptique de Bismut : valeurs propres • ; $\frac{1}{h^2}\times$ la valeur propre du Laplacien de Witten correspondante proche • ; $\varrho_h = e^{-\frac{\varepsilon}{h}}$.

L'idée initiale était d'obtenir des estimations spectrales fines pour le Laplacien hypoelliptique de Bismut en suivant la stratégie développée dans [LNV2] pour le Laplacien de Witten. Celle-ci localise l'analyse à l'aide de problèmes aux limites (conditions de Dirichlet et de Neumann) artificielles. La première idée consistait donc à étudier le Laplacien hypoelliptique avec des conditions aux limites. Motivés par cette problématique du cas à bord, nous avons commencé par revisiter l'inégalité sous-elliptique (voir [NSW1] ou Chapitre 2), avec une approche locale, différente de celle de Lebeau dans [Leb1] et [Leb2], qui utilise les opérateurs intégraux de Fourier. Par sa nature locale notre approche est plus robuste et a plus de chance d'être transposée au cas à bord. Notons qu'à l'heure actuelle les résultats d'analyse sur le Laplacien hypoelliptique à bord, hormis des cas scalaires spécifiques, restent limités à l'analyse fonctionnelle (voir [Nie] et [NiSh]) et ne disent rien sur le comportement asymptotique quand $(b,h)\to(0,0)$.
Une fois l'inégalité sous-elliptique obtenue, et avant même que nous ayons eu le temps de nous pencher sur le cas à bord, nous avons été informés des travaux de Z. Tao avec Q. Ren sur le mouvement brownien cinétique (voir [ReTa]), version très simplifiée et scalaire du Laplacien hypoelliptique pour laquelle ils ont introduit un problème de Grushin bien adapté à l'analyse asymptotique du bas du spectre. Il nous est alors apparu, avec nos estimations sous-elliptiques précises dans le cas sans bords, que nous pouvions étendre cette méthode pour faire l'analyse complète du bas du spectre dans l'asymptotique $(b,h)\to(0,0)$. Cela fait l'objet du deuxième article [NSW2] ou Chapitre 3. L'analyse asymptotique du Laplacien hypoelliptique avec conditions aux bords est un programme à continuer.

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1.1 La géométrie de la variété de base Q

On se place sur une variété Riemannienne compacte de dimension d (Q, g^{TQ}, ∇^{LC}), où g^{TQ} désigne la métrique sur le fibré tangent TQ et ∇^{LC} la connexion de Levi-Civita associée.
Soit E un fibré vectoriel sur Q, avec π_E la projection de E sur Q. Ce fibré est muni d'une connexion plate ∇^E ainsi que d'une métrique hermitienne h^E. Il n'est pas nécessaire de supposer que la connexion ∇^E et la métrique h^E sont compatibles, c'est à dire que nous ne supposons pas :

$$\nabla^{\mathscr{L}(E;\bar{E}^*)} h^E = 0.$$

Suivant l'usage nous notons $\Omega^k(Q;E)$ l'ensemble des formes différentielles lisses de degré k à valeurs dans le fibré E

$$\Omega^k(Q;E) = \mathscr{C}^\infty(Q; \Lambda^k T^*Q \otimes E).$$

Pour plus de précision sur la régularité des sections, nous utiliserons la notation $\mathscr{F}(Q; \Lambda^k T^*Q \otimes_Q E)$, où $\mathscr{F}$ pourra être L^2 ou $\tilde{\mathscr{W}}^s$, ce dernier étant défini aux paragraphes 1.5.2 .
La règle de Leibniz alternée, définie pour tout $k, \ell \in \{0, 1, \dots, d\}$, $s \in \Omega^k(Q;\mathbb{C})$ et $u \in \Omega^\ell(Q;E)$, s'écrit :

$$\nabla^E(s \wedge u) = d^Q s \wedge u + (-1)^k s \wedge \nabla^E u.$$

Elle permet d'étendre la connexion :

$$\nabla^E : \mathscr{C}^\infty(Q;E) \to \Omega^1(Q;E)$$

à toutes les formes à valeurs dans E :

$$\nabla^E : \Omega^\bullet(Q;E) \to \Omega^{\bullet+1}(Q;E).$$

1.1.1 Cohomologie à valeurs dans un fibré vectoriel plat

Le complexe ci-dessous

$$0 \longrightarrow \Omega^0(Q;E) \xrightarrow{\nabla^E} \Omega^1(Q;E) \xrightarrow{\nabla^E} \dots \xrightarrow{\nabla^E} \Omega^d(Q;E) \longrightarrow 0 \qquad (1.1.1.1)$$

est un complexe de cochaînes si et seulement si la courbure $R^E = 0$. La proposition suivante nous donne la raison de cette équivalence.

Proposition 1.1.1. *L'égalité ci-dessous est vérifiée*

$$\nabla^E \circ \nabla^E = R^E.$$

Une preuve de la Proposition 1.1.1 se trouve dans [BeGeVe].

Démonstration. En prenant les mêmes notations que ci-dessus, un calcul direct fournit :

$$\nabla^E \nabla^E s \wedge u = s \wedge \nabla^E \nabla^E u.$$

Cela montre que $\nabla^E \circ \nabla^E$ est $\Omega(Q)$−linéaire. Il suffit alors de vérifier cette relation sur les sur les 0-formes $u \in \mathscr{C}^\infty(Q;E)$. Pour ces formes, nous avons :

$$\left(\nabla^E \nabla^E u\right)(U,V) = \nabla^E_U \nabla^E_V u - \nabla^E_V \nabla^E_U u - \nabla^E_{[U,V]} u = R^E(U,V)u.$$

□

Comme la connexion ∇^E est supposée plate, le complexe $(\Omega^\bullet(Q;E),\nabla^E)$ est un complexe de cochaînes. Rappelons quelques terminologies associées à ce contexte :

Définition 1.1.2 (forme fermée/ forme exacte)**.** Soit u une forme à valeurs dans E, on dit que u est :

— Fermée si $u \in \mathrm{Ker}(\nabla^E)$, i.e. $\nabla^E u = 0$.
— Exacte si $u \in \mathrm{Im}(\nabla^E)$, i.e. il existe une forme $\tilde{u}$ tel que $u = \nabla^E \tilde{u}$.

Sachant que $\nabla^E\nabla^E = 0$, on en déduit que toutes les formes exactes sont fermées. Autrement dit :

$$\mathrm{Im}(\nabla^E) \subset \mathrm{Ker}(\nabla^E).$$

Définition 1.1.3 (Cohomologie à valeurs dans un fibré vectoriel plat)**.** La cohomologie de ce complexe de cochaînes est définie par :

$$H^\bullet(Q;E) = \frac{\mathrm{Ker}(\nabla^E)}{\mathrm{Im}(\nabla^E)}.$$

De manière plus précise :

$$\forall k \in \{0,\dots,d\}, \quad H^k(Q;E) = \frac{\mathrm{Ker}\left(\nabla^E : \Omega^k(Q;E) \to \Omega^{k+1}(Q;E)\right)}{\mathrm{Im}\left(\nabla^E : \Omega^{k-1}(Q;E) \to \Omega^k(Q;E)\right)},$$

avec les conventions suivantes :

$$\Omega^{-1}(Q;E) = \{0\} \quad \text{et} \quad \Omega^{d+1}(Q;E) = \{0\}.$$

Un exemple de tel fibré plat sur lequel nous revenons au paragraphe 1.1.3 ci-dessous est pour $h > 0$ le fibré vectoriel trivial hermitien $E = Q \times \mathbb{C}$, muni de la métrique hermitienne $h^E = e^{-2\frac{V}{h}} dz \otimes d\bar{z}$ avec V une fonction lisse et de la connexion plate $\nabla^E = d^Q$. Cela permet de voir le Laplacien de Witten comme un Laplacien de Hodge sur un tel fibré.

1.1.2 Théorie de Hodge

Dans la section précédente, la métrique g^{TQ} sur Q n'a pas été utilisée. Pour développer la théorie de Hodge, cette métrique permet de définir l'espace $L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E)$, qui contient les formes à valeurs dans E.

La métrique g^{TQ} induit un isomorphisme entre le fibré tangent TQ et le fibré cotangent T^*Q, défini comme suit :

$$\varphi_{g^{TQ}} : \begin{array}{ccc} TQ & \to & T^*Q \\ u & \mapsto & \iota_u g^{TQ}, \end{array}$$

où, pour $u \in T_qQ$, la forme $\iota_u g^{TQ} \in T_q^*Q$ est donnée par :

$$\forall v \in T_qQ, \quad \iota_u g^{TQ}(v) = g_q^{TQ}(u,v).$$

Par identification canonique, nous utilisons la même notation pour la section lisse $\varphi_{g^{TQ}}$ de $\mathrm{End}(TQ;T^*Q)$ et pour la forme bilinéaire g^{TQ}. Avec cette convention, l'égalité suivante est vérifiée :

$$\forall u,v \in T_qQ, \quad g_q^{TQ}(u,v) = \left\langle\, u \;\middle|\; g_q^{TQ}(v) \,\right\rangle.$$

On note $\varphi_{g^{\Lambda TQ}}$ l'extension de $\varphi_{g^{TQ}}$ sur l'algèbre extérieure par la règle :

$$\forall u,v \in \Lambda TQ, \quad \varphi_{g^{\Lambda TQ}}(u \wedge v) = \varphi_{g^{\Lambda TQ}} u \wedge \varphi_{g^{\Lambda TQ}} v.$$

De même, on définit $\varphi_{g^{\Lambda T^*Q}}$ à partir de $g^{T^*Q} = (g^{TQ})^{-1}$:

$$\forall u, v \in T^*Q, \quad g^{T^*Q}(u,v) = \left\langle\, g^{T^*Q}(u) \,\middle|\, v \,\right\rangle.$$

La métrique sur ΛT^*Q compatible avec g^{TQ} est donnée par :

$$\forall u, v \in \Lambda T^*Q, \quad g^{\Lambda T^*Q}(u,v) = \left\langle\, \varphi_{g^{\Lambda T^*Q}}(u) \,\middle|\, v \,\right\rangle.$$

En tensorisant la métrique $g^{\Lambda T^*Q}$ avec la forme hermitienne h^E, on obtient une forme hermitienne $h^{\Lambda T^*Q\otimes E}$ sur $\Lambda T^*Q \otimes E$, définie par :

$$\forall u, v \in \Omega^\bullet(Q), \forall e, f \in \mathscr{C}^\infty(Q;E), \quad h^{\Lambda T^*Q\otimes E}(u\otimes e, v\otimes f) = g^{\Lambda T^*Q}(u,v)h^E(e,f).$$

Définition 1.1.4 (Les espaces $L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E)$ et $L^2(Q, dvol_g^{TQ}; \Lambda^k T^*Q \otimes E)$)**.** Une section u de $\Lambda T^*Q \otimes E$ appartient à l'espace $L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E)$ si l'intégrale suivante est finie :

$$\|u\|^2_{L^2(Q,dvol_g^{TQ};\Lambda T^*Q\otimes E)} = \int_Q h^{\Lambda T^*Q\otimes E}(u,u)\, dvol_{g^{TQ}} < \infty.$$

Ici, $dvol_{g^{TQ}}$ désigne la mesure associée à la métrique g^{TQ}. Pour $k \in \{0,1,\ldots,d\}$, l'espace $L^2(Q, dvol_g^{TQ}; \Lambda^k T^*Q \otimes E)$ est constitué des éléments homogènes de degré k de $L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E)$. Nous avons ainsi la somme directe orthogonale suivante :

$$L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E) = \bigoplus_{0\le k\le d} L^2(Q, dvol_g^{TQ}; \Lambda^k T^*Q \otimes E).$$

Définition 1.1.5 (La codifférentielle)**.** L'adjoint formel de la différentielle ∇^E par rapport au produit scalaire sur $L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E)$, noté $\nabla^{E,*}$, est appelé codifférentielle :

$$\nabla^{E,*} : \ \Omega^\bullet(Q;E) \ \to \ \Omega^{\bullet-1}(Q;E) \ .$$

Elle est définie par :

$$\forall u, v \in \Omega(Q;E), \quad \left\langle\, u \,\middle|\, \nabla^E v \,\right\rangle_{L^2(Q;\Lambda T^*Q\otimes E)} = \left\langle\, \nabla^{E,*} u \,\middle|\, v \,\right\rangle_{L^2(Q;\Lambda T^*Q\otimes E)}.$$

Remarque 1.1.6. La codifférentielle satisfait également $\nabla^{E,*}\nabla^{E,*} = 0$, et le complexe $\Omega^\bullet(Q; \nabla^{E,*})$ est un complexe de chaînes.

Définition 1.1.7 (Le Laplacien de Hodge)**.** Le Laplacien de Hodge Δ^E est un opérateur différentiel défini par :

$$\Delta^E = \nabla^{E,*}\nabla^E + \nabla^E \nabla^{E,*}.$$

Ce Laplacien agit degré par degré, et nous notons $\Delta^{E,(k)}$ son action sur les $k-$formes pour $k \in \{0,1,\ldots,d\}$.

Rappelons que le Laplacien de Hodge est un opérateur auto-adjoint positif, elliptique d'ordre 2 dans le calcul pseudodifférentiel classique sur Q, avec une résolvante compacte. Le théorème suivant porte sur la décomposition de Hodge.

Théorème 1.1.1 (Décomposition de Hodge)**.** *L'espace $L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E)$ se décompose en somme directe orthogonale :*

$$L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E) \quad = \quad \mathrm{Ker}(\Delta^E) \oplus \overline{\mathrm{Ran}(\Delta^E)}. \tag{1.1.2.1}$$

On a également :

$$L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E) = \left(\mathrm{Ker}(\nabla^E) \cap \mathrm{Ker}(\nabla^{E,*})\right) \oplus \overline{\mathrm{Ran}(\nabla^E)} \oplus \overline{\mathrm{Ran}(\nabla^{E,*})}$$

où $\mathscr{H} = \mathrm{Ker}(\Delta^E) = \left(\mathrm{Ker}(\nabla^E) \cap \mathrm{Ker}(\nabla^{E,*})\right)$ *est l'espace des formes harmoniques.*

Ce résultat classique remonte au moins aux travaux de Gaffney [Gaf] et une version un peu plus générale pour des problèmes à bords en faible régularité est donnée dans [GMM], Propositions 2.3, 2.4 et Corollaire 2.5, et rappelée dans l'Appendice A de [LNV2].

1.1.3 Le Laplacien de Witten semi-classique comme cas particulier du Laplacien de Hodge

Considérons, pour $h > 0$, le fibré vectoriel trivial hermitien $E = Q \times \mathbb{C}$, muni de la métrique hermitienne $h^E = e^{-2\frac{V}{h}} dz \otimes d\bar{z}$ avec V une fonction lisse et de la connexion plate $\nabla^E = d^Q$.

Dans ce cadre, l'espace $L^2(Q, dqdp; \Lambda T^*Q \otimes E)$ s'identifie à $L^2(Q, e^{-\frac{V}{h}} dqdp; \Lambda T^*Q)$ par la transformation unitaire :

$$\Phi : \begin{array}{ccc} L^2(Q, dvol_g^{TQ}; \Lambda T^*Q \otimes E) & \rightarrow & L^2(Q, dvol_g^{TQ}; \Lambda T^*Q) \\ u & \mapsto & e^{\frac{V}{h}} u \end{array}.$$

On a alors

$$\Phi d^Q \Phi^{-1} = d^Q + \frac{dV}{h} = \frac{1}{h} d_{V,h},$$

où $d_{V,h} = hd^Q + d_V$ est la différentielle de Witten. Le Laplacien de Hodge Δ^E se transforme en :

$$\frac{1}{h^2} \Delta_{V,h},$$

avec :

$$\Delta_{V,h} = h^2 \Delta^{\mathbb{C}} + g^{TQ}(\nabla V, \nabla V) + h(\mathscr{L}_{\nabla V}^* + \mathscr{L}_{\nabla V}),$$

où $\Delta_{V,h}$ est le Laplacien de Witten semi-classique associé au potentiel V, $\Delta^{\mathbb{C}}$ le Laplacien de Hodge usuel sur les formes, et ∇V est le gradient de V pour la métrique g^{TQ}.

1.1.4 Connexion de Levi-Civita

La connexion de Levi-Civita est l 'unique connexion sur le fibré tangent TQ qui est compatible avec la métrique et sans torsion.
La compatibilité avec la métrique :

$$\forall u, v \in \mathscr{C}^\infty(Q; TQ), \quad d(g^{TQ}(u,v)) = g^{TQ}(\nabla^{LC} u, v) + g^{TQ}(u, \nabla^{LC} v).$$

L'absence de torsion :

$$\forall u, v \in \mathscr{C}^\infty(Q; TQ), \quad T(u,v) = \nabla_u^{LC} v - \nabla_v^{LC} u - [u,v] = 0.$$

La courbure de la connexion de Levi-Civita, notée $R^{TQ} \in \Omega^2(Q; \mathrm{End}(TQ))$, est une 2-forme à valeurs dans les endomorphismes de TQ. Elle est définie par :

$$\forall u, v, w \in \mathscr{C}^\infty(Q; TQ), \quad R^{TQ}(u,v)w = \nabla_u^{LC} \nabla_v^{LC} w - \nabla_v^{LC} \nabla_u^{LC} w - \nabla_{[u,v]}^{LC} w.$$

Proposition 1.1.8 (Symétrie du tenseur de courbure)**.** *La courbure R^{TQ} est anti-symétrique par rapport à la métrique g^{TQ} :*

$$\forall u, v, w, z \in \mathscr{C}^\infty(Q; TQ), \quad g^{TQ}(R^{TQ}(u,v)w, z) = -g^{TQ}(w, R^{TQ}(u,v)z).$$

Les symboles de Christoffel :
Si $(e_1, \ldots, e_d)$ un repère de TQ et $(e^1, \ldots, e^d)$ le repère dual. Les symboles de Christoffel (Γ_{ik}^j) associés à ce repère sont définis par :

$$\forall i, j \in \{1, \ldots, d\}, \quad \nabla_{e_i}^{LC} e_j = \Gamma_{ij}^k e_k.$$

— Si le repère est associé à un système de coordonnée alors les symboles de Christoffel satisfont :

$$\forall i,j,k \in \{1,\dots,d\}, \quad \Gamma^k_{ij} = \Gamma^k_{ji},$$

— Si c'est un repère orthonormé alors les symboles de Christoffel satisfont :

$$\forall i,j,k \in \{1,\dots,d\}, \quad \Gamma^k_{ji} = -\Gamma^i_{jk}.$$

1.2 L'espace total du fibré cotangent

L'espace total X du fibré cotangent de Q, est naturellement muni d'une projection $\pi_X : X = T^*Q \to Q$. C'est une variété symplectique exacte définie via la 1−forme tautologique τ :

$$\forall x \in X, \quad \tau_x = \pi_X^* x.$$

Il s'agit bien d'une 1−forme sur T^*X puisqu'elle est le tiré en arrière par π_X de la forme duale $x \in T^*Q$. La forme symplectique $\sigma = d\tau$ est définie comme la différentielle de τ.
Rappelons quelques quantités liées à cette stucture :

— Énergie cinétique :
La métrique g^{TQ} induit une métrique sur T^*Q, notée g^{T^*Q}, et définit une fonction sur l'espace total X par :

$$\mathscr{H}(x) = \frac{1}{2} g^{T^*Q}(x,x).$$

Cette fonction représente l'énergie cinétique, qui est la moitié du carré de la norme mesurée via g^{T^*Q} de la forme linéaire $x \in T^*Q$.

— Champs Hamiltonien :
Comme toute fonction sur X, l'énergie cinétique $\mathscr{H}$ induit un champ hamiltonien $\mathscr{Y}$ défini par l'équation

$$\iota_{\mathscr{Y}} \sigma + d\mathscr{H} = 0.$$

En coordonnées locales $x = (q^1,\dots,q^d,p_1,\dots,p_d)$, le champs Hamiltonien s'écrit :

$$\mathscr{Y}_x = -2g^{ij}(q)p_j \partial_{q^i|x} + \frac{\partial g^{ij}}{\partial q^k}(q) p_i p_j \partial_{p_k|x}.$$

1.2.1 Décomposition horizontale verticale du fibré tangent TX et du fibré cotangent T^*X

La connexion de Levi-Civita ∇^{LC} sur TQ permet de définir, par transport parallèle, une distribution horizontale $(TX)^H$ sur le fibré tangent TX, de sorte que :

$$d\pi_X : (TX)^H \longrightarrow TQ$$

est un isomorphisme de fibrés vectoriels. (voir Figure 1.4).

Plus précisément, si on considère un vecteur $\underline{e} \in T_qQ$ dans l'espace tangent au point $q \in Q$, alors il existe un chemin $\underline{\gamma} :]-\varepsilon,\varepsilon[\to Q$ définissant le vecteur $\underline{e} = \underline{\gamma}'(0)$. La connexion ∇^{LC} sur le fibré vectoriel T^*Q permet de faire un relèvement parallèle du chemin $\underline{\gamma}$ en un chemin γ passant par $x \in X = T_q^*Q$ définit sur l'espace total X du fibré cotangent de Q i.e :

$$\begin{cases} \gamma(0) = x & \\ \pi_X \circ \gamma = \underline{\gamma} \\ \nabla^{LC}_{\underline{\gamma}'} \gamma = 0 \end{cases}$$

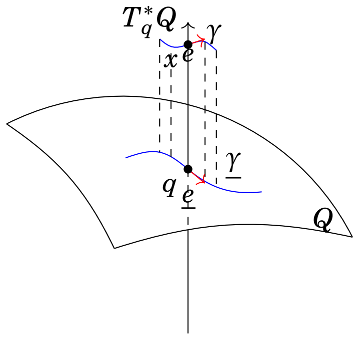

FIGURE 1.4 – Transport parallèle

Le vecteur $e \in T_x X$ tel que $(\pi_X)_* e = \underline{e}$ alors défini par $e = \gamma'(0)$, ne dépend pas du chemin $\underline{\gamma}$ choisi pour définir $\underline{e}$. La distribution horizontale $(TX)^H_x$ au point x est l'ensemble des vecteurs e ainsi construit. L'application suivante est un isomorphisme d'espaces vectoriels

$$(d\pi_X)_x\big|_{(TX)^H_x} : \begin{array}{ccc} (TX)^H_x & \to & T_q Q \\ e & \mapsto & \underline{e} \end{array} .$$

En choisissant un repère de TQ, on construit un repère de $(TX)^H$ via l'isomorphisme réciproque. Par exemple le repère $(\partial_{q^1}, \dots, \partial_{q^d})$ associé à un système de coordonnées sur Q nous donne $(e_1, \dots, e_d)$ comme repère sur TX^H défini par :

$$e_i = \partial_{q^i} - \Gamma^k_{ij} p_k \partial_{p_j} \quad \forall i \in \{1, \dots, d\}. \tag{1.2.1.1}$$

La distribution verticale $(TX)^V$ est définie canoniquement comme $(TX)^V = \ker(d\pi_X)$, et un repère de celui-ci est donné par $(\partial_{p_1}, \dots, \partial_{p_d})$.

Remarque 1.2.1. Le champ Hamiltonien $\mathscr{Y}$ associé à l'énergie cinétique $\mathscr{H} = \frac{1}{2} g^{ij} p_i p_j$ dans ce repère s'écrit :

$$\mathscr{Y} = g^{ij}(q) p_j e_i.$$

Il existe un isomorphisme de fibré vectoriel sur X entre $(TX)^H$ et $\pi_X^* TQ$ donnée par

$$\begin{array}{ccc} \pi_X^* TQ & \to & (TX)^H \\ (x, \underline{e}) & \mapsto & e = \underline{e}^H \end{array} ,$$

où $e = \underline{e}^H \in T_x X$ et $d\pi_X(e) = \underline{e} \in T_{\pi_X(x)} Q$.
De même, l'application suivante est un isomorphisme de fibré vectoriel :

$$\begin{array}{ccc} \pi_X^* T^* Q & \to & (TX)^V \\ (x, \underline{\hat{e}}) & \mapsto & \hat{e} = \underline{\hat{e}}^V \end{array} , \tag{1.2.1.2}$$

où $\underline{\hat{e}} = p_i dq^i \in T^*_{\pi_X(x)} Q$ et $\hat{e} = p_i \partial_{p_i} \in T_x X$. Un repère de $(TX)^V$ est donné par $(\hat{e}^1, \dots, \hat{e}^d)$, où :

$$\forall i \in \{1, \dots, d\}, \quad \hat{e}^i = \partial_{p_i}. \tag{1.2.1.3}$$

Le fibré tangent TX de X se décompose en somme directe :

$$TX = (TX)^H \oplus (TX)^V.$$

De même, le fibré cotangent T^*X de X se décompose en somme directe :

$$T^*X = (T^*X)^H \oplus (T^*X)^V,$$

avec :

$$\begin{array}{rcl} (T^*X)^H = ((TX)^H)^* & = & \{\ell \in T^*X, \forall e \in (TX)^V, \ell(e) = 0\} \\ (T^*X)^V = ((TX)^V)^* & = & \{\ell \in T^*X, \forall e \in (TX)^H, \ell(e) = 0\} \end{array}$$

définis par dualité. De plus, on a :

$$\begin{array}{rcl} (T^*X)^H & = & \mathrm{Vect}(e^1, e^2, \ldots, e^d) \\ (T^*X)^V & = & \mathrm{Vect}(\hat{e}_1, \ldots, \hat{e}_d) \end{array}$$

avec :

$$\forall i \in \{1, \ldots d\}, \quad \left\{ \begin{array}{rcl} e^i & = & dq^i \\ \hat{e}_i & = & dp_i + \Gamma^k_{ij} p_k dq^j. \end{array} \right.$$

Pour la suite de ce texte, on utilisera toujours un repère adapté pour la décomposition horizontale verticale pour le fibré tangent TX ou cotagent T^*X de l'espace total.

1.2.2 Métrique et connexion sur TX et sur T^*X

Comme TQ et T^*Q sont munis respectivement de la métrique g^{TQ} et g^{T^*Q}. On définit une métrique sur TX de sorte que la somme directe $TX = (TX)^H \oplus (TX)^V$ soit orthogonale et les isomorphismes $(TX)^V \simeq \pi_X^* T^*Q$ et $(TX)^H \simeq \pi_X^* TQ$ soient des isométries. Ainsi dans le repère $(e_1, \ldots, e_d, \hat{e}^1, \ldots, \hat{e}^d)$ la représentation matricielle de cette métrique $g^{TX} = g^{TQ} \oplus g^{T^*Q}$ s'exprime simplement par :

$$\begin{pmatrix} g^{TQ} & 0 \\ 0 & g^{T^*Q} \end{pmatrix}$$

Dans le repère $(e^1, \ldots, e^d, \hat{e}_1, \ldots, \hat{e}_d)$ de TX la connexion $\nabla^{TX} = \pi_X^*(\nabla^{TQ} \oplus \nabla^{T^*Q})$ sur TX s'exprime par :

$$\begin{array}{rclrcl} \nabla^{TX}_{e_i} e_j & = & \Gamma^k_{ij} e_k & \nabla^{TX}_{e_i} \hat{e}^j & = & -\Gamma^j_{ik} \hat{e}^k \\ \nabla^{TX}_{\hat{e}^i} e_j & = & 0 & \nabla^{TX}_{\hat{e}^i} \hat{e}^j & = & 0. \end{array}$$

Remarque 1.2.2. Cette connexion n'est pas la connexion de Levi-Civita associée à la métrique $g^{TX} = \pi_X^* g^{TQ} \oplus^\perp \pi_X^* g^{T^*Q}$ sur le fibré tangent.

— La torsion T^{TX} s'écrit :

$$\forall x = (q,p) \in X, \quad T^{TX}(e_i, e_j) = -(R^{TQ})^k_{ij\ell} p_k \partial_{p_\ell} = \underline{R^{T^*Q}(\partial_{q^i}, \partial_{q^j})p}^V \neq 0,$$

où $\underline{R^{T^*Q}(\partial_{q^i}, \partial_{q^j})p}^V$ désigne l'unique vecteur de $(TX)^V_x$ qui s'identifie à $R^{T^*Q}(\partial_{q^i}, \partial_{q^j})p \in T^*_q Q$ via l'isomorphisme donnée en (1.2.1.2).

— Elle est compatible avec la métrique :

$$dg^{TX}(U,V) = g^{TX}(\nabla^{TX} U, V) + g^{TX}(U, \nabla^{TX} V)$$

quels que soient les champs de vecteurs U, V considérés.

De même le fibré cotangent T^*X de X avec la décomposition $T^*X = (T^*X)^H \oplus (T^*X)^V$, où $(T^*X)^H \simeq \pi_X^* T^*Q$ et $(T^*X)^V \simeq \pi_X^* TQ$, permet de définir une métrique sur celui-ci

$$\begin{pmatrix} g^{T^*Q} & 0 \\ 0 & g^{TQ} \end{pmatrix}$$

et une connexion ∇^{T^*X} sur T^*X à l'aide du repère $(e^1,\dots,e^d,\hat{e}_1,\dots,\hat{e}_d)$ sur T^*X et du repère $(e_1,\dots,e_d,\hat{e}^1,\dots,\hat{e}^d)$ sur TX. Pour tout $i,j \in \{1,\dots,d\}$

$$\begin{array}{rclrcl} \nabla^{T^*X}_{e_i} e^j &=& -\Gamma^j_{ik} e^k & \nabla^{T^*X}_{e_i} \hat{e}_j &=& \Gamma^k_{ij} \hat{e}_k \\ \nabla^{T^*X}_{\hat{e}^i} e^j &=& 0 & \nabla^{T^*X}_{\hat{e}^i} \hat{e}_j &=& 0. \end{array}$$

1.3 Fibré tiré en arrière $\pi_X^* E = \mathscr{E}$

Le tiré en arrière du fibré vectoriel E sur la base Q définit un fibré vectoriel $\mathscr{E} = \pi_X^* E$ sur l'espace total du fibré cotangent X, avec le diagramme commutatif :

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{pr_E} & E \\ \downarrow{\scriptstyle \pi_{\mathscr{E}}} & & \downarrow{\scriptstyle \pi_E} \\ X & \xrightarrow{\pi_X} & Q. \end{array}$$

Ce fibré $\mathscr{E}$ est naturellement muni d'une structure de fibré plat hermitien. La métrique étant donnée par $h^{\mathscr{E}} = \pi_X^* h^E$, la tirée en arrière de la métrique sur E :

$$\forall u,v \in \mathscr{E}, \quad h^{\mathscr{E}}(u,v) = h^E(pr_E(u), pr_E(v)),$$

où pr_E désigne la projection de $\mathscr{E}$ sur E. La connexion tirée en arrière est définie par :

$$\forall i \in \{1,\dots,d\}, \quad \left\{ \begin{array}{rcl} \nabla^{\mathscr{E}}_{e_i} v &=& pr_E^*(\nabla^{LC}_{\partial_{q^i}} v) \\ \nabla^{\mathscr{E}}_{\hat{e}^i} v &=& 0 \end{array} \right. ,$$

pour toute section v de E qu'on identifie avec son tiré en arrière sur $\mathscr{E}$ et le repère $(e_1,\dots,e_d,\hat{e}^1,\dots,\hat{e}^d)$ défini par les égalités (1.2.1.1) et (1.2.1.3).

L'oscillateur harmonique verticale $\mathscr{O}$ est un opérateur différentiel défini globalement dont l'écriture locale en coordonnées $x = (q,p)$ et dans un repère du fibré $\Lambda T^*X \otimes \mathscr{E}$, adapté à la décomposition horizontale et verticale de T^*X et plat pour la connexion $\nabla^{\mathscr{E}}$ sur $\mathscr{E}$, par :

$$\mathscr{O} = \frac{1}{2}(-g_{ij}(q)\hat{e}^i\hat{e}^j) + g^{ij}(q)p_ip_j). \qquad (1.3.0.1)$$

L'oscillateur harmonique vertical s'applique composantes par composantes. la clôture de cet opérateur est auto-adjointe et constitue une intégrale directe, par rapport à $q \in U$ ouvert de carte de Q, d'oscillateurs harmonique en la variable du fibre p.
On définit un opérateur sur les fonctions, appelé le Laplacien horizontal Δ_H, par

$$\Delta_H = (e_i)^* g^{ij} e_j = \Delta_g - g_{ij}\hat{e}^i\hat{e}^j, \qquad (1.3.0.2)$$

où Δ_g désigne l'opérateur de Laplace-Beltrami sur les fonctions. Son action est définie par

$$\Delta_g f = \mathrm{div}(\nabla f),$$

avec div l'opérateur de divergence et ∇f le gradient de f associé à la métrique g^{TX}. Le laplacien horizontal est symétrique sur $\mathscr{C}_0^\infty(X;\mathbb{C})$ mais ne possède pas de réalisation auto-adjointe évidente (voir [BeBo]).

1.3.1 Une forme bilinéaire non dégénérée sur ΛT^*X

Grâce à la décomposition horizontale-verticale de TX, on peut définir la forme bilinéaire

$$\eta_b = g^{TQ} + b\sigma$$

sur TX. Par dualité, celle-ci induit une forme bilinéaire ϕ_b non dégénérée sur T^*X donnée par

$$\forall \omega, \rho \in T^*X, \quad \phi_b(\omega,\rho) = \langle\, \omega \mid \rho \,\rangle_{\eta_b} = \langle\, \omega \mid \eta_b^{-1}\rho \rangle_{T^*X\times TX}.$$

La forme bilinéaire sur ΛT^*X est donnée par :

$$\forall \omega, \rho \in \Lambda T^*X, \quad \phi_b(\omega,\rho) = \langle\, \omega \mid \rho \,\rangle_{\eta_b} = \langle\, \omega \mid \left(\wedge(\eta^{-1})\right)\rho \rangle_{\Lambda T^*X\times \Lambda TX},$$

où $\wedge(\eta^{-1})(\omega_1 \wedge \cdots \wedge \omega_d) = (\eta^{-1}\omega_1) \wedge \cdots \wedge (\eta^{-1}\omega_d)$.

1.4 Le Laplacien hypoelliptique de Bismut

Le Laplacien hypoelliptique de Bismut, dans le cadre de la théorie de De Rham est défini comme une déformation de l'opérateur de Dirac $d + d^*$. Plutôt que de définir l'adjoint de la différentielle extérieure à l'aide d'une forme bilinéaire symétrique définie positive, Bismut le définit dans [Bis05][BiLe] via une forme bilinéaire non dégénérée sur $L^2(X, dqdp; \Lambda T^*X \otimes \mathscr{E})$ reposant sur la forme η_b introduite précédemment et qui combine à la fois la structure métrique de (Q, g^{TQ}) et la forme symplectique σ sur le cotangent.

1.4.1 Forme bilinéaire non dégénérée sur $L^2(X, dqdp; \Lambda T^*X \otimes \mathscr{E})$

La forme bilinéaire définie sur $L^2(X, dqdp; \Lambda T^*X \otimes \mathscr{E})$ est une forme bilinéaire continue non symétrique

$$\int_X \langle\, u \mid v \,\rangle_{\eta_b \otimes h^{\mathscr{E}}} \, dqdp.$$

L'adjoint formel de la différentielle extérieure $d^{\mathscr{E}} = \nabla^{\mathscr{E}}$ par cette forme bilinéaire non dégénérée est noté $d^{\mathscr{E},*}_{\phi_b}$. On a évidement la relation $d^{\mathscr{E},*}_{\phi_b} \circ d^{\mathscr{E},*}_{\phi_b} = 0$.

1.4.2 Définition du Laplacien hypoelliptique de Bismut

Le Laplacien hypoelliptique de Bismut $\mathfrak{A}^2_b$, est un opérateur différentiel dépendant d'un paramètre $b > 0$, défini par :

$$\mathfrak{A}^2_b = [d^{\mathscr{E}}, d^{\mathscr{E},*}_{\phi_b}] = d^{\mathscr{E}} d^{\mathscr{E},*}_{\phi_b} + d^{\mathscr{E},*}_{\phi_b} d^{\mathscr{E}}.$$

C'est un opérateur différentiel. La formule de Weitzenböck donnée par Bismut dans [Bis05][BiLe] exprime l'opérateur sous la forme suivante :

$$\mathfrak{A}^2_b = \frac{1}{b^2}\alpha + \frac{1}{b}\beta + \gamma,$$

avec :

— L'opérateur différentiel α est donné par :

$$\alpha = \mathscr{O} + \frac{2N_V - d}{2},$$

où N_V est le nombre vertical.

— L'opérateur différentiel β contient à la fois la dérivation par rapport au champ $\mathscr{Y}$ ainsi qu'un terme provenant de la métrique hermitienne h^E.
— L'opérateur matriciel d'ordre 0, non borné γ contient des termes liés à la géométrie de la base comme

$$\langle R^{T^*Q}(p,e_i)p,e_j\rangle$$

et aussi des termes qui sont liée à la courbure du fibré vectoriel E sur Q

$$\nabla^E_{e_i}\omega(\nabla^E,g^E)(e_j).$$

Notons également que pour bien traiter la dualité de Poincaré, nous devons travailler avec deux versions du Laplaciens hypoelliptiques

$$\mathfrak{A}^2_{\pm,b}=\frac{1}{b^2}\alpha_\pm+\frac{1}{b}\beta_\pm+\gamma_\pm,$$

pour lesquelles l'analyse est essentiellement la même.
Nous renvoyons au Chapitre 3 et plus particulièrement à la Section 3.2.5 pour plus de détails.
Bismut montre aussi la formule suivante :

$$\pi_{\pm,0}\left(\gamma_\pm-\beta_\pm\alpha_\pm^{-1}\beta_\pm\right)\pi_{\pm,0}=\frac{1}{2}\Delta^E, \tag{1.4.2.1}$$

avec $\pi_{\pm,0}$ la projection sur le noyau de $\alpha_\pm$. qui est essentielle pour comparer le Laplacien hypoelliptique $\mathfrak{A}^2_{\pm,b}$ avec le Laplacien de Hodge Δ^E sur la base, ce point est étudié dans [BiLe] et repris dans notre deuxième article (voir Chapitre 3).

1.5 Les espaces de Hilbert

1.5.1 L'espace $L^2_w(X,dqdp;\Lambda T^*X\otimes\mathscr{E})$

L'espace $L^2_w(X,dqdp;\Lambda T^*X\otimes\mathscr{E})$, défini sur les formes différentielles à valeurs dans $\mathscr{E}$, est introduit en considérant une section w de $\mathrm{End}(\Lambda T^*X\otimes\mathscr{E})$. Le carré de w est appelé poids, bien que w soit en réalité une matrice.
Les sections u de $\Lambda T^*X\otimes\mathscr{E}$ appartiennent à $L^2_w(X,dqdp;\Lambda T^*X\otimes\mathscr{E})$) si et seulement si :

$$\|u\|^2_{L^2_w(X,dqdp;\Lambda T^*X\otimes\mathscr{E})}=\|wu\|^2_{L^2_{id}(X,dqdp;\Lambda T^*X\otimes\mathscr{E})}=\int_X g^{\Lambda T^*X\otimes\mathscr{E}}(wu,wu)\,dqdp<\infty.$$

Selon les textes et les situations, la matrice w peux prendre les formes suivantes :
— $w=id$ dans [NSW1],
— $w=\langle p\rangle^{\frac{N_V-N_H}{2}}\otimes id_{\mathscr{E}}$ dans [NiSh],
— $w=\langle p\rangle^{N_V}\otimes id_{\mathscr{E}}$ dans [Leb1] et [Leb2].

Pour une matrice w générale, celui-ci doit satisfaire des conditions pour garantir l'analyse qu'on fait.

Le produit scalaire sur cet espace est défini par :

$$\langle\, u\mid v\,\rangle_{L^2_w(X,dqdp;\Lambda T^*X\otimes\mathscr{E})}=\int_X g^{\Lambda T^*X\otimes\mathscr{E}}(wu,wv)dqdp.$$

L'application suivante est une isométrie évidente :

$$\begin{array}{ccc} L^2_w(X,dqdp;\Lambda T^*X\otimes\mathscr{E}) & \longrightarrow & L^2_{id}(X,dqdp;\Lambda T^*X\otimes\mathscr{E}) \\ u & \longmapsto & wu. \end{array}$$

Cette isométrie permet de ramener l'analyse sur l'espace non pondéré $L^2(X,dqdp;\Lambda T^*X\otimes\mathscr{E})=L^2_{id}(X,dqdp;\Lambda T^*X\otimes\mathscr{E})$.

1.5.2 L'échelle de Sobolev

Dans cette section, nous considérons une version simplifiée scalaire du cas général traité dans [NSW1]. L'espace total du fibré cotangent $X = T^*Q \simeq Q \times \mathbb{R}_p^d$ est non compacte. Par conséquent, les espaces de Sobolev globaux ne sont pas canoniquement définis et on doit préciser le comportement des fonctions à l'infini, ici dans l'asymptotique $p \to \infty$, des sections.

Localement X s'identifie à $T^*U \simeq U \times \mathbb{R}_p^d$ avec U un ouvert borné de $\mathbb{R}_q^d$, les opérations naturelles sur cet espace incluent la multiplication par q^i, la dérivation par q^i, la multiplication par p_i et la dérivation par p_i (pour tout i). Les transformations naturelles Ψ considérées sur le fibré cotangent sont des transformations qui préservent la structure du fibré vectoriel :

$$\Psi : \begin{array}{ccc} \mathbb{R}_q^d \times \mathbb{R}_p^d & \to & \mathbb{R}_q^d \times \mathbb{R}_p^d \\ (q,p) & \mapsto & (\psi(q), A(q)p), \end{array}$$

où $q \to A(q)$ est une application bornée de $\mathbb{R}_p^d$ à valeurs dans $\mathrm{Gl}(\mathbb{R}^d)$. Lorsque $A(q) = {}^t(d\psi(q))^{-1}$, on retrouve le cas particulier des transformations symplectiques sur $X = T^*Q$ induites par le changement de variable ψ sur la base Q.

Après une transformation $\Psi(q,p) = (\tilde{q},\tilde{p}) = \tilde{x}$, les opérations se transforment comme suit :

$$\begin{array}{rcl} (\Psi_*\partial_{q^i})\big|_{\tilde{x}} & = & \dfrac{\partial \tilde{q}^j}{\partial q^i}(q)\partial_{\tilde{q}^j}\big|_{\tilde{x}} + \dfrac{\partial A_j^k}{\partial q^i}(q)(A^{-1})_k^\ell(q)\tilde{p}_\ell \partial_{\tilde{p}_j}\big|_{\tilde{x}} \\ \Psi_*\partial_{p_i}\big|_{\tilde{x}} & = & A_j^i \partial_{\tilde{p}_j}\big|_{\tilde{x}} \\ \tilde{p} & = & A(q)p \\ \tilde{q} & = & \psi(q). \end{array}$$

Il est naturel de construire un espace de Sobolev qui préserve l'ordre de ces opérateurs après ce type de transformation :

$$\begin{array}{ccccccccccc} \mathrm{ord}(\partial_{\tilde{q}^\bullet}) & = & \mathrm{ord}(\partial_{q^\bullet}) & = & \mathrm{ord}\Psi_*\partial_{q^\bullet} & & \mathrm{ord}(\partial_{\tilde{p}_\bullet}) & = & \mathrm{ord}(\partial_{p_\bullet}) & = & \mathrm{ord}(\Psi_*\partial_{p_\bullet}) \\ & & \mathrm{ord}(q^\bullet) & = & \mathrm{ord}(\tilde{q}^\bullet) & & & & \mathrm{ord}(p_\bullet) & = & \mathrm{ord}(\tilde{p}_\bullet). \end{array}$$

Ces relations conduisent à définir les ordres des opérateurs comme suit :

opérateur	ordre
∂_{q^i}, e_i	1
$\partial_{p_i} = \hat{e}^i$	1/2
$\times p_i, \langle p \rangle$	1/2
champ hamiltonien $\mathscr{Y}$	3/2
oscillateur harmonique verticale $\mathscr{O}$	1

.

Ces ordres sont définis à homothétie près. Cependant, en fixant la convention $\mathrm{ord}(\partial_{q^\bullet}) = 1$, ils deviennent uniques.

L'espace de Sobolev $\tilde{\mathscr{W}}^s(X;\mathbb{C})$ est construit pour respecter ces considérations sur l'ordre des opérateurs. Une fonction u appartient à $\tilde{\mathscr{W}}^s(X;\mathbb{C})$ si, dans chaque ouvert de carte, elle satisfait la condition :

$$\| \langle p \rangle^k \partial_q^I \partial_p^J u \|_{L^2(X,dqdp;\mathbb{C})} < \infty,$$

quelque soit $I, J \in \mathbb{N}^d$ et $k \in \mathbb{N}$, avec la restriction

$$|I| + \frac{|J| + k}{2} \leq s.$$

Un choix légèrement différent est fait dans [Leb1], [Leb2] et nous reviendrons sur ce point dans la Section 1.8.

1.6 Opérateur de Kramers-Fokker-Planck géométrique

Le Laplacien hypoelliptique de Bismut s'inscrit dans la classe des opérateurs de Kramers-Fokker-Planck géométriques. Cette classe d'opérateurs a été introduite initialement par Lebeau dans [Leb1],[Leb2]. Nous en proposons ici une version qui permet de traiter le cas des formes différentielles à valeurs dans des fibrés généraux, comme c'est le cas dans la théorie du Laplacien hypoelliptique de Bismut [BiLe]. Cette version géométrique est définie sur le cotangent d'une variété riemannienne compacte, $X = T^*Q$, et généralise l'opérateur de Kramers-Fokker-Planck classique, qui s'écrit sur sur $\mathbb{R}^{2d}_{q,p} = T^*\mathbb{R}^d_q$ sous la forme :

$$p.\partial_q + \frac{-\Delta_p + |p|^2 - d}{2},$$

ou

$$p.\partial_q - \partial_q V(q).\partial_p + \frac{-\Delta_p + |p|^2 - d}{2}.$$

Soit E un fibré plat muni d'une métrique hermitienne h^E et d'une connexion ∇^E .

Définition 1.6.1 (Opérateur de Kramers-Fokker-Planck géométrique)**.** Un opérateur différentiel $A_{\pm,b}$, dépendant d'un paramètre $b > 0$, défini sur les sections du fibré pull-back $\mathscr{E} = \pi_X^* E$ sur X, est un opérateur de Kramers-Fokker-Planck Géométrique si celui-ci s'écrit localement sous la forme :

$$A_{\pm,b} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\nabla^{\mathscr{E}}_{\mathscr{Y}} + M_b,$$

où :
- $\mathscr{O}$ est l'oscillateur harmonique vertical,
- $\mathscr{Y}$ est le champ hamiltonien associé à l'énergie cinétique,
- $M_b = M_1^i(b)p_i + M_i^2(b)\partial_{p_i} + M_0(b)$, avec M_1^i, M_i^2 et M_0, des opérateurs de multiplications , satisfont des conditions spécifiques liées à la matrice w introduite dans la section 1.5.1.

Les conditions sur M_b sont :

1. $$\left\|\mathscr{O} w M_1(b) w^{-1}\right\|_{\mathscr{L}(\tilde{\mathscr{W}}^s(X;\mathscr{E});\tilde{\mathscr{W}}^s(X;\mathscr{E}))} \leq \frac{\nu_{1,s}(b)}{b},$$
2. $$\left\|w M_0(b) w^{-1}\right\|_{\mathscr{L}(\tilde{\mathscr{W}}^s(X;\mathscr{E});\tilde{\mathscr{W}}^s(X;\mathscr{E}))} \leq \nu_{0,s}(b)\left(1 + \frac{1}{b^2}\right),$$
3. Les constantes $\nu_{1,s}$ et $\nu_{0,s}$ sont telles que :
$$\forall v \in (0,+\infty), \nu_{1,s}^2(b) b^2 \leq \frac{C_s + 9\nu_{0,s}}{16}(1 + b^2).$$

Remarque 1.6.2. Les conditions imposées sur M_1^i, M_i^2 et M_0 garantissent plusieurs propriétés importantes pour les opérateurs considérés :
- La substitution locale de $\nabla^{\mathscr{E}}_{\mathscr{Y}}$ par la dérivation scalaire $\mathscr{Y}$ (définie lorsque l'on fixe un repère pour E) dans $L^2(X, dqdp; \mathscr{E})$ engendre une terme d'erreur R_b. Celui-ci est de la même forme que M_b, à savoir :
$$R_b = R_1^i(b)p_i + R_i^2(b)\partial_{p_i} + R_0(b),$$
avec R_1^i, R_i^2 et R_0 satisfont les mêmes conditions que M_1^i, M_i^2 et M_0.
- L'opérateur différentiel $A_{\pm,b}$, défini sur $\mathscr{S}(X;\mathscr{E}) \subset L^2_w(X, dqdp; \mathscr{E})$, est conjugué à un opérateur de même type mais par rapport à $w = id$.
- L'opérateur agissant sur $\tilde{\mathscr{W}}^s(X;\mathscr{E})$ peut être ramené, par conjugaison, à un opérateur de même type défini sur $L^2(X, dqdp; \mathscr{E})$.

1.7 Résultats

On rappelle que notre problématique consiste à étudier le spectre du Laplacien hypoelliptique de Bismut, rappelé en Section 1.4, en fonction des deux paramètres $b > 0$ (inverse de la friction) et $h > 0$ (proportionnel à la température). Plus particulièrement nous nous intéressons à la double asymptotique $b \to 0^+$ et $h \to 0^+$. La première étape consiste à écrire des estimations sous-elliptiques les meilleures possibles (exposant de gain de régularité optimal) et un contrôle des constantes dans les inégalités par rapport au jeu de paramètre b et h. En fait le paramètre h peut-être oublié dans un premier et on se concentre sur la classe générale des opérateurs de Kramers-Fokker-Planck Géométriques de Lebeau, rappelée en Section 1.6. C'est l'objet du premier travail dont le détail est au Chapitre 2 avec une stratégie sensiblement différente de [BiLe][Leb1][Leb2] que nous expliquons brièvement.
En adaptant les idées de Q. Ren et Z. Tao [ReTa] et en utilisant d'une part les estimations sous-elliptiques du premier travail et d'autre part des propriétés très spécifiques du Laplacien hypoelliptique, à savoir sa structure de Hodge et des propriétés de PT-symétrie, nous avons pu dans un deuxième texte aboutir à une justification complète de la description du bas du spectre du Laplacien hypoelliptique dans la double limite $b \to 0^+$ et $h \to 0^+$, correspondant à la figure 1.3.
Les deux paragraphes qui suivent résument les résultats et approches de ces deux articles.

1.7.1 Estimation sous-elliptique maximale pour les opérateur de Kramers-Fokker-Planck géométrique (voir Chapitre 2)

Dans ce paragraphe, nous nous plaçons dans le cadre des opérateurs de Kramers-Fokker-Planck géométriques introduits par Lebeau dans [Leb1] (voir Définition 1.6.1 ci-dessus), une classe d'opérateurs qui contient le Laplacien hypoelliptique de Bismut. Les inégalités sous-elliptiques présentées dans [Leb1] et [BiLe] ont été améliorées dans [Leb2] sous forme maximale. On rappelle que ces inégalités sous-elliptiques maximales s'expriment avec un gain de régularité d'exposant maximal, ici 2/3, en lien avec la notion de hypoellipticité maximale d'Helffer et Nourrigat dans [HeNo]. Par ailleurs, les estimations sous-elliptiques de [Leb2] incluent un contrôle des constantes dans l'asymptotique $b \to 0^+$ (limite grande friction), mais ne donnent aucune information dans le cas $b \to +\infty$.
Concernant les techniques employées, la méthode de [Leb2] repose sur une réduction microlocale via des opérateurs intégraux de Fourier, dans l'esprit du chapitre XXVII du livre de Hörmander [HormIII]. Un des objectifs initiaux de cette thèse était d'étendre ces résultats au cas d'une variété à bord. Dans ce contexte, l'usage d'opérateurs intégraux de Fourier s'avère délicate, ce qui nous a conduit à privilégier une méthode de démonstration purement locale, en reprenant d'abord le cas sans bord.
De plus, les résultats obtenus améliorent légèrement ceux de [Leb2], car nous obtenons un encadrement des constantes dans le régime $b \to \infty$, absent dans les travaux de Lebeau. Par ailleurs, nous avons pu associer l'échelle de Sobolev, naturelle pour analyser les opérateurs de Kramers-Fokker-Planck Géométriques, à un opérateur auto-adjoint géométriquement défini dans le cas scalaire. Plus précisément, il s'agit de l'opérateur :

$$W^2 = C - \Delta_H + C\mathcal{O}^2$$

où Δ_H est le Laplacien horizontal sur $X = T^*Q$ (voir (1.3.0.2) et le texte de J.P. Bourguigon et L. Bérard-Bergery [BeBo] sur le sujet), $\mathcal{O}$ l'oscillateur harmonique vertical, et C une constante positive suffisamment grande. Cet opérateur peut être intéressant pour étudier des modèles cinétiques écrits sur le cotangent d'une variété Riemannienne. À partir de cet opérateur scalaire, on introduit un opérateur ayant la même partie principale agissant sur les sections d'un fibré hermitien $\mathscr{E}$, où E est un fibré sur Q. Pour une partition quadratique de l'unité $(\theta_j)_{j\in J}$ en $q \in Q$, telle que

$$\forall q \in Q, \quad \sum_{j\in J} \theta_j^2(q) \equiv 1,$$

on définit un opérateur $W_\theta^2 = \sum_{j\in J} \theta_j W_{sc,j}^2 \theta_j$, où $W_{sc,j}^2$ représente l'action de l'opérateur W^2, composante par composante, dans un repère orthonormé du fibré hermitien au-dessus du support de θ_j.

On vérifie que W_θ^2 a une réalisation auto-adjointe et commute au sens fort avec l'oscillateur harmonique vertical $\mathscr{O}$. Nous introduisons alors les espaces de Sobolev définis par :

$$\tilde{W}^{s_1,s_2}(X;\mathscr{E}) = \{u \in \mathscr{S}', (\mathscr{O})^{s_1}(W_\theta^2)^{\frac{s_2}{2}} u \in L^2(X, dqdp;\mathscr{E})\},$$

avec la norme Hilbertienne associée :

$$\|u\|_{\tilde{\mathscr{W}}^{s_1,s_2}} = \|(\mathscr{O})^{s_1}(W_\theta^2)^{\frac{s_2}{2}} u\|_{L^2(X,dqdp;\mathscr{E})}.$$

Notons que le cas $s_1 = 0$ et $s_2 = s$ correspond aux espaces introduits dans la section 1.5.2.

Théorème 1.7.1. *Soit $P_{\pm,b,M} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\nabla_{\mathscr{Y}}^{\mathscr{E}} + M(b)$ un opérateur de Kramers-Fokker-Planck Géométrique. Pour tout $s \in \mathbb{R}$, il existe une constante $C_{g,s} \geq 1$ déterminée par les données géométriques $(g^{TQ}, E, h^E, \nabla^E)$, telle que l'opérateur $\frac{\kappa_b}{b^2} + P_{\pm,b,M}$ soit essentiellement maximal accrétif sur $\mathscr{C}_0^\infty(X;\mathscr{E})$ (ou $\mathscr{S}(X;\mathscr{E})$) dès lors que $\kappa_b \geq C_{g,s}(1+b^5)$. Notons par $\overline{P}_{\pm,b,M}^s$ la fermeture de $P_{\pm,b,M}$ dans $\tilde{\mathscr{W}}^s(X;\mathscr{E})$. L'inégalité suivante est vérifiée :*

$$\mathrm{Re}\, \langle (\frac{\kappa_b}{b^2} + \overline{P}_{\pm,b,M}^s)u \mid v \rangle_{\tilde{\mathscr{W}}^s(X;\mathscr{E})} \geq \frac{1}{8b^2}\left[\|u\|^2_{\tilde{\mathscr{W}}^{1,s}(X;\mathscr{E})} + \kappa_b \|u\|^2_{\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})}\right].$$

De plus,

$$\begin{aligned}\|(\overline{P}_{\pm,b,M} - \frac{i\lambda}{b})u\|_{\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})} &+ \frac{1}{b^2}\|u\|_{\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})} \\ &\geq \frac{1}{C_{g,s}(1+b^7)}\Big(\|\frac{\mathscr{O}}{b^2}u\|_{\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})} + \|\frac{1}{b}(\pm\nabla_{\mathscr{Y}}^{\mathscr{E}} - i\lambda)u\|_{\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})} \\ &\qquad + \frac{1}{b^{4/3}}\|u\|_{\tilde{\mathscr{W}}^{0,s+2/3}(X;\mathscr{E})} + \frac{1}{b^{4/3}}\|(\frac{|\lambda|}{\langle p \rangle})^{2/3} u\|_{\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})}\Big). \quad (1.7.1.1)\end{aligned}$$

Ces deux inégalités sont vraies pour toutes les sections u dans le domaine de $\overline{P}_{\pm,b,M}^s$ et pour tout $(\lambda, b) \in \mathbb{R} \times (0, +\infty)$.

L'inégalité ci-dessus montre que ces opérateurs sont hypoelliptiques avec un gain de régularité d'exposant 2/3. De plus l'inégalité

$$\|(\overline{P}_{\pm,b}^s - i\frac{\lambda}{b})u\|_{\tilde{\mathscr{W}}^{0,s}(X;\Lambda T^*X\otimes\mathscr{E})} + \frac{1}{b^2}\|u\|_{\tilde{\mathscr{W}}^{0,s}(X;\Lambda T^*X\otimes\mathscr{E})} \geq \frac{1}{C_g(1+b^7)}\frac{1}{b^{4/3}}\|(\frac{|\lambda|}{\langle p\rangle})^{2/3}u\|_{\tilde{\mathscr{W}}^{0,s}(X;\Lambda T^*X\otimes\mathscr{E})}$$

dit que l'opérateur est cuspidal au sens de [Nie], c'est-à-dire que l'on dispose de bonnes estimations de résolvante en dehors d'un cusp (voir Figure 1.3). Cette terminologie a été inspirée par celle des opérateurs sectoriels (voir [ReSi]).

Pour démontrer ce théorème, on conjugue l'opérateur $P_{\pm,b,M}$ par l'opérateur $(W_\theta^2)^{\frac{s}{2}}$ qui définit $\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})$. Nous sommes amenés à l'étude d'un opérateur $P_{\pm,b,M'} = (W_\theta^2)^{\frac{s}{2}} P_{\pm,b,M}(W_\theta^2)^{-\frac{s}{2}}$ sur l'espace $L^2(X, dqdp;\mathscr{E})$. L'opérateur M' est simplement une perturbation de la partie principale de l'opérateur :

$$P_{\pm,b} = P_{\pm,b,0} = \frac{1}{b^2}\mathscr{O} + \pm\frac{1}{b}\nabla_{\mathscr{Y}}^{\mathscr{E}}.$$

Il suffit donc de prouver le résultat dans le cas L^2 avec $M'(b) = 0$.

Une des difficultés pour l'étude de notre opérateur provient du fait que l'espace total X n'est pas compact. Les opérateurs de multiplication par $\mathscr{H}$ et p_i, pour tout i, sont des opérateurs non bornés.

On commence par localiser l'opérateur $P_{\pm,b}$ dans un ouvert de carte en utilisant une partition quadratique de l'unité $(\varrho_j)_{j\in J}$ sur Q, telle que :

$$\forall q \in Q, \quad \sum_{j\in J} \varrho_j^2(q) \equiv 1.$$

pour $j \in J$ fixé on est dans un ouvert de carte $\pi_X^{-1}(U) \simeq U \times \mathbb{R}^d$ avec U un ouvert de carte de Q. on utilise deux fonctions lisses $\theta, \tilde{\theta}$ satisfaisant :

$$\operatorname{supp}(\theta) \subset [\frac{1}{4}, 4], \quad \operatorname{supp}(\tilde{\theta}) \subset [0, 4],$$

et

$$\forall t \in \mathbb{R}_+, \quad \tilde{\theta}^2(4t^2) + \sum_{\ell=0}^{\infty} \theta^2(2^{-2\ell} t^2) \equiv 1.$$

À partir d'elles, on construit une famille de fonctions $(\theta_\ell)_{-1\leq \ell \leq +\infty}$ définie par :

$$\forall -1 \leq \ell \leq +\infty, \forall x \in X \quad \theta_\ell(x) = \begin{cases} \tilde{\theta}\big(8\mathscr{H}(x)\big) & \text{si } \ell = -1 \\ \theta\big(2^{-2\ell+1}\mathscr{H}(x)\big) & \text{si } \ell \geq 0 \end{cases}.$$

cette famille forme une partition quadratique et dyadique de l'unité :

$$\forall x \in X, \quad \sum_{\ell \geq -1} \theta_\ell^2(x) \equiv 1.$$

Voire la Figure 1.5 pour le support de la fonction θ_ℓ. L'étude de l'opérateur $P_{\pm,b}$ sur X est équivalent à l'étude de la famille dépendant de ℓ d'opérateur $P_{\pm,b}$ agissant sur l'ensemble :

$$\begin{array}{ll} \{x \in X, 2^{\ell-1} \leq \mathscr{H}(x) \leq 2^{\ell+1}\} & \text{si } \ell \neq -1 \\ \{x \in X, \mathscr{H} \leq \frac{1}{2}\} & \text{si } \ell = -1 \end{array}.$$

Pour ℓ fixé, on effectue un changement d'échelle dans la variable $p \in T_q^*Q$ du fibre. On se retrouve avec l'opérateur :

$$P_{\pm,b,\ell} = \frac{1}{2b^2}\left(g^{ij}(q)(2^\ell p_i)(2^\ell p_j) - g_{ij}(q)\frac{\partial_{p_i}}{2^\ell}\frac{\partial_{p_j}}{2^\ell}\right) \pm \frac{1}{b} g^{ij}(2^\ell p_j) e_i$$

agissant sur l'espace compact

$$\begin{array}{ll} \{x, \frac{1}{4} \leq \mathscr{H}(x) \leq 4\} & \text{si } \ell \neq -1 \\ \{x, \mathscr{H}(x) \leq 4\} & \text{si} \ell = -1 \end{array}.$$

Par la suite, nous découpons la variété de base $U \simeq \mathbb{R}_q^d$ en réseau et effectuons une partition quadratique et en réseau de l'unité, définie à partie d'une fonction $\Psi \in \mathscr{C}_0^\infty(\mathbb{R}^d; [0,1])$ telle que :

$$\forall q \in \mathbb{R}^d, \quad \sum_{m\in\mathbb{Z}^d} \Psi^2(q-m) \equiv 1.$$

Pour un paramètre $A > 0$ et un $\ell \geq -1$ fixé, on pose, pour tout $m \in \mathbb{Z}^d$:

$$\Psi_{m,\ell,A}(q) = \Psi\Big(\frac{q - A2^{-\ell}m}{A2^{-\ell}}\Big)$$

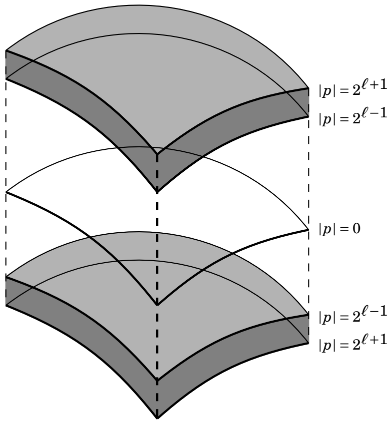

FIGURE 1.5 – Partition dyadique : représentation du support de θ_ℓ avec $\ell \neq 0$.

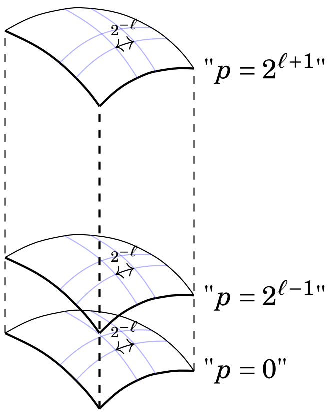

FIGURE 1.6 – Partition en réseau sur la base Q

de sorte que :

$$\forall q \in \mathbb{R}^d, \quad \sum_{m \in \mathbb{Z}^d} \Psi^2_{m,\ell,A}(q) \equiv 1.$$

Voire la Figure 1.6, qui illustre une cellule de la partition en réseau dans les différentes couches d'énergies dépendant de ℓ. À partir d'un certain rang en ℓ, la taille du réseau $A2^{-\ell}$ devient plus petite que le rayon d'injectivité de la variété Riemannienne Q. Par conséquent, le support de $\Psi_{m,\ell,A}$ est contenu dans le domaine d'une carte normale centrée en $A2^{-\ell}m$, pour $m \in \mathbb{Z}^d$.

Fixons $m \in \mathbb{Z}^d$, on se place alors dans la carte normale centrée en $A2^{-\ell}m$. Pour s'approcher du cas Euclidien, on effectue un changement de variable non symplectique, qui peut être interprété comme un changement d'échelle en q, donné par :

$$(q,p) \mapsto (2^{-\ell}q, g(2^{-\ell}q)p).$$

Remarquons que dans le cas $g = (\delta_{ij})_{1\leq i,j\leq d}$, la composition des deux changements de variables successifs $(q,p) \mapsto (q, 2^\ell p)$ et $(q,p) \mapsto (2^{-\ell}q, p)$ est le changement d'échelle symplectique, mais ce n'est plus le cas pour une métrique générale.

Après ce changement, l'opérateur s'écrit comme suit :

$$P_{\pm,b,\ell,m} = \tilde{P}_{\pm,b,\ell} + R_{b,\ell,m},$$

avec

— l'opérateur

$$\tilde{P}_{\pm,b,\ell} = \frac{1}{b^2}\mathscr{O}_\ell - \pm\frac{2^{2\ell}}{b}\delta^{ij}p_j\partial_{q^i},$$

— l'oscillateur harmonique $\mathscr{O}_\ell$ est unitairement équivalent au cas euclidien ($\ell = 0$)

$$\mathscr{O}_\ell = \frac{1}{2}\big(2^{2\ell}\delta^{ij}p_ip_j - 2^{-2\ell}\delta_{ij}\partial_{p_i}\partial_{p_j}\big).$$

L'étude du cas euclidien sur la base des inégalités sous-elliptiques pour l'opérateur d'Airy complexe, donne l'estimation uniforme en $\ell \geq -1$,

$$\forall u \in \mathscr{S}(\mathbb{R}^{2d}_{q,p};\mathbb{C}), \|\frac{1}{b^2}\mathscr{O}_\ell u\|_{L^2} \leq C\left[\|\tilde{P}_{\pm,b,\ell}u\|_{L^2} + \frac{1}{b^2}\|u\|_{L^2}\right].$$

— Le terme de perturbation $R_{b,\ell,m}$ contient différents termes dont l'un d'entre eux a la forme

$$\frac{1}{2b^2}[2^{2\ell}(g^{ij}(2^{-\ell}q) - \delta^{ij})p_ip_j - 2^{-2\ell}(g_{ij}(2^{-\ell}q) - \delta_{ij})\partial_{p_i}\partial_{p_j}],$$

avec $|2^{-\ell}q| \leq A2^{-\ell_0}$.
On voit alors (il en est de même pour les autres termes contenus dans $R_{\pm,b,\ell}$) que $R_{\pm,b,\ell}$ est une perturbation relativement bornée avec borne aussi petite que l'on veut, de $\tilde{P}_{\pm,b,\ell}$ pour $\ell \geq \ell_0$ avec $\ell_0 \gg 1$ assez grand.

Nous en déduisons pour $\ell \geq \ell_0$ avec $\ell_0 \gg 1$, l'inégalité

$$\left|\|(\frac{\kappa_b}{b^2} + P_{\pm,b,\ell,m} - i\frac{\lambda}{b})u\|_{L^2} - \|(\frac{\kappa_b}{b^2} + \tilde{P}_{\pm,b,\ell} - i\frac{\lambda}{b})u\|_{L^2}\right| \leq \frac{1}{2}\|(\frac{\kappa_b}{b^2} + \tilde{P}_{\pm,b,\ell} - i\frac{\lambda}{b})u\|_{L^2},$$

qui permet de ramener l'inégalité sous-elliptique pour $P_{\pm,b,\ell,m}$ à celle pour le modèle euclidien $\tilde{P}_{\pm,b,\ell}$ aisément déduite de celle pour l'opérateur d'Airy complexe.

Pour l'ensemble fini des $\ell \leq \ell_0$, les opérateurs de multiplication par $\mathscr{H}$ ou p_i (pour tout i) sont des opérateurs uniformément bornés et les inégalités d'hypoellipticité classiques pour $g^{ij}(q)p_i\partial_{q^j} - \frac{1}{2}\Delta_p$ suffisent.

1.7.2 Comparaison entre le Laplacien hypoelliptique de Bismut et le Laplacien de Witten (voir Chapitre 3)

Dans ce paragraphe, nous examinons le Laplacien hypoelliptique de Bismut, noté $\mathfrak{A}'^2_{b,\frac{V}{h}}$, défini sur les formes à valeurs dans le fibré tiré en arrière $\mathcal{E}$, du fibré en droites plat $E = Q \times \mathbb{C}$ muni d'une métrique Hermitienne $h^E = e^{-\frac{2V}{h}}d\bar{z}dz$. Après un changement unitaire

$$\begin{array}{ccc} L^2(X, dqdp; \Lambda T^*X \otimes \mathcal{E}) & \to & L^2(X, dqdp; \Lambda T^*X \otimes \mathbb{C}) \\ v & \mapsto & e^{-\frac{V}{h}}v, \end{array}$$

l'opérateur correspondant au Laplacien hypoelliptique $\mathfrak{A}'^2_{b,\frac{V}{h}}$ sera noté $B_{\pm,b,\frac{V}{h}}$. Dans [BiLe], il est montré que dans le régime grande friction ($b \to 0$), avec une température fixée $h = 1$, cet opérateur converge vers le Laplacien de Witten sur la base Q. Shen montre ensuite dans sa thèse [She] que, dans le régime grande friction et basse température avec condition $\frac{b^2}{h} \simeq 1$, et avec un potentiel V de Morse accompagné d'un choix de métrique g^{TQ} bien adaptée autour des points critiques du potentiel, l'opérateur converge également vers le Laplacien de Witten sur la base Q. Récemment, dans [ReTa], les auteurs Z.Tao et Q.Ren utilisent un problème de Grushin pour montrer la convergence, dans un cas plus simple, du Laplacien hypoelliptique scalaire sur la cosphère vers le Laplacien de Witten sur la base.

Inspirés par leurs travaux et en vue de notre texte [NSW1], nous avons envisagé d'adapter leurs idées à notre cas. En procédant ainsi, nous obtenons une comparaison quantitative des résolvantes entre le Laplacien hypoelliptique de Bismut $B_{\pm,b,\frac{V}{h}}$ sur l'espace total X et le Laplacien de Witten semi-classique sur la base Q. Comme opérateur différentiel ce dernier est défini sur l'espace total X comme suit :

$$\mathscr{S}(X;\Lambda T^*X\otimes\mathscr{E})\overset{U_\pm^*}{\twoheadrightarrow}\mathscr{S}(Q;\Lambda T^*Q\otimes E)\xrightarrow{\Delta_{V,h}}\mathscr{S}(Q;\Lambda T^*Q\otimes E)\overset{U_\pm}{\rightarrowtail}\mathscr{S}(X;\Lambda T^*X\otimes\mathscr{E}),$$

où :

— l'opérateur $\Delta_{V,h}$ est le Laplacien de Witten semi-classique sur la base Q,

— l'application $U_\pm$ envoie les formes en q définis sur la base Q dans le noyaux de $\alpha_\pm$. Dans le cas +, l'application est donnée par

$$\omega\mapsto e^{-\mathscr{H}}\omega,$$

pour le cas −, la formule un peu moins explicite vient de dualité de Poincaré,

— l'adjoint L^2 de $U_\pm$, $U_\pm^*$, s'interprète comme une intégrale à poids gaussien le long des fibres de T^*Q.

La résolvante mentionnée ici est définie de manière similaire. La comparaison quantitative nous permet d'effectuer une intégrale de contour, donnant une estimation de la différences entre les projecteurs spectraux définis par les deux opérateurs. Le Laplacien hypoelliptique de Bismut vérifie la PT-symétrie pour l'involution $r:(q,p)\mapsto(q,-p)$ et son action unitaire r^* sur les formes, c'est à dire

$$r^*B_{\pm,b,\frac{V}{h}}(r^*)^{-1}=B^*_{\pm,b,\frac{V}{h}}$$

une fois les domaines précisés. En combinant l'estimation quantitative et la PT-symétrie nous obtenons des informations sur les valeurs propres du Laplacien hypoelliptique de Bismut à partir du Laplacien de Hodge. On définit ϱ_h comme un trou spectral pour le Laplacien de Witten semi-classique, au sens suivant :

— Dans [CFKS], pour un potentiel de Morse V, les auteurs distinguent les valeurs propres du Laplacien de Witten semi-classique inférieures à $\varrho_h=h^{3/2}$ des autres.

— Dans [HeSj4], pour un potentiel de Morse V, les valeurs propres inférieures à $\varrho_h=ch$, $c<<1$, sont exponentiellement petites.

— Dans [LNV2], pour les potentiels avec un nombre fini de valeurs critiques, il existe une constante $C>0$ telle que pour tout $\varepsilon<C$ positif alors :

$$\forall\lambda\in\mathrm{Spec}(\Delta_{V,h}),\quad\left(\lambda\le\varrho_h=e^{-\frac{\varepsilon}{h}}\right)\Rightarrow\left(\lambda\le e^{-\frac{C}{h}}\right).$$

Théorème 1.7.2. *Soit g^{TQ} une métrique sur Q et V une fonction lisse avec un nombre fini de valeurs critiques. Dans les cas suivants, il existe une constante $C_0\ge 1$ assez grande de telle sorte que :*

a) dès que $bC_0\le h\varrho_h\le h$ alors toutes les valeurs propres de $B_{\pm,b,\frac{V}{h}}$ ayant une partie réelle inférieure à $\frac{\varrho_h}{h^2}$ sont des réels positifs, c'est-à-dire :

$$\mathrm{Spec}(B_{\pm,b,\frac{V}{h}})\cap\{z\in\mathbb{C},\mathrm{Re}\,z\le\frac{\varrho_h}{h^2}\}=\mathrm{Spec}(B_{\pm,b,\frac{V}{h}})\cap[0,\frac{\varrho_h}{h^2}],$$

où $D(0,\frac{\varrho_h}{h^2})$ désigne le disque centré en 0 de rayon $\frac{\varrho_h}{h^2}$ dans le plan complexe et $[0,\frac{\varrho_h}{h^2}]$ désigne le segment joignant l'origine 0 au point $(\frac{\varrho_h}{h^2},0)$.
De plus leur nombre, compté avec multiplicité, est lié au nombre de valeurs propres du Laplacien de Witten semi-classique sur la base, et dépend également du signe ±, qui traduit la dualité de Poincaré.

b) Sous une hypothèse plus forte $bA^4C_0 \leq h\varrho_h \leq h$ avec $A \geq C_0$, la comparaison entre le Laplacien de Witten semi-classique et le Laplacien hypoelliptique de Bismut sur les petites valeurs propres est donnée par :

$$\left(h^2 \frac{\lambda^{(p)}_{j,\pm,h}}{\tilde{\lambda}^{(p-\frac{d}{2}\pm\frac{d}{2})}_{\pm,j,h}(V)} \right)^{\pm 1} \leq 1 + C_0 A^{-1/2}$$

où

- *$\lambda^{(p)}_{j,\pm,h}$ est le j−ème valeurs propres du Laplacien hypoelliptique de Bismut $B_{\pm,b,\frac{V}{h}}$ sur les formes de degré p,*
- *$\tilde{\lambda}^{(p-\frac{d}{2}\pm\frac{d}{2})}_{\pm,j,h}(V)$ est le j-ème valeurs propres du Laplacien de Witten semi-classique sur les formes de degré $p - \frac{d}{2} \pm \frac{d}{2}$.*

Pour une illustration voir Figure 1.3.

c) Quand $bC_0 \leq h\varrho_h$, le semi-groupe $(e^{-tB_{\pm,b,\frac{V}{h}}})_{t>0}$ satisfait :

$$e^{-tB_{\pm,b,\frac{V}{h}}} = \sum_{p\in\{0,\dots,2d\}} \sum_j e^{-t\lambda^{(p)}_{\pm,j,h}} \left| u^{(p)}_{\pm,j,h} \right\rangle \left\langle v^{(p)}_{\pm,j,h} \right| + R_h(t),$$

où :

- *La norme du terme de reste $R_h(t)$ est contrôlée par :*

$$\|R_h(t)\|_{\mathscr{L}(\, L^2(X;\Lambda T^*X\otimes\mathscr{E})\, ;\, L^2(X;\Lambda T^*X\otimes\mathscr{E})\,)} \leq \frac{1}{b^2}(h^2 + \frac{1}{t})e^{-t\frac{\varrho_h}{h^2}}.$$

- *Les familles de fonctions propres normalisées $(u^{(p)}_{\pm,j,h})$ associées à la famille de valeurs propres $(\lambda^{(p)}_{\pm,j,h})$ et la famille $(v^{(p)}_{\pm,j,h})$ définie par dualité L^2 par rapport à la famille précédente, satisfont :*

$$\max\left(\|u^{(p)}_{\pm,j,h}\|_{L^2(X;\Lambda T^*X\otimes\mathscr{E})}\,,\, \|v^{(p)}_{\pm,j,h}\|_{L^2(X;\Lambda T^*X\otimes\mathscr{E})}\right) \leq C_0.$$

Il s'agit dans un premier temps de pouvoir utiliser les estimations sous-elliptiques du premier texte (cf Théorème 1.7.1, Chapitre 2) de façon uniforme par rapport à $h \in]0, h_0]$. Cela se fait par un argument de dilatation riemannienne. Pour simplifier, nous présentons maintenant la stratégie du deuxième texte (Chapitre 3) pour démontrer le Théorème 1.7.2 ci-dessus dans le cas $h = 1$. Inspirés par [ReTa], nous développons une approche de type problème de Grushin pour le Laplacien de Bismut (après conjugaison), $B_{\pm,b,V}$. Suivant la stratégie de Q.Ren et Z.Tao, nous introduisons une version modifiée, $B_{\pm,b,V} + Q_{A,L,V}$, de cet opérateur avec une inversibilité renforcée, où

$$Q_{A,L,V} = A^2 U_\pm \chi(\frac{C_d + \Delta_{V,1}}{(LA)^2}) U_\pm^{-1},$$

et χ est une fonction plateau satisfaisant :

$$\mathrm{supp}(\chi) \subset [-2, 2] \quad \text{et} \quad \chi(t) = 1 \quad \text{pour } t \in [-1, 1].$$

En fait d'inversibilité, l'opérateur $B_{\pm,b,V} + Q_{A,L,V} - z$ devient inversible pour $z \in B(0, \frac{A^2}{2})$ grâce à une nouvelle estimation sous-elliptique sans reste pour $B_{\pm,b,V} + Q_{A,L,V} - z$ déduite en partie du Théorème 1.7.1. L'opérateur matriciel

$$\begin{pmatrix} B_{\pm,b,\frac{V}{h}} + Q_{A,L,V,h} - z & R_- \\ R_+ & 0 \end{pmatrix}$$

avec $R_- = U_\pm$ et $R_+ = U_\pm^* \pi_{\pm,0}$ est alors inversible d'inverse

$$\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

Le complément de Schur donne

$$(B_{\pm,b,V} + Q_{A,L,V} - z)^{-1} = E - E_+ E_{-+}^{-1} E_-$$

et il s'agit de faire un raisonnement perturbatif pour en déduire des estimations pour $(B_{\pm,b,V})$. C'est là que le point de vue problème de Grushin fait gagner en efficacité par rapport à une application directe du complément de Schur comme elle est faite dans [BiLe]-Chap. XVII, puisqu'il s'agit finalement comme indiqué dans [SjZw] de faire une étude perturbative pour les matrices blocs inversibles et étudier

$$\begin{pmatrix} B_{\pm,b,V} - z & R_- \\ R_+ & 0 \end{pmatrix}^{-1} - \begin{pmatrix} B_{\pm,b,V} + Q_{A,L,V} - z & R_- \\ R_+ & 0 \end{pmatrix}^{-1}.$$

L'utilisation de la formule de Bismut (1.4.2.1) et la formule du complément de Schur

$$(B_{\pm,b,\frac{V}{h}} - z)^{-1} = \tilde{E} - \tilde{E}_+ \tilde{E}_{-+}^{-1} \tilde{E}_- \quad \text{avec} \quad \begin{pmatrix} B_{\pm,b,V} - z & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{E} & \tilde{E}_- \\ \tilde{E}_+ & \tilde{E}_{-+} \end{pmatrix}$$

conduit alors à

$$\|(B_{\pm,b,V} - z)^{-1} - U(\frac{1}{2}\Delta_{V,1} - z)^{-1} U^{-1}\|_{\mathscr{L}(\tilde{W}^s;\tilde{W}^s)} \leq \left[(1 + \frac{A^2}{\mathrm{dist}(z, \frac{1}{2}\Delta_{V,1})})^2 b A^{-1/2} + A^{-2}\right] \frac{C_s}{1 + b\sqrt{|\mathrm{Im}\, z|}}$$

sous les conditions

$$C_s \max(Ab, b, A^{-1}) \leq 1 \quad , \quad |\mathbb{R}e(z)| \leq \frac{A^2}{2} \quad \text{et} \quad \mathrm{dist}(z, \mathrm{Spec}(\frac{1}{2}\Delta_{V,1})) \geq C_s b A^4\ .$$

Nous renvoyons en particulier à la Proposition 3.4.4.

Pour résumer, une discussion détaillée du choix de A, cette fois avec le jeu des deux paramètres $b > 0$ et $h > 0$, donne alors, par intégration de contour des résolvantes, la comparaison des projecteurs spectraux :

$$\|\pi_{\Gamma, B_{\pm,b,V}} - U \pi_{\Gamma, \frac{1}{2}\Delta_V} U^{-1}\|_{\mathscr{L}(L^2;L^2)} < 1,$$

où Γ désigne un contour du plan complexe bien choisi (voir Section 3.5.1 du Chapitre 3 et en particulier la figure 3.5.1), $\pi_{\Gamma, B_{\pm,b,V}}$ est le projecteur spectral associé à l'opérateur $B_{\pm,b,V}$ pour ce contour Γ et $\pi_{\Gamma,\frac{1}{2}\Delta_V}$ est le projecteur spectral associé à l'opérateur $\frac{1}{2}\Delta_V$, pour ce même contour. A partir de là, on utilise de façon combinée, la propriété de PT-symétrie et la structure de Hodge de $B_{\pm,b,V}\big|_{\mathrm{Ran}(\pi_{\Gamma,B_{\pm,b,V}})}$ pour obtenir la comparaison précise de toutes les valeurs propres prises individuellement.

1.8 Compléments

1.8.1 Les différents espaces de Sobolev

Lebeau, pour le traitement à l'infini, construit ses espaces de Sobolev à l'aide d'une compactification projective fibre par fibre. La différence entre les espaces introduits par Lebeau dans [Leb1] [Leb2] et les espaces introduits dans la section 1.5.2 réside en un choix différent pour les opérateurs d'ordre 1 pour définir ces espaces.

Lebeau considère les opérateurs $\langle p\rangle\partial_p$, $\mathscr{H}$ et ∂_q comme des opérateurs d'ordre 1. Ces considérations conduisent à la définition suivante :

$$\forall s \in \mathbb{N}, \quad \big(u \in \mathscr{W}^s\big) \Longleftrightarrow \Big(\sum_{\substack{I,J\in\mathbb{N}^d, k\in\mathbb{N}\\ |I|+|J|+k\leq s}} \|\mathscr{H}^k(\langle p\rangle\partial_p)^J\partial_q^I u\|_{L^2} < \infty\Big).$$

La symétrie de l'oscillateur harmonique verticale $\mathscr{O}$, symétrique en p et ∂_p, n'est pas reflétée dans le choix des opérateurs fait par Lebeau.
Dans les travaux de Nier et Shen (voir [NiSh]), ils choisissent de traiter l'infini à l'aide de l'oscillateur harmonique :

$$\forall s \in \mathbb{N}, \quad \big(u \in \tilde{\mathscr{W}}^s\big) \Longleftrightarrow \Big(\sum_{\substack{I,J\in\mathbb{N}^d, \mathbb{N}\\ |I|+\frac{|J|+k}{2}\leq s}} \|\mathscr{H}^{\frac{k}{2}}\partial_p^J\partial_q^I u\|_{L^2} < \infty\Big).$$

Dans la même lignée, dans [NSW1], nous avons pu associer à ces espaces l'opérateur W_θ^2 , voir Proposition 2.3.5 , auto-adjoint qui les rend plus simples à manipuler.

Pour l'échelle $\mathscr{W}$, la multiplication par $\mathscr{H}$, la dérivation ∂_q et la multiplication par $\langle p\rangle_q\partial_p$ sont d'ordre 1. Pour l'échelle $\tilde{\mathscr{W}}$, on demande en plus que ∂_p^2 soit d'ordre 1, d'où l'inclusion :

$$\forall s \in \mathbb{N}, \quad \tilde{\mathscr{W}}^s \subset \mathscr{W}^s.$$

Dans l'autre sens, on a :

$$(\frac{1}{i}\partial_p)^2 \leq (\langle p\rangle\partial_p)^*(\langle p\rangle\partial_p),$$

d'où l'on déduit que :

$$\forall s \in \mathbb{N}, \quad \mathscr{W}^{2s} \subset \tilde{\mathscr{W}}^s.$$

On en conclut que :

$$\forall s \in 2\mathbb{N}, \quad \tilde{\mathscr{W}}^s \subset \mathscr{W}^s \subset \tilde{\mathscr{W}}^{\frac{s}{2}},$$

et ainsi l'espace de Schwartz des fonctions lisses à décroissance rapide par rapport à p égale

$$\mathscr{S} = \bigcap_{s\in\mathbb{N}} \tilde{\mathscr{W}}^s = \bigcap_{s\in\mathbb{N}} \mathscr{W}^s,$$

et par dualité, l'espace des distributions tempérées (la tempérance valant pour la variable p) est

$$\mathscr{S}' = \bigcup_{s\in\mathbb{N}} \tilde{\mathscr{W}}^{-s} = \bigcup_{s\in\mathbb{N}} \mathscr{W}^{-s}.$$

1.8.2 Une histoire de poids

La différence entre la dérivée de Lie et la dérivée covariante est un tenseur, opérateur d'ordre 0, donné par :

$$\mathscr{L}_{\mathscr{Y}} - \nabla_{\mathscr{Y}} = p_i M_1^i + M_0,$$

où M_1^i et M_0, pour tout i, sont des opérateurs bornés sur $L^2(X;\Lambda T^*X\otimes\mathscr{E})$, mais la différence est un opérateur de multiplication non borné sur cet espace. Il est nécessaire d'introduire un poids w^2, avec $w = \langle p\rangle^{\frac{N_V-N_H}{2}}$ ou $w = \langle p\rangle^{N_V}$, pour que la différence soit un opérateur borné sur $L^2_w(X;\Lambda T^*X\otimes\mathscr{E})$. Cela permet donc de remplacer la dérivée de Lie $\mathscr{L}_{\mathscr{Y}}$, qui ne préserve pas la décomposition horizontale/verticale, par la dérivée covariante $\nabla_{\mathscr{Y}}$, qui préserve cette décomposition.
Le Laplacien hypoelliptique de Bismut, en tant qu'opérateur non borné, est défini sur l'espace

$L^2_w(X;\Lambda T^*X\otimes\mathscr{E})$. Après conjugaison par w pour se ramener à l'espace $L^2(X;\Lambda T^*X\otimes\mathscr{E})$ sans poids, elle s'écrit sous la forme :

$$\tilde{P}_b+\frac{1}{b}\langle p\rangle^2 R,$$

où $\tilde{P}_b$ désigne un opérateur de Kramers-Fokker-Planck Géométrique et R un opérateur de multiplication borné.

Dans le régime de grande friction $b\to 0$, l'inégalité sous elliptique (1.7.1.1) nous dit :

$$\forall u\in D(\tilde{P}_b),\quad \|\frac{1}{b}\langle p\rangle^2 Ru\|_{L^2(X;\Lambda T^*X\otimes\mathscr{E})}\leq C_1 b\|\frac{1}{b^2}\mathscr{O}u\|_{L^2(X;\Lambda T^*X\otimes\mathscr{E})}\leq C_2 b\|\tilde{P}_b u\|_{L^2(X;\Lambda T^*X\otimes\mathscr{E})} \tag{1.8.2.1}$$

donc le Laplacien hypoelliptique est une perturbation relativement bornée d'un opérateur de Kramers-Fokker-Planck Géométrique, avec une borne aussi petite qu'on le souhaite, pourvu que b soit assez petit.

Conclusion : Dans la limite $b\to 0$ le choix d'un poids est sans effet sur l'analyse. En revanche pour la limite $b\to+\infty$, on doit faire face à deux difficultés. D'une part, le modèle dit de Kramers-Fokker-Planck géométrique est plus compliqué et demande une compréhension de la partie dominante de l'opérateur, à savoir le flot géodésique qui domine le comportement diffusif. D'autre part, le bon choix du poids w^2 est important pour assurer la comparaison du Laplacien hypoelliptique de Bismut avec le modèle de l'opérateur de Kramers-Fokker-Planck géométrique.

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Chapter 2

Global subelliptic estimates for geometric Kramers-Fokker-Planck operators on closed manifolds

Joint work with Francis Nier [1] and Francis Gilbert White [2].
Article en anglais.

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Abstract

In this article we reconsider the proof of subelliptic estimates for Geometric Kramers-Fokker-Planck operators, a class which includes Bismut's hypoelliptic Laplacian, when the base manifold is closed (no boundary). The method is significantly different from the ones proposed by Bismut-Lebeau in [BiLe] and Lebeau in [Leb1] and [Leb2]. As a new result we are able to prove maximal subelliptic estimates with some control of the constants in the two asymptotic regimes of high ($b \to 0$) and low ($b \to +\infty$) friction. After a dyadic partition in the momentum variable, the analysis is essentially local in the position variable, contrary to the microlocal reduction techniques of the previous works. In particular this method will be easier to adapt on manifolds with boundaries. A byproduct of our analysis is the introduction of a very convenient double exponent Sobolev scale associated with globally defined differential operators. Applications of this convenient parameter dependent functional analysis to accurate spectral problems, in particular for Bismut's hypoelliptic Laplacian with all its specific geometry, is deferred to subsequent works.

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2.1 Introduction

2.1.1 Background and Motivations

In [Bis041][Bis042][Bis05] J.M. Bismut introduced the hypoelliptic Laplacian which can be viewed alternatively as a deformation of Hodge theory on the cotangent bundle or as a generalization to the case of arbitrary degree differential forms of the Fokker-Planck equation (often specified as Kramers-Fokker-Planck equation in this case) associated with the Langevin process. Very rapidly J.M. Bismut and G. Lebeau made in [BiLe] a careful analysis of families of such hypoelliptic Laplacians indexed by a parameteer $b > 0$, proving in particular the spectral convergence as $b \to 0^+$ of the hypoelliptic Laplacian to the Hodge Laplacian on the base manifold. It was in the mood of those times to develop the accurate spectral analysis of parameter dependent non self-adjoint hypoelliptic operators and we refer the reader to [Dav][DSZ][HerNi][HHS] for works in this direction which still have many developments. In [BiLe] J.M. Bismut and G. Lebeau combined such involved microlocal and semiclassical spectral analysis with the heavy geometrical construction of the hypoelliptic Laplacian and this was continued by S. Shen in [She], for the double parameter asymptotics where the first limit $b \to 0^+$ allows to recover a semiclassical Witten Laplacian on the base manifold let say with a parameter $h \to 0^+$ which leads to Morse theory as in [Wit][HeSj4][Zha]. In [Leb1][Leb2], G. Lebeau introduced a general class of non self-adjoint hypoelliptic operators for which accurate subelliptic estimates, a corner stone of the spectral asymptotic analysis, can be proven.
About the Witten Laplacian on the base manifold, it was realized in [LNV1][LNV2] that the introduction of artificial boundary value problems, associated with a suitable cutting and gluing of the manifold, was a very convenient and robust way for the generalization of the accurate spectral asymptotic analysis with a potential which can be more general than a Morse function. Such accurate spectral analysis was motivated by various questions related with molecular dynamics. This raised the question of understanding similar problems for the Langevin process, where the arbitrary degree form formulation, involves boundary value problems for the hypoelliptic Laplacian. The functional analysis was started in [Nie][NiSh] for fixed hypoelliptic Laplacians, which

means with no parameter. The asymptotic spectral analysis with respect to one parameter $b>0$ or two parameters $(b,h)\in(0,+\infty)^2$ remains to be done.
In this direction, the microlocal reduction approach proposed by in [BiLe][Leb1][Leb2] appears to be not well adapted for a similar asymptotic analysis of boundary value problems. The present article proposes an alternative approach which relies on some local approximation of the hypoelliptic Laplacian on the cotangent T^*Q of a general compact Riemannian manifold Q, by the euclidean version. This euclidean approximation relies on the use of normal coordinates around a point $q_0\in Q$ while the Taylor expansion of the metric leads to controlled error terms in balls $B(q_0,r_{b,|p|})$ parametrized by the parameter $b>0$ and the size of momentum variable $p\in T^*_{q_0}Q$.
While doing so we are able to provide a control of constants in the global subelliptic estimates not only as $b\to 0^+$ but also as $b\to+\infty$ which can be of interest for further developments. The spectral analysis as $b\to 0^+$ proving the spectral convergence to the Hodge or Witten Laplacian on the base manifold, will be carried out in another text. Compared to the works of [BiLe][She], the approach proposed here makes possible an easier extension to boundary value problems and a more direct connexion with standard tools of spectral analysis like Grushin problems (see[SjZw]) and other related works with similar spectral problems studied in [ReTa][Dro][Smi]. In particular, global Sobolev scales $\tilde{\mathcal{W}}^{s_1,s_2}(T^*Q)$, $s_1,s_2\in\mathbb{R}$, are associated with a pair of commuting self-adjoint operators which make use of the scalar horizontal Laplacian associated with a Sasaki type metric. This synthetic formulation, of which the properties nevertheless rely on some global pseudodifferential techniques, was inspired by [ReTa] and [BeBo]. Again, it will allow a rather direct adaptation of techniques developed for simpler spectral problems, namely scalar operators on a compact total space within the framework of standard Sobolev spaces, to the more involved framework of the hypoelliptic Laplacian (and possibly other global analysis problems on the total space of the cotangent bundle $X=T^*Q$).

2.1.2 General Framework

In this text we shall consider geometric Kramers-Fokker-Planck operators on $X=T^*Q$, where (Q,g) is a smooth compact d-dimensional Riemannian manifold without boundary. Points in X will be denoted by x, and $\pi_X:X\to Q$ will denote the natural projection $T^*_qQ\ni x\mapsto q\in Q$. Local coordinates on Q will be denoted by $(q^1,\dots,q^d)$. We shall use Einstein's convention of summing over repeated up and down indices. If $q\in Q$ and $(U,q^1,\dots q^d)$ is a local coordinate system for Q, then an element of the fiber $p\in T^*_qQ$ will be written $p=p_j\,dq^j$, and $(q^1,\dots,q^d,p_1,\dots,p_d)$ will denote canonical coordinates in $U\times\mathbb{R}^d\sim T^*U\subset T^*Q$. If local canonical coordinates $(q^1,\dots,q^d,p_1,\dots,p_d)$ have been fixed in a neighborhood T^*U of a point $x\in X$, then we shall write $x=(q,p)$.

The metric g on Q, i.e. on the tangent bundle $\pi_{TQ}:TQ\to Q$, will be denoted by $g(q)={}^tg(q)=\left(g_{jk}(q)\right)_{1\le j,k\le d}$ or $g=g_{jk}(q)dq^j\,dq^k$. The corresponding dual metric on the cotangent bundle $\pi_{T^*Q}:T^*Q\to Q$ will be denoted by $g^{-1}(q)=(g^{jk}(q))_{1\le j,k\le d}$. Let ∇^{LC} be the Levi-Civita connection on the tangent bundle $\pi_{TQ}:TQ\to Q$ associated with the metric g. By abuse of notation, we shall also denote by ∇^{LC} the connection on the tensor bundle

$$\mathcal{T}^{(k,\ell)}TQ=\underbrace{TQ\otimes\cdots\otimes TQ}_{k\ \text{times}}\otimes\underbrace{T^*Q\otimes\cdots\otimes T^*Q}_{\ell\ \text{times}} \tag{2.1.2.1}$$

induced by the Levi-Civita connection on TQ for every $k,\ell\in\mathbb{N}$. If $q=(q^1,\dots,q^d)$ are local coordinates for Q, then the Christoffel symbols associated the Levi-Civita connection ∇^{LC} are given by

$$\Gamma^\ell_{jk}=\frac{1}{2}g^{\ell a}\left(\partial_{q^j}g_{ak}+\partial_{q^k}g_{aj}-\partial_{q^a}g_{jk}\right),\ \ 1\le j,k,\ell\le d. \tag{2.1.2.2}$$

The connection ∇^{LC} gives rise to a global decomposition

$$TX = T^H X \oplus T^V X\,, \tag{2.1.2.3}$$

where $T^V X = \ker(d\pi_X)$ is the vertical subbundle of TX associated with the projection π_X and $T^H X$ is the horizontal subbundle of TX defined at each point $x \in X$ by

$$\begin{aligned} T^H_x X = \{\gamma'(0) : &\text{ there exists } \epsilon > 0 \text{ and a smooth path } \gamma : (-\epsilon,\epsilon) \to X \text{ such that} \\ &\gamma(0) = x \text{ and } \nabla^{LC}_{d\pi_X(\gamma'(t))}\gamma(t) = 0 \text{ for all } -\epsilon < t < \epsilon\}\,. \end{aligned} \tag{2.1.2.4}$$

In terms of local coordinates for X, the subbundles $T^H X$ and $T^V X$ may be described as follows. If $(q^1,\ldots,q^d,p_1,\ldots,p_d)$ are local canonical coordinates for X, let

$$e_j = \frac{\partial}{\partial q^j} + \Gamma^\ell_{jk}(q) p_\ell \frac{\partial}{\partial p_k}, \quad 1 \le j \le d\,, \tag{2.1.2.5}$$

and let

$$\hat{e}^j = \frac{\partial}{\partial p_j}, \quad 1 \le j \le d\,. \tag{2.1.2.6}$$

Together these vector fields form a local frame $(e_1,\ldots,e_d,\hat{e}^1,\ldots,\hat{e}^d)$ for TX and locally we have

$$T^H X = \mathrm{span}(e_1,\ldots,e_d) \tag{2.1.2.7}$$

and

$$T^V X = \mathrm{span}\left(\hat{e}^1,\ldots,\hat{e}^d\right). \tag{2.1.2.8}$$

Since for every $x = (q,p) \in X$ the differential $d\pi_X|_x$ restricts to a linear isomorphism $T^H_x X \to T_q Q$ while $T^V_x X \cong T^*_q Q$, the decomposition (2.1.2.3) yields the identifications

$$TX \cong \pi_X^*(TQ \oplus T^*Q)\,. \tag{2.1.2.9}$$

The total tangent space $TX = T^H X \oplus T^V X$ is equiped with the metric $\pi_X^*(g \overset{\perp}{\oplus} g^{-1})$, simply written $g \oplus g^{-1}$, by using the above identification. Clearly horizontal (resp. vertical) vector fields on X are given as sections in $\mathscr{C}^\infty(X;T^H X)$ (resp. $\mathscr{C}^\infty(X;T^V X)$). Specific subspaces of horizontal (resp. vertical) sections are provided by the fact that (2.1.2.9) induces a natural imbedding

$$i_g : \mathscr{C}^\infty(Q;TQ \oplus T^*Q) \to \mathscr{C}^\infty(X;\pi_X^*(TQ \oplus T^*Q)) = \mathscr{C}^\infty(X;TX)$$

and we introduce

$$\begin{aligned} &\mathscr{C}^\infty_Q(X;T^H X) = i_g(\mathscr{C}^\infty(Q;TQ)) \subset \mathscr{C}^\infty(X;T^H X)\,, \\ \text{resp.}\quad &\mathscr{C}^\infty_Q(X;T^V X) = i_g(\mathscr{C}^\infty(Q;T^*Q)) \subset \mathscr{C}^\infty(X;T^V X)\,. \end{aligned}$$

These spaces $\mathscr{C}^\infty_Q(X;T^H X)$ and $\mathscr{C}^\infty_Q(X;T^V Q)$ are $\mathscr{C}^\infty(Q;\mathbb{R})$ modules. Additionally on $\mathscr{C}^\infty(Q;TQ \oplus T^*Q)$ a $\mathscr{C}^k$-norm can be fixed once and for all by using a finite partition of unity subordinate to an open chart covering $Q = \cup_{j=1}^J \Omega_j$ while changing the atlas and the partition of unity gives an equivalent norm. We therefore can speak of $\|T\|_{\mathscr{C}^k}$ for $T \in \mathscr{C}^\infty_Q(X;T^H X)$ and $T \in \mathscr{C}^\infty_Q(X;T^V X)$ without specifying its expression.

It will be convenient to use the following families of vector fields.

Definition 2.1.1. For any $N \in \mathbb{N}$ and any $k \in \mathbb{N}$, the set $\mathscr{T}^H_{N,k}$ (resp. $\mathscr{T}^V_{N,k}$) is defined by

$$\mathscr{T}^H_{N,k} = \left\{(T^H_1,\dots,T^H_N) \in \mathscr{C}^\infty_Q(X;T^H X)^N\,,\, \forall j \in \{1,\dots,N\}\,,\|T^H_j\|_{\mathscr{C}^k} \le 1\right\},$$

$$\text{resp.}\qquad \mathscr{T}^V_{N,k} = \left\{(T^V_1,\dots,T^V_N) \in \mathscr{C}^\infty_Q(X;T^V X)^N\,,\, \forall j \in \{1,\dots,N\}\,,\|T^V_j\|_{\mathscr{C}^k} \le 1\right\}.$$

By duality, we also have the identifications

$$T^*X \cong T^*Q \oplus TQ. \tag{2.1.2.10}$$

If $(q^1,\dots,q^d,p_1,\dots,p_d)$ are local canonical coordinates for X, we let

$$e^j = dq^j,\ \ 1 \le j \le d, \tag{2.1.2.11}$$

and

$$\widehat{e}_j = dp_j - \Gamma^\ell_{jk}(q)p_\ell dq^k,\ \ 1 \le j \le d. \tag{2.1.2.12}$$

It is clear that $(e^1,\dots,e^d,\widehat{e}_1,\dots,\widehat{e}_d)$ is a local coframe for T^*X, and locally it is true that

$$(T^H X)^* = \operatorname{span}\left(e^1,\dots,e^d\right) \tag{2.1.2.13}$$

and

$$(T^V X)^* = \operatorname{span}(\widehat{e}_1,\dots,\widehat{e}_d). \tag{2.1.2.14}$$

We also note that X is naturally a symplectic manifold with respect to the usual symplectic form σ given in local canonical coordinates (q,p) by

$$\sigma = \sum_{j=1}^d dp_j \wedge dq^j. \tag{2.1.2.15}$$

Since $\sigma^{\wedge d} \neq 0$, the manifold X is orientable, and we orient X so that every local canonical coordinate system (q,p) is positively oriented. The volume form on X for the metric $g \oplus g^{-1}$ is denoted by $d\mathrm{vol}_X$ and given locally by

$$d\mathrm{vol}_X = dq^1 \wedge \cdots \wedge dq^d \wedge dp_1 \wedge \cdots \wedge dp_d. \tag{2.1.2.16}$$

The volume form $d\mathrm{vol}_X$ is related to σ by

$$d\mathrm{vol}_X = \frac{1}{d!}(-1)^{\frac{d(d+1)}{2}}\sigma^{\wedge d}. \tag{2.1.2.17}$$

In particular, if $H \in C^\infty(X;\mathbb{R})$ and $\mathscr{Y}$ is the Hamilton vector field of H with respect to the symplectic form σ, i.e. $\mathscr{Y}$ is the unique smooth vector field on X such that $\iota_{\mathscr{Y}}\sigma = -dH$, then the flow $\Phi^t = \exp(t\mathscr{Y})$ on X generated by $\mathscr{Y}$ preserves $d\mathrm{vol}_X$. In this text, we will be primarily concerned with the situation in which H is the kinetic energy

$$H(q,p) = \frac{1}{2}|p|^2_q = \frac{1}{2}g^{jk}(q)p_jp_k. \tag{2.1.2.18}$$

In this case, the Hamilton vector field $\mathscr{Y}$ of H is given locally by

$$\mathscr{Y} = g^{jk}(q)p_je_k, \tag{2.1.2.19}$$

where e_k is as in (2.1.2.5), and the projections of the integral curves of $\mathscr{Y}$ to Q by π_X are precisely the smooth geodesic curves of the metric g. We will use also the metric-dependent Japanese bracket

$$\langle p\rangle_q = (1+|p|_q^2)^{1/2} = (1+g^{ij}(q)p_ip_j)^{1/2}\,, \tag{2.1.2.20}$$

while the notation

$$\langle p\rangle = (1+|p|^2)^{1/2} = (1+\delta^{ij}p_ip_j)^{1/2}(1+\sum_{i=1}^{d}p_i^2)^{1/2}\,, \tag{2.1.2.21}$$

will be used for the euclidean version.
Let $E \xrightarrow{\pi_E} Q$ be a smooth complex vector bundle over Q of complex dimension N that is equipped with an affine connection ∇^E and a Hermitian metric g^E. Let $\mathscr{E} := \pi_X^* E \xrightarrow{\pi_{\mathscr{E}}} X$ denote the pullback bundle of $E \xrightarrow{\pi_E} Q$ by the map $\pi_X : X \to Q$. Locally, smooth sections u of $\mathscr{E} \xrightarrow{\pi_{\mathscr{E}}} X$ have the form

$$u(x) = \sum_{\ell=1}^{N} u_\ell(x) f^\ell(q), \quad x=(q,p)\in X\,, \tag{2.1.2.22}$$

where $(f^1,\ldots,f^N)$ is a smooth local frame for $E \xrightarrow{\pi_E} Q$ and $u_1,\ldots,u_N$ are smooth locally defined complex-valued functions on X. We equip $\mathscr{E} \xrightarrow{\pi_{\mathscr{E}}} X$ with the pullback connection $\nabla^{\mathscr{E}}$, which is defined using the decomposition (2.1.2.9) of TX by the relations

$$\begin{aligned}\left(\nabla^{\mathscr{E}}_{e_j}u\right)(x) &= \sum_{\ell=1}^{N}\left[(e_ju_\ell)(x)f^\ell(q) + u_\ell(x)\nabla^E_{\frac{\partial}{\partial q^j}}f^\ell(q)\right],\\ \left(\nabla^{\mathscr{E}}_{\hat{e}^j}u\right)(x) &= \sum_{\ell=1}^{N}(\hat{e}^ju_\ell)(x)f^\ell(q) = \sum_{\ell=1}^{N}(\partial_{p_j}u_\ell)(x)f^\ell(q), \quad x=(q,p)\in X,\ \ 1\le j\le d\,,\end{aligned} \tag{2.1.2.23}$$

whenever $u\in C^\infty(X;\mathscr{E})$ is of the form (2.1.2.22). Because the connection $\nabla^{\mathscr{E}}_{T_V}$ is trivial for $T_V \in TX^V$, the covariant derivative with respect to a vertical vector field will be identified with the associated scalar first order differential operator. Accordingly the vertical Laplacian and the vertical harmonic oscillator, written locally as,

$$\Delta_p = \sum_{1\le j,k\le d}\frac{1}{2}g_{jk}(q)\nabla^{\mathscr{E}}_{\hat{e}^j}\nabla^{\mathscr{E}}_{\hat{e}^k} = \sum_{1\le j,k\le d}\frac{1}{2}g_{jk}(q)\partial_{p_j}\partial_{p_k} \tag{2.1.2.24}$$

$$\mathscr{O} = -\frac{1}{2}\Delta_p + \frac{1}{2}|p|_q^2\,. \tag{2.1.2.25}$$

are globally defined operators, which happen to be scalar differential operators in the sense that in any local frame $(f^1,\ldots,f^N)$ of $E \xrightarrow{\pi_E} Q$,

$$\Delta_p(\sum_{\ell=1}^{N}u_\ell(x)f^\ell(q)) = \sum_{\ell=1}^{N}(\Delta_pu_\ell)(x)f^\ell(q) \quad\text{and}\quad \mathscr{O}(\sum_{\ell=1}^{N}u_\ell(x)f^\ell(q)) = \sum_{\ell=1}^{N}(\mathscr{O}u_\ell)(x)f^\ell(q)\,.$$

We also equip the bundle $\mathscr{E}$ with the pulled back Hermitian metric $g^{\mathscr{E}}$ defined by

$$g^{\mathscr{E}}(u,u') = \sum_{\ell_1,\ell_2}\overline{u_{\ell_1}(x)}u'_{\ell_2}(x)g^E(f^{\ell_1}(q),f^{\ell_2}(q)), \quad x=(q,p)\in X\,, \tag{2.1.2.26}$$

where $u=\sum u_\ell(x)f^\ell(q)$ and $u'=\sum u'_\ell(x)f^\ell(q)$. Using the Hermitian metric $g^{\mathscr{E}}$ on $\mathscr{E}$ and the volume form $d\mathrm{vol}_X$ on X, we may introduce the Hilbert space $L^2(X;\mathscr{E})$ of square integrable sections of $\mathscr{E}$ as follows. The space $L^2(X;\mathscr{E})$ is the set of measurable sections u such that

$$\langle u\,,\,u\rangle_{L^2(X;\mathscr{E})} = \int_X g^{\mathscr{E}}_x(u(x),u(x))\,d\mathrm{vol}_X(x) < +\infty\,, \tag{2.1.2.27}$$

and it is a Hilbert space for the scalar product

$$\langle u_1 , u_2 \rangle_{L^2(X;\mathscr{E})} = \int_X g_x^{\mathscr{E}} \left(u_1(x), u_2(x) \right) d\mathrm{vol}_X(x) , \tag{2.1.2.28}$$

in which $\mathscr{C}_0^\infty(X;\mathscr{E})$ is dense.
By recalling $d\mathrm{vol}_X = dq\,dp$, the operator $\mathscr{O}$ is clearly self-adjoint with its maximal domain $D(\mathscr{O}) = \left\{ u \in L^2(X, \mathscr{E}), \mathscr{O}u \in L^2(X;\mathscr{E}) \right\}$, in which $\mathscr{C}_0^\infty(X;\mathscr{E})$ is dense with the graph norm. It is also bounded from below by $\frac{d}{2}$ and $\sqrt{\mathscr{O}}$ is well defined.
Let us now introduce some Sobolev type spaces, taking into account the different homogeneities of e_i , p_i and ∂_{p_i} .

Definition 2.1.2. For $k \in \mathbb{N}$ and u a sufficiently regular section of $\mathscr{E}$, we define

$$\| u \|_{\tilde{\mathscr{W}}^k} = \sup_{\substack{N_1 + \frac{N_2+N_3}{2} \leq k \\ (T_1^H, \ldots, T_{N_1}^H) \in \mathscr{T}_{N_1,k}^H \\ (T_1^H, \ldots, T_{N_2}^V) \in \mathscr{T}_{N_2,k}^V}} \left\| \langle p \rangle_q^{N_3} \nabla_{T_1^H}^{\mathscr{E}} \ldots \nabla_{T_{N_1}^H}^{\mathscr{E}} \nabla_{T_1^V}^{\mathscr{E}} \ldots \nabla_{T_{N_2}^V}^{\mathscr{E}} u \right\|_{L^2(X;\mathscr{E})} , \tag{2.1.2.29}$$

and we take

$$\tilde{\mathscr{W}}^k(X;\mathscr{E}) = \overline{C_0^\infty(X;\mathscr{E})}^{\|\cdot\|_{\tilde{\mathscr{W}}^k}} . \tag{2.1.2.30}$$

The space $\tilde{\mathscr{W}}^s(X;\mathscr{E})$ is then defined for all $s \geq 0$ by interpolation and for $s < 0$ by setting $\tilde{\mathscr{W}}^s(X;\mathscr{E}) = (\tilde{\mathscr{W}}^{-s}(X;\mathscr{E}))^*$.
Finally the space $\tilde{\mathscr{W}}^{1,s}(X;\mathscr{E})$ is the space

$$\tilde{\mathscr{W}}^{1,s}(X;\mathscr{E}) = \left\{ u \in \mathscr{W}^s(X;\mathscr{E}), \quad \sqrt{\mathscr{O}}u \in \mathscr{W}^s(X;\mathscr{E}) \right\}$$

endowed with the norm $\| \sqrt{\mathscr{O}}u \|_{\tilde{\mathscr{W}}^s}$.

Remark 2.1.3. The supremum norm over the families of vector fields ensure the geometrical global meaning of the functional spaces $\tilde{\mathscr{W}}^k(X;\mathscr{E})$ and therefore of $\tilde{\mathscr{W}}^s(X;\mathscr{E})$ and $\tilde{\mathscr{W}}^{1,s}(X;\mathscr{E})$. It is not the most convenient definition and in particular their Hilbert nature is not obvious here. A more convenient presentation in terms of local coordinates and then the use of a specific pseudo-differential calculus presented in Appendix 2.E is detailed in Section 2.3.
Although those spaces are modelled on Lebeau's spaces in [Leb1][Leb2] they slightly differ, e.g. the case $s = 1$ allows $N_2 = 2$ with two vertical derivatives bounded in L^2 .

We shall define geometric Kramers-Fokker-Planck operators as second order differential operators acting on sections of the pullback bundle $\mathscr{E}$ that depend on a parameter $b \in (0, \infty)$. Our definition will be slightly more general than that of Lebeau [Leb1][Leb2] but in the same spirit.

Definition 2.1.4 (Geometric Kramers-Fokker-Planck Operator)**.** A Geometric Kramers-Fokker-Planck (abbreviated as GKFP) operator is a b-dependent operator $P_{\pm,b} + M(b)$ acting on $C_0^\infty(X;\mathscr{E})$ or $\mathscr{S}(X;\mathscr{E})$ with

$$P_{\pm,b} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\nabla_{\mathscr{Y}}^{\mathscr{E}} ,$$
$$\text{and} \qquad \forall s \in \mathbb{R}, \quad M(b) \in \mathscr{L}(\tilde{\mathscr{W}}^{1,s}(X;\mathscr{E}); \tilde{\mathscr{W}}^s(X;\mathscr{E}))$$

where $\mathscr{Y}$ is Hamilton vector field of the kinetic energy (2.1.2.18) with respect to the symplectic form σ and $\mathscr{O}$ is the vertical harmonic oscillator.

Actually the term $M(b)$ will appear as a perturbative term for which the norm estimates of $\|M(b)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,s};\tilde{\mathcal{W}}^{s})}$ with respect to the parameter $b>0$ can be discussed afterwards. Actually all the analysis focuses on the case $M(b)=0$. The Hörmander Theorem about sum of squares and type II operators (see [Hor67]) provides the local hypoelliptic nature of the geometric Kramers-Fokker-Planck operator $P_{\pm,b}$ for every $b\in(0,\infty)$. By following the method of Lebeau in [Leb1][Leb2] our aim is to provide accurate subelliptic estimates with the best regularity exponents, that will also account for the behavior of $P_{\pm,b}$ as either $b\to 0^{+}$ (the large friction limit) or $b\to\infty$ (the low friction limit).

2.1.3 Statement of the Main Result

Remember the following notion.

Definition 2.1.5. In a Hilbert space $\mathfrak{H}$ and densely defined operator $A:D\mapsto\mathfrak{H}$ is called essentially maximal accretive, if it is accretive, therefore closable, and if it admits a unique maximal accretive extension equal to its closure $\overline{A}:D(\overline{A})\mapsto\mathfrak{H}$ with $D(\overline{A})=\overline{D}^{\|\ \|_A}$, $\|u\|_A^2=\|u\|_{\mathfrak{H}}^2+\|Au\|_{\mathfrak{H}}^2$.

The main result of this paper is the following subelliptic estimate for geometric Kramers-Fokker-Planck operators.

Theorem 2.1.6. *Let $P_{\pm,b}=\frac{1}{b^2}\mathcal{O}\pm\frac{1}{b}\nabla^{\mathcal{E}}_{\mathcal{Y}}$. There exists a constant $C_g\geq 1$ determined by the geometric data (g,E,g^E,∇^E) such that the operator $\frac{\kappa_b}{b^2}+P_{\pm,b}$ is essentially maximal accretive on $\mathcal{C}_0^\infty(X;\mathcal{E})$ (or on $\mathcal{S}(X;\mathcal{E})$), when $\kappa_b\geq C_g(1+b^5)$. If $\overline{P}_{\pm,b}$ denotes its closure, the inequalities*

$$\mathrm{Re}\langle u\,,(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b})u\rangle_{L^2}\geq\frac{1}{4b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_b\|u\|^2_{L^2}\right]. \tag{2.1.3.1}$$

and

$$\left\|\left(\overline{P}_{\pm,b}-\frac{i\lambda}{b}\right)u\right\|_{L^2}+\frac{1}{b^2}\|u\|_{L^2}\geq\frac{1}{C_g(1+b)^7}\Bigg(\left\|\frac{\mathcal{O}}{b^2}u\right\|_{L^2}+\left\|\frac{1}{b}\left(\pm\nabla^{\mathcal{E}}_{\mathcal{Y}}-i\lambda\right)u\right\|_{L^2} + \frac{1}{b^{4/3}}\left[||u||_{\tilde{\mathcal{W}}^{\frac{2}{3}}}+\left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{L^2}\right]\Bigg) \tag{2.1.3.2}$$

hold for every $u\in D(\overline{P}_{\pm,b})$ and every $(\lambda,b)\in\mathbb{R}\times(0,+\infty)$.

The proof of the Theorem 2.1.6 can be found in Section 2.6. Other results involving the realizations of $P_{\pm,b}$ in the Sobolev spaces $\mathcal{W}^s(X;\mathcal{E})$ or other perturbative results will be deduced as corollaries in Section 2.7.

2.1.4 Outline of the article

In Section 2.2 an elementary integration by part provides the first a priori lower bound for $\mathrm{Re}\langle u,P_{\pm,b}u\rangle$. This implies that the analysis of $P_{\pm,b}$ can be localized in the q-variable via partition of unity. Comparison of different connections can be done locally which reduces the problem to purely scalar operators and then the essential maximal accretivity on $\mathcal{C}_0^\infty(X;\mathcal{E})$ or $\mathcal{S}(X;\mathcal{E})$ is proved.
The Sobolev spaces $\tilde{\mathcal{W}}^{s_1,s_2}(X;\mathcal{E})$ are then studied in Section 2.3. After a localization via partition of unity, the Definition 2.1.2 is characterized in term of the suitable pseudodifferential calculus, local in the q-variable but global in the p-variable. The construction of this pseudodifferential calculus relying on standard techniques, nevertheless to be adapted, is detailed in

Appendix 2.E. Section 2.3 ends with a very convenient global characterization of these Sobolev spaces $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E})$ in terms of the functional calculus of two geometrically defined commuting self-adjoint operators, namely $\mathscr{O}$ and $W^2 = C - \Delta_H + C\mathscr{O}^2$ where Δ_H is a scalar horizontal Laplacian.

Section 2.4 is devoted to the localization process. A dyadic partition of unity in the p-variable is used and then once the parameter 2^j of the dyadic partition if fixed, a grid partition in the q-variable with the spacing 2^{-j} is introduced. Near point of the grid, a Taylor expansion of the metric in normal coordinates expresses the scalar GKFP operator as the euclidean one with a $(2^j, b)$-dependent error term.

In Section 2.5 the maximal subelliptic estimate, where the exponent 2/3 is obtained via the model problem of the one dimensional complex Airy operator, is recalled. Actually the uniform estimates with respect to the parameters $(b, 2^j, \lambda) \in (0, +\infty)^2 \times R$ are carefully checked.

Section 2.6 gathers the local comparison of the scalar GKFP operator with the euclidean model of Section 2.4 with the uniform estimates of the euclidean model. Error terms due to the two partition of unities (dyadic in p and 2^j-dependent grid in q) happen to be controlled by the lower bounds of the parameter dependent euclidean model. While doing this, intermediate parameters of the grid partition must be tuned carefully according to the two regimes $2^j >> 1$ or $2^j \leq C$.

Section 2.7 completes Theorem 2.1.6 with various consequences or precisions. In particular the b-dependence of the perturbation $M(b)$ in Definition 2.1.4, which allows the generalization of Theorem 2.1.6 is specified. A corollary is the $\tilde{W}^{0,s}(X;\mathscr{E})$ version of Theorem 2.1.6, where a simple conjugation reduces the perturbed operator in $L^2(X, dqdp;\mathscr{E})$.

The Appendices gathers known material. A rather long paragraph is about the global pseudodifferential calculus on the total space $X = T^*Q$. As already said, it follows the general approach but things have to be specified in particular for proving, via the Helffer-Sjöstrand formula, that functions of self-adjoint globally elliptic operators in this class are pseudodifferential operators with a good asymptotic expansion.

2.2 Reduction to a scalar operator

Here we write first a priori estimates for Geometric Kramers-Fokker-Planck (GKFP) operators coming from a simple integration by parts. The essential maximal accretivity of $\frac{\kappa_b}{b^2} + P_{\pm,b}$ is checked and all the perturbative terms coming from a partition of unity in the q-variable will be shown to be of lower order with a uniform control of the constants w.r.t b. Similarly a local change of connection happens to be of lower order and this reduces the problem to local scalar GKFP operators.

2.2.1 Integration by parts and maximal accretivity

Proposition 2.2.1. *Let $P_{\pm,b} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\nabla^{\mathscr{E}}_{\mathscr{Y}}$. There exists $C_0 \geq 1$, determined by the geometric data (g, E, ∇^E, g^E), such that for all $b > 0$, $\lambda \in \mathbb{R}$ and for $\kappa_b \geq C_0(1+b^2)$ the inequality*

$$\mathrm{Re}\langle u\,, (\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u\rangle_{L^2(X;\mathscr{E})} \geq \frac{1}{4b^2}\left[\|u\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})} + \kappa_b\|u\|^2_{L^2(X;\mathscr{E})}\right] \tag{2.2.1.1}$$

$$\text{and} \quad \left\|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u\right\|^2_{L^2(X;\mathscr{E})} \geq \frac{\kappa_b}{16b^4}\left[\|u\|^2_{\mathscr{W}^{1,0}(X;\mathscr{E})} + \kappa_b\|u\|^2_{L^2(X;\mathscr{E})}\right] \tag{2.2.1.2}$$

holds for all $u \in \mathscr{C}^\infty_0(X;\mathscr{E})$.

Before proving this result let us specify the formal adjoint of $\nabla^{\mathscr{E}}_{\mathscr{Y}}$. Start with the vector bundle $\pi_E : E \to Q$ and the data (∇^E, g^E) and consider the dual connection with respect to g^E given by

$$Xg^E(s, s') = g^E(s, \nabla^E_X s') + g^E(\nabla^{E,*}_X s, s').$$

The unitary connection

$$\nabla^{E,u} = \frac{\nabla^E + \nabla^{E,*}}{2}$$

differs from ∇^E by

$$\nabla^{E,u} - \nabla^E = \frac{1}{2}\omega(\nabla^E, g^E) \quad \in \mathscr{C}^\infty(Q; T^*Q \otimes \mathrm{End}(E))$$

With the pull back we obtain with $\mathscr{Y} = g^{ij}(q)p_i e_j$ written in a local canonical coordinates system

$$\nabla^{\mathscr{E},u}_{\mathscr{Y}} - \nabla^{\mathscr{E}}_{\mathscr{Y}} = \frac{1}{2}g^{ij}(q)p_i\omega(\nabla^E, g^E)(\frac{\partial}{\partial q^j}) = a^i(q)p_i \quad , \quad a^i(q) \in \mathrm{End}(E_q)$$

We also recall the formula

$$\forall v, w \in \mathscr{C}^\infty_0(X;\mathscr{E}), \forall T \in \mathscr{C}^\infty(X;TX), \int_X g^{\mathscr{E}}(v, \nabla^{\mathscr{E},u}_T w)\, d\mathrm{vol}_X = -\int_X g^{\mathscr{E}}(\nabla^{\mathscr{E},u}_T v, w) + \mathrm{div}(T) g^{\mathscr{E}}(v,w)\, d\mathrm{vol}_X$$

while here $d\mathrm{vol}_X = dqdp$ and $\mathrm{div}\mathscr{Y} = 0$.
We find that the formal adjoint of $\nabla^{\mathscr{E}}_{\mathscr{Y}}$ is nothing but

$$(\nabla^{\mathscr{E}}_{\mathscr{Y}})^* = -\nabla^{\mathscr{E}}_{\mathscr{Y}} - g^{ij}(q)p_i\omega(\nabla^E, g^E)(\frac{\partial}{\partial q^j}) = -\nabla^{\mathscr{E}}_{\mathscr{Y}} - a^i(q)p_i\,. \tag{2.2.1.3}$$

Proof of Proposition 2.2.1. Let $\cup_{j=1}^J \Omega_j = Q$ be a finite open chart covering of Q and let $\sum_{j=1}^J \varrho_j(q)^2 \equiv 1$ be a subordinate quadratic partition of unity, $\varrho_j \in \mathscr{C}^\infty_0(\Omega_j;[0,1])$. Because $\nabla^{\mathscr{E}}_{\mathscr{Y}}$, $P_{\pm,b}$, $\mathscr{O}$, are at most first order differential operators in q we get

$$\langle u, (\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u\rangle_{L^2(X;\mathscr{E})} = \sum_{j=1}^J \langle u_j, (\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u_j\rangle_{L^2(X;\mathscr{E})}$$

$$\underbrace{\|\sqrt{\mathscr{O}}u\|^2_{L^2(X;\mathscr{E})}}_{=\|u\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})}} + \kappa_b\|u\|^2_{L^2(X;\mathscr{E})} = \langle u, \mathscr{O}u\rangle_{\mathscr{L}^2(X;\mathscr{E})} + \kappa_b\|u\|^2_{L^2(X,\mathscr{E})} = \sum_{j=1}^J \underbrace{\|\sqrt{\mathscr{O}}u_j\|^2_{L^2(X;\mathscr{E})}}_{=\|u_j\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})}} + \kappa_b\|u_j\|^2_{L^2(X;\mathscr{E})}$$

for all $u \in \mathscr{C}^\infty_0(X;\mathscr{E})$, by setting $u_j = \varrho_j u \in \mathscr{C}^\infty_0(T^*\Omega_j;\mathscr{E})$.
With canonical local coordinates (q,p) in $T^*\Omega_j$, (2.2.1.3) implies

$$\begin{aligned}
\mathrm{Re}\langle u_j, (\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u_j\rangle_{L^2(X;\mathscr{E})} &= \langle u_j, \frac{2\kappa_b - \Delta_p + |p|^2_q}{2b^2}u_j\rangle_{L^2(X;\mathscr{E})} \mp \frac{1}{b}\langle u_j, a^i(q)p_i u_j\rangle_{L^2(X;\mathscr{E})} \\
&\geq \frac{1}{2b^2}\left[\|u_j\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})} + 2\kappa_b\|u\|^2_{L^2(X;\mathscr{E})}\right] - \frac{C_0'}{b}\|u_j\|_{L^2(X;\mathscr{E})}\|u_j\|_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})} \\
&\geq \frac{1}{2b^2}\left[\|u_j\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})} + 2\kappa_b\|u\|^2_{L^2(X;\mathscr{E})}\right] - \frac{1}{4b^2}\|u_j\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})} - 2{C_0'}^2\|u_j\|^2_{L^2(X;\mathscr{E})},
\end{aligned}$$

for some $C_0' > 0$ determined by the geometric data (g, E, g^E, ∇^E). With $\frac{\kappa_b}{2b^2} \geq C_0\frac{1+b^2}{2b^2} \geq \frac{C_0}{2}$, the first inequality (2.2.1.1) is proved for $C_0 \geq 4{C_0'}^2$.
Using Cauchy-Schwarz inequality in the left hand side of (2.2.1.1) yields

$$\|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u\|_{L^2(X;\mathscr{E})}\|u\|_{L^2(X;\mathscr{E})} \geq \frac{1}{4b^2}\left[\|u\|^2_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})} + \kappa_b\|u\|^2_{L^2(X;\mathscr{E})}\right].$$

We deduce at once $\|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u\|_{L^2(X;\mathscr{E})} \geq \frac{\kappa_b}{4b^2}\|u\|_{L^2(X,\mathscr{E})}$. The latter inequality multiplied by $\|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)u\|_{L^2(X;\mathscr{E})}$ yields (2.2.1.2). □

Corollary 2.2.2. *Let $P_{\pm,b}=\frac{1}{b^2}\mathscr{O}\pm\frac{1}{b}\nabla^{\mathscr{E}}_{\mathscr{Y}}$ and let $C_0\geq 1$ be determined by the geometric data (g,E,∇^E,g^E) according to Proposition 2.2.1. For $\kappa_b\geq C_0(1+b^2)$ the operator $\frac{\kappa_b}{b^2}+P_{\pm,b}$ is essentially maximal accretive on $\mathscr{C}^\infty_0(X;\mathscr{E})$ and therefore on $\mathscr{S}(X;\mathscr{E})$.*

Proof. Proposition 2.2.1 says that the operator $(\frac{\kappa_b}{b^2}+P_{\pm,b})$ and its formal adjoint $\frac{\kappa_b}{b^2}+P^*_{\pm,b}$ are accretive on $\mathscr{C}^\infty_0(X;\mathscr{E})$ with the lower bound (2.2.1.1).
It suffices to prove that the range $(\frac{\kappa_b}{b^2}+P_{\pm b})\mathscr{C}^\infty_0(X;\mathscr{E})$ is dense in $L^2(X;\mathscr{E})$. It is equivalent to

$$\left.\begin{array}{l} u\in L^2(X;\mathscr{E})\\ (\frac{\kappa_b}{b^2}+P^*_{\pm,b})u=0\in\mathscr{D}'(X;\mathscr{E})\end{array}\right\}\Rightarrow u=0.$$

With $P^*_{\pm,b}=P_{\mp,b}\mp a^i(q)b_i$ in local coordinates according to (2.2.1.3), Hörmander's hypoellipticity result for type II operators (see [Hor67]) implies $u\in\mathscr{C}^\infty(X;\mathscr{E})$. For $\chi\in\mathscr{C}^\infty_0(\mathbb{R};[0,1])$ such that $\chi\equiv 1$ in a neighborhood of 0 and for $\varepsilon>0$ set $u_\varepsilon=\chi(\varepsilon|p|^2_q)u$.
The above equation implies

$$(\frac{\kappa_b}{b^2}+P^*_{\pm,b})u_\varepsilon=-\left[P^*_{\pm,b}\,,\,\chi(\varepsilon|p|^2_q)\right]u=-\left[-\frac{\Delta_p}{2b^2}\,,\,\chi(\varepsilon|p|^2_q)\right]u$$

because $\mathscr{Y}f(|p|^2_q)=0$. The form of the last commutator allows to write

$$(\frac{\kappa_b}{b^2}+P^*_{\pm,b}u_\varepsilon)=-\left[-\frac{\Delta_p}{2b^2}\,,\,\chi(\varepsilon|p|^2_q)\right](1-\tilde{\chi}(\varepsilon|p|^2_q))u$$

where $\tilde{\chi}\in\mathscr{C}^\infty_0(\mathbb{R};R)$ has a support included a neighborhood of 0 where $\chi\equiv 1$ and its derivatives vanish, while $\tilde{\chi}\equiv 1$ in a smaller neighborhood of 0. By taking the scalar product with u_ε, the inequality (2.2.1.1) for $P^*_{\pm,b}$ implies

$$\frac{1}{4b^2}\left[\|u_\varepsilon\|^2_{\tilde{\mathscr{W}}^{1,0}(X,\mathscr{E})}+\kappa_b\|u_\varepsilon\|^2_{L^2(X;\mathscr{E})}\right]\leq C_{g,\chi}\|u_\varepsilon\|_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})}\|(1-\tilde{\chi}(\varepsilon|p|^2_q))u\|_{L^2(X;\mathscr{E})}$$

and

$$\sqrt{\frac{d}{2}}\|u_\varepsilon\|_{L^2(X;\mathscr{E})}\leq\|u_\varepsilon\|_{\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E})}\leq 4C_{g,\chi}b^2\|(1-\tilde{\chi}(\varepsilon)|p|^2_q)u\|_{L^2(X;\mathscr{E})}.$$

Lebesgue's theorem for the limit $\varepsilon\to 0$ gives

$$\sqrt{\frac{d}{2}}\|u\|_{L^2(X;\mathscr{E})}\leq 4C_{g,\chi}b^2\lim_{\varepsilon\to 0}\|(1-\tilde{\chi}(\varepsilon|p|^2_q))u\|_{L^2(X;\mathscr{E})}=0.$$

□

2.2.2 Localization

Proposition 2.2.3. *Let $P_{\pm,b}=\frac{1}{b^2}\mathscr{O}\pm\frac{1}{b}\nabla^{\mathscr{E}}_{\mathscr{Y}}$ and fix $Q=\cup_{j=1}^J\Omega_j$ a finite open chart covering of Q. Let $\sum_{j=1}^J\varrho_j(q)^2\equiv 1$ be a subordinate quadratic partition of unity, $\varrho_j\in\mathscr{C}^\infty_0(\Omega_j;[0,1])$. There exists $C_0\geq 1$, determined by the geometric data (g,E,∇^E,g^E), and now the partition of unity $(\varrho_j)_{1\leq j\leq J}$, such that for all $b>0$, $\lambda\in\mathbb{R}$ and for $\kappa_b=C_0(1+b^2)$ the following equivalence of norms*

$$\left(\frac{\left\|(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda)u\right\|^2_{L^2(X;\mathscr{E})}}{\sum_{j=1}^J\left\|(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda)(\varrho_j u)\right\|^2_{L^2(X;\mathscr{E})}}\right)^{\pm 1}\leq 4 \qquad (2.2.2.1)$$

holds for all $u\in\mathscr{C}^\infty_0(X;\mathscr{E})$.

Proof. It's a straightforward application of Corollary 2.C.2. We have to check the assumption (2.C.0.4) which says

$$\forall u \in \mathscr{C}_0^\infty(X;\mathscr{E}), \quad \frac{r}{2}\sum_{j\in J}\|(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda)\varrho_j u\|^2_{L^2(X;\mathscr{E})} \geq 2\sum_{j_1,j\in J}\|[(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda),\varrho_{j_1}]\varrho_j u\|^2_{L^2(X;\mathscr{E})}$$
$$+4\sum_{j_1,j_2,j\in J}\|[[(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda),\varrho_{j_2}],\varrho_{j_1}]\varrho_j u\|^2_{L^2(X;\mathscr{E})},$$

for some $r\in[0,1)$. Because the operator $P_{\pm,b}$ is a first-order differential operator in the q variable the first commutator equals

$$[\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda,\varrho_{j_1}] = \pm\frac{1}{b}\mathscr{Y}\varrho_{j_1} = \pm\frac{1}{b}g^{\ell k}(q)p_k\frac{\partial\varrho_{j_1}}{\partial q^\ell},$$

for $j_1\in J$ where the right-hand side is written local canonical coordinate (q,p). Moreover the double commutators indexed by $j_1,j_2\in J$ all vanish.
We deduce the existence of $C_0'>0$ such that

$$\forall u\in\mathscr{C}_0^\infty(X;\mathscr{E}),\quad \|[(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda),\varrho_{j_1}]\varrho_j u\|^2_{L^2(X;\mathscr{E})}\leq C_0'\frac{1}{b^2}\||p|_q\varrho_j u\|^2_{L^2(X;\mathscr{E})}.$$

and the summation over $j_1,j\in J$, combined with the inequality (2.2.1.2), yields

$$\forall u\in\mathscr{C}_0^\infty(X;\mathscr{E}),\quad \sum_{j_1,j\in J}\|[(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda),\varrho_{j_1}]\varrho_j u\|^2_{L^2(X;\mathscr{E})}\leq C_0'\,|J|\,\frac{16b^2}{\kappa_b}\sum_j\|(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda)\varrho_j u\|^2_{L^2(X,\mathscr{E})}.$$

With $C_0'\,|J|\,\frac{16b^2}{\kappa_b}\leq C_0'\,|J|\,\frac{16}{C_0}$, choosing $C_0\geq 1$ large enough guarantees the assumption (2.C.0.4) with $r=\frac{1}{2}$. □

2.2.3 Changing locally the connections

With Proposition 2.2.3 the analysis of $P_{\pm,b}$ can be localized in a chart open domain Ω_j. Additionally it can be assumed that there is a well defined local frame $(f^1(q),\ldots,f^N(q))$ of the restricted bundle $E\big|_{\Omega_j}$. In this frame a trivial connection $\nabla^{E,j}$ on $E\big|_{\Omega_j}$ and therefore a corresponding flat connection on $\mathscr{E}\big|_{T^*\Omega_j}$ is defined by pull-back. The operator $P^j_{\pm,b}$ defined locally can be identified with a scalar operator according to

$$P^j_{\pm,b}\left[\sum_{\ell=1}^N u_\ell f^\ell\right] = \left(\frac{1}{b^2}\mathscr{O}\pm\frac{1}{b}\nabla^{\mathscr{E},j}_Y\right)\left[\sum_{\ell=1}^N u_\ell f^\ell\right] \tag{2.2.3.1}$$
$$= \sum_{\ell=1}^N\left[\frac{-g_{ik}(q)\frac{\partial^2}{\partial p_i\partial p_k}+g^{ik}(q)p_ip_k}{2b^2}(u_\ell)\pm\frac{1}{b}g^{ik}(q)p_ie_k(u_\ell)\right]f^\ell. \tag{2.2.3.2}$$

Proposition 2.2.4. *Under the assumptions of Proposition 2.2.3 with and with the additional condition that E is trivialized by a local frame $(f^1(q),\ldots,f^N(q))$ over Ω_j for every $j\in\{1,\ldots,J\}$, let $P^j_{\pm,b}$ be defined by* (2.2.3.1). *There exists $C_0\geq 1$, determined by the geometric data (g,E,∇^E,g^E) and the partition of unity $(\varrho_j)_{1\leq j\leq J}$, such that for all $b>0$, $\lambda\in\mathbb{R}$ and for $\kappa_b=C_0(1+b^2)$ the following equivalence of norms*

$$\left(\frac{\left\|(\frac{\kappa_b}{b^2}+P_{\pm,b}-i\lambda)u\right\|^2_{L^2(X;\mathscr{E})}}{\sum_{j=1}^J\left\|(\frac{\kappa_b}{b^2}+P^j_{\pm,b}-i\lambda)(\varrho_j u)\right\|^2_{L^2(X;\mathscr{E})}}\right)^{\pm1}\leq 12 \tag{2.2.3.3}$$

holds for all $u\in\mathscr{C}_0^\infty(X;\mathscr{E})$.

Proof. For a given $j \in \{1,\ldots,J\}$ and for $v \in \mathscr{C}_0^\infty(\Omega_j;\mathscr{E})$ we have

$$(P_{\pm,b} - P^j_{\pm,b})(v) = \pm\frac{1}{b}(\nabla^{\mathscr{E}}_{\mathscr{Y}} - \nabla^{\mathscr{E},j}_{\mathscr{Y}})(v) = \pm\frac{1}{b} g^{ik}(q)p_i\pi_X^*(\nabla^E_{\frac{\partial}{\partial q^k}} - \nabla^{E,j}_{\frac{\partial}{\partial q^k}})(v)$$

where

$$(\nabla^E_{\frac{\partial}{\partial q^k}} - \nabla^{E,j}_{\frac{\partial}{\partial q^k}})[v_\ell(q,p)f^\ell(q)] = v_\ell(q,p)F^\ell_{\ell',k}(q)f^{\ell'}(q)$$

with $F^\ell_{\ell',k} \in \mathscr{C}^\infty(\Omega_j;\mathbb{R})$.
Applied to $v = \varrho_j u$ this gives the upper bound

$$\|(P_{\pm,b} - P^j_{\pm,b}))(\varrho_j u)\|^2 \leq \frac{C_0'}{b^2}\||p|_q(\varrho_j u)\|^2_{L^2(X;\mathscr{E})},$$

where C_0' depends only on the metric g^E and the connection ∇^E given on the vector bundle E and is uniform with respect to $j \in \{1,\ldots J\}$. The same argument as in the proof o Proposition 2.2.3 shows that

$$\|(P_{\pm,b} - P^j_{\pm,b})(\varrho_j u)\|^2_{L^2(X;\mathscr{E})} \leq \frac{1}{4}\|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})},$$

when the constant C_0 in $\kappa_b = C_0(1+b^2)$ is chosen large enough. The parallelogram identity implies

$$\|(\frac{\kappa_b}{b^2} + P^j_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})} \leq 3\|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})}$$
$$\text{and } \|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})} \leq 3\|(\frac{\kappa_b}{b^2} + P^j_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})}.$$

By summation over $j \in \{1,\ldots,J\}$ we obtain

$$\left(\frac{\sum_{j=1}^J \|(\frac{\kappa_b}{b^2} + P^j_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})}}{\sum_{j=1}^J \|(\frac{\kappa_b}{b^2} + P_{\pm,b} - i\lambda)(\varrho_j u)\|^2_{L^2(X;\mathscr{E})}}\right)^{\pm 1} \leq 3.$$

The inequality (2.2.3.3) is obtained by taking the product with the result (2.2.2.1) of Proposition 2.2.3. □

2.3 Sobolev spaces

Like for the operator $P_{\pm,b}$ we firstly reduce the characterization of $u \in \tilde{\mathscr{W}}^k(X;\mathscr{E})$, $k \in \mathbb{N}$, to a local problem with the possibility of replacing the connection $\nabla^{\mathscr{E}}$ by a trivial connection in a given local frame. Then $\mathscr{W}^s(X;\mathscr{C})$ and its norm will be expressed in terms of the functional calculus of pseudo-differential elliptic self-adjoint operator W^2 in the class $\mathrm{OpS}^2_\Psi(Q;\mathbb{C})$ presented in Appendix 2.E. Finally general spaces $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E})$, $s_1, s_2 \in \mathbb{R}$ are introduced by using the functional calculus of two commuting self-adjoint operators.

2.3.1 First properties of $\tilde{\mathscr{W}}^k$, $k \in \mathbb{N}$

We collect rather immediate consequences of the Definition 2.1.2. The norm $\|u\|_{\tilde{\mathscr{W}}^k}$ is

$$\|u\|_{\tilde{\mathscr{W}}^k} = \max_{N_1 + \frac{N_2+N_3}{2} \leq k} P_{k,N_1,N_2,N_3}(u) \tag{2.3.1.1}$$

where $P_{k,N_1,N_2,N_3}(u)$ is the smallest constant $C \geq 0$ such that

$$\forall (T_1^H, \dots T_{N_1}^H) \in \mathscr{C}_Q^\infty(X; T^H X)^{N_1}, \forall (T_1^V, \dots, T_{N_2}^V) \in \mathscr{C}_Q^\infty(X; T^V X)^{N_2},$$
$$\|\langle p\rangle_q^{N_3} \nabla^{\mathscr{E}}_{T_1^H} \dots \nabla^{\mathscr{E}}_{T_{N_1}^H} \nabla^{\mathscr{E}}_{T_1^V} \dots \nabla^{\mathscr{E}}_{T_{N_2}^V} u\|_{L^2} \leq C \prod_{n_1=1}^{N_1} \|T_{n_1}^H\|_{\mathscr{C}^k} \prod_{n_2=1}^{N_2} \|T_{n_2}^V\|_{\mathscr{C}^k}.$$

Because

$$\langle p\rangle_q^{N_3} \nabla^{\mathscr{E}}_{T_1^H} \dots \nabla^{\mathscr{E}}_{T_{N_1}^H} \nabla^{\mathscr{E}}_{T_1^V} \dots \nabla^{\mathscr{E}}_{T_{N_2}^V} u = \nabla^{\mathscr{E}}_{T_1^H} \dots \nabla^{\mathscr{E}}_{T_{N_1}^H} \langle p\rangle_q^{N_3} \nabla^{\mathscr{E}}_{T_1^V} \dots \nabla^{\mathscr{E}}_{T_{N_2}^V} u,$$

$\|u\|_{\tilde{\mathscr{W}}^k}$ also equals

$$\|u\|_{\tilde{\mathscr{W}}^k} = \max_{\substack{N_2+N_3=j\leq 2k\\ N_1\leq k-2j}} \sup_{(T_1^V,\dots T_{N_2}^V)\in\mathscr{C}_Q^\infty(X;T^V X)^{N_2}} \frac{P_{k,N_1,0,0}(\langle p\rangle_q^{N_3} \nabla^{\mathscr{E}}_{T_1^V} \dots \nabla^{\mathscr{E}}_{T_{N_2}^V} u)}{\prod_{n_2=1}^{N_2} \|T_{n_2}^V\|_{\mathscr{C}^k}}. \tag{2.3.1.2}$$

Proposition 2.3.1. *Let $k \in \mathbb{N}$, let $\theta \in \mathscr{C}^\infty(Q; \mathrm{End}(E))$ and fix any $\mathscr{C}^k$-norm on Q. The multiplication by $\pi_X^*\theta \in \mathscr{C}^\infty(X; \mathrm{End}(\mathscr{E}))$, $\pi_X^*\theta(x) = \pi_X^*(\theta(\pi_X(x)))$, is a bounded operator in $\tilde{\mathscr{W}}^k(X; \mathscr{E})$ with*

$$\forall u \in \tilde{\mathscr{W}}^k(X; \mathscr{E}), \quad \|(\pi_X^*\theta)u\|_{\tilde{\mathscr{W}}^k} \leq C_k \|\theta\|_{\mathscr{C}^k} \|u\|_{\tilde{\mathscr{W}}^k}. \tag{2.3.1.3}$$

Proof. Because $\langle p\rangle_q^{N_3} \nabla^{\mathscr{E}}_{T_1^V} \dots \nabla^{\mathscr{E}}_{T_{N_2}^V} (\pi_X^*\theta) = (\pi_X^*\theta) \langle p\rangle_q^{N_3} \nabla^{\mathscr{E}}_{T_1^V} \dots \nabla^{\mathscr{E}}_{T_{N_2}^V}$ and owing to (2.3.1.2) the problem is reduced to

$$P_{k,N_1,0,0}((\pi_X^*\theta)v) \leq C_k \|\theta\|_{\mathscr{C}^k} P_{k,N_1,0,0}(v)$$

for all $N_1 \in \{0, \dots, k\}$.
It is obviously true for $N_1 = 0$. If it is true for $N_1 \in \{0, \dots, k-1\}$ then for $T_{N_1+1}^H \in \mathscr{C}_Q^\infty(X; TX^H)$ with $T_{N_1+1} = \pi_{X,*} T_{N_1+1}^H \in \mathscr{C}^\infty(Q; TQ)$, we write

$$\nabla^{\mathscr{E}}_{T_{N_1+1}^H} (\pi_X^*\theta)v = (\pi_X^*(\theta) \nabla^{\mathscr{E}}_{T_{N_1+1}^H} v) + [\pi_X^*(\nabla^{\mathrm{End}(E)}_{T_{N_1+1}} \theta)]v.$$

We get

$$\begin{aligned} P_{k-1,N_1,0,0}(\nabla^{\mathscr{E}}_{T_{N+1}^H} (\pi_X^*\theta)v) &\leq C_{k-1} \left[\|\theta\|_{\mathscr{C}^{k-1}} P_{k-1,N_1,0,0}(\nabla^{\mathscr{E}}_{T_{N_1+1}^H} v) + \|\nabla^{\mathrm{End}(E)}_{T_{N_1+1}} \theta\|_{\mathscr{C}^{k-1}} P_{k-1,N_1,0,0}(v) \right] \\ &\leq 2C_{k-1} \|\theta\|_{\mathscr{C}^k} P_{k,N_1+1,0,0}(v). \end{aligned}$$

The obvious inequality

$$P_{k,N_1+1,0,0}(w) \leq \sup_{T_{N_1+1}^H \in \mathscr{C}_Q^\infty(X; T^H V)} \frac{P_{k-1,N_1,0,0}(\nabla^{\mathscr{E}}_{T_{N_1+1}^H} w)}{\|T_{N_1+1}^H\|_{\mathscr{C}^k}}$$

ends the proof by induction. □

The previous statement contains two particular cases which allow the local scalar characterization of $u \in \tilde{\mathscr{W}}^k(X; \mathscr{E})$ with a simpler norm.

Proposition 2.3.2. *Fix $k \in \mathbb{N}$ and consider two different connections $\nabla^{E,1}$ and $\nabla^{E,2}$ on the vector bundle $E \stackrel{\pi_E}{\to} Q$ with the associated connections $\nabla^{\mathscr{E},1}$ and $\nabla^{\mathscr{E},2}$ on $\mathscr{E} \stackrel{\pi_{\mathscr{E}}}{\to} X$. Let $\tilde{\mathscr{W}}^k_{\nabla^j}(X; \mathscr{E})$ and $\| \ \|_{\tilde{\mathscr{W}}^k_{\nabla^j}}$ be the corresponding $\tilde{\mathscr{W}}^k(X; \mathscr{E})$ spaces and norms according to Definition 2.1.2.*

1) *The space $\mathscr{W}^k(X;\mathscr{E})$ is a $\mathscr{C}^\infty(Q;\mathbb{R})$ module and for any finite atlas $Q=\bigcup_{j=1}^J \Omega_j$ and for any subordinate partition of unity $\sum_{j=1}^J \varrho_j(q)\equiv 1$, a section u belongs to $\mathscr{W}^k(X;\mathscr{E})$ if and only if, for every $1\leq j\leq J$, $\varrho_j u\in \mathscr{W}^k_{\Omega_j-\mathrm{comp}}(T^*\Omega_j;\mathscr{E}\big|_{T^*\Omega_j})$ and the norm $\|u\|_{\tilde{\mathscr{W}}^k}$ is equivalent to* $\max_{1\leq j\leq J}\|\varrho_j u\|_{\tilde{\mathscr{W}}^k}$.

2) *For two different connections $\nabla^{E,1}$ and $\nabla^{E,2}$, the two spaces $\tilde{\mathscr{W}}^k_{\nabla^1}(X;\mathscr{E})$ and $\tilde{\mathscr{W}}^k_{\nabla^2}(X;\mathscr{E})$ are equal and the norms $\|\ \|_{\tilde{\mathscr{W}}^k_{\nabla^1}}$ and $\|\ \|_{\tilde{\mathscr{W}}^k_{\nabla^2}}$ are equivalent.*

Proof. **1)** Simply apply Proposition 2.3.1 with $\theta\in\mathscr{C}^\infty(Q;\mathbb{R})$. The definition of $\tilde{\mathscr{W}}^k_{\Omega_j-\mathrm{comp}}(T^*\Omega_j;\mathscr{E}\big|_{T^*\Omega_j})$ and the other statements are explained in Appendix 2.D. Simply use the triangular inequality for $\|u\|_{\tilde{\mathscr{W}}^k}\leq J\max_{1\leq j\leq J}\|\varrho_j u\|_{\tilde{\mathscr{W}}^k}$.

2) Remember $\nabla^{E,2}_T-\nabla^{E,1}_T=R(T)\in \mathrm{End}(E)$ with $\|R(T)\|_{\mathscr{C}^k}\leq C_k\|T\|_{C^{k+1}}$ for any $T\in\mathscr{C}^\infty(Q;TQ)$. Let P^j_{k,N_1,N_2,N_3} be the norms involved in (2.3.1.1)(2.3.1.2) for the two associated connections $\nabla^{\mathscr{E},\ell}$ for $\ell=1,2$. We can make an induction proof with respect to $N_1\in\{0,\ldots,k\}$ like in Proposition 2.3.1 after reducing the problem to $N_2=N_3=0$ by noticing

$$\forall T^V\in\mathscr{C}^\infty_Q(X;TX^V),\quad \nabla^{\mathscr{E},2}_{T^V}=\nabla^{\mathscr{E},1}_{T^V}$$

and by using

$$\nabla^2_{T^H}-\nabla^1_{T^H}=\pi_X^*(R(T))$$

for any $T^H\in\mathscr{C}^\infty_Q(X;TX^H)$ with $\pi_{X,*}T^H=T\in\mathscr{C}^\infty(Q;TQ)$.
Actually the induction proof relies on

$$\begin{aligned}P^2_{k-1,N_1,0,0}(\nabla^{\mathscr{E},2}_{T^H_{N_1+1}}v)&\leq P^2_{k-1,N_1,0,0}(\nabla^{\mathscr{E},1}_{T^H_{N_1+1}}v)+P^2_{k-1,N_1,0,0}([\pi^*(R(T^H_{N_1+1})]v)\\ &\underbrace{\leq}_{\text{induction}} C_k\left[P^1_{k-1,N_1,0,0}(\nabla^{\mathscr{E},1}_{T^H_{N_1+1}}v)+P^1_{k-1,N_1,0,0}([\pi^*(R(T^H_{N_1+1})]v)\right]\\ &\underbrace{\leq}_{\text{Prop. 2.3.1}} C'_k\|T^H_{N_1+1}\|_{\mathscr{C}^k}P^1_{k,N_1+1,0,0}(v),\end{aligned}$$

and leads to $\|u\|_{\tilde{\mathscr{W}}^k_{\nabla^2}}\leq C''_k\|u\|_{\tilde{\mathscr{W}}^k{}_{\nabla^1}}$. The result follows by symmetry. □

The atlas $(\Omega_j)_{1\leq j\leq J}$ can be chosen such that $E\big|_{\Omega_j}$ is trivial with a local frame $(f^1_j,\ldots,f^N_j)$, $f^n_j\in\mathscr{C}^\infty(\Omega_j;E\big|_{\Omega_j})$. Above Ω_j the connection ∇^E can be replaced by the trivial connection $\nabla^{j,E}$ given by

$$\forall T\in\mathscr{C}^\infty(Q;TQ),\quad \nabla^j_T(\sum_{n=1}^N v_n(q)f^n_j)=\sum_{n=1}^N(Tv_n)f^n_j\,.$$

For a section $u=\sum_{n=1}^N u_{n,j}(x)f^n_j$ of $\mathscr{E}\big|_{T^*\Omega_j}$ we get

$$\nabla^{j,\mathscr{E}}_{T^H_1}\ldots\nabla^{j,\mathscr{E}}_{T^H_{N_1}}\nabla^{j,\mathscr{E}}_{T^V_1}\ldots\nabla^{j,\mathscr{E}}_{T^V_{N_2}}u=\sum_{n=1}^N(T^H_1\ldots T^H_{N_1}T^V_1\ldots T^V_{N_2}v_{n,j})f^n_j\,.$$

Proposition 2.3.2 now implies that the $\tilde{\mathscr{W}}^k$-norm of

$$u=\sum_{j=1}^J\varrho_j(q)u=\sum_{j=1}^J\sum_{n=1}^N u_{n,j}f^n_j$$

with $u_{n,j}\in\tilde{\mathscr{W}}^k_{\Omega_j-\mathrm{comp}}(T^*\Omega_j;\mathbb{C})\subset\tilde{\mathscr{W}}^k(X;\mathbb{C})$, is equivalent to

$$\max_{\substack{1\leq j\leq J\\1\leq n\leq N}}\|u_{n,j}\|_{\tilde{\mathscr{W}}^k(X;\mathbb{C})}\,.$$

The $\tilde{\mathscr{W}}^k$-spaces for $k\in\mathbb{N}$ and their norms is are thus fully understood in a local scalar setting.

2.3.2 Pseudo-differential definition of $\tilde{\mathcal{W}}^s$, $s \in \mathbb{R}$

The end of Subsection 2.3.1 reduced the description of $\tilde{\mathcal{W}}^k(X;\mathcal{E})$ to the local description of $\tilde{\mathcal{W}}^k(X;\mathbb{C})$. We can thus focus on scalar sections, and we now give a pseudo-differential and a global characterization.
We need the pseudo-differential calculus in $\bigcup_{m\in\mathbb{R}}\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ introduced in Appendix 2.E. We recall that $a \in S^m_\Psi(Q;\mathbb{C})$ if in doubly canonical coordinates (q,p,ξ,η) in $T^*(T^*\Omega)$ associated with the local coordinates $q=(q^1,\ldots,q^d)$ on a chart open set $\Omega\subset Q$, the uniform estimate

$$|\partial_q^\alpha\partial_p^\beta\partial_\xi^\gamma\partial_\eta^\delta a(q,p,\xi,\eta)| \le C_{\alpha,\beta,\gamma,\delta}(1+|\xi|^2+|p|^4+|\eta|^4)^{\frac{m-|\gamma|-\frac{|\beta|+|\delta|}{2}}{2}}$$

and that, the quantization is the standard one,given by the local kernel on $T^*\Omega\times T^*\Omega$:

$$[a(q,p,D_q,D_p)](q,p,q',p') = \int_{\mathbb{R}^{2d}} e^{i[(q-q').\xi+(p-p').\eta]}a(q,p,\xi,\eta)\,\frac{d\xi d\eta}{(2\pi)^{2d}}\,.$$

Actually the general quantization of $a\in S^m_\Psi(Q;\mathbb{C})$ is defined by introducing a partition of unity $\sum_{j=1}^J\hat{\varrho}_j(q)\equiv 1$ and cut-off functions $\hat{\chi}\in\mathcal{C}_0^\infty(\Omega_j;[0,1])$, $\hat{\chi}\equiv 1$ on $\mathrm{supp}\,\hat{\varrho}_j$, and by setting $a_{\hat{\varrho},\hat{\chi}}(x,D_x)=\sum_{j=1}^J(\hat{\varrho}_j(q)a)(x,D_x)\circ\hat{\chi}_j(q)$, and the set of pseudo-differential operators is

$$\mathrm{OpS}^m_\Psi = \left\{a_{\hat{\varrho},\hat{\chi}}(x,D_x)+R\,,a\in S^m_\Psi(Q;\mathbb{C}),R\in\mathcal{R}(Q;\mathbb{C})\right\},$$

with $\mathcal{R}(Q;\mathbb{C})=\mathcal{L}(\mathcal{S}'(T^*Q;\mathbb{C});\mathcal{S}(T^*Q;\mathbb{C}))$. It is proved in Appendix 2.E that this pseudo-differential calculus has the same properties as the usual pseudo-differential calculus, with a different homogeneity which takes into account the global estimates as $p\to\infty$.
In canonical coordinates (q,p) associated with the local coordinates $(q^1,\ldots,q^d)$ on Ω we know the two frames $(e_1,\ldots,e_d)$ (resp. $(\hat{e}^1,\ldots,\hat{e}^d)$) of $T(T^*\Omega)^H$ (resp. $T(T^*\Omega)^V$) given by

$$e_i = \frac{\partial}{\partial q^i}+\Gamma^k_{i\ell}(q)p_k\frac{\partial}{\partial p_\ell}$$

$$\text{resp.}\qquad \hat{e}^i = \frac{\partial}{\partial p_i}\,.$$

As differential operators, the locally defined operators e_i, ∂_{q^i}, $p_k\partial_{p_\ell}$, $\mathcal{O}=\frac{-g_{ij}(q)\partial_{p_i}\partial_{p_j}+g^{ij}(q)p_ip_j}{2}$ belong to $\mathrm{OpS}^1_{\Psi,\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$, while $p_k\times$, $\langle p\rangle_q\times$ and ∂_{p_k} belong to $\mathrm{OpS}^{1/2}_{\Psi,\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$.
Thus any $T^H\in\mathcal{C}^\infty_Q(X;TX^H)$ is a differential operator that belongs to $\mathrm{OpS}^1_\Psi(Q;\mathbb{C})$, and any $T^V\in\mathcal{C}^\infty_Q(X;TX^V)$ belongs to $\mathrm{OpS}^{1/2}_\Psi(Q;\mathbb{C})$.
Let us introduce another operator which involves the scalar horizontal Laplacian Δ_H. We follow [BeBo]: On X with the decomposition $TX=TX^H\oplus TX^V\sim TQ\oplus T^*Q$ given by the Levi-Civita connection associated with g we put the riemannian metric $g\oplus^\perp g^{-1}$ and consider the associated total Laplacian Δ_x. The projection $\pi_X: X=T^*Q\to Q$ is now a riemannian submersion with totally geodesic fibers and the horizontal Laplacian $\Delta_H=\Delta_x-\Delta_p$ equals in local canonical coordinates

$$\Delta_H = g^{ij}(q)(e_ie_j-\Gamma^k_{ij}(q)e_k)\,.$$

Because the volume of $g\oplus^\perp g^{-1}$ is equal to the symplectic volume $dqdp$ and Δ_x and Δ_p are symmetric on $\mathcal{S}(T^*Q;\mathbb{C})$, the operator Δ_H is symmetric on $\mathcal{S}(T^*Q;\mathbb{C})$ for the $L^2(T^*Q,dqdp;\mathbb{C})$ scalar product. By introducing the adjoint differential operator $e_i^*=-\partial_q^i-\Gamma^k_{ii'}p_k\partial_{p_{i'}}-\Gamma^{'}_{ii'}$ and owing to the symmetry or by explicit computations with

$$\partial_{q^i}g^{ij}=\partial_{q^i}g^{-1}(dq^i,dq^j)=g^{-1}(-\Gamma^i_{i,k}dq^k,dq^j)+g^{-1}(dq^i,-\Gamma^j_{i,k}dq^k)=-\Gamma^i_{ik}g^{kj}-\Gamma^j_{ik}g^{ik}\,,$$

the horizontal Laplacian is also given by

$$-\Delta_H = e_i^* g^{ij}(q) e_j\,,$$

without a divergence term because integrations are made with respect to the symplectic volume $dqdp$.

Definition 2.3.3. The operator W^2 is the closure in $L^2(X, dqdp;\mathbb{C})$ of the differential operator $C_g - \Delta_H + C_g\mathcal{O}^2 : \mathcal{S}(X;\mathbb{C}) \to \mathcal{S}(X;\mathbb{C}) \subset L^2(X, dqdp;\mathbb{C})$ for $C_g \geq 1$ large enough.

Notice that because the flow $\exp(te_i)$ sends isometrically T_q^*Q to $T^*_{\exp(t\partial_{q^i})q}Q$, the commutations

$$[e_i, -\Delta_p] = [e_i, |p|_q^2] = [e_i, \mathcal{O}] = [\Delta_H, \mathcal{O}] = [W^2, \mathcal{O}] = 0$$

hold true on $\mathcal{S}_{\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$.
As a consequence of Appendix 2.E we have a simple characterization of $\tilde{\mathcal{W}}^s(X;\mathbb{C})$, in the case when $E = Q \otimes \mathbb{C}$. The general case can then be deduced either by the localization at the end of the previous paragraph or by the approach proposed afterwards. Both are equivalent.

Proposition 2.3.4. *For any $s \in \mathbb{R}$, the space $\tilde{\mathcal{W}}^s(X;\mathbb{C})$ is characterized by*

$$\tilde{\mathcal{W}}^s(X;\mathbb{C}) = \left\{u \in \mathcal{S}'(X;\mathbb{C}), \forall A \in \mathrm{OpS}^s_\Psi(Q;\mathbb{C}),\ Au \in L^2(X, dqdp;\mathbb{C})\right\}$$

For $C_g \geq 1$ large enough, $C_g - \Delta_H + C_g\mathcal{O}^2 : \mathcal{S}(X;\mathbb{C}) \to L^2(X, dqdp;\mathbb{C})$ is a non negative essentially self-adjoint operator, with self-adjoint extension W^2 and $D(W^2) = \tilde{\mathcal{W}}^2(X;\mathbb{C})$.
For any $s \in \mathbb{R}$, $W^s = (W^2)^{s/2}$ is an elliptic operator in $\mathrm{OpS}^s_\Psi(Q;\mathbb{C})$ and the norm on $\tilde{\mathcal{W}}^s(X;\mathbb{C})$ can be chosen as $\|W^s u\|_{L^2(X,dqdp;\mathbb{C})}$.

Proof. **a)** If we start from the above definition of $\tilde{\mathcal{W}}^s(X;\mathbb{C})$ the problem is reduced to the ellipticity and the identification of the principal symbol of $W^2 = C_g - \Delta_H + C_g\mathcal{O}^2$. Because it is a differential operator $W^2 = \sum_{j=1}^J \hat{\varrho}_j(q) W^2 \hat{\chi}_j(q)$, and we obtain $W^2 = a_{\hat{\varrho},\hat{\chi}}(x, D_x)$ with $a - a_2 \in S^1_\Psi(Q;\mathbb{C})$ and

$$a_2(x,\Xi) = C_g + |\xi + \Gamma^k_{..}(q)p_k\eta|_q^2 + \frac{C_g}{4}(|p|_q^2 + |\eta|_q^2)^2 \geq C_g + |\xi|_q^2 - 2|\Gamma^k_{..}(q)p_k\eta|_q^2 + \frac{C_g}{4}(|p|_q^2 + |\eta|_q^2)^2$$

We used the notation $|\tau|_q^2 = g^{ij}(q)\tau_i\tau_j$ for $\tau = \xi + \Gamma^k_{..}(q)p_k\eta$, $\tau = p$ and $|\eta|_q^2 = g_{ij}(q)\eta^i\eta^j$. The ellipticity comes from

$$a_2(x,\Xi) \geq C_g + \varepsilon_g|\xi|^2 - \frac{1}{\varepsilon_g}|p|^2|\eta|^2 + \frac{C_g\varepsilon_g}{4}(|p|^4 + |\eta|^4)$$

for some $\varepsilon_g > 0$ given by g and the fixed open covering $\bigcup_{j=1}^J \Omega_j$. The ellipticity $a_2 \geq C_g + \varepsilon_g(|\xi|^2 + |p|^4 + |\eta|^4)$ holds true if $C_g\varepsilon_g^2 - 1 \geq 8\varepsilon_g^2$.
The operator W^2 is symmetric with

$$\langle u\,, W^2 u\rangle = C_g\|u\|_{L^2}^2 + \sum_{j=1}^J \int_{T^*\Omega_j} g^{ii'}(q)\left[\overline{(e_i\theta_j(q)u)}(e_{i'}\theta_j(q)u) - (\partial_{q^i}\theta_j)(\partial_{q^i}\theta_j)|u|^2\right]\, dqdp + C_g\|\mathcal{O}u\|_{L^2}^2 \tag{2.3.2.1}$$

for all $u \in \mathcal{S}(X;\mathbb{C})$ when $\sum_{j=1}^J \theta_j^2(q) \equiv 1$ with $\theta_j \in \mathcal{C}_0^\infty(\Omega_j;[0,1])$. It is bounded from below by 1 for $C_g > 0$ large enough.
It suffices to apply Proposition 2.E.22 of Appendix 2.E.

b) For the identification of the two definitions of $\tilde{\mathcal{W}}^s(X;\mathbb{C})$ and the equivalence of the norms, it suffices to consider the case $s = k \in \mathbb{N}$, because all the other cases will follow by interpolation and duality.

We start from the Definition 2.1.2 and Proposition 2.3.2-**1)** which says that the $\tilde{\mathscr{W}}^k(X;\mathbb{C})$ norm of u is equivalent to

$$\max_{1\leq j\leq J}\|\varrho_j u\|_{\tilde{\mathscr{W}}^k}$$

with $\sum_{j=1}^J \varrho_j(q)\equiv 1$, $\varrho_j\in\mathscr{C}_0^\infty(\Omega_j;[0,1])$. Additionally the definition of $\tilde{\mathscr{W}}^k_{\Omega_j-\mathrm{comp}}(\Omega_j;\mathbb{C})$ ensures that it is independent of the choice of a coordinate system $(q^1,\ldots,q^d)$ with equivalent norm for two different choices. So let us work on $T^*\Omega=\Omega\times\mathbb{R}^d\subset\mathbb{R}^d_q\times\mathbb{R}^d_p$ and let us consider functions $u\in L^2_{\Omega-\mathrm{comp}}(T^*\Omega;\mathbb{C})$ with a Ω-support included in a fixed compact set $K\subset\subset\Omega$ (a neighborhood of $\mathrm{supp}\,\varrho$ with $\varrho=\varrho_j$ when $\Omega=\Omega_j$). Any vector field $T^H\in\mathscr{C}^\infty_{\Omega-\mathrm{comp}}(T^*\Omega^d;TX^H)$ (resp. $T^V\in\mathscr{C}^\infty_{\Omega-\mathrm{comp}}(T^*\Omega^d;TX^V)$) can be written

$$T^H=\sum_{i=1}^d t^i(q)e_i\quad\text{resp.}\quad T^V=\sum_{i=1}^d t_i(q)\hat{e}^i$$

with $\max_{1\leq i\leq d}\|t^i\|_{\mathscr{C}^k}\asymp\|T^H\|_{\mathscr{C}^k}$ (resp. $\max_{1\leq i\leq d}\|t_i\|_{\mathscr{C}^k}\asymp\|T^V\|_{\mathscr{C}^k}$).

We deduce

$$\frac{\|\langle p\rangle_q^{N_3}T_1^H\ldots T_{N_1}^H T_1^V\ldots T_{N_2}^V u\|_{L^2}^2}{\prod_{n_1=1}^{N_1}\|T_{n_1}^H\|_{\mathscr{C}^k}\prod_{n_2=1}^{N_2}\|T_{n_2}^V\|_{\mathscr{C}^k}}\leq C_{K,k}\max_{\substack{1\leq i_1,\ldots,i_{N_1}\leq d\\1\leq j_1,\ldots,j_{N_2}\leq d}}\|\langle p\rangle_q^{N_3}e_{i_1}\ldots e_{i_{N_1}}\hat{e}^{j_1}\ldots\hat{e}^{j_{N_2}}u\|_{L^2}$$

where

$$\|\langle p\rangle_q^{N_3}e_{i_1}\ldots e_{i_{N_1}}\hat{e}^{j_1}\ldots\hat{e}^{j_{N_2}}u\|_{L^2}=\|\langle p\rangle_q^{N_3}(\chi e_{i_1})\ldots(\chi e_{i_{N_1}})(\chi\hat{e}^{j_1})\ldots(\chi\hat{e}^{j_{N_2}})u\|_{L^2}$$

for some $\chi=\chi(q)\in\mathscr{C}_0^\infty(\Omega;[0,1])$ such that $\chi\equiv 1$ in a neighborhood of $K\supset\Omega-\mathrm{supp}\,u$. Because $\chi(q)e_i\in\mathscr{C}^\infty_{\Omega-\mathrm{comp}}(T^*\Omega;TX^H)$ and $\chi(q)\hat{e}^j\in\mathscr{C}^\infty_{\Omega-\mathrm{comp}}(T^*\Omega;TX^V)$ the right-hand side is bounded by $C_{K,\chi,k}\|u\|_{\tilde{\mathscr{W}}^k}$.

By taking the supremum with respect to $N_1+\frac{N_2+N_3}{2}\leq k$, $T_1^H,\ldots,T_{N_1}^H\in\mathscr{C}^\infty_{\Omega-\mathrm{comp}}(T^*\Omega;TX^H)$ and $T_1^V,\ldots,T_{N_2}^V\in\mathscr{C}^\infty_{\Omega-\mathrm{comp}}(T^*\Omega;TX^V)$. The norm $\|u\|_{\tilde{\mathscr{W}}^k}$ for $u\in L^2(T^*\Omega;\mathbb{C})$ with Ω-support included in K, is thus equivalent to

$$\max_{\substack{1\leq i_1,\ldots,i_{N_1}\leq d\\1\leq j_1,\ldots,j_{N_2}\leq d\\N_1+\frac{N_2+N_3}{2}\leq k}}\|\langle p\rangle_q^{N_3}e_{i_1}\ldots e_{i_{N_1}}\hat{e}^{j_1}\ldots\hat{e}^{j_{N_2}}u\|_{L^2}\asymp\max_{\substack{1\leq i_1,\ldots,i_{N_1}\leq d\\N_1+\frac{|\beta|+N_3}{2}\leq k}}\|\langle p\rangle^{N_3}e_{i_1}\ldots e_{i_{N_1}}\partial_p^\beta u\|_{L^2}\tag{2.3.2.2}$$

where we have replaced $\langle p\rangle_q=(1+g^{ij}(q)p_ip_j)^{1/2}$ by the equivalent quantity $\langle p\rangle=(1+\sum_i p_i^2)^{1/2}$.

From

$$[e_i,f^k(q)p_k]=(\partial_{q^i}f^k)(q)p_k+(f^k\Gamma_{ik}^\ell)(q)p_\ell\,,$$

we get by induction

$$e_{i_1}\ldots e_{i_{N_1}}\partial_p^\beta-\partial_{q^{i_1}}\ldots\partial_{q^{i_{N_1}}}\partial_p^\beta=\sum_{\substack{|\alpha|\leq N_1-1\\|\alpha|+\frac{|\gamma|+|\beta'|}{2}=N_1+\frac{|\beta|}{2}}}f_{\alpha,\beta',\gamma}(q)p^\gamma\partial_q^\alpha\partial_p^{\beta'}$$

and we deduce

$$\varepsilon^{N_1}\|\langle p\rangle^{N_3}e_{i_1}\ldots e_{i_{N_1}}\partial_p^\beta u\|_{L^2}-\varepsilon^{N_1}\|\langle p\rangle_q^{N_3}\partial_{q^{i_1}}\ldots\partial_{q^{i_{N_1}}}\partial_p^\beta u\|_{L^2}$$
$$\leq C_{K,k}\varepsilon\max_{\substack{|\alpha|\leq N_1-1\\|\alpha|+\frac{N_3'+|\beta'|}{2}\leq N_1+\frac{|\beta|}{2}}}\varepsilon^{N_1-1}\|\langle p\rangle^{N_3'}\partial_q^\alpha\partial_p^{\beta'}u\|_{L^2}\,.$$

Choosing $\varepsilon = \varepsilon_{K,k}$ for $\varepsilon_{K,k} > 0$ small enough implies that the norm $\|u\|_{\tilde{\mathscr{W}}^k}$ is equivalent to

$$\max_{|\alpha|+\frac{N_3+|\beta|}{2}\leq k} \|\langle p\rangle^{N_3}\partial_q^\alpha\partial_p^\beta\chi(q)u\|_{L^2} \quad \text{or} \quad \sqrt{\sum_{|\alpha|+\frac{N_3+|\beta|}{2}\leq k} \|\langle p\rangle^{N_3}\partial_q^\alpha\partial_p^\beta\chi(q)u\|_{L^2}^2}\,.$$

But according Appendix 2.E and in particular Proposition 2.E.7, it is equivalent to the norm $\|W^k\chi(q)u\|_{L^2}$. □

Let us extend now this result to $\tilde{\mathscr{W}}^s(Q;\mathscr{E})$. For a self-adjoint non negative scalar operator $A \in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$, like W^2 with $m = 2$ or W^s, $s = m \in \mathbb{R}$, it is not possible to define directly its action on sections of $\mathscr{E}$. However a localization technique makes it possible, up to lower order corrections.
We fix, as we did in Appendix 2.E, the atlas covering $Q = \bigcup_{j=1}^J \Omega_j$ by assuming that for every $j \in \{1,\ldots,J\}$ the two properties are satisfied:

— the open set $\tilde{\Omega}_j = \bigcup_{\Omega_{j'}\cap\Omega_j\neq\emptyset}\Omega_{j'}$ is a chart open set;

— the restricted vector bundle $E\big|_{\tilde{\Omega}_j}$ admits an orthonormal frame $(f_j^1,\ldots,f_j^N)$ for the metric g_E.

If $\sum_{j=1}^J\theta_j^2(q)\equiv 1$ is a quadratic partition of unity with $\theta_j \in \mathscr{C}_0^\infty(\Omega_j;[0,1])$ we set

$$A_\theta = \sum_{j=1}^J \theta_j(q)\circ A_{\mathrm{sc}\,j}\circ\theta_j(q)\,,$$

where $A_{\mathrm{sc},j}$ is the scalar pseudo-differential operator in the othonormal local frame $(f_j^1,\ldots,f_j^N)$ above $\tilde{\Omega}_j$:

$$A_{\mathrm{sc,j}}(u_kf_j^k(q))(q,p) = [Au_k](q,p)f_j^k(q)\,.$$

When $A = a_{\varrho,\chi}(x,D_x)+R = \sum_{j_1=1}^J(\varrho_{j_1}(q)a)(x,D_x)\circ\chi_{j_1}(q)+R \in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ we obtain

$$A_\theta = \sum_{j_1=1}^J\varrho_{j_1}(q)\sum_{\Omega_j\cap\Omega_{j_1}\neq\emptyset}\theta_j(q)\circ[a(x,D_x)]_{\mathrm{sc},j}\circ\theta_j(q)\circ\chi_{j_1}(q)+R_\theta\,.$$

If $U_{j_1,j_2}(q)$ is the unitary matrix of $(f_{j_1}^1,\ldots,f_{j_1}^N)$ in the frame $(f_{j_2}^1,\ldots,f_{j_2}^N)$, the operator

$$\sum_{\Omega_j\cap\Omega_{j_1}\neq\emptyset}\theta_j(q)\circ[a(x,D_x)]_{\mathrm{sc},j}\circ\theta_j(q)$$

with $\bigcup_{\Omega_j\cap\Omega_{j_1}\neq\emptyset}\Omega_j\subset\tilde{\Omega}_{j_1}\subset\mathbb{R}^d$, is nothing but the operator $\tilde{a}_{j_1}(x,D_x)$ with $a_{j_1}\in S(\Psi^m,g_\Psi;\mathbb{C}^N)$ given by

$$\sum_{\Omega_j\cap\Omega_{j_1}\neq\emptyset}U_{j_1,j}(q)\sharp([(\theta_j(q)a)\sharp\theta_j(q)]\otimes\mathrm{Id}_{\mathbb{C}^N})\sharp U_{j,j_1}(q)\,,$$

where we recall $(a\sharp b)(x,D_x) = a(x,D_x)\circ b(x,D_x)$.
Owing to the exact chain rules $U_{j_3,j_1}(q) = U_{j_3,j_2}\circ U_{j_2,j_1}(q)$ and $U_{j_3,j_1}(q) = U_{j_3,j_2}(q)\sharp U_{j_2,j_1}(q)$ and the exact commutation $\theta_{j_3}(q)\sharp U_{j_2,j_1}(q) = U_{j_2,j_1}(q)\sharp\theta_{j_3}(q)$, we can write

$$A_\theta = \sum_{j_1=1}^J(\varrho_{j_1}(q)a_\theta)(x,D_x)\circ\chi_{j_1}(q)+R_\theta = (a_\theta)_{\varrho,\chi}(x,D_x)+R_\theta$$

with $a_\theta\in S^m_\Psi(Q;\mathrm{End},\mathscr{E})$ and $R_\theta\in\mathscr{R}(Q;\mathscr{E})$.
Additionally, if $A = a_{m,\varrho,\chi}(x,D_x)+A_{m-1}$, $A_{m-1}\in\mathrm{OpS}^{m-1}_\Psi(Q;\mathbb{C})$, then $A_\theta = (a_m\otimes\mathrm{Id}_{\mathscr{E}})_{\varrho,\chi}(x,D_x)+A_{\theta,m-1}$ with $A_{\theta,m-1}\in\mathrm{OpS}^{m-1}_\Psi(Q;\mathrm{End}\mathscr{E})$. In particular the principal symbol does not depend on

the chosen orthonormal frames $(f_j^1,\dots,f_j^N)$ $j=1,\dots,J$.
If A is self-adjoint and elliptic, the same holds for A_θ with

$$\begin{aligned} D(A_\theta)=\tilde{\mathscr{W}}^m(X;\mathscr{E}) &= \left\{u\in\mathscr{S}'(X;\mathscr{E}),\forall j\in\{1,\dots,J\},\quad u\big|_{\Omega_j}=u_k(x)f_j^k(q),u_k\in\tilde{\mathscr{W}}^m_{\Omega_j-\mathrm{loc}}(T^*\Omega_j;\mathbb{C})\right\}\\ &= \left\{u\in\mathscr{S}'(X;\mathscr{E}),\forall B\in\mathrm{OpS}^m_\Psi(Q;\mathrm{End}\mathscr{E}),Bu\in L^2(X,dqdp;\mathscr{E})\right\}. \end{aligned}$$

We can conclude with the following summary.

Proposition 2.3.5. *Once the quadratic partition of unity $\sum_{j=1}^J\theta_j^2(q)\equiv 1$ and the local orthonormal frames $(f_j^1,\dots,f_j^N)$, $1\le j\le J$, are fixed and for $C_g\ge C_{g,\theta}$ chosen large enough, the operator*

$$W_\theta^2=\sum_{j=1}^J\theta_j(q)(W^2)_{\mathrm{sc},j}\theta_j(q)\quad\text{with}\quad D(W_\theta^2)=\{s\in L^2(X,dqdp;\mathscr{E}),W_\theta^2s\in L^2(X,dqdp;\mathscr{E})\}$$

is self-adjoint and bounded from below by 1.
For $s\in\mathbb{R}$, the space $\tilde{\mathscr{W}}^s(X;\mathscr{E})$ introduced in Definition 2.1.2 for $s=k\in\mathbb{N}$ and then extended by interpolation and duality, equals

$$\begin{aligned} \tilde{\mathscr{W}}^s(X;\mathscr{E}) &= \{u\in\mathscr{S}'(X;\mathscr{E}),\forall B\in\mathrm{OpS}^s_\Psi(Q;\mathrm{End}\mathscr{E}),Bu\in L^2(X,dqdp;\mathscr{E})\}\\ &= \left\{u\in\mathscr{S}'(X;\mathscr{E}),\quad (W_\theta^2)^{s/2}u\in L^2(X,dqdp;\mathscr{E})\right\}\\ &= \left\{u\in\mathscr{S}'(X;\mathscr{E}),\forall j\in\{1,\dots,J\},u\big|_{\Omega_j}=u_k(x)f_j^k(q),W^su_k\in L^2_{\Omega_j-\mathrm{loc}}(T^*\Omega_j,dqdp;\mathscr{E})\right\}\\ &= \left\{u\in\mathscr{S}'(X;\mathscr{E}),\quad (C_s+(W^{|s|})_\theta)^{\mathrm{sign}\,s}u\in L^2(X,dqdp;\mathscr{E})\right\}, \end{aligned}$$

where the constant $C_s>0$ is chosen large enough.

Proof. We already know that $W^2=C_g-\Delta_H+C_g\mathscr{O}^2$ is elliptic, self-adjoint and bounded from below by 1 for $C_g\ge 1$ large enough with domain $D(W^2)=\tilde{\mathscr{W}}^2(X;\mathbb{C})$.
With the previous discussion this proves that W_θ^2 is elliptic and self-adjoint. The same computation as (2.3.2.1) shows that $W_\theta^2\ge 1$: Actually the derivatives of the unitary matrix associated with the change of frames, $\partial_{q^i}U_{j_1,j_2}(q)$, bring lower order terms which are absorbed if $C_g=C_{g,\theta}$ is chosen large enough.
For W_θ^2 with a scalar principal symbol $a_2\otimes\mathrm{Id}_{\mathscr{E}}\ge\frac{1}{\kappa}\Psi^2\otimes\mathrm{Id}_{\mathscr{E}}$, Proposition 2.E.22 applies and $(W_\theta^2)^{s/2}=f_s(W_\theta^2)$ with $f_s\in S(\langle t\rangle^{s/2},\frac{dt}{\langle t\rangle^2})$ is elliptic with the principal symbol $f_s(a_2)\otimes\mathrm{Id}_{\mathscr{E}}=a_2^{s/2}\otimes\mathrm{Id}_{\mathscr{E}}$.
The local characterization with $u\big|_{\Omega_j}=u_k(x)f_j^k(q)$ has been explained and with the reduction of the previous paragraph and Proposition 2.3.4 it shows that $D((W_\theta^2)^{k/2})$ coincides with $\tilde{\mathscr{W}}^k(X;\mathscr{E})$ when $k\in\mathbb{N}$. This ends the identifications of the general spaces $\tilde{\mathscr{W}}^s(X;\mathscr{E})$ for $s\in\mathbb{R}$.
Because $W^{|s|}=(W^2)^{|s|/2}=f_{|s|}(W^2)$ is elliptic with the principal symbol $a_2^{|s|/2}$ for $s\neq 0$, $(W^{|s|})_\theta$ is elliptic with the principal symbol $a_2^{|s|/2}\otimes\mathrm{Id}_{\mathscr{E}}$. It is self-adjoint with the same domain, $\tilde{\mathscr{W}}^{|s|}(X;\mathscr{E})$, as $(W_\theta^2)^{|s|/2}$. It is bounded from below by Garding inequality. Adding a constant C_s ensures that $(C_s+(W^{|s|})_\theta)$ is bounded from below by 1 and invertible. □

2.3.3 Spaces $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E})$

A priori $\mathscr{Y}=g^{ij}(q)p_ie_j$ belongs to $\mathrm{OpS}^{3/2}_\Psi(Q;\mathbb{C})$ but it has locally some specific structure made of $e_i\in\mathrm{OpS}^1_\Psi(\Omega;\mathbb{C})$ and followed by a multiplication by p_i. We start with a simple commutation result.

Proposition 2.3.6. *The self-adjoint operator $(W_\theta^2,D(W_\theta^2)=\tilde{\mathscr{W}}^2(X;\mathscr{E}))$ modelled on $W^2=C_g-\Delta_H+C_g\mathscr{O}^2$ and introduced in Proposition 2.3.5 and the vertical harmonic oscillator $\mathscr{O}$ with the maximal domain $D(\mathscr{O})=\{u\in L^2(X,dqdp;\mathscr{E}),\quad \mathscr{O}u\in L^2(X,dqdp;\mathbb{C})\}$ make a pair of strongly commuting self-adjoint operators: For any Borel functions $f,g:\mathbb{R}\to\mathbb{C}$, $f(W_\theta^2)g(\mathscr{O})=g(\mathscr{O})f(W_\theta^2)$ on the intersection of their domain.*

Proof. The space $L^2(X,dqdp;\mathscr{E})$ is isomorphic to the direct integral $\int_Q^{\oplus} L^2(\mathbb{R}^d,dp)d\mathrm{vol}_g(q)$ after the pointwise gauge transformation $(q,p)\mapsto(q,g(q)^{-1/2}.p)$. In this direct integral decomposition the operator $\mathscr{O}$ is nothing but $\int_Q^{\oplus} O\ d\mathrm{vol}_g(q)$ where $O=\sum_{j=1}^d \frac{-\partial_{p_j}^2+p_j^2}{2}$ is the euclidean harmonic oscillator.
The associated unitary group e^{itO} satisfies $e^{itO}\partial_{q^i}e^{-itO}=\partial_{q^i}$, $e^{itO}p_ie^{-itO}=\cos(t)p_i-\sin(t)D_{p_i}$ and $e^{itO}D_{p_i}e^{-itO}=\sin(t)p_i+\cos(t)D_{p_i}$. We deduce that for any $t\in\mathbb{R}$, e^{itO} is continuous from $\tilde{\mathscr{W}}^2(X;\mathbb{C})$ into itself, and therefore as a scalar operator from $\tilde{\mathscr{W}}^2(X;\mathscr{E})$ into itself.
Because the unitary transform $U_\Phi:L^2(X;\mathscr{E})\to\int_Q^{\oplus}L^2(\mathbb{R}^d)\,d\mathrm{vol}_g(q)$ given by $(U_\Phi u)(q,p')=u(q,g(q)^{1/2}.p)$ is a special case of Proposition 2.E.13 (it suffices to consider locally the effect on the scalar components). It is an isomorphism of $\mathscr{W}^2(X;\mathscr{E})=D(W_\theta^2)$ and $e^{it\mathscr{O}}$ is continuous from $D(W_\theta^2)$ into itself for any $t\in\mathbb{R}$.
Because the scalar operator W^2 and $\mathscr{O}$ commute on $\mathscr{S}(X;\mathbb{C})$ we deduce that $[W_\theta^2,\mathscr{O}]=0$ on $\mathscr{S}(X;\mathscr{E})$ which is a core for W_θ^2.
We have all the ingredients of [ABG] in order to conclude that

$$\mathrm{ad}_{\mathscr{O}}W_\theta^2=i\frac{d}{dt}e^{-it\mathscr{O}}W_\theta^2e^{it\mathscr{O}}\big|_{D(W_\theta^2)}=0\,,$$

and W_θ^2 and $\mathscr{O}$ strongly commute. □

This leads to the introduction of the following, double indexed, spaces.

Definition 2.3.7. For any $s_1,s_2\in\mathbb{R}$, the space $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E})$ is the space associated with the functional calculus of the two commuting self-adjoint operators $\mathscr{O}$ and W_θ^2 and endowed with the Hilbert norm

$$\|u\|_{\tilde{\mathscr{W}}^{s_1,s_2}}=\|\mathscr{O}^{s_1/2}(W_\theta^2)^{s_2/2}u\|_{L^2}\,.$$

In particular the space $\tilde{\mathscr{W}}^{1,s}(T^*Q;\mathscr{E})$ of Definition 2.1.2 with the norm

$$\|u\|_{\tilde{\mathscr{W}}^{1,s}}=\|\mathscr{O}^{1/2}u\|_{\tilde{\mathscr{W}}^s}=\|(W_\theta^2)^{s/2}\mathscr{O}^{1/2}u\|_{L^2}$$

is the particular case $s_1=1$, $s_2=s$. Clearly the spaces $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E})$ contain a finer description of the regularity properties. With

$$\|u\|^2_{\tilde{\mathscr{W}}^{s_1,s_2}}=\langle(W_\theta^2)^{s_2/2}\mathscr{O}^{(s_1-1)/2}u\,,\mathscr{O}(W_\theta^2)^{s_2/2}\mathscr{O}^{(s_1-1)/2}u\rangle_{L^2}\leq\|(W_\theta^2)^{(s_2+1/2)/2}\mathscr{O}^{(s_1-1)/2}u\|^2_{L^2}=\|u\|^2_{\tilde{\mathscr{W}}^{s_2+1/2,s_1-1}}\,.$$

for $s_1\geq1$, we deduce the continuous embeddings $\tilde{\mathscr{W}}^{0,s_2+s_1/2}(X;\mathscr{E})\subset\mathscr{W}^{s_1,s_2}(X;\mathscr{E})$ for $s_1\geq0$ and by duality $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E})\subset\tilde{\mathscr{W}}^{s_1,s_2+s_1/2}(X;\mathscr{E})$ for $s_1<0$. We will essentially work with $s_1\in\{0,1\}$.
As a first order differential operator with respect to q the operator $\nabla^{\mathscr{E}}_{\mathscr{Y}}$ can be written

$$\nabla^{\mathscr{E}}_{\mathscr{Y}}=\sum_{j=1}^J\theta_j(q)\nabla^{\mathscr{E}}_{\mathscr{Y}}\theta_j(q)=\sum_{j=1}^J\theta_j(q)[g^{ii'}(q)p_{i'}\nabla_{e_{i'}}\big|_{T^*\tilde{\Omega}_j}]\theta_j(q),$$

where $g^{ii'}(q)p_i\nabla^{\mathscr{E}}_{e_{i'}}\big|_{\mathscr{T}^*\tilde{\Omega}_j}$ is expressed with the local coordinates in $\tilde{\Omega}_j$.
With the cut-off function $\tilde{\chi}_j\in\mathscr{C}_0^\infty(\tilde{\Omega}_j;[0,1])$ such that $\tilde{\chi}_j\equiv1$ in a neighborhood of $\mathrm{supp}\,\theta_{j'}(q)$ when $\Omega_j\cap\Omega_{j'}\neq\emptyset$, we can introduce the local scalar operator

$$\hat{p}_{j,i}=\tilde{\chi}_j(q)p_i\otimes\mathrm{Id}_{\mathscr{E}}\quad,\quad\hat{D}_{j,i}=\tilde{\chi}_j(q)D_{p_i}\,,\qquad(2.3.3.1)$$

$$\text{while}\qquad\hat{E}_{j,i}=\theta_j(q)g^{ii'}(q)\nabla^{\mathscr{E}}_{e_{i'}}\theta_j(q)\in\mathrm{OpS}^1_\Psi(Q;\mathrm{End}\,\mathscr{E})\qquad(2.3.3.2)$$

$$\text{with}\qquad\hat{E}_{j,i}-(\theta_j(q)g^{ii'}(q)e_{i'}\theta_j(q)\otimes\mathrm{Id}_{\mathscr{E}})\in\mathrm{OpS}^0_\Psi(Q;\mathrm{End}\,\mathscr{E})\,.\qquad(2.3.3.3)$$

We have in particular

$$\nabla^{\mathscr{E}}_{\mathscr{Y}}=\sum_{j=1}^J\sum_{i=1}^d\hat{E}_{j,i}\circ\hat{p}_{j,i}\,.$$

Proposition 2.3.8. *Let $\hat{p}_{j,i}$, $\hat{D}_{j,i}$ and $\hat{E}_{j,i}$, $j \in \{1,\dots,J\}$, $i \in \{1,\dots,d\}$, be the operators defined by* (2.3.3.1) *and* (2.3.3.2). *For any $s \in \mathbb{R}$ we have the estimates:*

- $\|\hat{p}_{j,i}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,s};\tilde{\mathcal{W}}^{0,s})} + \|\hat{D}_{j,i}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,s};\tilde{\mathcal{W}}^{0,s})} \leq C_{g,s}$;
- $\|(W_\theta^2)^{s/2}\hat{p}_{j,i}(W_\theta^2)^{-s/2} - \hat{p}_{j,i}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};\tilde{\mathcal{W}}^{0,1})} + \|(W_\theta^2)^{s/2}\hat{D}_{j,i}(W_\theta^2)^{-s/2} - \hat{D}_{j,i}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};\tilde{\mathcal{W}}^{0,1})} \leq C_{g,s}$;
- $\|(W_\theta^2)^{s/2}\nabla^{\mathcal{E}}_{\mathcal{Y}}(W_\theta^2)^{-s/2} - \nabla^{\mathcal{E}}_{\mathcal{Y}}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};L^2)} \leq C_{g,s}$.

Proof. All the operators and commutators are well defined continuous operators onn the space of smooth rapidly decaying (w.r.t p) sections, $\mathcal{S}(X;\mathcal{E})$. The estimates are then extended by density. For $A = \hat{p}_{j,i}$ or $\hat{D}_{j,i}$ we know $A \in \mathrm{OpS}^{1/2}_{\Psi}(Q;\mathcal{E})$ while A and $(W_\theta^2)^{\pm s} \in \mathrm{OpS}^{\pm s}_{\Psi}(Q;\mathrm{End}\,\mathcal{E})$ have scalar principal symbols. We deduce

$$(W_\theta^2)^s A (W_\theta^2)^{-s} - A \in \mathrm{OpS}^{-1/2}_{\Psi}(Q;\mathrm{End}\,\mathcal{E}) \subset \mathcal{L}(\tilde{\mathcal{W}}^{1,0}(X;\mathcal{E});L^2(X;\mathcal{E})),$$

and $\|A\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,s};\tilde{\mathcal{W}}^{0,s})} = \|(W_\theta^2)^{s/2}A(W_\theta^2)^{-s/2}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};L^2)} \leq C_{g,s}$.
For the second estimate we need a more accurate decomposition of $(W_\theta^2)^{s/2}A(W_\theta^2)^{-s/2} - A$. Let us write $A = a(q,p,D_p)$ with the local coordinate writing, $a(q,p,\eta) = \tilde{\chi}_j(q)p_i$ when $A = \hat{p}_{j,i}$ and $a(q,p,\eta) = \tilde{\chi}_j(q)\eta_i$ when $A = \hat{D}_{j,i}$, and let $w(q,p,\xi,\eta) = (C_g + |\xi - \Gamma^k_{..}p_k\eta|_g^2 + C_g/4(|p|_g^2 + |\eta|_g^2)^2)^{1/2}$ be the principal scalar symbol of W_θ^2. If we forget the tensor product with $\mathrm{Id}_{\mathcal{E}}$, we have

$$(W_\theta^2)^{\pm s} - w^{\pm s}(q,p,D_q,D_p) = R^{\pm s-1} \in \mathrm{OpS}^{\pm s-1}_{\Psi}(Q;\mathrm{End}\,\mathcal{E})$$

and

$$(W_\theta^2)^s A (W_\theta^2)^{-s} - A = \underbrace{w^s(q,p,D_q,D_p)\circ A \circ w^{-s}(q,p,D_q,D_p) - A}_{=A_{1,s}} + A_{2,s} + R_s$$

with $\quad A_{2,s} = (W_\theta^2)^{s/2}R^{-s-1}A + R^{s-1}w^{-1}(q,p,D_q,D_p)A \in \mathcal{L}(\tilde{\mathcal{W}}^{1,0};\tilde{\mathcal{W}}^{0,1})$,
and $\quad R_s = (W_\theta^2)^{s/2}[A,R^{-s-1}] + R^{s-1}[A,w^{-s}(q,p,D_q,D_p)] \in \mathrm{OpS}^{-3/2}_{\Psi}(Q;\mathrm{End}\,\mathcal{E}) \subset \mathcal{L}(\tilde{\mathcal{W}}^{1,0};\tilde{\mathcal{W}}^{0,1})$.

By pseudo-differential calculus the symbol of $iA_{1,s}$ equals

$$\begin{aligned} &w^s\partial_\eta a.\partial_p(w^{-s}) - w^s\partial_p a.\partial_\eta(w^{-s}) - w^s\partial_q a.\partial_\xi w^{-s} + r_s \\ &\qquad = \frac{s}{2}\sum_{k=1}^{d} -(w^{-1}\partial_{\eta_k}a)\sharp(w^{-1}\partial_p w^2) + (w^{-1}\partial_{p_k}a)\sharp(w^{-1}\partial_{\eta_k}w^2) + w^s\partial_\xi w^{-s}\sharp\partial_q a + r'_s \end{aligned}$$

with $r_s, r'_s \in S^{-3/2}_{\Psi}(Q;\mathbb{C})$.
An explicit computation shows that the operators $(w^{-1}\partial_{p_k}w^2)(q,p,D_q,D_p)$, $(w^{-1}\partial_{p_k}w^2)(q,p,D_q,D_p)$ and $\partial_q a(q,p,D_p)$ belong to $\mathcal{L}(\tilde{\mathcal{W}}^{1,0}(X;\mathcal{E});L^2(X;\mathcal{E}))$.
The operators $(w^{-1}\partial_{\eta_k}a)(q,p,D_q,D_p)$, $(w^{-1}\partial_{p_k}a)(q,p,D_q,D_p)$ belong to $\mathcal{L}(L^2(X;\mathcal{E});\tilde{\mathcal{W}}^{0,1}(X;\mathcal{E}))$.
Finally the remainder $r'_s(q,p,D_q,D_p) \in \mathrm{OpS}^{-3/2}_{\Psi}(Q;\mathbb{C}) \subset \mathcal{L}(\tilde{\mathcal{W}}^{1,0}(X;\mathcal{E});\tilde{\mathcal{W}}^{0,1}(X;\mathcal{E}))$.
This ends the proof of

$$\|(W_\theta^2)^s A (W_\theta^2)^{-s} - A\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};\tilde{\mathcal{W}}^{0,1})} \leq C_{g,s} \quad \text{for } A = \hat{p}_{j,k} \text{ or } \hat{D}_{j,i}.$$

We split $(W_\theta^2)\nabla^{\mathcal{E}}_{\mathcal{Y}}(W_\theta^2)^{-s} - \nabla^{\mathcal{E}}_{\mathcal{Y}}$ into

$$\begin{aligned} \sum_{j=1}^{J}(W_\theta^2)^s\hat{E}_{j,i}\hat{p}_{j,i}(W_\theta^2)^{-s} - \hat{E}_{j,i}\hat{p}_{j,i} = \sum_{j=2}^{J}&\left[(W_\theta^2)^s\hat{E}_{j,i}(W_\theta^2)^{-s} - \hat{E}_{j,i}\right]\circ(W_\theta^2)^s\hat{p}_{j,i}(W_\theta^2)^{-s} \\ &+ \hat{E}_{j,i}\circ\left[(W_\theta^2)^s\hat{p}_{j,i}(W_\theta^2)^{-s} - \hat{p}_{j,i}\right] \end{aligned}$$

The factor $\left[(W_\theta^2)^s\hat{E}_{j,i}(W_\theta^2)^{-s} - \hat{E}_{j,i}\right]$ belongs to $\mathrm{OpS}^0_{\Psi}(Q;\mathrm{End}\,\mathcal{E}) \subset \mathcal{L}(L^2(X;\mathcal{E});L^2(X;\mathcal{E}))$ while the operator $(W_\theta^2)^s\hat{p}_{j,i}(W_\theta^2)^{-s}$ belongs to $\mathcal{L}(\tilde{\mathcal{W}}^{1,0}(X;\mathcal{E});L^2(X;\mathcal{E}))$.
The operator $(W_\theta^2)^s\hat{p}_{j,i}(W_\theta^2)^{-s} - \hat{p}_{j,i}$ belongs to $\mathcal{L}(\tilde{\mathcal{W}}^{1,0}(X;\mathcal{E});\tilde{\mathcal{W}}^{0,1}(X;\mathcal{E}))$ while the factor $\hat{E}_{j,i}$ belongs to $\mathrm{OpS}^1_{\Psi}(Q;\mathrm{End}\,\mathcal{E}) \subset \mathcal{L}(\tilde{\mathcal{W}}^{0,1}(X;\mathcal{E});L^2(X;\mathcal{E}))$. □

2.4 A priori estimates on the scalar GKFP operator

In this section we work directly with the localized scalar version of GKFP operators. The results of this section will then applied to the operators $P^j_{\pm,b}$'s of Subsection 2.2.3. From now on, we focus the analysis to the case $\pm = +$, because the other case $\pm = -$ is the same, and we write simply P^j_b and all the forthcoming related operators without the $\pm$ index.
The chart coordinates open set Ω in Q is fixed and any coordinate system allows the identification $T^*\Omega = \Omega\times\mathbb{R}^d \subset \mathbb{R}^{2d}_{q,p}$. The symplectic volume on $\Omega\times\mathbb{R}^d$, is the usual Lebesgue measure $dqdp$ and the corresponding $L^2(\Omega\times\mathbb{R}^d, dqdp;\mathbb{C})$-norm will be denoted simply by $\|\ \|_{L^2}$. We consider a scalar GKFP operator

$$\mathscr{P}_b = \frac{1}{b^2}\mathscr{O} + \frac{1}{b}\mathscr{Y},\quad b\in(0,\infty),$$

$$\text{with}\qquad \mathscr{Y} = g^{ij}(q)p_j e_i \quad,\quad \mathscr{O} = \frac{-g_{ij}(q)\partial_{p_i}\partial_{p_j} + g^{ij}(q)p_ip_j}{2},$$

with the domain

$$D(\mathscr{P}_b) = C_0^\infty(\Omega\times\mathbb{R}^d;\mathbb{C}). \tag{2.4.0.1}$$

By assuming $\Omega \subset\subset \Omega_1$ where Ω_1 is a bigger chart coordinates open subset of Q, we can assume $g\big|_{\Omega_{1/2}} = \tilde{g}\big|_{\Omega_{1/2}}$ where $\tilde{g}$ is a riemannian metric on $\mathbb{R}^d$ which is euclidean outside a compact set, and $\Omega_{1/2}$ is an open neighborhood of Ω such that $\Omega\subset\subset\Omega_{1/2}\subset\subset\Omega_1$.
Alternatively the local scalar GKFP operators $\mathscr{P}_b$ can be introduced directly on $\Omega\times\mathbb{R}^d\subset\mathbb{R}^{2d}$ with a metric g which is a compactly supported perturbation of the euclidean metric.

2.4.1 Dyadic partition of unity

By following [Leb1][Leb2] or [BCD], let $\theta,\tilde{\theta}\in C_0^\infty(\mathbb{R})$ be such that $\mathrm{supp}(\theta)\subset\left[\frac{1}{4},4\right]$, $\mathrm{supp}\left(\tilde{\theta}\right)\subset [0,4]$, and

$$\forall t\in[0,\infty),\qquad \tilde{\theta}^2(4t^2) + \sum_{\ell=0}^{\infty}\theta^2\left(2^{-2\ell}t^2\right) = 1. \tag{2.4.1.1}$$

For $x\in T^*\Omega$ and $\ell\in\mathbb{N}\cup\{-1\}$, set

$$\theta_\ell(x) = \begin{cases}\theta\left(2^{-2\ell}|p|_q^2\right), & \ell > -1,\\ \tilde{\theta}(4|p|_q^2), & \ell = -1.\end{cases} \tag{2.4.1.2}$$

The collection $\{\theta_\ell\}_{\ell=-1}^{\infty}$ constitutes a quadratic dyadic partition of unity for $T^*\Omega$ in the sense that

$$\forall x\in T^*\Omega,\quad \sum_{\ell=-1}^{\infty}\theta_\ell^2(x) = 1, \tag{2.4.1.3}$$

with

$$\mathrm{supp}(\theta_\ell)\subset\{2^{\ell-1}\le|p|_q\le 2^{\ell+1}\}\quad\text{whenever}\quad \ell>-1, \tag{2.4.1.4}$$

and

$$\mathrm{supp}(\theta_{-1})\subset\{0\le|p|_q\le 1\},\ \text{and}\ \theta_{-1}(x) = 1 \text{ for } 0\le|p|_q\le\frac{1}{2}. \tag{2.4.1.5}$$

Notice that because θ_ℓ is a function of $|p|_q^2$, θ_ℓ satisfies

$$\forall\ell\in\mathbb{N}\cup\{-1\},\quad \mathscr{Y}\theta_\ell\equiv 0. \tag{2.4.1.6}$$

When (q,p) are canonical coordinates on $T^*\Omega = \Omega\times\mathbb{R}^d$, we also observe

$$\forall\alpha\in\mathbb{N}^d,\ \exists C_\alpha>0,\ \forall\ell\in\mathbb{N}\cup\{-1\},\quad \sup_{x\in T^*\Omega}|\partial_p^\alpha\theta_\ell(x)|\le C_\alpha 2^{-|\alpha|\ell}. \tag{2.4.1.7}$$

Proposition 2.4.1. *There exists a constant $C_{g,\theta,\tilde{\theta}} \geq 1$ depending only on the metric g and the functions θ and $\tilde{\theta}$ so that*

$$\frac{1}{4}\sum_{\ell}\left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b}\right)\theta_\ell u\right\|_{L^2}^2 \leq \left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b}\right)u\right\|_{L^2}^2 \leq \frac{5}{2}\sum_{\ell}\left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b}\right)\theta_\ell u\right\|_{L^2}^2 \quad (2.4.1.8)$$

holds for all $u \in \mathscr{C}_0^\infty(\Omega\times\mathbb{R}^d;\mathbb{C})$ and all $(\lambda,b)\in\mathbb{R}\times\mathbb{R}_+$ when $\kappa_b = C_{g,\theta,\tilde{\theta}}(1+b^2)$.

Proof. Thanks to (2.4.1.6), we have the commutator identities

$$\left[\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b},\theta_{\ell_1}\right] = \frac{1}{b^2}[\mathscr{O},\theta_{\ell_1}] = -\frac{1}{b^2}g_{ij}(q)\frac{\partial\theta_{\ell_1}}{\partial p_i}\frac{\partial}{\partial p_j} - \frac{1}{2b^2}g_{ij}(q)\frac{\partial^2\theta_{\ell_1}}{\partial p_i\partial p_j}, \quad (2.4.1.9)$$

and

$$\left[\left[\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b},\theta_{\ell_1}\right],\theta_{\ell_2}\right] = \frac{1}{b^2}[[\mathscr{O},\theta_{\ell_1}],\theta_{\ell_2}] = -\frac{1}{b^2}g_{ij}(q)\frac{\partial\theta_{\ell_1}}{\partial p_i}\frac{\partial\theta_{\ell_2}}{\partial p_j}. \quad (2.4.1.10)$$

for any $\ell_1,\ell_2 \in \mathbb{N}\cup\{-1\}$. From (2.4.1.7), (2.4.1.9), (2.4.1.10) and the integration by parts inequality of Proposition 2.2.1, we deduce that there is a constant $C'_{g,\theta,\tilde{\theta}} \geq 1$, depending only on the metric g and the functions θ and $\tilde{\theta}$ such that $\kappa_b = C_{g,\theta,\tilde{\theta}}(1+b^2)$, with $C_{g,\theta,\tilde{\theta}} = C_0 + 32C'_{g,\theta,\tilde{\theta}}$ and $C_0 \geq 1$ fixed in Proposition 2.2.1, implies

$$\begin{aligned}\sum_{\ell,\ell_1}\left\|\left[\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b},\theta_{\ell_1}\right]\theta_\ell u\right\|_{L^2}^2 &\leq C'_{g,\theta,\tilde{\theta}}\left(\sum_\ell\left(\frac{1}{b^4}\|D_p\theta_\ell u\|_{L^2}^2+\frac{1}{b^4}\|\theta_\ell u\|_{L^2}^2\right)\right)\\ &\leq C'_{g,\theta,\tilde{\theta}}\left(\frac{1}{\kappa_b}+\frac{1}{\kappa_b^2}\right)\sum_\ell\left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b}\right)\theta_\ell u\right\|_{L^2}^2\end{aligned} \quad (2.4.1.11)$$

and

$$\sum_{\ell,\ell_1,\ell_2}\left\|\left[\left[\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b},\theta_{\ell_1}\right],\theta_{\ell_2}\right]\theta_l\right\|_{L^2}^2 \leq \frac{C'_{g,\theta,\tilde{\theta}}}{b^4}\|u\|_{L^2}^2 \leq \frac{C'_{g,\theta,\tilde{\theta}}}{\kappa_b^2}\sum_\ell\left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b}\right)\theta_\ell u\right\|_{L^2}^2 \quad (2.4.1.12)$$

for all $\lambda\in\mathbb{R}$, $b>0$, and $u \in C_0^\infty(T^*\Omega;\mathbb{C})$. The equivalence (2.4.1.8) then follows from Corollary 2.C.2 with $r=\frac{1}{2}$. □

For every $\ell \geq -1$, we define the change of variable

$$\begin{array}{cccc}\Phi_\ell: & \Omega\times\mathbb{R}^d & \to & \Omega\times\mathbb{R}^d \\ & (q,p) & \mapsto & (q,2^\ell p)\end{array}.$$

The change of variable in the integral give

$$\|(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b})\theta_\ell u\|_{L^2} = \|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u_\ell\|_{L^2} \quad (2.4.1.13)$$

with $u_\ell(q,p) = 2^{\frac{\ell d}{2}}\theta(|p|_q^2)u(q,2^\ell p)$ (for $\ell=-1$ replace θ by $\tilde{\theta}$) and $\mathscr{P}_{b,\ell} = \Phi_\ell^*\mathscr{P}_b(\Phi_\ell^{-1})^*$. After the change of variable, operators are changed by

$$\mathscr{P}_{b,\ell} = \frac{1}{b^2}\mathscr{O}_\ell + \frac{1}{b}\mathscr{Y}_\ell, \quad (2.4.1.14)$$

$$\mathscr{O}_\ell = \Phi_\ell^*\mathscr{O}(\Phi_\ell^{-1})^* = \frac{1}{2}(2^{-2\ell}g_{ij}(q)D_{p_i}D_{p_j} + 2^{2\ell}g^{ij}(q)p_ip_j) \quad (2.4.1.15)$$

$$\text{and}\quad \mathscr{Y}_\ell = \Phi_\ell^*\mathscr{Y}(\Phi_\ell^{-1})^* = 2^\ell g^{ij}(q)p_j(\frac{\partial}{\partial q^i} + \Gamma_{ik}^m(q)p_m\frac{\partial}{\partial p_k}). \quad (2.4.1.16)$$

The equivalence (2.4.1.8) can be rewritten as

$$\forall u \in \mathscr{C}_0^\infty(\Omega\times\mathbb{R}^d;\mathbb{C}),\quad \frac{1}{4}\sum_{\ell}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u_\ell\|_{L^2}^2 \leq \|(\frac{\kappa_b}{b^2}+\mathscr{P}_b-\frac{i\lambda}{b})u\|_{L^2}^2 \leq \frac{5}{2}\sum_{\ell}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u_\ell\|_{L^2}^2, \tag{2.4.1.17}$$

where now $u_\ell \in \mathscr{C}_0^\infty(S_{\ell,2};\mathbb{C})$ with

$$S_{\ell,R} = \begin{cases} \{x=(q,p)\in\Omega\times\mathbb{R}^d\,,\frac{1}{R}<|p|_q<R\} & \text{for } \ell\geq 0\\ \{x=(q,p)\in\Omega\times\mathbb{R}^d\,,|p|_q<1\} & \text{for } \ell=-1,\end{cases} \tag{2.4.1.18}$$

for any fixed $R>1$.

2.4.2 Partition with a 2^ℓ-dependent grid in the open set $\Omega\subset\mathbb{R}^d_q$

Here the integer $\ell\geq -1$ is fixed and the localization will be done with some translation invariance in $\mathbb{R}^d$ by using a regular grid with a spacing of size $A2^{-\ell}$.
Let us start with the translation invariant partition of unity

$$\sum_{m\in\mathbb{Z}^d}\psi^2(q-m)\equiv 1 \quad \text{with } \psi\in\mathscr{C}_0^\infty(\mathbb{R}^d;[0,1]). \tag{2.4.2.1}$$

For any $A>0$ and $\ell\in\mathbb{Z}$, $\ell\geq -1$, it can be written

$$\sum_{m\in\mathbb{Z}^d}\psi^2\left(\frac{q-q_{m,\ell,A}}{A2^{-\ell}}\right)\equiv 1 \quad \text{with } q_{m,\ell,A}=A2^{-\ell}m.$$

Accordingly we set

$$\psi_{m,\ell,A}(q)=\psi\left(\frac{q-q_{m,\ell,A}}{A2^{-\ell}}\right)$$

and we get

$$\mathrm{Supp}(\psi_{m,\ell,A}) = q_{m,\ell,A}+A2^{-\ell}\mathrm{Supp}(\psi), \tag{2.4.2.2}$$

$$\forall\alpha,\beta\in\mathbb{N}^d,\quad |\alpha|>0\Longrightarrow D_p^\alpha D_q^\beta\psi_{m,\ell,A}=0, \tag{2.4.2.3}$$

$$\forall\beta\in\mathbb{N}^d,\exists C_{\beta,\psi}>0,\quad |A^{|\beta|}2^{-\ell|\beta|}D_q^\beta\psi_{m,\ell,A}|\leq C_{\beta,\psi}. \tag{2.4.2.4}$$

Proposition 2.4.2. *Let $\mathscr{P}_{b,\ell}$ and $S_{\ell,2}$ be defined respectively by* (2.4.1.14) *and* (2.4.1.18) *for $\ell\in\mathbb{Z}$, $\ell\geq -1$. There exists a constant $C_{g,\psi}>0$ depending only on the metric g and the function ψ such that*

$$\frac{1}{2}\sum_{m\in\mathbb{Z}^d}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})\psi_{m,\ell,A}u\|_{L^2}^2-\frac{C_{g,\psi}}{A^2b^2}\left\|2^{2\ell}\psi_{m,\ell,A}u\right\|_{L^2}^2\leq\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|_{L^2}^2$$

and

$$\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|_{L^2}^2\leq 2\sum_{m\in\mathbb{Z}^d}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})\psi_{m,\ell,A}u\|_{L^2}^2+\frac{C_{g,\psi}}{A^2b^2}\left\|2^{2\ell}\psi_{m,\ell,A}u\right\|_{L^2}^2$$

holds for all $u\in\mathscr{C}_0^\infty(S_{\ell,2};\mathbb{C})$ and for all $(\lambda,b)\in\mathbb{R}\times(0,+\infty)$.

Proof. The computation of commutators gives

$$[\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b},\psi_{m_1,\ell,A}]=\frac{1}{b}[\mathscr{Y}_\ell,\psi_{m_1,\ell,A}]=\frac{1}{b}g^{ij}(q)2^\ell p_j\frac{\partial\psi_{m_1,\ell,A}}{\partial q^i}(q), \tag{2.4.2.5}$$

$$[[\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b},\psi_{\ell,m_1,A}],\psi_{m_2,\ell,A}]=\frac{1}{b}[[\mathscr{Y}_\ell,\psi_{m_1,\ell,A}],\psi_{m_2,\ell,A}]=0. \tag{2.4.2.6}$$

Because $\operatorname{supp} u \subset S_{\ell,2} \subset \{x = (q,p), |p|_q \leq 2\}$, the estimate (2.4.2.4) of the derivatives of $\psi_{m,\ell,A}$ implies that the right-hand side of (2.4.2.5) satisfies

$$\sum_{m_1 \in \mathbb{Z}^d} \left\| \frac{1}{b} g^{ij}(q) 2^{\ell} p_j \frac{\partial \psi_{m_1,\ell,A}}{\partial q^i}(q) \psi_{m_2,\ell,A} u \right\|_{L^2}^2 \leq \frac{\tilde{C}_{g,\psi} 2^{4\ell}}{(Ab)^2} \left\| \psi_{m_2,\ell,A} u \right\|_{L^2}^2$$

for all $m_1, m_2 \in \mathbb{Z}^d$ with a constant $\tilde{C}_{g,\psi} > 0$ depending only on g and ψ. Because the double commutator vanishes, we apply the formulas (2.C.0.2) and (2.C.0.3). This yields the result with the constant $C_{g,\psi} = 4\tilde{C}_{g,\psi}^2$. □

Since

$$\{x = (q,p) \in S_{\ell,2}, q \in \operatorname{supp}(\psi_{m,\ell,A})\} \subset (B(q_{m,\ell,A}, \hat{C}_{g,\psi} A 2^{-\ell}) \times \mathbb{R}^d) \cap S_{\ell,2} \tag{2.4.2.7}$$

when $u \in C_0^\infty(S_{\ell,2};\mathbb{C})$, the problem is thus reduced to finding lower bounds for $\left\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})v\right\|_{L^2}^2$ when $v \in C_0^\infty\left(B\left(q_{m,\ell,A}, \hat{C}_{g,\psi} A 2^{-\ell}\right) \times \mathbb{R}^d \cap S_{\ell,2}\right)$. When $A2^{-\ell}$ is small enough, (2.4.2.7) is contained in a q-ball with radius below the injectivity radius of the metric $\tilde{g}$ on $\mathbb{R}^d$, and normal coordinates around $q_{m,\ell,A}$ can be used. Note also that the ball $B(q_{m,\ell,A}, \hat{C}_{g,\psi} A 2^{-\ell})$ can be equivalently taken for the euclidean metric or the metric $\tilde{g}$ by possibly adapting the constant $\hat{C}_{g,\psi}$.

2.4.3 Use of normal coordinates

Due to (2.4.2.7), we are interested in $\left(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b}\right) v$ when $v \in C_0^\infty(B(q_{m,\ell,A}, \hat{C}_{g,\psi} A 2^{-\ell}) \times \mathbb{R}^d) \cap S_{\ell,2}$. For $2^{-\ell} A \leq \varepsilon_{g,\psi}$ with $\varepsilon_{g,\psi}$ small enough determined by the pair (g,ψ), we may introduce normal coordinates $\tilde{q} = (\tilde{q}^1, \ldots, \tilde{q}^d)$ centered at $q_{m,\ell,A}$. The associated canonical coordinates on $B(0, \hat{C}_{g,\psi} A 2^{-\ell}) \times \mathbb{R}^d \subset T^*\mathbb{R}^d$ will be denoted by $(\tilde{q}, \tilde{p})$ with $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_d)$. Since $\mathscr{P}_{b,\ell}$ maintains the same form (2.4.1.14) under the coordinate change $(q,p) \mapsto (\tilde{q}, \tilde{p})$, we may assume without loss of generality when considering

$$\left\| \left(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b} \right) v \right\|_{L^2} \tag{2.4.3.1}$$

for $2^{-\ell} A \leq \varepsilon_{g,\psi}$ and $v \in C_0^\infty(B(q_{m,\ell,A}, \hat{C}_{g,\psi} A 2^{-\ell}) \times \mathbb{R}^d \cap S_{\ell,2};\mathbb{C})$ that $q_{m,\ell,A} = 0$ and that the metric g satisfies

$$\forall \alpha \in \mathbb{N}^d, \ \ \partial_q^\alpha \left(g_{ij}(q) - \delta_{ij}\right) = \mathscr{O}(|q|^{(2-|\alpha|)_+}). \tag{2.4.3.2}$$

We note that since the Christoffel symbols for the metric g are given by

$$\Gamma_{ik}^{\ell}(q) = \frac{1}{2} g^{\ell j}(q) \left(\frac{\partial g_{ji}}{\partial q^k}(q) + \frac{\partial g_{jk}}{\partial q^i}(q) - \frac{\partial g_{ik}}{\partial q^j}(q) \right), \tag{2.4.3.3}$$

we have

$$\forall \alpha \in \mathbb{N}^d, \ \ \partial_q^\alpha \Gamma_{ik}^{\ell}(q) = \mathscr{O}(|q|^{(1-|\alpha|_+)}). \tag{2.4.3.4}$$

in these coordinates. By taking $\varepsilon_{g,\psi}$ smaller if necessary, we may restrict our attention to functions $v \in C_0^\infty(B(0, \hat{C}'_{g,\psi} A 2^{-\ell}) \times \mathbb{R}^d \cap S'_{\ell,4};\mathbb{C})$, where $B(0, \hat{C}'_{g,\psi} A 2^{-\ell})$ denotes the Euclidean ball in $\mathbb{R}^d$ of radius $\hat{C}'_{g,\psi} A 2^{-\ell}$ centered at the origin $0 \in \mathbb{R}^d$, $\hat{C}'_{g,\psi} > 0$ is a constant depending only on g and ψ, and $S'_{\ell,4} = \left\{(q,p) \in \mathbb{R}^{2d} : \frac{1}{4} < |p| < 4\right\}$. Here $|p| = (p_1^2 + \cdots + p_d^2)^{1/2}$.

Let $g(q) = \left(g_{ij}(q)\right)_{1 \leq i,j \leq d}$ extended to a function defined on the whole space $\mathbb{R}^d$ with the same properties. We now introduce the following non-symplectic change of coordinates on $\mathbb{R}^{2d}$:

$$\varphi_{\ell,m} : (q,p) \mapsto \left(2^{-\ell} q, g(2^{-\ell} q) p\right), \tag{2.4.3.5}$$

and the associated unitary map

$$\begin{array}{rcl}\mathscr{U}_{\ell,m}: \ L^2(\mathbb{R}^{2d};\mathbb{C}) & \to & L^2(\mathbb{R}^{2d};\mathbb{C})\\ v & \mapsto & 2^{-\frac{\ell d}{2}}\sqrt{\det(g(2^{-\ell}q))}(v\circ\varphi_{\ell,m})\,,\end{array}$$

which sends $L^2(B(0,\hat{C}'_{g,\psi}A2^{-\ell})\times\mathbb{R}^d;\mathbb{C})$ into $L^2(B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d;\mathbb{C})$.
By taking $\varepsilon_{g,\psi}$ smaller if necessary, we can assume that the unitary map $\mathscr{U}_{\ell,m}$ and the pull-back $\varphi^*_{\ell,m}$ send $C_0^\infty((B(0,\hat{C}'_{g,\psi}A2^{-\ell})\times\mathbb{R}^d)\cap S'_{\ell,4};\mathbb{C})$ into $\mathscr{C}_0^\infty((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8};\mathbb{C})$.
The change of variables in the L^2-norm gives

$$\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})v\|_{L^2}=\|\left(\mathscr{U}_{\ell,m}(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})\mathscr{U}^*_{\ell,m}\right)\mathscr{U}_{\ell,m}v\|_{L^2}\,, \tag{2.4.3.6}$$

where

$$\mathscr{U}_{\ell,m}\mathscr{P}_{b,\ell}\mathscr{U}^*_{\ell,m}=\sqrt{\det(g(2^{-\ell}q))}\mathscr{P}_{b,\ell,m}\frac{1}{\sqrt{\det(g(2^{-\ell}q))}}\,,$$

with

$$\mathscr{P}_{b,\ell,m}=\varphi^*_{\ell,m}\mathscr{P}_{b,\ell}(\varphi^{-1}_{\ell,m})^*\,.$$

A straightforward computation shows that

$$\mathscr{P}_{b,\ell,m}=\frac{1}{b^2}\mathscr{O}_{\ell,m}+\frac{1}{b}\mathscr{Y}_{\ell,m}\,, \tag{2.4.3.7}$$

$$\mathscr{U}_{\ell,m}\mathscr{P}_{b,\ell}\mathscr{U}^*_{\ell,m}=\mathscr{P}_{b,\ell,m}+\left[\sqrt{\det(g(2^{-\ell}q)},\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b}\right]\frac{1}{\sqrt{\det(g(2^{-\ell}q)}} \tag{2.4.3.8}$$

where

$$\mathscr{O}_{\ell,m}:=\varphi^*_{\ell,m}\mathscr{O}_\ell(\varphi^{-1}_{\ell,m})^*=\mathscr{U}_{\ell,m}\mathscr{O}_\ell\mathscr{U}^*_{\ell,m}=\frac{1}{2}\sum_{i,j}\left(g^{ij}(2^{-\ell}q)\frac{D_{p_i}}{2^\ell}\frac{D_{p_j}}{2^\ell}+g_{ij}(2^{-\ell}q)2^\ell p_i2^\ell p_j\right) \tag{2.4.3.9}$$

and

$$\mathscr{Y}_{\ell,m}:=\varphi^*_{\ell,m}\mathscr{Y}_\ell(\varphi^{-1}_{\ell,m})^*=2^{2\ell}\delta^{ij}p_j\frac{\partial}{\partial q^i}+f^{ij}_k(q,\ell)p_ip_j\frac{\partial}{\partial p_k}\,, \tag{2.4.3.10}$$

$$f^{ij}_k(q,\ell):=2^\ell\sum_{n'}g_{n',j}(2^{-\ell}q)\left(\frac{\partial g^{kn'}}{\partial q^i}(2^{-\ell}q)+\sum_n\Gamma^{n'}_{in}(2^{-\ell}q)g^{nk}(2^{-\ell}q)\right)\,, \tag{2.4.3.11}$$

$$\mathscr{U}_{\ell,m}\mathscr{Y}_\ell\mathscr{U}^*_{\ell,m}=\mathscr{Y}_{\ell,m}+\left[\sqrt{\det(g(2^{-\ell}q)},\mathscr{Y}_{\ell,m}\right]\frac{1}{\sqrt{\det(g(2^{-\ell}q)}}\,. \tag{2.4.3.12}$$

From (2.4.3.2) and (2.4.3.4) we know that

$$\sup_{q\in B(0,\hat{C}'_{g,\psi}A)}|f^{ij}_k(q,\ell)|\leq C^{(1)}_{g,\psi}A\,. \tag{2.4.3.13}$$

Since $\det(g)\circ\varphi_{\ell,m}=\det(g(2^{-\ell}q))$, we have

$$\begin{aligned}\left[\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b},\sqrt{\det(g)\circ\varphi_{\ell,m}}\right]&=\frac{1}{b}\left[\mathscr{Y}_{\ell,m},\sqrt{\det(g(2^{-\ell}q))}\right]\\&=\frac{1}{b}\varphi^*_{\ell,m}[\mathscr{Y}_\ell,\sqrt{\det(g)}](\varphi^{-1}_{\ell,m})^*\\&=\frac{1}{b}[\mathscr{Y}_\ell,\sqrt{\det(g)}]\circ\varphi_{\ell,m}\\&=\frac{2^\ell}{b}\delta^{ki}p_k\frac{\partial\sqrt{\det(g)}}{\partial q^i}\circ\varphi_{\ell,m}\,.\end{aligned} \tag{2.4.3.14}$$

is bounded by some constant $C^{(2)}_{g,\psi}\frac{A}{b}$ on $(B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8}$. With the identities (2.4.3.6) and (2.4.3.8) we deduce

$$\Big|\,\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})\mathscr{U}_{\ell,m}v\|_{L^2}-\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})v\|_{L^2}\,\Big|\leq C^{(3)}_{g,\psi}\frac{A}{b}\|\mathscr{U}_{\ell,m}v\|_{L^2}. \tag{2.4.3.15}$$

Proposition 2.4.3. *There exists a constant $C_{g,\psi}\geq 1$, determined by the metric g and the function ψ, such that the inequalities*

$$\frac{1}{4}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})\mathscr{U}_{\ell,m}v\|^2_{L^2}\leq\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})v\|^2_{L^2}\leq 4\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})\mathscr{U}_{\ell,m}v\|^2_{L^2}, \tag{2.4.3.16}$$

hold for all $v\in\mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A2^{-\ell})\times\mathbb{R}^d)\cap S'_{\ell,4};\mathbb{C})$, all $(\lambda,b)\in\mathbb{R}\times(0,\infty)$ when $2^{-\ell}A\leq\frac{1}{C_{g,\psi}}$ and $\frac{\kappa_b}{b^2}>C_{g,\psi}\frac{A}{b}$.
Additionally, we recall

$$\mathscr{U}_{\ell,m}v\in\mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8};\mathbb{C}),$$

for all $v\in\mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A2^{-\ell})\times\mathbb{R}^d)\cap S'_{\ell,4};\mathbb{C})$.

Proof. The same integration by parts as in the proof of Proposition 2.2.1 now gives

$$\begin{aligned}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-i\lambda)u\|_{L^2}\|u\|_{L^2}&\geq\frac{1}{4b^2}\left[C_g\|2^{-\ell}D_pu\|^2_{L^2}+C_g\|2^{\ell}|p|u\|^2_{L^2}+\kappa_b\|u\|^2_{L^2}\right]\\&\quad+\frac{1}{b}\mathrm{Re}\langle u\,,f^{ij}_k(q,\ell)p_ip_j\frac{\partial}{\partial p_k}u\rangle\\&\geq(\frac{\kappa_b}{4b^2}-C^{(4)}_{g,\psi}\frac{A}{b})\|u\|^2_{L^2}\end{aligned}$$

for all $u\in\mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8};\mathbb{C})$, owing to (2.4.3.13). For $\kappa_b\geq 8C^{(4)}_{g,\psi}Ab$ we deduce

$$\forall u\in\mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8};\mathbb{C}),\quad\frac{\kappa_b}{8b^2}\|u\|\leq\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})u\|.$$

The inequality (2.4.3.15) now implies

$$\Big|\,\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})\mathscr{U}_{\ell,m}v\|_{L^2}-\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})v\|_{L^2}\,\Big|\leq\frac{8C^{(3)}_{g,\psi}Ab}{\kappa_b}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})\mathscr{U}_{\ell,m}v\|_{L^2}. \tag{2.4.3.17}$$

when $\kappa_b\geq 8C^{(4)}_{g,\psi}Ab$. It suffices to take $C_{g,\psi}=\max(8C^{(4)}_{g,\psi},16C^{(3)}_{g,\psi})$. □

The next step is to replace the $\mathscr{O}_{\ell,m}$ with the euclidean version defined by

$$\tilde{\mathscr{O}}_\ell=\frac{1}{2}(\delta_{ij}2^{-2\ell}D_{p_i}D_{p_j}+2^{2\ell}|p|^2), \tag{2.4.3.18}$$

and we set according to (2.4.3.10)

$$\tilde{\mathscr{P}}_{b,\ell,m}=\frac{1}{b^2}\tilde{\mathscr{O}}_\ell+\frac{1}{b}\mathscr{Y}_{\ell,m}\,, \tag{2.4.3.19}$$

$$\text{with}\quad\mathscr{Y}_{\ell,m}=2^{2\ell}\delta^{ij}p_j\frac{\partial}{\partial q^i}+f^{ij}_k(q,\ell)p_ip_j\frac{\partial}{\partial p_k}. \tag{2.4.3.20}$$

Proposition 2.4.4. *There is a constant $C_{g,\psi}\geq 1$ determined by the metric g and the function ψ, such that the following inequalities*

$$\|\left(\frac{\kappa_b}{b^2}+\tilde{\mathscr{P}}_{b,\ell,m}-\frac{i\lambda}{b}\right)u\|_{L^2}-C_{g,\psi}A^2 2^{-2\ell}\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|_{L^2}\leq\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b}\right)u\|_{L^2}$$

and

$$\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b}\right)u\|_{L^2}\leq\|\left(\frac{\kappa_b}{b^2}+\tilde{\mathscr{P}}_{b,\ell,m}-\frac{i\lambda}{b}\right)u\|_{L^2}+C_{g,\psi}A^2 2^{-2\ell}\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|_{L^2}$$

hold for all $u\in\mathscr{C}_0^\infty((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8},\mathbb{C})$, all $\ell\in\mathbb{Z}$, $\ell\geq -1$, and all $(\lambda,b)\in\mathbb{R}\times(0,+\infty)$.

Proof. From the equality

$$\mathscr{P}_{b,\ell,m}-\tilde{\mathscr{P}}_{b,\ell,m}=\frac{1}{b^2}\left(\mathscr{O}_{\ell,m}-\tilde{\mathscr{O}}_\ell\right). \tag{2.4.3.21}$$

The difference between $\mathscr{O}_{\ell,m}$ and $\tilde{\mathscr{O}}_{\ell,m}$ is given by

$$\mathscr{O}_{\ell,m}-\tilde{\mathscr{O}}_\ell=\frac{1}{2}\left[\left(g^{ij}(2^{-\ell}q)-\delta_{ij}\right)2^{-2\ell}D_{p_i}D_{p_j}+2^{2\ell}\left(g_{ij}(2^{-\ell}q)-\delta^{ij}\right)p_ip_j\right]. \tag{2.4.3.22}$$

A unitary change of scale replaces $(2^\ell p,2^{-\ell}D_p)$ by (p,D_p) and the problem is reduce to the comparison of harmonic oscillator hamiltonians in Proposition 2.A.1 relying on global ellipticity. With $|g^{ij}(2^{-\ell}q)-\delta^{ij}|+|g_{ij}(2^{-\ell}q)-\delta_{ij}|\leq C_{g,\psi,1}A^2 2^{-2\ell}$, we obtain

$$\|\frac{1}{b^2}\left(\mathscr{O}_{\ell,m}-\tilde{\mathscr{O}}_\ell\right)u\|_{L^2}\leq C_{g,\psi}A^2 2^{-2\ell}\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|_{L^2}, \tag{2.4.3.23}$$

for some constant $C_{g,\psi}>0$ determined by (g,ψ). The two inequalities of this proposition follow. □

We set $h=\frac{1}{2^{2\ell}b}$,

$$\frac{\kappa_b}{b^2}+\tilde{\mathscr{P}}_{b,\ell,m}-\frac{i\lambda}{b}=2^{2\ell}\left(\frac{\kappa_b h}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right) \tag{2.4.3.24}$$

where

$$\hat{\mathscr{P}}_{b,h,f}=\frac{1}{2}\delta_{ij}(hD_{p_i})(hD_{p_j})+\frac{|p|^2}{2b^2}+\frac{1}{b}\delta^{ij}p_j\frac{\partial}{\partial q^i}+hf_k^{ij}p_ip_j\frac{\partial}{\partial p_k}. \tag{2.4.3.25}$$

The problem is now reduced to a careful study of the operator $\hat{\mathscr{P}}_{b,h,f}$ acting on $\mathscr{C}_0^\infty((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8},\mathbb{C})$. We will firstly consider the euclidean case in Section 2.5, where $f=(f_k^{ij})_{1\leq i,j,k\leq d}=0$, and the results for the general case will be obtained by some accurate perturbative argument in Subsection 2.6.1. Passing from the local to the global estimates will be developed in the end of Section 2.6 but preliminary results are collected in the next paragraph.

2.4.4 From the local models to the global estimates for a fixed ℓ

We work in the framework of Subsection 2.4.2 and Subsection 2.4.3. Remember the notations and assumptions:

— the partition of unity $\sum_{m\in\mathbb{Z}^d}\psi^2(q-m)\equiv 1$ and the notations $\psi_{m,\ell,A}(q)=\psi(\frac{q-q_{m,\ell,A}}{2^{-\ell}A})$, $q_{\ell,m,A}=A2^{-\ell}m$;

— $\mathscr{P}_{b,\ell}=\frac{1}{b^2}\mathscr{O}_\ell+\frac{1}{b}\mathscr{Y}_\ell$ introduced in (2.4.1.14)(2.4.1.15)(2.4.1.16);

— the condition $2^{-\ell}A\leq\varepsilon_{g,\psi}=\frac{1}{C_{g,\psi}}$ for $\varepsilon_{g,\psi}$ small enough which allows in Subsection 2.4.3 the use of normal coordinates centered at $q_{\ell,m,A}=A2^{-\ell}m$, $m\in\mathbb{Z}^d$, and the comparison between the metric g with the euclidean metric;

— the unitary transform $\mathscr{U}_{\ell,m}$ associated with the change of variables $\varphi_{\ell,m}:(q,p)\mapsto\big(2^{-\ell}q, g(2^{-\ell}q)p\big)$ written in normal coordinates;
— the operator $\mathscr{P}_{b,\ell,m}=\frac{1}{b^2}\mathscr{O}_{\ell,m}+\frac{1}{b}\mathscr{Y}_{\ell,m}$ introduced in (2.4.3.7).

Proposition 2.4.5. *With the above notations and assumptions, in particular $2^{-\ell}A\leq\frac{1}{C_{g,\psi}}$, set for $u\in\mathscr{C}_0^\infty(S_{\ell,2},\mathbb{C})$ and for any $m\in\mathbb{Z}^d$,*

$$u_{\ell,m}=\mathscr{U}_{\ell,m}(\psi_{m,\ell,A}u)\ \in\mathscr{C}_0^\infty((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8},\mathbb{C}).$$

The following inequalities

$$\sum_{m\in\mathbb{Z}^d}\frac{1}{8}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}-\frac{C_{g,\psi}}{A^2b^2}\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}\leq\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})},$$

$$\text{and}\qquad\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})}\leq\sum_{m\in\mathbb{Z}^d}8\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}+\frac{C_{g,\psi}}{A^2b^2}\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}$$

hold true as soon as $\kappa_b\geq C_{g,\psi}Ab$ and the constant $C_{g,\psi}\geq1$ is chosen large enough.

Proof. Simply combine Proposition 2.4.2 and Proposition 2.4.3. □

A similar result can be written for the hamiltonian vector field alone.

Proposition 2.4.6. *With the same notations and assumptions as in Proposition 2.4.5 the following inequalities*

$$\sum_{m\in\mathbb{Z}^d}\frac{1}{8}\|\frac{1}{b}(\mathscr{Y}_{\ell,m}-i\lambda)u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}-C_{g,\psi}(\frac{1}{A^2b^2}+\frac{1}{b^22^{2\ell}})\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}\leq\|\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)u\|^2_{L^2(\mathbb{R}^{2d})},$$

$$\text{and}\qquad\|\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)u\|^2_{L^2(\mathbb{R}^{2d})}\leq\sum_{m\in\mathbb{Z}^d}8\|\frac{1}{b}(\mathscr{Y}_{\ell,m}-i\lambda)u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}+C_{g,\psi}(\frac{1}{A^2b^2}+\frac{1}{b^22^{2\ell}})\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}$$

hold true as soon as the constant $C_{g,\psi}\geq1$ is chosen large enough.

Proof. In Proposition 2.4.2 the operator $(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})$ can be replaced by $\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)$ and this produces the same error term $\frac{C_{g,\psi}}{A^2b^2}\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2}$. In the same way, the calculation leading to (2.4.3.15) can be done with $(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})$ replaced by $\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)$. This second step leads to

$$\|\frac{1}{b}(\mathscr{Y}_{\ell,m}-i\lambda)\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}\leq2\|\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)v\|^2_{L^2(\mathbb{R}^{2d})}+2[C^{(3)}_{g,\psi}]^2\frac{A^2}{b^2}\|\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}$$

and

$$\|\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)v\|^2_{L^2(\mathbb{R}^{2d})}\leq2\|\frac{1}{b}(\mathscr{Y}_{\ell,m}-i\lambda)\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}+2[C^{(3)}_{g,\psi}]^2\frac{A^2}{b^2}\|\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}.$$

Since $2^{-\ell}A\leq\frac{1}{C_{g,\psi}}\leq1$, we conclude that

$$\frac{A^2}{b^2}\|\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}\leq\frac{2^{-2\ell}A^2}{b^22^{2\ell}}\|2^{2\ell}\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}\leq\frac{1}{b^22^{2\ell}}\|2^{2\ell}\mathscr{U}_{\ell,m}v\|^2_{L^2(\mathbb{R}^{2d})}.$$

□

The last result of this paragraph is about the comparison between local and global estimates of the $\tilde{\mathscr{W}}^{2/3}$-norm appearing in the lower bound of Theorem 2.1.6. According to Proposition 2.E.7-**ii)** in Appendix 2.E, the $\tilde{\mathscr{W}}^{2/3}$-norm of $u\in\mathscr{C}_0^\infty(\Omega\times\mathbb{R}^d;\mathbb{C})$ can be written

$$\|u\|^2_{\tilde{\mathscr{W}}^{2/3}}=\left\|(\tilde{\mathscr{O}}_1)^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\||D_q|^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})},$$

expressed with operators constructed in the euclidean case. The change of scale $(p,D_p)\to(2^\ell p,2^{-\ell}D_p)$ leads to consider

$$\|u\|^2_{\tilde{\mathscr{W}}^{2/3},\ell}=\left\|(\tilde{\mathscr{O}}_\ell)^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\||D_q|^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})}. \qquad (2.4.4.1)$$

Proposition 2.4.7. *With the same notations and assumptions as in Proposition 2.4.5 and with* $\|u\|_{\mathscr{W}^{2/3},\ell}$ *defined by* (2.4.4.1)*, the following inequalities*

$$\|u\|^2_{\tilde{\mathscr{W}}^{2/3},\ell}\le C_{g,\psi}\sum_{m\in\mathbb{Z}^d}\|(\tilde{\mathscr{O}}_\ell)^{2/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}+\||2^\ell D_q|^{2/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{A^{4/3}}\left\|2^{2\ell/3}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}$$

hold true as soon as $\kappa_b\ge C_{g,\psi}Ab$ *and the constant* $C_{g,\psi}\ge 1$ *is chosen large enough.*

Proof. With $\left\|\langle D_q\rangle^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})}=\langle u\,,(1-\Delta_q)^{2/3}u\rangle$ and $(1-\Delta_q)^{2/3}=\left[(1+|\xi|^2)^{2/3}\right]^{Weyl}(q,D_q)$. and $\sum_{m\in\mathbb{Z}^d}\psi^2(q-m)\equiv 1$, Weyl-Hörmander pseudo-differential calculus with the standard metric $dq^2+\frac{d\xi^2}{\langle\xi\rangle^2}$ (see [HormIII]-Chap XVIII) , provides a constant $C_\psi\ge 1$ such that

$$\|\langle D_q\rangle^{\frac23}v\|^2_{L^2}\le C_\psi\sum_{m\in\mathbb{Z}^d}\|\langle D_q\rangle^{\frac23}\psi(.-m)v\|^2_{L^2}.$$

By setting $h=A2^{-\ell}$, $\psi(\frac{q-hm}{h})=\psi_{m,\ell,A}(q)$, $v(q,p)=h^{d/2}u(hq,p)$ and

$$v_{m,\ell,A}(q,p)=\psi_{m,\ell,A}(q)u(q,p)=h^{-d/2}\psi(\frac{q-hm}{h})v(h^{-1}q,p),$$

the above inequality becomes

$$\||hD_q|^{\frac23}u\|^2_{L^2}\le C_\psi\sum_{m\in\mathbb{Z}^d}\|\langle hD_q\rangle^{\frac23}v_{m,\ell,A}\|^2_{L^2}\le 2C_\psi\left[\sum_{m\in\mathbb{Z}^d}\||hD_q|^{\frac23}v_{m,\ell,A}\|^2_{L^2}+\|v_{m,\ell,A}\|^2_{L^2(\mathbb{R}^{2d})}\right].$$

By multiplying both sides of the inequality by $h^{-\frac43}=\frac{2^{4\ell/2}}{A^{4/3}}$, we get

$$\||D_q|^{\frac23}u\|^2_{L^2}\le 2C_\psi\sum_{m\in\mathbb{Z}^d}\||D_q|^{\frac23}v_{m,\ell,A}\|^2_{L^2}+\frac{1}{A^{4/3}}\|2^{2\ell/3}v_{m,\ell,A}\|^2_{L^2(\mathbb{R}^{2d})}.$$

By adding $\|(\tilde{\mathscr{O}}_\ell)u\|^2_{L^2(\mathbb{R}^{2d})}=\sum_{m\in\mathbb{Z}^d}\|(\tilde{\mathscr{O}}_\ell)v_{m,\ell,A}\|^2_{L^2(\mathbb{R}^{2d})}$, we deduce

$$\|u\|^2_{\tilde{\mathscr{W}}^{2/3},\ell}\le C_\psi\left[\sum_{m\in\mathbb{Z}^d}\|v_{m,\ell,A}\|^2_{\tilde{\mathscr{W}}^{2/3},\ell}+\frac{1}{A^{4/3}}\|2^{2\ell/3}v_{m,\ell,A}\|^2_{L^2(\mathbb{R}^{2d})}\right],$$

which is not exactly the seeked inequality expressed in terms of the $u_{m,\ell,A}$. By setting $\tilde{v}_{m,\ell,A}=2^{\ell d/2}v_{m,\ell,A}(q,2^\ell p)$ we have on one side

$$\|v_{m,\ell,A}\|^2_{\tilde{\mathscr{W}}^{2/3},\ell}=\|\tilde{v}_{m,\ell,A}\|^2_{\tilde{\mathscr{W}}^{2/3},1}$$

while on the other side

$$\begin{aligned}\tilde{v}_{m,\ell,A}(q,p)&=2^{\ell d/2}\psi_{m,\ell,A}(q)u(q,2^\ell p)=2^{\ell d/2}[\mathscr{U}^{-1}_{\ell,m}u_{m,\ell,A}](q,2^\ell p)\\&=\frac{2^{\ell d}}{\sqrt{\det(g(2^\ell q))}}u_{m,\ell,A}(2^\ell q,g(2^\ell q)^{-1}2^\ell p)=[\hat{\mathscr{U}}_{\ell,m}\hat{u}_{m,\ell,A}](q,p)\\ \text{with}\quad&\hat{u}_{m,\ell,A}(q,p)=2^{\ell d}u_{m,\ell,A}(2^\ell q,2^\ell p)\quad,\quad[\hat{\mathscr{U}}_{\ell,m}w](q,p)=\frac{1}{\sqrt{\det(g(q))}}w(q,g(q)^{-1}p).\end{aligned}$$

Proposition gives $\|\hat{\mathscr{U}}_{\ell,m}w\|_{\tilde{\mathscr{W}}^{2/3},1}\le C^{(1)}_{g,\psi}\|w\|_{\tilde{\mathscr{W}}^{2/3},1}$ and

$$\|\tilde{v}_{m,\ell,A}\|^2_{\tilde{\mathscr{W}}^{2/3},1}\le[C^{(1)}_{g,\psi}]^2\|\hat{u}_{m,\ell,A}\|^2_{\tilde{\mathscr{W}}^{2/3},1}\le[C^{(1)}_{g,\psi}]^2\|(\tilde{\mathscr{O}}_\ell)^{2/3}u_{m,\ell,A}\|^2_{L^2(\mathbb{R}^{2d})}+\||2^\ell D_q|^{2/3}u_{m,\ell,A}\|^2_{L^2(\mathbb{R}^{2d})},$$

while

$$\|v_{m,\ell,A}\|_{L^2(\mathbb{R}^{2d})}=\|\tilde{v}_{m,\ell,A}\|_{L^2(\mathbb{R}^{2d})}=\|\hat{u}_{m,\ell,A}\|_{L^2(\mathbb{R}^{2d})}=\|u_{m,\ell,A}\|_{L^2(\mathbb{R}^{2d})}.$$

This ends the proof after choosing the constant $C_{g,\psi}\le 1$ large enough. □

2.5 Euclidean Case

In this section we consider the scalar euclidean case indexed by two parameters $b,h>0$ with the Kramers-Fokker-Planck operator

$$\hat{\mathscr{P}}_{b,h,0}=\underbrace{\frac{1}{2}(-h^2\Delta_p+\frac{|p|^2}{b^2})}_{=\hat{\mathscr{O}}_{b,h}}+\frac{1}{b}\underbrace{\sum_{j=1}^{d}p_j\frac{\partial}{\partial q^j}}_{=ip\cdot D_q} \tag{2.5.0.1}$$

on $\mathbb{R}^{2d}=\mathbb{R}^d_q\times\mathbb{R}^d_p$ where $(q,p)=(q^1,\ldots,q^d,p_1,\ldots,p_d)$.

Proposition 2.5.1. *There exists a universal constant $C\geq 1$ such that the inequality*

$$\begin{aligned}C\|(\frac{h}{b}+\hat{\mathscr{P}}_{b,h,0}-ih\lambda)u\|^2_{L^2(\mathbb{R}^{2d})}\geq\|(\frac{h}{b}+\hat{\mathscr{O}}_{b,h})u\|^2_{L^2(\mathbb{R}^{2d})}+\|(\frac{1}{b}p\cdot D_q-h\lambda)u\|^2_{L^2(\mathbb{R}^{2d})}\\+\|(|\frac{h}{b}D_q|^{\frac{2}{3}}+\frac{h}{b})u\|^2_{L^2(\mathbb{R}^{2d})}+\left\|\left(h^2\frac{|\lambda|}{\sqrt{hb}+|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}\end{aligned} \tag{2.5.0.2}$$

holds for all $u\in C_0^\infty(\mathbb{R}^{2d};\mathbb{C})$, all $\lambda\in\mathbb{R}$ and all $(b,h)\in(0,\infty)^2$.

Proof. A unitary change of scale transforms (q,p,D_q,D_p) into $(\sqrt{\frac{b}{h}}q,\sqrt{bh}p,\sqrt{\frac{h}{b}}D_q,\frac{1}{\sqrt{hb}}D_p)$ and $\hat{\mathscr{P}}_{b,h,0}$ into $\frac{h}{b}\hat{\mathscr{P}}_{1,1,0}$. The problem is thus reduced to the case $b=h=1$ and the final result will be obtained after multiplying both sides of the specific inequality by h^2/b^2.
Let us introduce some simplified notations:

— The partial Fourier transform with respect to the variable q is normalized as

$$\mathscr{F}_{q\mapsto\xi}u(\xi,p)=\int_{\mathbb{R}^d}e^{-iq\cdot\xi}u(q,p)\,dq,\quad u\in C_0^\infty(\mathbb{R}^{2d}). \tag{2.5.0.3}$$

It is unitary from $L^2(\mathbb{R}^{2d}_{q,p},dqdp;\mathbb{C})$ onto $L^2(\mathbb{R}^{2d}_{\xi,p},\frac{d\xi}{(2\pi)^d}dp;\mathbb{C})$.

— The operator $\hat{\mathscr{P}}_{1,1,0}$ is simply denoted by $\hat{P}$ and we set

$$\hat{P}=\hat{\mathscr{P}}_{1,1,0}=\underbrace{\frac{1}{2}(-\Delta_p+|p|^2)}_{=\mathscr{O}}+ip\cdot D_q$$

$$\widetilde{P}:=\mathscr{F}_{q\mapsto\xi}\circ\hat{P}\circ\left(\mathscr{F}_{q\mapsto\xi}\right)^{-1}=\int^{\oplus}_{\mathbb{R}^d}\underbrace{\frac{1}{2}(-\Delta_p+|p|^2)+i(p\cdot\xi)}_{=\widetilde{P}_\xi}\frac{d\xi}{(2\pi)^d}, \tag{2.5.0.4}$$

in the direct integral decomposition $L^2(\mathbb{R}^{2d}_{\xi,p},\frac{d\xi}{(2\pi)^d}dp;\mathbb{C})=\int^{\oplus}_{\mathbb{R}^d}L^2(\mathbb{R}^d,dp;\mathbb{C})\,\frac{d\xi}{(2\pi)^d}$.

— The harmonic oscillator hamiltonian $\mathscr{O}=\frac{-\Delta_p+|p|^2}{2}$ is decomposed as $\mathscr{O}=\sum_{j=1}^d\mathscr{O}_j$ with

$$\mathscr{O}_j=-\frac{1}{2}\frac{\partial^2}{\partial p_j^2}+\frac{1}{2}p_j^2.$$

— For any fixed $\xi\in\mathbb{R}^d$, there exists an orthogonal matrix $\mathscr{R}_\xi\in O(d)$ such that

$$\mathscr{R}^T_\xi(\xi)=(|\xi|,0,\ldots,0). \tag{2.5.0.5}$$

The corresponding unitary pullback on functions $(\mathscr{R}_\xi^T)^* : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)$ is given by

$$(\mathscr{R}_\xi^T)^* u(p) = u(\mathscr{R}_\xi^T p), \quad u \in L^2(\mathbb{R}^d),$$

and we let $(\mathscr{R}_\xi)^* := \left(\left(\mathscr{R}_\xi^T\right)^*\right)^{-1}$ be the inverse of $\left(\mathscr{R}_\xi^T\right)^*$. For $\xi \in \mathbb{R}^d$, let

$$\widetilde{P}_{\xi,\mathscr{R}} := (\mathscr{R}_\xi^T)^* \circ \widetilde{P}_\xi \circ (\mathscr{R}_\xi)^* = \frac{1}{2}\left(-\Delta_p + |p|^2\right) + i p_1 |\xi| \tag{2.5.0.6}$$

be the conjugation of $\widetilde{P}_\xi$ by $(\mathscr{R}_\xi^T)^*$. From (2.5.0.6), it is clear that

$$\widetilde{P}_{\xi,\mathscr{R}} = \mathscr{O}_1 + i p_1 |\xi| + \sum_{j \neq 1} \mathscr{O}_j. \tag{2.5.0.7}$$

We now turn our attention to the topic of obtaining lower bounds for the quantity

$$\left\|(1 + \hat{P} - i\lambda)u\right\|_{L^2(\mathbb{R}^{2d})} \tag{2.5.0.8}$$

when $u \in C_0^\infty(\mathbb{R}^{2d})$ and $\lambda \in \mathbb{R}$. We begin by observing, via a straightforward calculation, that

$$\begin{aligned}\|(1 + \mathscr{O}_1 + i(p_1|\xi| - \lambda))u\|^2_{L^2(\mathbb{R}^d)} = &\left\|\left(1 - \frac{1}{2}\frac{\partial^2}{\partial p_1^2} + i(p_1|\xi| - \lambda)\right)u\right\|^2_{L^2(\mathbb{R}^d)} \\ &+ \left\|\frac{p_1^2}{2}u\right\|^2_{L^2_p} + \|p_1 u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\left\|p_1 \frac{\partial}{\partial p_1}u\right\|^2_{L^2(\mathbb{R}^d)} - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^d)}\end{aligned} \tag{2.5.0.9}$$

for any $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Meanwhile, an integration by parts argument gives

$$\|(1 - \frac{1}{2}\frac{\partial^2}{\partial p_1^2} + i(p_1|\xi| - \lambda))u\|^2_{L^2(\mathbb{R}^d)} \geq \|u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\frac{\partial}{\partial p_1}u\|^2_{L^2(\mathbb{R}^d)} \tag{2.5.0.10}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. On the other hand, from Proposition 2.B.1, we know that there is a universal constant $C_0 > 0$, such that

$$\begin{aligned}C_0\|1 - \frac{1}{2}\Delta_{p_1} + i(p_1|\xi| - \lambda)u\|^2_{L^2(\mathbb{R}^d)} \geq &\|\frac{1}{2}\Delta_{p_1}u\|^2_{L^2(\mathbb{R}^d)} + \|(p_1|\xi| - \lambda)u\|^2_{L^2(\mathbb{R}^d)} \\ &+ (|\xi|^{\frac{2}{3}} + 1)^2\|u\|^2_{L^2(\mathbb{R}^d)} + \left\|\left(\frac{|\lambda|}{1 + |p_1|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^d)}\end{aligned} \tag{2.5.0.11}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Since

$$\begin{aligned}\|(1 + \mathscr{O}_1)u\|^2_{L^2(\mathbb{R}^d)} = &\frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^d)} + \|p_1 u\|^2_{L^2(\mathbb{R}^d)} \\ &+ \|\frac{p_1^2}{2}u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|p_1\frac{\partial}{\partial p_1}u\|^2_{L^2(\mathbb{R}^d)} + \|\frac{\partial}{\partial p_1}u\|^2_{L^2(\mathbb{R}^d)} + \|\frac{1}{2}\frac{\partial^2}{\partial p_1^2}u\|^2_{L^2(\mathbb{R}^d)}\end{aligned}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, we deduce from (2.5.0.9), (2.5.0.10), and (2.5.0.11) that there is a universal constant $C_1 \geq 1$ such that

$$\begin{aligned}C_1\|(1 + \mathscr{O}_1 + i(p_1|\xi| - \lambda))u\|^2_{L^2(\mathbb{R}^d)} \geq &\|(1 + \mathscr{O}_1)u\|^2_{L^2(\mathbb{R}^d)} + \|(p_1|\xi| - \lambda)u\|^2_{L^2(\mathbb{R}^d)} \\ &+ (|\xi|^{\frac{2}{3}} + 1)^2\|u\|^2_{L^2(\mathbb{R}^d)} + \left\|\left(\frac{|\lambda|}{1 + |p_1|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^d)}\end{aligned} \tag{2.5.0.12}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$.

Next, we observe that elementary algebraic manipulations give the following identity:

$$(1+\mathscr{O})^2 - (1+\mathscr{O}_1)^2 = \left(\sum_{j\neq 1} \mathscr{O}_j\right)^2 + 2(1+\mathscr{O}_1)\sum_{j\neq 1}\mathscr{O}_j \geq \left(\sum_{j\neq 1}\mathscr{O}_j\right)^2. \tag{2.5.0.13}$$

A straightforward computation using (2.5.0.13) gives that

$$\begin{aligned}(1+\widetilde{P}_{\xi,\mathscr{R}} - i\lambda)^*(1+\widetilde{P}_{\xi,\mathscr{R}} - i\lambda) &= (1+\mathscr{O}_1 - i(p_1|\xi|-\lambda))(1+\mathscr{O}_1 + i(p_1|\xi|-\lambda)) \\ &\quad + (1+\mathscr{O})^2 - (1+\mathscr{O}_1)^2\end{aligned} \tag{2.5.0.14}$$

holds for every $\xi \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$. As a consequence, we have

$$\left\|(1+\widetilde{P}_{\xi,\mathscr{R}} - i\lambda)u\right\|^2_{L^2(\mathbb{R}^d)} \geq \|(1+\mathscr{O}_1 + i(p_1|\xi|-\lambda))u\|^2_{L^2(\mathbb{R}^d)} + \left\|\sum_{j\neq 1}\mathscr{O}_j u\right\|^2_{L^2(\mathbb{R}^d)} \tag{2.5.0.15}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Combining (2.5.0.12) and (2.5.0.15) then leads to

$$\begin{aligned} C_1\|(1+\widetilde{P}_{\xi,\mathscr{R}} - i\lambda)u\|^2_{L^2(\mathbb{R}^d)} &\geq \|\sum_{j\neq 1}\mathscr{O}_j u\|^2_{L^2(\mathbb{R}^d)} + \|(1+\mathscr{O}_1)u\|^2_{L^2(\mathbb{R}^d)} + \|(p_1|\xi|-\lambda)u\|^2_{L^2(\mathbb{R}^d)} \\ &\quad + (|\xi|^{\frac{2}{3}}+1)^2\|u\|^2_{L^2(\mathbb{R}^d)} + \left\|\left(\frac{|\lambda|}{1+|p|}\right)^{\frac{2}{3}} u\right\|^2_{L^2(\mathbb{R}^d)} \\ &\geq \frac{1}{2}\|(1+\mathscr{O})u\|^2_{L^2(\mathbb{R}^d)} + \|(p_1|\xi|-\lambda)u\|^2_{L^2(\mathbb{R}^d)} \\ &\quad + (|\xi|^{\frac{2}{3}}+1)^2\|u\|^2_{L^2(\mathbb{R}^d)} + \left\|\left(\frac{|\lambda|}{1+|p|}\right)^{\frac{2}{3}} u\right\|^2_{L^2(\mathbb{R}^d)} \end{aligned} \tag{2.5.0.16}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. From (2.5.0.5), (2.5.0.6), (2.5.0.16), and the unitarity of $\left(\mathscr{R}_\xi^T\right)^*$, we see that there is a universal constant $C = 2C_1 \geq 1$ such that

$$\begin{aligned} C\|(1+\widetilde{P}_\xi - i\lambda)u\|^2_{L^2(\mathbb{R}^d)} &\geq \|(1+\mathscr{O})u\|^2_{L^2(\mathbb{R}^d)} + \|(p\cdot\xi-\lambda)u\|^2_{L^2(\mathbb{R}^d)} \\ &\quad + (|\xi|^{\frac{2}{3}}+1)^2\|u\|^2_{L^2(\mathbb{R}^d)} + \left\|\left(\frac{|\lambda|}{1+|p|}\right)^{\frac{2}{3}} u\right\|^2_{L^2(\mathbb{R}^d)} \end{aligned} \tag{2.5.0.17}$$

for every $u \in C_0^\infty(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}$. Using (2.5.0.4), (2.5.0.17) and the unitarity of $\mathscr{F}_{q\to\xi}$, we obtain

$$\begin{aligned} C\|(1+\hat{P} - i\lambda)u\|^2_{L^2(\mathbb{R}^{2d})} &\geq \|(1+\mathscr{O})u\|^2_{L^2(\mathbb{R}^{2d})} + \|(p\cdot D_q-\lambda)u\|^2_{L^2(\mathbb{R}^{2d})} \\ &\quad + \|(|D_q|^{\frac{2}{3}}+1)u\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\left(\frac{|\lambda|}{1+|p|}\right)^{\frac{2}{3}} u\right\|^2_{L^2(\mathbb{R}^{2d})} \end{aligned}$$

for all $u \in C_0^\infty(\mathbb{R}^{2d})$ and $\lambda \in \mathbb{R}$. The change of scale introduced in the beginning of this proof says for λ replaced by $b\lambda$

$$\begin{aligned} C\|(1+\frac{b}{h}\hat{\mathscr{P}}_{b,h,0} - ib\lambda)u\|^2_{L^2(\mathbb{R}^{2d})} &\geq \|(1+\frac{b}{h}\hat{\mathscr{O}}_{b,h})u\|^2_{L^2(\mathbb{R}^{2d})} + \|(\frac{1}{h}p\cdot D_q - b\lambda)u\|^2_{L^2(\mathbb{R}^{2d})} \\ &\quad + \|(|\sqrt{\frac{b}{h}}D_q|^{\frac{2}{3}}+1)u\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\left(\frac{|b\lambda|}{1+\frac{|p|}{\sqrt{hb}}}\right)^{\frac{2}{3}} u\right\|^2_{L^2(\mathbb{R}^{2d})} \end{aligned}$$

for all $u \in C_0^\infty(\mathbb{R}^{2d})$ and $\lambda \in \mathbb{R}$. Multiplying both sides by $(\frac{h}{b})^2$ ends the proof. □

2.6 Final proof of Theorem 2.1.6

We now collect all the information given by the localization techniques of Section 2.4 and the accurate estimates for the euclidean model in Section 2.5. The first result will be the derivation of the subelliptic estimate for the local model $\tilde{\mathscr{P}}_{b,\ell,m} = \frac{1}{b^2}\tilde{\mathscr{O}}_\ell + \frac{1}{b}\mathscr{Y}_{\ell,m}$ introduced in (2.4.3.19) at the end of Subsection 2.4.3 from the subelliptic estimates for the euclidean model. The second result is about the other local operator $\mathscr{P}_{b,\ell,m} = \frac{1}{b^2}\mathscr{O}_{\ell,m} + \frac{1}{b}\mathscr{Y}_{\ell,m}$ introduced in (2.4.3.7)(2.4.3.9)(2.4.3.10). These preliminary results hold for all momenta $p \sim 2^\ell$ and arbitrary values of the intermediate parameter $A > 0$ introduced in the grid partition. Then, in the third paragraph, we consider the case of large momenta or large ℓ and the summation with respect to the grid index $m \in \mathbb{Z}^d$ will hold for $A \geq 1$ large enough. Here the intermediate parameter A will be fixed to $A = A_\infty(b) \geq 1$ large enough according to the value of $b > 0$ and the geometric data.
Once $A_\infty(b) \geq 1$ is fixed, the fourth paragraph collects the information when the momentum $p \sim 2^\ell$ is bounded by $C_{A_\infty(b),b}$. For this part the term $p \times p \times \partial_p$, estimated by $O(C_{A_\infty(b),b})\partial_p$, is controlled by a simple integration by parts argument provided κ_b is large enough. The summation with respect to the grid index $m \in \mathbb{Z}^d$ will be done by choosing another value for the intermediate parameter $A = A_0(b)$ with $A_0(b) > 0$ small enough according to the value of $b > 0$ and the geometric data.
Finally all of the summations with respect to $\ell \geq -1$ are carried out.

2.6.1 Comparison of the local model $\widetilde{\mathscr{P}}_{b,\ell,m}$ with the euclidean case

We write general local subelliptic estimates for the local operator $\tilde{\mathscr{P}}_{b,\ell,m} = \frac{1}{b^2}\tilde{\mathscr{O}}_\ell + \frac{1}{b}\mathscr{Y}_{\ell,m}$ introduced in (2.4.3.19) at the end of Subsection 2.4.3 and parametrized by the dyadic scale 2^ℓ, $b > 0$, the grid index $m \in \mathbb{Z}^d$ and the constant grid scaling $A > 0$.

Proposition 2.6.1. *Let $C \geq 1$ be the universal constant given by Proposition 2.5.1 for the euclidean metric.*
There exists a constant $C^{(3)}_{g,\psi} > 0$, depending only on the metric g and the function ψ, such that for all $(A,b) \in (0,+\infty)^2$, $\kappa_b \geq C^{(3)}_{g,\psi}Ab + 1$ implies the inequalities

$$\left\|\left(\frac{\kappa_b}{b^2} + \tilde{\mathscr{P}}_{b,\ell,m} - \frac{i\lambda}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{2^{16}b^4}\|2^{2\ell}u\|^2_{L^2(\mathbb{R}^{2d})} \tag{2.6.1.1}$$

and

$$\begin{aligned}4C\left(1+C^{(3)}_{g,\psi}b^2A^2\right)\left\|\left(\frac{\kappa_b}{b^2} + \tilde{\mathscr{P}}_{b,\ell,m} - \frac{i\lambda}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} &\geq \left\|\frac{1}{b^2}\left(\kappa_b + \tilde{\mathscr{O}}_\ell\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\frac{1}{b}\left(\mathscr{Y}_{\ell,m} - i\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \left\|\left(\left|\frac{2^\ell}{b^2}D_q\right|^{\frac{2}{3}} + \frac{1}{b^2}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\left(\frac{1}{b^2}\frac{|\lambda|}{1+2^\ell|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})},\end{aligned} \tag{2.6.1.2}$$

when either $\quad u \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A) \times \mathbb{R}^d) \cap S'_{0,8}, \mathbb{C})$ *and* $(\lambda,\ell) \in \mathbb{R} \times \mathbb{N}$,
or $\quad u \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A) \times \mathbb{R}^d) \cap S'_{-1,8}, \mathbb{C})$ *and* $\lambda \in \mathbb{R}$ *for* $\ell = -1$.

Remember the notations of Subsection 2.4.3
— $h = \frac{1}{b2^{2\ell}}$,
— $\frac{\kappa_b}{b^2} + \tilde{\mathscr{P}}_{b,\ell,m} - \frac{i\lambda}{b} = 2^{2\ell}\left(\frac{\kappa_b h}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda\right) = \frac{1}{bh}\left(\frac{\kappa_b h}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda\right)$,

— $\tilde{\mathscr{O}}_\ell = \frac{b}{h}\hat{\mathscr{O}}_{b,h}$ with $\hat{\mathscr{O}}_{b,h} = \frac{-h^2\Delta_p + \frac{|p|^2}{b^2}}{2}$,

— $\frac{1}{b}\mathscr{Y}_{\ell,m} = 2^{2\ell}(\frac{1}{b}p\cdot\partial_q + hf_k^{ij}(q,\ell)p_ip_j\partial_{p_k}) = \frac{1}{bh}(\frac{1}{b}p\cdot\partial_q + hf_k^{ij}(q,\ell)p_ip_j\partial_{p_k})$,

— $S'_{1,8} = \{(q,p)\in\mathbb{R}^{2d}, \frac{1}{8}\leq |p|\leq 8\}$ and $S'_{0,8} = \{(q,p)\in\mathbb{R}^{2d}, |p|\leq 8\}$.

The result of Proposition 2.6.1 is actually deduced from the same results for the operator $\hat{\mathscr{P}}_{b,h,f}$. This will be done in two steps.

Proposition 2.6.2. *There exists a constant $C^{(2)}_{g,\psi}>0$, depending on the metric g and the function ψ, such for all $(A,b)\in(0,+\infty)^2$ and for $\kappa_b\geq 1+C^{(2)}_{g,\psi}Ab$ the inequalities*

$$\mathrm{Re}\left\langle\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u,u\right\rangle_{L^2}\geq\frac{1}{2^7b^2}\|u\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2}\sum_j\|hD_{p_j}u\|^2_{L^2(\mathbb{R}^{2d})}\quad(2.6.1.3)$$

$$\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u\right\|^2_{L^2}\geq\frac{1}{2^{14}b^4}\|u\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2^8b^2}\sum_j\left\|hD_{p_j}u\right\|^2_{L^2(\mathbb{R}^{2d})}.\quad(2.6.1.4)$$

hold true

when either $u\in\mathscr{C}_0^\infty((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{0,8},\mathbb{C})$ *and* $(\lambda,\ell)\in\mathbb{R}\times\mathbb{N}$,

or $u\in\mathscr{C}_0^\infty((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{-1,8},\mathbb{C})$ *and* $\lambda\in\mathbb{R}$ *for* $\ell=-1$.

Proof. A straightforward computation gives

$$\mathrm{Re}\left\langle\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u,u\right\rangle_{L^2}=\frac{h\kappa_b}{b}\|u\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2}\sum_j\left\|hD_{p_j}u\right\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2}\left\|\frac{|p|}{b}u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$+\mathrm{Re}\left\langle hf_k^{ij}p_ip_j\frac{\partial u}{\partial p_k},u\right\rangle_{L^2},\quad(2.6.1.5)$$

and, owing to $|p|\leq 8$ and $|f^k_{ij}(q)|\leq C^{(1)}_{g,\psi}A$ according to (2.4.3.13),

$$\left|\mathrm{Re}\left\langle hf_k^{ij}p_ip_j\frac{\partial u}{\partial p_k},u\right\rangle_{L^2}\right|=\left|\frac{h}{2}\left\langle\left[f_k^{ij}p_ip_j,\frac{\partial}{\partial p_k}\right]u,u\right\rangle_{L^2}\right|=\left|-\frac{h}{2}\left\langle\sum_k\left(f_k^{ik}p_i+f_k^{kj}p_j\right)u,u\right\rangle_{L^2}\right|$$
$$\leq 8hC^{(1)}_{g,\psi}A\|u\|^2_{L^2(\mathbb{R}^{2d})}\quad(2.6.1.6)$$

for any u is supported in $(B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{\ell,8}$ for $\ell=0$ or $\ell=-1$.
When $\ell=-1$ and $h=\frac{4}{b}$ we simply use

$$\mathrm{Re}\left\langle\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u,u\right\rangle_{L^2}\geq 4\frac{\kappa_b-8C^{(1)}_{g,\psi}Ab}{b^2}\|u\|^2_{L^2}+\frac{1}{2}\sum_j\left\|hD_{p_j}u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$\geq\frac{2}{b^2}\|u\|^2_{L^2}+\frac{1}{2}\sum_j\left\|hD_{p_j}u\right\|^2_{L^2(\mathbb{R}^{2d})},$$

if $\kappa_b\geq 1+C^{(2)}_{g,\psi}Ab$ and $C^{(2)}_{g,\psi}=16C^{(1)}_{g,\psi}$.
When $\ell\geq 0$ and $\mathrm{supp}\,u\subset(B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S_{0,8}$, we deduce with the lower bound $|p|\geq 2^{-3}$ the inequality

$$\mathrm{Re}\left\langle\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u,u\right\rangle_{L^2}\geq(\frac{h\kappa_b}{b}-\frac{hC^{(2)}_{g,\psi}A}{2}+\frac{1}{2^7b^2})\|u\|^2_{L^2}+\frac{1}{2}\sum_j\left\|hD_{p_j}u\right\|^2_{L^2(\mathbb{R}^{2d})},$$

where again $C^{(2)}_{g,\psi} = 16C^{(1)}_{g,\psi}$ is determined by the metric g and the function ψ. The assumption $\kappa_b \geq C^{(2)}_{g,\psi} A b$ implies

$$\mathrm{Re}\left\langle \left(\frac{h\kappa_b}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda\right)u, u\right\rangle_{L^2} \geq \left(\frac{h\kappa_b}{2b} + \frac{1}{2^7 b^2}\right)\|u\|^2_{L^2(\mathbb{R}^{2d})} + \frac{1}{2}\sum_j \|hD_{p_j}u\|^2_{L^2(\mathbb{R}^{2d})}$$

and this ends the proof of (2.6.1.3).

The second inequality (2.6.1.4) is deduced from (2.6.1.3) via the Cauchy-Schwarz inequality like in the proof of Proposition 2.2.1.

□

We are now able to give a lower bound for the operator $\hat{\mathscr{P}}_{b,h,f}$

Proposition 2.6.3. *Let $C \geq 1$ be the universal constant given by Proposition 2.5.1 for the euclidean metric.*
There exists a constant $C^{(3)}_{g,\psi} > 0$, depending only on the metric g and the function ψ, such that for all $(A,b) \in (0,+\infty)^2$, and $\kappa_b \geq C^{(3)}_{g,\psi} A b + 1$ the inequality

$$\begin{aligned} C\left(1 + C^{(3)}_{g,\psi} A^2 b^2\right) \left\| \left(\frac{h\kappa_b}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} &\geq \frac{1}{4}\left\|\left(\frac{\kappa_b h}{b} + \hat{\mathscr{O}}_{b,h}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \frac{1}{4}\left\|\left(\frac{1}{b}p\cdot\partial_q + hf^{ij}_k(q,\ell)p_i p_j \partial_{p_k} - ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \frac{1}{2}\left\|\left(\left|\frac{h}{b}D_q\right|^{\frac{2}{3}} + \frac{h}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \frac{1}{2}\left\|\left(h^2\frac{|\lambda|}{\sqrt{hb} + |p|}\right)^{\frac{2}{3}} u\right\|^2_{L^2(\mathbb{R}^{2d})} \end{aligned} \tag{2.6.1.7}$$

holds

$$\begin{aligned} &\textit{when either} \quad u \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{0,8},\mathbb{C}) \quad \textit{and} \quad (\lambda,\ell) \in \mathbb{R}\times\mathbb{N}, \\ &\qquad\quad \textit{or} \quad u \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{-1,8},\mathbb{C}) \quad \textit{and} \quad \lambda \in \mathbb{R} \quad \textit{for} \quad \ell = -1. \end{aligned}$$

Proof. We start with the inequality

$$\left\|\left(\frac{h\kappa_b}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{2}\left\|\left(\frac{h\kappa_b}{b} + \hat{\mathscr{P}}_{b,h,0} - ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} - \left\|hf^{ij}_k p_i p_j \frac{\partial u}{\partial p_k}\right\|^2_{L^2(\mathbb{R}^{2d})} \tag{2.6.1.8}$$

and

$$\|hf^{ij}_k p_i p_j \frac{\partial u}{\partial p_k}\|^2_{L^2(\mathbb{R}^{2d})} \leq [64 C^{(1)}_{g,\psi} A]^2 \sum_j \|hD_{p_j}u\|^2_{L(\mathbb{R}^{2d})}, \tag{2.6.1.9}$$

which comes from $|p| \leq 8$ and (2.4.3.13). With $\kappa_b \geq 1$ and $C \geq 1$ given by Proposition 2.5.1, the inequality (2.5.0.2) combined with

$$\begin{aligned} \left\|\left(\frac{h\kappa_b}{b} + Q - ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} &= \left\|\left(\frac{h}{b} + Q - ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\frac{h(\kappa_b - 1)}{b}u\right\|^2_{L^2(\mathbb{R}^{2d})} \\ &\quad + \underbrace{2\left\langle\frac{h(\kappa_b-1)}{b}u, \left(\frac{h}{b} + \hat{\mathscr{O}}_{b,h}\right)u\right\rangle}_{\geq 0} \end{aligned}$$

$$\text{for} \quad Q = \hat{\mathscr{P}}_{b,h,0} \text{ or } Q = \hat{\mathscr{O}}_{b,h} = \frac{1}{2}\left(-h^2\Delta_p + \frac{|p|^2}{b^2}\right),$$

implies

$$C\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,0}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}\geq\frac{1}{2}\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{O}}_{b,h}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\|\left(\frac{1}{b}p\cdot\partial_q-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$+\left\|\left(\left|\frac{h}{b}D_q\right|^{\frac{2}{3}}+\frac{h}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\|\left(h^2\frac{|\lambda|}{\sqrt{hb}+|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}. \tag{2.6.1.10}$$

Combining (2.6.1.8)(2.6.1.9) and (2.6.1.10) implies

$$C\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}\geq\frac{1}{4}\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{O}}_{b,h}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2}\left\|\left(\frac{1}{b}p\cdot\partial_q-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$+\frac{1}{2}\left\|\left(\left|\frac{h}{b}D_q\right|^{\frac{2}{3}}+\frac{h}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2}\left\|\left(h^2\frac{|\lambda|}{\sqrt{hb}+|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$-C[64C^{(1)}_{g,\psi}A]^2\sum_j\|hD_{p_j}u\|^2_{L^2(\mathbb{R}^{2d})}. \tag{2.6.1.11}$$

Putting

$$\left\|\left(\frac{1}{b}p\cdot\partial_q-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}\geq\frac{1}{2}\left\|\left(\frac{1}{b}p\cdot\partial_q+hf^{ij}_kp_ip_j\partial_{p_k}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$-\left\|hf^{ij}_kp_ip_j\partial_{p_k}u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$\geq\frac{1}{2}\left\|\left(\frac{1}{b}p\cdot\partial_q+hf^{ij}_kp_ip_j\partial_{p_k}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$-[64C^{(1)}_{g,\psi}A]^2\sum_j\|hD_{p_j}u\|^2_{L^2(\mathbb{R}^{2d})}$$

into (2.6.1.11) gives

$$C\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}\geq\frac{1}{4}\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{O}}_{b,h}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{4}\left\|\left(\frac{1}{b}p\cdot\partial_q+hf^{ij}_kp_ip_j\partial_{p_k}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$+\frac{1}{2}\left\|\left(\left|\frac{h}{b}D_q\right|^{\frac{2}{3}}+\frac{h}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{2}\left\|\left(h^2\frac{|\lambda|}{\sqrt{hb}+|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$-2C[64C^{(1)}_{g,\psi}A]^2\sum_j\|hD_{p_j}u\|^2_{L^2(\mathbb{R}^{2d})}. \tag{2.6.1.12}$$

When $\kappa_b\geq C^{(2)}_{g,\psi}Ab$ the inequality (2.6.1.4) provides

$$\|hD_{p_j}u\|^2_{L^2(\mathbb{R}^{2d})}\leq 2^9b^2\left\|\left(\frac{h\kappa_b}{b}+\hat{\mathscr{P}}_{b,h,f}-ih\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}$$

and taking

$$C^{(3)}_{g,\psi}=\max(C^{(2)}_{g,\psi},2^{22}[C^{(1)}_{g,\psi}]^2)\geq 1\quad,\quad\kappa_b\geq C^{(3)}_{g,\psi}Ab+1,$$

yields the result. □

2.6.2 Comparison of the local models $\mathscr{P}_{b,\ell,m}$ and $\widetilde{\mathscr{P}}_{b,\ell,m}$

We now deduce subelliptic estimates for the local operator $\mathscr{P}_{b,\ell,m} = \frac{1}{b^2}\mathscr{O}_{\ell,m} + \frac{1}{b}\mathscr{Y}_{\ell,m}$ introduced in (2.4.3.7)(2.4.3.9)(2.4.3.10) from the one obtained in the previous paragraph for $\tilde{\mathscr{P}}_{b,\ell,m}$. It is a consequence of the upper bounds for the differences $(\tilde{\mathscr{P}}_{b,\ell,m} - \mathscr{P}_{b,\ell,m})$ and $(\tilde{\mathscr{O}}_\ell - \mathscr{O}_{\ell,m})$ studied in Proposition 2.4.4.

Proposition 2.6.4. *There exists a constant $C^{(4)}_{g,\psi} \geq 1$ such that for any $A, b > 0$, $\kappa_b \geq C^{(4)}_{g,\psi}(1+A)(1+b)$ and $2^{2\ell} \geq C^{(4)}_{g,\psi}(1+A)(1+b)A^2$ imply*

$$C^{(4)}_{g,\psi}\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{b^4}\|2^{2\ell}u\|^2_{L^2(\mathbb{R}^{2d})} \tag{2.6.2.1}$$

and

$$\begin{aligned} C^{(4)}_{g,\psi}(1+A)^2(1+b)^2\left\|\left(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} &\geq \left\|\frac{1}{b^2}\left(\kappa_b + \mathscr{O}_{\ell,m}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\frac{1}{b}\left(\mathscr{Y}_{\ell,m} - i\lambda\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \left\|\frac{1}{b^2}\left(\kappa_b + \tilde{\mathscr{O}}_\ell\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\left(\left|\frac{2^\ell}{b^2}D_q\right|^{\frac{2}{3}} + \frac{1}{b^2}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\left(\frac{1}{b^2}\frac{|\lambda|}{1+2^\ell|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}, \end{aligned} \tag{2.6.2.2}$$

when

$$\begin{aligned} &\textit{when either} && u \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{0,8},\mathbb{C}) \quad \textit{and} \quad (\lambda,\ell)\in\mathbb{R}\times\mathbb{N}, \\ &\textit{or} && u \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{-1,8},\mathbb{C}) \quad \textit{and} \quad \lambda\in\mathbb{R} \quad \textit{for} \quad \ell=-1. \end{aligned}$$

Proof. **a)** For the inequality (2.6.2.1) we must reconsider the proof of Proposition 2.6.2 after noticing

$$(bh)\mathscr{P}_{b,\ell,m} = \hat{\mathscr{P}}_{b,h,f} + \frac{-h^2(g_{ij}(2^{-\ell}q)-\delta_{ij})\partial_{p_i}\partial_{p_j} + (g^{ij}(2^{-\ell}q)-\delta^{ij})p_ip_j/b^2}{2}.$$

With $|g(2^{-\ell}q) - \mathrm{Id}| \leq C_{g,\psi}A^2 2^{-2\ell}$ we get

$$\begin{aligned} &|\langle u\,, \frac{-h^2(g_{ij}(2^{-\ell}q)-\delta_{ij})\partial_{p_i}\partial_{p_j} + (g^{ij}(2^{-\ell}q)-\delta^{ij})p_ip_j/b^2}{2} u\rangle| \\ &\qquad\qquad \leq C'_{g,\psi}A^2 2^{-2\ell}\left[\|hD_pu\|^2_{L^2(\mathbb{R}^{2d})} + \frac{1}{b^2}\|u\|^2_{L^2(\mathbb{R}^{2d})}\right] \end{aligned}$$

and

$$\begin{aligned} &\left|\langle u\,, (bh)(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - i\frac{\lambda}{b})u\rangle - \langle u\,, (\frac{h\kappa_b}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda)u\rangle\right| \\ &\qquad\qquad \leq C''_{g,\psi}A^2 2^{-2\ell}\mathrm{Re}\langle u\,, (\frac{h\kappa_b}{b} + \hat{\mathscr{P}}_{b,h,f} - ih\lambda)u\rangle. \end{aligned}$$

The lower bound for $\mathrm{Re}\langle u\,, (bh)(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - i\frac{\lambda}{b})u\rangle$ and by Cauchy-Schwarz for $\|(bh)(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - i\frac{\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})}$ are thus deduced from Proposition 2.6.2 when $2^{2\ell} \geq 2C''_{g,\psi}A^2$. The conditions and the inequality (2.6.2.1) are thus satisfied by taking $C^{(4)}_{g,\psi} \geq 2C''_{g,\psi}$ large enough.
b) Let us consider (2.6.2.2). We recall the inequality (2.4.3.23)

$$\|\frac{1}{b^2}\left(\mathscr{O}_{\ell,m} - \tilde{\mathscr{O}}_\ell\right)u\|_{L^2(\mathbb{R}^{2d})} \leq C_{g,\psi}A^2 2^{-2\ell}\|\left(\frac{\kappa_b}{b^2} + \frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|_{L^2(\mathbb{R}^{2d})},$$

which implies

$$\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|^2_{L^2(\mathbb{R}^{2d})}\geq\frac{1}{2}\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\mathscr{O}_{\ell,m}\right)u\|^2_{L^2(\mathbb{R}^{2d})}$$

$$\text{and}\quad\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|^2_{L^2(\mathbb{R}^{2d})}\geq\frac{1}{4}\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\mathscr{O}_{\ell,m}\right)u\|^2_{L^2(\mathbb{R}^{2d})}+\frac{1}{4}\|\left(\frac{\kappa_b}{b^2}+\frac{1}{b^2}\tilde{\mathscr{O}}_\ell\right)u\|^2_{L^2(\mathbb{R}^{2d})}$$

as soon as $2^{2\ell}\geq\frac{C_{g,\psi}A^2}{\sqrt{2}-1}$.
Proposition 2.6.1 holds under the sufficient condition $\kappa_b\geq C^{(3)}_{g,\psi}Ab+1$ which can be simplified into $\kappa_b\geq C^{(3)}_{g,\psi}(1+A)(1+b)$ while the left-hand side of (2.6.1.2) can be replaced by

$$4CC^{(3)}_{g,\psi}(1+A)^2(1+b)^2\left\|\left(\frac{\kappa_b}{b^2}+\tilde{\mathscr{P}}_{b,\ell,m}-\frac{i\lambda}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}.$$

Therefore Proposition 2.6.1 and Proposition 2.4.4 say

$$\begin{aligned}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-i\frac{\lambda}{b})u\|_{L^2(\mathbb{R}^{2d})} &\geq \left(1-2\sqrt{CC^{(3)}_{g,\psi}}(1+A)(1+b)C_{g,\psi}A^22^{-2\ell}\right)\|(\frac{\kappa_b}{b^2}+\tilde{\mathscr{P}}_{b,\ell,m}-i\frac{\lambda}{b})u\|_{L^2(\mathbb{R}^{2d})}\\ &\geq \frac{1}{2}\|(\frac{\kappa_b}{b^2}+\tilde{\mathscr{P}}_{b,\ell,m}-i\frac{\lambda}{b})u\|_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

as soon as

$$2^{2\ell}\geq 4\sqrt{CC^{(3)}_{g,\psi}}C_{g,\psi}(1+A)(1+b)A^2.$$

When all these conditions are satisfied this proves the seeked inequality with the upper bound

$$64CC^{(3)}_{g,\psi}(1+A)^2(1+b)^2\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-i\frac{\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})}.$$

The result is thus proved by choosing

$$C^{(4)}_{g,\psi}\geq\max(4\sqrt{CC^{(3)}_{g,\psi}}C_{g,\psi},64CC^{(3)}_{g,\psi})$$

where the right-hand side is larger than $C^{(3)}_{g,\psi}$ and for which $C^{(4)}_{g,\psi}(1+A)(1+b)A^2\geq\frac{C_{g,\psi}A^2}{\sqrt{2}-1}$. □

2.6.3 Estimate for $|p|\sim 2^\ell$ large

We now prove subelliptic estimates for $\mathscr{P}_{b,\ell}$ by summing the local subelliptic estimates for $\mathscr{P}_{b,\ell,m}$. All the error terms coming from the partition of unity $\sum_{m\in\mathbb{Z}^d}\psi^2(.-m)\equiv 1$ and studied in Subsections 2.4.2, 2.4.3 and 2.4.4, happen to be relatively small enough when the parameter $A=A_\infty(b)$ is much larger than 1 and ℓ is large enough so that $2^{2\ell}\gg[A_\infty(b)]^3\times(1+b)$.

Proposition 2.6.5. *Let $\mathscr{P}_{b,\ell}$ and $S_{\ell,2}=S_{0,2}\subset\Omega\times\mathbb{R}^d$ be defined respectively by* (2.4.1.14) *and* (2.4.1.18) *for $\ell\in\mathbb{N}$.*
There exists a constant $C^{(5)}_{g,\psi}\geq 1$ such that $A_\infty(b)=C^{(5)}_{g,\psi}(1+b)$, $\kappa_b\geq C^{(5)}_{g,\psi}A_\infty(b)\times(1+b)\geq[C^{(5)}_{g,\psi}]^2(1+b)^2$ and $2^{2\ell}\geq C^{(5)}_{g,\psi}[A_\infty(b)]^3(1+b)=[C^{(5)}_{g,\psi}]^4(1+b)^4$ imply

$$\begin{aligned}[C^{(5)}_{g,\psi}]^3(1+b)^4\left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})}\geq\left\|\frac{1}{b^2}(\kappa_b+\mathscr{O}_\ell)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\|\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)u\right\|^2_{L^2(\mathbb{R}^{2d})}\\ +\frac{1}{b^{8/3}}\left[\left\|(\tilde{\mathscr{O}}_\ell)^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\||D_q|^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}\right]+\left\|\left(\frac{1}{b^2}\frac{|\lambda|}{1+2^\ell|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})},\end{aligned}\quad(2.6.3.1)$$

when $u\in\mathscr{C}^\infty_0(S_{0,2};\mathbb{C})$, $\lambda\in\mathbb{R}$ and $b>0$.

Proof. Our conditions and $A = A_\infty(b) = C^{(5)}_{g,\psi}(1+b) \geq 1$ and $2^{2\ell} \geq C^{(5)}_{g,\psi}A^3(1+b) \geq C^{(5)}_{g,\psi}A^2$ imply $2^{-\ell}A \leq \frac{1}{\sqrt{C^{(5)}_{g,\psi}}} \leq \frac{1}{C_{g,\psi}}$ when $C_{g,\psi} \geq 1$ is the constant of Proposition 2.4.5 and $C^{(5)}_{g,\psi}$ is chosen larger than $C^2_{g,\psi}$. With $\kappa_b \geq C^{(5)}_{g,\psi}A(1+b) \geq C_{g,\psi}Ab$, Proposition 2.4.5 says

$$\sum_{m\in\mathbb{Z}^d} \frac{1}{8}\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} - \frac{C_{g,\psi}}{A^2b^2}\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})} \leq \|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})},$$

with

$$\forall m \in \mathbb{Z}^d, \quad u_{\ell,m} = \mathscr{U}_{\ell,m}(\psi_{m,\ell,A}u) \in \mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A) \times \mathbb{R}^d) \cap S'_{0,8}, \mathbb{C}).$$

By choosing $C^{(5)}_{g,\psi}$ larger than the constant $2\times C^{(4)}_{g,\psi}$ of Proposition 2.6.4, the condition $\kappa_b \geq C^{(5)}_{g,\psi}A(1+b) \geq C^{(4)}_{g,\psi}(1+A)(1+b)$ and the inequality (2.6.2.1) imply

$$\forall m \in \mathbb{Z}^d, \quad C^{(4)}_{g,\psi}\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{b^4}\|2^{2\ell}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})},$$

With $\frac{C^{(4)}_{g,\psi}C_{g,\psi}b^2}{A^2} \leq \frac{1}{16}$ when $A \geq C^{(5)}_{g,\psi}(1+b)$ and $C^{(5)}_{g,\psi}$ is chosen larger than $4\sqrt{C^{(4)}_{g,\psi}C_{g,\psi}}$, we obtain

$$\frac{1}{16}\sum_{m\in\mathbb{Z}^d}\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \leq \|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})},$$

and the two lower bounds of $\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}$ of Proposition 2.6.4 can be used for every $m \in \mathbb{Z}^d$. The first one (2.6.2.1) already used gives now

$$\frac{1}{32}\sum_{m\in\mathbb{Z}^d}\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} + \frac{1}{C^{(4)}_{g,\psi}b^4}\|2^{2\ell}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \leq \|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})}.$$

The second one (2.6.2.2) with here $2A \geq (1+A)$ and combined with the second inequality of Proposition 2.4.6 it implies

$$\begin{aligned}
2^{10}C^{(4)}_{g,\psi}A^3(1+b)\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \quad &\geq \|\frac{1}{b^2}(\kappa_b + \mathscr{O}_\ell)u\|^2_{L^2(\mathbb{R}^{2d})} + \|\frac{1}{b}(\mathscr{Y}_\ell - i\lambda)u\|^2_{L^2(\mathbb{R}^{2d})} \\
&+ \left\|\left(\frac{1}{b^2}\frac{|\lambda|}{1+2^\ell|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})} \\
&+ \sum_{m\in\mathbb{Z}^d}\left(\frac{1}{C^{(4)}_{g,\psi}b^4} - \frac{C_{g,\psi}}{A^2b^2} - \frac{C_{g,\psi}}{b^2 2^{2\ell}}\right)\|2^{2\ell}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \\
&+ \left\|\frac{1}{b^2}(\kappa_b + \tilde{\mathscr{O}}_\ell)u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})} + \frac{1}{b^{8/3}}\left\|\left|2^\ell D_q\right|^{\frac{2}{3}}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}.
\end{aligned}$$

Moreover $\kappa_b \geq C^{(5)}_{g,\psi}A(1+b) \geq (1+b)^2$ implies $\frac{1}{b^2}(\kappa_b + \tilde{\mathscr{O}}_\ell) \geq 1$ and interpolation gives

$$\left\|\frac{1}{b^2}(\kappa_b + \tilde{\mathscr{O}}_\ell)u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{b^{8/3}}\left\|(\kappa_b + \tilde{\mathscr{O}}_\ell)^{2/3}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{1}{b^{8/3}}\left\|\tilde{\mathscr{O}}_\ell^{2/3}u_{\ell,m}\right\|_{L^2(\mathbb{R}^{2d})}.$$

Proposition 2.4.7 implies

$$\begin{aligned}
\sum_{m\in\mathbb{Z}^d}\|\tilde{\mathscr{O}}_\ell^{2/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} + \||2^\ell D_q|^{2/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \quad &\geq \quad \frac{1}{C_{g,\psi}}\left[\|\tilde{\mathscr{O}}_\ell^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})} + \||D_q|^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}\right] \\
&- \sum_{m\in\mathbb{Z}^d}\frac{1}{A^{4/3}}\underbrace{\|2^{2\ell/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}}_{\leq\|2^{2\ell}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}}
\end{aligned}$$

The complete error term in the lower bound is now bounded from below by

$$\frac{1}{b^4}\sum_{m\in\mathbb{Z}^d}\left(\frac{1}{C^{(4)}_{g,\psi}}-\frac{C_{g,\psi}b^2}{A^2}-\frac{C_{g,\psi}b^2}{2^{2\ell}}-\frac{b^{4/3}}{A^{4/3}}\right)\|2^{2\ell}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}$$

which is non negative when

$$\begin{aligned} & A=C^{(5)}_{g,\psi}(1+b)\\ \text{and}\qquad & 2^{2\ell}\geq C^{(5)}_{g,\psi}A^3(1+b)\geq[C^{(5)}_{g,\psi}]^4(1+b)^4\geq C^{(5)}_{g,\psi}(1+b)^2\end{aligned}$$

with $C^{(5)}_{g,\psi}$ large enough. The inequality (2.6.3.1) is then obtained by taking $C^{(5)}_{g,\psi}\geq 2^8C^{(4)}_{g,\psi}C_{g,\psi}$. □

2.6.4 Estimate for $|p|\sim 2^\ell\leq C_{b,g,\psi}$

We have fixed the value of $A_\infty(b)=C^{(5)}_{g,\psi}(1+b)\gg 1$ in Proposition 2.6.5 in order to get a result for all $\ell\geq\ell_{b,g,\psi}+1$ with

$$2^{2\ell_{b,g,\psi}+2}\geq[C^{(5)}_{g,\psi}]^4(1+b)^4\geq 2^{2\ell_{b,g,\psi}}\,. \tag{2.6.4.1}$$

We now consider all the bounded values of $\ell\in\{-1,0,1,\ldots,\ell_{b,g,\psi}\}$. Here the error terms related with the partition of unity $\sum_{m\in\mathbb{Z}^d}\psi^2(.-m)\equiv 1$ will be relatively small by taking a new, small enough, value for the intermediate parameter $A>0$ denoted by $A_0(b)$ and the parameter $\kappa_b\gg 1$.

Proposition 2.6.6. *Let $\ell_{b,g,\psi}\in\mathbb{N}$ be such that* (2.6.4.1) *is satisfied. For all $\ell\in\{-1,0,1,\ldots,\ell_{b,g,\psi}\}$ let $\mathscr{P}_{b,\ell}$ and $S_{\ell,2}\subset\Omega\times\mathbb{R}^d$ be defined respectively by* (2.4.1.14) *and* (2.4.1.18).
Take $A_0(b)=\frac{1}{2C'_{g,\psi}(1+b)}$ where $C'_{g,\psi}=\max(C_{g,\psi},C^{(4)}_{g,\psi})$, $C_{g,\psi}\geq 1$ is given by Proposition 2.4.5 and Proposition 2.4.7 while $C^{(4)}_{g,\psi}\geq 1$ is given by Proposition 2.6.4.
There exists a constant $C^{(6)}_{g,\psi}\geq 1$ such that $\kappa_b\geq C^{(6)}_{g,\psi}(1+b)^5$ implies

$$\begin{aligned} C^{(6)}_{g,\psi}(1+b)^2\left\|\left(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b}\right)u\right\|^2_{L^2(\mathbb{R}^{2d})} &\geq\left\|\frac{1}{b^2}(\kappa_b+\mathscr{O}_\ell)u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\|\frac{1}{b}(\mathscr{Y}_\ell-i\lambda)u\right\|^2_{L^2(\mathbb{R}^{2d})}\\ &+\frac{1}{b^{8/3}}\left[\left\|(\tilde{\mathscr{O}}_\ell)^{2/3}u\right\|^2_{L^2(\mathbb{R}^{2d})}+\left\||D_q|^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}\right]+\left\|\left(\frac{1}{b^2}\frac{|\lambda|}{1+2^\ell|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})}, \end{aligned}\tag{2.6.4.2}$$

when $u\in\mathscr{C}^\infty_0(S_{\ell,2};\mathbb{C})$, $\ell\in\{-1,0,1,\ldots,\ell_{b,g,\psi}\}$, $\lambda\in\mathbb{R}$ and $b>0$.

Proof. The proof has the same structure as the one of Proposition 2.6.5. Choose now $A=A_0(b)=\frac{1}{2C'_{g,\psi}(1+b)}$, where the constant $C'_{g,\psi}$ will be fixed later in the proof.
With $2^{-\ell}A\leq 2A\leq\frac{1}{C'_{g,\psi}(1+b)}\leq\frac{1}{C_{g,\psi}}$ which combined with $\kappa_b\geq C^{(6)}_{g,\psi}(1+b)^6\geq\frac{b}{2(1+b)}\geq C_{g,\psi}Ab$ in Proposition 2.4.5, implies

$$\sum_{m\in\mathbb{Z}^d}\frac{1}{8}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell,m}-\frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}-\underbrace{\frac{C_{g,\psi}}{A^2b^2}}_{\leq\frac{4[C'_{g,\psi}]^3(1+b)^2}{b^2}}\underbrace{\left\|2^{2\ell}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}}_{\times[C^{(5)}_{g,\psi}]^8(1+b)^8\|u\|^2_{L^2(\mathbb{R}^{2d})}}\leq\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})},$$

with

$$u_{\ell,m}=\mathscr{U}_{\ell,m}(\psi_{m,\ell,A}u)\in\mathscr{C}^\infty_0((B(0,\hat{C}'_{g,\psi}A)\times\mathbb{R}^d)\cap S'_{0,8},\mathbb{C}).$$

for all $m \in \mathbb{Z}^d$ and all $\ell \in \{-1,0,1,\dots,\ell_{b,g,\psi}\}$.
The same integration by parts argument as for Proposition 2.2.1 says that for $\kappa_b \geq C^{(6)}_{g,\psi}(1+b)^6 \geq C^{(6)}_{g,\psi}(1+b)^2$ with $C^{(6)}_{g,\psi} \geq 1$ large enough

$$\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{\kappa_b^2}{16b^4}\|u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{[C^{(6)}_{g,\psi}]^2(1+b)^{10}}{16b^2}\|u\|^2_{L^2(\mathbb{R}^{2d})}.$$

The bound $2^{2\ell} \leq [C^{(5)}_{g,\psi}]^4(1+b)^4$ implies

$$\begin{aligned}\frac{[C^{(6)}_{g,\psi}]^2(1+b)^2}{b^2}\sum_{m\in\mathbb{Z}}\|2^{2\ell}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} &\leq \frac{[C^{(6)}_{g,\psi}]^2[C^{(5)}_{g,\psi}]^8(1+b)^{10}}{b^2}\|u\|^2_{L^2(\mathbb{R}^{2d})} \\ &\leq 16[C^{(5)}_{g,\psi}]^8\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})}.\end{aligned}$$

The additional condition $[C^{(6)}_{g,\psi}]^2 \geq 8[C'_{g,\psi}]^3$ thus implies

$$16[C^{(5)}_{g,\psi}]^8\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \geq \frac{4[C'_{g,\psi}]^3[C^{(5)}_{g,\psi}]^8(1+b)^{10}}{b^2}\|u\|^2_{L^2(\mathbb{R}^{2d})} + \underbrace{8[C^{(5)}_{g,\psi}]^8}_{\geq 1+4}\|(\frac{\kappa_b}{b^2}+\mathscr{P}_{b,\ell}-\frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})}$$

and

$$\begin{aligned}16[C^{(5)}_{g,\psi}]^8\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{m\in\mathbb{Z}^d} &\frac{1}{8}\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \frac{\kappa_b^2}{4b^4}\left\|u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

Because $2^\ell \geq 1/2 \geq C^{(4)}_{g,\psi}(1+b)A \geq C^{(4)}_{g,\psi}(1+A)(1+b)A^2$, the condition of Proposition 2.6.4 are satisfied. Multiplying the above inequality by $8 \times C^{(4)}_{g,\psi}4(1+b)^2$ which is larger than $8C^{(4)}_{g,\psi}(1+A)^2(1+b)^2 \geq 1$, leads to

$$\begin{aligned}2^9C^{(4)}_{g,\psi}(1+b)^2[C^{(5)}_{g,\psi}]^8\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \geq \sum_{m\in\mathbb{Z}^d} &C^{(4)}_{g,\psi}(1+A)^2(1+b)^2\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell,m} - \frac{i\lambda}{b})u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \frac{2\kappa_b^2(1+b)^2}{b^4}\left\|u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

Proposition 2.6.4 combined with Proposition 2.4.6 leads to

$$\begin{aligned}2^9C^{(4)}_{g,\psi}(1+b)^2[C^{(5)}_{g,\psi}]^8\|(\frac{\kappa_b}{b^2} + \mathscr{P}_{b,\ell} - \frac{i\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} \quad &\geq \|\frac{1}{b^2}(\kappa_b + \mathscr{O}_\ell)u\|^2_{L^2(\mathbb{R}^{2d})} + \|\frac{1}{b}(\mathscr{Y}_\ell - i\lambda)u\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \left\|\left(\frac{1}{b^2}\frac{|\lambda|}{1+2^\ell|p|}\right)^{\frac{2}{3}}u\right\|^2_{L^2(\mathbb{R}^{2d})} \\ + \sum_{m\in\mathbb{Z}^d}&\left(\frac{2\kappa_b^2(1+b)^2}{b^4} - \frac{C_{g,\psi}2^{4\ell}}{A^2b^2} - \frac{C_{g,\psi}2^{4\ell}}{b^2 2^{2\ell}}\right)\|u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} \\ &+ \frac{1}{b^{8/3}}\left[\left\|\tilde{\mathscr{O}}_\ell^{2/3}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})} + \left\|\left|2^\ell D_q\right|^{\frac{2}{3}}u_{\ell,m}\right\|^2_{L^2(\mathbb{R}^{2d})}\right]\end{aligned}$$

where we used that $\frac{\kappa_b+\tilde{\mathcal{O}}_\ell}{b^2}\geq 1$ when $\kappa_b\geq C^{(6)}_{g,\psi}(1+b)^6\geq C^{(6)}_{g,\psi}(1+b)^2$, as in the proof of Proposition 2.6.5. Proposition 2.4.7 implies

$$\begin{aligned}\sum_{m\in\mathbb{Z}^d}\|\tilde{\mathcal{O}}_\ell^{2/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}+\||2^\ell D_q|^{2/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})} &\geq \frac{1}{C_{g,\psi}}\left[\|\tilde{\mathcal{O}}_\ell^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}+\||D_q|^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}\right]\\ &\quad -\sum_{m\in\mathbb{Z}^d}\frac{1}{A^{4/3}}\|2^{2\ell/3}u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}\\ &\geq \frac{1}{C_{g,\psi}}\left[\|\tilde{\mathcal{O}}_\ell^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}+\||D_q|^{2/3}u\|^2_{L^2(\mathbb{R}^{2d})}\right]\\ &\quad -\underbrace{2^{4/3}[C'_{g,\psi}]^{4/3}(1+b)^{4/3}[C^{(5)}_{b,\psi}]^{8/3}(1+b)^{8/3}}_{\leq 4[C'_{g,\psi}]^3[C^{(5)}_{g,\psi}]^8(1+b)^4}\|u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

The complete error term in the lower bound is now bounded from below by

$$\sum_{m\in\mathbb{Z}^d}\left(\frac{\kappa_b^2(1+b)^2}{b^4}-4[C'_{g,\psi}]^3[C^{(5)}_{g,\psi}]^8\left(\frac{(1+b)^{10}+(1+b)^8}{b^2}+\frac{(1+b)^4}{b^{8/3}}\right)\right)\|u_{\ell,m}\|^2_{L^2(\mathbb{R}^{2d})}$$

which is non negative as soon as

$$\frac{\kappa_b^2(1+b)^2}{b^4}\geq 12[C'_{g,\psi}]^3[C^{(5)}_{g,\psi}]^8\frac{(1+b)^{10}}{b^2}.$$

A sufficient condition is $\kappa_b\geq C^{(6)}_{b,\psi}(1+b)^5$ with $C^{(6)}_{g,\psi}\geq 1$ large enough.
For the final writing of the the inequality we also take $C^{(6)}_{g,\psi}\geq 2^9C_{g,\psi}C^{(4)}_{g,\psi}[C^{(5)}_{g,\psi}]^8$. □

2.6.5 Lower bound for $\|(P_b-i\lambda/b)u\|_{L^2}+1/b^2\|u\|_{L^2}$

After the dyadic partition of unity for $\sum_{\ell=-1}^{\infty}\theta_\ell^2(q,p)\equiv 1$ on $\Omega\times\mathbb{R}^d$ of Subsection 2.4.1 and by setting

$$u_\ell(q,p)=2^{\ell d/2}\theta_\ell(q,2^\ell p)u(q,2^\ell p)\in\mathcal{C}_0^\infty(S_{2,\ell};\mathbb{C})\quad\text{for } u\in\mathcal{C}_0^\infty(\Omega\times\mathbb{R}^d;\mathbb{C}),$$

the results of Proposition 2.6.5 and Proposition 2.6.6 give after a rescaling

$$\begin{aligned}C^{(7)}_{g,\psi}(1+b)^4\|(\frac{\kappa_b}{b^2}+\mathcal{P}_b-i\frac{\lambda}{b})\theta_\ell u\|^2_{L^2(\mathbb{R}^{2d})} &\geq \|\frac{(\kappa_b+\mathcal{O})}{b^2}\theta_\ell u\|^2_{L^2(\mathbb{R}^{2d})}+\|\frac{1}{b}(\mathcal{Y}-i\lambda)\theta_\ell u\|^2_{L^2(\mathbb{R}^{2d})}\\ &\quad+\frac{1}{b^{8/3}}\|\theta_\ell u\|^2_{\tilde{\mathcal{W}}^{2/3}}+\frac{1}{b^{8/3}}\|\frac{|\lambda|}{\langle p\rangle}\theta_\ell u\|^2_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

for all $\ell\in\mathbb{Z}$, $\ell\geq -1$, as soon as $\kappa_b\geq C^{(7)}_{g,\psi}(1+b)^5$ and $C^{(7)}_{g,\psi}\geq 1$ is chosen large enough. By summation with respect to $\ell\geq -1$, Proposition 2.4.1 and the same result with $\mathcal{P}_b-i\lambda/b$ replaced by $\frac{1}{b^2}\mathcal{O}$ and Proposition 2.E.7 for $\|\ \|_{\tilde{\mathcal{W}}^{2/3}}$ imply

$$\begin{aligned}C^{(8)}_{g,\psi}(1+b)^4\|(\frac{\kappa_b}{b^2}+\mathcal{P}_b-i\frac{\lambda}{b})u\|^2_{L^2(\mathbb{R}^{2d})} &\geq \|\frac{(\kappa_b+\mathcal{O})}{b^2}u\|^2_{L^2(\mathbb{R}^{2d})}+\|\frac{1}{b}(\mathcal{Y}-i\lambda)u\|^2_{L^2(\mathbb{R}^{2d})}\\ &\quad+\frac{1}{b^{8/3}}\|u\|^2_{\tilde{\mathcal{W}}^{2/3}}+\frac{1}{b^{8/3}}\|\frac{|\lambda|}{\langle p\rangle}u\|^2_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

for all $u\in\mathcal{C}_0^\infty(\Omega\times\mathbb{R}^d;\mathbb{C})$ as soon as $\kappa_b\geq C^{(8)}_{g,\psi}(1+b)^5$ with $C^{(8)}_{g,\psi}\geq 1$ large enough.
By taking $C_{g,\mathcal{M}}\geq C^{(8)}_{g,\psi}$ large enough so that the comparison results of Proposition 2.2.4 and Proposition 2.E.7-**ii)** can be applied with

$$\begin{aligned}C_{g,\mathcal{M}}(1+b)^2\|(\frac{\kappa_b}{b^2}+P_b-i\frac{\lambda}{b})u\|_{L^2(\mathbb{R}^{2d})} &\geq \|\frac{(\kappa_b+\mathcal{O})}{b^2}u\|_{L^2(\mathbb{R}^{2d})}+\|\frac{1}{b}(\nabla^{\mathcal{E}}_{\mathcal{Y}}-i\lambda)u\|_{L^2(\mathbb{R}^{2d})}\\ &\quad+\frac{1}{b^{4/3}}\|u\|_{\tilde{\mathcal{W}}^{2/3}}+\frac{1}{b^{4/3}}\|\frac{|\lambda|}{\langle p\rangle}u\|_{L^2(\mathbb{R}^{2d})}\end{aligned}$$

for all $u \in \mathscr{C}_0^\infty(X;\mathscr{E})$, when $\kappa_b \geq C_{g,\mathscr{M}}(1+b)^5$
By writing

$$C_{g,\mathscr{M}}(1+b)^2\kappa_b\left(\|(P_b - i\frac{\lambda}{b})u\|_{L^2(\mathbb{R}^{2d})} + \frac{1}{b^2}\|u\|_{L^2(\mathbb{R}^{2d})}\right) \geq C_{g,\mathscr{M}}(1+b)^2\|(\frac{\kappa_b}{b^2} + P_b - i\frac{\lambda}{b})u\|_{L^2(\mathbb{R}^{2d})}$$

and by noticing that the factor of the left-hand side can be

$$C_{g,\mathscr{M}}(1+b)^2 \times C_{g,\mathscr{M}}(1+b)^5 = C_{g,\mathscr{M}}(1+b)^7$$

this ends the proof of the inequality (2.1.3.2) in Theorem 2.1.6.
The essential maximal accretivity result for $\kappa_b \geq C_0(1+b^2)$, $C_0 \geq 1$ determined by (g, E, g^E, ∇^E), was proved in Proposition 2.2.1 and Corollary 2.2.2.

2.7 Consequences and optimality of Theorem 2.1.6

In this subsection we will discuss the optimality of the constants appearing in Theorem 2.1.6 as well as several consequences and extensions of the subelliptic estimate.

2.7.1 About b-dependent constants

Obviously the constant $\frac{1}{C_g(1+b)^7}$ or the condition $\kappa_b \geq C_g(1+b)^5$ of Theorem 2.1.6 can be replaced by a uniform constant and a uniform lower bound for κ_b when $b \in]0, b_0]$ for some $b_0 > 0$. The question arises about the regime $b \to +\infty$. We do not claim that neither our lower bound nor the condition on κ_b are optimal with respect to b as $b \to +\infty$, but they cannot be written with uniform constants.
Actually, we show here that scalar GKFP operators admit quasimodes at $\lambda = 0$ of size $\mathscr{O}(b^{-2})$ as $b \to \infty$.

Proposition 2.7.1. *Let $P_{\pm,b} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\mathscr{Y}$ be the scalar GKFP operator on $X = T^*Q$ where the Hermitian bundle is $E = Q \times \mathbb{C}$ with ∇^E the trivial connection, g^E is the usual pointwise Hermitian inherited from $\mathbb{C}$, and $\mathscr{M}(b) = 0$. Then there exists $u \in C_0^\infty(X;\mathscr{E}) = C_0^\infty(X;\mathbb{C})$ with $\|u\|_{L^2(X;\mathscr{E})} = 1$ satisfying*

$$\left\|P_{\pm,b}u\right\|_{L^2(X;\mathscr{E})} \leq Cb^{-2}, \quad b \in (0,\infty), \tag{2.7.1.1}$$

for some constant $C > 0$ independent of b.

Proof. Let $u \in C^\infty(X;\mathbb{C})$ be any function of the form

$$u(q,p) = \varphi(|p|_q^2), \quad (q,p) \in X, \tag{2.7.1.2}$$

where $\varphi \neq 0$ belongs to $\mathscr{C}_0^\infty(\mathbb{R};\mathbb{C})$. By multiplying u by a non-zero constant if necessary, we may ensure that $\|u\|_{L^2(X;\mathbb{C})} = 1$. Since u is a function of the kinetic energy, we have

$$\mathscr{Y}u \equiv 0. \tag{2.7.1.3}$$

Hence

$$\left\|P_{\pm,b}u\right\|_{L^2(X;\mathbb{C})} \leq \frac{1}{b^2}\|\mathscr{O}u\| + \frac{1}{b}\|\mathscr{Y}u\| \leq Cb^{-2}, \quad b \in (0,\infty). \tag{2.7.1.4}$$

□

An immediate consequence of Proposition 2.7.1 is that the best possible constant appearing on the right-hand side of (2.1.6) fails to be independent of b in general. Indeed, let $P_{\pm,b} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\mathscr{Y}$ be a scalar GKFP operator as in Proposition 2.7.1 and let $C(b) > 0$ be the largest constant such that

$$\left\|\left(P_{\pm,b} - \frac{i\lambda}{b}\right)u\right\|_{L^2} + \frac{1}{b^2}\|u\|_{L^2} \geq C(b)\left(\left\|\frac{\mathscr{O}}{b^2}u\right\|_{L^2} + \left\|\frac{1}{b}\left(\nabla^{\mathscr{E}}_{\mathscr{Y}} - i\lambda\right)u\right\|_{L^2} + \frac{1}{b^{4/3}}\left[\|u\|_{\tilde{\mathscr{W}}^{\frac{2}{3}}} + \left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{L^2}\right]\right) \tag{2.7.1.5}$$

holds for every $u \in C_0^\infty(X;\mathbb{C})$. Taking $\lambda = 0$ in (2.7.1.5) gives

$$\|P_{\pm,b}u\|_{L^2(X;\mathbb{C})} + \frac{1}{b^2}\|u\|_{L^2(X;\mathbb{C})} \geq \frac{C(b)}{b^{4/3}}\|u\|_{\tilde{\mathscr{W}}^{\frac{2}{3}}(X;\mathbb{C})} \tag{2.7.1.6}$$

If $u \in C_0^\infty(X;\mathbb{C})$ is as in Proposition 2.7.1, then (2.7.1.1) and (2.7.1.6) together give

$$C(b) \leq Cb^{-\frac{2}{3}}, \quad b \in (0,\infty), \tag{2.7.1.7}$$

for some $C > 0$ independent of b. Because $b^{-2/3} \to 0$ as $b \to \infty$, there cannot exist a constant $C_0 > 0$ such that $C(b) \geq C_0$ for $b \gg 1$. In particular, $C(b)$ cannot be constant with respect to b.

2.7.2 Perturbative Estimate

In this subsection, we consider the stability of the subelliptic estimate (2.1.3.2) under a general class of perturbations.

Proposition 2.7.2. *Let $P_{\pm,b} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\nabla^{\mathscr{E}}_{\mathscr{Y}}$, let $M(b) \in \mathscr{L}(\tilde{\mathscr{W}}^{1,0}(X;\mathscr{E}); L^2(X, dqdp;\mathscr{E}))$ satisfy*

$$M(b) = M_1(b) + M_0(b)$$
$$\|M_1(b)\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)} \leq \frac{\nu_1(b)}{b} \quad \textit{and} \quad \|M_0(b)\|_{\mathscr{L}(L^2;L^2)} \leq \nu_0\left(1 + \frac{1}{b^2}\right),$$

and set

$$P_{\pm,b,M} = P_{\pm,b} + M_1(b) + M_0(b).$$

We assume that $\nu_1(b)$ and ν_0 satisfy

$$\forall b \in (0,+\infty), \quad \nu_1^2(b)b^2 \leq \frac{C_g + 8\nu_0}{16}(1 + b^2),$$

where $C_g \geq 1$ is the constant determined by (g, E, g^E, ∇^E) in Theorem 2.1.6. For $\kappa_b \geq (C_g + 16\nu_0)(1 + b^5)$, the operator $\frac{\kappa_b}{b^2} + P_{\pm,b,M}$ is closable and its closure $\frac{\kappa_b}{b^2} + \overline{P}_{\pm,b,M}$ is maximal accretive with $D(\overline{P}_{\pm,b,M}) = D(\overline{P}_{\pm,b})$ and

$$\forall u \in D(\overline{P}_{\pm,b,M}), \quad \mathrm{Re}\langle u, (\frac{\kappa_b}{b^2} + \overline{P}_{\pm,b,M})u\rangle \geq \frac{1}{8b^2}\left[\|u\|^2_{\tilde{\mathscr{W}}^{1,0}} + \kappa_b\|u\|^2_{L^2}\right].$$

Moreover, the inequalities

$$\left\|(\overline{P}_{\pm,b,M} - \frac{i\lambda}{b})u\right\|_{L^2} + \frac{1+b^2}{b^2}\|u\|_{L^2} \geq \frac{(1+b)^{-7}}{8(C_g + 16\nu_0)^2}\Big(\|\frac{\mathscr{O}}{b^2}u\|_{L^2} + \|\frac{1}{b}(\pm\nabla^{\mathscr{E}}_{\mathscr{Y}} - i\lambda)u\|_{L^2}$$
$$+ \frac{1}{b^{4/3}}\left[\|u\|_{\tilde{\mathscr{W}}^{\frac{2}{3}}} + \left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{L^2}\right]\Big)$$

and

$$\left\|(\overline{P}_{\pm,b,M}-\frac{i\lambda}{b})u\right\|_{L^2}+\frac{2\kappa_b}{b^2}\|u\|_{L^2}\geq\frac{1}{4C_g(1+b)^7}\Big(\|\frac{\mathcal{O}}{b^2}u\|_{L^2}+\|\frac{1}{b}(\pm\nabla^{\mathcal{E}}_{\mathcal{Y}}-i\lambda)u\|_{L^2}$$
$$+\frac{1}{b^{4/3}}\left[||u||_{\tilde{\mathcal{W}}^{\frac{2}{3}}}+\left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{L^2}\right]\Big)$$

hold for every $u\in D(\overline{P}_{\pm,b,M})$ *and every* $(\lambda,b)\in\mathbb{R}\times(0,+\infty)$.

Proof. Let us first check the accretivity of $P_{\pm,b,M}$ on $\mathcal{C}^\infty_0(X;\mathcal{E})$. For $u\in\mathcal{C}^\infty_0(X;\mathcal{E})$, write

$$\begin{aligned}
\mathrm{Re}\langle u\,,[\frac{\kappa_b}{b^2}+P_{\pm,b}+M_1(b)+M_0(b)]u\rangle_{L^2} &\geq \frac{1}{4b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_b\|u\|^2_{L^2}\right]-\frac{\nu_1(b)}{b}\|u\|_{L^2}\|u\|_{\tilde{\mathcal{W}}^{1,0}}\\
&\quad -\nu_0\frac{1+b^2}{b^2}\|u_0\|^2_{L^2}\\
&\geq \frac{1}{4b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_b\|u\|^2_{L^2}\right]-\frac{1}{8b^2}\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}\\
&\quad -(2\nu_1^2(b)+\nu_0\frac{1+b^2}{b^2})\|u\|^2_{L^2}\\
&\geq \frac{1}{8b^2}\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\frac{\kappa_b+(C_g+8\nu_0)(1+b^2)-16\nu_1^2(b)b^2}{8b^2}\|u\|^2_{L^2}\\
&\geq \frac{1}{8b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_b\|u\|^2_{L^2}\right].
\end{aligned}$$

This proves the accretivity of $P_{\pm,b,M}$ which is therefore closable.
From the inequality (3.2.7.8) for $\overline{P}_{\pm,b}$ we deduce for any $\lambda\in\mathbb{R}$

$$\|(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b}-i\lambda)u\|_{L^2}\|u\|_{L^2}\geq\mathrm{Re}\langle u\,,(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b})u\rangle_{L^2}\geq\frac{1}{4b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_b\|u\|^2_{L^2}\right]$$

and for all $t>0$

$$\frac{t}{2}\|(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b}-i\lambda)u\|^2_{L^2}\geq\frac{1}{4b^2}\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\frac{\kappa_b-2t^{-1}b^2}{4b^2}\|u\|^2_{L^2}.$$

This inequality with $t=\frac{1}{8\nu_1^2(b)}$ and

$$\kappa_b-2t^{-1}b^2=\kappa_b-16\nu_1^2(b)b^2\geq(C_g+16\nu_0)(1+b^2)-16\nu_1^2(b)b^2\geq0$$

gives

$$\frac{1}{4}\|(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b}-i\lambda)u\|^2_{L^2}\geq\frac{\nu_1^2(b)}{b^2}\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}\geq\|M_1(b)u\|^2_{L^2}.$$

Actually the inequality (3.2.7.8) also gives

$$\frac{1}{4}\|(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b}-i\lambda)u\|_{L^2}-\|M_0(b)u\|_{L^2}\geq\frac{\kappa_b}{16b^2}\|u\|_{L^2}-\frac{\nu_0(1+b^2)}{b^2}\|u\|_{L^2}\geq\frac{(C_g-16\nu_0)(1+b^2)}{16b^2}\|u\|_{L^2}\geq0.$$

Used with $\lambda=0$, we have two closed accretive operators $A=(\frac{\kappa_{\pm,b}}{b^2}+\overline{P}_{\pm,b})$ and $C=(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b,M})$ such that $D=\mathcal{C}^\infty_0(X;\mathcal{E})$ is dense in $D(A)=D(\overline{P}_{\pm,b})$ and $D(C)=D(\overline{P}_{\pm,b,M})$ while

$$\forall u\in D,\quad \|(A-C)u\|_{L^2}\leq\frac{3}{4}\|Au\|_{L^2},$$

with $\frac{3}{4}<1$ and A maximal accretive. Theorem X.50 of [ReSi] tells us that C is maximal accretive as well with $D(C)=D(A)$.
We can take $\kappa_b=(C_g+16\nu_0)(1+b^2)$, and $\|(A-C)u\|_{L^2}\leq\frac{3}{4}\|(A-i\lambda)u\|_{L^2}$ leads to

$$\begin{aligned}2(C_g+16\nu_0)\big[\|(\overline{P}_{\pm,b,M}-i\lambda)u\|_{L^2}+\frac{1+b^2}{b^2}\|u\|_{L^2}\big] &\geq \|(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b,M}-i\lambda)u\|_{L^2}+\frac{\kappa_b}{b^2}\|u\|_{L^2}\\ &\geq \frac{1}{4}\|(\frac{\kappa_b}{b^2}+\overline{P}_{\pm,b}-i\lambda)u\|_{L^2}+\frac{\kappa_b}{b^2}\|u\|_{L^2}\\ &\geq \frac{1}{4}\big[\|(\overline{P}_{\pm,b}-i\lambda)u\|_{L^2}+\frac{1}{b^2}\|u\|_{L^2}\big],\end{aligned}$$

and

$$\|(\overline{P}_{\pm,b,M}-i\lambda)u\|_{L^2}+\frac{2\kappa_b}{b^2}\|u\|_{L^2}\geq\frac{1}{4}\big[\|(\overline{P}_{\pm,b}-i\lambda)u\|_{L^2}+\frac{1}{b^2}\|u\|_{L^2}\big]$$

where, in both inequalities, the right-hand side is bounded from below by (3.2.7.9) of Theorem 2.1.6. □

2.7.3 $\widetilde{\mathcal{W}}^s$-versions

The subellitpic estimate of Theorem 2.1.6 wich is concerned with the case $L^2(X,dqp;\mathcal{E})=\widetilde{\mathcal{W}}^0(X;\mathcal{E})$ can be extended to $\widetilde{\mathcal{W}}^s(X;\mathcal{E})$ subelliptic estimates for any $s\in\mathbb{R}$ as follows.

Proposition 2.7.3. *Let $P_{\pm,b,M}=\frac{1}{b^2}\mathcal{O}\pm\frac{1}{b}\nabla^{\mathcal{E}}_{\mathcal{Y}}+M_1(b)+M_0(b)$ satisfy*

$$\forall s\in\mathbb{R},\quad \|M_1(b)\|_{\mathcal{L}(\widetilde{\mathcal{W}}^{1,s};\widetilde{\mathcal{W}}^s)}\leq\frac{\nu_{1,s}(b)}{b}\quad,\quad \|M_0(b)\|_{\mathcal{L}(\widetilde{\mathcal{W}}^s;\widetilde{\mathcal{W}}^s)}\leq\nu_{0,s}\frac{1+b^2}{b^2}$$

whenever

$$\forall b\in(0,+\infty),\quad \nu_{1,s}^2(b)b^2\leq\frac{C_g+8\nu_{0,s}}{16}(1+b^2).$$

For every $s\in\mathbb{R}$, there exists $C_{g,s}\geq 1$ determined by the geometric data (g,E,g^E,∇^E) and the pair $(\nu_{0,s},\nu_{1,s}(.))$ such that, for $\kappa_b\geq C_{g,s}(1+b^5)$, the operator $\frac{\kappa_b}{b^2}+P_{\pm,b,M}$ is closable in $\widetilde{\mathcal{W}}^s(X;\mathcal{E})$ and its closure $\frac{\kappa_b}{b^2}+\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b,M}$ is maximal accretive with $D(\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b,M})=D(\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b})=W_\theta^{-s}D(\overline{P_{\pm,b}}^{L^2})$ and

$$\forall u\in D(\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b,M}),\quad \mathrm{Re}\langle u\,,(\frac{\kappa_b}{b^2}+\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b,M})u\rangle_{\widetilde{\mathcal{W}}^s}\geq\frac{1}{8b^2}\left[\|u\|^2_{\widetilde{\mathcal{W}}^{1,s}}+\kappa_b\|u\|^2_{\mathcal{W}^s}\right].$$

Moreover, the inequalities

$$\begin{aligned}\left\|(\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b,M}-\frac{i\lambda}{b})u\right\|_{\widetilde{\mathcal{W}}^s}+\frac{1+b^2}{b^2}\|u\|_{\widetilde{\mathcal{W}}^s}\geq\frac{(1+b)^{-7}}{8C_{g,s}^2}\Bigg(&\|\frac{\mathcal{O}}{b^2}u\|_{\widetilde{\mathcal{W}}^s}+\|\frac{1}{b}(\pm\nabla^{\mathcal{E}}_{\mathcal{Y}}-i\lambda)u\|_{\widetilde{\mathcal{W}}^s}\\ &+\frac{1}{b^{4/3}}\left[||u||_{\widetilde{\mathcal{W}}^{s+\frac{2}{3}}}+\left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{\widetilde{\mathcal{W}}^s}\right]\Bigg)\end{aligned}$$

and

$$\begin{aligned}\left\|(\overline{P}^{\widetilde{\mathcal{W}}^s}_{\pm,b,M}-\frac{i\lambda}{b})u\right\|_{\widetilde{\mathcal{W}}^s}+\frac{\kappa_b}{b^2}\|u\|_{\widetilde{\mathcal{W}}^s}\geq\frac{1}{4C_g(1+b)^7}\Bigg(&\|\frac{\mathcal{O}}{b^2}u\|_{\widetilde{\mathcal{W}}^s}+\|\frac{1}{b}(\nabla^{\mathcal{E}}_{\mathcal{Y}}-i\lambda)u\|_{\widetilde{\mathcal{W}}^s}\\ &+\frac{1}{b^{4/3}}\left[||u||_{\widetilde{\mathcal{W}}^{s+\frac{2}{3}}}+\left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{\widetilde{\mathcal{W}}^s}\right]\Bigg)\end{aligned}$$

holds for every $u\in D(\overline{P}_{\pm,b,M})$ and every $(\lambda,b)\in\mathbb{R}\times(0,+\infty)$.

Proof. We use the pseudodifferential operator W_θ^2 introduced in Propostion 2.3.5 which is self-adjoint with $D(W_\theta^2) = \tilde{\mathscr{W}}^2(X;\mathscr{E})$ with an elliptic scalar principal symbol in $S^2_\Psi(Q;\mathrm{End}\mathscr{E})$ and for which we can write

$$\|u\|_{\tilde{\mathscr{W}}^{s_1,s_2}} = \|\mathscr{O}^{s_1/2}(W_\theta^2)^{s_2/2}u\|_{L^2},$$

for any $(s_1,s_2)\in\mathbb{R}^2$ according to Definition 2.3.7.
Because $\mathscr{O}$ and $(W_\theta^2)^{s/2}$ commute on $\mathscr{S}(X;\mathscr{E})$, according to Proposition 2.3.6, when considering the operator $P_{\pm,b,M}:\mathscr{S}(X;\mathscr{E})\to\tilde{\mathscr{W}}^s(X;\mathscr{E})$ we may instead work with the operator

$$(W_\theta^2)^{-s/2}P_{\pm,b,M}(W_\theta^2)^{s/2} = \frac{1}{b^2}\mathscr{O} \pm \frac{1}{b}\nabla^{\mathscr{E}}_{\mathscr{Y}} \pm \frac{1}{b}(W_\theta^2)^{-s/2}[\nabla^{\mathscr{E}}_{\mathscr{Y}},(W_\theta^2)^{s/2}] + (W_\theta^2)^{-s/2}(M_1(b)+M_0(b))(W_\theta^2)^{s/2}$$

initially defined from $\mathscr{S}(X;\mathscr{E})\to L^2(X;\mathscr{E})$.
The assumptions ensure

$$\|(W_\theta^2)^{-s/2}(M_1(b)(W_\theta^2)^{s/2}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)} \le \frac{\nu_{1,s}(b)}{b} \quad\text{and}\quad \|(W_\theta^2)^{-s/2}(M_0(b)(W_\theta^2)^{s/2}\|_{\mathscr{L}(L^2;L^2)} \le \nu_{0,s}(1+\frac{1}{b^2}).$$

With $(W_\theta^2)^{-s/2}[\nabla^{\mathscr{E}}_{\mathscr{Y}},(W_\theta^2)^{s/2}] = (W_\theta^2)^{-s/2}\nabla^{\mathscr{E}}_{\mathscr{Y}}(W_\theta^2)^{s/2} - \nabla^{\mathscr{E}}_{\mathscr{Y}}$, Proposition 2.3.8 tells us

$$\|(W_\theta^2)^{-s/2}[\nabla^{\mathscr{E}}_{\mathscr{Y}},(W_\theta^2)^{s/2}]\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)} \le \tilde{C}_{g,s} \tag{2.7.3.1}$$

for some constant $\tilde{C}_{g,s}\ge 1$ determined by (g,E,g^E,∇^E). It then suffices to apply Proposition 2.7.2 with $M_1(b)$ replaced by $M_1(b)\pm(W_\theta^2)^{-s/2}[\nabla^{\mathscr{E}}_{\mathscr{Y}},(W_\theta^2)^{s/2}]$, $\nu_1(b)$ replaced by $\nu_{1,s}(b)+\tilde{C}_{g,s}$ and ν_0 by $\nu_{0,s}+2\tilde{C}_{g,s}$. □

Appendix

2.A Comparison of harmonic oscillator hamiltonians

For a positive definite symmetric matrix $g = (g_{ij})_{1\leq i,j\leq d} \in \mathcal{M}_{dd}(\mathbb{R})$ with $g^{-1} = (g^{ij})_{1\leq i,j\leq d}$ let $\mathcal{O}_g$ denote the harmonic oscillator hamiltonian

$$\begin{aligned}\mathcal{O}_g &= \frac{-g_{ij}\partial_{p_i}\partial_{p_j} + g^{ij}p_ip_j}{2}\\ D(\mathcal{O}_g) &= \{u \in L^2(\mathbb{R}^d, dp), \forall \alpha,\beta \in \mathbb{N}^d, |\alpha|+|\beta| \leq 2, p^\alpha \partial_p^\beta u \in L^2(\mathbb{R}^d, dp)\} = D(\mathcal{O}_{\mathrm{Id}}).\end{aligned}$$

The following result is a consequence of the ellipticity of $\mathcal{O}_g$ in the Hörmander class $S(\langle p,\eta\rangle^2, \frac{dp^2+d\eta^2}{\langle p,\eta\rangle^2})$(see [HormIII]-Chap XVIII) combined with $\|\mathcal{O}_g u\|_{L^2} \geq \frac{d}{2}\|u\|_{L^2}$.

Proposition 2.A.1. *For two positive definite symmetric matrices g_1 and g_2 there exist two constants $C_{g_1,g_2} > 0$ and $C_{g_1} > 0$ such that*

$$\begin{aligned} &\left(\frac{\left\|\mathcal{O}_{g_1}u\right\|_{L^2}}{\left\|\mathcal{O}_{g_2}u\right\|_{L^2}}\right)^{\pm 1} \leq C_{g_1,g_2},\\ \textit{and}\quad &\left\|(\mathcal{O}_{g_2} - \mathcal{O}_{g_1})u\right\|_{L^2} \leq C_{g_1}\|g_2 - g_1\|_{\mathcal{M}_{dd}(\mathbb{R})}\left\|\mathcal{O}_{g_1}u\right\|_{L^2}\end{aligned}$$

hold for all $u \in D(\mathcal{O}_{\mathrm{Id}})$.

2.B Complex Airy Operator

We consider here the case of the one dimensional euclidean case of which the properties are due to the fact that the complex Airy operator has a compact resolvent and an empty spectrum. Set

$$P_1(\xi,\lambda) = i(p_1\xi - \lambda) - \frac{1}{2}\Delta_{p_1}$$

with $\xi,\lambda \in \mathbb{R}$. It is maximal accretive with $D(P_1(\xi,\lambda)) = \left\{u \in L^2(\mathbb{R}, dp), P_1(\xi,\lambda)u \in L^2(\mathbb{R}, dp)\right\}$ in which $\mathcal{C}_0^\infty(\mathbb{R})$ is dense (with respect to the graph norm).

Proposition 2.B.1. *There exists $C_0 \geq 1$, such that for all $(\xi,\lambda) \in \mathbb{R}^2$ the inequality*

$$C_0\|(1+P_1(\xi,\lambda))u\| \geq \|\frac{1}{2}\Delta_p u\| + \|(p_1\xi-\lambda)u\| + (|\xi|^{2/3}+1)\|u\| + \|\left(\frac{|\lambda|}{1+|p_1|}\right)^{\frac{2}{3}} u\|, \qquad (2.B.0.1)$$

holds for all $u \in D(P_1(\xi,\lambda))$.

Proof. The lower bound

$$\forall u\in D(P_1(\xi,\lambda))\,,\,\|(1+P_1(\xi,\lambda))u\|^2\geq\|u\|^2+\|P_1(\xi,\lambda)u\|^2$$

is due to the accretivity of $P_1(\xi,\lambda)$.
With

$$\begin{aligned}\|P_1(\xi,\lambda)u\|^2 &\geq \|\frac{1}{2}\Delta_{p_1}u\|^2+\|(\lambda-p_1\xi)u\|^2-|\langle\xi u\,,D_{p_1}u\rangle|\\ &\geq \|\frac{1}{2}\Delta_{p_1}u\|^2+\|(\lambda-p_1\xi)u\|^2-[c_4\||\xi|^{2/3}u\|^2+\frac{1}{8}\|\Delta_{p_1}u\|^2]\\ &\geq \frac{1}{2}\|\frac{1}{2}\Delta_{p_1}u\|^2+\|(\lambda-p_1\xi)u\|^2-c_4'\||\xi|^{2/3}u\|^2\,.\end{aligned}$$

Consider now the lower bound of $\|P_1(\xi,\lambda)u\|$ by $\||\xi|^{2/3}u\|$. It is obviously true for $\xi=0$. For $\xi\neq 0$, the operator

$$P_1(\xi,\lambda)=i(p-p_0)\xi-\frac{1}{2}\Delta_{p_1}\quad,\quad p_0=\frac{\lambda}{\xi}\,,$$

is unitarily equivalent to $|\xi|^{2/3}P_1(1,0)=|\xi|^{2/3}(ip_1-\frac{1}{2}\Delta_{p_1})$ and there exists $c_1>0$ such that

$$\forall u\in D(P_1(\xi,\lambda)),\quad \|P_1(\xi,\lambda)u\|\geq c_1|\xi|^{2/3}\|u\|\,.$$

There exists a constant $C_1\geq 1$ such that

$$\forall(\xi,\lambda)\in\mathbb{R}^2\,,\forall u\in D(P_1(\xi,\lambda)),\quad C_1\|P_1(\xi,\lambda)u\|^2\geq\|u\|^2+\|\frac{1}{2}\Delta_{p_1}u\|^2+\|(p_1\xi-\lambda)u\|^2+\||\xi|^{2/3}u\|^2\,.$$

The lower bound (2.B.0.1) is then obviously true for $|\lambda|\leq 1$ and it suffices to consider the case $\lambda\geq 1$.
Take a dyadic partition of unity $\chi_0^2(\varepsilon p_1)+\sum_{\ell=1}^{\infty}\chi^2(\varepsilon 2^{-\ell}p_1)\equiv 1$ with $\operatorname{supp}\chi_0\cup\operatorname{supp}\chi\subset[-4,4]$ and for all $\ell\in\mathbb{N}$, we define χ_ℓ by

$$\forall t\in\mathbb{R}_+,\quad \chi_\ell(t)=\begin{cases}\chi(\varepsilon 2^{-\ell}t) & \text{if } \ell\neq 0\\ \chi_0(\varepsilon t) & \text{if } \ell=0\,.\end{cases}$$

We get

$$\|P_1(\xi,\lambda)u\|^2-\sum_{\ell=0}^{\infty}\|P_1(\xi,\lambda)\chi_\ell u\|^2\leq C_\chi\varepsilon^2\left[\|\partial_{p_1}u\|^2+\|u\|^2\right]\leq 16C_1C_\chi\varepsilon^2\|P_1(\xi,\lambda)u\|^2$$

for some constant $C_\chi>0$ determined by the pair (χ_0,χ). By taking $\varepsilon\leq\frac{1}{8\sqrt{C_1C_\chi}}$ it suffices to consider

$$\|P_1(\xi,\lambda)(\chi_\ell u)\|=\frac{1}{2^{2\ell}}\|P_1(\xi 2^{3\ell},\lambda 2^{2\ell})u_\ell\|$$

with $u_\ell(p_1)=2^{\ell/2}(\chi_\ell u)(2^\ell p_1)$ and $\operatorname{supp}u_\ell\subset[-4\varepsilon^{-1},4\varepsilon^{-1}]$.
There are two cases

— $|\lambda 2^{2\ell}|\geq 2(4\varepsilon^{-1}2^{3\ell}|\xi|)$ and then

$$\forall p_1\in\operatorname{supp}u_\ell,\quad |\lambda 2^{2\ell}-p_1\xi 2^{3\ell}|\geq|\lambda|2^{2\ell}-4\varepsilon^{-1}2^{3\ell}|\xi|\geq\frac{|\lambda|2^{2\ell}}{2}$$

This implies

$$C_1\|\frac{1}{2^{2\ell}}P_1(\xi 2^{3\ell},\lambda 2^{2\ell})u_\ell\|^2\geq\|\frac{|\lambda|2^{2\ell}}{2\times 2^{2\ell}}u_\ell\|^2\geq\|\frac{|\lambda|^2}{2}u_\ell\|^2$$

Finally with $|\lambda| \geq 1$ and $|\lambda| \geq |\lambda|^{2/3} \geq \left(\frac{|\lambda|}{2^\ell}\right)^{2/3}$ we obtain

$$4C_1 \| \frac{1}{2^{2\ell}} P(\xi 2^{3\ell}, \lambda 2^{2\ell}) u_\ell \|^2 \geq \| \left(\frac{|\lambda|}{2^\ell}\right)^{2/3} u_\ell \|^2 ,$$

in this case.

— $|\lambda 2^{2\ell}| \leq 2(4\varepsilon^{-1} 2^{3\ell} |\xi|)$ and then the lower bound

$$C_1 \| \frac{1}{2^{2\ell}} P_1(\xi 2^{3\ell}, \lambda 2^{2\ell}) u_\ell \|^2 \geq \| |\xi|^{2/3} u_\ell \|^2$$

implies with $|\xi| \geq \frac{4^{-1}\varepsilon|\lambda|}{2^{\ell+1}}$

$$C_1 \| \frac{1}{2^{2\ell}} P_1(\xi 2^{3\ell}, \lambda 2^{\ell}) u_\ell \|^2 \geq \frac{4^{-4/3}\varepsilon^{4/3}}{16} \| \left(\frac{|\lambda|}{2^\ell}\right)^{2/3} u_\ell \|^2$$

We conclude with the uniform equivalence

$$C_\varepsilon^{-1} \langle p \rangle \leq 2^\ell \leq C_\varepsilon \langle p \rangle$$

on $\operatorname{supp} u_\ell$ where $\varepsilon \leq \frac{1}{8\sqrt{C_1 C_\psi}}$ and all the other constants are actually universal constants once the pair (χ_0, χ) for the dyadic partition of unity is fixed. □

2.C Result used to localize the operator

In this appendix M is a manifold endowed with a volume density $d\mathrm{vol}$ and $\pi_E : E \to M$ is a smooth complex vector bundle endowed with a hermitian metric g^E, so that $L^2(M;E)$ is well defined with the norm $\| \; \|_{L^2(M;E)}$ simply denoted by $\| \; \|$.

For a differential operator P acting on $\mathscr{C}_0^\infty(M;E)$ and a function $\chi \in \mathscr{C}^\infty(M)$, the equality $\chi P = P\chi - [P, \chi]$ and the triangular inequality give

$$\forall u \in \mathscr{C}_0^\infty(M;E), \quad ||P\chi u|| - ||[P,\chi]u|| \leq ||\chi P u|| \leq ||P\chi u|| + ||[P,\chi]u|| .$$

It then follows that

$$\forall u \in \mathscr{C}_0^\infty(M;E), \quad \frac{1}{2}||P\chi u||^2 - ||[P,\chi]u||^2 \leq ||\chi P u||^2 \leq 2||P\chi u||^2 + 2||[P,\chi]u||^2.$$

After three iterations with the additional assumptions that third order commutators vanish, which is relevant for differential operators of order less or equal to 2, we get the following statement.

Proposition 2.C.1. *Let P be a differential operator acting on $\mathscr{C}_0^\infty(M;E)$ and let χ_1, χ_2 and χ_3 be three $\mathscr{C}^\infty$ functions such that*

$$[[[P,\chi_1],\chi_2],\chi_3] = 0 . \tag{2.C.0.1}$$

The following inequalities hold for u in $\mathscr{C}_0^\infty(M;E)$

$$||\chi_1\chi_2\chi_3 P u||^2 \leq 2||\chi_2\chi_3 P\chi_1 u||^2 + 4||\chi_3[P,\chi_1]\chi_2 u||^2 + 8||[[P,\chi_1],\chi_2]\chi_3 u||^2$$

and

$$||\chi_1\chi_2\chi_3 P u||^2 \geq \frac{1}{2}||\chi_2\chi_3 P\chi_1 u||^2 - 2||\chi_3[P,\chi_1]\chi_2 u||^2 - 4||[[P,\chi_1],\chi_2]\chi_3 u||^2.$$

Applying the above proposition with a locally finite quadratic partition of unity $(\chi_\ell)_{\ell\in L}$, $\chi_\ell \in \mathscr{C}_0^\infty$, $\sum_{\ell\in L}\chi_\ell^2 \equiv 1$, and summing over theindices ℓ_1,ℓ_2 and ℓ_3 leads to

$$\|Pu\|^2 \leq 2\sum_{\ell_1}\|P\chi_{\ell_1}u\|^2 + 4\sum_{\ell_1,\ell_2}\|[P,\chi_{\ell_2}]\chi_{\ell_1}u\|^2 + 8\sum_{\ell_1,\ell_2,\ell_3}\|[[P,\chi_{\ell_2}],\chi_{\ell_3}]\chi_{\ell_1}u\|^2 \tag{2.C.0.2}$$

and

$$\|Pu\|^2 \geq \frac{1}{2}\sum_{\ell_1}\|P\chi_{\ell_1}u\|^2 - 2\sum_{\ell_1,\ell_2}\|[P,\chi_{\ell_2}]\chi_{\ell_1}u\|^2 - 4\sum_{\ell_1,\ell_2,\ell_3}\|[[P,\chi_{\ell_2}],\chi_{\ell_3}]\chi_{\ell_1}u\|^2 \tag{2.C.0.3}$$

for all $u \in \mathscr{C}_0^\infty(M;E)$.
With (2.C.0.2) and (2.C.0.3) we have

Corollary 2.C.2. *Let $(\chi_\ell)_{\ell\in L}$ be a family of functions such that*

$$\sum_{\ell\in L}\chi_\ell^2 = 1$$

and let P be a second order differential operator such that

$$\forall u \in \mathscr{C}_0^\infty(M;E), \quad \frac{r}{2}\sum_{\ell_1}\|P\chi_{\ell_1}u\|^2 \geq 2\sum_{\ell_1,\ell_2}\|[P,\chi_{\ell_2}]\chi_{\ell_1}u\|^2 + 4\sum_{\ell_1,\ell_2,\ell_3}\|[[P,\chi_{\ell_2}],\chi_{\ell_3}]\chi_{\ell_1}u\|^2 \tag{2.C.0.4}$$

for some $r \in (0,1)$. Then

$$\forall u \in \mathscr{C}_0^\infty(M;E), \quad (2+r)\sum_{\ell\in L}\|P\chi_\ell u\|^2 \geq \|Pu\|^2 \geq \frac{1-r}{2}\sum_{\ell}\|P\chi_\ell u\|^2. \tag{2.C.0.5}$$

2.D $N-\mathrm{loc}$ and $N-\mathrm{comp}$ functional spaces

Let $f: M \to N$ be a $\mathscr{C}^\infty$ map from the manifold M to the manifold N, let $E \overset{\pi_E}{\to} M$ be a vector bundle and $\mathscr{F}(M;E)$ be a locally convex space of sections continuously embedded in $\mathscr{D}'(M;E)$ (abbreviated as a functional space of sections) such that for any $\chi \in \mathscr{C}_0^\infty(N;\mathbb{R})$ the multiplication by $\chi\circ f$ is a continuous endomorphism of $\mathscr{F}(M;E)$. The notation $\mathscr{F}_{f-\mathrm{loc}}(M;E)$ will denote the set of sections s of E such that

$$\forall \chi \in \mathscr{C}_0^\infty(N;\mathbb{R}), \quad [\chi\circ f]s \in \mathscr{F}(M;E).$$

Once $\mathscr{F}_{f-\mathrm{loc}}(M;E)$ is defined $\mathscr{F}_{f-\mathrm{comp}}(M;E)$ is the set of sections $s \in \mathscr{F}_{f-\mathrm{loc}}(M;E)$ such that there exists $\chi \in \mathscr{C}_0^\infty(N;\mathbb{R})$ with $s = [\chi\circ f]s$. For $s \in \mathscr{F}_{f-\mathrm{loc}}(M;E)$ the f-support of s is defined by

$$f-\mathrm{supp}\, s = \bigcap_{\substack{F\subset N \\ F \text{ closed} \\ s|_{f^{-1}(N\setminus F)}\equiv 0}} F = \overline{f(\mathrm{supp}\, s)}.$$

For a compact subset K of N and $\mathscr{F}_{f-K}(M;E) = \left\{s \in \mathscr{F}_{f-\mathrm{loc}}(M;E), f-\mathrm{supp}\, s \subset K\right\}$ and

$$\mathscr{F}_{f-\mathrm{comp}}(M;E) = \bigcup_{K \text{ compact in } N} \mathscr{F}_{f-K}(M;E).$$

When the topology on $\mathscr{F}(M;E)$ is known, the topology on $\mathscr{F}_{f-\mathrm{loc}}(M;E)$ is the initial topology for the collection of maps $(s \mapsto [\chi\circ f]s)_{\chi\in\mathscr{C}_0^\infty(N;\mathbb{R})}$. This induces the topology on $\mathscr{F}_{f-K}(M;\mathscr{E})$ and the topology on $\mathscr{F}_{f-\mathrm{comp}}(M;E)$ is the inductive limit topology.

We will use this in a particular case.

Definition 2.D.1. Let $M = T^*N$ or $M = T^*(T^*N)$ be endowed with the natural projection $\pi : M \to N$ and let $\mathscr{F}(M;E) \subset \mathscr{D}'(M;E)$ be a functional space of sections of the vector bundle $E \overset{\pi_E}{\to} M$ which is a $\mathscr{C}_0^\infty(N;\mathbb{R})$-module. We will use in both cases the notation $\mathscr{F}_{N-\text{loc}}(M;E) = \mathscr{F}_{\pi-\text{loc}}(M;E)$, $N - \text{supp}\, s = \pi - \text{supp}\, s \subset N$, $\mathscr{F}_{N-K}(M;\mathscr{E}) = \mathscr{F}_{\pi-K}(M;E)$, $\mathscr{F}_{N-\text{comp}}(M;E) = \mathscr{F}_{\pi-\text{loc}}(M;E)$.

When N is a locally compact manifold, introducing a locally finite atlas and a subordinate partition of unity $\sum_{i\in\mathscr{I}} \chi_i(q) \equiv 1$ reduces the characterization of $s \in \mathscr{F}_{N-\text{loc}}(M;E)$, $M = T^*N$ or $M = T^*(T^*N)$ to the meaning of $s \in \mathscr{F}_{\Omega-\text{loc}}(M;E)$, $M = T^*\Omega = \Omega \times \mathbb{R}^d$ or $M = T^*(T^*\Omega) = \Omega \times \mathbb{R}^{3d}$ and the invariance of $\mathscr{F}_{\Omega-\text{loc}}(M;E)$ via a diffeomorphism $\phi : \Omega \to \Omega$. With the extension by 0 and the restriction, the embeddings $\mathscr{F}_{\Omega-\text{comp}}(\Omega;E) \subset \mathscr{F}_{N-\text{comp}}(N;E) \subset \mathscr{F}_{\Omega'-\text{loc}}(\Omega';E)$ hold for two different open sets Ω and Ω' in N.

Example:
The spaces $\mathscr{S}_{N-\text{comp}}(T^*N;\mathbb{C})$, $\mathscr{S}'_{N-\text{comp}}(T^*N;\mathbb{C})$ and their respective duals $\mathscr{S}'_{N-\text{loc}}(T^*N;\mathbb{C})$ and $\mathscr{S}_{N-\text{loc}}(T^*N;\mathbb{C})$ are well defined for any locally compact manifold N.
If additionally N is compact $\mathscr{S}_{N-\text{loc}}(T^*N;\mathbb{C}) = \mathscr{S}_{N-\text{comp}}(T^*N;\mathbb{C})$ (resp. $\mathscr{S}'_{N-\text{loc}} = \mathscr{S}'_{N-\text{comp}}$) will be simply denoted $\mathscr{S}(T^*N;\mathbb{C})$ (resp. $\mathscr{S}'(T^*N;\mathbb{C})$).
The Schwartz kernel theorem for continuous maps $\mathscr{C}_0^\infty(T^*N;\mathbb{C}) \to \mathscr{D}'(T^*N;\mathbb{C})$ implies that any continuous map from $A : \mathscr{S}_{N-\text{comp}}(T^*N;\mathbb{C}) \to \mathscr{S}'_{N-\text{loc}}(T^*N;\mathbb{C})$ admits a kernel in $K_A \in \mathscr{S}'_{N\times N-\text{loc}}(T^*(N\times N);\mathbb{C})$. Additionally A is continous from $\mathscr{S}'_{N-\text{comp}}(T^*N;\mathbb{C})$ to $\mathscr{S}_{N-\text{loc}}(T^*N;\mathbb{C})$ if and only if its kernel K_A belongs to $\mathscr{S}_{N\times N-\text{loc}}(T^*(N\times N);\mathbb{C})$.
Other $N - \text{loc}$ and $N - \text{comp}$ spaces are introduced in the text.

2.E Some pseudo-differential calculus on $X = T^*Q$

The manifold Q is either a compact manifold which can be endowed with any riemannian metric or $\mathbb{R}^d$. The total space of the cotangent bundle is denoted by $X = T^*Q$ and symbols of pseudo-differential operators are defined as functions of $T^*X = T^*(T^*Q)$.
The pseudo-differential calculus presented here and, of which the global geometrical meaning is checked, implements the idea that ∂_{q^i} is an operator of order 1 while $p_i\times$ and ∂_{p_i} are of order 1/2 as presented in [Leb1][Leb2]. However our presentation, like our definition of the spaces $\tilde{\mathscr{W}}^k(X;\mathscr{E})$ in Definition 2.1.2 slightly differ from Lebeau's approach (see Remark 2.1.3).

2.E.1 Definitions and properties

We give here the definitions and state the main properties but their global geometrical meaning will be consequences of the subsequent paragraphs.

Definition 2.E.1. For any coordinate system $(q^1,\dots,q^d)$ on a chart open set $\Omega \subset Q$, the associated canonical coordinates on $T^*\Omega \subset X$ are $(q^1,\dots,q^d,p_1,\dots,p_d)$ with $p = p_i dq^i \in T_q^*Q$. Accordingly doubly canonical coordinates on $T^*(T^*\Omega) \subset T^*X$ associated with the coordinates $(q^1,\dots,q^d)$ on Ω will be (q,p,ξ,η) with $(\xi,\eta) = \xi_i dq^i + \eta^i dp_i \in T^*_{q,p}X$ for the canonical coordinates (q,p) associated with the coordinates $(q^1,\dots,q^d)$. Those coordinates will be abbreviated as $X = (x,\Xi) \in \Omega \times \mathbb{R}^{3d}$ with $x = (q,p) \in \Omega \times \mathbb{R}^d$ and $\Xi = (\xi,\eta) \in \mathbb{R}^{2d}$.

Definition 2.E.2. The set $S^m_\Psi(Q;\mathbb{C})$ is the class of functions $a \in \mathscr{C}^\infty(T^*X;\mathbb{C})$ such that for any doubly canonical coordinate system (q,p,ξ,η) on $T^*(T^*\Omega)$, where Ω is a chart open set in Q, the following inequalities hold:

$$\forall K \subset\subset \Omega, \forall(\alpha,\beta,\gamma,\delta) \in \mathbb{N}^d, \exists C_{K,\alpha,\beta,\gamma,\delta} > 0, \forall(q,p,\xi,\eta) \in K \times (\mathbb{R}^d)^3,$$
$$|\partial_q^\alpha \partial_p^\beta \partial_\xi^\gamma \partial_\eta^\delta a(q,p,\xi,\eta)| \le C_{K,\alpha,\beta,\gamma,\delta}(1 + |\xi|^2 + |p|^4 + |\eta|^4)^{\frac{m-|\gamma|-\frac{|\beta|+|\delta|}{2}}{2}}. \quad (2.E.1.1)$$

The intersection $\bigcap_{m\in\mathbb{R}} S^m_\Psi(Q;\mathbb{C})$ is denoted by $S^{-\infty}_\Psi(Q;\mathbb{C})$. The topology on $S^m_\Psi(Q;\mathbb{C})$ is given by the seminorms $p_{K,\alpha,\beta,\gamma,\delta}(a)$ which are the best constant $C_{K,\alpha,\beta,\gamma,\delta}$ in the above inequality. For any open set $\Omega\subset Q$, the spaces $S^m_{\Psi,\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$ and $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ are defined according to Appendix 2.D. Finally the equivalence relation $a_1\sim a_2$ means $a_1-a_2\in S^{-\infty}_\Psi(Q;\mathbb{C})$.

Actually $S^{-\infty}_\Psi(Q;\mathbb{C})=\mathscr{S}_{Q-\mathrm{loc}}(T^*(T^*Q);\mathbb{C})=\mathscr{S}(T^*(T^*Q);\mathbb{C})$ with the notations of Appendix 2.D.

Definition 2.E.3. The quantization of a symbol $a\in S^m_{\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ is given by

$$a(q,p,D_q,D_p)u=\int_{\Omega\times\mathbb{R}^{3d}} e^{i[\xi(q-q')+\eta.(p-p')]}a(q,p,\xi,\eta)u(q',p')\,dq'dp'd\xi d\eta\in\mathscr{S}'_{\Omega-\mathrm{comp}}(T^*\Omega;\mathbb{C})$$

for any $u\in\mathscr{S}'_{\Omega-\mathrm{comp}}(T^*\Omega;\mathbb{C})$.
The global definition of $a(q,p,D_q,D_p)$ for $a\in S^m_\Psi(Q;\mathbb{C})$ is given by

$$a(q,p,D_q,D_p)u=\sum_{n=1}^{N}(\chi_n(q)a)(q,p,D_q,D_p)(\tilde{\chi}_n(q)u) \tag{2.E.1.2}$$

for some partition of unity $\sum_{n=1}^N\chi_n\equiv 1$ on Q subordinate to a finite atlas $Q=\bigcup_{n=1}^N\Omega_n$ with $\tilde{\chi}_n\in\mathscr{C}^\infty_0(\Omega_n;[0,1])$, $\tilde{\chi}_n\equiv 1$ in a neighborhood of $\mathrm{supp}\,\chi_n$.
The set $\mathscr{R}(Q;\mathbb{C})$ of regularizing operators is $\mathscr{L}(\mathscr{S}'(T^*Q;\mathbb{C});\mathscr{S}(T^*Q;\mathbb{C}))$.
The set $\left\{a(q,p,D_q,D_p)+R\,,a\in S^m_\Psi(Q;\mathbb{C}),R\in\mathscr{R}(Q;\mathbb{C})\right\}$ is denoted $\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ and $\mathrm{OpS}^{-\infty}_\Psi(Q;\mathbb{C})=\mathscr{R}(Q;\mathbb{C})$ with the equivalence relation $A_1\sim A_2$ in $\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ iff $A_1=A_2+R$ with $R\in\mathscr{R}(Q;\mathbb{C})$.

This pseudo-differential calculus has the same properties listed below as the classical pseudo-differential calculus and our approach relies on the global pseudo-differential calculus when $Q=\mathbb{R}^d$ recalled in the next paragraph.
Properties:

- **a)** For any vector bundle $\mathscr{C}^\infty$ isomorphism $\Phi:T^*Q\to T^*Q'$ given by

  $$(q',p')=\Phi(q,p)=(\phi(q),L(q).p)\quad\text{with } L(q)\in\mathrm{GL}(T_qQ;T_{\phi(q)}Q'),$$

  the pull-back Φ^* defined by $[\Phi^*a](x,\Xi)=a(\Phi(x),{}^td\Phi^{-1}(x).\Xi)$ with $x=(q,p)$ and $\Xi=(\xi,\eta)$ defines a continuous isomorphism from $S^m_\Psi(Q';\mathbb{C})$ to $S^m_\Psi(Q;\mathbb{C})$.
  Any $a\in S^m_\Psi(Q;\mathbb{C})$ equals $\sum_{n=1}^N\chi_n(q)a$ for any finite partition of unity $\sum_{n=1}^N\chi_n(q)\equiv 1$. The particular case where $L(q)={}^td\phi(q)^{-1}$ says that $\chi_n a\in S^m_{\Psi,\Omega_n-\mathrm{comp}}(\Omega_n,\mathbb{C})$ is independent of the choice of the coordinate system $(q^1,\ldots,q^d)$.
- **b)** When $a\in S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ and $\tilde{\chi}_1$ and $\tilde{\chi}_2$ are two elements of $\mathscr{C}^\infty_0(\Omega;[0,1])$ such that $\tilde{\chi}_1-\tilde{\chi}_2\equiv 0$ in a neighborhood of $\Omega-\mathrm{supp}\,a$, the operator $a\circ[\tilde{\chi}_1-\tilde{\chi}_2]$ belongs to $\mathrm{OpS}^{-\infty}_\Psi(Q;\mathbb{C})$. This property ensures that the quantization (2.E.1.2) is actually independent of the choice of the $\tilde{\chi}_n$ cut-off functions as a map from $S^m_\Psi(Q;\mathbb{C})/S^{-\infty}_\Psi(Q;\mathbb{C})$ to $\mathrm{OpS}^m_\Psi(Q;\mathbb{C})/\mathrm{OpS}^{-\infty}_\Psi(Q;\mathbb{C})$.
- **c)** For any $a\in S^m_\Psi(Q;\mathbb{C})$ the operator $a(q,p,D_q,D_p)$ defined by (2.E.1.2) is continuous from $\mathscr{S}(X;\mathbb{C})$ to $\mathscr{S}(X;\mathbb{C})$ and from $\mathscr{S}'(X;\mathbb{C})$ to $\mathscr{S}'(X;\mathbb{C})$.
- **d)** The set $\bigcup_{m\in\mathbb{R}}\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ is an algebra for the composition product with

  $$a_1(q,p,D_q,D_p)\circ a_2(q,p,D_q,D_p)=[a_1a_2](q,p,D_q,D_p)\qquad\mathrm{mod}\,\mathrm{OpS}^{m_1+m_2-1}_\Psi(Q;\mathbb{C}),$$
  $$\left[a_1(q,p,D_q,D_p),a_2(q,p,D_q,D_p)\right]=[\frac{1}{i}\{a_1a_2\}](q,p,D_q,D_p)\qquad\mathrm{mod}\,\mathrm{OpS}^{m_1+m_2-2}_\Psi(Q;\mathbb{C}),$$

  for $a_k\in S^{m_k}_\Psi(Q;\mathbb{C})$ and $\{a_1,a_2\}=\partial_\xi.a_1\partial_qa_2+\partial_\eta a_1.\partial_pa_2-\partial_qa_1.\partial_\xi a_2-\partial_pa_1.\partial_\eta a_2$ in doubly canonical coordinates.

e) For any family $(a_j)_{j\in\mathbb{N}}$ with $a_j \in S^{m-j}_\Psi(Q;\mathbb{C})$ there exists $a \in S^m_\Psi(Q;\mathbb{C})$ such that $a - \sum_{j=0}^J a_j \in S^{m-J-1}_\Psi(Q;\mathbb{C})$, which is simply written $a \sim \sum_{j\in\mathbb{N}} a_j$.

In particular for any $a \in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ which is elliptic ($|a(q,p,\xi,\eta)| \geq C^{-1}(1+|\xi|^2+|p|^4+|\eta|^4)^{m/2}$) there exists $b \sim \sum_{n=0}^\infty b_j$ with $b_0 = \frac{1}{a}$ such that $b(q,p,D_q,D_p)\circ a(q,p,D_q,D_p) \sim a(q,p,D_q,D_p)\circ b(q,p,D_q,D_p) \sim \mathrm{Id}$.

f) For any $a \in S^0_\Psi(Q;\mathbb{C})$, $a(q,p,D_q,D_p) \in \mathscr{L}(L^2(X,dqdp;\mathbb{C}))$ with

$$\|a(q,p,D_q,D_p)\|_{\mathscr{L}(L^2)} \leq C \sup_{|\alpha|+|\beta|+|\gamma|+|\delta|\leq N_d} p_{\alpha,\beta,\gamma,\delta}(a)$$

for some $N_d \in \mathbb{N}$ determined by $d = \dim Q$.

g) When $\Phi : T^*Q \to T^*Q'$ is a $\mathscr{C}^\infty$-isomorphism like in **a)** and $U_\Phi : L^2(X',dq'dp';\mathbb{C}) \to L^2(X,dqdp;\mathbb{C})$ is the unitary map defined by $[U_\Phi u](q,p) = \sqrt{|\det(d\phi(q))\det(L(q))|}u(\phi(q),L(q).p)$ then for any $a \in S^m_\Psi(Q';\mathbb{C})$, $U_\Phi a(q',p',D_{q'},D_{p'})U_\Phi^{-1} = b(q,p,D_q,D_p) \in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ with $b \sim \sum_{j=0}^\infty b_j$ and $b_0 = (\Phi^* a)$.

When $L(q) = {}^t d\phi(q)^{-1}$ and $q \in \Omega$ a chart open set in Q, this result contains the fact that the space of pseudo-differential operator $\mathrm{OpS}^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ does not depend on the choice of coordinates on Q with a functorial transformation of the principal symbol. With the local definition of the quantization (2.E.1.2) and **b)**, the space $\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ has a global geometric meaning.

h) The vector bundle version of pseudo-differential operators $a(q,p,D_q,D_p) \in \mathrm{OpS}^m_\Psi(Q;\pi_2^*(\mathrm{End}(E)))$ acting on sections of $\pi_X^*(E)$ where $X = T^*Q \overset{\pi_X}{\to} Q$ and $T^*X \overset{\pi_2}{\to} Q$ are the natural projection and $E \overset{\pi_E}{\to} Q$ is a vector bundle over Q, is reduced to the case of matricial pseudo-differential operators acting on $\mathbb{C}^N$-valued sections via the localized definition (2.E.1.2). It has the same properties as the scalar pseudo-differential operators except for the principal symbol of a commutator. We will use the abbreviation $\mathrm{OpS}^m_\Psi(Q;\mathrm{End}\mathscr{E})$ for $\mathrm{OpS}^m_\Psi(Q;\pi_2^*(\mathrm{End}(E)))$.

i) The seminorm topology on $\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ and the continuity properties of $(A_1,A_2) \to A_1\circ A_2$ and of $A \to U_\Phi \circ A \circ U_\Phi^{-1}$ are discussed in Subsection 2.E.4.

2.E.2 Global calculus when $Q = \mathbb{R}^d$

Let $T^*(T^*\mathbb{R}^d) = \mathbb{R}^{4d}_{q,p,\xi,\eta}$ and consider the function

$$\Psi(q,p,\xi,\eta) = \sqrt{1+|\xi|^2+|\eta|^4+|p|^4}.$$

The symbol class $S(m,g_\Psi)$ is the set of $a \in \mathscr{C}^\infty(\mathbb{R}^{4d};\mathbb{C})$ such that:

$$\forall \alpha,\beta,\gamma,\delta \in \mathbb{N}^d, \exists C_{\alpha,\beta,\gamma,\delta} > 0, \forall X = (q,p,\xi,\eta) \in \mathbb{R}^{4d}, \quad |\partial_q^\alpha\partial_p^\beta\partial_\xi^\gamma\partial_\eta^\delta a(X)| \leq C_{\alpha,\beta,\gamma,\delta}m(X)\Psi(X)^{-|\gamma|-\frac{|\beta|+|\delta|}{2}}$$

and the topology on $S(m,g_\Psi;\mathbb{C})$ is given by the family of the seminorms

$$p_{m,k}(a) = \sum_{\substack{|\alpha|+|\beta|+|\gamma|+|\delta|\leq k \\ X\in\mathbb{R}^{4d}}} \left| \frac{\partial_q^\alpha\partial_p^\beta\partial_\xi^\gamma\partial_\eta^\delta a(X)}{m(X)\Psi^{-|\gamma|-\frac{|\beta|+|\delta|}{2}}(X)} \right|.$$

We follow the terminology of [Bon].

Proposition 2.E.4. *The metric $g_\Psi = dq^2 + \frac{d\xi^2}{\Psi^2} + \frac{dp^2}{\Psi} + \frac{d\eta^2}{\Psi}$ on $T^*\mathbb{R}^{2d}_{q,p} = \mathbb{R}^{4d}_{q,p,\xi,\eta}$ is a splitted Hörmander metric with the gain function $\Psi(q,p,\xi,\eta)$.*
Additionally it is geodesically temperate with $g^\sigma_\Psi = \Psi^2 g_\Psi$.
For any $s \in \mathbb{R}$, the function Ψ^s is a g_Ψ-weight for any $s \in \mathbb{R}$ and an elliptic symbol in $S(\Psi^s,g_\Psi)$.

Remember that $\sigma=\sum_{j=1}^{2d} d\Xi_j\wedge dx_j$ denotes the canonical symplectic form on $\mathbb{R}^{4d}_{x,\Xi}$, with here $x=(q,p)$ and $\Xi=(\xi,\eta)$.
We follow the usual abusive convention for the presentation of Weyl-Hörmander calculus and write shortly $g_{\Psi,X}(T)$ for the quadratic form applied to the tangent vector $T=(t_x,t_\Xi)$ instead of $g_{\Psi,X}(T,T)$.

Proof. The properties $g_\Psi^\sigma=\Psi^2 g_\Psi$ and $g_{\Psi,X}(t_x,-t_\Xi)=g_{\Psi,X}(t_x,t_\Xi)$ (g_Ψ is splitted), $\Psi\geq 1$ (Hörmander uncertainty condition) and $\Psi^s\in S(\Psi^s,g_\Psi;\mathbb{C})$ are obvious.
The inequality

$$\left(\frac{g_{\Psi,X}}{g_{\Psi,X'}}\right)^{-\pm 1}\leq \max\left\{1,\left(\frac{\Psi(X)}{\Psi(X')}\right)^{\pm 2},\left(\frac{\Psi(X)}{\Psi(X')}\right)^{\pm 1}\right\}, \tag{2.E.2.1}$$

says that the slowness and (geodesic) temperance are proved when Ψ is a slow and (geodesically) tempered weight for g_Ψ.
Slowness: Set $X=(q,p,\xi,\eta)$ and $X'=(q',p',\xi',\eta')$.For $g_{\Psi,X}(X'-X)\leq\frac{1}{R^2}$ let us prove $\left(\frac{\Psi(X)}{\Psi(X')}\right)^{-1}\leq R^2$ for $R>1$ large enough.
The assumption implies

$$|\tilde\xi-\tilde\xi'|\leq\frac{\sqrt{1+|\tilde\xi|^2}}{R}=\frac{\langle\tilde\xi\rangle}{R}\quad\text{with}\left\{\begin{array}{l}\tilde\xi=\frac{1}{(1+|p|^4+|\eta|^4)^{1/2}}\xi\\ \tilde\xi'=\frac{1}{(1+|p|^4+|\eta|^4)^{1/2}}\xi'.\end{array}\right.$$

We deduce

$$|\tilde\xi'|^2\leq 2|\tilde\xi|^2+2|\tilde\xi'-\tilde\xi|^2\leq(1+\frac{2}{R^2})\langle\tilde\xi\rangle^2,$$

and

$$|\tilde\xi|^2\leq 2\langle\tilde\xi'\rangle^2+\frac{2}{R^2}\langle\tilde\xi\rangle^2.$$

This gives for $R\geq 2$,

$$(1-\frac{2}{R^2})\langle\tilde\xi\rangle^2\leq 2\langle\tilde\xi'\rangle^2\leq 2(1+\frac{2}{R^2})\langle\tilde\xi\rangle^2$$

and

$$\left(\frac{\Psi(q,p,\xi,\eta)}{\Psi(q',p,\xi',\eta)}\right)^{\pm 1}\leq 2(1+\frac{2}{R^2}).$$

Let us consider now the quantity

$$\left(\frac{\Psi(q',p,\xi',\eta)}{\Psi(q',p',\xi',\eta')}\right)^{\pm 1}$$

while noticing that the first result and the assumption $g_{\Psi,X}(X-X')\leq\frac{1}{R^2}$ implies

$$\begin{aligned}
&|p-p'|\leq\frac{1}{R}\Psi(q,p,\xi,\eta)^{1/2}\leq\frac{\sqrt{2(1+2/R^2)}}{R}\Psi(q',p,\xi',\eta)^{1/2}\leq\frac{2}{R}(1+|\xi'|^2+|p|^4+|\eta|^4)^{1/4}\\
\text{and}\quad &|\eta-\eta'|\leq\frac{\sqrt{2(1+2/R^2)}}{R}\Psi(q',p,\xi',\eta)^{1/2}\leq\frac{2}{R}(1+|\xi'|^2+|p|^4+|\eta|^4)^{1/4}\\
\text{with}\quad &\frac{2}{R}(1+|\xi'|^2+|p|^4+|\eta|^4)^{1/4}\leq\frac{2\langle\xi'\rangle^{1/2}(1+|\tilde p|^2+|\tilde\eta|^2)^{1/2}}{R}
\end{aligned}$$

by setting $\tilde p=\langle\xi'\rangle^{-1/2}p$ and $\tilde\eta=\langle\xi'\rangle^{-1/2}\eta$. By using the same normalization for $(\tilde p',\tilde\eta')$ we deduce that the vectors $Y=(\tilde p,\tilde\eta)$ and $Y'=(\tilde p',\tilde\eta')$ satisfy

$$|Y-Y'|\leq\frac{2}{R}\langle Y\rangle$$

and again

$$\left(\frac{\Psi(q',p,\xi',\eta)}{\Psi(q',p',\xi',\eta')}\right)^{\pm 1/2} = \left(\frac{\langle Y\rangle}{\langle Y'\rangle}\right)^{\pm 1} \leq 2(1+\frac{8}{R^2})$$

when $R/2 > 2$.
We deduce the uniform inequality

$$\left(\frac{\Psi(X)}{\Psi(X')}\right)^{\pm 1} = \left(\frac{\Psi(q,p,\xi,\eta)}{\Psi(q',p,\xi',\eta)}\right)^{\pm 1} \times \left(\frac{\Psi(q',p,\xi',\eta)}{\Psi(q',p',\xi',\eta')}\right)^{\pm 1} \leq 2(1+\frac{2}{R^2})4(1+\frac{8}{R^2})^2 \leq 2^{12} \leq R$$

as soon as $g_{\Psi,X}(X'-X) \leq \frac{1}{R^2}$ if $R \geq 2^{12}$.

Geodesic Temperance: With $g^\sigma_\Psi \geq dq^2 + d\xi^2 + dp^2 + d\eta^2$, we get $g^\sigma_{\Psi,X}(X-X') \geq |X-X'|^2$ and the same inequality holds for the geodesic distance for g^σ_Ψ, $d^\sigma_\Psi(X,X') \geq \left|X-X'\right|$.
From

$$\Psi(q',p',\xi',\eta')^2 \leq 1+|\xi'|^2+(|p'|^2+|\eta'|^2)^2 \leq 1+2|\xi|^2+2|\xi'-\xi|^2+\left(2|p|^2+2|p'-p|^2+2|\eta|^2+2|\eta'-\eta|\right)^2$$

we deduce

$$\Psi(q',p',\xi',\eta')^2 \leq 64\Psi(q,p,\xi,\eta)^2(1+|\xi'-\xi|^2)(1+|p'-p|^2+|\eta'-\eta|^2)^2$$

and the symmetric version results from the exchange $X \leftrightarrow X'$. We obtain

$$\left(\frac{\Psi(X)}{\Psi(X')}\right)^{\pm 2} \leq 64(1+|q-q'|^2+|\xi-\xi'|^2+|p'-p|^2+|\eta'-\eta|^2)^3 \leq 64(1+|X-X'|^2)^3.$$

With $|X-X'|^2 \leq \min(g^\sigma_{\Psi,X}(X-X'); d^\sigma_\Psi(X,X')^2)$, this proves that the weight Ψ, and the metric g_Ψ owing to (2.E.2.1), are geodesically tempered. □

All the result of [HormIII]-Chap XVIII can be applied for the Weyl quantization $a^W(q,p,D_q,D_p)$ when $a \in S(m,g_\Psi)$ and m is a g_Ψ-weight. Because g_Ψ is splitted the Weyl and standard quantizations are equivalent and we recall $a(x,D_x) = b^W(x,D_x)$ with

$$a = e^{iD_x.D_\Xi/2}b = \sum_{n=0}^{N-1}\frac{(iD_x.D_\Xi/2)^n}{n!}b + R_{N,+}(b)$$
$$b = e^{-iD_x.D_\Xi/2}a = \sum_{n=0}^{N-1}\frac{(-iD_x.D_\Xi/2)^n}{n!}a + R_{N,-}(a)$$

where every n-th term is continuous from $S(m,g_\Psi)$ to $S(m\Psi^{-n},g_\Psi)$ while the remainders are continous from $S(m,g_\Psi)$ to $S(m\Psi^{-N},g_\Psi)$. Accordingly if $a_1\sharp^W a_2$ (resp. $a_1\sharp a_2$) denote the symbols of $a_1^W(x,D_x)\circ a_2^W(x,D_x)$ (resp. $a_1(x,D_x)\circ a_2(x,D_x)$) we know

$$\begin{aligned} a_1\sharp^W a_2(X) &= e^{i\sigma(D_{X_1},D_{X_2})/2}a_1(X_1)a_2(X_2)\big|_{X_1=X_2=X} \\ &= \sum_{n=0}^{N-1}\frac{(i\sigma(D_{X_1},D_{X_2})/2)^n}{n!}a_1(X_1)a_2(X_2)\big|_{X_1=X_2=X} + R^W_N(a_1,a_2), \end{aligned}$$

and respectively

$$\begin{aligned} a_1\sharp a_2(X) = e^{iD_{\Xi_1}D_{x_2}}a_1(X_1)a_2(X_2)\big|_{X_1=X_2=X} &= \sum_{n=0}^{N-1}\frac{(iD_{\Xi_1}D_{x_2})^n}{n!}a_1(X_1)a_2(X_2)\big|_{X_1=X_2=X} + R_N(a_1,a_2) \\ &= \sum_{n=0}^{N-1}\sum_{|\alpha|\leq n}\frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_\Xi a_1\partial^\alpha_x a_2 + R_N(a_1,a_2), \end{aligned}$$

where every n-th term is is bilinear continuous from $S(m_1,g_\Psi)\times S(m_2,g_\Psi)$ to $S(m_1m_2\Psi^{-n},g_\Psi)$ while the remainder is bilinear continous from $S(m_1,g_\Psi)\times S(m_2,g_\Psi)$ to $S(m_1m_2\Psi^{-N},g_\Psi)$. Two differences: $a^W(x,D_x)^*=(\bar{a})^W(x,D_x)$ remains true only modulo $S(m\Psi^{-1},g_\Psi)$ for the classical quantization while $f(x)a(x,D_x)=(fa)(x,D_x)$ remains true only modulo $S(m\Psi^{-1},g_\Psi)$ for the Weyl quantization.
In [BoCh] were introduced the general Sobolev spaces $H(m,g)$ for any Hörmander metric g and g-weight m as Hilbert spaces with the norms $\|u\|_{H(m,g)}=\|M^W(x,D_x)u\|_{L^2}$, where $M\in S(m,g)$ is any fixed elliptic invertible operator.
We are concerned here with a simple case.

Definition 2.E.5. For $s\in\mathbb{R}$ the space $\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})$ is nothing but $H(\Psi^s,g_\Psi)$ with the norm

$$\|u\|_{\tilde{\mathscr{W}}^s}=\|(M_s)^W(x,D_x)u\|_{L^2}.$$

with $M_s=(C_s+\Psi^{|s|})^{\mathrm{sign}\,s}$ for some $C_s\geq 1$.

The invertibility of $M_s(x,D_x):\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})\to L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})$ comes from $(C_s+\Psi^{|s|})^{-1}\sharp^W(C_s+\Psi^{|s|})=1+\mathscr{O}(1/C_s)$ in $S(1,g_\Psi)$.
Because g_Ψ is geodesically tempered with $g_\Psi^\sigma=\Psi^2 g_\Psi$, J.M. Bony provides us a simple version of Beals criterion in [Bon]. In our particular case the symbol class $S^+(1,g_\Psi)$ is nothing but the set of $a\in\mathscr{C}^\infty(\mathbb{R}^{2d};\mathbb{C})$ such that for any $i\in\{1,\ldots,d\}$ $\partial_{q^i}a\in S(\Psi,g_\Psi)$, $\partial_{\xi_i}a\in S(1,g_\Psi)$, while $\partial_{p_i}a$ and $\partial_{\eta^i}a$ belong to $S(\Psi^{1/2};g_\Psi)$. An operator A equals $a^W(x,D_x)$ with $a\in S(1,g_\Psi)$ if and only if all the commutators $\mathrm{ad}_{b_1^W(x,D_x)}\ldots\mathrm{ad}_{b_N^W(x,D_x)}A$ for $b_n\in S^+(1,g_\Psi)$ are bounded in $\mathscr{L}(L^2)$.
Alternatively, the simpler and original version of Beals criterion in [Bea] works here according [BoCh] (see [NaNi] for a detailed version of Remark 5.6 in [BoCh]) owing to the three properties

— The metric is diagonal in the canonical basis $\mathscr{B}$ of $\mathbb{R}^{4d}=T^*(\mathbb{R}^{2d})$, written as

$$\mathscr{B}=\left\{\partial_{q^i},\partial_{p_i},\partial_{\xi_i},\partial_{\eta^i},1\leq i\leq d\right\},$$

while the convex hull $C_{X,\mathscr{B}}=\left\{\sum_{e\in\mathscr{B}}t_e g_{\Psi,X}(e)^{-1/2}e,(t_e)_{e\in\mathscr{B}}\in[-1,1]^{\sharp\mathscr{B}}\right\}$ satisfies

$$\exists r\in]0,1],\forall X\in\mathbb{R}^{4d}=T^*(\mathbb{R}^{2d}),\quad B_{g_{\Psi,X}}(0,r)\subset C_{X,\mathscr{B}}\subset B_{g_{\Psi,X}}(0,2).$$

— If $L(e)=(\sigma(e,X))^W$ for $e\in\mathscr{B}$ with $\sigma=d\eta\wedge dp+d\xi\wedge dq$ and X the radial vector field, we get $L(\partial_{q^i})=-D_{q^i}$, $L(\partial_{p_i})=-D_{p_i}$, $L(\partial_{\xi_i})=q^i$ and $L(\partial_{\eta^i})=p^i$.

— For a finite familly $E=(e_n)_{1\leq n\leq N}$ of elements of $\mathscr{B}$ the weight $m_E(X)$ equals

$$m_E(X)=\prod_{n=1}^{N_E}g_{\Psi,X}(e_k)^{1/2}=\Psi(X)^{-N_1-N_2/2}\quad\text{with}\quad\begin{cases}N_1=\sharp\{k,e_k\in\{\partial_{\xi_i}\}\}\\ N_2=\sharp\{k,e_k\in\{\partial_{p_i},\partial_{\eta_i}\}\}\end{cases}$$

The Beals criterion thus says that $A=a^W(q,p,D_q,D_p)$ with $a\in S(\Psi^s,g_\Psi)$ for some g_Ψ-weight m, if and only if all the commutators

$$\mathrm{ad}_q^\alpha\mathrm{ad}_p^\beta\mathrm{ad}_{D_q}^\gamma\mathrm{ad}_{D_p}^\delta A$$

initially defined as continuous operators on $\mathscr{S}(\mathbb{R}^{2d};\mathbb{C})$ (or on $\mathscr{S}'(\mathbb{R}^{2d};\mathbb{C})$) actually belongs to

$$\mathscr{L}(\tilde{\mathscr{W}}^{s_0}(\mathbb{R}^{2d};\mathbb{C});\tilde{\mathscr{W}}^{s_0-s+|\alpha|+\frac{|\beta|+|\delta|}{2}}(\mathbb{R}^{2d};\mathbb{C}))$$

for some $s_0\in\mathbb{R}$ (and equivalently for all $s_0\in\mathbb{R}$). Additionally the topology on $\mathscr{S}(\Psi^s;g_\Psi)$ is equivalently defined by the family of seminorms $(q_{\Psi^s,k})_{k\in\mathbb{N}}$ or $(\tilde{q}_{\Psi^s,k})_{k\in\mathbb{N}}$,

$$q_{\Psi^s,k}(A)=\max_{|\alpha|+|\beta|+|\gamma|+|\delta|\leq k}\|\mathrm{ad}_q^\alpha\mathrm{ad}_p^\beta\mathrm{ad}_{D_q}^\gamma\mathrm{ad}_{D_p}^\delta A\|_{\mathscr{L}(L^2;\tilde{\mathscr{W}}^{-s+|\alpha|+\frac{|\beta|+|\delta|}{2}})}.$$

Beals criterion is especially convenient for the link between a global pseudo-differential calculus and functional analysis.

Proposition 2.E.6. *Let $A = a^W(x,D_x)$ be a self-adjoint operator in $L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})$ with an elliptic (and real) symbol $a \in S(\Psi^\mu,g_\Psi)$ (ellipticity means here $a \geq \frac{1}{C}\Psi^\mu$ uniformly on $\mathbb{R}^{2d}$) such that $D(A) = \tilde{\mathcal{W}}^\mu(\mathbb{R}^{2d};\mathbb{C})$. Then for any $f \in S(\langle t\rangle^s, \frac{dt^2}{\langle t\rangle^2};\mathbb{C})$, the operators $f(A)$ and $f(A) - f(a)^W(q,p,D_q,D_p)$ are pseudo-differential operators with symbols respectively in $S(\Psi^{\mu s},g_\Psi)$ and $S(\Psi^{\mu s-1},g_\Psi)$.*
If additionally $A \geq C\mathrm{Id}_{L^2}$ with $C > 0$, then the same result holds for A^s and $A^s - (a^s)^W(q,p,D_q,D_p)$.

Proof. The proof of Bony in [Bon]-Theorem 3.8 relies on Helffer-Sjöstrand functional calculus formula very convenient with Beals criterion (seminorms on $S(\Psi^s,g_\Psi)$ are expressed in terms of norms of commutators). We refer the reader to [Bon][DiSj][HeSj] or to the end of Subsection 2.E.4 for a more detailed use of Helffer-Sjöstrand formula. The only thing which was not verified in [Bon], because it is about a more general framework, is the principal symbol statement. Actually we can focus on $\mu \geq 0$ and when one knows that $(z-A)^{-1} = b_z^W(x,D_x)$ with seminorms of b_z estimated by $p_{\Psi^{-\mu},k}(b) \leq C_k \frac{\langle z\rangle^{N_k}}{|\mathrm{Im}\, z|^{N_k}}$ it suffices to write $\left(\frac{1}{z-a}\right)^W \circ (z-A) = 1 + r_z^W(x,D_X)$ with seminorms of $r_z \in S(\Psi^{-1},g_\Psi)$ estimated by $p_{\Psi^{-1},k}(r_z) \leq C'_k \frac{\langle z\rangle^{N'_k}}{|\mathrm{Im} z|^{N'_k+1}}$. We deduce $\left(\frac{1}{z-a}\right)^W - (z-A)^{-1} = c_z^W(x,D_x)$ with $p_{\Psi^{-\mu-1},k}(c_z)$ estimated by $\frac{\langle z\rangle^{N''_k}}{|\mathrm{Im} z|^{N''_k+1}}$. Inserting this into Helffer-Sjöstrand formula proves the result for $s < 0$, by simple integration. For $s \geq 0$, write $f(A) = (i+A)^N f_N(A)$ with $f_N(t) = (i+t)^{-N} f(t) \in S(\langle t\rangle^{s-N}, \frac{dt^2}{\langle t\rangle^2})$ and N large enough. □

The former result provides us an easy way for comparing various simple definitions of the spaces $\tilde{\mathcal{W}}^s(\mathbb{R}^{2d};\mathbb{C})$ and the equivalence of the norms.

Proposition 2.E.7. *Consider the self-adjoint operator $A = 1 - \Delta_q^2 + \left(\frac{-\Delta_p + |p|^2}{2}\right)^2$ in $L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})$ with domain $D(A) = \tilde{\mathcal{W}}^2(\mathbb{R}^{2d};\mathbb{C})$. Let $\sum_{\ell=-1}^{\infty} \theta_\ell^2(t) \equiv 1$ on $[0,+\infty)$ be a quadratic dyadic partition of unity like in (2.4.1.1). Then the following squared norms on $\tilde{\mathcal{W}}^s(\mathbb{R}^{2d};\mathbb{C})$ equivalent with $\|\ \|^2_{\tilde{\mathcal{W}}^s}$:*

- **i)** $\|A^{s/2}u\|^2_{L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})}$ *for any $s \in \mathbb{R}$;*
- **ii)** $\|(\frac{-\Delta_p+|p|^2}{2})^s u\|^2_{L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})} + \||D_s|^s u\|^2_{L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})}$ *for $s \geq 0$;*
- **iii)** $\sum_{\ell=-1}^{\infty} \|\theta_\ell(|p|^2)u\|^2_{\tilde{\mathcal{W}}^s}$;
- **iv)** *for $s = k \in \mathbb{N}$, $\sum_{|\alpha|+\frac{|\beta|+|\gamma|}{2} \leq k} \|\partial_q^\alpha p^\beta \partial_p^\gamma u\|^2_{L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})}$;*
- **v)** *for $s = k \in \mathbb{N}$, $\sum_{|\alpha|+\frac{N_3+|\gamma|}{2} \leq k} \|\langle p\rangle^{N_3} \partial_q^\alpha \partial_p^\gamma u\|^2_{L^2(\mathbb{R}^{2d},dqdp:\mathbb{C})}$;*

Proof. The equivalence with **i)** is a direct application of Proposition 2.E.6 because $C+A = a^W(q,p,D_q,D_p)$ with $a = C + 1 + |\xi|^2 + \frac{1}{4}(|p|^2 + |\eta|^2)^2 \mod S(\Psi^1,g_\Psi)$ which and a is elliptic for $C > 0$ large enough.
For $s \in \mathbb{R}$ the functional calculus says that $\|A^{s/2}u\|_{L^2}$ is equivalent to $\|(C+A)^{s/2}u\|_{L^2}$. But for $C \geq C_s > 0$ large enough $(C+A)^{s/2} = a_s^W(q,p,D_q,D_p)$ with a_s elliptic in $S(\Psi^s,g_\psi)$ and $\|(C+A)^{s/2}u\|_{L^2}$ is equivalent to $\|u\|_{\tilde{\mathcal{W}}^s}$.
The statement **ii)** is actually a consequence of the functional calculus with $(1+t^2+t'^2)^s \asymp t^{2s} + t'^{2s}$ for all $t \geq 0, t' \geq d/2$ when $s \geq 0$ is fixed.
For $s = k$ the squared norms of **iv)** and **v)** are equivalent to $\langle u\,, B_{iv}u\rangle_{L^2}$ and $\langle u\,, B_v u\rangle$ with

$$B_{iv} = C + \sum_{|\alpha|+\frac{|\beta|+|\gamma|}{2} \leq k} D_p^\gamma D_q^{2\alpha} p^{2\beta} D_p^\gamma \quad , \quad B_v = C + \sum_{|\alpha|+\frac{N_3+|\gamma|}{2} \leq k} D_p^\gamma D_q^{2\alpha} \langle p\rangle^{2N_3} D_p^\gamma$$

which both have a symbol elliptic in $S(\Psi^{2k},g_\Psi)$ for $C > 0$ large enough. By Proposition 2.E.6, the operators $(C+B_{iv})^{1/2}$ and $(C+B_v)^{1/2}$ have an elliptic symbol in $S(\Psi^k,g_\Psi)$ for $C > 0$ large enough and the two norms of **iv)** and **v)** are equivalent to $\|u\|_{\tilde{\mathcal{W}}^k}$ for $k \in \mathbb{N}$.

Finally for **iii)**, the partial Fourier $F_{q\to\xi}$ transform with respect to q sends $L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})$ onto the direct integral $\int_{\mathbb{R}^d} L^2(\mathbb{R}^d,dp;\mathbb{C})\,\frac{d\xi}{(2\pi)^d}$ and for $s=k\in\mathbb{N}$ the squared norm in $\tilde{\mathcal{W}}^k(\mathbb{R}^2;\mathbb{R})$ is equivalent to

$$N_{C,k}(u)^2 = C\|u\|_{L^2}^2 + \sum_{|\alpha|+\frac{|\beta|+|\gamma|}{2}\leq k} \|\xi^\alpha p^\beta \partial_p^\gamma u\|_{L^2}^2$$

and to $\sum_{2|\alpha|+|\beta|+|\gamma|\leq 2} N_{C,k-1}(\xi^\alpha p^\beta \partial_p^\gamma u)^2$. We are considering operators of order 2 in ∂_p and Corollary 2.C.2 can be applied with a recurrence with respect to $k\in\mathbb{N}$ and gives

$$\frac{1}{C_k}\sum_{\ell=-1}^{\infty} N_{C,k}(\theta_\ell(|p|^2)u) \leq N_{C,k}(u)^2 \leq C_k \sum_{\ell=-1}^{\infty} N_{C,k}(\theta_\ell(|p|^2)u)^2,$$

for all $u\in\mathcal{C}_0^\infty(\mathbb{R}^{2d};\mathbb{C})$ and by density for all $u\in\tilde{\mathcal{W}}^k(\mathbb{R}^{2d};\mathbb{C})$. The result for $s\in\mathbb{R}$ follows by interpolation and duality. □

Remark 2.E.8. The equivalence with **ii)** could be done with a semi-classical calculus $a^W(\sqrt{h}p,\sqrt{h}D_p)$ with the semiclassical parameter $h=\frac{1}{\sqrt{C+|\xi|^2}}$ with the symbol classes $S(\langle p,\eta\rangle^{\mu_1}\langle p\rangle^{\mu_2}\langle\eta\rangle^{\mu_3},\frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$ with $1+\frac{h^2}{4}(-\Delta_p+|p|^2)^2 = a_1(\sqrt{h}p,\sqrt{h}D_p)$, a_1 elliptic in $S(\langle p,\eta\rangle^4,\frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$ and $\chi_\ell(|p|^2)\in S(1,\frac{dp^2}{\langle p\rangle^2}+\frac{d\eta^2}{\langle\eta\rangle^2})$ (see e.g. [Rob] or [NaNi]). The necessity of two Hörmander metrics suggests the link with the second microlocalization of [BoLe]. The chosen elementary method suffices here.

2.E.3 Localization and geometric invariance

For the localization in $\Omega\times\mathbb{R}^{3d}$, $q\in\Omega$, Ω open set of $\mathbb{R}^d$, it is more convenient to work with the classical quantization:

$$[a(q,p,D_q,D_p)](q,p,q',p') = \int_{\mathbb{R}^{2d}} e^{i[\xi.(q-q')+\eta.(p-p')}a(q,p,\xi,\eta)\,\frac{d\xi d\eta}{(2\pi)^{2d}},$$

for which $\varrho(q)a(x,D_x) = [\varrho(q)a](x,D_x)$.
Every $a\in\mathcal{S}'_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^{3d};\mathbb{C})$ gives rise to a kernel in $\mathcal{S}'_{\Omega\times\Omega-\mathrm{loc}}(\Omega\times\Omega\times\mathbb{R}^{2d};\mathbb{C})$ and therefore a continuous operator A_a from $\mathcal{S}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$ to $\mathcal{S}'_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^d;\mathbb{C})$ and $a\mapsto A_a$ is a bijection. Consider the symbol classe $S^m_{\Psi,\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$ characterized by (2.E.1.1) with the associated space $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ and the spaces $\tilde{\mathcal{W}}^s{}_{\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$, $\tilde{\mathcal{W}}^s_{\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$. For two open sets Ω and Ω' of $\mathbb{R}^d$ and $\varrho\in\mathcal{C}_0^\infty(\Omega;\mathbb{C})$ we have the following continuous embeddings when the letter E in $E(\Omega)$ stands for S^m_Ψ, $\tilde{\mathcal{W}}^s$, $\mathcal{S}$ or $\mathcal{S}'$:

$$E_{\Omega-\mathrm{comp}}(\Omega)\subset E_{\mathbb{R}^d-\mathrm{comp}}(\mathbb{R}^d)\subset E_{\Omega'-\mathrm{loc}}(\Omega')$$
$$\varrho(q)E_{\Omega'-\mathrm{comp}}(\Omega')\subset \varrho(q)E_{\mathbb{R}^d-\mathrm{comp}}(\mathbb{R}^d)\subset E_{\Omega-\mathrm{comp}}(\Omega).$$

Notice also for $\bullet=$ loc or comp:

$$\bigcap_{s\in\mathbb{R}}\tilde{\mathcal{W}}^s{}_{\Omega-\bullet}(\Omega\times\mathbb{R}^d;\mathbb{C}) = \mathcal{S}_{\Omega-\bullet}(\Omega\times\mathbb{R}^d;\mathbb{C}) \quad\text{and}\quad \bigcup_{s\in\mathbb{R}}\tilde{\mathcal{W}}^s{}_{\Omega-\bullet}(\Omega\times\mathbb{R}^d;\mathbb{C}) = \mathcal{S}'_{\Omega-\bullet}(\Omega\times\mathbb{R}^d;\mathbb{C})$$

In particular symbols $a\in S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ can be viewed as symbols $a\in S(\Psi^m,g_\Psi)$. Therefore $a(x,D_x)$ defines a continuous operator from $\tilde{\mathcal{W}}^s_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$ to $\tilde{\mathcal{W}}^s_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$. For $\chi\in\mathcal{C}_0^\infty(\Omega;[0,1])$ such that $\chi\equiv1$ on a neighborhood Ω_χ of $\Omega-\mathrm{supp}\,a$ we have

$$a(x,D_x)\chi(q):\tilde{\mathcal{W}}^s_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^d;\mathbb{C})\to\tilde{\mathcal{W}}^s_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$$
$$\text{and}\quad a(x,D_x)\chi(q)\big|_{\tilde{\mathcal{W}}^s_{\Omega_\chi-\mathrm{comp}}} = a(x,D_x)\big|_{\tilde{\mathcal{W}}^s_{\Omega_\chi-\mathrm{comp}}}.$$

For two different choices of χ and χ' which are equal to 1 in a neighborhood of $\Omega-\mathrm{supp}\, a$, $a \in S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$, the differences $a(x,D_x)\chi(q)-a(x,D_x)\chi'(q) = a(x,D_x)(\chi(q)-\chi'(q)) = b(x,D_x)$ with $b\in S(\Psi^{-\infty},g_\Psi)$ and therefore is continuous from $\mathscr{S}'_{\Omega'-\mathrm{loc}}(\Omega'\times\mathbb{R}^d;\mathbb{C})$ to $\mathscr{S}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$, where $\Omega'=\Omega$ or any other open subset of $\mathbb{R}^d$.

Definition 2.E.9. For an open set Ω the set of regularizing operators is

$$\mathscr{R}(\Omega;\mathbb{C}) = \mathscr{L}(\mathscr{S}'_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^d;\mathbb{C});\mathscr{S}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})).$$

For two operators A,B continuous from $\mathscr{S}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$ to $\mathscr{S}'_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C})$ the equivalence $A\sim B$ is defined by $A-B\in\mathscr{R}(\Omega;\mathbb{C})$.
The set $\mathrm{OpS}^m_\Psi(\Omega;\mathbb{C})$ is the set of sums $a(x,D_x)\chi(q)+R$ with $a\in S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})\subset S(\Psi^m,g_\Psi)$, $\chi\in\mathscr{C}^\infty_0(\Omega;[0,1])$ with $\chi\equiv 1$ on a neighborhood of $\Omega-\mathrm{supp}\,a$, and $R\in\mathscr{R}(\Omega;\mathbb{C})$. The set $\mathrm{OpS}^{-\infty}_\Psi(\Omega;\mathbb{C})$ is $\mathscr{R}(\Omega;\mathbb{C})$.

With the previous remarks $\bigcup_{m\in\mathbb{R}}\mathrm{OpS}^m_\Psi(\Omega;\mathbb{C})$ is an algebra and clearly $\bigcap_{m\in\mathbb{R}}\mathrm{OpS}^m_\Psi(\Omega;\mathbb{C}) = \mathscr{R}(\Omega;\mathbb{C}) = \mathrm{OpS}^{-\infty}_\Psi(\Omega;\mathbb{C})$. Moreover if $A_j = a_j(x,D_x)\chi_j(q)+R_j\in\mathrm{OpS}^m_\Psi(\Omega_j;\mathrm{C})$ for $j=1,2$ then $A_1\circ A_2\in\mathrm{OpS}^m_\Psi(\Omega_1\cup\Omega_2;\mathbb{C})$ with $A_1\circ A_2\sim a_1(x,D_x)\circ a_2(x,D_x)\chi(q) = (a_1\sharp a_2)(x,D_x)\chi(q)$ for any $\chi\in\mathscr{C}^\infty_0(\Omega_1\cap\Omega_2;\mathbb{C})$ such that $\chi\equiv 1$ on a neighborhood of $\Omega-\mathrm{supp}\,a_1\cap\Omega-\mathrm{supp}\,a_2$.

Definition 2.E.10. For $A\in\mathscr{L}(\mathscr{S}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C});\mathscr{S}'_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^d;\mathbb{C}))$, the notation $A\sim\sum_{j=0}^\infty a_j(x,D_x)$ is thought of as a localized asymptotic sum, for a sequence $a_j\in S^{m_j}_{\Psi,\Omega-\mathrm{loc}}(\Omega;\mathbb{C})$ with $\lim_{j\to\infty}m_j = -\infty$ and $(m_j)_{j\in\mathbb{N}}$ decreasing. It means that for any pair $\varrho,\chi\in\mathscr{C}^\infty_0(\Omega;[0,1])$, with $\chi\equiv1$ on a neighborhood of $\Omega-\mathrm{supp}\,\varrho$, there exists $a_\varrho\in S^{m_0}_\Psi(\Omega;\mathbb{C})$ such that $\varrho(q)A\sim a_\varrho(x,D_x)\chi(q)$ and for any $J\in\mathbb{N}$, $a_\varrho-\sum_{j=0}^J\varrho(q)a_j\in S^{m_{J+1}}_\Psi(\Omega;\mathbb{C})$.

Notice that if $A=\tilde\varrho(q)A\tilde\chi(q)$ for some $\tilde\varrho,\tilde\chi\in\mathscr{C}^\infty_0(\Omega;\mathbb{C})$ and $a_j=\tilde\varrho(q)a_j$ for all $j\in\mathbb{N}$, the above condition is reduced to $a-\sum_{j=0}^J a_j\in S^{m_{J+1}}(\Omega;\mathbb{C})$ with a independent of ϱ. Moreover the localized definition means that we can always consider this simpler case.
The previous definition is justified by the following standard result.

Proposition 2.E.11. *For any sequence $a_j\in S(\Psi^{m_j},g_\Psi)$ with m_j decreasing and $\lim_{j\to\infty}m_j=-\infty$, there exists $a\in S(\Psi^{m_0},g_\Psi)$ such that $a-\sum_{j=0}^J a\in S(\Psi^{m_{J+1}},g_\Psi)$.*
For $A\in\mathscr{L}(\mathscr{S}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C});\mathscr{S}'_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^d;\mathbb{C}))$, $A\sim\sum_{j=0}^\infty a_j(x,D_x)$ is equivalent to the apparently weaker condition $A-\sum_{j=0}^J a_j(x,D_x)\in\mathscr{L}(\tilde{\mathscr{W}}^{-\mu_J}_{\Omega-\mathrm{comp}}(\Omega\times\mathbb{R}^d;\mathbb{C});\tilde{\mathscr{W}}^{\mu_J}_{\Omega-\mathrm{loc}}(\Omega\times\mathbb{R}^d;\mathbb{C}))$ with $\lim_{J\to\infty}\mu_J = -\infty$.

Proof. The first statement can be reduced to the case where $m_j=m_0-j$ by putting together $b_n=\sum_{m_0-n-1<m_j\le m_0-n}a_j$. Then use the standard Borel summation in $S(\Psi^{m_0},g_\Psi)$ by taking $a=\sum_{n=0}^\infty(1-\chi)\left(\frac{\Psi}{N_n(1+p_{\Psi^{m_0-n},n}(b_n))}\right)b_n$ for $\chi\in\mathscr{C}^\infty_0(\mathbb{R};[0,1])$ equal to 1 in a neighborhood of 0 and the sequence $(N_n)_{n\in\mathbb{N}}$ being increasing fast enough such that for every $k\in\mathbb{N}$,

$$\sum_{n=k}^\infty p_{\Psi^{m_0-k},k}\left[(1-\chi)\left(\frac{\Psi}{N_n(1+p_{\Psi^{m_0-n},n}(b_n))}\right)b_n\right]<+\infty.$$

For the second statement, fix $\varrho,\chi\in\mathscr{C}^\infty(\Omega;[0,1])$ with $\chi\equiv1$ in a neighborhood of ϱ. Then take $a_\varrho\in S(\Psi^{m_0},g_\Psi)$ such that $a_\varrho-\sum_{j=0}^j\varrho(q)a_j\in S(\Psi^{m_{J+1}},g_\Psi)$. For any $J\in\mathbb{N}$, the difference $D = \varrho(q)A\chi(q)-a_\varrho(x,D_x)\chi(q)$ equals

$$D=\varrho(q)A\chi(q)-a_\varrho(x,D_x)\chi(q)=\varrho(q)(A-\sum_{j=0}^J a_j(x,D_x))\chi(q)+(a_\varrho(x,D_x)-\sum_{j=0}^{m_j}(\varrho(q)a_j)(x,D_x))\chi(q)$$

and belongs to $\mathscr{L}(\tilde{\mathscr{W}}^{-\mu_J}(\mathbb{R}^{2d};\mathbb{C});\tilde{\mathscr{W}}^{\mu_J}(\mathbb{R}^{2d};\mathbb{C}))$ for all $J \in \mathbb{N}$ with $D = \tilde{\varrho}(q)D\tilde{\chi}(q)$ for some pair $\tilde{\varrho},\tilde{\chi} \in \mathscr{C}_0^\infty(\Omega;[0,1]) \subset \mathscr{C}^\infty(\mathbb{R}^d;[0,1])$. This implies that $D = \varrho(q)A\chi(q) - a_\varrho(x,D_x)\chi(q)$ is continuous from $\mathscr{S}'(\mathbb{R}^{2d};\mathbb{C})$ to $\mathscr{S}(\mathbb{R}^{2d};\mathbb{C})$ and with $D = \tilde{\varrho}(q)D\tilde{\chi}(q)$ it means $D \in \mathscr{R}(\Omega;\mathbb{C})$. □

In particular when $A_k \in \mathrm{OpS}^{m_k}_\Psi(\Omega;\mathbb{C})$ with $A_k = a_k(x,D_x)\chi_k(q) + R_k$, $a_k \in S^{m_k}_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ we can write as usual

$$A_1 \circ A_2 \sim \sum_{\alpha\in\mathbb{N}^{2d}} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha a_1 \partial_x^\alpha a_2 .$$

With the previous localization method, the global differential calculus of Subsection 2.E.2 is well defined with all its properties, if the following two conditions are satisfied:

- the class of symbols $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C}) \subset S^m_{\Psi,\mathbb{R}^d-\mathrm{comp}}(\mathbb{R}^d;\mathbb{C})$ is sent onto $S^m_{\Psi,\phi(\Omega)-\mathrm{comp}}(\phi(\Omega;\mathbb{C}) \subset S^m_{\Psi,\mathbb{R}^d-\mathrm{comp}}(\mathbb{R}^d;\mathbb{C})$ by the canonical transformation $\Phi_* : \mathbb{R}^{4d}_{q,p,\xi,\eta} \to R^{4d}_{q,p,\xi,\eta}$ induced by a diffeomorphism $\phi : \mathbb{R}^d \to \mathbb{R}^d$ with $\Phi(q,p) = (\phi(q), {}^t d\phi(q)^{-1}.p)$ and $\Phi_* : T^*\mathbb{R}^{2d} \to T^*\mathbb{R}^{2d}$;
- when U_Φ is the unitary transform in $L^2(\mathbb{R}^{2d}, dqdp;\mathbb{C})$ given by

$$(U_\Phi u)(q,p) = u \circ \Phi(q,p) = u(\phi(q), {}^t d\phi(q)^{-1}.p)$$

  satisfies $U_\Phi a(x,D_x)U_\Phi^{-1} \sim \sum_{n=0}^\infty b_n(x,D_x)$ in $\mathrm{OpS}^m_\Psi(\Omega,\mathbb{C})$ with $b_n \in S^{m-n}_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$, $b_0 = a \circ \Phi_* = \Phi^* a$ in $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$, for any $a \in S^m_{\Psi,\phi(\Omega)-\mathrm{comp}}(\phi(\Omega);\mathbb{C})$.

In particular all the sets $\Omega \times \mathbb{R}^d_p$ and $\Omega \times \mathbb{R}^{3d}_{p,\xi,\eta}$ can be replaced by $T^*\Omega$ and $T^*(T^*\Omega)$, with a natural geometrical meaning.
Actually we will consider more general changes of variables on $\mathbb{R}^{2d}$ which can be viewed as vector bundle isomorphisms of $\mathbb{R}^{2d} = T^*\mathbb{R}^d$. We consider the following change of variables

$$(\tilde{q},\tilde{p}) = \Phi(q,p) = (\phi(q), L(q).p) \tag{2.E.3.1}$$

which is a $\mathscr{C}^\infty$-diffeomorphism, a bijection such that $d\phi(q)$ and $L(q)$ belong to $GL_d(\mathbb{R})$ for all q with the following estimates

$$\forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0, \quad \|\partial_q^\alpha d\phi\|_{L^\infty} + \|\partial_q^\alpha (d\phi)^{-1}\|_{L^\infty} + \|\partial_q^\alpha L\|_{L^\infty} + \|\partial_q^\alpha L^{-1}\|_{L^\infty} \leq C_\alpha . \tag{2.E.3.2}$$

Note that Φ^{-1} takes the same form with

$$\Phi^{-1}(\tilde{q},\tilde{p}) = (q,p) = (\phi^{-1}(\tilde{q}), [L(\phi^{-1}(\tilde{q}))]^{-1}\tilde{p}).$$

It is given by a change of variable $\phi : \mathbb{R}^d_q \to \mathbb{R}^d_q$ when $L(q) = {}^t d\phi(q)^{-1}$.
Its differential is given by

$$d\Phi(q,p).\begin{pmatrix} t_q \\ t_p \end{pmatrix} = \begin{pmatrix} d\phi(q).t_q \\ (dL(q).p).t_q + L(q).t_p \end{pmatrix} = \begin{pmatrix} d\phi(q) & 0 \\ dL(q).p & L(q) \end{pmatrix}\begin{pmatrix} t_q \\ t_p \end{pmatrix}.$$

and

$${}^t d\Phi(q,p)^{-1} = \begin{pmatrix} {}^t d\phi(q)^{-1} & -{}^t d\phi(q)^{-1\,t}[L(q).p]^t L(q)^{-1} \\ 0 & {}^t L(q)^{-1} \end{pmatrix} = \begin{pmatrix} L_1(q) & (L_2(q).p) \\ 0 & L_3(q) \end{pmatrix},$$

where all linear maps L_1, L_2 and the bilinear map L_2 have uniformly bounded derivatives of any order with respect to $q \in \mathbb{R}^d$.
The canonical transformation $\Phi_* : T^*\mathbb{R}^{2d} \to T^*\mathbb{R}^{2d}$ is thus given by

$$\begin{pmatrix} \tilde{q} \\ \tilde{p} \\ \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \Phi_* \begin{pmatrix} q \\ p \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi(q) \\ L(q).p \\ L_1(q).\xi + L_2(q).p.\eta \\ L_3(q).\eta \end{pmatrix}$$

With the transformation Φ given by (2.E.3.1) we associate the unitary transform U_Φ in $L^2(\mathbb{R}^{2d}, dqdp;\mathbb{C})$:

$$(U_\Phi u)(q,p) = |\det(d\phi(q))\det(L(q))|^{1/2} u(\phi(q), L(q)p) = J^{1/2}(q) u(\phi(q), L(q)p)\,. \qquad (2.E.3.3)$$

Let us consider the simplest versions of spaces of functions.

Proposition 2.E.12. *For the tranformation Φ given by* (2.E.3.1) *the following properties hold for any $s \in \mathbb{R}$:*

- **i)** *The operator U_Φ (resp. U_Φ^{-1}) is continuous from $\mathscr{S}(\mathbb{R}^{2d};\mathbb{C})$ and from $\mathscr{S}'(\mathbb{R}^{2d};\mathbb{C})$ onto itself. For any open subset $\Omega \in \mathbb{R}^d$ it is continuous from $\mathscr{S}^\dagger_{\phi(\Omega)-\bullet}(\phi(\Omega)\times\mathbb{R}^d;\mathbb{C})$ (resp. $\mathscr{S}^\dagger_{\Omega-\bullet}(\Omega\times\mathbb{R}^d;\mathbb{C})$) onto $\mathscr{S}^\dagger_{\Omega-\bullet}(\Omega\times\mathbb{R}^d;\mathbb{C})$ (resp. $\mathscr{S}^\dagger_{\phi(\Omega)-\bullet}(\phi(\Omega)\times\mathbb{R}^d;\mathbb{C})$) where, with respective correspondence, $\mathscr{S}^\dagger$ stands for $\mathscr{S}$ or $\mathscr{S}'$ and $\bullet$ means* loc *or* comp.
- **ii)** *The map $a \mapsto a_\Phi = a\circ\Phi_*$ is continuous from $S(\Psi^s, g_\Psi)$ onto itself. For any open subset Ω in $\mathbb{R}^d$ and any $m \in \mathbb{R}$, it is continuous from $S^m_{\Psi,\phi(\Omega)-\mathrm{comp}}(\Phi(\Omega);\mathbb{C})$ onto $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{R})$.*

Proof. **i)** It suffices to consider $J^{-1/2}(q)U_\Phi$ because $|\partial_q^\alpha J^{\pm 1/2}| \le C_\alpha$ for all $\alpha \in \mathbb{N}^d$. We write

$$\partial_{q^i}(J^{-1/2}U_\Phi u)(q,p) = \partial_{q^i}\phi^j(q)(\partial_{q^i}u)(\phi(q), L(q).p) + (\partial_{q^i}L(q).p)_j\partial_{p_j}u(\phi(q),L(q).p)$$
$$\partial_{p_j}(J^{-1/2}U_\Phi u)(q,p) = L(q)^j_k(\partial_{p_k}u)(\phi(q),L(q).p)$$
$$\frac{1}{C_\Phi}(1+|q|^2+|p|^2) \le 1+|\phi(q)|^2+|L(q).p|^2 \le C_\Phi(1+|q|^2+|p|^2),$$

where the last inequality is a consequence of $1+|\phi(q)| \le C_\phi(1+|q|^2)$ owing to $|d\phi| \le C_0$ and its reverse inequality for ϕ^{-1}.
ii) The formula for $a\circ\Phi_*$ is

$$a\circ\Phi_* = a(\phi(q), L(q).p, L_1(q).\xi + L_2(q).p.\eta, L_3(q).\eta)\,.$$

Let us first compare Ψ and $\Psi\circ\Phi_*$:

$$\begin{aligned}\Psi^2\circ\Phi_*(q,p,\xi,\eta) &= 1+|L(q).p|^4 + |L_1(q).\xi + L_2(q).p.\eta|^2 + \left|L_3(q).\eta\right|^4 \\ &\le C_\Phi(1+|p|^4+|\xi|^2+|p|^2|\eta|^2+|\eta|^4) \\ &\le 2C_\Phi\Psi^2(q,p,\xi,\eta)\,.\end{aligned}$$

Applied to $\Psi^2\circ(\Phi^{-1})_*$, this provides the equivalence

$$\frac{1}{C'_\Phi}\Psi \le \Psi\circ(\Phi^{\pm 1})_* \le C'_\Phi\Psi\,.$$

The operators ∂_{q^i}, ∂_{p_i}, ∂_{ξ_i}, ∂_{η^i} applied to $a\circ\Phi_*$ are equivalent to the following $\mathscr{C}^\infty_b(\mathbb{R}^d_q;\mathbb{R})$ linear combinations (abbreviated as L.C) of elementary operators acting on a:

∂_{q^i}**:** L.C of ∂_{q^j}, $p_k\partial_{p_j}$, $\xi_i\partial_{\xi_j}$, $p_i\eta^j\partial_{\xi_\ell}$, ∂_{η_j}, which all are continuous from $S(\Psi^s, g_\Psi)$ to $S(\Psi^s, g_\Psi)$.

∂_{p_i}**:** L.C. of ∂_{p_j}, $\eta^j\partial_{\xi_k}$, which are continuous from $S(\Psi^s, g_\Psi)$ to $S(\Psi^{s-1/2}, g_\Psi)$.

∂_{ξ_i}**:** L.C. of ∂_{ξ_j} which are all continuous from $S(\Psi^s, g_\Psi)$ to $S(\Psi^{s-1}, g_\Psi)$.

∂_{η^i}**:** L.C. of $p_j\partial_{\xi_k}$ and of ∂_{η^i} which are all continuous from $S(\Psi^s, g_\Psi)$ to $S(\Psi^{s-1/2}, g_\Psi)$.

This proves the continuity of $a \mapsto a\circ\Phi_*$ from $S(\Psi^s, g_\Psi)$ to $S(\Psi^s, g_\Psi)$. □

Let us consider the functoriality of the transformation of the quantization rule $a \mapsto a(x, D_x)\chi(q) + R$ with $a \in S^m_{\Psi,\phi(\Omega)-\mathrm{comp}}(\Omega;\mathbb{C})$, $\chi \equiv 1$ in a neighborhood of $\phi(\Omega) - \mathrm{supp}\,a$ and $R \in \mathscr{R}(\phi(\Omega);\mathbb{C})$.

Proposition 2.E.13. *For any* $A = a(x,D_x)\chi(q)+R \in \mathrm{OpS}^m_\Psi(\phi(\Omega);\mathbb{C})$*, the operator* $U_\Phi A U_\Phi^*$ *is equal to* $b_\Phi(x,D_x)\chi(\Phi(q))+R_\Phi$ *with* $R_\Phi \in \mathscr{R}(\phi(\Omega);\mathbb{C})$ *and* $b_\Phi \in S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$ *and satisfies*

$$U_\Phi A U_\Phi \sim \sum_{n=0}^{\infty} b_n(x,D_x)$$

according to Definition 2.E.10 with $b_0 = a\circ\Phi_*$. *More precisely when* Ω *is a bounded open subset, with* $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})\subset S(\Psi^m,g_\psi)$ *and* $S^m_{\Psi,\phi(\Omega)-\mathrm{comp}}(\phi(\Omega);\mathbb{C})\subset S(\Psi^m,g_\psi)$,

$$b_\Phi = \sum_{n=0}^{N-1} b_n + r_{\Phi,N}(a)$$

with a continuous map $r_{\Phi,N}: S(\Psi^m,g_\psi)\to S(\Psi^{m-N},g_\psi)$ *for every* $N\in\mathbb{N}$.

Remark 2.E.14. Except for the principal symbol this result does not say that the transformation $a\mapsto b_\Phi$ corresponds $b_\Phi = a\circ\Phi_*$. It works exactly only for functions $a(q,p)$ and in particular for the cut-off functions with respect to q. But when Ω is a bounded open subset of $\mathbb{R}^d$, $U_\Phi a(x,D_x)\chi(q)U_\Phi^{-1} = b_\phi(x,D_x)\chi(\phi(q))$ defines a continuous operator from $S^m_{\Psi,\phi(\Omega)-\mathrm{comp}}(\phi(\Omega);\mathbb{C})$ to $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$.

Proof. With the localization we can assume $\Omega=\mathbb{R}^d$, $a\in S(\Psi^m;g_\Psi)$, $\mathbb{R}^d-\mathrm{supp}\,a$ compact, and $R=0$. We introduce another cut-off function $\tilde\chi\in\mathscr{C}^\infty_0(\mathbb{R}^d;[0,1])$ equal to 1 on $\Phi(\Omega)$ when Ω is bounded and, for a more general choice of Ω, equal to 1 in a neighborhood of $\mathrm{supp}\,\chi$.
Because the function $J^{\pm 1/2}(q)\in S(1,g_\Psi)$, the problem is reduced to the study of the operator

$$(J^{-1/2}(q)U_\Phi)\tilde\chi(q)a(x,D_x)\tilde\chi(q)(J^{-1/2}(q)U_\Phi)^{-1}$$

of which the Schwarz kernel is given by the oscillating integral

$$K(x,y)=\int_{\mathbb{R}^{2d}} e^{i[\xi.(\phi(q)-\phi(q'))+\eta.(L(q).p-L(q').p')]}\tilde\chi(\phi(q))[a(\phi(q),L(q).p,\xi,\eta)]\tilde\chi(\phi(q'))\,\frac{d\xi d\eta}{(2\pi)^{2d}}.$$

Metrics and cut-off: On $\mathbb{R}^{6d}_{x,x',\Xi}$ with $x=(q,p)$, $x'=(q',p')$ and $\Xi=(\xi,\eta)$ we consider the metrics

$$G_f = dq^2+dq'^2+\frac{dp^2}{f}+\frac{dp'^2}{f}+\frac{d\xi^2}{f^2}+\frac{d\eta^2}{f}$$

for $\quad f=\Psi(q,p,\xi,\eta)=(1+|\xi|^2+|p|^4+|\eta|^4)^{1/2}\quad$ and $\quad f=\widetilde\Psi=(1+|\xi|^2+|p|^4+|p'|^4+|\eta|^4)^{1/2}$.

The metrics G_Ψ and $G_{\tilde\Psi}$ are slow (same proof as for g_Ψ). But the slowness of g_ψ implies

$$\left(\frac{\Psi(q,p,\xi,\eta)}{\Psi(q',p',\xi,\eta)}\right)^{\pm1}\le C_\Psi \quad\text{when}\quad |p-p'|\le C_\Psi^{-1}\Psi(q,p,\xi,\eta)^{1/2}$$

and therefore with the above notations

$$|p-p'|\le\frac{1}{C'_\Psi}\Psi^{1/2}\Rightarrow\left(\frac{\Psi}{\tilde\Psi}\right)^{\pm1}\le C'_\Psi$$

for some large enough constant $C'_\Psi\ge1$, and

$$c\in S(\Psi^m,G_\Psi)\Leftrightarrow c\in S(\widetilde\Psi^m,G_{\tilde\Psi})\quad\text{when }\mathrm{supp}\,c\subset\left\{(x,y,\Xi)\in\mathbb{R}^{6d},\,|p-p'|<\frac{1}{C'_\Psi}\Psi^{1/2}\right\}.$$

For $\theta \in \mathscr{C}_0^\infty(\mathbb{R};[0,1])$ and $\varepsilon > 0$, $\varepsilon \leq \frac{1}{C'_\Psi}$ fixed later consider the two cut-off functions

$$\Theta_1(x,x',\Xi) = \Theta_1(q,q') = \theta\Big(\frac{|q-q'|^2}{\varepsilon^2}\Big) \quad \text{and} \quad \Theta_2(x,x',\Xi) = \theta\Big(\frac{|p-p'|^2}{\varepsilon^2 \Psi}\Big).$$

By looking at the region $2^{n+10} \leq \widetilde{\Psi} \leq 2^{n+12}$ contained in a fixed shell for the rescaled variable $(\tilde{p},\tilde{p}',\tilde{\xi},\tilde{\eta}) = (2^{-n/2}p, 2^{-n/2}p', 2^{-n}\xi, 2^{-n/2}\eta)$, a homogeneity argument gives $\Theta_2 \in S(1,G_{\widetilde{\Psi}}) \cap S(1,G_\Psi)$. The following properties become obvious when $a \in S(\Psi^m, g_\Psi)$

$$\begin{aligned}
&b_{\tilde{\chi},\phi} = \tilde{\chi}(\phi(q))[a(\phi(q),L(q).p,\xi,\eta)]\tilde{\chi}(\phi(q')) \in S(\Psi^m, G_\Psi),\\
&\Theta_1, \Theta_2, 1-\Theta_2 \in S(1,G_{\widetilde{\Psi}}) \cap S(1,G_\Psi).
\end{aligned}$$

We now write the kernel $K(x,x')$ or the operator $K : \mathscr{S}_{\mathbb{R}^d-\mathrm{loc}}(\mathbb{R}^{2d};\mathbb{C}) \to \mathscr{S}'_{\mathbb{R}^d-\mathrm{comp}}(\mathbb{R}^{2d};\mathbb{C})$ as

$$\begin{aligned}
&K = K_{\mathrm{diag}} + K_1 + K_2\\
\text{with}\quad &K_1(x,x') = \int_{\mathbb{R}^{2d}} e^{i[\xi.(\phi(q)-\phi(q'))+\eta.(L(q).p-L(q').p')]}(1-\Theta_1(q,q'))b_{\tilde{\chi},\phi}(x,x',\Xi)\,\frac{d\xi d\eta}{(2\pi)^{2d}}\\
&K_2(x,x') = \int_{\mathbb{R}^{2d}} e^{i[\xi.(\phi(q)-\phi(q'))+\eta.(L(q).p-L(q').p')]}\Theta_1(q,q')(1-\Theta_2(x,x',\Xi))b_{\tilde{\chi},\phi}(x,x',\Xi)\,\frac{d\xi d\eta}{(2\pi)^{2d}}\\
&K_{\mathrm{diag}(x,x')} = \int_{\mathbb{R}^{2d}} e^{i[\xi.(\phi(q)-\phi(q'))+\eta.(L(q).p-L(q').p')]}\Theta_1(q,q')\Theta_2(x,x',\Xi)b_{\tilde{\chi},\phi}(x,x',\Xi)\,\frac{d\xi d\eta}{(2\pi)^{2d}},
\end{aligned}$$

and use the same symbol K_{diag}, $K_{1,2}$ for the associated operators $\mathscr{S}_{\mathbb{R}^d-\mathrm{loc}}(\mathbb{R}^{2d};\mathbb{C}) \to \mathscr{S}'_{\mathbb{R}^d-\mathrm{comp}}(\mathbb{R}^{2d};\mathbb{C})$ with uniformly controlled supports.
Non stationary phase in q: For a given $k \in \mathbb{N}$, $N \geq N_{k,d}$ integrations by parts with

$$\Big[\frac{1}{|\phi(q)-\phi(q')|^2}(\phi(q)-\phi(q')).D_\xi\Big]^N e^{i\xi.(\phi(q)-\phi(q'))} = e^{i\xi.(\phi(q)-\phi(q'))},$$

and the lower bound

$$\forall (q,q') \in \mathrm{supp}(1-\Theta_1) \cap V_{\tilde{\chi}}^2, \quad |\phi(q)-\phi(q')| \geq \frac{1}{C_{\tilde{\chi},\phi}}, \tag{2.E.3.4}$$

for $\varepsilon \leq \frac{1}{C_{\tilde{\chi},\phi}}$, small enough, and $V_{\tilde{\chi}}$ a compact neighborhood of $\mathrm{supp}\,\tilde{\chi}$, implies that for all $(\alpha_j,\beta_j,\gamma_j) \in \mathbb{N}^{3d}$, $2|\alpha_j| + |\beta_j| + |\gamma_j| \leq k$, for $j = 1,2$, the kernel

$$(-1)^{\alpha_2+\gamma_2}\partial_q^{\alpha_1}p^{\beta_1}\partial_p^{\gamma_1}\partial_{q'}^{\alpha_2}(p')^{\beta_2}\partial_{p'}^{\gamma_2}K_1(x,x')$$

of $(\partial_q^{\alpha_1}p^{\beta_1}\partial_p^{\gamma_1}) \circ K_1 \circ (\partial_q^{\alpha_2}p^{\beta_2}\partial_p^{\gamma_2})$ belongs to $L^2(\mathbb{R}^{4d}, dqdpdq'dp';\mathbb{C})$. Actually the estimates with powers of p' is deduced from our estimates with powers of p via integration by parts with

$$D_\eta e^{i\eta.(L(q).p-L(q').p')} = (L(q).p - L(q').p')e^{i\eta.(L(q).p-L(q').p')}.$$

We deduce that K_1 is Hilbert-Schmidt and therefore bounded operator from $\tilde{\mathscr{W}}^{-k}(\mathbb{R}^{2d};\mathbb{C})$ to $\tilde{\mathscr{W}}^{k}(\mathbb{R}^{2d};\mathbb{C})$ for any $k \in \mathbb{C}$. With the fixed compact Ω-support, $K_1 \in \mathscr{L}(\mathscr{S}'(\mathbb{R}^{2d};\mathbb{C});\mathscr{S}(\mathbb{R}^{2d};\mathbb{C}))$. It has a symbol in $\mathscr{S}(\mathbb{R}^{4d};\mathbb{C}) \subset S(\Psi^{-\infty}, g_\psi)$ with a compact support in Ω.

Kuranishi's trick: Write

$$\xi.(\phi(q)-\phi(q')) + \eta.(L(q)p - L(q')p') = (\xi,\eta).(\Phi(x)-\Phi(x')) = (\xi,\eta).\left[\int_0^1 d\Phi((1-t)x+tx')\,dt\right]\begin{pmatrix} q-q'\\ p-p'\end{pmatrix}$$

and remember with $x_t=(1-t)x+tx'$, $q_t=(1-t)q+tq'$, $p_t=(1-t)p+tp'$

$$\int_0^1 d\Phi(x_t)\,dt=\int_0^1\begin{pmatrix} d\phi(q_t) & 0\\ dL(q_t).p_t & L(q_t)\end{pmatrix}dt=\begin{pmatrix}\int_0^1 d\phi(q_t)\,dt & 0\\ [\int_0^1(1-t)dL(q_t)\,dt].p+[\int_0^1 t dL(q_t)\,dt]p' & \int_0^1 L(q_t)\,dt\end{pmatrix}$$

Because $(\mathbb{R}^{2d}_{q,q'}-\mathrm{supp}K)\subset\mathrm{supp}\,\tilde{\chi}\times\mathrm{supp}\,\tilde{\chi}$, we can fix $\varepsilon\leq\frac{1}{C_{\tilde{\chi},\phi}}$ small enough so that the inequalities

$$\forall(q,q')\in\mathrm{supp}\,\Theta_1\cap V_{\tilde{\chi}}^2,\ \forall A\in\mathrm{Conv}(d\phi([q,q']))\cup\mathrm{Conv}(L([q,q'])),\quad |\det(A)|\geq\frac{1}{C_{\tilde{\chi},\Phi}},\qquad(2.E.3.5)$$

where $\mathrm{Conv}(M([q,q']))$ stands for the convex hull in $\mathcal{M}_d(\mathbb{C})$ of the set $M([q,q'])=\{M(q_t,t\in[0,1])\}\subset M_d(\mathbb{C})$, while (2.E.3.4) remains valid. We obtain for (q,q')

$$[\xi.(\phi(q)-\phi(q'))+\eta.(L(q).p-L(q').p')]=\left[\begin{pmatrix}A(q,q') & B(q,q').p+C(q,q').p'\\ 0 & D(q,q')\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right]\cdot\begin{pmatrix}q-q'\\ p-p'\end{pmatrix}$$

with $\quad E(x,x')=\begin{pmatrix}A(q,q') & B(q,q').p+C(q,q').p'\\ 0 & D(q,q')\end{pmatrix}^{-1}=\begin{pmatrix}A(q,q')^{-1} & B'(q,q').p+C'(q,q').p'\\ 0 & D(q,q')^{-1}\end{pmatrix}$

and $\quad A,B,C,D,A^{-1},B^{-1},C',D'\in S(1,dq^2+dq'^2;\mathcal{M}_d(\mathbb{R}))$.

We obtain

$$K_2(x,x')=\int_{\mathbb{R}^{2d}}e^{i[\xi.(q-q')+\eta.(p-p')]}[\Theta_1(1-\Theta_2)b_{\tilde{\chi},\phi}](x,x',E(x,x').\Xi)\,|\det(E^{-1})(q,q')|\,\frac{d\xi d\eta}{(2\pi)^{2d}},$$

$$K_{\mathrm{diag}}(x,x')=\int_{\mathbb{R}^{2d}}e^{i[\xi.(q-q')+\eta.(p-p')]}[\Theta_1\Theta_2 b_{\tilde{\chi},\phi}](x,x',E(x,x').\Xi)\,|\det(E^{-1})(q,q')|\,\frac{d\xi d\eta}{(2\pi)^{2d}},$$

where choosing $\varepsilon\leq\frac{1}{C_{\psi,\phi,\tilde{\chi}}}$, small enough, ensures

$$\Theta_1(x,x',E(x,x').\Xi)=\theta\Big(\frac{|q-q'|^2}{\varepsilon^2}\Big)=\Theta_1(q,q')\in S(1,G_{\tilde{\Psi}}),$$

$$\Theta_2(x,x',E(x,x').\Xi)=\theta\left(\frac{|p-p'|^2}{\varepsilon^2\Psi(0,p,A(q,q').\xi+B'(q,q').p.\eta+C'(q,q').p'.\eta,D(q,q')^{-1}.\eta)}\right)\in S(1,G_{\tilde{\Psi}}),$$

$$b_{\tilde{\chi},\phi}(x,x',E(x,x').\Xi)=\tilde{\chi}(q)\tilde{\chi}(q')a(\phi(q),L(q).p,A(q,q').\xi+B'(q,q').p.\eta+C'(q,q').p'.\eta,D(q,q')^{-1}.\eta)$$

$$[(1-\Theta_2)b_{\tilde{\chi},\phi}](x,x',E(x,x').\Xi)\in S(\tilde{\Psi}^{|m|},dx^2+dx'^2+d\Xi^2)\quad,\quad[\Theta_2 b_{\tilde{\chi},\phi}](x,x',E(x,x').\Xi)\in S(\tilde{\Psi}^m,G_{\tilde{\Psi}}).$$

Actually it suffices to check

$$\Psi^2(0,p,E(x,x').\Xi)\leq C(1+(|\xi|+|p||\eta|+|p'||\eta|)^2+|p|^4+|p'|^4+|\eta|^4)\leq C'\tilde{\Psi}^2$$

with the symmetric version by applying the same result to $\Psi^2(0,p,E(x,x')^{-1}.\Xi)$ and then to use the equivalence $\left(\frac{\tilde{\Psi}}{\Psi}\right)^{\pm1}\leq C_\varepsilon$ when $|p-p'|\leq\varepsilon\tilde{\Psi}^{1/2}$ owing to the slowness of $G_{\tilde{\Psi}}$.

Non stationary phase in p: Despite the bad a priori estimate of $[(1-\Theta_2)b_{\tilde{\chi},\phi}](x,x',E(x,x').\Xi)$, $N\geq N_{k,d}$ integrations by parts for a given $k\in N$ with

$$\left(\frac{1}{|p-p'|^2}(p-p').D_\eta\right)^N e^{i[\xi.(q-q')+\eta.(p-p')]}=e^{i[\xi.(q-q')+\eta.(p-p')]}$$

$$\text{and}\qquad\forall(x,x')\in\mathrm{supp}(1-\Theta_2)(.,.,E(.,.).),\quad|p-p'|\geq\frac{1}{C_{\Psi,\phi,\tilde{\chi}}}\tilde{\Psi}^{1/2},$$

leads to the property that the kernel of $(\partial_q^{\alpha_1}p^{\beta_1}\partial_p^{\gamma_1})\circ K_2\circ(\partial_q^{\alpha_2}p^{\beta_2}\partial_p^{\gamma_2})$ is Hilbert-Schmidt and therefore bounded in $L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})$ for $|\alpha_j|+\frac{|\beta_j|+|\gamma_j|}{2}\leq k$. We conclude as we did for K_1 that K_2

belongs to $\mathscr{L}(\mathscr{S}'(\mathbb{R}^{2d};\mathbb{C});\mathscr{S}(\mathbb{R}^{2d};\mathbb{C}))$. It has a symbol in $\mathscr{S}(\mathbb{R}^{4d};\mathbb{C})\subset S(\Psi^{-\infty},g_\Psi)$ with a compact support in Ω.

Gauss transform : The kernel of K_{diag} can be written

$$K_{\text{diag}}(x,x')=b(x,D_x)\quad\text{with}\quad b(x,\Xi)=e^{iD_\Xi.D_{x'}}\left[[\Theta_1\Theta_2 b_{\tilde{\chi},\phi}](x,x',E(x,x')\Xi)\right]\big|_{x=x'},$$

where the metric $G_{\widetilde{\Psi}}$ is slow on $\mathbb{R}^{6d}$, the B-dual metric of $G_{\widetilde{\Psi}}$ for $B=\begin{pmatrix}0&0&0\\0&0&\frac{1}{2}\mathrm{Id}_{\mathbb{R}^{2d}}\\0&\frac{1}{2}\mathrm{Id}_{\mathbb{R}^{2d}}&0\end{pmatrix}$ is the degenerate metric $G^B=\widetilde{\Psi}^2dq'^2+\widetilde{\Psi}dp'^2+d\xi^2+\widetilde{\Psi}d\eta^2$. Fortunately $G_{\widetilde{\Psi}}$ is G^B-temperate along the vector space $V_0=\left\{(x,x',\Xi)\in\mathbb{R}^{6d}\,,x=x'\right\}$ and $\widetilde{\Psi}\big|_{V_0}$ can be replaced with Ψ. Theorem 18.4.11 of [HormIII] tells us that $b\in S(\Psi^m,g_\Psi)$ with the asymptotic expansion

$$b(x,D_x)\sim\sum_{n=0}^{\infty}\underbrace{b_n(x,D_x)}_{b_n\in S(\Psi^{m-n},g_\psi)}$$

and the first term $b_0(x,\Xi)=b_{\phi,\tilde{\chi}}(x,x,\Xi)=a\circ\Phi_*(x,\Xi)$. □

Remark 2.E.15. We could have used the general theory of global Fourier integral operators of J.M. Bony in [Bon]. At least when $\phi-\mathrm{Id}_{\mathbb{R}^d}$ and $L-\mathrm{Id}_{\mathbb{R}^d}$ have a compact support, this describes U_Φ as a Fourier integral operator of which the global symbol is a section of the fiber bundle of affine metaplectic operators $\mathscr{M}\to\mathscr{G}_\phi$ above the graph of ϕ_*, $\mathscr{G}_\Phi=\left\{(X,\Phi_*(X)),\quad X\in\mathbb{R}^{4d}\right\}$ with a value above $(X_0,\Phi(X_0))$, $x_0=(q_0,p_0,\xi_0,\eta_0)$ which is the composition $\tau_{\phi(X_0)}U_{X_0}\tau_{-x_0}$ where $\tau_{(x_1,\Xi_1)}$ is the phase translation $e^{i(\Xi_1.D_x-x_1.D_\Xi)}$ and U_{X_0} is a metaplectic representation of the linear symplectic map $d\Phi_*(X_0)$.
The proposed methods mimics the classical techniques of pseudo-differential calculus for the metric $dq^2+\frac{d\xi^2}{\langle\xi\rangle^2}$ modulo the localization process only in the q-variable and for specific linear transformations in the p-variable. It is informative from this point of view and provides a more explicit formulation for the functoriality of the principal symbol.

From Proposition 2.E.12 and the definition of $\tilde{\mathscr{W}}^s_{\Omega-\bullet}(\Omega;\mathbb{C})$, $\bullet=\text{loc or comp}$, deduced from Definition 2.E.5, we obtain the following result.

Proposition 2.E.16. *Consider the unitary map $U_\Phi:L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})\to L^2(\mathbb{R}^{2d},dqdp;\mathbb{C})$ given by* (2.E.3.1)(2.E.3.2)(2.E.3.3) *with the additional assumption that $\phi-\mathrm{Id}_{\mathbb{R}^d}$ and $L-\mathrm{Id}_{\mathbb{R}^d}$ have a compact support. Then for any $s\in\mathbb{R}^d$, U_Φ and U_Φ^{-1} are isomorphisms from $\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})$ into itself, with $\mathbb{R}^d_q-\operatorname{supp}U_\Phi^{\pm1}u=\phi^{\mp1}(\mathbb{R}^d_q-\operatorname{supp}u)$ for every $u\in\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})$.*

Proof. The support property is obvious from the definition (2.E.3.3) of U_Φ. With the additional support assumption on $\Phi-\mathrm{Id}_{\mathbb{R}^{2d}}$, we can write for any $u\in\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})$, $U_\Phi u=U_\Phi\chi_1(\phi(q))u+\chi_2(q)u$ for $\chi_1,\chi_2\in\mathscr{C}^\infty(\mathbb{R}^d;[0,1])$, $\chi_1+\chi_2\equiv1$ and $\operatorname{supp}\chi_1$ compact. From the Definition 2.E.5 and the global pseudo-differential calculus in $S(\Psi^{2s},g_\psi)$ remember

$$C_s^{-1}\mathrm{Re}\langle u\,,M_{2s}(x,D_x)u\rangle_{L^2}\leq\|u\|^2_{\tilde{\mathscr{W}}^s}=\|M_s^W(x,D_x)u\|^2_{L^2}\leq C_s\mathrm{Re}\langle u\,,M_{2s}(x,D_x)u\rangle_{L^2},$$

for $M_s=(C_s+\Psi^{|s|})^{\mathrm{sign}\,s}$ with $C_s\geq1$ large enough.
We deduce

$$\|U_\Phi\chi_1(\phi(q))u\|^2_{\tilde{\mathscr{W}}^s}\leq C_s\mathrm{Re}\langle u\,,U_\phi^*(\chi_1(q)M_{2s})(x,D_x)U_\Phi\chi_1(\phi(q))u\rangle.$$

By Proposition 2.E.13 we know

$$U_\phi^*(\chi_1(q)M_{2s})(x,D_x)U_\Phi\chi_1(\phi(q))=b_{2s}(x,D_x)\circ\chi_1(\phi(q))=c_{2s}^W(x,D_x)\quad b_{2s},c_{2s}\in S(\Psi^{2s},g_\Psi),$$

and the pseudo-differential calculus in $S(\Psi^{2s},g_\Psi)$ gives

$$\|U_\Phi\chi_1(\phi(q))u\|^2_{\tilde{\mathscr{W}}^s}\leq C'_s\|u\|^2_{\tilde{\mathscr{W}}^s}\,.$$

By the triangular inequality $U_\Phi:\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})\to\tilde{\mathscr{W}}^s(\mathbb{R}^{2d};\mathbb{C})$ is continuous and we conclude with $U_\Phi^{-1}=U_{\Phi^{-1}}$. □

All this section gives a meaning to $\mathscr{S}^\dagger_{\Omega-\bullet}(T^*\Omega;\mathbb{C})$, $S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathbb{C})$, $\mathscr{R}(\Omega;\mathbb{C})$, $\mathrm{OpS}^m_\Psi(\Omega;\mathbb{C})$ and $\tilde{\mathscr{W}}^s_{\Omega-\bullet}(T^*\Omega;\mathbb{C})$ (with $\mathscr{S}^\dagger=\mathscr{S}$ or $\mathscr{S}'$ and $\bullet$ meaning loc or comp) when Ω is a chart open set in the compact manifold Q.
The topology, the continuity properties and the global ellipticity that we need will be better discussed in the global setting which avoids considerations of inductive limit topologies.

2.E.4 Globalization on Q and applications

Let us fix, an atlas covering of Q, $Q=\cup_{j=1}^J\Omega_j$ such that $\tilde{\Omega}_j=\cup_{\Omega_{j'}\cap\Omega_j\neq\emptyset}\Omega_{j'}$ is still a chart open set. We take a finite partition of unity $\sum_{j=1}^J\varrho_j(q)\equiv 1$ subordinate with the atlas covering $Q=\cup_{j=1}^J\Omega_j$ and cut-off functions $\chi_j\in\mathscr{C}^\infty_0(\Omega_j;[0,1])$ such that $\chi_1\equiv 1$ in a neighborhood of $\mathrm{supp}\,\varrho_j$. Notice that $\chi_j\chi_{j'}\neq 0$ implies $\Omega_i\cap\Omega_j\neq 0$ and in this case $\varrho_j,\varrho_{j'},\chi_j,\chi_{j'}$ are cut-off functions in the coordinate charts $\tilde{\Omega}_j$ and $\tilde{\Omega}_{j'}$.
The spaces $S^m_\Psi(Q;\mathbb{C})$ (resp. $\widetilde{\mathscr{W}}^s(T^*Q;\mathbb{C})$) are the sets $a=\sum_{j=1}^J\varrho_j(q)a$ with $\varrho_j(q)a\in S^m_{\Psi,\Omega_j-\mathrm{comp}}(\Omega_j;\mathbb{C})\subset S(\Psi^m,g_\Psi)$ (resp. $\varrho_ja\in\widetilde{\mathscr{W}}^s_{\Omega_j-\mathrm{comp}}(T^*\Omega_j;\mathbb{C})$) with the topologies given by

$$p_{m,k}(a)=\sum_{j=1}^J p_{\tilde{\Omega}_j,\Psi^m,k}(\varrho_j(q)a)\,,\quad k\in\mathbb{N}$$

$$\text{resp.}\qquad \|a\|^2_{\widetilde{\mathscr{W}}^s(Q;\mathbb{C})}=\sum_{j=1}^J\|\varrho_j(q)a\|^2_{\widetilde{\mathscr{W}}^s(T^*\Omega;\mathbb{C})}\,.$$

The subscript $_{\tilde{\Omega}_j}$ in $p_{\tilde{\Omega}_j,\Psi^m,k}$ refers to the choice of some local coordinates in $\tilde{\Omega}_j$. But Proposition 2.E.12 and Proposition 2.E.13 for the conjugation $a\mapsto U_\Phi a(x,D_x)\chi(q)U_\Phi^{-1}$ with $\Phi(q,p)=(\phi(q),{}^td\phi(q)^{-1}.p)$, says that the seminorms $p_{\tilde{\Omega}_j,\Psi^m,k}(\varrho_j(q)a)$ can be replaced by $p_{\tilde{\Omega}_{j'},\Psi^m,k}(\varrho_j(q)a)$ for any $j'\in\{1,\ldots,J\}$ such that $\Omega_j\subset\tilde{\Omega}_{j'}$.
A vector bundle isomorphism on T^*Q, written locally as $\Phi:(q,p)\mapsto(\phi(q),L(q).p)$ with the associated unitary operator U_Φ, gives rise to an isomorphism of the space $\tilde{\mathscr{W}}^s(Q;\mathbb{C})$ according to the local result of Proposition 2.E.16. This gives a first application, which is not exactly due to the pseudo-differential calculus

Proposition 2.E.17.
For any riemannian metric $g=g_{ij}(q)dq^idq^j$ on TQ with the dual metric $g^{ij}(q)dp_idp_j$ on T^*Q, if $\sum_{\ell=-1}^\infty\theta_\ell^2(t)\equiv 1$ is a quadratic dyadic partition of unity like (2.4.1.1) and $|p|_q^2=g^{ij}(q)p_ip_j$, then for every $s\in\mathbb{R}$ the squared norm $\|u\|^2_{\tilde{\mathscr{W}}^s(Q;\mathbb{C})}$ is equivalent to $\sum_{\ell=-1}^\infty\|\theta_\ell(|p|_q^2)u\|^2_{\tilde{\mathscr{W}}^s(Q;\mathbb{C})}$.

Proof. It suffices to use the local gauge transform given by $\Phi:(q,p)\mapsto(q,g^{-1/2}(q).p)$ with $|p|_q^2={}^tpg^{-1}(q)p=|g^{-1/2}(q).p|^2$ and to write

$$\sum_{\ell=-1}^\infty\|\theta_\ell(|p|_q^2)u\|^2_{\tilde{\mathscr{W}}^s}\asymp\sum_{\ell=-1}^\infty\|U_\Phi^{-1}\theta_\ell(|p|_q^2)u\|^2_{\tilde{\mathscr{W}}^s}\asymp\sum_{\ell=-1}^\infty\|\theta_\ell(|p|^2)U_\Phi^{-1}u\|^2_{\tilde{\mathscr{W}}^s}$$

and Proposition 2.E.7 gives

$$\sum_{\ell=-1}^\infty\|\theta_\ell(|p|_q^2)u\|^2_{\tilde{\mathscr{W}}^s}\asymp\|U_\Phi^{-1}u\|^2_{\tilde{\mathscr{W}}^s}\asymp\|u\|^2_{\tilde{\mathscr{W}}^s}\,.$$

□

The intersection $\bigcap_{s\in\mathbb{R}}\widetilde{\mathcal{W}}^s(T^*Q;\mathbb{C})$ is nothing but $\mathscr{S}(T^*Q;\mathbb{C})$.
On $\mathscr{R}(Q;\mathbb{C})=\mathscr{L}(\mathscr{S}'(T^*Q;\mathbb{C});\mathscr{S}(T^*Q;\mathbb{C}))\sim\mathscr{S}(T^*Q\times T^*Q;\mathbb{C})$, the Fréchet space topology is equivalently defined by the family of (semi)norms

$$q_k(R)=\|R\|_{\mathscr{L}(\widetilde{\mathcal{W}}^{-k}(T^*Q;\mathbb{C});\widetilde{\mathcal{W}}^{k}(T^*Q;\mathbb{C}))}\,,\quad k\in\mathbb{N}.$$

Before giving an explicit family of seminorms on

$$\mathrm{OpS}^m_\Psi(Q;\mathbb{C})=\left\{\sum_{j=1}^J(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)+R\,,a\in S^m_\Psi(\Omega;\mathbb{C}),R\in\mathscr{R}(\Omega;\mathbb{C})\right\},$$

let us check that any $A\in\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ admits a canonical decomposition after fixing some cut-off function on $Q\times Q$.
Attention must be payed to the following point: although $\varrho_j(q)a_{j'}=\varrho_{j'}(q)a_j$ for all pairs (j,j') allows to define $a(x,\Xi)=\sum_{j'=1}^J\varrho_{j'}(q)a_{j'}(x,\Xi)$, the equality $\sum_{j=1}^J(\varrho_j(q))a_j(x,D_x)\circ\chi_j(q)-\sum_{j=1}^J(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)=R\in\mathscr{R}(Q;\mathbb{C})$ is true with $R=0$ only under the equality of the operators

$$(\varrho_{j'}(q)a_j)(x,D_x)\circ\chi_{j'}(q)=(\varrho_j(q)a_{j'})(x,D_x)\circ\chi_j(q)$$

for all pairs (j,j').
With the subset $\tilde{\Omega}_j=\bigcup_{\Omega_{j'}\cap\Omega_j\neq\emptyset}\Omega_{j'}$ take a cut-off function $\tilde{\chi}_j\in\mathscr{C}^\infty_0(\tilde{\Omega}_j;[0,1])$ such that $\tilde{\chi}_j\equiv1$ on a neighborhood of

$$\bigcup_{\Omega_{j'}\cap\Omega_j\neq\emptyset}\mathrm{supp}\,\chi_j\supset\bigcup_{\Omega_{j'}\cap\Omega_j\neq\emptyset}\mathrm{supp}\,\varrho_j$$

Because for all $j\in\{1,\ldots,J\}$, $\varrho_j(q)\chi_j(q')$ and $\tilde{\chi}_j(q)1_{Q\setminus\tilde{\Omega}_j}(q')$ vanish in a neighborhood of the diagonal $\Delta_Q=\{(q,q),\,q\in Q\}$, there exists $\Theta_1\in\mathscr{C}^\infty(Q\times Q;[0,1])$ such that $\Theta_1\equiv1$ in a neighborhood of Δ_Q and

$$\begin{aligned}\varrho_j(q)\Theta_1(q,q') &= \varrho_j(q)\Theta_1(q,q')\chi_j(q')=\varrho_j(q)\tilde{\chi}_j(q)\Theta_1(q,q')\chi_j(q)\\ &= \varrho_j(q)\left[\sum_{j'=1}^J\varrho_{j'}(q)\tilde{\chi}_j(q)\right]\Theta_1(q,q')\chi_j(q')\\ &= \varrho_j(q)\left[\sum_{j'=1}^J\varrho_{j'}(q)\tilde{\chi}_{j'}(q)\Theta_1(q,q')\right]\chi_j(q'),\end{aligned}\tag{2.E.4.1}$$

where the equalities hold as multiplication operators on $\mathscr{S}'(T^*\Omega_j\times T^*\Omega_j;\mathbb{C})$. Additionally the function Θ_1 can be chosen symmetric: $\Theta_1(q,q')=\Theta_1(q',q)$, and we set

$$\Theta_2(q,q')=1-\Theta_1(q,q').$$

For any $K\in\mathscr{L}(\mathscr{S}(T^*Q;\mathbb{C});\mathscr{S}'(T^*Q;\mathbb{C}))$, identified with its Schwartz kernel $K(x,x')\in\mathscr{S}'(T^*Q\times T^*Q;\mathbb{C})$, we set

$$K_{\mathrm{diag}}(x,x')=\Theta_1(q,q')K(x,x')\quad,\quad K_{\mathrm{off}}(x,x')=\Theta_2(q,q')K(x,x')\quad,\quad K=K_{\mathrm{diag}}+K_{\mathrm{off}}.\tag{2.E.4.2}$$

Notice that $K\mapsto(K_{\mathrm{diag}},K_{\mathrm{off}})$ is an isomorphism between $\mathscr{S}'(T^*Q\times T^*Q;\mathbb{C})$ and the closed set of $\mathscr{S}'(T^*Q\times T^*Q;\mathbb{C})\times\mathscr{S}'(T^*Q\times T^*Q;\mathbb{C})$

$$\left\{(K_1,K_2)\in\mathscr{S}'(T^*Q\times T^*Q;\mathbb{C})\times\mathscr{S}'(T^*Q\times T^*Q;\mathbb{C}),\,\Theta_1(q,q')K_2(x,x')-\Theta_2(q,q')K_1(x,x')=0\right\}.$$

With (2.E.4.1), we have the additional properties

$$K_{\mathrm{diag}}=\sum_{j=1}^{J}\varrho_j(q)\circ K_{\mathrm{diag}}=\sum_{j=1}^{J}\varrho_j(q)\circ\left[\sum_{j'=1}^{J}(\varrho_{j'}\tilde{\chi}_{j'})(q)\circ K\right]_{\mathrm{diag}}\circ\chi_j(q)$$

$$\text{and}\sum_{j=1}^{J}\left[(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)\right]_{\mathrm{diag}}=\sum_{j=1}^{J}\varrho_j(q)\circ\left[\sum_{j'=1}^{J}[(\varrho_{j'}\tilde{\chi}_{j'})(q)a](x,D_x)\right]_{\mathrm{diag}}\circ\chi_j(q)$$

for some $\tilde{\chi}_j\in\mathscr{C}_0^\infty(\Omega_j;[0,1])$ such that $\tilde{\chi}_j\equiv 1$ in a neighborhood of $\mathrm{supp}\,\chi_j$.

Proposition 2.E.18. *Every* $A=\sum_{j=1}^{J}(\varrho_j(q)a)(x,D_x)\chi_q+R\in\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ *admits a unique decomposition*

$$A=\underbrace{\sum_{j=1}^{J}(\varrho_j(q)a_A(x,D_x))\chi_j(q)}_{=A_{\mathrm{diag}}}+\underbrace{R_A}_{=A_{\mathrm{off}}},$$

with $a_A\in S^m_\Psi(Q;\mathbb{C})$ *and* $R_A\in\mathscr{R}(Q;\mathbb{C})$.
Additionally this provides a topological direct sum on $\mathrm{OpS}^m(Q;\mathbb{C})$*, because the map*

$$\begin{array}{rcl} S^m_\Psi(Q;\mathbb{C})\times\mathscr{R}(Q;\mathbb{C}) & \to & S^m_\Psi(Q;\mathbb{C})\times\mathscr{R}(Q;\mathbb{C})\\ (a,R) & \mapsto & (a_A,R_A)\quad,\quad A=\sum_{j=1}^{J}(\varrho_j(q)a)(x,D_x)\chi_j(q)+R\end{array}$$

is continuous when $S^m_\Psi(Q;\mathbb{C})\times\mathscr{R}(Q;\mathbb{C})$ *is endowed with the seminorms* $(p_{m,k}(a)+q_k(R))_{k\in\mathbb{N}}$.

Remark 2.E.19. When A is a differential operator or more generally a local operator with respect to q-variable, then we can write $A=A_{\mathrm{diag}}$ with a vanishing remainder $R_A=0$.

Proof. The decomposition of $A=\sum_{j=1}^{J}(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)+R=A_a+R$

$$\begin{aligned}
&A=(A_a+R)_{\mathrm{diag}}+(A_a+R)_{\mathrm{off}}=A_{a,\mathrm{diag}}+R_{\mathrm{diag}}+A_{a,\mathrm{off}}+R_{\mathrm{off}},\\
\text{with}\quad &R_{\mathrm{off}}(x,x')=\Theta_2(q,q')R(x,x')\in\mathscr{S}(T^*Q\times T^*Q;\mathbb{C}),\\
&A_{a,\mathrm{off}}(x,x')=\sum_{j=1}^{J}A_{j,\mathrm{off}}(x,x')\quad,\quad A_{j,\mathrm{off}}(x,x')=\Theta_2(q,q')[(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)](x,x')\\
&R_{\mathrm{diag}}=\sum_{j=1}^{J}\varrho_j(q)\circ\left[\sum_{j'=1}^{J}(\varrho_{j'}\tilde{\chi}_{j'})(q)\circ R\right]_{\mathrm{diag}}\circ\chi_j(q),\\
\text{and}\quad &A_{a,\mathrm{diag}}=\sum_{j=1}^{J}\varrho_j(q)\circ\left[\sum_{j'=1}^{J}[(\varrho_{j'}\tilde{\chi}_{j'})(q)a](x,D_x)\right]_{\mathrm{diag}}\circ\chi_j(q).
\end{aligned}$$

The kernel of $A_{j,\mathrm{off}}$ with coordinates in Ω_j is

$$\int_{\mathbb{R}^{2d}}e^{i[\xi.(q-q')+\eta.(p-p')}\Theta_2(q,q')\varrho_j(q)a(q,p,\xi,\eta)\chi_j(q')\,\frac{d\xi d\eta}{(2\pi)^d}.$$

A non stationary phase argument with $\frac{(q-q')}{|q-q'|^2}D_\xi e^{i[\xi.(q-q')+\eta.(p-p')}=e^{i[\xi.(q-q')+\eta.(p-p')]}$ with the factors $\varphi_j(q)\theta_j(q')$ implies that the map $a\mapsto A_{j,\mathrm{off}}(x,x')$ is continuous from $S(\Psi^m,g_\Psi)$ to $\mathscr{S}(\mathbb{R}^{4d};\mathbb{C})$. Again with the controlled support, the map $a\mapsto A_{j,\mathrm{off}}$ is continuous from $S^m_\Psi(Q;\mathbb{C})$ to $\mathscr{R}(Q;\mathbb{C})$. This proves that the map

$$(a,A)\mapsto A_{\mathrm{off}}=R_{\mathrm{off}}+\sum_{j=1}^{J}A_{j,\mathrm{off}}$$

is continuous from $S^m_\Psi(Q;\mathbb{C})\times\mathscr{R}(Q;\mathbb{C})$ to $\mathscr{R}(Q;\mathbb{C})$.
For the diagonal part, let us first notice at the operator level that the sum with respect to j' is introduced for

$$\varrho_{j'}(q)\left[(\varrho_j\tilde{\chi}_j)(q)M\right]_{\mathrm{diag}}=\varrho_{j'}(q)\varrho_j(q)\tilde{\chi}_j(q)\tilde{\chi}_{j'}(q)[M]_{\mathrm{diag}}\chi_j(q)\chi_{j'}(q)$$

with $M=R$ or $M=(\tilde{\chi}_j\tilde{\chi}_{j'}(q)a)(x,D_x)$.
It remains to identify these operators as the quantization of symbols. Every term indexed by (j,j') for a fixed $j'\in\{1,\dots,J\}$, has a kernel with a $Q\times Q$ support that is compact in $\tilde{\Omega}_{j'}\times\tilde{\Omega}_{j'}$. We choose coordinates in $\tilde{\Omega}_{j'}$ in order to compare the terms of the double sum for different values of $j\in\{1,\dots,J\}$. For $[(\varrho_{j'}\tilde{\chi}_{j'})(q)R]_{\mathrm{diag}}$ the kernel

$$(\varrho_{j'}\tilde{\chi}_{j'})(q)R(x,x')\Theta_1(q,q')=\varrho_{j'}(q)R(x,x')\Theta_1(q,q')\chi_{j'}(q')$$

belongs to $\mathscr{S}(\Omega_{j'}\times\Omega_j\times\mathbb{R}^{2d};\mathbb{C})$. With the factors $\varrho_{j'}(q)\chi_{j'}(q')$, it can be written $a_{j',\mathrm{reg}}(x,D_x)$ with $a_{j',\mathrm{reg}}\in\mathscr{S}(\mathbb{R}^{4d};\mathbb{C})\subset S(\Psi^{-\infty},g_\Psi)$.
For $A_{a,\mathrm{diag}}$, the kernel is localized in the same way and equals

$$\int_{\mathbb{R}^{2d}}e^{i[\xi.(q-q')+\eta.(p-p')]}\varrho_{j'}(q)a(q,p,\xi,\eta)\Theta_1(q,q')\chi_{j'}(q')\frac{d\xi d\eta}{(2\pi)^{2d}},$$

We obtain

$$\left[[\varrho_{j'}\tilde{\chi}_{j'}(q)a](x,D_x)\right]_{\mathrm{diag}}=b_{j'}(x,D_x)$$

where $b_{j'}$ is given by the Gauss transform

$$b_{j'}(x,\xi)=e^{iD_\Xi.D_{x'}}\varrho_{j'}(q)a(q,p,\xi,\eta)\Theta_1(q,q')\chi_{j'}(q')\big|_{X=X'}=e^{iD_\xi.D_{q'}}\varrho_{j'}(q)a(q,p,\xi,\eta)\Theta_1(q,q')\chi_{j'}(q')\big|_{q'=q},$$

where the variables (p,η) are now simple parameters. We deduce that the map $a\mapsto b_{j'}$ is continuous from $S(\Psi^m,g_\Psi)$ to $S(\Psi^m,g_\Psi)$. With $b_{j'}(x,D_x)=\tilde{\chi}_{j'}(q)b_{j'}(x,D_x)$ we deduce that $b_{j'}\in S^m_{\Psi,\tilde{\Omega}_j-\mathrm{comp}}(\tilde{\Omega}_j;\mathbb{C})\subset S^m_\Psi(Q;\mathbb{C})$.
So we have written

$$\begin{aligned}
&R_{\mathrm{diag}}=\sum_{j=1}^J(\varrho_j(q)b_{j,\mathrm{reg}})(x,D_x)\circ\chi_j(q)\quad,\quad A_{\mathrm{diag}}=\sum_{j=1}^J(\varrho_j(q)b_j)(x,D_x)\circ\chi_j(q)\\
\text{with}\quad&b_{j,\mathrm{reg}}\in S^{-\infty}_\Psi(Q;\mathbb{C})\quad,\quad b_j\in S^m_\Psi(Q;\mathbb{C})\\
\text{and}\quad&\forall j,j'\in\{1,\dots,J\},\quad\varrho_{j'}(q)(b_j+b_{j,\mathrm{reg}})=\varrho_j(q)(b_{j'}+b_{j',\mathrm{reg}}),
\end{aligned}$$

The last identity follows the same strategy as at the operator level except that we consider only left multiplications by functions of q, which commute with the Gauss transform. □

Definition 2.E.20. According to Proposition 2.E.18 the topology on $\mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ is equivalently defined by the family of (semi)norms $(q_{m,k})_{k\in\mathbb{N}}$ and $(\tilde{q}_{m,k})_{k\in\mathbb{N}}$ with

$$q_{m,k}(A)=p_{m,k}(a_A)+q_k(R_A)\quad\text{with }A=\underbrace{\sum_{j=1}^J\varrho_j(q)a_A(x,D_x)\circ\chi_j(q)}_{=A_{\mathrm{diag}}}+\underbrace{R_A}_{=A_{\mathrm{off}}}$$

and

$$\tilde{q}_{m,k}(A)=\inf\left\{p_{m,k}(a)+q_k(R),A=\sum_{j=1}^J(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)+R\,,a\in S^m_\Psi(Q;\mathbb{C},R\in\mathscr{R}(Q;\mathbb{C}))\right\}.$$

Definition 2.E.21. We now write $a_{\varrho,\chi}(x,D_x)=\sum_{j=1}^{J}(\varrho_j(q)a)(x,D_x)\circ\chi_j(q)$ for $a\in S^m_\Psi(Q;\mathbb{C})$. A symbol $a\in S^m_\Psi(Q;\mathbb{C})$ is said to be elliptic if there exists $\kappa\geq 1$ such that $|a|\geq \frac{1}{\kappa}\Psi^m$ for $\Psi\geq\kappa$. An operator $A=a_{\varrho,\chi}(x,D_x)+R\in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ is said to be elliptic if it admits a symbol a which is elliptic.

The product of two operators $A=a_{\varrho,\chi}(x,D_x)+R, A'=a'_{\varrho,\chi}(x,D_x)+R'\in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ equals

$$A\circ A'=a_{\varrho,\chi}(x,D_x)\circ a'_{\varrho,\chi}(x,D_x)+\underbrace{a_{\varrho,\chi}(x,D_x)\circ R'+R\circ a_{\varrho,\chi}(x,D_x)+R\circ R'}_{\in\mathscr{R}(Q;\mathbb{C})}$$

and the treatment of every non vanishing term $\varrho_j(q)a(x,D_x)\chi_j(q)\varrho_{j'}(q)a(x,D_x)\chi_{j'}(q)$ of the product $a_{\varrho,\chi}(x,D_x)\circ a'_{\varrho,\chi}(x,D_x)$ can be studied in the chart open set $\tilde{\Omega}_j$ with the expansion of the Gauss transform $e^{iD_{\Xi_1}.D_{x_2}}a_1(X_1)a_2(X_2)\big|_{X_1=X_2=X}$ in $S(\Psi^{mm'},g_\psi)$. We deduce that

$$A\circ A'=\sum_{n=1}^{N}b_n(a,a')_{\varrho,\chi}(x,D_x)+R_{N+1}(A,A')$$

with $p_{m+m'-n,k}(b_n(a,a'))\leq C_{m,m',n,k}p_{m,\ell_{n,k}}(a)p_{m',\ell_{n,k}}(A')$ and the remainder estimated by

$$q_{m+m'-N-1,k}(R_{N+1}(A,A'))\leq C'_{m,m',N+1,k}q_{m,\ell_{N+1,k}}(A)q_{m,\ell_{N+1,k}}(A'),$$

when $p_{m,\ell}(a)\leq C_{m,\ell}q_{m,\ell}(A)$ and $p_{m,\ell}(a')\leq C_{m,\ell}q_{m,\ell}(A')$. According to Remark 2.E.19, differential operators provide a wide family of examples where the latter condition holds true. And this can be extended for operators of which the Schwartz kernel is explicitly localized in a small neighborhood of the Q-diagonal.
The rough version of this continuity property says

$$A\circ A'=\sum_{n=1}^{N}B_n(A,A')+R_{N+1}(A,A')$$

with $q_{m+m'-n,k}(B_n(A,A'))\leq C_{m,m',n,k}q_{m,\ell_{n,k}}(a)q_{m',\ell_{n,k}}(A')$, and

$$q_{m+m'-N-1,k}(R_{N+1}(A,A'))\leq C'_{m,m',N+1,k}q_{m,\ell_{N+1,k}}(A)q_{m,\ell_{N+1,k}}(A').$$

Similarly for a vector bundle isomorphism $\Phi:T^*Q\to T^*Q$, the conjugation by the associated unitary transform $A=a_{\varrho,\chi}(x,D_x)+R\mapsto U_\Phi AU_\Phi^{-1}$ can be written

$$U_\Phi AU_\Phi^{-1}=\sum_{n=1}^{N}[b_{n,\Phi}(a)]_{\varrho,\chi}(x,D_x)+R_{\Phi,N+1}(A)$$

with continuity estimates gathered from the local model treated in Proposition 2.E.13. It can be written more roughly as

$$\begin{aligned}&U_\Phi AU_\Phi^{-1}=\sum_{n=1}^{N}B_{n,\Phi}(A)+R_{\Phi,N+1}(A)\\ \text{with}\quad & q_{m-n,k}(B_{n,\Phi}(A))\leq C_{\Phi,m,k}q_{m,\ell_{n,k}}(A)\\ \text{and}\quad & q_{m-N-1,k}(R_{\Phi,N+1}(A))\leq C'_{\Phi,m,N+1}q_{m,\ell_{N+1},k}(A).\end{aligned}$$

Because the function $(q,q')\mapsto\tilde{\chi}_j(q)\tilde{\chi}_j(q')\left[\sum_{\Omega_{j'}\cap\Omega_j\neq\emptyset}\varrho_{j'}(q)\Theta_1(q,q')\chi_{j'}(q')-\chi_{j'}(q)\Theta_1(q,q')\varrho_{j'}(q')\right]$, which is symmetric if $\Theta_1(q,q')=\Theta_1(q',q)$, has a compact support away from the diagonal Δ, the decomposition of the formal adjoint can be reduced to the local model with the formula $b(x,D_x)^*=$

$[e^{iD_\Xi.D_x}\overline{b}](x,D_x)$. The formal adjoint A^* of the operator $A = a_{\varrho,\chi}(x,D_x)+R \in \mathrm{OpS}^m_\Psi(Q;\mathbb{C})$ can be written

$$A^* = \sum_{n=0}^{N}[b_n]_{\varrho,\chi}(x,D_x)+R_{N+1}(A) = \sum_{n=0}^{N} B_N(A)+R_{N+1}(A)$$

with $b_0 = \overline{a}$ and the estimates $p_{m-n,k}(b_n) \le C_{m,n,k}p_{m,\ell_{n,k}}(a)$ and $q_{m-n,k}(B_n) \le C_{m,n,k}q_{m,\ell_{n,k}}(A)$ and $q_{m-N-1,k}(R_{N+1}) \le C_{m,N,k}q_{m,\ell_{N+1},k}(A)$.
All the classical estimates can be decomposed in this way by going back to the $Q = \mathbb{R}^d$-model. The condition $p_{m,\ell}(a) \le C_{m,\ell}q_{m,\ell}(A)$ holds true if one starts with an operator $A = a_{\varrho,\chi}(x,D_x)+0$, exactly given by the local quantization rule with no regularizing global remainder, and $a \notin S^{-\infty}_\Psi(Q;\mathbb{C})$. For example this is the case if A is a differential operator. The estimates are then propagated via the operations.
The local Calderon-Vaillancourt theorem and our choice of the norm q_k on $\mathscr{R}(Q;\mathbb{C})$ gives at once the existence of a $k_d \in \mathbb{N}$ determined by the dimension of Q, such that $\|A\|_{\mathscr{L}(L^2)} \le Cq_{0,k_d}(A)$.
Similarly the Garding inequality says that if $A \in S^m_\Psi(Q;\mathbb{C})$ has an elliptic non negative symbol $a \ge \frac{1}{\kappa}\Psi^m$ for $\Psi \ge \kappa$, there exists $C_\kappa \ge 1$ and $k_1 \in \mathbb{N}$ such that

$$\forall u \in \mathscr{S}(T^*Q;\mathbb{C}), \quad \mathrm{Re}\langle u\,, Au\rangle_{L^2} \ge \frac{1}{C_\kappa}\|u\|^2_{\tilde{\mathscr{W}}^{m/2}} - C_\kappa q_{m,k_1}(A)\|u\|^2_{\tilde{\mathscr{W}}^{(m-1)/2}}.$$

All these properties extend to $\mathrm{OpS}^m_\Psi(Q;\mathrm{End}\mathscr{E})$ with the following constraint for the symbol of the adjoint: The reduction to $a \in S^m_{\Psi,\Omega-\mathrm{comp}}(\Omega;\mathscr{M}_d(\mathbb{C}))$ is done by chosing $\tilde{\Omega}_j$ such that the vector bundle $E\big|_{\tilde{\Omega}_j}$ admits an orthonormal frame $(f^1_j,\ldots,f^N_j)$ for the metric g_E. Then the adjoint of $A = a_{m,\varrho,\chi}(x,D_x)+A_{m-1} \in \mathrm{OpS}^m_\Psi(Q;\mathrm{End}\mathscr{E})$ can be written

$$A^* = (a^*_m)_{\varrho,\chi}(x,D_x)+A'_{m-1} \quad \text{with} \quad A'_{m-1} \in \mathrm{OpS}^{m-1}_\Psi(Q;\mathrm{End}\mathscr{E})$$

and this property is invariant by a change of orthonormal frame.
Actually if $U(q) \in \mathscr{U}_N(\mathbb{C})$ is the unitary matrix which represent another orthonormal frame $(\tilde{f}^N_j,\ldots,\tilde{f}^N_j)$ in the frame $(f^N_j,\ldots,f^N_j)$ for $E\big|_{\tilde{\Omega}_j}$, the symbol of $a_{m,\varrho,\chi}(x,D_x)+A_{m-1}$ equals

$$b_m(x,\Xi) = U(q)a_m(x,\Xi)U^{-1}(q) = U(q)a_m(x,\Xi)U^*(q) \quad \text{with} \quad b^*_m(x,\Xi) = U(q)a^*_m(x,\Xi)U^*(q).$$

The norms $(q_{m,k})_{k\in\mathbb{N}}$ are very convenient for handling the ellipticity as it is done in the case of the global pseudo-differential calculus on $\mathbb{R}^d$. We focus here on the case of non negative elliptic operators.

Proposition 2.E.22.
Let $A \in \mathrm{OpS}^m_\Psi(Q;\mathrm{End}\mathscr{E})$, $m > 0$, be an elliptic operator with $A = (a_m \otimes \mathrm{Id}_\mathscr{E})_{\varrho,\chi}(x,D_x)+A_{m-1}$, $a_m \ge \frac{1}{\kappa}\Psi^m$ for $\Psi \ge \kappa$, $A_{m-1} \in \mathrm{OpS}^{m-1}_\Psi(Q;\mathrm{End}\mathscr{E})$. If additionally A is symmetric on $\mathscr{S}(T^*Q;\mathscr{E})$ then it is self-adjoint with $D(A) = \tilde{\mathscr{W}}^m(T^*Q;\mathscr{E})$, bounded from below, and its resolvent is compact.
In the case when $m = 2$, if $A = a_{\varrho,\chi}(x,D_x)+R \in \mathrm{OpS}^2_\Psi(Q;\mathrm{End}\mathscr{E})$ fulfills the above conditions, then for every $f \in S(\langle t\rangle^s, \frac{dt^2}{\langle t\rangle^2})$, $s \in \mathbb{R}$, the operator $f(A) - f(a_2)_{\varrho,\chi}(x,D_x)$ belongs to $\mathrm{OpS}^{2s-1}_\Psi(Q;\mathrm{End}\mathscr{E})$ while $f(a_2) \in S^{2s}_\Psi(Q;\mathrm{End}\mathscr{E})$.

Proof. The first results are the standard ones.
We just show how Helffer-Sjöstrand formula can be used in this framework and we focus on the case $m = 2$. We write a in the form $a = a_2 \otimes \mathrm{Id}_\mathscr{E} + a_1$ with $a_1 \in S^1_\Psi(Q;\mathrm{End}\mathscr{E})$ and $a_2 \otimes \mathrm{Id}_\mathscr{E}$ is simply written a_2.
By the Leibniz formula applied to $1 = (z-a_2)^{-1} \times (z-a_2)$ for $z \in \mathbb{C}\setminus\mathbb{R}$, the seminorms $p_{-2,k}(\frac{1}{z-a_2})$ are estimated by

$$p_{-2,k}\left(\frac{1}{z-a_2}\right) \le C_k p_{2,k}(a)\frac{\langle z\rangle^k}{|\mathrm{Im}\, z|^{k+1}}.$$

By taking $B_z = \left[\frac{1}{z-a_2} + \frac{a_1}{(z-a_2)^2}\right]_{\varrho,\chi}(x,D_x) + 0$, the second term of the expansion of $B_z \circ (z-A)$, to be studied in $S^1_\Psi(Q;\mathrm{End}\mathscr{E})$, can be reduced to

$$-\frac{1}{(z-a_2)^2} a_1 \circ a_1 + \sum_{|\alpha_1|+|\alpha_2|=1} \frac{1}{\alpha_1!\alpha_2!} \partial_\xi^{\alpha_1}\partial_\eta^{\alpha_2} \frac{1}{(z-a_2)} . \partial_q^{\alpha_1}\partial_p^{\alpha_2}[a_2 + a_1]$$

and it is of order -2 with seminorms estimated like the seminorms $p_{-2,k}(\frac{1}{z-a_2})$. We deduce

$$B_z \circ (z-A) = zB_z - B_z \circ A = (1 - r_L(z))$$

with $q_{-2,k}(r_L(z)) \leq C_k \frac{\langle z\rangle^{N_k}}{|\mathrm{Im}\, z|^{N_k+1}}$. By left multiplying with $\sum_{n=0}^M r_L(z)^n$ we obtain

$$\sum_{n=0}^M r_L(z)^n \circ B_z \circ (z-A) = 1 - r_L(z)^{M+1},$$

with $q_{(-M-1)2,k}[r_L(z)^{M+1}] \leq C_{M,k} \frac{\langle z\rangle^{N_{M,k}}}{|\mathrm{Im}\, z|^{N_{M,k}+1}}$ and $\|r_L(z)^{M+1}\|_{\mathscr{L}(L^2;\widetilde{\mathscr{W}}^{(Mm-c_d)})} \leq C'_M \frac{\langle z\rangle^{N_M}}{|\mathrm{Im}\, z|^{N_M+1}}$. In the right-hand side of

$$(z-A)^{-1} = \sum_{n=0}^M r_L(z)^n \circ B(z) + (z-A)^{-1} \circ r_L(z)^{M+1},$$

all terms except the remainder term $(z-A)^{-1} \circ r_L(s)^{M+1}$ are known to be pseudo-differential operators $r_L(z)^n \circ B(z) \in \mathrm{OpS}_\Psi^{-2(n+1)}(Q;\mathrm{End}\mathscr{E})$ and $q_{-2(n+1),k}(r_L(z)^n \circ B(z)) \leq C_k \frac{\langle z\rangle^{N_{n,k}}}{|\mathrm{Im}\, z|^{N_{n,k}+1}}$. But we can do the same for the right-multiplication and obtain similarly:

$$(z-A)^{-1} = \sum_{n=0}^M B_z \circ r_D(z)^n + r_D(z)^{M+1}(z-A)^{-1},$$

with the same upper bounds.
We deduce

$$(z-A)^{-1} = \sum_{j=1}^J \left[\varrho_j(q)(\frac{1}{z-a_2} + \frac{a_1}{(z-a_2)^2})\right](x,D_x)\chi_j(q) + \sum_{n=1}^{2M} b_n(z) + \underbrace{r_D(z)^{M+1}(z-A)^{-1}r_L(z)^{M+1}}_{=r_M(z)} \quad (2.E.4.3)$$

$$\text{with} \quad q_{-2(n+1),k}(b_n) \leq C_{n,k}\frac{\langle z\rangle^{N_{n,k}}}{|\mathrm{Im}\, z|^{N_{n,k}+1}} \quad , \quad \|r_M\|_{\mathscr{L}(\widetilde{\mathscr{W}}^{-M+c_d},\widetilde{\mathscr{W}}^{M-c_d})} \leq C_M \frac{\langle z\rangle^{N_M}}{|\mathrm{Im}\, z|^{N_M+2}}.$$

Inserting (2.E.4.3) into the Helffer-Sjöstrand formula (see [HeSj][DiSj]) gives

$$\begin{aligned}
& f(A) = \frac{1}{2i\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde f(z)(z-A)^{-1}\, dz\wedge d\bar z \\
\text{while}\quad & f(a_2) = \frac{1}{2i\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde f(z)(z-a_2)^{-1}\, dz\wedge d\bar z \quad , \quad f'(a_2) = \frac{1}{2i\pi}\int_{\mathbb{C}} \partial_{\bar z}\tilde f(z)(z-a_2)^{-2}\, dz\wedge d\bar z, \\
\text{with}\quad & \tilde f \in \mathscr{C}^\infty(\mathbb{C};\mathbb{C}) \quad , \quad \mathrm{supp}\,\tilde f \subset \{z\in\mathbb{C},\, |\mathrm{Im}\, z| \leq C_f\langle z\rangle\} \quad , \quad \tilde f\big|_{\mathbb{R}} = f, \\
\text{and}\quad & \forall N\in\mathbb{N}, \exists C_N > 0, |\partial_{\bar z}\tilde f(z)| \leq C_N \frac{|\mathrm{Im}\, z|^N}{\langle z\rangle^N}\langle z\rangle^{s-1}
\end{aligned}$$

when $f \in S(\langle t\rangle^s, \frac{dt^2}{\langle t\rangle})$, $s<0$, we obtain by integration of the respective terms by choosing $N \geq \max\{N_{n,k}, N_M\}$

$$f(A) = [f(a_2)]_{\varrho,\chi}(x,D_x) + [f'(a_2)a_1]_{\varrho,\chi}(x,D_x) + \sum_{n=1}^{2M}[\beta_n]_{\varrho,\chi}(x,D_x) + R_M$$

with $\beta_n \in S_\Psi^{-2(n+1)}(Q;\mathrm{End}\mathscr{E})$ for $n \geq 1$ while $f(a) \in S_\Psi^{-2s}(Q;\mathrm{End}\mathscr{E})$.

Let us first conclude for $s \in [-3/2, 0[$. Take $\beta \in S^{-4}(Q;\mathrm{End}\mathscr{E})$ such that $\beta \sim \sum_{n=1}^{\infty} \beta_n$. For every $M \in \mathbb{N}$,

$$f(A) = [f(a_2)]_{\varrho,\chi}(x,D_x) + [f'(a_2)a_1]_{\varrho,\chi}(x,D_x) + \beta_{\varrho,\chi}(x,D_x) + R_{\beta,2M+1} + R_M$$

with $R_{\beta,2M+1} \in \mathrm{OpS}_\Psi^{-2(2M+1)}(Q;\mathrm{End}\mathscr{E})$ and $R_M \in \mathscr{L}(\widetilde{\mathscr{W}}^{-M+c_d};\widetilde{\mathscr{W}}^{M-c_d})$. By taking M arbitrarily large, this says that $f(A) - [f(a_2)]_{\varrho,\chi}(x,D_x) - [f'(a_2)a_1]_{\varrho,\chi}(x,D_x) - \beta_{\varrho,\chi}(x,D_x)$ belongs to $\mathscr{R}(Q;\mathrm{End}\mathscr{E})$. Because $s \in [-3/2, -0[$, we know $f(a_2) \in S_\Psi^{2s}(Q;\mathrm{End}\mathscr{E})$ with $2s \geq -3$ and $f'(a_2) \in S^{2(s-1)}(Q;\mathrm{End}\mathscr{E})$, $f'(a_2)a_1 \in S_\Psi^{2s-1}(Q;\mathrm{End}\mathscr{E})$, while $\beta_{\varrho,\chi} \in S_\Psi^{-4}(Q;\mathrm{End}\mathscr{E})$, $-4 \leq -3-1 \leq 2s-1$.

Now for a general $s < 0$, simply write $\langle t\rangle^s = \langle t\rangle^{s_1 n_1}$ with $s_1 \in [-3/2, 0[$ and $n_1 \in \mathbb{N}$. The composition of pseudo-differential operators says that the principal symbol of $\langle A\rangle^s = \langle A\rangle^{s_1} \circ \ldots \circ \langle A\rangle^{s_1}$ is $\langle a_2\rangle^s$. Any power $\langle t\rangle^s$, $s \in \mathbb{R}$, can be written $\langle t\rangle^{2N}\langle t\rangle^{s'}$ with $s' < 0$ and $N \in \mathbb{N}$. With $\langle t\rangle^{2N} = (1+t^2)^N \in \mathbb{R}[t]$, $\langle A\rangle^s$ is a pseudo-differential operator with principal symbol $\langle a_2\rangle^s$ for any $s \in \mathbb{R}$. Finally a general function $f \in S(\langle t\rangle^s; \frac{dt^2}{\langle t\rangle^2})$ is written $\langle t\rangle^{s+3/2} f_s(t)$ with $f_s \in S(\langle t\rangle^{-3/2}, \frac{dt^2}{\langle t\rangle^2})$.

For the End$\mathscr{E}$ version it suffices to notice that all the explicit computations above, are done essentially with scalar symbols. $\square$

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Chapter 3

A Grushin problem for Bismut's hypoelliptic Laplacian

Joint work with Francis Nier [1] and Francis Gilbert White [2].
Article en anglais.

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Abstract

The name “Grushin problem” refers here to the variation of Schur complement technique introduced by J. Sjöstrand in his early works, which is now a commonly used tool in spectral analysis. Recently Q. Ren and Z. Tao proposed such an approach for the analysis of the low lying eigenvalues in the large friction limit for a simple scalar kinetic model. Inspired by this recent work and with the introduction of functional spaces adapted to the analysis of geometric Kramers-Fokker-Planck operators in a previous article, the authors now study the combined asymptotic analysis of Bismut’s hypoelliptic Laplacian, in the high friction $b \to 0^+$ and possibly low temperature $h \to 0^+$ regimes.

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Keywords: Bismut’s hypoelliptic Laplacian, Grushin problem and spectral convergence, multi-scale analysis, large friction, low temperature.

Contents

3.1 Introduction

3.1.1 Problem and motivations

In [Bis041][Bis042], J.M. Bismut introduced the hypoelliptic Laplacian which allows to extend to p-forms the generator of the semigroup associated with the Langevin process. The Witten Laplacian, self-adjoint and elliptic, corresponds to the simpler description of the Brownian motion, proposed by Einstein and where the temperature denoted by $h > 0$ is essentially the only parameter. The full Langevin process written here in the euclidean case:

$$dq = p dt \quad , \quad dp = -\frac{1}{h}\partial_q V dt - \frac{1}{b} p dt + \frac{1}{\sqrt{b}} dW$$

involves actually the two independent parameters $h > 0$ and $b > 0$, where $h > 0$ can be interpreted (after rescaling) as a temperature and $\frac{1}{b}$ as a friction parameter. The low temperature limit $h \to 0^+$ is known as the semiclassical limit for the semiclassical Witten Laplacian $\Delta_{V,h}$ and many works have been devoted to its analysis after the seminal articles [Wit][HeSj4][CFKS] or to its consequences for the theory of topological invariant of manifolds (see e.g. [Zha][BiZh]). It also has obvious relationships with all the asymptotic results of Freidlin-Wentzell theory (see e.g. [FrWe]) and its development in the study of simulated annealing in the late 70’s (see e.g. [HKS] or [Mic]). We refer the reader to [Ber] for additional references and a historical background and to [LeSt] for the presentation of more recent applications and issues for the design of effective algorithms in molecular dynamics.
It was rapidly shown in [BiLe] that the large friction limit $b \to 0^+$ (and $h > 0$ fixed) of Bismut’s hypoelliptic Laplacian , is related to the Witten or Hodge Laplacian on the base manifold. It is summarized by the commonly used terminology of “overdamped Langevin process” for Einstein’s description of the brownian motion.
Motivated by the applications to molecular dynamics or kinetic theory many works have been

devoted in the last twenty years to the accurate computations of small eigenvalues of such operators, with the aim of providing quantitative information about the trend to the equilibrium. Even in the elliptic, self-adjoint and purely semiclassical framework of the semiclassical Witten Laplacian new questions arouse concerned with the accurate computation of spectral element in various geometrical or topological landscape and possibly with boundary value problems. In particular, in [LNV2] it was proved that when the potential $V \in \mathscr{C}^\infty(Q;\mathbb{R})$ has finitely many critical values and Q is a closed manifold, dim $Q = d$, the various exponentially small scales of low-lying spectrum of the semiclassical Witten Laplacian's $\Delta_{V,h}$ in the limit $h \to 0^+$ are determined by a topological object: the persistent homology bar code of the function V .
One question addressed in this text is whether a similar result holds for Bismut's hypoelliptic Laplacian in the limit $b \to 0^+$ and $h \to 0^+$.
Before giving our main result, let us recall, what is known about similar problems:

— The accurate description of exponentially small eigenvalues for semi-classical Witten Laplacians, in connection with Eyring-Kramers asymptotics, the generalized Arrhenius law, or the study of quasistationnary distributions, has been studied or used in [BEGK] [BGK] [HKN] [HeNi2] [Lep1] [LeNi] [DLLN] [LeNe] [LNV1] [LNV2] and references therein.
— For the Langevin process, the semiclassical regime, which after a rescaling corresponds to $b \propto \sqrt{h}$ and $h \to 0^+$, for functions (0-forms) in the euclidean space with a Morse potential V was considered in [HHS]. An accurate study of the tunnel effect, with microlocal analytic techniques, led to a full asymptotic description of the bottom spectrum under the above assumptions.
— A similar asymptotic framework was considered in the Ph.D thesis of S. Shen (in [She]) for Bismut's hypoelliptic Laplacian on the cotangent of a closed riemannnian manifold with $b \propto \sqrt{h}$, $h \to 0^+$, the potential V is a Morse function and the metric is euclidean in Morse coordinates around critical points.
— In [BLM] the analysis of Hérau-Hitrik-Sjöstrand was extended to a more general class of still scalar (0-forms) semiclassical non self-adjoint and subelliptic operators. Such methods have been developped in [Nor1][Nor2] for other relevant kinetic scalar models where the diffusive part is no longer given by a harmonic oscillator hamiltonian but by a possibly non local operator in the momentum variable.
— In [BFLS] the authors considered the scalar operator for the Langevin dynamics in the euclidean space but with rather general kinetic energy and potential function. They discuss according to the friction and temperature parameter, the size of the spectral gap (or resolvent estimate). Their variational (so called "hypocoercive") method is combined with a Schur complement method which is reminiscent of the formal calculations of [BiLe]-Chap 17.
— In [ReTa] Ren and Tao developed a Grushin problem approach for a simple kinetic model in a high friction limit $\gamma = \frac{1}{b} \to +\infty$. Their operator is $\mathscr{Y} - \gamma\Delta_V^{\mathbb{S}}$ on the cosphere bundle $S^*Q = \left\{(q,p) \in T^*Q, g^{ij}(q)p_i p_j = 1\right\}$, where $\mathscr{Y}$ is the hamiltonian vector field of the geodesic flow and $\Delta_V^{\mathbb{S}}$ is the vertical Laplace-Beltrami operator on the spherical fiber.

The results of S. Shen in [She] are up to now the only accurate asymptotic results on p-forms for Bismut's hypoelliptic Laplacian in the combined limit $b \to 0^+$ and $h \to 0^+$, and it is done under some restricted assumptions. We note that the works of [BiLe] and [She] are also concerned with the convergence of generalized determinants in connection with Ray-Singer metrics on determinant bundles and other topological invariants, by developing the strategy of [BiZh][Zha].
Additionally it is known from the various studies of the elliptic case, i.e. the semiclassical Witten Laplacian, that 1-forms can be extremely useful even if one is only interested in the scalar case (degree 0). This is due to the supersymmetric argument: if ω is an eigenvector in degree p then for a Hodge type operator, $(\mathbf{d} + \mathbf{d}^*)^2 = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$, $\mathbf{d}\omega \neq 0$ (resp. $\mathbf{d}^*\omega \neq 0$) , $\mathbf{d}\omega$ (resp. $\mathbf{d}^*\omega$) is an

eigenvector of degree $p+1$ (resp. $p-1$). For these reasons, it is very natural to explore the accurate description of the small eigenvalues in the combined asymptotic regimes $b\to 0^+$ and $h\to 0^+$. Although our previous work, was initially intended to the study of Bismut's hypoelliptic Laplacian with boundary conditions, it rapidly appeared after we heard of Ren and Tao article [ReTa], that our functional framework should allow a rather straightforward transposition of their method. Briefly said, it suffices to replace the total Laplacian $\Delta_{q,p}$ on the total space S^*Q by the operator W_θ^2 introduced in [NSW] for the definition of global Sobolev spaces adapted to the analysis of Bismut's hypoelliptic Laplacian. This combined with various explicit geometric formulas in [Bis05][BiLe] finally convinced us that an accurate description in the double asymptotics $b\to 0^+$ and $h\to 0^+$ (it works for $h=1$) and for a general potential $V\in\mathscr{C}^\infty(Q;\mathbb{R})$ should be accessible.

3.1.2 Main result and comments

We are concerned with the spectral and semigroup properties of Bismut's hypoelliptic Laplacian, denoted here by $B_{\pm,b,\frac{1}{h}V}$ on $X=T^*Q$, where $b,h>0$ are parameters. Actually, the operator $B_{\pm,b,V}$ is equal to $2(\mathfrak{A}'_{\phi_b,\pm\mathscr{H}})^2$ with the presentation of [BiLe]-Section 2 and the additional parameter $h>0$ is introduced by replacing V by $\frac{1}{h}V$. We refer to Subsection 3.2.5 below for a detailed presentation. The semiclassical Witten Laplacian on the closed base manifold Q is given by $\Delta_{V,h}=(d_{V,h}+d_{V,h}^*)^2$ with $d_{V,h}=e^{-\frac{V}{h}}(hd)e^{\frac{V}{h}}=hd+dV\wedge$ and we refer the reader to Subsection 3.2.3 and Subsection 3.2.6 for various unitarily equivalent presentations adapted to our problem. We use the $h-$dependent version of the double exponent Sobolev spaces $\tilde{\mathscr{W}}_h^{s_1,s_2}$ introduced in Definition 3.2.2 for $h=1$ and in Definition 3.2.6 for $h\in]0,1]$.

The data of our problem are the spectrum of the semiclassical Witten Laplacian $\mathrm{Spec}(\Delta_{V,h})=\mathrm{Spec}(\Delta_{V^h,1})$, where $V^h(q)=\frac{1}{h}V(hq)$ is defined on a dilated manifold (see Subsection 3.2.6), and the parameters $b,h\in]0,1]$.

The following definition makes sense if one considers asymptotic regimes where $h\to 0^+$ and when $V\in\mathscr{C}^\infty(Q;\mathbb{R})$ has a finite number of critical values.

Definition 3.1.1. The parameter $\varrho_h\in]0,1]$ parametrized by $h\in]0,1]$ measures a spectral gap for $\Delta_{V,h}$ according to

$$\begin{aligned}&\mathrm{Spec}(\frac{1}{2}\Delta_{V,h})\cap[0,\varrho_h]\subset[0,e^{-\frac{\varepsilon}{h}}]\subset[0,\frac{\varrho_h}{2}]\\ \text{and}\qquad&\mathrm{Spec}(\frac{1}{2}\Delta_{V,h})\cap]\varrho_h,+\infty[\subset[4\varrho_h,+\infty[\end{aligned}$$

for all $h\in]0,1]$. We call $\mathscr{N}_\pm(V)$ the rank of $1_{[0,\varrho_h]}(\frac{1}{2}\Delta_{V,h})$ and $\mathscr{N}_\pm^{(p)}(V)$ the rank of $1_{[0,\varrho_h]}(\frac{1}{2}\Delta_{V,h}^{(p)})$ for $p\in\{0,\dots,d\}$, where the $\pm$ sign refers to the choice of the line bundle $F_+=Q\times\mathbb{C}$ or $F_-=(Q\times\mathbb{C})\otimes\mathbf{or}_Q$.
For every $p\in\{0,\dots,d\}$ the eigenvalues of $\frac{1}{2}\Delta_{V,h}^{(p)}$ in $[0,\varrho_h]$, repeated with multiplicity, are labelled by $\tilde{\lambda}_{\pm,j,h}^{(p)}(V)$, $1\le j\le\mathscr{N}_\pm(V)$, in the increasing order.

It was proved in [HeSj4] (resp. in [LNV2]) that one can take $\varrho_h=ch$ with $c>0$ (resp. $\varrho_h=e^{-\frac{\varepsilon}{h}}$ with $\varepsilon>0$ arbitrarily small) when $V\in\mathscr{C}^\infty(Q;\mathbb{R})$ is a Morse function (resp. has a finite number of critical values). Additionally the number of eigenvalues of $\frac{1}{2}\Delta_{V,h}^{(p)}$ counted with multiplicities, in $[0,\varrho_h]$, is fixed for $h\in]0,h_0]$, $h_0>0$ small enough, and determined by the topological properties of the sublevel sets of V, via Morse theory, or more generally via the barcode of persistent homology. We note also that the Poincaré duality implies $\mathscr{N}_+^{(p)}(V)=\mathscr{N}_-^{(d-p)}(-V)$ for every $p\in\{0,\dots,d\}$.

Definition 3.1.2. For every $p \in \{0,\dots,2d\}$, the eigenvalues of $B^{(p)}_{\pm,b,\frac{V}{h}}$ lying in $D(0,\frac{\varrho_h}{h^2})$, repeated according to their algebraic multiplicity, will be denoted by $(\lambda^{(p)}_{\pm,j,h})_{1\leq j\leq \mathcal{N}^{(p)}_{\pm}}$. The characteristic space

$$E^{(p)}_{\pm,b,h} = \mathrm{Ran}\Big(\frac{1}{2i\pi}\int_{|z|=\frac{\varrho_h}{h^2}}(z-B^{(p)}_{\pm,b,\frac{V}{h}})^{-1}\,dz\Big)$$

has the dimension $\mathcal{N}^{(p)}_{\pm} = \dim(E^{(p)}_{\pm,b,h})$. When $B^{(p)}_{\pm,b,\frac{V}{h}}\big|_{E^{(p)}_{\pm,b,h}}$ is diagonalizable (see Theorem 3.1.3-**a)**), a basis of eigenvectors is written $(u^{(p)}_{\pm,j,h})_{1\leq j\leq \mathcal{N}^{(p)}_{\pm}}$ and its L^2 dual basis is denoted by $(v^{(p)}_{\pm,j,h})_{1\leq j\leq \mathcal{N}^{(p)}_{\pm}}$.

Theorem 3.1.3. *Let g be a metric on Q and let $V\in\mathscr{C}^\infty(Q;\mathbb{R})$ be a potential function with finitly many critical values. In the following statements $C_s\geq 1$ denotes a large enough constant determined by $s\in\mathbb{R}$.*

a) *When $bC_0\leq h\varrho_h\leq h$ all the eigenvalues of $B_{\pm,b,\frac{V}{h}}$ with real part below $\frac{\varrho_h}{h^2}$ are real and non negative:*

$$\mathrm{Spec}(B_{\pm,b,\frac{V}{h}})\cap\{z\in\mathbb{C},\mathrm{Re}z\leq\frac{\varrho_h}{h^2}\} = \mathrm{Spec}(B_{\pm,b,\frac{V}{h}})\cap[0,\frac{\varrho_h}{h^2}].$$

In degree $p\in\{0,\dots,2d\}$, their number, counted with multiplicity, is given by $\mathcal{N}^{(p)}_{\pm}=\mathcal{N}^{(p-\frac{d}{2}\pm\frac{d}{2})}_{\pm}(V)$, which is 0 if $p>d$ (resp. $p<d$) in the $+$ case (resp. $-$ case). Poincaré duality implies $\lambda^{(p)}_{+,j,h}=\lambda^{(2d-p)}_{-,j,h}$. Additionally the restricted operator $B^{(p)}_{\pm,b,\frac{V}{h}}\big|_{E^{(p)}_{\pm,b,h}}$ is diagonalizable.

b) *Under the stronger assumption $bA^4C_0\leq h\varrho_h\leq h$ with $A\geq C_0$, the comparison between the Witten Laplacian and Bismut's hypoelliptic Laplacian of the low lying spectrum, is given by*

$$\forall p\in\{0,\dots,2d\},\forall j\in\{1,\dots,\mathcal{N}^{(p)}_{\pm}\},\quad (1+C_0A^{-1/2})^{-1}\frac{\tilde{\lambda}^{(p-\frac{d}{2}\pm\frac{d}{2})}_{\pm,j,h}(V)}{h^2}\leq\lambda^{(p)}_{j,\pm,h}\leq(1+C_0A^{-1/2})\frac{\tilde{\lambda}^{(p-\frac{d}{2}\pm\frac{d}{2})}_{\pm,j,h}(V)}{h^2}.$$

c) *When $bC_s\leq h\varrho_h$, the semigroup $(e^{-tB_{\pm,b,\frac{V}{h}}})_{t>0}$ satisfies:*

$$e^{-tB_{\pm,b,\frac{V}{h}}} = \sum_{p\in\{0,\dots,2d\}}\sum_j e^{-t\lambda^{(p)}_{\pm,j,h}}|u^{(p)}_{\pm,j,h}\rangle\langle v^{(p)}_{\pm,j,h}| + R_h(t)$$

$$\text{with}\qquad \|R_h(t)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s}_h;\tilde{\mathcal{W}}^{0,s}_h)}\leq\frac{1}{b^2}(h^2+\frac{1}{t})e^{-t\frac{\varrho_h}{h^2}}$$

$$\text{and}\qquad \max(\|u^{(p)}_{\pm,j,h}\|_{\tilde{\mathcal{W}}^{0,s}_h},\|v^{(p)}_{\pm,j,h}\|_{\tilde{\mathcal{W}}^{0,s}_h})\leq C_s,$$

for suitably normalized basis of eigenvectors $(u^{(p)}_{\pm,j,h})_{1\leq j\leq\mathcal{N}^{(p)}_{\pm}}$.

The second statement **b)** says in particular that in the limit $h\to0^+$ the eigenvalues of Bismut's hypoelliptic Laplacian have the same exponentially small asymptotic behaviour as the eigenvalues of the Witten Laplacian. The latter were shown in [LNV2] to be related to the bar codes of persistent homology.

Corollary 3.1.4. *When $V\in\mathscr{C}^\infty(Q;\mathbb{R})$ has finitely many critical values and under the condition $C_0^5b\leq h\varrho_h$, the eigenvalues $(\lambda^{(p)}_{+,j,h})_{1\leq j\leq\mathcal{N}^{(p)}_{+}}$ satisfy $\lim_{h\to0^+}-h\log(\lambda^{(p)}_{+,j,h})=2\ell^{(p)}_j$, where $\ell^{(p)}_j$ is the length of a bar, indexed by j, with a degree p endpoint in the bar code associated with V. The $-$ case is obtained by Poincaré duality with $\lambda^{(p)}_{-,j,h}=\lambda^{(2d-p)}_{+,j,h}$.*

Comments:

— Actually all these results are related with resolvent estimates of which the accurate formulation is given in Proposition 3.4.4, after introducing other intermediate quantities.

— In order to write a general statement, we preferred to express things in terms of the non explicit spectral gap ϱ_h of Definition 3.1.1. For a general function $V \in \mathscr{C}^\infty(Q;\mathbb{R})$ with finitely many critical values the result of [LNV2] says that one can take $\varrho_h = e^{-\frac{\varepsilon}{h}}$ with $\varepsilon > 0$ arbitrarily small. But when one knows better the geometry of the critical sets an algebraic expression $\varrho_h = h^\nu$ can be obtained. The basic example is when $V(x) = x^n$ in $\mathbb{R}$, in which case a simple rescaling argument gives $\varrho_h = h^{2\frac{n-1}{n}}$.

— When the potential V is a Morse function, the condition $C_0 b \le \varrho_h h$ says $b \le ch^2$, which is stronger than the condition $b \le c\sqrt{h}$ suggested by the works of S. Shen [She] and Hérau-Hitrik-Sjöstrand [HHS], where they considered $b \propto \sqrt{h}$. Actually our method relies on the elimination of the potential term, as a perturbative term, after the rescaling $\phi_h : Q \mapsto Q^h = \frac{1}{h}Q$ of Subsection 3.2.6. A similar analysis of what is proposed here, could be developed with better treatment of the Morse potential function. Instead of the above dilation take $\phi_{\sqrt{h}} : Q \mapsto Q^{\sqrt{h}} = \frac{1}{\sqrt{h}}Q$ and use on $Q^{\sqrt{h}}$ a partition of unity in riemannian balls of radius $M\sqrt{h}$ with $M \ge 1$ large enough. With a more inclusive description of the scalar quadratic model in every ball, which takes better into account the quadratic Taylor approximation of the potential $V^{\sqrt{h}}(q) = \frac{1}{\sqrt{h}}V(\sqrt{h}q)$, subelliptic estimates of [NSW] can be improved in particular by using the accurate quantitative estimates of [BNV] for quadratic Kramers-Fokker-Planck operators in the euclidean space. In the end the rescaling leads to the comparison of the spectral gap for $\Delta_{V^{\sqrt{h}},1}$ on $Q^{\sqrt{h}}$, which by unitary equivalence is equal to $\frac{1}{h}\varrho_h \propto 1$ and the rescaled parameter $\frac{b}{\sqrt{h}}$ instead of $\frac{b}{h}$. One then recovers the natural condition $b \le c\sqrt{h}$. This is just a sketch and an accurate spectral analysis remains to be done. In this article, we preferably considered a $\mathscr{C}^\infty$-function without assuming that it is a Morse function, in order to highlight the generality of the Grushin problem approach.

— In [LNV2] results were given for non smooth potentials, in particular when V is a Lipschitz subanalytic function. This more general case is not considered here and it would require a specific analysis, which could follow partly the strategy presented here.

— Theorem 3.1.3 is a digest of what can be deduced from the Grushin problem method. Many intermediate resolvent estimates can be used and maybe improved for other purposes.

— Finally, we have not considered as in [BiLe] and [She] the convergence of generalized determinants. Actually, for topological invariants which do not depend on the riemannian metric, the simplifying assumptions of [She] suffice for a general treatment. It is not clear that a more accurate and general analysis would bring relevant improvements.

3.1.3 Outline of the article

The geometric framework, the operators, the various scalings and the functional spaces are defined in Section 3.2. Remember that Bismut's hypoelliptic Laplacian is a second order non self-adjoint and non elliptic operator acting on differential forms defined on the total space X of the cotangent bundle T^*Q of the closed riemannian manifold (Q,g). The definition of the hypoelliptic Laplacian in[Bis05][BiLe], the associated Weitzenböck formula, and the introduction of adapted functional spaces, strongly relies on the horizontal and vertical decomposition $T(T^*Q) = TX = T^H X \oplus T^V X$ recalled in subsection 3.2.2. The exact definition of the Witten Laplacian and the hypoelliptic Laplacian are given in Subsections 3.2.3 and 3.2.5. An h-dependent change of scale is introduced in Subsection 3.2.6. This allows to get easily uniform constants with respect to $h \in]0,1]$ in all the subelliptic estimates which are used in the text. The first of these subelliptic estimates in Subsection 3.2.7 is an adaptation of the general results of [NSW] to the present framework. Although Theorem 3.1.3 is expressed for the operator $B_{\pm,b,\frac{1}{h}V} = B_{\pm,(Q,g,\frac{V}{h},b)}$ on $X = T^*Q$, all the analysis of this text is carried out on the dilated geometry of Subsection 3.2.6

with the operator $B_{\pm,b',V^h} = B_{\pm,(Q^h,g^h,V^h,b')}$, $b' = \frac{b}{h}$ and where the $'$ is dropped afterward) where the h-dependence is easier to track.

In Section 3.3, various perturbations or modifications of the operator $B_{\pm,b,V^h}$ are considered, which have no spectrum around 0. For such new operators, resolvent and possibly subelliptic estimates are specified. One of them, denoted by $B_{\pm,b,V^h} + Q_{A,L,V^h}$, is directly inspired from the work [ReTa] of Q. Ren and Z. Tao. In a crucial way, this section aims at providing subelliptic estimates for $B_{\pm,b,V^h} + Q_{A,L,V^h}$ with a uniform lower bound with respect to $b,h \in]0,1]$ for a large enough new additional parameter $A \geq 1$. Due to the new complexity of our problem but also in order to improve Ren-Tao lower bounds, this is done in two steps with the intermediate operator $B_{\pm,b,V^h} + A^2\pi_{0,\pm}$ easier to handle, especially if one uses the maximal subelliptic estimates.

The writing of a Grushin problem in Section 3.4 allows an accurate comparison of the resolvents $(B_{\pm,b,V^h} + Q_{A,L,V^h} - z)^{-1}$, $(B_{\pm,b,V^h} - z)^{-1}$, $(\Delta_{V^h,1} - z)^{-1}$ and $(\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}$. Although the resolvent estimates of Section 3.3 can be written with uniform constants which are independent of the Sobolev exponent $s \in \mathbb{R}$, the range of validity for the parameters $b,h,A > 0$ actually depends on this Sobolev exponent s. Attention must be paid to the formal calculations which are not done in the general distributional setting but rather in an arbitrarily fixed range of Sobolev exponents $s \in [s_{\min}, s_{\max}]$.

The proof of Theorem 3.1.3 is achieved in Section 3.5. The resolvent comparison in Section 3.4 and the spectral information of the semiclassical Witten Laplacian, summarized in Definition 3.1.1, provide the first accurate localization of the spectrum of $B_{\pm,b,V^h}$ around 0, with new accurate estimates for the resolvent and the semigroup. We finally use the Hodge structure and the PT-symmetry property, $r^* B_{\pm,b,V^h} r^* = B^*_{\pm,b,V^h}$ with r^* a unitary involution, in order to make an accurate comparison between the eigenvalues of $B_{\pm,b,V^h}$, identified now as the squared singular values of a restricted differential, and the eigenvalues of the Witten Laplacian $\Delta_{V^h,1}$, with $\mathrm{Spec}(\Delta_{V^h,1}) = \mathrm{Spec}(\Delta_{V,h})$.

3.2 Framework

3.2.1 Total space X of the cotangent bundle

Let (Q, g^{TQ}) be a closed Riemannian manifold of dimension d, ∇^{LC} the associated Levi-Civita connection and let $X = T^*Q$ be the total space of the cotangent bundle. On one side, the total space X is a symplectic manifold with the canonical symplectic form σ. One the other side, the kinetic energy function is globally defined by

$$\forall x = (q,p) \in T^*_q Q, \quad \mathcal{H}(x) = \frac{|p|_q^2}{2} = \frac{1}{2} g^{T^*Q}(p,p). \tag{3.2.1.1}$$

Then the hamiltonian vector field $\mathcal{Y}$ of the geodesic flow is given by

$$d^X \mathcal{H} + \mathbf{i}_{\mathcal{Y}} \sigma = 0. \tag{3.2.1.2}$$

The scalar vertical harmonic oscillator $\mathcal{O}$ is the self-adjoint differential operator defined with its maximal domain in $L^2(X, dqdp; \mathbb{C})$ by

$$\frac{1}{2}(g^{TQ}(D_p, D_p) + g^{T^*Q}(p,p)) \geq \frac{d}{2}\mathrm{Id} \quad \text{with } D_p = \frac{1}{i}\partial_p \tag{3.2.1.3}$$

where $-\Delta_{\mathrm{Vert}} = g^{TQ}(D_p, D_p)$ is the fiberwise vertical Laplacian.

3.2.2 The horizontal-vertical decomposition

The Levi-Civita connection induces a splitting of the tangent space and cotangent space of X given by

$$TX = (TX)^H \oplus (TX)^V \simeq \pi^*(TQ \oplus T^*Q) \quad ; \quad T^*X = (T^*X)^H \oplus (T^*X)^V \simeq \pi^*(T^*Q \oplus TQ) \quad (3.2.2.1)$$

where $\pi : X = T^*Q \to Q$ is the natural projection, $(TX)^H \simeq \pi^*(TQ)$ is the horizontal distribution and $(TX)^V = \ker(d\pi) \simeq \pi^*(T^*Q)$ is the vertical distribution. Once a frame $u_1, u_2, \ldots, u_d$ and the associated coframe $u^1, \ldots, u^d$ are locally chosen, we take a copy of those two frames $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_d$ and $\hat{u}^1, \ldots, \hat{u}^d$ where the above identification is written

$$(TX)^H \simeq \mathrm{Span}(u_1, \ldots, u_d) \quad ; \quad (TX)^V \simeq \mathrm{Span}(\hat{u}^1, \ldots, \hat{u}^d)$$

and

$$(T^*X)^H \simeq \mathrm{Span}(u^1, \ldots, u^d) \quad ; \quad (T^*X)^V \simeq \mathrm{Span}(\hat{u}_1, \ldots, \hat{u}_d).$$

In the rest of the text we will use the above identification with the following additional conventions

— When u_i's are associated to a coordinate system on Q i.e. $u_i = \frac{\partial}{\partial q^i}$ for all $i \in \{1, \ldots, d\}$ then we use the notation

$$\pi^*(u_i) = e_i = \frac{\partial}{\partial q^i} + \Gamma_{ij}^k p_k \frac{\partial}{\partial p_j} \in (TX)^H \quad ; \quad \pi^*(\hat{u}^i) = \hat{e}^i = \frac{\partial}{\partial p_i} \in (TX)^V$$

and

$$\pi^*(u^i) = e^i = dq^i \in (T^*X)^H \quad ; \quad \pi^*(\hat{u}_i) = \hat{e}_i = dp_i - \Gamma_{ij}^k p_k dq^j \in (T^*X)^V.$$

Where Γ_{ij}^k denote the Christoffel symbol for the Levi-Civita connection, defined by $\nabla^{LC}_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \Gamma_{ij}^k(q) \frac{\partial}{\partial q^k}$.

— When u_i's is an (local) orthonormal frame of TQ we will use the notation

$$\pi^*(u_i) = f_i \in (TX)^H \quad : \quad \pi^*(\hat{u}^i) = \hat{f}^i \in (TX)^V \quad (3.2.2.2)$$

$$\text{and} \quad \pi^*(u^i) = f^i \in (T^*X)^H \quad ; \quad \pi^*(\hat{u}_i) = \hat{f}_i \in (T^*X)^V. \quad (3.2.2.3)$$

Passing from one writing to another is simply given by a section P of the fiber bundle $GL(TQ)$ above Q. Indeed for all $i \in \{1, \ldots, d\}$

$$u_i = P(q)_i^j \frac{\partial}{\partial q^j}.$$

The following relations hold on TX and T^*X

$$f_i = P(q)_i^j e_j \qquad \hat{f}^i = (P(q)^{-1})_j^i \hat{e}^j,$$

$$\text{and} \quad f^i = (P(q)^{-1})_j^i e^j \qquad \hat{f}_i = P(q)_i^j \hat{e}_j.$$

With this decomposition

— The metric g^{TX} on TX is defined as $g^{TX} = g^{TQ} \oplus^\perp g^{T^*Q}$ with respect to the decomposition (3.2.2.1). The frame $f_1, \ldots, f_d, \hat{f}^1, \ldots, \hat{f}^d$ is an orthonormal frame with respect to g^{TX}. Similarly we define the metric $g^{T^*X} = g^{T^*Q} \oplus^\perp g^{TQ}$ on the cotangent space T^*X of X. For the exterior algebra we use $\Lambda T^*X \simeq (\Lambda T^*Q) \otimes (\Lambda TQ)$ as a vector space and $g^{\Lambda T^*X} = g^{\Lambda T^*X} \otimes g^{\Lambda TX}$. With the orthonormal frames $f^1, \ldots, f^d, \hat{f}_1, \ldots, \hat{f}^d$, an orthonormal frame of ΛT^*X is given by $(f^I \wedge \hat{f}_J)_{I, J \subset \{1, \ldots, d\}}$.

— The hamiltonian vector field $\mathscr{Y}$ defined by (3.2.1.2) can be written

$$\mathscr{Y} = g^{T^*Q}(e^i, e^j) p_j e_i = \sum_{i=1}^{d} \tilde{p}_i f_i$$

where $p = p_i dq^i = \tilde{p}_i f^i \in T^*_q Q$.

— The vertical Laplacian equals

$$\Delta^V = g^{TQ}(e_i, e_j) \hat{e}^i \hat{e}^j = \sum_{i=1}^{d} (\hat{f}^i)^2.$$

— The tautological connection on TX and T^*X, extended to ΛT^*X or $\Lambda T^*X \otimes \pi^* \mathbf{or}(Q)$, is defined by the following formula

$$\begin{array}{rcl} \nabla^{TX}_{e_i} e_j &=& \Gamma^k_{ij} e_k \quad ; \quad \nabla^{TX}_{e_i} \hat{e}^j = -\Gamma^j_{ik} \hat{e}^k, \\ \nabla^{TX}_{\hat{e}^i} e_j &=& 0 \quad ; \quad \nabla^{TX}_{\hat{e}^i} \hat{e}^j = 0 \\ \text{and} \quad \nabla^{T^*X}_{e_i} e^j &=& -\Gamma^j_{ik} e^k \quad ; \quad \nabla^{T^*X}_{e^i} \hat{e}_j = \Gamma^k_{ij} \hat{e}_k, \\ \nabla^{T^*X}_{\hat{e}^i} e^j &=& 0 \quad ; \quad \nabla^{T^*X}_{\hat{e}^i} \hat{e}_j = 0. \end{array}$$

3.2.3 Hermitian trivial bundle F over Q

Although Bismut's theory of the hypoelliptic Laplacian in [Bis041][Bis042][Bis05] works in a much more general framework, we focus here on the simpler case which makes the connection with the standard semiclassical Witten Laplacian on the base manifold Q. Namely we work with the trivial bundle $F = Q \times \mathbb{C}$ on the base manifold Q, equiped with the hermitian metric $g^F = \exp(-2V(q)) d\bar{z} \otimes dz$ and the trivial connection $\nabla^F = d^Q$. When smooth duality arguments are used on a non-oriented manifold Q, the trivial bundle $Q \times \mathbb{C}$ must be replaced by $(Q \times \mathbb{C}) \otimes \mathbf{or}_Q$ where $\mathbf{or}_Q$ is the orientation bundle on Q. Locally nothing is changed.
By following Bismut's notations, set

$$\omega(\nabla^F, g^F) = (g^F)^{-1} \nabla^F g^F = -2dV,$$

which is here a real scalar 1−form on Q. The adjoint connection ∇^{F*} of ∇^F with respect to g^F equals

$$\nabla^{F*} = d^Q - 2dV$$

and the associated unitary connection $\nabla^{F,u}$ is

$$\nabla^{F,u} = d^Q - dV.$$

Contrary to the general case studied in [Bis05][BiLe], here the unitary connection $\nabla^{F,u}$ is flat since its curvature R^F is given by $R^F = -\frac{1}{4}\omega(\nabla^F, g^F) \wedge \omega(\nabla^F, g^F) = -dV \wedge dV = 0$.
In Bismut work and more generally for a probabilistic approach, the natural L^2-space is $L^2(Q, d\mathrm{Vol}_g; \Lambda T^*Q \otimes F)$ where the notation recalls the non trivial metric $g^F = e^{-2V(q)}$ on $F \simeq Q \otimes C$, in the L^2-scalar product

$$\langle u, v \rangle_{L^2(Q, d\mathrm{Vol}_g; \Lambda T^*Q \otimes F)} = \int_Q g^{\Lambda T^*Q}(\overline{u}, v)\, e^{-2V(q)}\, d\mathrm{Vol}_g(q).$$

For the accurate spectral analysis it is simpler to work in the standard L^2-space, $L^2(Q, d\mathrm{Vol}_g; \Lambda T^*Q \otimes \mathbb{C})$, with the scalar product

$$\langle u, v \rangle_{L^2(Q, d\mathrm{Vol}_g; \Lambda T^*Q \otimes \mathbb{C})} = \int_Q g^{\Lambda T^*Q}(\overline{u}, v)\, d\mathrm{Vol}_g(q).$$

Functional space	$L^2(Q,d\mathrm{Vol}_g;\Lambda^{\cdot}T^*Q\otimes F)$	$L^2(Q,d\mathrm{Vol}_g;\Lambda^{\cdot}T^*Q\otimes\mathbb{C})$
Sections	$v=e^V u$	$u=e^{-V}v$
metric	$g^F=exp(-2V)$	1
Connection	$\nabla^F=d^Q$	d^Q+dV
Endomorphism ω	$\omega(\nabla^F,g^F)=-2dV$	$\omega(\nabla^F,g^F)=-2dV$
Adjoint connection	$\nabla^{F*}=d^Q-2dV$	d^Q-dV
unitary connection	$\nabla^{F,u}$	d^Q
differential	d^Q	$d^Q+dV\wedge=:d_{V,1}$
codifferential	$d^{Q,F,*}=e^{2V}d^{Q,*}e^{-2V}=d^{Q,*}+2\mathbf{i}_{\nabla V}$	$d^{Q,*}+\mathbf{i}_{\nabla V}=:d^*_{V,1}$
Hodge/Witten Laplacian	$\square^{Q,F}=(d^Q+d^{Q,F,*})^2$	$\Delta_{V,1}=(d_{V,1}+d^*_{V,1})^2$

Table 3.2.1 – Correspondance of L^2 spaces

Passing from one formulation to the other via the unitary multiplication by $e^{\pm\frac{V(q)}{h}}$ is summarized by the following table.

We recall the formulas

$$\begin{aligned}
\square^{Q,F} &= (d^Q d^{Q,*}+d^{Q,*}d^Q)+2\mathscr{L}_{\nabla V}\,,\\
d_{V,1} &= e^{-V}(d)e^V=d+dV\wedge \quad , \quad d^*_{V,1}=e^V(d^*)e^{-V}=(d_{V,1})^*=d^*+\mathbf{i}_{\nabla V}\,,\\
\Delta_{V,1} &= (d_{V,1}+d^*_{V,1})^2=(d_{V,1}d^*_{V,1}+d^*_{V,1}d_{V,1})=(d^Qd^{Q,*}+d^{Q,*}d^Q)+|\nabla V|^2+(\mathscr{L}_{\nabla V}+\mathscr{L}^*_{\nabla V})\,.
\end{aligned}$$

The subscript 1 in $d_{V,1}$, $d^*_{V,1}$ and $\Delta_{V,1}$ refers to the specific case $h=1$ for the semiclassical Witten differential, codifferential and Laplacian:

$$\begin{aligned}
&d_{V,h}=e^{-\frac{V}{h}}(hd)e^{\frac{V}{h}} \quad , \quad d^*_{V,h}=e^{\frac{V}{h}}(hd)^*e^{-\frac{V}{h}}\\
&\Delta_{V,h}=(d_{V,h}+d^*_{V,h})^2=h^2(dd^*+d^*d)+|\nabla V|^2+h(\mathscr{L}_{\nabla V}+\mathscr{L}^*_{\nabla V})\,.
\end{aligned}$$

Within the presentation of Table 3.2.1 the semiclassical regime can be introduced by simply replacing V by $\frac{V}{h}$ and by choosing the metric $g^F=e^{-\frac{2V}{h}}$. This actually leads to

$$\square^F=(d^Qd^{Q,*}+d^{Q,*}d^Q)+\frac{2}{h}\mathscr{L}_{\nabla V}$$

in $L^2(Q,d\mathrm{Vol}_g;\Lambda T^*Q\otimes F)$, transformed in the $L^2(Q;d\mathrm{Vol}_g;\Lambda T^*Q\otimes\mathbb{C})$ picture into

$$(d^Qd^{*,Q}+d^{*,Q}d^Q)+\frac{1}{h^2}|\nabla V|^2+\frac{1}{h}(\mathscr{L}_{\nabla V}+\mathscr{L}^*_{\nabla V})=\frac{1}{h^2}\Delta_{V,h}\,.$$

We will explain in the specific Subsection 3.2.6 how the semiclassical asymptotic regime, or more generally $h\in]0,1]$, can be easily introduced in the analysis of geometric Kramers-Fokker-Planck operators of [NSW], where the parameter $h\in]0,1]$ was actually not considered.

3.2.4 Functional spaces on X

The isomorphism of vector bundles $\mathscr{E}$ and $\pi^*(\underbrace{\Lambda T^*Q\otimes\Lambda TQ\otimes F}_{=E})$ is provided by the horizontal-vertical decomposition (3.2.2.1) of $\Lambda T^*X\otimes\pi^*F$. With this identification, the vector bundle $\mathscr{E}$ is endowed with the metric $\pi^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ}\otimes g^F)$ where we recall $F=Q\times\mathbb{C}$ (or possibly $F=(Q\times\mathbb{C})\otimes$

$\mathbf{or}_Q$) and $g^F = e^{-\frac{2V(q)}{h}} d\bar{z}\otimes dz$. The pulled back vector bundle will be denoted by $\mathcal{E} = \Lambda T^*X \otimes \pi^* F$ and depending on the case $\mathcal{E}_+ = \Lambda T^*X \otimes \mathbb{C}$ and $\mathcal{E}_- = \Lambda T^*X \otimes \mathbb{C} \otimes \pi^*(\mathbf{or}_Q)$. With the symplectic volume denoted by $dqdp = d\mathrm{Vol}_{g\oplus^{\perp} g^{-1}}$, the associated L^2 space, denoted by $L^2(X, dqdp;\mathcal{E})$ and equal to $L^2(X, e^{-\frac{2V(q)}{h}} dqdp;\mathcal{E}_\pm)$, is given by the hermitian scalar product

$$\langle u, v\rangle_{L^2(X,dqdp;\mathcal{E})} = \int_X g^{\Lambda T^*X}(\overline{u}, v) e^{-\frac{2V(q)}{h}}\, dqdp\ .$$

After setting $\tilde{u} = e^{-\frac{V(q)}{h}} u$ and $\tilde{v} = e^{-\frac{V(q)}{h}} v$, it can be replaced by the standard $L^2(X, dqdp;\mathcal{E}_\pm)$ with the scalar product

$$\langle \tilde{u}, \tilde{v}\rangle = \int_X g^{\Lambda T^*X}(\overline{\tilde{u}}, \tilde{v})\, dqdp = \langle u, v\rangle_{L^2(X,dqdp;\mathcal{E}_\pm)}\ .$$

Those L^2 spaces and the Schwartz space of rapidly decaying (as $p\to\infty$) smooth sections, $\mathcal{S}(X;\mathcal{E})$ and $\mathcal{S}(X;\mathcal{E}_\pm)$, coincide with the obvious density result. The first L^2-norm depends on $h>0$ while the second L^2-norm does not change with $h>0$ and is more convenient here.

We work in $L^2(X, dqdp;\mathcal{E}_\pm)$.

When necessary, formulas of [Bis05][BiLe] written in $L^2(X, dqdp;\mathcal{E})$ with the corresponding scalar product and duality, will be translated later by extending the general rules of Table 3.2.1

For the analysis it is more convenient to work with a local presentation on the base manifold Q of the functional spaces and associated differential operators.

Definition 3.2.1. Let $\sum_{j=1}^J \theta_j^2(q) \equiv 1$ be a quadratic partition of unity on Q, such that above a neighborhood $\mathcal{V}_{\theta,j}$ of every $\mathrm{supp}\,\theta_j$, there are smooth dual orthonormal frames $(u_j^1,\ldots,u_j^d)$ of $T^*Q\big|_{\mathcal{V}_{\theta,j}}$ and $(u_{j,1},\ldots,u_{j,d})$ of $TQ\big|_{\mathcal{V}_{\theta,j}}$. Set $f^i = \pi^*(u^i)\in (T^*X)^H$ and $\hat{f}_i = \pi^*(u_i)\in (T^*X)^V$ according to (3.2.2.3).
For $F = Q\times\mathbb{C}$ or $F = (Q\times \mathbb{C})\otimes \mathbf{or}_Q$, let $\mathcal{I}_{\theta,Q}$ and $\mathcal{I}_{\theta,X}$ denote the product of isometries:

$$\mathcal{I}_{\theta,Q} : L^2(Q, d\mathrm{Vol}_g;\Lambda T^*Q\otimes F) \to \overset{\perp}{\underset{1\leq j\leq J}{\oplus}} L^2(\mathcal{V}_{\theta,j}, d\mathrm{Vol}_g; (\Lambda T^*Q\otimes F)\big|_{\mathcal{V}_{\theta,j}}) \to \overset{\perp}{\underset{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}{\oplus}} L^2(\mathcal{V}_{\theta,j}, d\mathrm{Vol}_g;\mathbb{C}) \tag{3.2.4.1}$$

$$s \mapsto (\theta_j s)_{1\leq j\leq J} \mapsto \mathcal{I}_{\theta,Q} s = (s_{j,I})_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}} \tag{3.2.4.2}$$

with

$$\theta_j s = \sum_{I\subset\{1,\ldots,d\}} s_{j,I}(q) u_j^I \quad,\quad u^{j,I} = u_j^{i_1}\wedge\ldots\wedge u_j^{i_{|I|}}\,,$$

and

$$\mathcal{I}_{\theta,X} : L^2(X, dqdp;\mathcal{E}_\pm) \to \overset{\perp}{\underset{1\leq j\leq J}{\oplus}} L^2(\pi^*(\mathcal{V}_{\theta,j}), dqdp;\mathcal{E}_\pm\big|_{\pi^*(\mathcal{V}_{\theta,j})}) \to \overset{\perp}{\underset{\substack{1\leq j\leq J\\ I,K\subset\{1,\ldots,d\}}}{\oplus}} L^2(\pi^*(\mathcal{V}_{\theta,j}), dqdp;\mathbb{C}) \tag{3.2.4.3}$$

$$s \mapsto (\theta_j s)_{1\leq j\leq J} \mapsto (s_{j,I}^K)_{\substack{1\leq j\leq J\\ I,K\subset\{1,\ldots,d\}}}\,, \tag{3.2.4.4}$$

with

$$\theta_j s = \sum_{I,K\subset\{1,\ldots,d\}} s_{j,I}^K(q) f_j^I\wedge \hat{f}_{j,K} \quad,\quad f_j^I = f_j^{i_1}\wedge\ldots\wedge f_j^{i_{|I|}} \quad,\quad \hat{f}_{j,K} = \hat{f}_{j,k_1}\wedge\ldots\wedge\hat{f}_{j,k_{|K|}}$$

Let us gather obvious properties of the isometries $\mathscr{I}_{\theta,Q}$ and $\mathscr{I}_{\theta,X}$:

— The adjoints of $\mathscr{I}_{\theta,Q}$ and $\mathscr{I}_{\theta,X}$ are given by

$$\mathscr{I}_{\theta,Q}^{*}\left[(s_{j,I})_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\right]=\sum_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\theta_j(q)s_{j,I}u_j^I\,,$$

$$\text{and}\qquad \mathscr{I}_{\theta,X}^{*}\left[(s_{j,I}^K)_{\substack{1\leq j\leq J\\ I,K\subset\{1,\ldots,d\}}}\right]=\sum_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\theta_j(q)s_{j,I}^K f_j^I\wedge\hat{f}_{j,K}\,.$$

— The isometry $\mathscr{I}_{\theta,Q}$ is continuous from $\mathscr{C}^\infty(Q;\Lambda T^*Q\otimes F)$ to $\oplus_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots d\}}}\mathscr{C}_0^\infty(\mathscr{V}_{\theta,j};\mathbb{C})$, resp. $\mathscr{I}_{\theta,X}$ is continuous from $\mathscr{S}(X:\mathscr{E}_\pm)$ to $\oplus_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots d\}}}\mathscr{S}(\pi^*(\mathscr{V}_{\theta,j});\mathbb{C})$ while the supports satisfy $\mathrm{supp}\, s_{j,I}^K\subset\pi^*(\mathrm{supp}\,\theta_j)$, with

$$\mathscr{I}_{\theta,Q}^*\mathscr{I}_{\theta,Q}=\mathrm{Id}_{L^2}\quad,\quad \mathscr{I}_{\theta,Q}^*\mathscr{I}_{\theta,Q}\big|_{\mathscr{C}^\infty(Q;\Lambda T^*Q\otimes F)}=\mathrm{Id}_{\mathscr{C}^\infty(Q;\Lambda T^*Q\otimes F)}\,,$$

$$\text{and}\qquad \mathscr{I}_{\theta,X}^*\mathscr{I}_{\theta,X}=\mathrm{Id}_{L^2}\quad,\quad \mathscr{I}_{\theta,X}^*\mathscr{I}_{\theta,X}\big|_{\mathscr{S}(X;\mathscr{E}_\pm)}=\mathrm{Id}_{\mathscr{S}(X;\mathscr{E}_\pm)}\,.$$

— The vertical harmonic oscillator hamiltonian given by (3.2.1.3) satisfies as a self-adjoint operator

$$\mathscr{O}=\mathscr{I}_{\theta,X}^*\left[\frac{-\Delta_{\mathrm{Vert}}+|p|_q^2}{2}\otimes\mathrm{Id}_{\mathbb{C}^{J\times 2^{2d}}}\right]\mathscr{I}_{\theta,X}\tag{3.2.4.5}$$

with the functional calculus given by

$$f(\mathscr{O})=\mathscr{I}_{\theta,X}^*\left[f(\frac{-\Delta_{\mathrm{Vert}}+|p|_q^2}{2})\otimes\mathrm{Id}_{\mathbb{C}^{J\times 2^{2d}}}\right]\mathscr{I}_{\theta,X}\tag{3.2.4.6}$$

for any Borel function $f:\mathbb{R}\to\mathbb{C}$.

The vertical degree N^V written locally as $\sum_{i=1}^d\hat{f}_{j,i}\wedge\mathbf{i}_{\hat{f}_{j,i}}$ is diagonal according to

$$N^V=\mathscr{I}_{\theta,X}^*\left[\overset{\perp}{\underset{K\subset\{1,\ldots,K\}}{\oplus}}|K|\mathrm{Id}_{\mathbb{C}^{J\times 2^d}}\right]\mathscr{I}_{\theta,X}\,.\tag{3.2.4.7}$$

We recall now the general definition of the global Sobolev spaces $\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E}_\pm)$, $(s_1,s_2)\in\mathbb{R}^2$, introduced in [NSW].

With the horizontal-vertical decomposition (3.2.2.1) and the metric g^{TQ}, the horizontal scalar Laplacian (see [BeBo]) is given by

$$\Delta_H=g^{ij}(q)(e_ie_j-\Gamma_{ij}^k e_k)=(e_i)^*\circ g^{ij}(q)\circ e_j\,,$$

while the vertical scalar harmonic oscillator operator $\mathscr{O}$ has already been introduced in (3.2.1.3). The scalar operator W^2 is defined as the closure in $L^2(X,dqdp;\mathbb{C})$ of the differential operator $C_g-\Delta_H+C_g\mathscr{O}^2:\mathscr{S}(X;\mathbb{C})\to\mathscr{S}(X;\mathbb{C})$ for $C_g\geq 1$ large enough. The operator W^2 is self-adjoint and $(W^2,\mathscr{O})$ is a pair of commuting self-adjoint operators.

The non scalar version W_θ^2 is modelled on the scalar version after using the quadratic partition of unity on Q, $\sum_{j=1}^J\theta_j^2(q)\equiv 1$ and the isometry $\mathscr{I}_{\theta,X}$. It is given by

$$W_\theta^2=\mathscr{I}_{\theta,X}^*\left[W^2\otimes\mathrm{Id}_{\mathbb{C}^{J\times 2^{2d}}}\right]\mathscr{I}_{\theta,X}=\sum_{j=1}^J\theta_j(q)\circ W_{sc,j}^2\circ\theta_j(q)\,.\tag{3.2.4.8}$$

where $W^2_{sc,j}$ is defined by using the connection ∇^j which is trivial in the orthonormal frame $(f_j^1,..,f_j^d,\hat{f}_{j,1},..,\hat{f}_{j,d})$.
Again for $C_g \geq 1$ large enough, $(W^2_\theta,\mathscr{O})$ is a pair of strongly commuting self-adjoint operators in $L^2(X,dqdp;\mathscr{E}_\pm)$. We refer the reader to [NSW] for details.

Definition 3.2.2 (Sobolev Spaces)**.** For all $s_1,s_2\in\mathbb{R}$, the double exponent Sobolev space $\tilde{\mathscr{W}}^{s_1,s_2}(X,dqdp;\mathscr{E}_\pm)$ is defined by

$$\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E}_\pm)=\{u\in\mathscr{S}'(X;\mathscr{E}_\pm),\mathscr{O}^{\frac{s_1}{2}}(W^2_\theta)^{s_2/2}u\in L^2(X,dqdp;\mathscr{E}_\pm)\}.$$

The norm is defined as $\|u\|_{\tilde{\mathscr{W}}^{s_1,s_2}(X;\mathscr{E}_\pm)}=\|\mathscr{O}^{\frac{s_1}{2}}(W^2_\theta)^{\frac{s_2}{2}}u\|_{L^2}$. For simplicity, those spaces will often be denoted by $\tilde{\mathscr{W}}^{s_1,s_2}$.

The pseudodifferential calculus associated with W^2_θ was introduced in [NSW] where the order of operators is recalled here:

$$p_i,D_{p_i}\ (1/2)\quad,\quad \mathscr{O},e_i\ (1)\quad,\quad \nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}\ (3/2)\quad,\quad (W^2_\theta)^{s/2}\ (s),$$

and it says in particular

$$\begin{aligned}\tilde{\mathscr{W}}^{0,s_2+\frac{s_1}{2}}\subset\tilde{\mathscr{W}}^{s_1,s_2}\subset\tilde{\mathscr{W}}^{0,s_2},&\\ \tilde{\mathscr{W}}^{0,s_2}\subset\tilde{\mathscr{W}}^{-s_1,s_2}\subset\tilde{\mathscr{W}}^{0,s_2-\frac{s_1}{2}}&\qquad\text{for } s_1\geq 0,s_2\in\mathbb{R},\end{aligned}$$

and

$$\bigcap_{s_2\in\mathbb{R}}\tilde{\mathscr{W}}^{s_1,s_2}=\mathscr{S}(X;\mathscr{E}_\pm)\qquad\bigcup_{s_2\in\mathbb{R}}\tilde{\mathscr{W}}^{s_1,s_2}=\mathscr{S}'(X;\mathscr{E}_\pm)\quad\text{for all } s_1\in\mathbb{R}.$$

We end this section by adding some notations and by recalling some functional analysis properties.

Definition 3.2.3. For a continuous operator $A:\mathscr{S}(X;\mathscr{E}_\pm)\to\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$, which is closable in the Hilbert space $\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$, its closure will be denoted by $\overline{A}^s$ while $\overline{A}=\overline{A}^0$.
Its formal adjoint for the $\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$-scalar product will be written $A'^{,s}:\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E})\to\mathscr{S}'(X;\mathscr{E}_\pm)$, with $A'=A'^{,0}$. The same notation will be used for its restriction to $\mathscr{S}(X;\mathscr{E}_\pm)$ instead of $A'^{,s}\big|_{\mathscr{S}(X;\mathscr{E}_\pm)}$.
Its adjoint for the $\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$ will be denoted by $A^{*,s}:D(A^{*,s})\to\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$ with $u\in D(A^{*,s})$ characterized by

$$\exists C_u\geq 0,\quad \forall v\in\mathscr{S}(X;\mathscr{E}),|\langle u\,,Av\rangle_{\tilde{\mathscr{W}}^{0,s}}|\leq C_{u,s}\|v\|_{\tilde{\mathscr{W}}^{0,s}}.$$

Again the simpler notation $A^*=A^{*,0}$ is reserved for the case $s=0$.

Because $(W^2_\theta)^{s/2}:\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)\to L^2(X,dqdp;\mathscr{E}_\pm)$ is unitary, while it is a continuous automorphism of $\mathscr{S}(X;\mathscr{E}_\pm)$ (resp. $\mathscr{S}'(X;\mathscr{E}_\pm)$), the study of $A:\mathscr{S}(X;\mathscr{E}_\pm)\to\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$ is equivalent to the one of

$$A_s=(W^2_\theta)^{s/2}A(W^2_\theta)^{-s/2}:\mathscr{S}(X;\mathscr{E}_\pm)\to L^2(X,dpdp;\mathscr{E}_\pm).$$

This is in particular convenient when $A:\mathscr{S}(X;\mathscr{E})\to\mathscr{S}(X;\mathscr{E})$ is continuous. Actually $\overline{A}^s=(W^2_\theta)^{-s/2}\overline{A_s}^0(W^2_\theta)^{s/2}$ and we can simply work in $L^2(X,dqdp;\mathscr{E}_\pm)$ with the family of densely defined operators $(A_s)_{s\in\mathbb{R}}$ as we already did in the proof of Proposition 3.2.9.
We deduce in particular the formulas:

$$\begin{aligned}A'^{,s}&=\left[(W^2_\theta)^{-s/2}A_s(W^2_\theta)^{s/2}\right]'^{,s}=(W^2_\theta)^{-s}\left[(W^2_\theta)^{-s/2}A_s(W^2_\theta)^{s/2}\right]'(W^2_\theta)^s &(3.2.4.9)\\&=(W^2_\theta)^{-s/2}A'_s(W^2_\theta)^{s/2}\\(A'^{,s})_s&=A'_s &(3.2.4.10)\\\text{and}\quad(A^{*,s})_s&=A^*_s. &(3.2.4.11)\end{aligned}$$

In all of our cases the operator A and its formal adjoint A' are continuous from $\mathscr{S}(X;\mathscr{E}_\pm)$ to itself. Alternatively A is continous from $\mathscr{S}(X;\mathscr{E}_\pm)$ to itself and from $\mathscr{S}'(X;\mathscr{E}_\pm)$ to itself. We always have

$$\overline{A^{\prime,s}}^s \subset A^{*,s} \quad \text{and} \quad \overline{A'_s} \subset A^*_s$$

in the sense that $\overline{A^{\prime,s}}^s$ is the minimal extension of $A^{\prime,s}\big|_{\mathscr{S}(X;\mathscr{E}_\pm)}$ while $A^{*,s}$ is its maximal extension. Under the above assumption the case of equality is treated via the equivalence

$$\left(\overline{A^{\prime,s}}^s = A^{*,s}\right) \Leftrightarrow \left(A^{'}_s = A^*_s\right).$$

Remember that accretive operators are closable and with an additional positive constant they are one to one and have a closed range. Essential maximal accretivity, under the above assumptions, means exactly $A^{\prime,s} = A^{*,s}$ or, equivalently, $A'_s = A^*_s$.

3.2.5 Bismut's hypoelliptic Laplacian

We do present here neither the construction of the hypoelliptic Laplacian as a deformed Hodge type operator, nor the various various versions of it which are presented in [Bis05][BiLe]. We directly start with the Weitzenböck formula for the version denoted by $2\mathfrak{A}'_{\phi_b,\pm\mathscr{H}}$ in [BiLe]-p 32 formulas (2.3.12)(2.3.13). According to [BiLe]-p32 (see formula (2.3.14)) it makes sense as an operator acting on $\mathscr{S}(X;\mathscr{E})$ and the formal adjoints are computed with the scalar product of $L^2(X,dqdp;\mathscr{E}) = L^2(X,e^{-2V(q)}dqdp;\mathscr{E}_\pm)$. For this presentation the parameter $h\in]0,1]$ is not yet considered but it suffices to replace like in Subsection 3.2.3 the potential V by $\frac{V}{h}$ and various equivalent representations are explained in Subsection 3.2.6.
Formulas (2.3.12)(2.3.13) of [BiLe] say for a local orthonormal frame $f_1,\dots,f_d$ of TQ:

$$2\mathfrak{A}'^2_{\phi_b,\pm\mathscr{H}} = \frac{1}{b^2}\alpha'_\pm + \frac{1}{b}\beta'_\pm + \gamma'_\pm \tag{3.2.5.1}$$

where

$$\alpha'_\pm = \frac{1}{2}(-\Delta^V + |p|^2_g \pm (2\hat{f}_i\mathbf{i}_{\hat{f}^i} - d)), \tag{3.2.5.2}$$

$$\beta'_\pm = -(\pm\nabla^{\Lambda^\cdot T^*X\otimes\pi^*F,u}_{\mathscr{Y}} - (f_iV)\nabla^{\Lambda^\cdot T^*X}_{\hat{f}^i}), \tag{3.2.5.3}$$

$$\gamma'_\pm = -\frac{1}{4}\left\langle R^{TQ}(f_i,f_j)f_k,f_\ell\right\rangle(f^i-\hat{f}_i)(f^j-\hat{f}_j)\mathbf{i}_{f_k+\hat{f}^k}\mathbf{i}_{f_\ell+\hat{f}^\ell} \tag{3.2.5.4}$$

$$\qquad -\left(\pm\left\langle R^{TQ}(p,f_i)p,f_j\right\rangle - (f_i(f_jV) + \tilde{\Gamma}^k_{ij}f_kV)\right)(f^i-\hat{f}_i)\mathbf{i}_{f_j+\hat{f}^j}, \tag{3.2.5.5}$$

and $\tilde{\Gamma}^k_{ij}(q) = f^k(\nabla^{TQ}_{f_i}f_j)$ the Christoffel symbol expressed in this frame.
In order to have good duality arguments when the base manifold Q is not oriented, the vector bundle $\mathscr{E}$ must be $\Lambda T^*X\otimes\mathbb{C}$ in the + case and $\Lambda T^*X\otimes\mathbb{C}\otimes\pi^*(\mathbf{or}_Q)$ in the − case.
In [Bis05], Propositions 3.14 also provides the formula

$$\pi_{0,\pm}(\gamma'_\pm - \beta'_\pm\alpha'^{,-1}_\pm\beta'_\pm)\pi_{0,\pm} = \frac{\square^{Q,F}}{2}, \tag{3.2.5.6}$$

where $\pi_{0,\pm}$ is the orthogonal projection on the kernel of $\ker(\alpha'_\pm)$. In the formula (3.2.5.6), there is an identification between operators acting on $\mathrm{Ran}\,\pi_{0,\pm}$ and operators defined on the base manifold Q which is detailed below. Let us keep for the moment the notations of [Bis05][BiLe].

In our framework, i.e. when we work in $L^2(X,dqdp;\mathscr{E}_\pm)$, it suffices to conjugate all the operators according to $A\mapsto e^{-V}Ae^V$. We obtain

$$B_{\pm,b,V} = 2e^{-V}\mathfrak{A}'^2_{\phi_b,\pm\mathscr{H}}e^V = \frac{1}{b^2}\alpha_\pm + \frac{1}{b}\beta_\pm + \gamma_\pm \tag{3.2.5.7}$$

where

$$\alpha_\pm = \alpha'_\pm = \frac{1}{2}(-\Delta^V + |p|_g^2 \pm (2\hat{f}_i \mathbf{i}_{\hat{f}^i} - d)), \tag{3.2.5.8}$$

$$\beta_\pm = e^{-V(q)}\beta'_\pm e^{V(q)} = -e^{-V(q)}(\pm\nabla^{\Lambda T^*X\otimes\pi^*F,u}_{\mathcal{Y}} - (f_iV)\nabla^{\Lambda T^*X}_{\hat{f}^i})e^{V(q)}, \tag{3.2.5.9}$$

$$\gamma_\pm = \gamma'_\pm = -\frac{1}{4}\left\langle R^{TQ}(f_i,f_j)f_k,f_\ell\right\rangle(f^i - \hat{f}_i)(f^j - \hat{f}_j)\mathbf{i}_{f_k+\hat{f}^k}\mathbf{i}_{f_\ell+\hat{f}^\ell} \tag{3.2.5.10}$$

$$-\left(\pm\left\langle R^{TQ}(p,f_i)p,f_j\right\rangle - (f_i(f_jV) + \tilde{\Gamma}^k_{ij}f_kV)\right)(f^i - \hat{f}_i)\mathbf{i}_{f_j+\hat{f}^j}. \tag{3.2.5.11}$$

The only non trivial calculation is for $\beta_\pm$. According to Table 3.2.1, $e^{-V(q)}\nabla^{F,u}e^{V(q)} = d^Q$ and we obtain

$$e^{-V(q)}\nabla^{\Lambda T^*X\otimes\pi^*F,u}e^{-V(q)} = \nabla^{\mathcal{E}_\pm}$$

where ∇^{T^*X} is the tautological connection on T^*X associated with the Levi-Civita connection on TQ and $\nabla^{\mathcal{E}_\pm}$ is the exterior algebra extension.
We obtain

$$\beta_\pm = -(\pm\nabla^{\mathcal{E}_\pm}_{\mathcal{Y}} - (f_iV)\nabla^{\mathcal{E}_\pm}_{\hat{f}^i}) \tag{3.2.5.12}$$

Because $\alpha'_\pm$ commutes with $e^{\pm V(q)}$ the kernel of $\alpha_\pm$ and the orthogonal projections $\pi_{0,\pm}$ are not changed. By Table 3.2.1 we also know

$$e^{-V(q)}\square^{Q,F}e^{V(q)} = \Delta_{V,1}.$$

The formula (3.2.5.6) becomes

$$\pi_{0,\pm}(\gamma_\pm - \beta_\pm\alpha_\pm^{-1}\beta_\pm)\pi_{0,\pm} = \frac{1}{2}\Delta_{V,1},$$

and when the potential V is replaced by $\frac{V}{h}$, $h \in]0,1]$,

$$\pi_{0,\pm}(\gamma_\pm - \beta_\pm\alpha_\pm^{-1}\beta_\pm)\pi_{0,\pm} = \frac{1}{2h^2}\Delta_{V,h}, \tag{3.2.5.13}$$

where $\Delta_{V,h} = (d_{V,h} + d^*_{V,h})^2$ is the semiclassical Witten Laplacian.

Another property proved in [Bis05][BiLe] which will be useful for proving $\mathrm{Spec}(B_{\pm,b,V}) \subset [0,+\infty[$ for $b > 0$ small enough, is related with the Hodge structure of $B_{\pm,b,V} = 2\mathfrak{A}'^2_{\phi_b,\pm\mathcal{H}}$ that we briefly recall here.

Definition 3.2.4.

- The tensorial operations λ_0 and μ_0, expressed in the orthonormal frames $(f_i,\hat{f}^i,f^i,\hat{f}_i)_{1\le i\le d}$, are $\lambda_0 = f^i \wedge \mathbf{i}_{\hat{f}^i}$ (resp. $\mu_0 = \hat{f}_i \wedge \mathbf{i}_{f_i}$) which increases the horizontal (resp. vertical) degree by 1 and decreases the vertical (resp. horizontal) degree by 1. As nilpotent elements of $\mathrm{End}(\Lambda T^*X)$, their exponential $e^{\pm\lambda_0}$ (resp. $e^{\pm\mu_0}$) are polynomials.
- For $a \in \mathbb{R}$, $r_a : X \to X$ is given by $r_a(q,p) = (q,ap)$ and $r_a^* : \mathcal{S}(X;\mathcal{E}_\pm) \to \mathcal{S}(X;\mathcal{E}_\pm)$ is the natural pull-back. The simpler notations r and r^* will be used for the isometric involutions obtained for $a = -1$. The linear map $K_a : \mathcal{S}(X;\mathcal{E}_\pm) \to \mathcal{S}(X;\mathcal{E}_\pm)$ is given by $K_a(s_I^J(q,p)f^I\hat{f}_J) = a^{d/2}s_I^J(q,ap)(f^I\hat{f}_J)\big|_x$ with a trivial action in the bases $(f^I,\hat{f}_J)_{I,J\subset\{1,\dots,d\}}$, at $x = (q,p)$ and $(f^I,\hat{f}_J)_{I,J\subset\{1,\dots,d\}}$, at $x = (q,ap)$.
- The hermitian form $\langle\,,\,\rangle_r$ on $\mathcal{S}(X;\mathcal{E}_\pm)$ is given by

$$\langle u\,,v\rangle_r = \langle u\,,r^*v\rangle.$$

The operator $\mathfrak{A}'_{\pm,b}$ equals

$$B_{\pm,b,V}=2\mathfrak{A}'^2_{\phi_b,\pm\mathscr{H}}=2(\frac{\delta_{\pm,b,V}+\delta^{*,r}_{\pm,b,V}}{2})^2=\frac{1}{2}[\delta_{\pm,b,V}\delta^{*,r}_{\pm,b,V}+\delta^{*,r}_{\pm,b,V}\delta_{\pm,b,V}], \quad (3.2.5.14)$$

with $$\delta_{\pm,b,V}=K_be^{-\mu_0}e^{-V}d^X_{\pm\frac{1}{b^2}\mathscr{H}}e^Ve^{\mu_0}K_b^{-1}=e^{-\mu_0}e^{\mp\mathscr{H}-V}(K_bd^XK_b^{-1})e^{\pm\mathscr{H}+V}e^{\mu_0}, \quad (3.2.5.15)$$

and $$\delta^{*,r}_{\pm,b,V}=e^{-\lambda_0}e^{\pm\mathscr{H}+V}K_bd^{X,*}K_b^{-1}e^{\mp\mathscr{H}-V}e^{+\lambda_0}\,, \quad (3.2.5.16)$$

where $d^{X,*}$ stand for the standard Hodge codifferential for the metric $\pi^*(g\oplus g^{-1})$ on $TX=T(T^*Q)$.
The operator $\delta^{*,r}_{\pm,b,V}$ is actually the $\langle\,,\,\rangle_r$-formal adjoint of $\delta_{\pm,b,V}$:

$$\forall u,v\in\mathscr{S}(X;\mathscr{E}_\pm),\quad \langle u\,,r^*(\delta^{*,r}_{\pm,b,V})v\rangle=\langle\delta_{\pm,b,V}u\,,r^*v\rangle\,.$$

The important properties for us are $\delta^2_{\pm,b,V}=0$, $(\delta^{*,r}_{\pm,b,V})^2=0$ and the fact that $\frac{1}{2}B_{\pm,b,h}$ is the square of the $\langle\,,\,\rangle_r$ symetric operator $\mathfrak{A}'_{\phi_b,\pm\mathscr{H}}$.
Let us explain how (3.2.5.14)(3.2.5.15) and (3.2.5.16) written in our setting are deduced from the formulas (2.1.23) (2.1.24) and (2.1.28) of [BiLe] (see also Section 2 and Section 3 of [Bis05]):

— The factors $e^{\pm V}$ come from our choice of scalar product $\langle\,,\,\rangle$ instead of $\langle\,,\,\rangle_{L^2(X,dqdp;\mathscr{E})}$ and the correspondance of Table 3.2.1. Once this is settled, this factor can be forgotten for the comparison with the formulas of [BiLe].
— For a general $b>0$, the formula (2.1.28)-[BiLe]

$$\mathfrak{A}'_{\phi_b,\pm\mathscr{H}}=K_b\mathfrak{A}'_{\phi_1,\pm r^*_{\frac{1}{b}}\mathscr{H}}K_b^{-1}=K_b\mathfrak{A}'_{\phi_1,\pm\frac{1}{b^2}\mathscr{H}}K_b^{-1}$$

allows to extend the formulas (2.1.22)(2.1.23)-[BiLe] written for the case $b=1$ to the general case. Because K_b commutes with μ_0 (and λ_0) this provides the formula (3.2.5.15). Because K_b commutes with r^* it implies that $\delta^{*,r}_{\pm,b,V}$ is the $\langle\,,\,\rangle_r$-formal adjoint of $\delta_{\pm,b,V}$.
— Finally the explicit expression of $\delta_{\pm,b,V}$ is obtained after using the property that λ_0 is the $\langle\,,\,\rangle$-formal adjoint of μ_0, and the involutive identity $r^*\lambda_0r^*=-\lambda_0$.

Like in [BiLe]-page 32 but with now the $L^2(X,dqdp;\Lambda T^*X)$ scalar product $\langle\,,\,\rangle$, we recall the elementary functional properties of $\alpha_\pm$ and $\beta_\pm$. Meanwhile, we make the identifications hidden in (3.2.5.6) and (3.2.5.13) more explicit by using the isometries $\mathscr{I}_{\theta,X}$ and $\mathscr{I}_{\theta,Q}$ of Definition 3.2.1.

— The operator $\alpha_\pm=\mathscr{O}\pm(N^V-d/2)$ is self-adjoint on its domain $\tilde{\mathscr{W}}^{2,0}(X;\mathscr{E}_\pm)$,. By using the fiberwise change of variable $\tilde{p}_i=(\sqrt{g(q)})^{ij}p_j$ the Hilbert space $L^2(X,dqdp;\mathscr{E}_\pm)$ can be written as the direct integral

$$L^2(X,dqdp;\mathscr{E}_\pm)=\int_Q^\oplus L^2(\mathbb{R}^d,d\tilde{p};\mathbb{C}^{2^{2d}})\,d\mathrm{Vol}_g(q)$$

if we notice $dqdp=|\det(g(q))|^{1/2}\,dqd\tilde{p}$. In this direct integral representation, $\alpha_\pm$ is nothing but

$$\alpha_\pm=\int_Q^\oplus\frac{-\Delta_{\tilde{p}}+|\tilde{p}|^2}{2}\otimes\mathrm{Id}_{\mathbb{C}^{2^{2d}}}\pm(N_V-d/2)\,d\mathrm{Vol}_g(q)$$

where $\frac{-\Delta_{\tilde{p}}+|\tilde{p}|^2}{2}=\sum_{i=1}^d\frac{-\partial^2_{\tilde{p}_j}+\tilde{p}_j^2}{2}$ is the euclidean scalar harmonic oscillator. Therefore the spectrum of $\alpha_\pm$ equals $\mathbb{N}$. The kernel of α_+ is given by horizontal forms times $\exp\left(-\frac{|p|_q^2}{2}\right)$ and the kernel of α_- is given by the exterior product of horizontal forms with a top vertical form times $\exp\left(-\frac{|p|_q^2}{2}\right)$.

— With the orthogonal projection $\pi_{0,\pm}$ on the kernel of $\alpha_\pm$ and $1-\pi_{0,\pm}=\pi_{\perp,\pm}$ its orthogonal complement, we have

$$\begin{aligned} &L^2(X,dqdp;\mathcal{E}_\pm)=\ker\alpha_\pm \overset{\perp}{\oplus}\mathrm{Ran}\,\alpha_\pm=\mathrm{Ran}\,\pi_{0,\pm}\overset{\perp}{\oplus}\mathrm{Ran}\,\pi_{\perp,\pm}\\ \text{and}\quad &\mathcal{S}(X;\mathcal{E}_\pm)=\left(\mathrm{Ran}\,\pi_{0,\pm}\cap\mathcal{S}(X;\mathcal{E}_\pm)\right)\oplus\left(\mathrm{Ran}\,\pi_{\perp,\pm}\cap\mathcal{S}(X;\mathcal{E}_\pm)\right)\end{aligned}$$

while the functional calculus says that $\alpha_\pm:\mathrm{Ran}\pi_{\perp,\pm}\cap\tilde{\mathcal{W}}^{2,0}(X;\mathcal{E}_\pm)\to\mathrm{Ran}\,\pi_{\perp,\pm}$ is invertible with the norm of $\alpha_\pm^{-1}\pi_{\perp,\pm}$ equal to 1.

— The differential operator $\beta_\pm$ maps $\ker\alpha_\pm\cap\mathcal{S}(X;\mathcal{E}_\pm)=\mathrm{Ran}\,\pi_{0,\pm}\cap\mathcal{S}(X;\mathcal{E}_\pm)$ into $\mathrm{Ran}\,\alpha_\pm\cap\mathcal{S}(X;\mathcal{E}_\pm)=\mathrm{Ran}\,\pi_{\perp,\pm}\cap\mathcal{S}(X;\mathcal{E}_\pm)$.

— With (3.2.4.5)(3.2.4.6)(3.2.4.7) we can write

$$f(\alpha_\pm)=\mathcal{I}_{\theta,X}^*\left[\bigoplus_{K\subset\{1,\ldots,d\}} f\left(\frac{-\Delta_{\mathrm{Vert}}+|p|_q^2}{2}\pm(|K|-d/2)\right)\otimes\mathrm{Id}_{\mathbb{C}^{J\times 2d}}\right]\mathcal{I}_{\theta,X}$$

for any Borel function $f:\mathbb{R}\to\mathbb{C}$. In particular for $f=1_{\{0\}}$ we obtain

$$\pi_{0,\pm}=\mathcal{I}_{\theta,X}^*\left[1_{\{0\}}\left(\frac{-\Delta_{\mathrm{Vert}}+|p|_q^2-d}{2}\right)1_{\{0\}}\left(|K|-d/2\pm d/2\right)\right]\mathcal{I}_{\theta,X}\,.$$

The kernel of the harmonic oscillator $\frac{-\Delta_{\mathrm{Vert}}+|p|_q^2-d}{2}$ equals $\mathbb{C}\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}$ with

$$\int_{\mathbb{R}^d}\left|\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}\right|^2\,dp=|\det(g(q)|^{1/2}\,.$$

We deduce that

$$U_{+,\theta}=\mathcal{I}_{\theta,X}^*\left[\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}\times\right]\mathcal{I}_{\theta,Q}\tag{3.2.5.17}$$

is a unitary transform $U_{+,\theta}:L^2(Q,d\mathrm{Vol}_g;\Lambda T^*Q\otimes\mathbb{C})\to\mathrm{Ran}\,\pi_{+,0}=\ker(\alpha_+)$ in the + case. In the − case, we choose $\eta\in\mathcal{C}^\infty(X;\Lambda^d(T^*X)^V\otimes\pi^*(\mathbf{or}_Q))$ to be a normalized non vanishing section, which can be written locally as $\hat{f}_{j,1}\wedge\ldots\wedge\hat{f}_{j,d}$ with the suitable orientation. Then the unitary transform $U_{-,\theta}:L^2(Q,d\mathrm{Vol}_g;\Lambda T^*Q\otimes\mathbb{C}\otimes\mathbf{or}_Q)\to\mathrm{Ran}\,\pi_{-,0}=\ker(\alpha_-)$ is given by

$$U_{-,\theta}=\mathcal{I}_{\theta,X}^*\left[\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}\times\right](\mathcal{I}_{\theta,Q}\wedge\eta).\tag{3.2.5.18}$$

When $\mathcal{I}_{\theta,Q}(s)=(s_{j,I})_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}$ for $s\in L^2(Q,d\mathrm{Vol}_g;\Lambda T^*Q\otimes F)$ ($F=Q\times\mathbb{C}$ in the + case and $F=(Q\times\mathbb{C}\otimes\mathbf{or}_Q)$ in the − case) we get

$$U_{+,\theta}s=\sum_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\theta_j(q)s_{j,I}(q)\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}f_j^I\tag{3.2.5.19}$$

$$\text{and}\quad U_{-,\theta}s=\sum_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\theta_j(q)s_{j,I}(q)\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}f_j^I\wedge\hat{f}_{j,1}\wedge\ldots\wedge\hat{f}_{j,d}\,.\tag{3.2.5.20}$$

When $\mathscr{I}_{\theta,X}(s')=(s^{K}_{j,I})_{\substack{1\leq j\leq J\\ I,K\subset\{1,\ldots,d\}}}$ for $s'\in L^2(X,dqdp;\mathscr{E}_\pm)$ we get

$$U_{+,\theta}^{-1}s'=U_{+,\theta}^{*}s'=\sum_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\theta_j(q)\left(\int_{T_q^*Q}\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}s^{\emptyset}_{j,I}(q,p)\,dp\right)u_j^I \tag{3.2.5.21}$$

$$\text{and}\quad U_{-,\theta}^{-1}s'=U_{-,\theta}s'=\sum_{\substack{1\leq j\leq J\\ I\subset\{1,\ldots,d\}}}\theta_j(q)\left(\int_{T_q^*Q}\frac{e^{-\frac{|p|_q^2}{2}}}{\pi^{d/4}}s^{\{1,\ldots,d\}}_{j,I}(q,p)\,dp\right)u_j^I\wedge\hat{f}_{j,1}\wedge\ldots\wedge\hat{f}_{j,d}\,. \tag{3.2.5.22}$$

The unitary map $U_{\pm,\theta}:L^2(Q,d\mathrm{Vol}_g;\Lambda T^*Q\otimes F)\to\mathrm{Ran}\,\pi_{0,\pm}=\ker(\alpha_\pm)$ clearly induces an isomorphism depending on the case:

$$\begin{aligned}U_{+,\theta}\ &:\ \mathscr{C}^\infty(Q;\Lambda^{\cdot}T^*Q\otimes\mathbb{C})\to\mathscr{S}(X;\mathscr{E}_+)\cap\ker\alpha_+\\ U_{-,\theta}\ &:\ \mathscr{C}^\infty(Q;\Lambda^{\cdot}T^*Q\otimes\mathbb{C}\otimes\mathbf{or}_Q)\to\mathscr{S}(X;\mathscr{E}_-)\cap\ker\alpha_-\,.\end{aligned}$$

Other functional spaces can be considered. With those notations, formula (3.2.5.13) means precisely

$$U_{\pm,\theta}^{-1}[\pi_{0,\pm}(\gamma_\pm-\beta_\pm\alpha_\pm^{-1}\beta_\pm)\pi_{0,\pm}]U_{\pm,\theta}=\frac{1}{2h^2}\Delta_{V,h}\,. \tag{3.2.5.23}$$

Notice also that $e_i(e^{-\frac{|p|_q^2}{2}}a(q))=e^{-\frac{|p|_q^2}{2}}\frac{\partial a}{\partial q^i}(q)$ implies

$$U_{\pm,\theta}^{-1}[\pi_{0,\pm}W_\theta^2\pi_{0,\pm}]U_{\pm,\theta}=\mathscr{I}_{\theta,Q}^{*}\left[(C+C\frac{d^2}{4}-\frac{1}{2}\Delta_Q)\otimes\mathrm{Id}_{\mathbb{C}^{J\times 2^d}}\right]\mathscr{I}_{\theta,Q} \tag{3.2.5.24}$$

where Δ_Q is the scalar Laplace-Beltrami operator on Q .

Lemma 3.2.5. *There exists a constant $C_{g,\theta}\geq 1$ determined by the metric g and the quadratic partition of unity $\sum_{j=1}^J\theta_j^2(q)\equiv 1$ such that*

$$C_{g,\theta}^{-1}\|u\|^2_{\tilde{\mathscr{W}}^{0,1}}\leq\|u\|^2_{L^2}+\|\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}u\|^2_{L^2}\leq C_{g,\theta}\|u\|^2_{\tilde{\mathscr{W}}^{0,1}}$$

holds for all $u\in\mathscr{S}(X;\mathscr{E}_\pm)\cap\ker\alpha_\pm$.

Proof. We first notice that for any connection ∇ and all $u\in\mathscr{S}(X;\mathscr{E}_\pm)\cap\ker\alpha_\pm$,

$$\nabla_{\mathscr{Y}}u=e^{-\frac{|p|_q^2}{4}}\nabla_{\mathscr{Y}}e^{\frac{|p|_q^2}{4}}\pi_{0,\pm}u\,.$$

Because $\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}$ is a first order differential operator we have

$$\|\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}u\|^2_{L^2}=\sum_{j=1}^J\|\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}(\theta_j(q)u)\|^2_{L^2}-\sum_{j=1}^J\|(\mathscr{Y}\theta_j)u\|^2_{L^2}\,.$$

With the local formula

$$\mathscr{Y}\theta_j=g^{ik}(q)p_k(\partial q^i\theta_j)(q)=e^{-\frac{|p|_q^2}{4}}g^{ik}p_k(\partial q^i\theta_j)(q)e^{\frac{|p|_q^2}{4}}$$

and $\|e^{\frac{|p|_q^2}{4}}\pi_0\|_{\mathscr{L}(L^2)}\leq C_g$ we obtain

$$C_{g,\theta,1}^{-1}\sum_{j=1}^J\|\theta_j(q)u\|^2_{L^2}+\|\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}(\theta_j(q)u)\|^2_{L^2}\leq\|u\|^2_{L^2}+\|\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}u\|^2_{L^2}\leq\sum_{j=1}^J\|\theta_j(q)u\|^2_{L^2}+\|\nabla^{\mathscr{E}_\pm}_{\mathscr{Y}}(\theta_j(q)u)\|^2_{L^2}\,.$$

Above the neighborhood $\mathcal{V}_{\theta,j}\supset \mathrm{supp}\theta_j$, we use the connection ∇^j which is trivial in the local orthonormal frame $(f_j^1\ldots f_j^d,\hat{f}_{j,1},\ldots,\hat{f}_{j,d})$.
The relation

$$\nabla^{\mathcal{E}_\pm}_{\mathcal{Y}}-\nabla^j_{\mathcal{Y}}=e^{-\frac{|p|_q^2}{4}}g^{ik}(q)p_k(\nabla^{\mathcal{E}_\pm}_{e_i}-\nabla^j_{e_i})e^{\frac{|p|_q^2}{4}}$$

allows the same comparison which leads to

$$C_j^{-1}\left(\|\theta_ju\|^2_{L^2}+\|\nabla^{\mathcal{E}_\pm}_{\mathcal{Y}}(\theta_ju)\|^2_{L^2}\right)\leq \|\theta_ju\|^2_{L^2}+\|\nabla^j_{\mathcal{Y}}(\theta_ju)\|^2_{L^2}\leq C_j\left(\|\theta_ju\|^2_{L^2}+\|\nabla^{\mathcal{E}_\pm}_{\mathcal{Y}}(\theta_ju)\|^2_{L^2}\right).$$

With the trivial connection ∇^j in the orthonormal frame $(f_j^1,\ldots,f_j^d,\hat{f}_{j,1},\ldots,\hat{f}_{j,d})$ the estimate of the middle term is reduced to the computation of $\|\theta_ju\|^2_{L^2}+\|\mathcal{Y}(\theta_ju)\|^2_{L^2}$ for $u=\pi^{-d/4}e^{-\frac{|p|_q^2}{2}}a(q)$ for $a\in\mathcal{C}_0^\infty(\mathcal{V}_{\theta,j};\mathbb{C})$. We compute

$$\|\mathcal{Y}(\pi^{-d/4}e^{-\frac{|p|_q^2}{2}}\theta_ja)\|^2_{L^2}=\int_Q\left[\int_{\mathbb{R}^d}|g^{ik}(q)p_k(\partial_{q^i}(\theta_ja)|^2\pi^{-d/2}e^{-g^{ik}(q)p_ip_k}\,dp\right]\,dq=\frac{1}{2}\int_Q|\nabla^g_q(\theta_ja)|^2d\mathrm{Vol}_g(q).$$

Similarly the definition $W_\theta^2=\mathcal{I}^*_{\theta,X}\left[W^2\otimes\mathrm{Id}_{\mathbb{C}^{J\times 2^{2d}}}\right]\mathcal{I}_{\theta,X}$ reduces the problem to the computation of

$$\langle\theta_ju\,,W^2\theta_ju\rangle=\langle\theta_ju\,,[C-\Delta_H+C\mathcal{O}^2]\theta_ju\rangle$$

with $u=\pi^{-d/4}e^{-\frac{|p|_q^2}{2}}a(q)$, $a\in\mathcal{C}_0^\infty(\mathcal{V}_{\theta,j};\mathbb{C})$. We obtain like in (3.2.5.24)

$$\langle\theta_ju\,,W^2\theta_ju\rangle=(C+C\frac{d^2}{4})\|\theta_ja\|^2_{L^2}+\frac{1}{2}\int_Q|\nabla^g_q(\theta_ja)|^2d\mathrm{Vol}_g(q)$$

and this ends the proof. □

3.2.6 Scalings

When we consider semiclassical Witten Laplacians, it is natural to introduce semiclassical Sobolev spaces. Accordingly the space $\tilde{\mathcal{W}}^{0,s}(X;\mathcal{E}_\pm)$ has to be defined with an h-dependent norm. There are various transformations on the operators, Witten's and Bismut's Laplacian, and on the functional spaces which allow to reduce the h-dependent problem, $h\in]0,1]$, to the case $h=1$. This simplifies the asymptotic analysis with respect to the pair of parameters $b>0,h>0$. In particular, the initial subelliptic estimates of [NSW], where only the parameter $b>0$ was considered, can be easily translated into a b,h-dependent version.
Semiclassical Witten Laplacian: The semiclassical Witten Laplacians $\Delta_{V,h}$ on the riemannian manifold (Q,g) can be given several equivalent presentations. It is better to think in terms of the four data (Q,g,V,h) where (Q,g) is the riemannian manifold $V\in\mathcal{C}^\infty(Q;\mathbb{R})$ is the potential function and $h\in]0,1]$ is the semiclassical parameter.
The semiclassical Witten Laplacian equals

$$\Delta_{V,h}=\Delta_{(Q,g,V,h)}=(d_{V,h}+d^{*,g}_{V,h})^2$$

$$\text{where}\qquad d_{V,h}=e^{-\frac{V}{h}}(hd)e^{\frac{V}{h}}=hd+dV\wedge\quad d^{*,g}_{V,h}=e^{-\frac{V}{h}}(hd^{*,g})e^{\frac{V}{h}}=hd^{*,g}+\mathbf{i}_{\nabla_gV}\,,$$

where the subscripts recall that the Hodge star operator, the codifferential and the gradient all depend on the chosen metric g.
Relations between the following differential operators acting on $\mathcal{C}^\infty(Q;\Lambda T^*Q\otimes F_\pm)$ can be written:

$$d^{*,\frac{1}{h^2}g}=h^2d^{*,g}\quad,\quad\nabla_{\frac{g}{h^2}}V=h^2\nabla_gV$$

$$d_{V,h}=hd_{\frac{V}{h},1}\quad,\quad d^{*,g}_{V,h}=\frac{1}{h}d^{*,\frac{g}{h^2}}_{\frac{V}{h},1}\quad,\quad\Delta_{(Q,g,V,h)}=\Delta_{(Q,\frac{g}{h^2},\frac{V}{h},1)}.$$

For the L^2-spaces we note that

$$dVol_{\frac{1}{h^2}g} = h^{-d} dVol_g \quad \int_Q \langle s,s'\rangle_{\frac{1}{h^2}g}\, dVol_{\frac{1}{h^2}g} = \int_Q (h^2)^{\mathrm{deg}s-d/2}\langle s,s'\rangle_g\, dVol_g$$

and the map $s\mapsto h^{d/2-\mathrm{deg}s}s$ is a unitary map from $L^2_g(Q;\Lambda T^*Q\otimes F_\pm)$ onto $L^2_{\frac{1}{h^2}g}(Q;\Lambda T^*Q\otimes F_\pm)$. Semiclassical Sobolev spaces are defined by replacing derivatives of vector fields with g-norms bounded by 1 , by vector fields with $\frac{1}{h^2}g$-norms bounded by 1 or g-norms of size $\mathcal{O}(h)$. By using a Laplace type operator $\Delta_{(Q,g,0,1)}$ or $H_{0,g}=\sum_{j=1}^J\theta_j(q)\Delta_{sc,g}\theta_j(q)$ the semiclassical Sobolev norms are given by

$$\|u\|_{H^{s,h}_g(Q)} = \|(1+h^2H_{0,g})^{s/2}u\|_{L^2_g(Q)} = \|h^{\frac{d}{2}-\mathrm{deg}}(1+H_{0,\frac{1}{h^2}g})^{s/2}u\|_{L^2_{\frac{g}{h^2}}(Q)}.$$

Another introduction of the scaling relies on the fact that (Q,g) can be isometrically embedded in the euclidean space $(\mathbb{R}^{d_Q};g_{\mathbb{R}^{d_Q}})$, according to Nash embedding theorem (see e.g. [Gro]). This isometric embedding can be done such that $d_{g^{\mathbb{R}^Q}}(0,Q)=1$ and one may consider the homothetic transformation of Q with center 0 and ratio $\frac{1}{h}$, $Q^h=\frac{1}{h}Q$ or $Q^h=\phi_hQ$ with $\phi_h(q)=\frac{1}{h}q$ for $q\in\mathbb{R}^{d_Q}$. The tangent and conormal vector bundle TQ , $N^*Q=\left\{v\in T^*\mathbb{R}^{d_Q}\big|_Q,\quad \forall t\in TQ, v.t=0\right\}$ are well defined and the euclidean metric $g^{\mathbb{R}^{d_Q}}$ allows to identify

$$T^*Q=\left\{v\in T^*\mathbb{R}^{d_Q}\big|_Q,\quad \forall w\in N^*Q,\quad (g^{\mathbb{R}^{d_Q}})^{-1}(v,w)=0\right\}$$

. The same can be done with Q^h which is endowed with the metric $g^h=g^{\mathbb{R}^{d_Q}}\big|_{TQ^h\times TQ^h}$. Then the riemannian manifold (Q^h,g^h) is isometric to $(Q,\frac{1}{h^2}g)$ and when $H_{0,g^h}=\sum_{j=1}^J\theta_j(h.)\Delta_{sc,g^h}\theta_j(h.)$ and $V^h(q)=\frac{1}{h}V(hq)$ we obtain

$$\phi_h^*\Delta_{(Q^h,g^h,V^h,1)}\phi_{h,*}=\Delta_{(Q,\frac{1}{h^2}g,\frac{1}{h}V,1)}=\Delta_{(Q,g,V,h)}$$
$$\|u\|_{H^{s,h}_g(Q)}=\|h^{\frac{d}{2}-\mathrm{deg}}(1+H_{0,\frac{1}{h^2}g})^{s/2}u\|_{L^2_{\frac{1}{h^2}g}(Q)}=\|h^{\frac{d}{2}-\mathrm{deg}}(1+H_{0,g^h})^{s/2}\phi_{h,*}u\|_{L^2_{g^h}(Q^h)}.$$

If instead of the quadratic partition of unity $\sum_{j=1}^J\theta_j^2(q)\equiv 1$ on Q one takes an h-dependent partition of unity $\sum_{j=1}^{J_h}\theta_{j,h}^2(q)\equiv 1$ subordinate to an atlas $\cup_{j=1}^{J_h}\Omega_{j,h}=Q$ with $\mathrm{diam}_g(\Omega_{j,h})\le Ch$ or $(\mathrm{diam}_{\frac{1}{h^2}g}(\Omega_{j,h})\le C)$ and an intersection number uniformly bounded with respect to h , one sees that g^h in a coordinates system in $\phi_h\Omega_{j,h}$ satisfies

$$\|\partial_q^\alpha g^h\|_{L^\infty(\phi_h\Omega_{j,h})}+\|\partial_q^\alpha(g^h)^{-1}\|_{L^\infty(\phi_h\Omega_{j,h})}\le C_\alpha,.$$

Although the volume of Q^h , $\mathrm{Vol}(Q^h)=h^{-d}\mathrm{Vol}(Q)$ increases as $h\to 0$, the above quantity $\|(1+H_{0,g^h})^s v\|_{L^2_{g^h}(Q^h)}$ correspond to the standard Sobolev space norm on Q^h with a uniform control of the local variations of the metric while $\nabla_{g^h}V^h$ is uniformly bounded as well as its covariant derivatives with respect to vector fields with a bounded g^h-norms. We will use the short notation $H^s(Q^h;\Lambda T^*Q^h\otimes F_\pm)$ for $H^{s,1}_{g^h}(Q^h;\Lambda T^*Q^h\otimes F_\pm)$.

Bismut hypoelliptic Laplacian: We do the same kind of scalings as above for the Bismut hypoelliptic Laplacian. Actually we will start from the expression (3.2.5.7)(3.2.5.8)(3.2.5.9)(3.2.5.10)

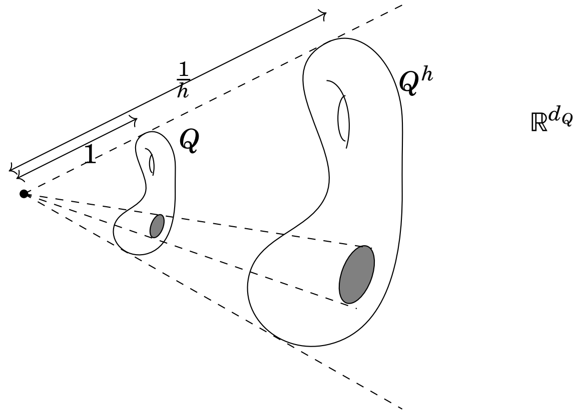

Figure 3.2.1 – The grey areas represent on Q a ball of radius 1 (resp. h) for the metric $\frac{1}{h^2}g$ (resp. g) and on Q^h the isometric ball of radius 1 for the metric g^h.

of the operator $B_{\pm,b,\frac{V}{h}}$ which is actually determined by the data $(Q,g,\frac{V}{h},b)$ where (Q,g) is the base riemannian manifold, $V \in \mathscr{C}^\infty(Q;\mathbb{R})$ is the potential function and $b,h>0$ are the two parameters:

$$B_{\pm,b,\frac{V}{h}} = B_{\pm,(Q,g,\frac{V}{h},b)} = \frac{1}{b^2}\alpha_{\pm,(Q,g)} + \frac{1}{b}\beta_{\pm,(Q,g,\frac{V}{h})} + \gamma_{\pm,(Q,g,\frac{V}{h})}.$$

By mimicking what we observed for the Witten Laplacian, we firstly want to establish a simple relation between $B_{\pm,(Q,g,\frac{V}{h},*)}$ and $B_{\pm,(Q,\frac{1}{h^2}g,\frac{V}{h},*)}$. We notice

$$(\frac{1}{h^2}g)\oplus(\frac{1}{h^2}g)^{-1} = (\frac{1}{h^2}g)\oplus(h^2g^{-1})$$

while the Christoffel symbols $\Gamma_{ij}^k(q)$ are the same for the metric g and the rescaled metric $\frac{1}{h^2}g$ and the Levi-Civita connection is not changed. If local orthonormal frames given by (3.2.2.2) and (3.2.2.3) are denoted by $(f_{i,g})_{1\leq i\leq d}$,$(\hat{f}_g^i)_{1\leq i\leq d}$ $(f_g^i)_{1\leq i\leq d}$ and $(\hat{f}_{i,g})_{1\leq i\leq d}$ for the metric g, orthonormal frames for the metric $\frac{1}{h^2}g$ are given by

$$f_{i,\frac{1}{h^2}g} = hf_{i,g} \quad , \quad \hat{f}^i_{\frac{1}{h^2}g} = \frac{1}{h}\hat{f}^i_g \quad ,$$
$$\text{and} \qquad f^i_{\frac{1}{h^2}g} = \frac{1}{h}f^i_g \quad , \quad \hat{f}_{i,\frac{1}{h^2}g} = h\hat{f}_{i,g}\,.$$

Other simple relations for the kinetic energy and the hamiltonian vector field $\mathscr{Y}$ are:

$$\frac{|p|^2_{q,g}}{2} = \frac{g^{ij}(q)p_ip_j}{2} = \frac{1}{2h^2}(\frac{1}{h^2}g)^{ij}p_ip_j = \frac{|p|^2_{q,\frac{1}{h^2}g}}{2h^2} \quad \text{and} \quad \mathscr{Y}_g = \frac{1}{h^2}\mathscr{Y}_{\frac{1}{h^2}g}$$

while the symplectic form $\sigma = dp\wedge dq$ on T^*Q is not changed.
Although $B_{\pm,(Q,g,\frac{V}{h},b)}$ preserves the total degree $|I|+|J|$, it mixes the horizontal degree $|I|$ and vertical degree $|J|$ for sections $s_I^J(x)f^I\hat{f}_J$. The different homogeneities in the conformal change of metric from g to $\frac{1}{h^2}g$ must be considered carefully as well.
Because $X = T^*Q$ is a vector bundle on Q, while $\mathscr{E}_\pm = \pi^*(\Lambda T^*Q\otimes\Lambda TQ\otimes F_\pm)$, we define the mapping $\Psi_h : \mathscr{S}(X;\mathscr{E}_\pm)\to\mathscr{S}(X;\mathscr{E}_\pm)$ by

$$\Psi_h(s_I(q,p)^Jf^I_{\frac{1}{h^2}g}\hat{f}_{J,\frac{1}{h^2}g}) = h^{-\frac{d}{2}+|I|-|J|}s_I^J(q,h^{-1}p)f^I_{\frac{1}{h^2}g}\hat{f}_{J,\frac{1}{h^2}g} = h^{-\frac{d}{2}}s_I^J(q,h^{-1}p)f^I_g\hat{f}_{J,g}\,.$$

We obtain

$$\Psi_h^{-1}(\alpha_{\pm,(Q,g)})\Psi_h = \frac{1}{2}\left(-h^{-2}\Delta_g^V + h^2|p|^2_{q,g} \pm (2\hat{f}_{i,g}\mathbf{i}_{\hat{f}_g^i} - d)\right) = \alpha_{\pm,(Q,\frac{1}{h^2}g)}$$

and $\Psi_h^{-1}(\beta_{\pm,(Q,g,\frac{V}{h})})\Psi_h = -(\pm h\nabla^{\mathcal{E}_\pm}_{\mathcal{Y}_g} - \frac{1}{h^2}(f_{i,g}V)\nabla^{\mathcal{E}_\pm}_{\hat{f}_g^i}) = -\frac{1}{h}(\pm\nabla^{\mathcal{E}_\pm}_{\mathcal{Y}_{\frac{1}{h^2}g}} - (f_{i,\frac{1}{h^2}g}\frac{V}{h})\nabla^{\mathcal{E}_\pm}_{\hat{f}^i_{\frac{1}{h^2}g}}) = \frac{1}{h}\beta_{\pm,(Q,\frac{1}{h^2}g,\frac{V}{h})}$.

For $\Psi_h^{-1}\gamma_{\pm,Q,g}\Psi_h$ firstly notice that the Riemann curvature tensors are compared according to

$$R_g^{TQ} = h^2 R^{TQ}_{\frac{1}{h^2}g}$$

while the coefficient $\tilde{\Gamma}^k_{ij}(q) = f^k(\nabla^{TQ}_{f_i} f_j)$ satisfies

$$\tilde{\Gamma}^k_{ij,g}(q) = \frac{1}{h}\tilde{\Gamma}^k_{ij,\frac{1}{h^2}g}.$$

The definition of the mapping Ψ_h ensures the identity of the tensorial operations

$$\Psi_h^{-1}\begin{pmatrix} f_g^i\wedge \\ \hat{f}_{i,g}\wedge\end{pmatrix}\Psi_h = \begin{pmatrix} f^i_{\frac{1}{h^2}g}\wedge \\ \hat{f}_{i,\frac{1}{h^2}g}\wedge\end{pmatrix} \quad\text{and}\quad \Psi_h^{-1}\begin{pmatrix}\mathbf{i}_{f_{i,g}} \\ \mathbf{i}_{\hat{f}_g^i}\end{pmatrix}\Psi_h = \begin{pmatrix}\mathbf{i}_{f_{i,\frac{1}{h^2}g}} \\ \mathbf{i}_{\hat{f}^i_{\frac{1}{h^2}g}}\end{pmatrix}.$$

We deduce

$$\Psi_h^{-1}\gamma_{\pm,(Q,g,\frac{V}{h})}\Psi_h = \frac{1}{h^2}\gamma_{\pm,(Q,\frac{1}{h^2}g,\frac{V}{h})}.$$

We have proved

$$\Psi_h^{-1}B_{\pm,(Q,g,\frac{V}{h},b)}\Psi_h = \frac{1}{h^2}\left[\frac{h^2}{b^2}\alpha_{\pm,(Q,\frac{1}{h^2}g)} + \frac{h}{b}\beta_{\pm,(Q,\frac{1}{h^2g},\frac{V}{h})} + \gamma_{\pm,(Q,\frac{1}{h^2}g,\frac{V}{h})}\right] = \frac{1}{h^2}B_{\pm,(Q,\frac{1}{h^2}g,\frac{V}{h},\frac{b}{h})}$$

Let us consider now what happens on the functional spaces.
The linear map Ψ_h is actually a unitary transform from $L^2_{\frac{1}{h^2}g}(X,dqdp;\mathcal{E}_\pm)$ to $L^2_g(X,dqdp;\mathcal{E}_\pm)$.
The h-dependent norms for the $\tilde{W}^{s_1,s_2}(X;\mathcal{E}_\pm)_g$ were not studied in [NSW] but we follow the dilatation trick presented for the Witten Laplacian in order to reduce the problem to uniform estimates in the case $h=1$.

Definition 3.2.6. On the cotangent space $X = T^*Q$ where Q is endowed with the riemannian metric g, the h-dependent norm, $h\in]0,1]$, of $\tilde{\mathcal{W}}^{s_1,s_2}(X;\mathcal{E}_\pm)$ is given by

$$\|u\|_{\tilde{\mathcal{W}}_h^{s_1,s_2}} = \|\mathcal{O}^{s_1/2}(W^2_{\theta,h})^{s_2/2}u\|_{L^2(X;\mathcal{E}_\pm)}$$

where

$$W^2_{\theta,h} = \sum_{j=1}^J \theta_j(q)(C_g - h^2\Delta_H + C_g\mathcal{O}^2)\theta_j(q).$$

Remember that the operator $W^2_{\theta,h}$ is an elliptic self-adjoint operator for any fixed (Q,g,h) with $h\in]0,1]$ when $C_g\geq 1$ is chosen large enough. However the uniformity of the subelliptic estimates for $B_{\pm,(Q,g,\frac{V}{h},b)}$ with these h-dependent norms requires some explanation.
We keep track of the change of riemannian metrics with subscripts like before and write $\Delta_H = \Delta_{H,g}$ and $W^2_{\theta,h} = W^2_{\theta,h,g}$. Actually the definition of $\Delta_{H,\frac{1}{h^2}g}$ gives $\Delta_{H,\frac{1}{h^2}g} = h^2\Delta_{H,g}$ and

$$\Psi_h^{-1}W^2_{\theta,h,g}\Psi_h = (C_g - \Delta_{H,\frac{1}{h^2}g} + C_g(\mathcal{O}_{\frac{1}{h^2}g})^2) = W^2_{\theta,1,\frac{1}{h^2}g}.$$

After the riemannian embedding $Q \to \mathbb{R}^{d_Q}$ and the identfication of $X = T^*Q$ as a subbundle of $(T^*\mathbb{R}^{d_q})\big|_Q$ the dilatation $\Phi_h : q \mapsto \frac{q}{h}$ in $\mathbb{R}^{d_Q}$ with $\Phi_{h,*} : X = T^*Q \to X^h = T^*Q^h$ is a symplectic map, and an isometry from $(X, (\frac{1}{h^2}g) \oplus^\perp (h^2 g^{-1}))$ to $(X^h, g^h \oplus^\perp (g^h)^{-1})$. We obtain

$$\Phi_{h,*}\Psi_h^{-1}(W^2_{\theta,h,g})\Psi_h\Phi_h^* = W^2_{\theta,1,g^h}\,,$$

and

$$\begin{aligned}\|u\|_{\tilde{\mathcal{W}}^{s_1,s_2}_{h,g}(X;\mathcal{E}_\pm)} &= \|\mathcal{O}_g^{s_1/2}(W^2_{\theta,h,g})^{s_2/2}u\|_{L^2_g(X;\mathcal{E}_\pm)} = \|\mathcal{O}_{g^h}^{s_1/2}(W^2_{\theta,1,g^h})^{s_2/2}\Phi_{h,*}\Psi_h^{-1}u\|_{L^2_{g^h}(X_h;\Phi_{h,*}\mathcal{E}_\pm)}\\ &= \|\Phi_{h,*}\Psi_h^{-1}u\|_{\tilde{\mathcal{W}}^{s_1,s_2}_{1,g^h}(X_h;\Phi_{h,*}\mathcal{E}_\pm)}\,.\end{aligned}$$

The above discussion can be summarized by the following statement.

Proposition 3.2.7. *With the above notation $\Phi_{h,*}\Psi_h^{-1}$ is a unitary map from $\tilde{\mathcal{W}}^{s_1,s_2}_{h,g}(X,\mathcal{E}_\pm)_g$ to $\tilde{\mathcal{W}}^{s_1,s_2}_{1,g^h}(X^h;\Phi_{h,*}\mathcal{E}_\pm)$ for all $s_1,s_2 \in \mathbb{R}$ with*

$$\begin{aligned}&\Phi_{h,*}\Psi_h^{-1}B_{\pm,(Q,g,\frac{V}{h},b)}\Psi_h\Phi_h^* = \frac{1}{h^2}B_{\pm,(Q^h,g^h,V^h,\frac{b}{h})}\\ &\Phi_{h,*}\Psi_h^{-1}\alpha_{\pm,(Q,g)}\Psi_h\Phi_h^* = \alpha_{\pm,(Q^h,g^h)}\\ &\Phi_{h,*}\Psi_h^{-1}\beta_{\pm,(Q,g,\frac{V}{h})}\Psi_h\Phi_h^* = \frac{1}{h}\beta_{\pm,(Q^h,g^h,V^h)} \quad , \quad \Phi_{h,*}\Psi_h^{-1}\gamma_{\pm,(Q,g,\frac{V}{h})}\Psi_h\Phi_h^* = \frac{1}{h^2}\gamma_{\pm,(Q^h,g^h,V^h)}\\ \textit{and}\quad &\Phi_{h,*}\Psi_h^{-1}W^2_{\theta,g,h}\Psi_h\Phi_h^* = W^2_{\theta(h.),g^h,1}\,,\end{aligned}$$

with $V^h(q) = \frac{1}{h}V(hq)$, $\theta_j(h.)(q) = \theta_j(hq)$ for $q \in Q^h$. Additionally $\nabla_{g^h}V^h$ and g^h and $(g^h)^{-1}$, expressed in the coordinates associated with the atlas $Q^h = \cup_{j=1}^J \frac{1}{h}\Omega_j$ (or $\frac{1}{h}$ times the coordinates associated with $Q = \cup_{j=1}^J \Omega_j$) have uniformly bounded derivatives.

This result and what we recalled just above for the semiclassical Witten Laplacian, allow to eliminate the parameter $h \in]0,1]$ in the analysis. Actually it suffices to make the analysis for (Q^h, g^h, V^h) and $(Q^h, g^h, V^h, \frac{b}{h})$ where the parameter b is replaced by $\frac{b}{h}$ and to use the uniform control of all the norm estimates with respect to $h \in]0,1]$ on the dilated manifolds Q^h and $X^h = T^*Q^h$.

3.2.7 The Hypoelliptic Laplacian as a perturbed Geometric Kramers-Fokker-Planck operator

Although we are ultimately interested in Bismut's hypoelliptic Laplacian $B_{\pm,b,\frac{V}{h}} = B_{\pm,(Q,g,\frac{V}{h},b)}$, Proposition 3.2.7 with

$$\Phi_{h,*}\Psi_h^{-1}B_{\pm,(Q,g,\frac{V}{h},b)}\Psi_h\Phi_h^* = \frac{1}{h^2}B_{\pm,(Q^h,g^h,V^h,\frac{b}{h})}$$

allows to reduce the analysis to $B_{\pm,\frac{b}{h},V^h} = B_{\pm,(Q^h,g^h,V^h,\frac{b}{h})}$ with uniform controls of the local derivatives (in coordinate charts) of $\nabla_{g^h}V^h$, g^h and $(g^h)^{-1}$. For the sake of simplicity we replace $\frac{b}{h}$ by $b > 0$ and we write $B_{\pm,b,V^h} = B_{\pm,(Q^h,g^h,V^h,b)}$.
We will use the short notations $\mathcal{E}^h_\pm = \Phi_{h,*}\mathcal{E}_\pm$ for the vector bundle above $X^h = T^*Q^h$ and the connection $\nabla^{\mathcal{E}_\pm,h}$ will be the connection on $\mathcal{E}^h_\pm$ associated with the metric g^h on Q^h.

With these modifications, Bismut's hypoelliptic Laplacian can be written as

$$B_{\pm,b,V^h} = P_{\pm,b} + R_{0,h} + R_{2,h} + \frac{1}{b}R_{1,\perp,h}, \tag{3.2.7.1}$$

where the principal part is

$$P_{\pm,b} = \frac{1}{b^2}\alpha_{\pm,g^h} \mp \frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}} \tag{3.2.7.2}$$

and the three lower order corrections are

$$\begin{aligned}
R_{0,h} &= -\frac{1}{4}\langle R^{TQ}_{g^h}(f_{i,g^h},f_{j,g^h})f_{k,g^h},f_{\ell,g^h}\rangle(f^i_{g^h}-\hat{f}_{i,g^h})(f^j_{g^h}-\hat{f}_{j,g^h})\mathbf{i}_{f_{k,g^h}+\hat{f}^k_{g^h}}\mathbf{i}_{f_{\ell,g^h}+\hat{f}^\ell_{g^h}} + (f_{i,g^h}(f_{j,g^h}V^h) \\
&\qquad + \tilde{\Gamma}^k_{ij,g^h}f_{k,g^h}V^h)(f^i_{g^h}-\hat{f}_{i,g^h})\mathbf{i}_{f_{j,g^h}+\hat{f}^j_{g^h}} \\
R_{1,\perp,h} &= R_1 = (f_{i,g^h}V^h)\nabla^{\mathscr{E}_\pm,h}_{\hat{f}^i_{g^h}} \\
R_{2,h} &= \mp\langle R^{TQ}_{g^h}(p,f_{i,g^h})p,f_{j,g^h}\rangle(f^i_{g^h}-\hat{f}_{i,g^h})\mathbf{i}_{f_{j,g^h}+\hat{f}^j_{g^h}}
\end{aligned}$$

where we have neglected the $\pm$ sign in the notations $R_{0,h},R_{1,\perp,h},R_{2,h}$.
These notations are motivated by the following conditions, for a given differential operator A, indexed by $i=0,1,2$:

$$\forall s\in\mathbb{R}, \exists C_{s,i}>0, \|A\|_{\mathscr{L}(\tilde{\mathscr{W}}^{i,0};L^2)} + \|A\|_{\mathscr{L}(L^2;\tilde{\mathscr{W}}^{-i,0})} + \|(W^2_\theta)^{s/2}A(W^2_\theta)^{-s/2}-A\|_{\mathscr{L}(L^2;L^2)} \leq C_{s,i}\,, \tag{3.2.7.3}$$

$$\pi_{0,\pm}A\pi_{0,\pm} = 0\,. \tag{3.2.7.4}$$

The collection of operators $R_{0,h},R_{1,h}=R_{1,\perp,h},R_{2,h}$ are differential operators in the class $\mathrm{OpS}^1_\Psi(Q^h;\mathrm{End}(\mathscr{E}^h_\pm))$ introduced in [NSW] while $(W^2_\theta)^{s/2}\in\mathrm{OpS}^s_\Psi(Q^h;\mathrm{End}(\mathscr{E}^h_\pm))$ has a scalar principal symbol. We recall that actually $W^2_\theta = W^2_{\theta,g^h}$ and that, because of the uniform bounds of Proposition 3.2.7, all of the local seminorms of symbols are uniformly controlled with respect to $h\in]0,1]$. Therefore $R_{0,h}$, $R_{1,\perp,h}$, $R_{2,h}$ all satisfy, uniformly with respect to $h\in]0,1]$, $\|(W^2_\theta)^{s/2}R_{i,h}(W^2_\theta)^{-s/2}-R_{i,h}\|_{\mathscr{L}(L^2;L^2)}\leq C_{s,i}$, while the inequality $\|R_{i,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{i,0};L^2)}+\|R_{i,h}\|_{\mathscr{L}(L^2;\tilde{\mathscr{W}}^{-i,0})}\leq C_i$ according to the index $i=0,1,2$ is straigthforward.
Finally the index $\perp$ in $R_{1,\perp,h}$ recalls that $R_{1,h}=R_{1,\perp,h}$ satisfies the condition (3.2.7.4).
The following result allows to reduce the analysis of Bismut's hypoelliptic Laplacians in any $\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm)$ space to the case $s=0$.

Proposition 3.2.8. *The conditions* (3.2.7.3) *and* (3.2.7.4) *are left invariant by a conjugation by* $(W^2_\theta)^{s'/2}$ *for any* $s'\in\mathbb{R}$*, or by taking the formal adjoint for the* L^2*-scalar product. Namely if* A *satisfy condition* (3.2.7.3) *(resp.*(3.2.7.4) *) then* $(W^2_\theta)^{s'/2}A(W^2_\theta)^{-s'/2}$ *and the formal* L^2*-adjoint* A' *also satisfy* (3.2.7.3) *(resp.* (3.2.7.4)*).*

The conjugation of Bismut's hypoelliptic Laplacian by $(W^2_\theta)^{s'/2}$, $s'\in\mathbb{R}$*, equals*

$$(W^2_\theta)^{s'/2}B_{\pm,b,V^h}(W^2_\theta)^{-s'/2} = P_{\pm,b}+R^{s'}_{0,h}+R^{s'}_{2,h}+\frac{1}{b}R^{s'}_{1,\perp,h}$$

$$\text{with}\qquad R^{s'}_{0,h}=(W^2_\theta)^{s'/2}R_{0,h}(W^2_\theta)^{-s'/2}\quad,\quad R^{s'}_{2,h}=(W^2_\theta)^{s'/2}R_{2,h}(W^2_\theta)^{-s'/2}$$

$$\text{and}\qquad R^{s'}_{1,\perp,h}=R^{s'}_{1,h}=(W^2_\theta)^{s'/2}R_{1,h}(W^2_\theta)^{-s'/2}\mp\left[(W^2_\theta)^{s'/2}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}(W^2_\theta)^{-s'/2}-\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}\right]$$

where $R^{s'}_{0,h},R^{s'}_{1,h},R^{s'}_{2,h}$ *satisfy the condition* (3.2.7.3)*, uniformly with respect to* $h\in]0,1]$*, for the respective values of* $i=0,1,2$ *and* $R^{s}_{1,\perp,h}=R^{s'}_{1,h}$ *satisfies the condition* (3.2.7.4)*. Additionally* $(R_{0,h}=0$ *and* $R_{2,h}=0)\Rightarrow(R^{s'}_{0,h}=0$ *and* $R^{s'}_{2,h}=0)$.
Finally the formal adjoint $B'^{,s'}_{\pm,b,V^h}$ *for the* $\tilde{\mathscr{W}}^{0,s'}(X^h;\mathscr{E}^h_\pm)$ *scalar product, according to Definition 3.2.3 satisfies*

$$(W^2_\theta)^{s'/2}B_{\pm,b,V^h}(W^2_\theta)^{-s'/2} = (P_{\pm,b})'^{,0}+(R^{s'}_{0,h})'+(R^{s'}_{2,h})'+\frac{1}{b}(R^{s'}_{1,\perp,h})' \tag{3.2.7.5}$$

$$\text{with}\qquad (P_{\pm,b})' = \frac{1}{b^2}\alpha_{\pm,g^h}\pm\frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}\quad:\quad L^2(X^h,dqdp;\mathscr{E}^h_\pm)\longrightarrow\mathscr{S}'(X^h;\mathscr{E}^h_\pm)\,. \tag{3.2.7.6}$$

Proof. The invariance of (3.2.7.3) actually comes from the continuous imbeddings $\tilde{\mathscr{W}}^{i,0} \subset L^2 \subset \tilde{\mathscr{W}}^{-i,0}$ for $i = 0,1,2$.
The invariance of (3.2.7.4) is due to the commutation of $(W_\theta^2)^{s'/2}$ with $\pi_{0,\pm} = 1_{\{0\}}(\alpha_{\pm,g^h})$: Actually W_θ^2 strongly commutes with $\mathscr{O}_{g^h}$ and preserves the vertical degree. It therefore commutes with $\alpha_{\pm,g^h}$ and with any functions of $\alpha_{\pm,g^h}$.
For the last property it suffices to check that $A_{\mathscr{Y}} = \left[(W_\theta^2)^{s'/2}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}(W_\theta^2)^{-s'/2} - \nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}\right]$ satisfies the two conditions (3.2.7.3) for $i = 1$ and (3.2.7.4).
The estimate

$$\left\| \left[(W_\theta^2)^{s'/2}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}(W_\theta^2)^{-s'/2} - \nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}\right] \right\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)} \leq C_{s'} \tag{3.2.7.7}$$

was proved in [NSW]-Proposition 3.8, where the uniform constant C_s' with respect to $h \in]0,1]$ is made possible by the uniform control of the derivatives of g^h and $(g^h)^{-1}$ recalled in Proposition 3.2.7. By duality and because $\mathscr{A}_{\mathscr{Y}}^* + \mathscr{A}_{\mathscr{Y}} \in \mathscr{L}(L^2)$ we deduce as well

$$\left\| \left[(W_\theta^2)^{s'/2}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}(W_\theta^2)^{-s'/2} - \nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}\right] \right\|_{\mathscr{L}(L^2;\tilde{\mathscr{W}}^{-1,0})} \leq C_{s'}.$$

Because $\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}} \in \mathrm{OpS}^{3/2}_\Psi(Q^h;\mathscr{E}^h_\pm)$, while $(W_\theta^2)^{s/2} \in \mathrm{OpS}^s_\Psi(Q^h;\mathscr{E}^h_\pm)$ with a scalar principal symbol, we deduce that $A_{\mathscr{Y}} \in \mathrm{OpS}^{1/2}_\Psi(Q^h;\mathscr{E}^h_\pm)$, with local seminorm of symbols uniformly bounded with respect to $h \in]0,1]$. Therefore $(W_\theta^2)^s A_{\mathscr{Y}_{g^h}}(W_\theta^2)^{-s} - A_{\mathscr{Y}_{g^h}} \in \mathrm{OpS}^{-1/2}_\Psi(Q^h;\mathrm{End}(\mathscr{E}^h_\pm))$ is a bounded operator in $L^2(X^h,dqdp;\mathscr{E}^h_\pm)$ with norm uniformly bounded with respect to $h \in]0,1]$.
The condition (3.2.7.4) is due to the identity $\pi_{0,\pm}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}\pi_{0,\pm} = 0$ as continuous operator on $\mathscr{S}'(X^h;\mathscr{E}^h_\pm)$.
For the formal adjoint $B'^{,s'}_{\pm,b,V^h}$, it suffices to apply (3.2.7.5)(3.2.7.6) after noticing that $B_{\pm,b,V^h}$ is continuous as an operator $\mathscr{S}(X^h;\mathscr{E}^h_\pm) \to \mathscr{S}(X^h;\mathscr{E}^h_\pm)$ and $\mathscr{S}'(X^h;\mathscr{E}^h_\pm) \to \mathscr{S}'(X^h;\mathscr{E}^h_\pm)$. The expression of $P'_{\pm,b}$ comes from the fact that $\alpha_{\pm,g^h}$ is self-adjoint and $\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}}$ is anti-adjoint because the connection $\nabla^{\mathscr{E}_\pm,h}$ is unitary. □

Proposition 3.2.9. *There exists a constant $C_g \geq 1$ determined by the metric g and, for any $s \in \mathbb{R}$, a constant $C_{g,V,s} \geq 1$ determined by $s \in \mathbb{R}$, the metric g and the potential function $V \in \mathscr{C}^\infty(Q;\mathbb{R})$ such that the following properties hold when $0 < b < \frac{1}{C_g}$ and $\kappa_s \geq C_{g,V,s}$:*
The operator $\frac{\kappa_s}{b^2} + B_{\pm,b,V^h}$, as an unbounded operator in $\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm)$, is essentially maximal accretive on $\mathscr{C}_0^\infty(X^h;\mathscr{E}^h_\pm)$ (or on $\mathscr{S}(X^h;\mathscr{E}^h_\pm)$).
If $\overline{B_{\pm,b,V^h}}^s$ denotes its closure according to Definition 3.2.3, the inequalities

$$\mathbb{R}\mathrm{e}\langle u\,,(\frac{\kappa_s}{b^2} + \overline{B_{\pm,b,V^h}}^s)u\rangle_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{16b^2}\left[\|u\|^2_{\tilde{\mathscr{W}}^{1,s}} + \kappa_s\|u\|^2_{\tilde{\mathscr{W}}^{0,s}}\right], \tag{3.2.7.8}$$

and

$$\begin{aligned}\left\|\left(\overline{B_{\pm,b,V^h}}^s - \frac{i\lambda}{b}\right)u\right\|_{\tilde{\mathscr{W}}^{0,s}} + \frac{2\kappa_s}{b^2}\|u\|_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{C_g}\Bigg(&\left\|\frac{\mathscr{O}_{g^h}}{b^2}u\right\|_{\tilde{\mathscr{W}}^{0,s}} + \left\|\frac{1}{b}\left(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}_{g^h}} - i\lambda\right)u\right\|_{\tilde{\mathscr{W}}^{0,s}}\\ &+ \frac{1}{b^{4/3}}\left[||u||_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}} + \left\|\left(\frac{|\lambda|}{\langle p\rangle_q}\right)^{2/3}u\right\|_{\tilde{\mathscr{W}}^{0,s}}\right] + \left(\frac{|\lambda|^{1/2}}{b^{3/2}}\right)\|u\|_{\tilde{\mathscr{W}}^{0,s}}\Bigg)\end{aligned} \tag{3.2.7.9}$$

hold for every $u \in D(\overline{B_{\pm,b,V^h}}^s)$ and every $\lambda \in \mathbb{R}$.
The formal adjoint $B'^{,s}_{\pm,b,V^h}$ and adjoint $B^{*,s}_{\pm,b,V^h}$ of Definition 3.2.3 satisfy $\overline{(B'^{,s}_{\pm,b,V^h}\big|_{\mathscr{S}(X^h;\mathscr{E}^h_\pm)})}^s = B^{*,s}_{\pm,b,h}$ while the formal adjoint $B'^{,s}_{\pm,b,V^h} = (W_\theta^2)^{-s}B'_{\pm,b,V^h}(W_\theta^2)^s$ satisfies (3.2.7.5)(3.2.7.6).

Remark 3.2.10. It will be checked after Proposition 3.3.1 that $C_{g,V,s}+\overline{B_{\pm,b,V^h}}^{s}$ is maximal accretive with

$$\forall u\in D(\overline{B_{\pm,b,V^h}}^{s}),\quad \mathbb{R}\mathrm{e}\,\langle u\,,B_{\pm,b,V^h}u\rangle_{\tilde{\mathcal{W}}^{0,s}}\geq 0\,.$$

Proof. By Proposition 3.2.8 the problem is reduced to the case $s=0$ for the operator

$$P_{\pm,b}+R^s_{0,h}+R^s_{2,h}+\frac{1}{b}R^s_{1,\perp,h}=\frac{1}{b^2}\mathcal{O}_{g^h}\mp\nabla^{\mathcal{E}_\pm,h}_{\mathcal{Y}_{g^h}}+M_{0,s}(b,h)+M_{1,s}(b,h)+R_{2,h}$$

with

$$\begin{aligned}
M_{0,s}(b,h)&=\frac{\pm(2\hat{f}_{i,g^h}\mathbf{i}_{\hat{f}^i_{g^h}}-d)+d}{2b^2}+R^s_{0,h}+R^s_{2,h}-R_{2,h}\quad,\quad \|M_{0,s}(b,h)\|_{\mathcal{L}(L^2;L^2)}\leq\frac{\nu_{0,s}}{b^2}\\
M_{1,s}(b,h)&=\frac{1}{b}R^s_{1,\perp,h}\quad,\quad \|M_1(b,h)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};L^2)}\leq\frac{\nu_{1,s}}{bh}\leq\frac{C_g+8\nu_{0,s}}{16b^2}(1+b^2),\\
R_{2,h}&=\mp\langle R^{TQ}_{g^h}(p,f_{i,g^h})p,f_{j,g^h}\rangle(f^i_{g^h}-\hat{f}_{i,g^h})\mathbf{i}_{f_{j,g^h}+\hat{f}^{j,g^h}}\quad,\quad \|R_2\|_{\mathcal{L}(\tilde{\mathcal{W}}^{2,0};L^2)}\leq\nu_g\,.
\end{aligned}$$

Actually

$$R^s_{2,h}-R_{2,h}=(W^2_\theta)^{s/2}R_2(W^2_\theta)^{-s/2}-R_{2,h}\in\mathrm{OpS}^{1-1}_{\Psi}(Q^h;\mathrm{End}(\mathcal{E}^h_\pm))\subset\mathcal{L}(L^2;L^2)$$

and the above inequalities hold true for suitably well chosen s-dependent values of $\nu_{0,s}>0$ and $\nu_{1,s}>0$ when $0<b\leq 1$, uniformly with respect to $h\in]0,1]$. The last result concerned with the equality of the minimal and maximal extension of the formal adjoint results from the essential maximal accretivity, as it is recalled after Definition 3.2.3.

When the final term $R_{2,h}$ is replaced by 0, the result is actually given by Proposition 7.2 in [NSW] with the following changes:

- the lower bound $\frac{1}{8b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_s\|u\|^2_{L^2}\right]$ in (3.2.7.8)
- the coefficient $\frac{1}{4C_g(1+b)^7}$ in the right-hand side of (3.2.7.9), under the sufficient condition $\kappa_s\geq(C_g+16\nu_{0,s})(1+b^2)$;
- the term $\left(\frac{|\lambda|^{1/2}}{b^{3/2}}\right)\|u\|_{L^2}$ in the right-hand side of (3.2.7.9) which is not written in [NSW].

For the last term of $\left(\frac{|\lambda|^{1/2}}{b^{3/2}}\right)\|u\|_{L^2}$ in (3.2.7.9), it suffices to notice the interpolation inequality

$$\begin{aligned}
\left(\frac{|\lambda|^{1/2}}{b^{3/2}}\right)\|u\|_{L^2}&\leq 3\Big[\frac{1}{b^{4/3}}\|\frac{|\lambda|}{\langle p\rangle_q}u\|_{L^2}+\frac{1}{b^2}\|\langle p\rangle^2_q u\|_{L^2}\Big]\\
&\leq 12\Big[\frac{1}{b^{4/3}}\|\frac{|\lambda|}{\langle p\rangle_q}u\|_{L^2}+\frac{1}{b^2}\|\mathcal{O}_{g^h}u\|_{L^2}\Big]\,.
\end{aligned}$$

Because $0<b\leq 1$, it suffices to replace the constant $C_{g,old}$ of Proposition 7.2 in [NSW] by $C_g=C_{g,new}2^9\times 13C_{g,old}$ and then to choose $C_{g,s}=2(C_{g,new}+16\nu_{0,s})$.

Let us consider now the case with the final term $R_{2,h}=\mp\langle R^{TQ}_{g^h}(p,f_{i,g^h})p,f_{j,g^h}\rangle(f^i_{g^h}-\hat{f}_{i,g^h})\mathbf{i}_{f_{j,g^h}+\hat{f}^j_{g^h}}$.

We set $A_s(b,h)=P_{\pm,b}+M_{0,s}(b,h)+M_{1,s}(b,h)$ and we now consider $A_s(b,h)+R_{2,h}$ by perturbative arguments. The accretivity of $A_s(b,h)+R_{2,h}$ is due to

$$\left|\mathbb{R}\mathrm{e}\langle u\,,R_{2,h}u\rangle_{L^2}\right|\leq C'_g\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}$$

while we know

$$\mathbb{R}\mathrm{e}\langle u\,,A_s(b,h)u\rangle_{L^2}\geq\frac{1}{8b^2}\left[\|u\|^2_{\tilde{\mathcal{W}}^{1,0}}+\kappa_s\|u\|^2_{L^2}\right].$$

It thus suffices to assume $0 < b \leq \frac{1}{4\sqrt{C'_g}}$. The second inequality (3.2.7.9) for $A_s(b,h)$ implies

$$\forall u \in D(\overline{A_s(b,h)}), \quad \|\overline{A_s(b,h)}u\|_{L^2} + \frac{2\kappa_s}{b^2}\|u\|_{L^2} \geq \frac{1}{C_g b^2}\|\mathcal{O}_{g^h}u\|_{L^2} \geq \frac{1}{C_g \nu_g b^2}\|R_{2,h}u\|_{L^2}.$$

Therefore $R_{2,h}$ is a relatively bounded perturbation of $\overline{A_s(b,h)}$ with relative bound $C_g\nu_g b^2 \leq 1/4 < 1$ provided that $0 < b \leq \frac{1}{2\sqrt{C_g\nu_g}}$. By [ReSi]-Theorem X.50, $\overline{A_s(b,h)+R_{2,h}}$ is maximal accretive with the same domain as $\overline{A_s(b,h)}$. This relative boundedness also implies

$$\|(\overline{B_{\pm,b,V^h}} - i\lambda)u\|_{L^2} + \frac{2\kappa_s}{b^2}\|u\|_{L^2} \geq \frac{3}{4}\left[\|(\overline{A_s(b,h)} - i\lambda)u\|_{L^2} + \frac{2\kappa_s}{b^2}\|u\|_{L^2}\right]$$

and the subelliptic estimate (3.2.7.9), with the coefficient $\frac{3}{4C_g}$ in the right-hand side follows.
We end the proof by adjusting a new value of C_g according to $C_{g,new} = \max(4/3C_g, \sqrt{4C'_g}, 2\sqrt{C_g\nu_g})$.

□

Let us recall a few consequences of Proposition 3.2.9:

1. For any $s \in \mathbb{R}$ and $z \in \mathbb{C}$, the compact imbedding: $\tilde{\mathcal{W}}^{0,s+2/3}(X^h;\mathcal{E}^h_\pm) \subset \tilde{\mathcal{W}}^{0,s}(X^h;\mathcal{E}^h_\pm)$ implies $\overline{B_{\pm,b,V^h}}^s - z : D(\overline{B_{\pm,b,V^h}}^s) \to \tilde{\mathcal{W}}^s(X^h;\mathcal{E}^h_\pm)$ is a Fredholm operator with index 0. Therefore the spectrum of $\overline{B_{\pm,b,V^h}}^s$ is discrete.
2. By a bootstrap argument when $z \notin \mathrm{Spec}(\overline{B_{\pm,b,V^h}}^s)$ the resolvent $(\overline{B_{\pm,b,V^h}}^s - z)^{-1}$ sends continuously $\mathcal{S}(X^h;\mathcal{E}^h_\pm)$ to $\mathcal{S}(X^h;\mathcal{E}^h_\pm)$ and the same holds for $(B^{*,s}_{\pm,b,V^h} - z)^{-1}$. Hence for two different $s,s' \in \mathbb{R}$ the resolvent $(\overline{B_{\pm,b,V^h}}^s - z)^{-1}$ and $(\overline{B_{\pm,b,V^h}}^{s'} - z)^{-1}$ coincide as $\mathcal{L}(\mathcal{S}(X^h;\mathcal{E}^h_\pm);\mathcal{S}'(X^h;\mathcal{E}^h_\pm))$-valued meromophic functions and $\mathrm{Spec}(\overline{B_{\pm,b,V^h}}^s)$ does not depend on $s \in \mathbb{R}$ as well.
3. The subelliptic estimate (3.2.7.9) ensures that $\overline{B_{\pm,b,V^h}}^s$ is cuspidal according to the terminology of [Nie] (see also [HerNi][HeNi][EcHa][BiLe]) and the integral representation

$$e^{-t\overline{B_{\pm,b,V^h}}^s} = \frac{1}{2i\pi}\int_{\Gamma_b} e^{-tz}(z - \overline{B_{\pm,b,h}}^s)^{-1}\,dz$$

   is a convergent integral for $t > 0$ when

$$\Gamma_b = \left\{z \in \mathbb{C}, \mathbb{R}\mathrm{e}\, z = \frac{1}{C_b}\langle \mathrm{Im}\, z\rangle^{1/2} - C_b\right\}$$

   and $e^{-t\overline{B_{\pm,b,V^h}}^s} : \mathcal{S}'(X^h;\mathcal{E}^h_\pm) \mapsto \mathcal{S}(X^h;\mathcal{E}^h_\pm)$.
   This implies that the poles of the resolvent $(z - B_{\pm,b,V^h})^{-1}$ are continuous finite rank operators from $\mathcal{S}'(X^h;\mathcal{E}^h_\pm)$ to $\mathcal{S}(X^h;\mathcal{E}^h_\pm)$.

4. Changing the contour Γ_b above allows to isolate the main contribution to $e^{-t\overline{B_{\pm,b,V^h}}^s}$ associated with eigenvalues with small real part from the others with exponentially smaller remainder as $t \to +\infty$.
5. With the scaling and Proposition 3.2.7 all these functional properties can be transferred to the operator $B_{\pm,b,\frac{V}{h}}$ associated with $(Q,g,\frac{V}{h},b)$ after replacing the condition $0 < b \leq \frac{1}{C_g}$ by $0 < \frac{b}{h} \leq \frac{1}{C_g}$, the spaces $\tilde{\mathcal{W}}^{s_1,s_2}(X^h;\mathcal{E}^h_\pm)$ by the spaces $\tilde{\mathcal{W}}^{s_1,s_2}_h(X;\mathcal{E}_\pm)$ according to Definition 3.2.6 and by multiplying the spectral parameter by $\frac{1}{h^2}$ or the time by h^2.

3.3 Improved lower bounds for modified operators

In this whole section we work with the rescaled Bismut Laplacian $B_{\pm,b,V^h}$ associated with the scaled data (Q^h, g^h, V^h, b) and the Sobolev spaces $\tilde{\mathcal{W}}^{s_1,s_2}(X^h;\mathcal{E}^h_\pm) = \tilde{\mathcal{W}}^{s_1,s_2}_1(X^h;\mathcal{E}^h_\pm)$. Although the connection, the vector field $\mathcal{Y}$, the terms $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$, and some other related quantities depend on h or the metric g^h, we will drop the corresponding subscript notations for the sake of simplicity. This is especially relevant owing to the uniform estimates Proposition 3.2.7 and of Proposition 3.2.9. For further comparisons, we keep the memory of the h-parameter only via the notations V^h, Q^h, X^h, $\mathcal{E}^h_\pm$ and $\nabla^{\mathcal{E}_\pm,h}$.
For the accurate spectral asymptotic analysis we need subelliptic estimates for the operator $\overline{B_{\pm,b,V^h}}^s$ itself without adding the constant $\frac{\kappa_s}{b^2}$ in order to study the spectrum around 0. Because $\alpha_\pm$ and possibly $\overline{B_{\pm,b,V^h}}^s$ have a non trivial kernel, resolvent estimates must be given for operators modified in such a way that the singularity of the resolvent at $z=0$ is removed with a good control as the parameter b tends to 0 (uniform with respect to $h\in]0,1]$). The first modification consists in adding $A^2\pi_{0,\pm}$ with $A=A(b)$ suitably chosen according to b, the second modification consists in looking at $\pi_{\perp,\pm}\overline{B_{\pm,b,V^h}}^s\pi_{\perp,\pm}$ with $\pi_{\perp,\pm}=1-\pi_{0,\pm}$. Finally the third one consists in adding $A^2\pi_{0,\pm}\chi\big(\frac{2W_\theta^2}{(LA)^2}\big)\pi_{0,\pm}$ with $\chi\in\mathcal{C}^\infty_0(\mathbb{R};[0,1])$ instead of $A^2\pi_{0,\pm}$.

3.3.1 The first modified operator $B_{\pm,b,V^h}+A^2\pi_{0,\pm}$

The main result of this paragraph is about a subelliptic estimate for $B_{\pm,b,V^h}+A^2\pi_{0,\pm}$ without adding a remainder term $\frac{\kappa_{b,h}}{b^2}$ and where the lower bound has coefficients which can be fixed large, independently of $b\to 0^+$. With this aim, the maximal subelliptic exponent 2/3 is replaced by the lower value 2/9 as a result of interpolation.

Proposition 3.3.1. *There exist two constants $C, C_s\geq 1$, which are respectively uniform and s-dependent, $s\in\mathbb{R}$, such that the condition $C_s\max(Ab,b,A^{-1})\leq 1$ implies that $\overline{B_{\pm,b,V^h}+A^2\pi_{0,\pm}}^s$ with $D(\overline{B_{\pm,b,V^h}+A^2\pi_{0,\pm}-\frac{A^2}{2}}^s)=D(\overline{B_{\pm,b,V^h}}^s)\subset\tilde{\mathcal{W}}^{0,s}(X^h;\mathcal{E}^h_\pm)$ is maximal accretive with*

$$\begin{aligned} C\left\|(B_{\pm,b,V^h}+A^2\pi_{0,\pm}-z)u\right\|_{\tilde{\mathcal{W}}^{0,s}} \quad\geq\quad & A^2\|u\|_{\tilde{\mathcal{W}}^{0,s}}+A^2\|\mathcal{O}u\|_{\tilde{\mathcal{W}}^{0,s}}+A^2b\left\|(\nabla^{\mathcal{E}_\pm,h}_{\mathcal{Y}}-i\mathrm{Im}\,z)u\right\|_{\tilde{\mathcal{W}}^{0,s}}\\ &+bA^2|\mathrm{Im}\,z|^{1/2}\|u\|_{\tilde{\mathcal{W}}^{0,s}}+A^2b^{\frac{2}{3}}\|u\|_{\tilde{\mathcal{W}}^{0,s+\frac{2}{3}}}+\frac{A^{\frac{8}{5}}}{\langle b|\mathrm{Im}\,z|^{1/2}\rangle^{\frac{4}{5}}}\|u\|_{\tilde{\mathcal{W}}^{0,s+\frac{2}{5}}}\\ &+A^{\frac{16}{9}}\|u\|_{\tilde{\mathcal{W}}^{0,s+\frac{2}{9}}} \end{aligned}\qquad(3.3.1.1)$$

for all $u\in\mathcal{S}(X^h;\mathcal{E}^h_\pm)$ and all $z\in\mathbb{C}$ such that $\mathrm{Re}z\leq\frac{A^2}{2}$, and where we recall that the operators $\mathcal{O},\mathcal{Y}$ and the Sobolev spaces $\tilde{W}^{s_1,s_2}$ depend on the metric g^h.

Remark 3.3.2. The constants $C, C_s\geq 1$ in Proposition 3.3.1 are obtained after several steps, and at every step the values of the constants C, C_s are suitably tuned. We will often conclude such an intermediate analysis at step n with the sentence "Choose $(C_{new}, C_{R,s,new})$ = Expression of $(C_{old}, C_{R,s,old})$", where $_{old}$ refers to the values obtained at step $n-1$ and $_{new}$ to the conclusion for the step n.

Before starting a proof let us verify the maximal accretivity announced in Remark 3.2.10.

Corollary 3.3.3. *For all $s\in\mathbb{R}$ there exists $C_s\geq 1$ such that $C_s+\overline{B_{\pm,b,V^h}}^s$ is maximal accretive when $C_sb\leq 1$ and $h\in]0,1]$:*

$$\forall u\in D(\overline{B_{\pm,b,V^h}}^s),\quad \mathbb{R}\mathrm{e}\,\langle u,(C_s+B_{\pm,b,V^h})u\rangle_{\tilde{\mathcal{W}}^{0,s}}\geq 0\,.$$

Proof. It suffices to notice

$$\mathbb{R}\mathrm{e}\,\langle u\,,(A^2+B_{\pm,b,V^h})u\rangle_{\tilde{\mathcal{W}}^{0,s}}\geq \mathbb{R}\mathrm{e}\,\langle u\,,(B_{\pm,b,V^h}+A^2\pi_{0,\pm})u\rangle_{\tilde{\mathcal{W}}^{0,s}}\geq 0$$

when $C_{s,old}\max(Ab,b,\frac{1}{A})\leq 1$, to choose $A=C_{s,old}$, $b\leq\frac{1}{C_{s,old}^2}$ and to take $C_{s,new}=C_{s,old}^2$. □

We start the proof of Proposition 3.3.1 with the simpler operator

$$P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm} \tag{3.3.1.2}$$

where A is a positive number and $P_{\pm,b}$ and $R_{1,\perp,h}$ are defined as in (3.2.7.1). Remember that the conjugated operator $(W_\theta^2)^{\frac{s}{2}}[P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}](W_\theta^2)^{-\frac{s}{2}}$ with $(W_\theta^2)^{\frac{s}{2}}=(W^2_{\theta,g^h})^{\frac{s}{2}}$, takes the same form $P_{\pm,b}+\frac{1}{b}\tilde{R}_{1,s,\perp,h}+A^2\pi_{0,\pm}$ with a new s-dependent remainder term $\frac{1}{b}R_{1,\perp,h}$ with the same uniform estimates. After this we will consider

$$B_{\pm,b,V^h}=[P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}]+R_{0,h}+R_{2,h}$$

by a simple perturbative argument.

We use the notations $u_0=\pi_{0,\pm}(u)$ and $u_\perp=\pi_{\perp,\pm}u=u-u_0$ for $u\in\mathcal{S}'(X^h;\mathcal{E}^h_\pm)$. The following properties are obvious

— The equality $\mathcal{O}u_0=\mathcal{O}_{g^h}u_0=\frac{d}{2}u_0$ holds and therefore

$$\|u_0\|^2_{\tilde{\mathcal{W}}^{1,0}}=\frac{d}{2}\|u_0\|^2_{L^2}\geq\frac{1}{2}\|u_0\|^2_{L^2}.$$

— With $\alpha_\pm=\mathcal{O}\pm(N_v-d/2)$ we have $\mathcal{O}+d/2\geq\alpha_\pm\geq\mathcal{O}-d/2$ and

$$\|u_\perp\|^2_{\tilde{\mathcal{W}}^{1,0}}+\frac{d}{2}\|u_\perp\|^2_{L^2}\geq\langle u_\perp\,,\alpha_\pm u_\perp\rangle\geq\|u_\perp\|^2_{\tilde{\mathcal{W}}^{1,0}}-\frac{d}{2}\|u_\perp\|^2_{L^2} \tag{3.3.1.3}$$

while we know $\quad\langle u_\perp\,,\alpha_\pm u_\perp\rangle\geq\|u_\perp\|^2_{L^2}\,.$ (3.3.1.4)

We begin with the following integration by parts.

Proposition 3.3.4. *For all $A,b\in\mathbb{R}_+^*$, the inequality*

$$\mathrm{Re}\langle(P_{\pm,b}+A^2\pi_{0,\pm})u,u\rangle_{L^2}\geq\frac{2}{(d+2)b^2}\|u_\perp\|^2_{\tilde{\mathcal{W}}^{1,0}}+A^2\|u_0\|_{L^2} \tag{3.3.1.5}$$

holds for all $u\in\mathcal{S}(X^h;\mathcal{E}^h_\pm)$.

Proof. Just use $\mathrm{Re}\langle P_{\pm,b}u\,,u\rangle=\frac{1}{b^2}\langle u_\perp\,,\alpha_\pm u_\perp\rangle$ and (3.3.1.3)(3.3.1.4). □

Proposition 3.3.5. *There is a positive constant $c_R>0$, such that for all $\varepsilon>0$, the inequality*

$$c_R|\mathrm{Re}\left\langle R_{1,\perp,h}u,u\right\rangle|\leq\varepsilon\|u_0\|^2_{L^2}+(1+\frac{1}{\varepsilon})\|u_\perp\|^2_{\tilde{\mathcal{W}}^{1,0}}$$

holds for all $u\in\mathcal{S}(X^h;\mathcal{E}^h_\pm)$.

Proof. From conditions (3.2.7.3), (3.2.7.4) we deduce

$$R_{1,\perp,h}=\pi_{0,\pm}R'_{1,h}\pi_{\perp,\pm}+\pi_{\perp,\pm}R''_{1,h}\pi_{0,\pm}+\pi_{\perp,\pm}R'''_{1,h}\pi_{\perp,\pm},$$

with $R'_{1,h}, R''_{1,h}, R'''_{1,h} \in \mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)$. The triangular and Cauchy-Schwarz inequalities yield

$$\begin{aligned}|\mathrm{Re}\langle R_{1,\perp,h}u,u\rangle| &\leq |\langle R'_{1,h}u_\perp,u_0\rangle|+|\langle R''_{1,h}u_0,u_\perp\rangle|+|\langle R'''_{1,h}u_\perp,u_\perp\rangle| \\ &\leq C_R\left(\|u_\perp\|_{\tilde{\mathscr{W}}^{1,0}}\|u_0\|_{L^2}+\|u_0\|_{L^2}\|u_\perp\|_{L^2}+\|u_\perp\|_{\tilde{\mathscr{W}}^{1,0}}\|u_\perp\|_{L^2}\right),\end{aligned}$$

where $0<\frac{1}{2c_R}=C_R=\sup_{h\in]0,1]}\max(\|R'_{1,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)},\sqrt{\frac{d}{2}}\|R''_{1,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)},\|R'''_{1,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)})<\infty$ by our hypothesis on $R_{1,\perp,h}$.

$$c_R|\mathrm{Re}\langle R_{1,\perp,h}u,u\rangle|\leq\|u_\perp\|_{\tilde{\mathscr{W}}^{1,0}}\|u_0\|_{L^2}+\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,0}}.$$

The result follows when we apply the inequality

$$\forall a,b,\varepsilon\in\mathbb{R}_+^*,\quad 2ab\leq\varepsilon a^2+\frac{1}{\varepsilon}b^2,$$

with $a=\|u_0\|_{L^2}$ and $b=\|u_\perp\|_{\tilde{\mathscr{W}}^{1,0}}$. □

The following proposition is a consequence of Proposition 3.3.4 and Proposition 3.3.5.

Proposition 3.3.6. *There is a constant $C_{R,s}\geq 1$, which depends $s\in\mathbb{R}$, such that the condition $\max(Ab,b,\frac{1}{A})\leq\frac{1}{C_{R,s}}$ implies the inequalities*

$$\mathrm{Re}\langle(P_{\pm,b}+\tfrac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm})u,u\rangle_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{(d+2)b^2}\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,s}}+\frac{3A^2}{4}\|u_0\|^2_{\tilde{\mathscr{W}}^{0,s}} \quad (3.3.1.6)$$

$$\left\|(P_{\pm,b}+\tfrac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-i\lambda)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{3A^2}{4}\|u\|_{\tilde{\mathscr{W}}^{0,s}}, \quad (3.3.1.7)$$

and $$\left\|(P_{\pm,b}+\tfrac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-i\lambda)u\right\|^2_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{3A^2}{4(d+2)b^2}\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,s}}+\frac{9A^4}{16}\|u_0\|^2_{\tilde{\mathscr{W}}^{0,s}} \quad (3.3.1.8)$$

for all $u\in\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ and all $\lambda\in\mathbb{R}$. Moreover under the above condtion, $(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm})$ is essentially maximal accretive on $\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ in $\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm)$.

Proof. We begin with the case $s=0$, Proposition 3.3.5 gives

$$\mathrm{Re}\langle(P_{\pm,b}+A^2\pi_{0,\pm}+\frac{1}{b}R_{1,\perp,h})u,u\rangle_{L^2}\geq\left(\frac{2}{(d+2)b^2}-\frac{1}{bc_R}(1+\frac{1}{\varepsilon})\right)\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,0}}+\left(A^2-\frac{\varepsilon}{bc_R}\right)\|u_0\|^2_{L^2},$$

for all $\varepsilon>0$. Choosing $\varepsilon=Ab\sqrt{d+2}$ and the sufficient conditions

$$b\leq\frac{c_R}{2(d+2)}\quad\text{and}\quad\frac{4\sqrt{d+2}}{c_R}\leq A$$

imply

$$\Big(\frac{2}{(d+2)b^2}-\underbrace{\frac{1}{bc_R}}_{\leq\frac{1}{2(d+2)b^2}}-\underbrace{\frac{1}{bc_R\varepsilon}}_{\leq\frac{1}{2(d+2)b^2}}\Big)\geq\frac{1}{(d+2)b^2}\quad\text{and}\quad\Big(A^2-\frac{\varepsilon}{bc_R}\Big)\geq\frac{3A^2}{4}.$$

This proves (3.3.1.6) under the condition $\max(Ab,b,\frac{1}{A})\leq\frac{1}{C_{R,0}}$, with $C_{R,0}=\max(\frac{2(d+2)}{c_R},\sqrt{d+2},\frac{4\sqrt{d+2}}{c_R})$.

With $\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,0}}\geq\frac{d}{2}\|u_\perp\|^2_{L^2}$ and

$$\mathrm{Re}\langle(P_{\pm,b}+A^2\pi_{0,\pm}+\frac{1}{b}R_{1,\perp,h})u,u\rangle_{L^2}=\mathrm{Re}\langle(P_{\pm,b}+A^2\pi_{0,\pm}+\frac{1}{b}R_{1,\perp,h}-i\lambda)u,u\rangle_{L^2}$$

the Cauchy-Schwarz inequality combined with (3.3.1.6) gives

$$\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-i\lambda)u\|_{L^2}\|u\|_{L^2}\geq\frac{d}{2(d+2)b^2}\|u_\perp\|_{L^2}^2+\frac{3A^2}{4}\|u_0\|_{L^2}^2\geq\frac{3A^2}{4}\|u\|_{L^2}^2 \quad (3.3.1.9)$$

as soon as $\frac{d}{(d+2)b^2}\geq A^2$, which is implied by $\sqrt{d+2}Ab\leq C_{R,0}Ab\leq 1$. This yields (3.3.1.7). The inequality (3.3.1.8) is a consequence of (3.3.1.7) and (3.3.1.9).

For the maximal accretivity property, the decomposition

$$P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}=\left[\frac{C'}{b^2}+\frac{1}{b^2}\mathscr{O}\mp\frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}+\frac{1}{b}R_{1,\perp,h}\right]+\left[A^2\pi_0-\frac{C'}{b^2}\pm\frac{1}{b^2}(N_V-d/2)\right]$$

shows that $(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm})$ is a bounded perturbation of

$$\frac{C'}{b^2}+P_{\pm,b,M}=\frac{C'}{b^2}+\frac{1}{b^2}\mathscr{O}\mp\frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}+M_1$$

where $M_1=\frac{1}{b}R_{1,\perp,h}$ fulfills the assumptions of Proposition 7.2 in [NSW] when $C_{R,0}b\leq 1$. Then Proposition 7.2 in [NSW] says that $\frac{C'}{b^2}+P_{\pm,b,M}$ is essentially maximal accretive on $\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ for $C'>0$ chosen large enough.

Finally, the case with a general $s\in\mathbb{R}$ amounts to the case $s=0$ owing to

$$(W_\theta^2)^{\frac{s}{2}}(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm})(W_\theta^2)^{-\frac{s}{2}}=P_{\pm,b}+\frac{1}{b}R^s_{1,\perp,h}+A^2\pi_{0,\pm}.$$

□

Below we give a first global subelliptic estimate without remainder for $P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_0$.

Proposition 3.3.7. *There exist two constants $C,C_{R,s}\geq 1$, which are respectively uniform and s-dependent, $s\in\mathbb{R}$, such that the inequality*

$$C\left\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-z)u\right\|_{\tilde{\mathscr{W}}^{0,s}}\geq A^2\|u\|_{\tilde{\mathscr{W}}^{0,s}}+A^2\|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}}+A^2b\left\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}-i\mathrm{Im}\,z)u\right\|_{\tilde{\mathscr{W}}^{0,s}}$$
$$+A^2b^{\frac{2}{3}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}}+A^2b|\mathrm{Im}z|^{1/2}\|u\|_{\tilde{\mathscr{W}}^{0,s}} \quad (3.3.1.10)$$

holds for all $u\in\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ and all $z\in\mathbb{C}$, such that $\mathrm{Re}z\leq\frac{A^2}{2}$ *as soon as* $C_{R,s}\max(Ab,b,A^{-1})\leq 1$.

Proof. Owing to the accretivity of Proposition 3.3.6, the case of a general $z\in\mathbb{C}$, $\mathrm{Re}z\leq\frac{A^2}{2}$, is reduced to the case $z=i\lambda$, $\lambda\in\mathbb{R}$. Actually the accretivity of $P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-\frac{A^2}{2}$ implies

$$\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-z)u\|_{\tilde{\mathscr{W}}^{0,s}}\geq\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-\frac{A^2}{2}-i\mathrm{Im}z)u\|_{\tilde{\mathscr{W}}^{0,s}}$$

when $\mathrm{Re}z\leq\frac{A^2}{2}$. But the inequality (3.3.1.7) also says

$$\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-\frac{A^2}{2}-i\lambda)u\|_{\tilde{\mathscr{W}}^{0,s}}\geq\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-i\lambda)u\|_{\tilde{\mathscr{W}}^{0,s}}-\frac{A^2}{2}\|u\|_{\tilde{\mathscr{W}}^{0,s}}$$
$$\geq\frac{1}{3}\|(P_{\pm,b}+\frac{1}{b}R_{1,\perp,h}+A^2\pi_{0,\pm}-i\lambda)u\|_{\tilde{\mathscr{W}}^{0,s}}.$$

So we focus on the case $z = i\lambda$, $\lambda \in \mathbb{R}$.
In the case $s = 0$ we refer again to Proposition 7.2 in [NSW]. Actually with $R_{1,\perp,h} = 0$, we set $M_0 = A^2\pi_{0,\pm} \pm \frac{1}{b^2}(N_V - d/2)$ and

$$P_{\pm,b} + A^2\pi_{0,\pm} = \frac{1}{b^2}\mathscr{O} + \frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} + \left[A^2\pi_{0,\pm} \pm \frac{1}{b^2}(N_V - d/2)\right] = P_{\pm,b,M_0}$$

where the right-hand side refers to the notation introduced in [NSW]. The operator $M_0 = A^2\pi_{0,\pm} \pm \frac{1}{b^2}(N_V - d/2)$ fulfills the assumptions of Proposition 7.2 in [NSW] with $\nu_1 = 0$ and a uniform $\nu_0 > 0$. It provides us the subelliptic estimate

$$C_1'\Big(\left\|(P_{\pm,b,M_0} - i\frac{\lambda}{b})u\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big) \geq \frac{C_0'}{b^2}\|u\|_{L^2} + \frac{1}{b^2}\|\mathscr{O}u\|_{L^2} + \frac{1}{b}\left\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - i\lambda)u\right\|_{L^2} + \frac{1}{b^{\frac{4}{3}}}\|u\|_{\tilde{\mathscr{W}}^{0,\frac{2}{3}}} + \frac{|\lambda|^{1/2}}{b^{3/2}}\|u\|_{L^2},$$

or after replacing $\frac{\lambda}{b}$ by λ,

$$C_1'\Big(\left\|(P_{\pm,b,M_0} - i\lambda u)\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big) \geq \frac{C_0'}{b^2}\|u\|_{L^2} + \frac{1}{b^2}\|\mathscr{O}u\|_{L^2} + \left\|(\frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - i\lambda)u\right\|_{L^2} + \frac{1}{b^{\frac{4}{3}}}\|u\|_{\tilde{\mathscr{W}}^{0,\frac{2}{3}}} + \frac{|\lambda|^{1/2}}{b}\|u\|_{L^2}, \tag{3.3.1.11}$$

for fixed uniform constants $C_1' \geq 1$ and $C_0' \geq 1$ when $b \leq 1$. Interpolation or the functional calculus tells us

$$\|u\|_{\tilde{\mathscr{W}}^{1,s}} \leq \|u\|^{1/2}_{\tilde{\mathscr{W}}^{0,s}}\|u\|^{1/2}_{\tilde{\mathscr{W}}^{2,s}} = \|u\|^{1/2}_{\tilde{\mathscr{W}}^{0,s}}\|\mathscr{O}u\|^{1/2}_{\tilde{\mathscr{W}}^{0,s}} \leq \delta\|u\|_{\tilde{\mathscr{W}}^{0,s}} + \delta^{-1}\|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}}. \tag{3.3.1.12}$$

Applied here with $s = 0$ and $\delta = \sqrt{C_0'}$, this implies

$$\|\frac{1}{b}R_{1,\perp,h}u\|_{L^2} \leq \frac{C_R}{b}\|u\|_{\tilde{\mathscr{W}}^{1,0}} \leq \frac{C_R}{b} \times \frac{C_1'b^2}{\sqrt{C_0'}}\Big(\left\|P_{\pm,b,M_0}u\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big)$$

and

$$C_1'\Big(\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big) \geq C_1'\left(1 - \frac{C_RC_1'b}{\sqrt{C_0'}}\right)\Big(\left\|(P_{\pm,b,M_0} - i\lambda)u\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big).$$

By assuming $C_{R,0}\max(Ab, b, \frac{1}{A}) \leq 1$ the inequality (3.3.1.7) implies

$$C_1'\left(1 + \frac{2C_0'}{b^2A^2}\right)\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u\right\|_{L^2} \geq C_1'\left(1 - \frac{C_RC_1'b}{\sqrt{C_0'}}\right)\Big(\left\|(P_{\pm,b,M_0} - i\lambda)u\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big).$$

With $b \leq \frac{\sqrt{C_0'}}{2C_RC_1'}$ after a multiplication by $2A^2b^2$ we obtain

$$\begin{aligned}C_1'\left(2A^2b^2 + 4C_0'\right)\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u\right\|_{L^2} &\geq A^2b^2C_1'\Big(\left\|(P_{\pm,b,M_0} - i\lambda)u\right\|_{L^2} + \frac{C_0'}{b^2}\|u\|_{L^2}\Big)\\ &\geq C_0'A^2\|u\|_{L^2} + A^2\|\mathscr{O}u\|_{L^2} + A^2b\left\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)u\right\|_{L^2} + A^2b^{2/3}\|u\|_{\tilde{\mathscr{W}}^{0,\frac{2}{3}}} + bA^2|\lambda|^{1/2}\|u\|_{L^2}.\end{aligned}$$

Because $A^2b^2 \leq \frac{1}{(C_{R,0})^2} \leq C_0'$ we deduce

$$\begin{aligned}6C_1'C_0'\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u\right\|_{L^2} \geq{}& A^2\|u\|_{L^2} + A^2\|\mathscr{O}u\|_{L^2} + A^2b\left\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)u\right\|_{L^2}\\ &+ A^2b^{2/3}\|u\|_{\tilde{\mathscr{W}}^{0,\frac{2}{3}}} + bA^2|\lambda|^{1/2}\|u\|_{L^2}.\end{aligned}$$

We have proved the result for $s=0$ if we take $C = 6C'_1 C'_0$, after replacing the initial value of $C_{R,0} = C_{R,0,old}$ by $C_{R,0,new} = \max(C_{R,0,old}, \frac{2C'_1 C_R}{\sqrt{C'_0}})$.
Let us now consider the case of a general $s \in \mathbb{R}$. We apply the inequality (3.3.1.11) in the case $s=0$ to the operator $P_{\pm,b} + A^2\pi_{0,\pm} + \frac{1}{b}R^s_{1,\perp,h} = (W^2_\theta)^{\frac{s}{2}}(P_{\pm,b} + A^2\pi_{0,\pm} + \frac{1}{b}R_{1,\perp,h})(W^2_\theta)^{-\frac{s}{2}}$ and the function $v = (W^2_\theta)^{\frac{s}{2}} u$, with the constant $C = 6C'_1C'_0$. Because $R^s_{1,\perp,h}$ satisfies the same estimates uniform with respect to $h \in]0,1]$ as $R_{1,\perp,h}$, there exists a constant $C_{R,s} \geq 1$ such that when $\max(Ab, b, \frac{1}{A}) \leq \frac{1}{C_{R,s}}$ we have

$$6C'_1C'_0 \left\|(P_{\pm,b} + A^2\pi_{0,\pm} + \frac{1}{b}R_{1,\perp,h} - i\lambda)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq A^2\|u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2\|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2 b\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)v\|_{L^2} + A^2 b^{2/3}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}} + A^2 b|\lambda|^{1/2}\|u\|_{\tilde{\mathscr{W}}^{0,s}}. \quad (3.3.1.13)$$

We use again (3.2.7.7) which gives the uniform bound $\|[\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}, (W^2_\theta)^{\frac{s}{2}}](W^2_\theta)^{-\frac{s}{2}}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)} < C_s$. Thus the decomposition $\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} v = (W^2_\theta)^{\frac{s}{2}}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} u + [\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}, (W^2_\theta)^{\frac{s}{2}}](W^2_\theta)^{-\frac{s}{2}} v$ entails

$$\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)u\|_{\tilde{\mathscr{W}}^{0,s}} \leq \|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)v\|_{L^2} + C_s\|u\|_{\tilde{\mathscr{W}}^{1,s}}.$$

The interpolation inequality (3.3.1.12) used with $\delta = 1$ tells us $\|u\|_{\tilde{\mathscr{W}}^{1,s}} \leq \|u\|_{\tilde{\mathscr{W}}^{0,s}} + \|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}}$ while (3.3.1.13) gives $\|u\|_{\tilde{\mathscr{W}}^{1,s}} \leq \frac{12C'_0C'_1}{A^2}\|(P_{\pm,b} + A^2\pi_{0,\pm} + \frac{1}{b}R_{1,\perp,h})u\|_{\tilde{\mathscr{W}}^{0,s}}$.
We finally obtain

$$\begin{aligned}6C'_0C'_1(1+2C_s b)\left\|(P_{\pm,b} + A^2\pi_{0,\pm} + \frac{1}{b}R_{1,\perp,h} - i\lambda)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq & A^2\|u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2\|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}} \\ & + A^2 b\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)u\|_{\tilde{\mathscr{W}}^{0,s}} \\ & + A^2 b^{2/3}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}} + A^2 b|\lambda|^{1/2}\|u\|_{\tilde{\mathscr{W}}^{0,s}}.\end{aligned}$$

It now suffices to take $C = 18C'_0C'_1$, while $C_{R,s,new} = \max(C_{R,s,old}, 2C_s)$ for the result concerned with $z = i\lambda$, and to take $C = 3 * 18C'_0C'_1$ for a general $z \in \mathbb{C}$ such that $\mathrm{Re} z \leq \frac{A^2}{2}$. □

The subelliptic estimate (3.3.1.10) is not yet satisfactory because the norm $\|u\|_{\tilde{\mathscr{W}}^{0,s+2/3}}$ appears in the right-hand side with the factor $A^2b^{2/3}$ which is too small as $b \to 0$. By possibly reducing the 2/3-gain of regularity, we seek a factor of the form A^α, $\alpha > 0$. In order to do this we write for $u \in \mathscr{S}(X^h; \mathscr{E}^h_\pm)$

$$\big(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda\big)u = f$$

where we focus again on the case $z = i\lambda$ and decompose u and the right-hand side f according to

$$u = u_0 + u_\perp = \pi_{0,\pm}u + \pi_{\perp,\pm}u \quad , \quad f = f_0 + f_\perp = \pi_{0,\pm}f + \pi_{\perp,\pm}f.$$

Lemma 3.3.8. *There is a constant $C_{R,s} \geq 1$, which depends on $s \in \mathbb{R}$, such that the inequality*

$$\|u_\perp\|_{\tilde{\mathscr{W}}^{1,s}} \leq \frac{2(d+2)b^2}{\varepsilon}\|f\|_{\tilde{\mathscr{W}}^{0,s}} + \varepsilon\|u_0\|_{\tilde{\mathscr{W}}^{0,s}}, \quad (3.3.1.14)$$

holds true for all $u \in \mathscr{S}(X^h; \mathscr{E}^h_\pm)$ and all $\varepsilon \in]0,1]$ as soon as $C_{R,s}\max(Ab, b, A^{-1}) \leq 1$.

Proof. In this case inequality (3.3.1.6) and orthogonality give

$$\frac{1}{(d+2)b^2}\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,s}} \leq |\langle f_0, u_0\rangle_{\tilde{\mathscr{W}}^{0,s}} + \langle f_\perp, u_\perp\rangle_{\tilde{\mathscr{W}}^{0,s}}| \leq \|f_0\|_{\tilde{\mathscr{W}}^{0,s}}\|u_0\|_{\tilde{\mathscr{W}}^{0,s}} + \|f_\perp\|_{\tilde{\mathscr{W}}^{0,s}}\|u_\perp\|_{\tilde{\mathscr{W}}^{0,s}}.$$

By using $\alpha\beta \le \frac{\alpha^2+\beta^2}{2}$ with $(\alpha,\beta) = (\frac{b}{\varepsilon_0}\|f_0\|_{\tilde{\mathscr{W}}^{0,s}}, \frac{\varepsilon_0}{b}\|u_0\|_{\tilde{\mathscr{W}}^{0,s}})$ and $(\alpha,\beta) = (\sqrt{\frac{2d+4}{d}}b\|f_\perp\|_{\tilde{\mathscr{W}}^{0,s}}, \sqrt{\frac{d}{2d+4}}\frac{1}{b}\|u_\perp\|_{\tilde{\mathscr{W}}^{0,s}})$ we obtain

$$\frac{1}{(d+2)b^2}\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,s}} \le \frac{b^2}{2\varepsilon_0^2}\|f_0\|^2_{\tilde{\mathscr{W}}^{0,s}} + \frac{\varepsilon_0^2}{2b^2}\|u_0\|^2_{\tilde{\mathscr{W}}^{0,s}} + \frac{(d+2)b^2}{d}\|f_\perp\|^2_{\tilde{\mathscr{W}}^{0,s}} + \frac{d}{4(d+2)b^2}\underbrace{\|u_\perp\|^2_{\tilde{\mathscr{W}}^{0,s}}}_{\le \frac{2}{d}\|u_\perp\|_{\tilde{\mathscr{W}}^{1,s}}}.$$

Multiplied by $2(d+2)b^2$, it becomes

$$\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,s}} \le \frac{(d+2)b^4}{\varepsilon_0^2}\|f_0\|^2_{\tilde{\mathscr{W}}^{0,s}} + \varepsilon_0^2(d+2)\|u_0\|^2_{\tilde{\mathscr{W}}^{0,s}} + \frac{2(d+2)^2b^4}{d}\|f_\perp\|^2_{\tilde{\mathscr{W}}^{0,s}}.$$

By choosing $\varepsilon_0 = \sqrt{\frac{1}{2(d+2)}}\varepsilon$, $\varepsilon \in]0,1] \subset]0,\sqrt{d}]$, we obtain

$$\|u_\perp\|^2_{\tilde{\mathscr{W}}^{1,s}} \le \frac{2(d+2)^2b^4}{\varepsilon^2}\|f\|^2_{\tilde{\mathscr{W}}^{0,s}} + \varepsilon^2\|u_0\|^2_{\tilde{\mathscr{W}}^{0,s}} \le \left(\frac{2(d+2)b^2}{\varepsilon}\|f\|_{\tilde{\mathscr{W}}^{0,s}} + \varepsilon\|u_0\|_{\tilde{\mathscr{W}}^{0,s}}\right)^2.$$

□

Lemma 3.3.9. *There exist two constants $\tilde{C}, C_{R,s} \ge 1$, which are respectively uniform and s-dependent, $s \in \mathbb{R}$, such that $C_{R,s}\max(Ab, b, A^{-1}) \le 1$ implies*

$$\frac{1}{\tilde{C}}\|u_0\|_{\tilde{\mathscr{W}}^{0,s+1}} \le \langle b|\lambda|^{1/2}\rangle^2(b+\frac{1}{A})\|f\|_{\tilde{\mathscr{W}}^{0,s}} + b^2\|f\|_{\tilde{\mathscr{W}}^{0,s+1}}, \tag{3.3.1.15}$$

for all $u \in \mathscr{S}(X^h, \mathscr{E}^h_\pm)$.

Proof. We start with the proof in the $s=0$ case.
Let $\pi_{i,\pm}$ denote the spectral projection on the eigenspace of the operator $\alpha_\pm$ associated with the eigenvalue $i \in \{0,1,2\}$. By projecting the equation $f = \big(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda\big)u$ on $\mathrm{Ran}(\pi_{1,\pm})$, we obtain

$$(\frac{1}{b^2} - i\lambda)u_1 \mp \frac{1}{b}\pi_1\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}(u_0+u_2) + \frac{1}{b}\pi_1 R_{1,\perp,h}u = f_1,$$

where $u_i = \pi_{i,\pm}u$ and $f_i = \pi_{i,\pm}f$ for $i \in \{0,1,2\}$. By isolating $\pi_1\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}u_0 = \nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}u_0$, it gives:

$$\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}u_0 = bf_1 - (\frac{1}{b} - ib\lambda)u_1 - \pi_1\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}u_2 - \pi_1 R_{1,\perp}u.$$

An upper bound of $\left\|\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}u_0\right\|_{L^2}$ is thus given by

$$\left\|\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}u_0\right\|_{L^2} \le b\|f_1\|_{L^2} + \frac{1}{b}\langle b|\lambda|^{1/2}\rangle^2\|u_1\|_{L^2} + \|u_2\|_{\mathscr{W}^{0,1}} + \left\|R_{1,\perp}u\right\|_{L^2}. \tag{3.3.1.16}$$

We now use the inequality (3.3.1.14) with the two regularity exponents $s=0$ and $s=1$ and with different values of $\varepsilon \in]0,1]$. It makes sense under the following constraints

$$\max(C_{R,0}, C_{R,1})\max(Ab, b, A^{-1}) \le 1.$$

We obtain

- $\|u_1\|_{L^2} \le \frac{2}{d}\|\mathscr{O}^{1/2}u_1\|_{L^2} \le 2\|u_\perp\|_{\tilde{\mathscr{W}}^{1,0}} \le \frac{4(d+2)b^2}{\varepsilon_0}\|f\|_{L^2} + 2\varepsilon_0\|u_0\|_{L^2}$;
- $\|u_2\|_{\tilde{\mathscr{W}}^{0,1}} \le 2\|u_\perp\|_{\tilde{\mathscr{W}}^{1,1}} \le \frac{4(d+2)b^2}{\varepsilon_1}\|f\|_{\tilde{\mathscr{W}}^{0,1}} + 2\varepsilon_1\|u_0\|_{\tilde{\mathscr{W}}^{0,1}}$;
- $\left\|R_{1,\perp}u\right\|_{L^2} \le C'_{R,0}\|u\|_{\tilde{\mathscr{W}}^{1,0}} \le C'_{R,0}(\|u_\perp\|_{\tilde{\mathscr{W}}^{1,0}} + \sqrt{\frac{d}{2}}\|u_0\|_{L^2}) \le C'_{R,0}(\frac{2(d+2)b^2}{\varepsilon_2}\|f\|_{L^2} + (\sqrt{\frac{d}{2}}+\varepsilon_2)\|u_0\|^2_{L^2})$;

where $C'_{R,0} = \sup_{h\in]0,1]} \|R_{1,\perp,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};L^2)} < \infty$. The upper bound (3.3.1.16) becomes

$$\left\|\nabla_{\mathscr{Y}}^{\mathscr{E}_\pm,h} u_0\right\|_{L^2} \leq \left(b + \frac{4(d+2)b}{\varepsilon_0}\langle b|\lambda|^{1/2}\rangle^2 + \frac{C'_{R,0}2(d+2)b^2}{\varepsilon_2}\right)\|f\|_{L^2}$$
$$+\left(\frac{2\varepsilon_0}{b}\langle b|\lambda|^{1/2}\rangle^2 + C'_{R,0}(\sqrt{\frac{d}{2}}+\varepsilon_2)\right)\|u_0\|_{L^2} + \frac{4(d+2)b^2}{\varepsilon_1}\|f\|_{\tilde{\mathscr{W}}^{0,1}} + 2\varepsilon_1\|u_0\|_{\tilde{\mathscr{W}}^{0,1}}\,.$$

On $\mathscr{S}(X^h;\mathscr{E}^h_\pm)\cap \operatorname{Ran}\pi_{0,\pm} = \mathscr{S}(X^h;\mathscr{E}^h_\pm)\cap\ker\alpha_\pm$, Lemma 3.2.5 provides the equivalence

$$\frac{1}{\tilde{C}_0}\|u_0\|_{\tilde{\mathscr{W}}^{0,1}} \leq \left\|\nabla_{\mathscr{Y}}^{\mathscr{E}_\pm,h}u_0\right\|_{L^2} + \|u_0\|_{L^2} \leq \tilde{C}_0\|u_0\|_{\tilde{\mathscr{W}}^{0,1}}$$

for some uniform $\tilde{C}_0 \geq 1$ so that

$$\left(\frac{1}{\tilde{C}_0} - 2\varepsilon_1\right)\|u_0\|_{\tilde{\mathscr{W}}^{0,1}} \leq \left(b + \frac{4(d+2)b}{\varepsilon_0}\langle b|\lambda|^{1/2}\rangle^2 + \frac{C'_{R,0}2(d+2)b^2}{\varepsilon_2}\right)\|f\|_{L^2}$$
$$+\left(1+\frac{2\varepsilon_0}{b}\langle b|\lambda|^{1/2}\rangle^2 + C'_{R,0}(\sqrt{\frac{d}{2}}+\varepsilon_2)\right)\|u_0\|_{L^2} + \frac{4(d+2)b^2}{\varepsilon_1}\|f\|_{\tilde{\mathscr{W}}^{0,1}}\,.$$

The integration by parts inequality (3.3.1.7) of Proposition 3.3.6 says

$$\|u_0\|_{L^2} \leq \|u\|_{L^2} \leq \frac{2}{A^2}\|f\|_{L^2}\,.$$

We have proved that $(\frac{1}{\tilde{C}_0} - 2\varepsilon_1)\|u_0\|_{\tilde{\mathscr{W}}^{0,1}}$ is less than

$$\left(b + \frac{4(d+2)b}{\varepsilon_0}\langle b|\lambda|^{1/2}\rangle^2 + \frac{2(d+2)C'_{R,0}b^2}{\varepsilon_2} + \frac{2}{A^2}\big(1+\frac{2\varepsilon_0}{b}\langle b|\lambda|^{1/2}\rangle^2 + C'_{R,0}(\sqrt{\frac{d}{2}}+\varepsilon_2)\big)\right)\|f\|_{L^2}$$
$$+\frac{4(d+2)b^2}{\varepsilon_1}\|f\|_{\tilde{\mathscr{W}}^{0,1}}\,,$$

and we choose

$$\varepsilon_0 = \varepsilon_2 = \sqrt{d+2}\,bA \leq C_{R,0}Ab \leq 1 \quad , \quad \varepsilon_1 = \frac{1}{4\tilde{C}_0} \leq 1\,.$$

This implies that $\frac{1}{2\tilde{C}_0}\|u_0\|_{\tilde{\mathscr{W}}^{0,1}}$ is bounded by

$$\left(b + \frac{2\sqrt{d+2}}{A}\langle b|\lambda|^{1/2}\rangle^2 + \frac{2\sqrt{d+2}C'_{R,0}b}{A} + \frac{2}{A^2} + \frac{2\sqrt{d+2}}{A}\langle b|\lambda|^{1/2}\rangle^2 + \frac{\sqrt{2d}C'_{R,0}}{A^2} + \frac{2\sqrt{d+2}C'_{R,0}b}{A}\right)\|f\|_{L^2}$$
$$+16\tilde{C}_0(d+2)b^2\|f\|_{\tilde{\mathscr{W}}^{0,1}}$$

and

$$\frac{1}{2\tilde{C}_0}\|u_0\|_{\tilde{\mathscr{W}}^{0,1}} \leq \left(b + \frac{4\sqrt{d+2}}{A}\langle b|\lambda|^{1/2}\rangle^2 + \frac{4\sqrt{d+2}C'_{R,0}b}{A} + \frac{2}{A^2} + \frac{\sqrt{2d}C'_{R,0}}{A^2}\right)\|f\|_{L^2} + 16\tilde{C}_0(d+2)b^2\|f\|_{\tilde{\mathscr{W}}^{0,1}}\,.$$

The condition $C'_{R,0}\max(b,A^{-1}) \leq 1$ ensures

$$\frac{1}{2\tilde{C}_0}\|u_0\|_{\tilde{\mathscr{W}}^{0,1}} \leq \big(b + \frac{1}{A}(6\sqrt{d+2} + \frac{2}{A} + \sqrt{2d})\big)\langle b|\lambda|^{1/2}\rangle^2\|f\|_{L^2} + 16\tilde{C}_0(d+2)b^2\|f\|_{\tilde{\mathscr{W}}^{0,1}}\,.$$

We conclude the proof of the case $s = 0$ by choosing new values of the constants $C_{R,0}$ and $\tilde{C}$ according to

$$\begin{aligned} C_{R,0,new} &= \max(C'_{R,0}, C_{R,0,old}, C_{R,1}), \\ \sqrt{\tilde{C}} &= \max(2\tilde{C}_0, 6\sqrt{d+2}+\sqrt{2d}+2, 16\tilde{C}_0(d+2)) \geq 1. \end{aligned}$$

For general $s \in \mathbb{R}$, writing $(W_\theta^2)^{s/2} f = (W_\theta^2)^{s/2}\big(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm}\big)(W_\theta^2)^{-s/2}(W_\theta^2)^{s/2}u$ in the form

$$(W_\theta^2)^{s/2} f = \big(P_{\pm,b} + \frac{1}{b}R^s_{1,\perp,h} + A^2\pi_{0,\pm}\big)(W_\theta^2)^{s/2}u$$

reduces the problem to the case $s = 0$ with $R_{1,\perp,h}$ replaced again by

$$R^s_{1,\perp,h} = (W_\theta^2)^{s/2}R_{1,\perp,h}(W_\theta^2)^{-s/2} + (W_\theta^2)^{s/2}\nabla^{\mathcal{E}_\pm,h}_{\mathcal{Y}}(W_\theta^2)^{-s/2} - \nabla^{\mathcal{E}_\pm,h}_{\mathcal{Y}}.$$

□

Lemma 3.3.10. *There exist two constants $\tilde{C}, C_{R,s} \geq 1$, which are respectively uniform and s-dependent, $s \in \mathbb{R}$, such that $C_{R,s}\max(Ab, b, A^{-1}) \leq 1$ implies*

$$\frac{1}{\tilde{C}}\|u_\perp\|_{\tilde{\mathcal{W}}^{1,s+1}} \leq \langle b|\lambda|^{1/2}\rangle^2\left(b + \frac{1}{A}\right)\|f\|_{\tilde{\mathcal{W}}^{0,s}} + b^2\|f\|_{\tilde{\mathcal{W}}^{0,s+1}} \tag{3.3.1.17}$$

for all $u \in \mathcal{S}(X^h;\mathcal{E}^h_\pm)$.

Proof. Apply the inequality (3.3.1.14) with s replaced by $s+1$ and $\varepsilon = 1$:

$$\|u_\perp\|_{\tilde{\mathcal{W}}^{1,s+1}} \leq 2(d+2)b^2\|f\|_{\tilde{\mathcal{W}}^{0,s+1}} + \|u_0\|_{\tilde{\mathcal{W}}^{0,s+1}}.$$

With (3.3.1.15) we deduce

$$\|u_\perp\|_{\tilde{\mathcal{W}}^{1,s+1}} \leq b^2\big(2(d+2)+\tilde{C}\big)\|f\|_{\tilde{\mathcal{W}}^{0,s+1}} + \tilde{C}\left(b + \frac{1}{A}\right)\langle b|\lambda|^{1/2}\rangle^2\|f\|_{\tilde{\mathcal{W}}^{0,s}}.$$

Finally choose $\tilde{C}_{new} = 2(d+2) + \tilde{C}_{old}$ and $C_{R,s,new} \geq \max(C_{R,s,Lemma\ 3.3.8}, C_{R,s,Lemma\ 3.3.9})$. □

We now decompose the equation

$$f = (P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u$$

into terms which adapt the low and high-frequency analysis of [ReTa]: The frequency truncations are actually replaced by spectral truncations associated with (W_θ^2).

Let us set:

- $f_L = 1_{\{W_\theta^2 \leq \frac{1}{b^2}\}}f$ the orthogonal projection of f on the low lying spectral part of (W_θ^2);
- $f_H = 1_{\{W_\theta^2 > \frac{1}{b^2}\}}f$ the spectral projection of f corresponding to high energies of (W_θ^2);
- $u_L = (P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)^{-1}f_L$ the preimage of f_L by $P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda$;
- $u_H = (P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)^{-1}f_H$ the preimage of f_H by $P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda$.

Lemma 3.3.11. *Under the condition $C_{R,s}\max(Ab, b, A^{-1}) \leq 1$ with constants $\tilde{C}, C_{R,s} \geq 1$, which are respectively uniform and s-dependent, $s \in \mathbb{R}$, the inequalities*

$$\|u_H\|_{\tilde{\mathcal{W}}^{0,s+1}} \leq \frac{\tilde{C}b\langle b|\lambda|^{1/2}\rangle^2}{A}\|f_H\|_{\tilde{\mathcal{W}}^{0,s+1}}, \tag{3.3.1.18}$$

$$\|u_L\|_{\tilde{\mathcal{W}}^{0,s+1}} \leq \frac{\tilde{C}\langle b|\lambda|^{1/2}\rangle^2}{A}\|f_L\|_{\tilde{\mathcal{W}}^{0,s}}. \tag{3.3.1.19}$$

hold for all $u \in \mathcal{S}(X^h;\mathcal{E}^h_\pm)$.

Proof. By the triangular inequality and the continuity of the inclusion $\mathscr{W}^{1,s+1} \hookrightarrow \mathscr{W}^{0,s+1}$, we know

$$\|u\|_{\tilde{\mathscr{W}}^{0,s+1}} \leq \|u_0\|_{\tilde{\mathscr{W}}^{0,s+1}} + \sqrt{\frac{2}{d}}\,\|u_\perp\|_{\tilde{\mathscr{W}}^{1,s+1}}\,.$$

Let us fix

$$\tilde{C} \geq \tilde{C}_{Lemma\ 3.3.9} + \sqrt{\frac{2}{d}}\tilde{C}_{Lemma\ 3.3.10} \quad , \quad C_{R,s} \geq \max(C_{R,s+1,Lemma\ 3.3.9}, C_{R,s+1,Lemma\ 3.3.10})\,.$$

We can now apply inequality (3.3.1.15) to $\|u_0\|_{\tilde{\mathscr{W}}^{0,s+1}}$ and inequality (3.3.1.17) to $\|u_\perp\|_{\tilde{\mathscr{W}}^{1,s+1}}$ with

$$\frac{1}{\tilde{C}}\,\|u\|_{\tilde{\mathscr{W}}^{0,s+1}} \leq \langle b|\lambda|^{1/2}\rangle^2 (b + \frac{1}{A})\,\|f\|_{\tilde{\mathscr{W}}^{0,s}} + b^2\,\|f\|_{\tilde{\mathscr{W}}^{0,s+1}}\,. \tag{3.3.1.20}$$

The functional calculus gives

$$\begin{cases} \|f_L\|_{\tilde{\mathscr{W}}^{0,s+1}} & \leq & \frac{1}{b}\,\|f_L\|_{\tilde{\mathscr{W}}^{0,s}}\,, \\ \|f_H\|_{\tilde{\mathscr{W}}^{0,s}} & \leq & b\,\|f_H\|_{\tilde{\mathscr{W}}^{0,s+1}}\,. \end{cases}$$

By combining (3.3.1.20) for $u = u_H$ and $u = u_L$ respectively, we deduce

$$\begin{cases} \|u_L\|_{\tilde{\mathscr{W}}^{0,s+1}} & \leq & 2\tilde{C}\langle b|\lambda|^{1/2}\rangle^2 (b + \frac{1}{A})\,\|f_L\|_{\tilde{\mathscr{W}}^{0,s}}\,, \\ \|u_H\|_{\tilde{\mathscr{W}}^{0,s+1}} & \leq & 2\tilde{C}b\langle b|\lambda|^{1/2}\rangle^2 (b + \frac{1}{A})\,\|f_H\|_{\tilde{\mathscr{W}}^{0,s+1}}\,. \end{cases}$$

Since we assumed $C_{R,s}Ab \leq 1$ it suffices to take $\tilde{C}_{new} = 4\tilde{C}_{old}$. □

Lemma 3.3.12. *Under the condition $C_{R,s}\max(Ab, b, A^{-1}) \leq 1$ with constants $\tilde{C}, C_{R,s} \geq 1$, which are respectively uniform and s-dependent, $s \in \mathbb{R}$, the inequalities*

$$\begin{cases} \|u_L\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} & \leq & \frac{\tilde{C}\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}{A^{\frac{8}{5}}}\,\|f_L\|_{\tilde{\mathscr{W}}^{0,s}}\,, \\ \|u_H\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} & \leq & \frac{\tilde{C}\langle b|\lambda|^{1/2}\rangle^{4/5}}{A^{\frac{8}{5}}}\,\|f_H\|_{\tilde{\mathscr{W}}^{0,s}} \end{cases}$$

hold for all $u \in \mathscr{S}(X^h; \mathscr{E}^h_\pm)$.

Proof. When we interpolate between integration by parts inequality (3.3.1.7)

$$\|u_L\|_{\tilde{\mathscr{W}}^{0,s}} \leq \frac{2}{A^2}\,\|f_L\|_{\tilde{\mathscr{W}}^{0,s}}\,,$$

and (3.3.1.19)

$$\|u_L\|_{\tilde{\mathscr{W}}^{0,s+1}} \leq \frac{\tilde{C}\langle b|\lambda|^{1/2}\rangle^2}{A}\,\|f_L\|_{\tilde{\mathscr{W}}^{0,s}}\,.$$

We obtain

$$\|u_L\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} \leq \frac{2^{\frac{3}{5}}\tilde{C}^{\frac{2}{5}}\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}{A^{\frac{8}{5}}}\,\|f_L\|_{\tilde{\mathscr{W}}^{0,s}}\,.$$

By doing the same with the subelliptic estimate (3.3.1.10), we get

$$\|u_H\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}} \leq \frac{C}{A^2 b^{\frac{2}{3}}}\,\|f_H\|_{\tilde{\mathscr{W}}^{0,s}}\,,$$

and (3.3.1.18)

$$\|u_H\|_{\tilde{\mathscr{W}}^{0,s}} \leq \frac{\tilde{C}b\langle b|\lambda|^{1/2}\rangle^2}{A}\,\|f_H\|_{\tilde{\mathscr{W}}^{0,s}}\,,$$

and thus

$$\|u_H\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} \leq C^{\frac{3}{5}}\tilde{C}^{\frac{2}{5}}\frac{\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}{A^{\frac{8}{5}}}\|f_H\|_{\tilde{\mathscr{W}}^{0,s}}.$$

Take $\tilde{C}_{new} = \max(2^{\frac{3}{5}}, C^{\frac{3}{5}})\tilde{C}^{\frac{2}{5}}$. The result follows for some large enough constants $C_s, C_{R,s} \geq 1$. □

Proposition 3.3.13. *There exist two constants $C, C_{R,s} \geq 1$, which are respectively uniform and s-dependent, $s \in \mathbb{R}$, such that $C_{R,s}\max(Ab, b, A^{-1}) \leq 1$ implies*

$$\begin{aligned} C\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - z)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq\ & A^2\|u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2\|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2 b\left\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \\ & + A^2 b^{\frac{2}{3}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}} + \frac{A^{\frac{8}{5}}}{\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} \\ & + A^2 b|\lambda|^{1/2}\|u\|_{\tilde{\mathscr{W}}^{0,s}} + A^{\frac{16}{9}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{9}}} \end{aligned} \tag{3.3.1.21}$$

for all $u \in \mathscr{S}(X^h;\mathscr{E}^h_\pm)$ and all $z \in \mathbb{C}$ such that $\mathrm{Re}z \leq \frac{A^2}{2}$

Proof. When $z = i\lambda$ the result follows from Proposition 3.3.7 and the inequality

$$\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} \leq \|u_L\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} + \|u_H\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} \leq \frac{\tilde{C}\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}{A^{\frac{8}{5}}}(\|f_L\|_{\tilde{\mathscr{W}}^{0,s}} + \|f_H\|_{\tilde{\mathscr{W}}^{0,s}}) \leq \frac{\sqrt{2}\tilde{C}\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}{A^{\frac{8}{5}}}\|f\|_{\tilde{\mathscr{W}}^{0,s}}.$$

For a general $z \in \mathbb{C}$, $\mathrm{Re}z \leq \frac{A^2}{2}$, we use again the inequality

$$\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - z)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{3}\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\mathrm{Im}z)u\right\|_{\tilde{\mathscr{W}}^{0,s}},$$

which is a consequence of Proposition 3.3.6.
Taking $C = 3\sqrt{2}(C_{old}+1)\tilde{C}$, the inequality

$$\begin{aligned} C\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - z)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq\ & A^2\|u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2\|\mathscr{O}u\|_{\tilde{\mathscr{W}}^{0,s}} + A^2 b\left\|(\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} - ib\lambda)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \\ & + A^2 b^{\frac{2}{3}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{3}}} + \frac{A^{\frac{8}{5}}}{\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}} \\ & + A^2 b|\lambda|^{1/2}\|u\|_{\tilde{\mathscr{W}}^{0,s}} \end{aligned} \tag{3.3.1.22}$$

is proved. The Inequality (3.3.1.22) above implies

$$C\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - z)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{A^{\frac{8}{5}}}{\langle b|\lambda|^{1/2}\rangle^{\frac{4}{5}}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{5}}}$$

$$\text{and}\qquad C\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - z)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq A^2\langle b|\lambda|^{1/2}\rangle\|u\|_{\tilde{\mathscr{W}}^{0,s}}.$$

Interpolation gives

$$C\left\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - z)u\right\|_{\tilde{\mathscr{W}}^{0,s}} \geq A^{\frac{16}{9}}\|u\|_{\tilde{\mathscr{W}}^{0,s+\frac{2}{9}}}.$$

Finally, taking $C_{new} = 2C$, the inequality (3.3.1.21) is satisfied. □

End of the proof of Proposition 3.3.1. We use

$$B_{\pm,b,V^h} = [P_{\pm,b} + \frac{1}{b}R_{1,\perp,h}] + R_{0,h} + R_{2,h}\,,$$

where Proposition 3.3.6 (resp. Proposition 3.3.13) provides the result about the maximal accretivity (resp. the desired subelliptic estimate) with some constants $C, C_{R,s} \geq 1$ when $R_{0,h} = 0$ and $R_{2,h} = 0$.
The property (3.2.7.3) of $R_{0,h}$ and $R_{2,h}$ implies the uniform inequality

$$\|R_{0,h}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})} + \|R_{2,h}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{2,s};\tilde{\mathcal{W}}^{0,s})} + \|R_{2,h}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,s};\tilde{\mathcal{W}}^{-1,s})} \leq C_s^{(1)}$$

For the accretivity of $\overline{B_{b,\pm,V^h} + A^2\pi_{0,\pm} - \frac{A^2}{2}}^s$ we write

$$\begin{aligned}\mathrm{Re}\,\langle u\,,(B_{\pm,b,V^h} + A^2\pi_0 - \frac{A^2}{2})u\rangle_{\tilde{\mathcal{W}}^{0,s}} &\geq \frac{1}{(d+2)b^2}\|u_\perp\|^2_{\tilde{\mathcal{W}}^{1,s}} + \frac{3A^2}{4}\|u_0\|^2_{\tilde{\mathcal{W}}^{0,s}} - 2C_s^{(1)}\|u\|^2_{\tilde{\mathcal{W}}^{1,s}} - \frac{A^2}{2}\|u\|^2_{\tilde{\mathcal{W}}^{0,s}} \\ &\geq [\frac{1}{(d+2)b^2} - 2C_s^{(1)} - \frac{A^2}{4}]\|u_\perp\|^2_{\tilde{\mathcal{W}}^{1,s}} + [\frac{A^2}{4} - dC_s^{(1)}]\|u_0\|_{\tilde{\mathcal{W}}^{0,s}} \geq 0\end{aligned}$$

where the inequality $A^2 \geq 4dC_s^{(1)}$ holds when $C_s \geq 1$ is large enough, and where the inequality $\frac{1}{(d+2)b^2} - 2C_s^{(1)} \geq \frac{A^2}{4}$ holds as soon as $C_s \max(Ab, b, \frac{1}{A}) \leq 1$.
For the subelliptic estimate, (3.3.1.21) implies

$$\begin{aligned}\|(R_{0,h} + R_{2,h})u\|_{\tilde{\mathcal{W}}^{0,s}} \leq C_s^{(1)}(\|u\|_{\tilde{\mathcal{W}}^{0,s}} + \|u\|_{\tilde{\mathcal{W}}^{2,s}}) &\leq \frac{CC_s^{(1)}}{A^2}\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u\|_{\tilde{\mathcal{W}}^{0,s}} \\ &\leq \frac{2CC_s^{(1)}}{A^2}\|(P_{\pm,b} + \frac{1}{b}R_{1,\perp,h} + A^2\pi_{0,\pm} - i\lambda)u\|_{\tilde{\mathcal{W}}^{0,s}}\,,\end{aligned}$$

when

$$C_{R,s}\max(Ab, b, A^{-1}) \leq 1\,.$$

We conclude by choosing A such that $\frac{2CC_s^{(1)}}{A^2} \leq \frac{1}{2} \leq \frac{C}{2}$, which is ensured by the new value

$$C_s = \max(C_{R,s}, \sqrt{4CC_s^{(1)}}),$$

and provides the result of Proposition 3.3.1 with the new value $C_{new} = 2C$. □

3.3.2 Second modified operator $\pi_{\perp,\pm}B_{\pm,b,V^h}\pi_{\perp,\pm}$

Another lower bound without the term $\frac{\kappa_b}{b^2}$ can be obtained after considering the block diagonal restriction of $B_{b,\pm,V^h}$ to $\mathrm{Ran}\,\pi_{\perp,\pm} = \ker(\alpha_\pm)$. Actually it is not a subelliptic estimate because there is no gain of regularity. But the strong accretivity inequality with a lower bound $\frac{1}{b^2}$ will be strong enough for our applications. Because W_θ^2 commutes with $\pi_{0,\pm}$ and $\pi_{\perp,\pm}$ the same notions of closure, formal adjoint and adjoint, as in Definition 3.2.3 can be considered with $\tilde{\mathcal{W}}^{0,s}(X^h;\mathcal{E}^h_\pm)$, $\mathcal{S}(X^h;\mathcal{E}^h_\pm)$ and $\mathcal{S}'(X^h;\mathcal{E}^h_\pm)$ replaced respectively by

$$\pi_{\perp,\pm}\tilde{\mathcal{W}}^{0,s}(X^h;\mathcal{E}^h_\pm) \quad , \quad \pi_{\perp,\pm}\mathcal{S}(X^h;\mathcal{E}^h_\pm) \quad \text{and} \quad \pi_{\perp,\pm}\mathcal{S}'(X^h;\mathcal{E}^h_\pm).$$

All the discussion after Definition 3.2.3 and the properties of Proposition 3.2.8 can be adapted with

$$\begin{aligned}&B_{\pm,b,V^h,\perp} = \pi_{\perp,\pm}B_{\pm,b,V^h}\pi_{\perp,\pm}\,,\\ &(W_\theta^2)^{s/2}B_{\pm,b,V^h,\perp}(W_\theta^2)^{-s/2} = \pi_{\perp,\pm}\left[P_{\pm,b} + R^s_{0,h} + \frac{1}{b}R^s_{1,\perp,h} + R^s_{2,h}\right]\pi_{\perp,\pm}\\ &B'^{,s}_{\pm,b,V^h,\perp} = \pi_{\perp,\pm}(B'^{,s}_{\pm,b,V^h})\pi_{\perp,\pm} = \pi_{\perp,\pm}\left[P'_{\pm,b} + (R^s_{0,h})' + \frac{1}{b}(R^s_{1,\perp,h})' + (R^s_{2,h})'\right]\pi_{\perp,\pm}\,.\end{aligned}$$

Proposition 3.3.14. *There exists a constant $C_s \geq 1$ which depends on $s \in \mathbb{R}$ such that the condition $C_s b \leq 1$ guaranties the following results.*
The densely defined operator $B_{\pm,b,V^h,\perp} = \pi_{\perp,\pm} B_{\pm,b,V^h} \pi_{\perp,\pm}$ in $\pi_{\perp,\pm}\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm)$ with domain $\pi_{\perp,\pm}\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ is essentially maximal accretive with the inequality

$$\forall u \in \pi_{\perp,\pm}\mathscr{S}(X^h;\mathscr{E}^h_\pm), \quad \mathrm{Re}\langle u\,, [\pi_{\perp,\pm} B_{\pm,b,V^h} \pi_{\perp,\pm}] u\rangle_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{12b^2}\|u\|^2_{\tilde{\mathscr{W}}^{0,s}}\,. \tag{3.3.2.1}$$

Addtionally the resolvent of its closure $\overline{B_{\pm,b,V^h,\perp}}^s = \overline{[\pi_{\perp,\pm} B_{\pm,b,V^h} \pi_{\perp,\pm}]}^{\pi_{\perp,\pm}\tilde{\mathscr{W}}^{0,s}}$ satisfies

$$\forall z \in \mathbb{C}, \mathrm{Re}\, z \leq \frac{1}{24b^2}\,, \quad \|(z - \overline{B_{\pm,b,V^h,\perp}}^s)^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq 24b^2\,.$$

Finally the closure of the $\pi_{\perp,\pm}\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm)$ formal adjoint $B'^{,s}_{\pm,b,V^h,\perp}$ satisfies

$$\overline{B'^{,s}_{\pm,b,V^h,\perp}\big|_{\mathscr{S}(X^h;\mathscr{E}^h_\pm)}}^s = B'^{,*}_{\pm,b,V^h,\perp}\,.$$

For the proof, we check firstly the accretivity in Proposition 3.3.15 and secondly the injectivity of the adjoint $B^{*,s}_{\pm,b,V^h,\perp} = (\overline{B_{\pm,b,V^h,\perp}})^{*,s}$ in Proposition 3.3.16. The final statement above is just a consequence of the essential maximal accretivity.

Proposition 3.3.15. *There exists a constant $C_s \geq 1$ which depends on $s \in \mathbb{R}$ such that, under the assumption $C_s b \leq 1$, the operator $B_{\pm,b,V^h,\perp} = \pi_{\perp,\pm} B_{\pm,b,V^h} \pi_{\perp,\pm}$ is accretive in $\pi_{\perp,\pm}\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm)$, with*

$$\forall u \in \pi_{\perp,\pm}\mathscr{S}(X^h;\mathscr{E}^h_\pm), \quad \mathrm{Re}\langle u\,, B_{\pm,b,V^h,\perp} u\rangle_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{2(d+2)b^2}\|u\|^2_{\tilde{\mathscr{W}}^{1,s}} \geq \frac{d}{4(d+2)b^2}\|u\|^2_{\tilde{\mathscr{W}}^{0,s}} \geq \frac{1}{12b^2}\|u\|^2_{\tilde{\mathscr{W}}^{0,s}}\,. \tag{3.3.2.2}$$

Hence its closure $\overline{B_{\pm,b,V^h,\perp}}^s$ is one to one with a closed range.

Proof. Take $u = u_\perp$, since $\pi_{\perp,\pm}\pi_{0,\pm}\pi_{\perp,\pm} = 0$ the inequality (3.3.1.6) holds at least with an arbitrary $A > 0$ that satisfies the required condition. Integration by parts gives

$$\begin{aligned}\mathrm{Re}\langle B_{\pm,b,V^h} u\,, u\rangle_{\tilde{\mathscr{W}}^{0,s}} &= \frac{1}{b^2}\mathrm{Re}\langle (P_{\pm,b} + \frac{1}{b}R_{1,\perp,h})u\,, u\rangle_{\tilde{\mathscr{W}}^{0,s}} + \mathrm{Re}\langle (R_{0,h} + R_{2,h})u\,, u\rangle_{\tilde{\mathscr{W}}^{0,s}} \\ &\geq \frac{1}{(d+2)b^2}\|u\|^2_{\tilde{\mathscr{W}}^{1,s}} - \|R_{0,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}\|u\|^2_{\tilde{\mathscr{W}}^{0,s}} - \|R_{2,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,s};\tilde{\mathscr{W}}^{-1,s})}\|u\|^2_{\tilde{\mathscr{W}}^{1,s}}\,.\end{aligned}$$

The condition (3.2.7.3) ensures $R_{0,h} \in \mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})$ and by interpolation $R_{2,h} \in \mathscr{L}(\tilde{\mathscr{W}}^{1,s};\tilde{\mathscr{W}}^{-1,s})$. Remember that $\frac{d}{2}\|u\|^2_{\tilde{\mathscr{W}}^{0,s}} \leq \|u\|^2_{\tilde{\mathscr{W}}^{1,s}}$. Finally

$$\begin{aligned}\mathrm{Re}\langle B_{\pm,b,V^h} u\,, u\rangle_{\tilde{\mathscr{W}}^{0,s}} &\geq \frac{1}{b^2}\left(\frac{1}{(d+2)} - \frac{2\|R_{0,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}b^2}{d} - \|R_{2,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,s};\tilde{\mathscr{W}}^{-1,s})}b^2\right)\|u\|^2_{\tilde{\mathscr{W}}^{1,s}} \\ &\geq \frac{1}{b^2}\left(\frac{1}{d+2} - (\frac{2}{d}+1)\tilde{C}_R b^2\right)\|u\|_{\tilde{\mathscr{W}}^{1,s}}\end{aligned}$$

with $\tilde{C}_R = \sup_{h\in]0,1]}\left[\|R_{0,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} + \|R_{2,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,s};\tilde{\mathscr{W}}^{-1,s})}\right]$.
Taking $C_s > \frac{\sqrt{2\tilde{C}_R}}{d}(d+2)$ and $C_{R,s}$ as in Proposition 3.3.6 , we have

$$\frac{1}{d+2} - \frac{2\|R_{0,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}b^2}{d} - \|R_{2,h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,s};\tilde{\mathscr{W}}^{-1,s})}b^2 \geq \frac{1}{2(d+2)}$$

as soon as $C_s b \leq 1$. □

Proposition 3.3.16. *There is a constant $C_s \geq 1$ which depends on $s \in \mathbb{R}$, such that the adjoint $B^{*,s}_{\pm,b,V^h,\perp} = (\overline{B_{\pm,b,V^h,\perp}}^s)^{*,s}$ is one to one and therefore $\overline{B_{\pm,b,V^h,\perp}}^s$ is maximal accretive, as soon as $C_s b \leq 1$.*

Proof. Assume $w \in \ker(B^{*,s}_{\pm,b,V^h,\perp})$ and let us prove $w = 0$. By setting $w = (W_\theta^2)^{-s/2}v$, the assumption is equivalent to $(W_\theta^2)^{s/2}B^{*,s}_{\pm,b,V^h,\perp}(W_\theta^2)^{-s/2}v = (W_\theta^2)^{-s/2}B^{*}_{\pm,b,V^h,\perp}(W_\theta^2)^{s/2}v = 0$ and $v \in L^2(X^h, dqdp; \mathscr{E}_\pm^h)$. The problem is reduced to

$$\left.\begin{array}{c}\pi_{\perp,\pm}\left[P^*_{\pm,b} + (R^s_{0,h})' + \frac{1}{b}(R^s_{1,\perp,h})' + (R^s_{2,h})'\right]\pi_{\perp,\pm}v = 0 \quad \text{in} \quad \mathscr{S}'(X^h;\mathscr{E}^h_\pm)\\ v \in L^2(X^h, dqdp; \mathscr{E}^h_\pm)\end{array}\right\} \Rightarrow v = 0\,.$$

We set $\tilde{B}_{\pm,b,V^h,\perp,s} = P'_{\pm,b} + (R^s_{0,h})' + \frac{1}{b}(R^s_{1,\perp,h})' + (R^s_{2,h})'$ which has the same form as $B_{\pm,b,V^h}$ with a changed sign before $\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}}$ and remainder terms $R_{0,h}, R_{1,\perp,h}, R_{2,h}$ respectively replaced by the s-dependent versions $(R^s_{0,h})', (R^s_{1,\perp,h})', (R^s_{2,h})'$. In particular $\pi_{\perp,\pm}\tilde{B}_{\pm,b,V^h,s}\pi_{\perp,\pm}$ is accretive on $\pi_{\perp,\pm}\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ for the $L^2(X^h, dqdp; \mathscr{E}^h_\pm)$ scalar product, with the same lower bounds as in (3.3.2.2), as soon as $C^1_s b \leq 1$ for some $C^1_s \geq 1$.
We take two cut-off functions $\chi, \tilde{\chi} \in C^\infty_0(\mathbb{R}, [0,1])$ such that

- χ and $\tilde{\chi}$ is equal to 1 near 0 ,
- $\text{supp}(\tilde{\chi}) \subset \chi^{-1}(\{1\})$,

and we recall that $f(W_\theta^2)$ is continuous from $\mathscr{S}'(X^h;\mathscr{E}^h_\pm)$ to $\mathscr{S}(X^h;\mathscr{E}^h_\pm)$ and commutes with $\pi_{\perp,\pm}$ and more generally with any function of $\alpha_\pm$ for all $f \in \mathscr{C}^\infty_0(\mathbb{R};\mathbb{C})$. For $\varepsilon > 0$ we set $v_\varepsilon = \chi(\varepsilon W_\theta^2)v \in \mathscr{S}(X^h;\mathscr{E}^h_\pm)$. A straightforward computation shows

$$\begin{aligned}\pi_{\perp,\pm}\tilde{B}_{\pm,b,V^h,s}\pi_{\perp,\pm}v_\varepsilon &= \pi_{\perp,\pm}[\tilde{B}_{\pm,b,V^h,s}, \chi(\varepsilon W_\theta^2)]\pi_{\perp,\pm}v\\ &= \pi_{\perp,\pm}[\underbrace{\pm\frac{1}{b}\nabla^{\mathscr{E}_\pm,h}_{\mathscr{Y}} + (R^s_{0,h})' + \frac{1}{b}(R^s_{1,\perp,h})' + (R^s_{2,h})'}_{=D}, \chi(\varepsilon W_\theta^2)]\pi_{\perp,\pm}v,\end{aligned} \tag{3.3.2.3}$$

where D is a differential operator in $\text{OpS}^{3/2}_\Psi(Q^h;\text{End}\,\mathscr{E}^h_\pm)$. The Helffer-Sjöstrand formula for the commutator gives

$$[D, \chi(\varepsilon W_\theta^2)] = [D, W_\theta^2]W_\theta^{-2}\chi'(\varepsilon W_\theta^2)\varepsilon W_\theta^2 + r_\varepsilon,$$

where

$$r_\varepsilon = -\frac{1}{2i\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\tilde{\chi}(z)\varepsilon^2(z - \varepsilon W_\theta^2)^{-1}[[D, W_\theta^2], W_\theta^2](z - \varepsilon W_\theta^2)^{-2}\, dz \wedge d\bar{z}\,.$$

By pseudo differential calculus in $\text{OpS}^*_\Psi(Q^h;\text{End}\,\mathscr{E}^h_\pm)$, the commutator $[[D, W_\theta^2], W_\theta^2]$ is a pseudo differential operator of order $\frac{3}{2} + (2-1) + (2-1) = \frac{7}{2}$. This implies that $[[D, W_\theta^2], W_\theta^2](W_\theta^2)^{-\frac{7}{4}}$ is bounded. The inequality

$$\|\varepsilon^{7/4}(W_\theta^2)^{-\frac{7}{4}}(z - \varepsilon W_\theta^2)^{-2}\| \leq \sup_{\lambda > 0}\left|\frac{\lambda^{7/8}}{|z - \lambda|}\right|^2 \leq C\left[\sup_{\lambda > 0}\frac{1 + \lambda}{|z - \lambda|}\right]^2 \leq C'\frac{\langle z\rangle^2}{|\text{Im}\, z|^2}$$

yields

$$\|r_\varepsilon\|_{\mathscr{L}(L^2;L^2)} \leq C_{R,s}\varepsilon^{1/4}\,, \tag{3.3.2.4}$$

where again the constant $C_{R,s}$ is uniform with respect to $h \in]0,1]$.
The obvious equality $\chi'(t) = (1 - \tilde{\chi}(t))\chi'(t)$ for all $t \in \mathbb{R}$ implies

$$[D, \chi(\varepsilon W_\theta^2)] = [D, W_\theta^2]W_\theta^{-2}\chi'(\varepsilon W_\theta^2)\varepsilon W_\theta^2(1 - \tilde{\chi}(\varepsilon W_\theta^2)) + r_\varepsilon. \tag{3.3.2.5}$$

Let us consider the following scalar product by using (3.3.2.3) and (3.3.2.5)

$$\begin{aligned}\mathrm{Re}\langle v_\varepsilon,\pi_{\perp,\pm}B_{\pm,b,V^h,s}\pi_{\perp,\pm}v_\varepsilon\rangle_{L^2} &= \mathrm{Re}\langle v_\varepsilon\,,[D,W_\theta^2]W_\theta^{-2}\chi'(\varepsilon W_\theta^2)\varepsilon W_\theta^2(1-\tilde\chi(\varepsilon W_\theta^2))v\rangle_{L^2}+\mathrm{Re}\langle v_\varepsilon\,,r_\varepsilon v\rangle_{L^2}\\ &= \mathrm{Re}\langle \varepsilon W_\theta^2\chi'(\varepsilon W_\theta^2)W_\theta^{-2}[W_\theta^2,D']v_\varepsilon\,,(1-\tilde\chi(\varepsilon W_\theta^2))v\rangle_{L^2}+\mathrm{Re}\langle v_\varepsilon\,,r_\varepsilon v\rangle_{L^2}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the right-hand side of this equality yields

$$\mathrm{Re}\langle v_\varepsilon,\pi_{\perp,\pm}\tilde B^*_{\pm,b,V^h,s}\pi_{\perp,\pm}v_\varepsilon\rangle_{L^2}\le\|\varepsilon W_\theta^2\chi'(\varepsilon W_\theta^2)W_\theta^{-2}[W_\theta^2,D']v_\varepsilon\|_{L^2}\|(1-\tilde\chi(\varepsilon W_\theta^2))v\|_{L^2}+\|v_\varepsilon\|_{L^2}\|r_\varepsilon v\|_{L^2}. \tag{3.3.2.6}$$

By functional calculus the operator $\varepsilon W_\theta^2\chi'(\varepsilon W_\theta^2)$ is a bounded operator

$$\|\varepsilon W_\theta^2\chi'(\varepsilon W_\theta^2)\|_{\mathcal{L}(L^2;L^2)}\le M=\sup_{t\in[0,+\infty[}|t\chi'(t)|<\infty,$$

while (3.2.7.7) and (3.2.7.3) ensure

$$W_\theta^{-2}[W_\theta^2,D']=\mp\underbrace{W_\theta^{-2}[W_\theta^2,\frac{1}{b}\nabla_{\mathcal{Y}}^{\mathcal{E}_\pm,h}]}_{\in\mathcal{L}(\tilde{\mathcal{W}}^{1,0};L^2)}+\underbrace{W_\theta^{-2}[W_\theta^2,R^s_{0,h}+\frac{1}{b}R^s_{1,\perp,h}+R^s_{2,h}]}_{\in\mathcal{L}(L^2;L^2)}.$$

The left-hand side of (3.3.2.6) is thus bounded by

$$\begin{aligned}\mathrm{Re}\langle v_\varepsilon,\pi_{\perp,\pm}\tilde B_{\pm,b,V^h,s}\pi_{\perp,\pm}v_\varepsilon\rangle_{L^2}\le\ & MC_{R,\mathcal{Y},s}\|v_\varepsilon\|_{\tilde{\mathcal{W}}^{1,0}}\|(1-\tilde\chi(\varepsilon W_\theta^2))v\|_{L^2}+C_{R,s}\varepsilon^{1/4}\|v\|^2_{L^2}\\ \le\ & \frac{1}{2}\varepsilon' MC_{R,\mathcal{Y},s,b}\|v_\varepsilon\|^2_{\tilde{\mathcal{W}}^{1,0}}+\frac{MC_{R,\mathcal{Y},s,b}}{2\varepsilon'}\|(1-\tilde\chi(\varepsilon W_\theta^2))v\|^2_{L^2}+C_{R,s}\varepsilon^{1/4}\|v\|^2_{L^2}\end{aligned}$$

where

$$C_{R,\mathcal{Y},s,b}=\frac{1}{b}\sup_{h\in]0,1]}\max(\|W_\theta^{-2}[W_\theta^2,\nabla_{\mathcal{Y}}^{\mathcal{E}_\pm,h}]\|_{\mathcal{L}(\tilde{\mathcal{W}}^{1,0};L^2)},W_\theta^{-2}[W_\theta^2,R^s_{i=0,1,2;h}]\|_{\mathcal{L}(L^2;L^2)}).$$

By Proposition 3.3.15 and (3.3.2.2) applied now to $\pi_{\perp,\pm}\tilde B_{\pm,b,V^h,s}\pi_{\perp,\pm}$, the left hand-side is bounded from below by $\frac{1}{2(d+2)b^2}\|v_\varepsilon\|^2_{\tilde{\mathcal{W}}^{1,0}}$ when $C_sb\le 1$.
By choosing $\varepsilon'=\frac{1}{2(d+2)b^2MC_{R,\mathcal{Y},s,b}}$, we obtain

$$\frac{d}{8(d+2)b^2}\|v_\varepsilon\|^2_{L^2}\le\frac{1}{4(d+2)b^2}\|v_\varepsilon\|^2_{\tilde{\mathcal{W}}^{1,0}}\le 2(d+2)b^2M^2C^2_{R,\mathcal{Y},s,b}\|(1-\tilde\chi(\varepsilon W_\theta^2))v\|^2_{L^2}+C_{R,s}\varepsilon^{1/4}\|v\|^2_{L^2}.$$

When ε goes to 0, the spectral theorem and the dominated convergence Theorem imply

$$\lim_{\varepsilon\to0}\|v_\varepsilon\|^2_{L^2}=\|v\|^2_{L^2}\quad\text{and}\quad\lim_{\varepsilon\to0}\|1-\tilde\chi(\varepsilon W_\theta^2))v\|^2_{L^2}=0.$$

We have proved $v=0$ and $B^{*,s}_{\pm,b,V^h,\perp}$ is one to one.

□

3.3.3 Final modifications with a frequency or spectral truncation

For $\chi\in\mathcal{C}_0^\infty(\mathbb{R},[0,1])$ such that

$$\mathrm{Supp}(\chi)\subset[-2,2]\quad\text{and}\quad\chi(t)=1\quad\text{for}\quad t\in[-1,1].$$

let $Q_{A,L}=A^2\pi_{0,\pm}\chi(\frac{2W_\theta^2}{(LA)^2})\pi_{0,\pm}$ for $L,A\ge1$, where we drop subscript $_h$ although it depends on $h\in]0,1]$, and consider the operator

$$B_{\pm,b,V^h}+Q_{A,L}.$$

Remember that the operator $U_{\pm,\theta}=U_{\pm,\theta,h}:L^2(Q^h,d\mathrm{Vol}_{g^h};\Lambda T^*Q^h\otimes F_\pm)\to\ker(\alpha_\pm)=\ker(\alpha_{\pm,g^h})$, with $F_+=Q\times\mathbb{C}$ and $F_-=Q\times\mathbb{C}\times\mathbf{or}_Q$, introduced in (3.2.5.17)(3.2.5.18)(3.2.5.19)(3.2.5.20), is unitary and satisfies

$$2\pi_{0,\pm}(W_\theta^2)\pi_{0,\pm}=U_{\pm,\theta}(\underbrace{2C+C\frac{d^2}{2}}_{C_d}+H_0)U_{\pm,\theta}^{-1}$$

where $H_0=H_{0,h}$ is the non negative elliptic, Laplace type operator, $H_0=-\sum_{j=1}^J\theta_j(hq)(\Delta_{Q,g^h})_{sc}\theta_j(hq)$ and where Δ_{Q,g^h} is the Laplace Beltrami operator, with a scalar realization in the orthonormal frame $(u^I_{j,g^h})_{I\subset\{1,\ldots,d\}}$ for every $j\in\{1,\ldots,J\}$.
Owing to the uniform estimates of g^h and V^h stated in Proposition 3.2.7 the operators $(d^{Q^h}+d^{Q^h,*_{g^h}})^2$ and the Witten Laplacian $\Delta_{V^h,1}=(d^{Q^h}+d^{Q^h,*_{g^h}})^2+|\nabla V_h(q)|^2+(\mathcal{L}_{\nabla V^h}+\mathcal{L}^*_{\nabla V^h})$ are elliptic operators in the classical space $\mathrm{OpS}^2(Q^h;\mathcal{E}^h_\pm)$ with the same scalar principal symbol as H_0 and with uniformly controlled lower order corrections.
By choosing the above constant $C\geq C_{g,V}\geq 1$ large enough and by setting again $C_d=2C+C\frac{d^2}{2}$, we deduce for every $s\in\mathbb{R}$ the equivalence of norms

$$\forall u\in H^s(Q^h;\Lambda T^*Q^h\otimes F_\pm),\quad\left(\frac{\|u\|_{H^s}}{\|(C_d+\Delta_{V^h,1})^{s/2}u\|_{L^2}}\right)^{\pm1}=\left(\frac{\|(C_d+H_0)^{s/2}u\|_{L^2}}{\|(C_d+\Delta_{V^h,1})^{s/2}u\|_{L^2}}\right)^{\pm1}\leq C_s\,.$$

With the operator $U_{\pm,\theta}$ we also have

$$\begin{aligned}\|U_{\pm,\theta}u\|_{\tilde{\mathcal{W}}^{s_1,s_2}}&=\|(\mathcal{O})^{s_1/2}(W_\theta)^{s/2}U_{\pm,\theta}u\|_{L^2}\\&=\left(\frac{d}{2}\right)^{s_1}\|(C_d+H_0)^{s_2/2}u\|_{L^2}\\&=\left(\frac{d}{2}\right)^{s_1}\|u\|_{H^{s_2}}\asymp\|(C_d+\Delta_{V^h,1})^{s_2/2}u\|_{L^2}\,.\end{aligned}$$

Because we aim at clarifying the relations between $\mathrm{Spec}(B_{\pm,b,V^h})$ and $\mathrm{Spec}(\Delta_{V^h,1})=\mathrm{Spec}(\Delta_{(Q^h,g^h,V^h),1})=\mathrm{Spec}\,(\Delta_{(Q,g,V,h)})$ (see Subsection 3.2.6), we consider the two perturbations of $B_{\pm,b,V^h}$

$$Q_{A,L}=A^2\pi_{0,\pm}\circ\chi\big(\frac{2W_\theta^2}{(LA)^2}\big)\circ\pi_{0,\pm}=A^2U_{\pm,\theta}\circ\chi\big(\frac{C_d+H_0}{(LA)^2}\big)\circ U_{\pm,\theta}^{-1}\tag{3.3.3.1}$$

$$\text{and}\qquad Q_{A,L,V^h}=A^2U_{\pm,\theta}\circ\chi\big(\frac{C_d+\Delta_{V^h,1}}{(LA)^2}\big)\circ U_{\pm,\theta}^{-1}\,.\tag{3.3.3.2}$$

The comparison of $Q_{A,L}$ and Q_{A,L,V^h} is easier to understant while staying on the base manifold Q^h and we also use the following notations.

Definition 3.3.17. The operators $\tilde{Q}_{A,L}:\mathcal{E}'(Q^h;\Lambda T^*Q^h\otimes F_\pm)\to\mathcal{C}^\infty(Q^h;\Lambda T^*Q^h\otimes F_\pm)$ and $\tilde{Q}_{A,L,V^h}:\mathcal{E}'(Q^h;\Lambda T^*Q^h\otimes F_\pm)\to\mathcal{C}^\infty(Q^h;\Lambda T^*Q^h\otimes F_\pm)$ are defined by

$$\tilde{Q}_{A,L}=A^2\chi(\frac{C_d+H_0}{(LA)^2})\quad\text{and}\quad\tilde{Q}_{A,L,V^h}=A^2\chi(\frac{C_d+\Delta_{V^h,1}}{(LA)^2}).$$

These two operators are bounded as well as $A^2\pi_{0,\pm}-Q_{A,L}$ and $A^2\pi_{0,\pm}-Q_{A,L,V^h}$. We will prove the following result.

Proposition 3.3.18. *There are constants $L\geq 1$, $C_s\geq 1$ and $C_{\chi,s}\geq 1$ respectively uniform, $s\in\mathbb{R}$-dependent and (χ,s)-dependent, such that inequality*

$$\frac{1}{4}\|\overline{(B_{\pm,b,V^h}+A^2\pi_{0,\pm}-z)}^s u\|_{\tilde{\mathcal{W}}^{0,s}}\leq\|\overline{(B_{\pm,b,V^h}+Q_{A,L,V^h}-z)}^s u\|_{\tilde{\mathcal{W}}^{0,s}}\leq\frac{9}{4}\|\overline{(B_{\pm,b,V^h}+A^2\pi_{0,\pm}-z)}^s u\|_{\tilde{\mathcal{W}}^{0,s}}$$

holds for all $u \in D(\overline{B_{\pm,b,V^h}}^s)$ *and all* $z \in \mathbb{C}$*, such that* $\mathrm{Re} z \leq \frac{A^2}{2}$*, as soon as*

$$C_s \max(b, Ab, \frac{1}{A}) \leq 1. \tag{3.3.3.3}$$

Therefore the subelliptic estimate (3.3.1.1) *holds true with* $B_{\pm,b,V^h} + A^2\pi_{0,\pm}$ *replaced by* $B_{\pm,b,V^h} + Q_{A,L,V^h}$ *and the constant* $C \geq 1$ *replaced by* $4C$.

Proof. **1)** We start with the simpler perturbation $Q_{A,L}$ instead of Q_{A,L,V^h} and we write:

$$\begin{aligned} B_{\pm,b,V^h} + Q_{A,L} - z &= B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z + (A^2\pi_{0,\pm}(1-\chi)(\frac{2W_\theta^2}{(LA)^2})\pi_{0,\pm}) \\ &= (1 + A^2\pi_{0,\pm}(1-\chi)(\frac{2W_\theta^2}{(LA)^2})\pi_{0,\pm}(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z)^{-1})(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z). \end{aligned}$$

According to Proposition 3.3.1 and under the condition 3.3.3.3, the resolvent $(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z)^{-1}$ is continuous from $\tilde{\mathscr{W}}^{0,s}$ to $\tilde{\mathscr{W}}^{0,s+\frac{2}{9}}$ with norm less than $\frac{C}{A^{\frac{16}{9}}}$. Because $\chi \equiv 1$ on $[-1,1]$, the operator $(1-\chi)(\frac{2W_\theta^2}{(LA)^2})$ is a bounded operator from $\tilde{\mathscr{W}}^{0,s+\frac{2}{9}}$ to $\tilde{\mathscr{W}}^{0,s}$ with norm less than $\frac{1}{(LA)^{\frac{2}{9}}}$. When $L \geq (2C)^{\frac{9}{2}}$ we obtain

$$\|A^2\pi_{0,\pm}(1-\chi)(\frac{2W_\theta^2}{(LA)^2})\pi_{0,\pm}(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z)^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq \frac{C}{L^{\frac{2}{9}}} \leq \frac{1}{2}.$$

Therefore the operator

$$1 + A^2\pi_{0,\pm}(1-\chi)(\frac{2W_\theta^2}{(LA)^2})\pi_{0,\pm}(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z)^{-1}$$

is invertible by Neumann series and the norm of its inverse is less than 2, while its norm is bounded by 3/2. We have proved

$$\frac{1}{2}\|\overline{(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z)}^s u\|_{\tilde{\mathscr{W}}^{0,s}} \leq \|\overline{(B_{\pm,b,V^h} + Q_{A,L} - z)}^s u\|_{\tilde{\mathscr{W}}^{0,s}} \leq \frac{3}{2}\|\overline{(B_{\pm,b,V^h} + A^2\pi_{0,\pm} - z)}^s u\|_{\tilde{\mathscr{W}}^{0,s}}, \tag{3.3.3.4}$$

for all $u \in D(\overline{B_{\pm,b,V^h}}^s)$ and all $z \in \mathbb{C}$ such that $\mathrm{Re} z \leq \frac{A^2}{2}$.

2) We now use the similar perturbative argument for

$$B_{\pm,b,V^h} + Q_{A,L,V^h} - z = B_{\pm,b,V^h} + Q_{A,L} - z - (Q_{A,L} - Q_{A,L,V^h}).$$

The inequality (3.3.3.8) of Lemma 3.3.19 below gives

$$\|Q_{A,L} - Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} = \|\tilde{Q}_{A,L} - \tilde{Q}_{A,L,V^h}\|_{\mathscr{L}(H^s;H^s)} \leq \frac{C_{\chi,s}A^2}{(LA)^2} \tag{3.3.3.5}$$

when $L, A \geq 1$, uniformly with respect to $h \in]0,1]$.
The subelliptic inequality (3.3.1.1) combined with (3.3.3.4), leads to

$$\begin{aligned} \big|\|(B_{\pm,b,V^h} + Q_{A,L,V^h} - z)u\|_{\tilde{\mathscr{W}}^{0,s}} - \|(B_{\pm,b,V^h} + Q_{A,L} - z)u\|_{\tilde{\mathscr{W}}^{0,s}}\big| &\leq C_{\chi,s}\frac{A^2}{(LA)^2}\|u\|_{\tilde{\mathscr{W}}^{0,s}} \\ &\leq \frac{2CC_{\chi,s}}{(LA)^2}\|(B_{\pm,b,V^h} + Q_{A,L})u\|_{\tilde{\mathscr{W}}^{0,s}}. \end{aligned}$$

By taking $C_{s,new} \geq 1$ such that $C_{s,new}\max(b, Ab, \frac{1}{A}) \leq 1$ implies $A \geq \sqrt{4CC_{\chi,s}}$ and $(LA)^2 \geq 4CC_{\chi,s}$, the right-hand side is less than $\frac{1}{2}\|(B_{\pm,b,V^h} + Q_{A,L})u\|_{\tilde{\mathscr{W}}^{0,s}}$.
We conclude by stating the result with $C_s = C_{s,new} = \max(\sqrt{4CC_{\chi,s}}, C_{s,old})$. □

Lemma 3.3.19. *For all $s, s' \in \mathbb{R}$ there exists $C_{\chi,s,s'} \geq 1$ such that*

$$\|\chi(\frac{C_d + \Delta_{V^h,1}}{(LA)^2})\|_{\mathscr{L}(H^s;H^{s'})} + \|\chi(\frac{C_d + H_0}{(LA)^2})\|_{\mathscr{L}(H^s;H^{s'})} \leq C_{\chi,s,s'}(LA)^{(s'-s)} \tag{3.3.3.6}$$

$$\forall z \in \mathbb{C}, \mathbb{R}\mathrm{e}\, z \leq \frac{A^2}{2}, \ \|\frac{1}{2}(\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}\|_{\mathscr{L}(H^s;H^s)} \leq \frac{2}{A^2 - \mathbb{R}\mathrm{e}\, z + |\mathrm{Im}\, z|} \leq \frac{4}{A^2 + 2|\mathrm{Im}\, z|} \tag{3.3.3.7}$$

and

$$\|\chi(\frac{C_d + H_0}{(LA)^2}) - \chi(\frac{C_d + \Delta_{V^h,1}}{(LA)^2})\|_{\mathscr{L}(H^s;H^s)} \leq \frac{C_{\chi,s,s}}{(LA)^2} \tag{3.3.3.8}$$

hold as soon as $\frac{A}{C_s}, L \geq 1$ and for all $h \in]0,1]$.

Proof. The two first inequalities (3.3.3.6) and (3.3.3.7) are straightforward applications of the functional calculus, because the H^s-norm is equivalently evaluated with $\|(C_d + H_0)^{s/2} u\|_{L^2}$ or with $\|(C_d + \Delta_{V^h,1})^{s/2} u\|_{L^2}$.
For (3.3.3.8), the difference

$$R_{A,L,h} = (C_d + \Delta_{V^h,1}) - (C_d + H_0) \tag{3.3.3.9}$$

satisfies $\|R_{A,L,h}\|_{\mathscr{L}(H^s;H^s)} \leq C^{(1)}_{\chi,s}$ uniformly with respect to $h \in]0,1]$.
The Helffer-Sjöstrand formula gives

$$\chi(\frac{C_d + \Delta_{V^h,1}}{(LA)^2}) - \chi(\frac{C_d + H_0}{(LA)^2}) = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{1}{(LA)^2} (\partial_{\bar{z}} \tilde{\chi})(\frac{z}{(LA)^2})(z - C_d - \Delta_{V^h,1})^{-1} R_{A,L,h} (z - C_d - H_0)^{-1}\, dz \wedge d\bar{z},$$

where $\tilde{\chi} \in \mathscr{C}_0^\infty(\mathbb{C};\mathbb{C})$ is an almost analytic extension of χ with

$$|\partial_{\bar{z}} \tilde{\chi}(z)| \leq C_{\chi,N} |\mathrm{Im}, z|^N$$

while $\partial_{\bar{z}} \tilde{\chi} \equiv 0$ in a neighborhood of 0. The $\mathscr{L}(H^s;H^s)$-norm of this difference in given by the $\mathscr{L}(L^2;L^2)$-norm of

$$(C_d + \Delta_{V^h,1})^{-s/2} \left[\chi(\frac{C_d + \Delta_{V^h,1}}{(LA)^2}) - \chi(\frac{C_d + H_0}{(LA)^2}) \right] (C_d + H_0)^{s/2}$$

or, by setting $\tilde{R}_{A,L,h} = (C_d + \Delta_{V^h,1})^{-s/2} R_{A,L,h} (C_d + H_0)^{s/2}$, of

$$\frac{1}{2i\pi} \int_{\mathbb{C}} \frac{1}{(LA)^2} (\partial_{\bar{z}} \tilde{\chi})(\frac{z}{(LA)^2})(z - C_d - \Delta_{V^h,1})^{-1} \tilde{R}_{A,L,h} (z - C_d - H_0)^{-1}\, dz \wedge d\bar{z}.$$

With

$$\|\tilde{R}_{A,L,h}\|_{\mathscr{L}(L^2;L^2)} \leq C^{(2)}_{\chi,s} \|R_{A,L,h}\|_{\mathscr{L}(H^s;H^s)},$$

and the inequalities

$$\|(z - C_d - \Delta_{V^h,1})^{-1}\|_{\mathscr{L}(L^2;L^2)} \|(z - C_d - H_0)^{-1}\|_{\mathscr{L}(L^2;L^2)} \leq \frac{1}{|\mathrm{Im}\, z|^2},$$

$$\text{and} \quad \left| \partial_{\bar{z}} \tilde{\chi}(\frac{z}{(LA)^2}) \right| \leq C_{\chi,2} \frac{|\mathrm{Im}\, z|^2}{(LA)^4},$$

a simple integration yields the result. □

3.4 The Grushin problem

3.4.1 Functional analysis of the Grushin problem

We consider the operators

$$\mathscr{P}_z = \begin{pmatrix} B_{\pm,b,V^h} + Q_{A,L,V^h} - z & U_{\pm,\theta} \\ U_{\pm,\theta}^{-1}\pi_0 & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{P}'_z = \begin{pmatrix} B'_{\pm,b,V^h} + Q_{A,L,V^h} - \bar{z} & U_{\pm,\theta} \\ U_{\pm,\theta}^{-1}\pi_0 & 0 \end{pmatrix}. \tag{3.4.1.1}$$

Remember that $B_{\pm,b,V^h} \in \mathrm{OpS}^{3/2}_{\Psi}(Q^h, \mathrm{End}(\mathscr{E}^h_\pm))$ while $Q_{A,L,V^h} : \mathscr{S}'(X^h;\mathscr{E}^h_\pm) \to \mathscr{S}(X^h;\mathscr{E}^h_\pm)$ and $U_{\pm,\theta}$ is an isomorphism from $H^s(Q^h;\Lambda T^*Q^h \otimes F_\pm)$ to $\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm) \cap \ker(\alpha_\pm)$ for all $s \in \mathbb{R}$, with $F_+ = Q \times \mathbb{C}$ and $F_- = (Q \times \mathbb{C}) \otimes \mathbf{or}_{Q^h}$. In particular, the operators $\mathscr{P}_z$, $\mathscr{P}'_z$ are bounded

$$\mathscr{P}_z, \mathscr{P}'_z : \tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm) \oplus H^s(Q^h;\Lambda T^*Q^h \otimes F_\pm) \longrightarrow \tilde{\mathscr{W}}^{0,s-3/2}(X^h;\mathscr{E}^h_\pm) \oplus H^s(Q^h;\Lambda T^*Q^h \otimes F_\pm) \tag{3.4.1.2}$$

for all $s \in \mathbb{R}$. With

$$\bigcap_{s\in\mathbb{R}} \tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm) \oplus H^s(Q^h;\Lambda T^*Q^h \otimes F_\pm) = \mathscr{S}(X^h;\mathscr{E}^h_\pm) \oplus \mathscr{C}^\infty(Q^h;\Lambda T^*Q^h \otimes F_\pm), \tag{3.4.1.3}$$

$$\bigcup_{s\in\mathbb{R}} \tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm) \oplus H^s(Q^h;\Lambda T^*Q^h \otimes F_\pm) = \mathscr{S}'(X^h;\mathscr{E}^h_\pm) \oplus \mathscr{E}'(Q^h;\Lambda T^*Q^h \otimes F_\pm) \tag{3.4.1.4}$$

the continuity also holds with these spaces endowed with their usual topology.
We will use the following abbreviations

$$\begin{aligned} \tilde{\mathscr{W}}^{0,s} \oplus H^{s'} &= \tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm) \oplus H^{s'}(Q^h;\Lambda T^*Q^h \otimes F_\pm), \\ \mathscr{S} \oplus \mathscr{C}^\infty &= \mathscr{S}(X^h;\mathscr{E}^h_\pm) \oplus \mathscr{C}^\infty(Q^h;\Lambda T^*Q^h \otimes F_\pm), \\ B_{Q,z} &= B_{\pm,b,V^h} + Q_{A,L,V^h} - z \quad \text{and} \quad B'_{Q,z} = B'_{\pm,b,V^h} + Q_{A,L,V^h} - \bar{z}. \end{aligned}$$

We recall

$$\begin{aligned} & B_{\pm,b,V^h} = \frac{1}{b^2}\alpha_\pm + \frac{1}{b}\beta_\pm + \gamma_\pm \\ \text{with} \quad & \pi_{0,\pm} B_{Q,z} \pi_{\perp,\pm} = \pi_{0,\pm}(\frac{1}{b}\beta_\pm + \gamma_\pm)\pi_{\perp,\pm}, \\ & \pi_{\perp,\pm} B_{Q,z} \pi_{0,\pm} = \pi_{\perp,\pm}(\frac{1}{b}\beta_\pm + \gamma_\pm)\pi_{0,\pm}, \\ \text{and} \quad & \pi_{0,\pm} B_{Q,z} \pi_{0,\pm} = \pi_{0,\pm}(\gamma_\pm + Q_{A,L,V^h} - z)\pi_{0,\pm}. \end{aligned}$$

We check that $\mathscr{P}_z$ and $\mathscr{P}'_z$ are invertible in a weak sense under suitable conditions and then we deduce via the Schur complement formula an explicit expression of $\overline{(B_{\pm,b,V^h} + Q_{A,L,V^h}}^s - z)^{-1}$ as an operator from $\tilde{\mathscr{W}}^{0,s}(X^h;\mathscr{E}^h_\pm) \to \tilde{\mathscr{W}}^{0,s-3}(X^h;\mathscr{E}^h_\pm)$. Because the condition $b \le \frac{1}{C_s}$, which ensures the invertibility of $\overline{\pi_{\perp,\pm}(B_{\pm,b,V^h} - z)\pi_{\perp,\pm}}^s$ in Proposition 3.3.14, depends on s, there is no choice of parameters which guarantees the meaning of some formulas simultaneously for all $s \in \mathbb{R}$, but only for all s in a fixed interval $[s_{\min}, s_{\max}] \subset \mathbb{R}$. Therefore not all the compositions of operators in what follows make sense with the topologies of (3.4.1.3) and (3.4.1.4) and products or inverses must be handled carefully. In particular, although we make product of continuous operators between different spaces, they do not necessarily have a closed range and we must distinguish clearly left-inverses and right-inverses. Alternatively it is better handled by the separate studies of the uniqueness and of the existence of solutions for linear systems.

Proposition 3.4.1. *Assume that the condition $b \le \frac{1}{C_s}$ of Proposition 3.3.14 holds true for all $s \in [-s_{max}, s_{max}]$, for some $s_{max} \in]3, +\infty[$.*

1) *When* $|s| \le s_{max}$ *and* $\mathbb{R}\mathrm{e} z \le \frac{1}{24b^2}$*, the operator* $\mathscr{P}_z : \mathscr{S} \oplus \mathscr{C}^\infty \to \tilde{\mathscr{W}}^{\pm s} \oplus H^{\pm s+1}$ *admits a left-inverse*

$$\mathscr{G}_z = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \quad \in \mathscr{L}(\tilde{\mathscr{W}}^{0,\pm s} \oplus H^{\pm s+1}; \tilde{\mathscr{W}}^{0,\pm s} \oplus H^{\pm s-1}).$$

The same result holds of $\mathscr{P}'_z : \mathscr{S} \oplus \mathscr{C}^\infty \to \tilde{\mathscr{W}}^{\mp s} \oplus H^{\mp s+1}$ *with the left inverse*

$$\mathscr{G}'_z = \begin{pmatrix} E' & E'_+ \\ E'_- & E'_{-+} \end{pmatrix} \in \mathscr{L}(\tilde{\mathscr{W}}^{0,\mp s} \oplus H^{\mp s+1}; \tilde{\mathscr{W}}^{0,\mp s} \oplus H^{\mp s-1}).$$

2) *For* $|s| \le s_{max} - 3/2$ *and* $\mathbb{R}\mathrm{e} z \le \frac{1}{24b^2}$*, the relations*

$$\mathscr{G}_z \circ \mathscr{P}_z = i_{\tilde{\mathscr{W}}^{0,\pm s} \oplus H^{\pm s} \to \tilde{\mathscr{W}}^{0,\pm s-3/2} \oplus H^{\pm s-5/2}}$$
$$\text{and} \qquad \mathscr{G}'_z \circ \mathscr{P}'_z = i_{\tilde{\mathscr{W}}^{0,\mp s} \oplus H^{\mp s} \to \tilde{\mathscr{W}}^{0,\mp s-3/2} \oplus H^{\mp s-5/2}}$$

make sense as the products $A \circ B$*, with* $A \in \mathscr{L}(\tilde{\mathscr{W}}^{0,\pm s-3/2} \oplus H^{\pm s-1/2}; \tilde{\mathscr{W}}^{0,\pm s-3/2} \oplus H^{\pm s-5/2})$ *and* $B \in \mathscr{L}(\tilde{\mathscr{W}}^{0,\pm s} \oplus H^{\pm s}; \tilde{\mathscr{W}}^{0,\pm s-3/2} \oplus H^{\pm s}) \subset \mathscr{L}(\tilde{\mathscr{W}}^{0,\pm s} \oplus H^{\pm s}; \tilde{\mathscr{W}}^{0,\pm s-3/2} \oplus H^{\pm s-1/2})$.

3) *For* $|s| \le s_{max} - 3/2$*,* $z \in \mathbb{C} \setminus \sigma(B_{\pm,b,V^h} + Q_{A,L,V^h})$ *and* $\mathbb{R}\mathrm{e} z < \frac{1}{24b^2}$*, the operator* $E_{-+} \in \mathscr{L}(H^{s-1/2}; H^{s-5/2}) \subset \mathscr{L}(\mathscr{C}^\infty; \mathscr{E}')$ *admits a right-inverse* $(E_{-+})_r^{-1} \in \mathscr{L}(\mathscr{C}^\infty; \mathscr{C}^\infty)$ *and a left-inverse* $(E_{-+})_\ell^{-1} \in \mathscr{L}(\mathscr{E}'; \mathscr{E}')$ *with* $(E_{-+})_\ell^{-1} \in \mathscr{L}(H^{s'}; H^{s'+2/3})$ *for all* $s' \in \mathbb{R}$ *and* $(E_{-+})_\ell^{-1}\big|_{\mathscr{C}^\infty} = (E_{-+})_r^{-1}$.

4) *For* $|s| \le s_{max} - 4$ *the equality*

$$(\overline{B_{\pm,b,V^h} + Q_{A,L,V^h}}^s - z)^{-1} = E - E_+ (E_{-+})_\ell^{-1} E_- \tag{3.4.1.5}$$

holds in the sense of $\mathscr{L}(\tilde{\mathscr{W}}^{0,s}; \tilde{\mathscr{W}}^{0,s-3})$*-valued meromorphic functions in* $\left\{z \in \mathbb{C}, \mathbb{R}\mathrm{e}\, z < \frac{1}{24b^2}\right\}$.

Proof. **1)** The range of $\mathscr{P}_z\big|_{\mathscr{S} \oplus \mathscr{C}^\infty}$ is included in $\mathscr{S} \oplus \mathscr{C}^\infty$. Let us check that for $\begin{pmatrix} f \\ f_+ \end{pmatrix} \in \mathscr{S} \oplus \mathscr{C}^\infty$ the equation $\mathscr{P}_z \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} f \\ f_+ \end{pmatrix}$ admits at most one solution in $\tilde{\mathscr{W}}^{0,s} \oplus H^{s-1}$ when $|s| \le s_{max}$.
By applying $\pi_{\perp,\pm}$ to the first line of the system

$$\begin{cases} B_{Q,z} u + U_{\pm,\theta} u_- & = f \\ U_{\pm,\theta}^{-1} \pi_{0,\pm} u & = f_+ \end{cases}$$

we must have

$$\pi_{\perp,\pm} B_{Q,z} \pi_{\perp,\pm} u + \pi_{\perp,\pm}(\frac{1}{b}\beta_\pm + \gamma_\pm) U_{\pm,\theta} f_+ = \pi_{\perp,\pm} f\,.$$

When $\pi_{\perp,\pm} B_{Q,z} \pi_{\perp,\pm} = \pi_{\perp,\pm}(B_{\pm,b,V^h} - z)\pi_{\perp,\pm}$ is invertible

$$u = \pi_{\perp,\pm} u + \pi_{0,\pm} u = [\pi_\perp(\mathscr{B}_{\pm,b,V^h} - z)\pi_\perp]^{-1}(\pi_{\perp,\pm} f - \pi_{\perp,\pm}(\frac{1}{b}\beta_\pm + \gamma_\pm) U_{\pm,\theta} f_+) + U_{\pm,\theta} f_+$$

With the conditions $b \le \frac{1}{C_s}$ and $\mathbb{R}\mathrm{e} z \le \frac{1}{24b^2}$ and by noticing $\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm + \gamma_\pm) U_{\pm,\theta} \in \mathscr{L}(H^{s+1}; \tilde{\mathscr{W}}^{0,s})$, Proposition 3.3.14 actually says

$$u = Ef + E_+ f_+$$
$$\text{with} \quad E = (\overline{B_{\pm,b,V^h,\perp}}^s - z)^{-1} \pi_{\perp,\pm} \in \mathscr{L}(\tilde{\mathscr{W}}^{0,s}; \tilde{\mathscr{W}}^{0,s}), \tag{3.4.1.6}$$
$$\text{and} \quad E_+ = U_{\pm,\theta} - (\overline{B_{\pm,b,V^h,\perp}}^s - z)^{-1} \pi_{\perp,\pm}(\frac{1}{b}\beta_\pm + \gamma_\pm) U_{\pm,\theta} \in \mathscr{L}(H^{s+1}; \tilde{\mathscr{W}}^{0,s}) \tag{3.4.1.7}$$

By applying the projection $\pi_{0,\pm}$ on the first line of the system, we must have

$$\pi_{0,\pm}f=\pi_{0,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm+Q_{A,L,V^h}-z)u+U_{\pm,\theta}u_-=\pi_{0,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)\pi_{\perp,\pm}u+\pi_{0,\pm}(\gamma_\pm+Q_{A,L,V^h}-z)U_{\pm,\theta}f_++U_{\pm,\theta}u_-$$

and $\pi_{0,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm+Q_{A,L,V^h}-z)\in\mathscr{L}(\tilde{\mathscr{W}}^{0,s};H^{s-1})$ now gives

$$u_-=E_-f+E_{-+}f_+,$$

with
$$E_-=U_{\pm,\theta}^{-1}\pi_{0,\pm}-U_{\pm,\theta}^{-1}\pi_{0,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)(\overline{B_{\pm,b,V^h,\perp}}^s-z)^{-1}\pi_{\perp,\pm}\in\mathscr{L}(\tilde{\mathscr{W}}^{0,s};H^{s-1}),\tag{3.4.1.8}$$

$$E_{-+}=U_{\pm,\theta}^{-1}\pi_0(z-Q_{A,L,V^h}-\gamma_\pm)U_{\pm,\theta}+U_{\pm,\theta}^{-1}\pi_0(\frac{1}{b}\beta_\pm+\gamma_\pm)(\overline{B_{\pm,b,V^h,\perp}}^s-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)U_{\pm,\theta},$$

and
$$E_{-+}\in\mathscr{L}(H^{s+1};H^{s-1}).\tag{3.4.1.9}$$

The result for $\mathscr{P}'_z$ is straightforward because

$$\mathscr{P}'_z=\begin{pmatrix}B'_{\pm,b,V^h}+Q_{A,L,V^h}-\overline{z} & U_{\pm,\theta}\\ U_{\pm,\theta}^{-1}\pi_0 & 0\end{pmatrix}$$

and $(\overline{B'_{\pm,b,V^h,\perp}}^{-s}-\overline{z})=(W_\theta^2)^s(\overline{B_{\pm,b,V^h,\perp}}^s-z)^{*,s}(W_\theta^2)^{-s}$.

2) By **1)** and $|s|\leq s_{max}-3/2$, we know that $\mathscr{G}_z\in\mathscr{L}(\tilde{\mathscr{W}}^{0,s-3/2}\oplus H^{s-1/2};\tilde{\mathscr{W}}^{0,s-3/2}\oplus H^{s-5/2})$ is a left-inverse on $\mathscr{S}\oplus\mathscr{C}^\infty$ and we deduce

$$\forall\begin{pmatrix}u\\u_-\end{pmatrix}\in\mathscr{S}\oplus\mathscr{C}^\infty,\quad \mathscr{G}_z\circ\mathscr{P}_z\begin{pmatrix}u\\u_-\end{pmatrix}=\begin{pmatrix}u\\u_-\end{pmatrix}.$$

The density of $\mathscr{S}\oplus\mathscr{C}^\infty$ in $\tilde{\mathscr{W}}^{0,s}\oplus H^s$, combined with $\mathscr{P}_z\in\mathscr{L}(\mathscr{S}\oplus\mathscr{C}^\infty;\mathscr{S}\oplus\mathscr{C}^\infty)\cap\mathscr{L}(\tilde{\mathscr{W}}^{0,s}\oplus H^s;\tilde{\mathscr{W}}^{0,s-3/2}\oplus H^{s-1/2})$ and $\mathscr{G}_z\in\mathscr{L}(\tilde{\mathscr{W}}^{0,s-3/2}\oplus H^{s-1/2})$, yields the result.

3) Let us prove firstly that for $u_-\in\mathscr{C}^\infty\subset H^{s-5/2}$ the equation $E_{-+}f_+=u_-$ admits at least one solution in $f_+\in\mathscr{C}^\infty\subset H^{s-1/2}$ when $|s|\leq s_{\max}-3/2$, $z\notin\sigma(B_{\pm,b,V^h}+Q_{A,L,V^h})$ and $\mathbb{R}\mathrm{e}\,z\leq\frac{1}{b^2}$. Take $w=(\overline{B_{\pm,b,V^h}}^{s'}+Q_{A,L,V^h}-z)^{-1}U_{\pm,\theta}u_-$ which belongs to $D(\overline{B_{\pm,b,V^h}}^{s'})\subset\tilde{\mathscr{W}}^{0,s'+2/3}$ according to Proposition 3.2.9 where the condition $0<b\leq h\leq\frac{1}{C_g}$ does not depend on $s'\in\mathbb{R}$. Set $u=\pi_{0,\pm}w\in\pi_{0,\pm}\left(\cap_{s'\in\mathbb{R}}\tilde{\mathscr{W}}^{0,s'}\right)=U_{\pm,\theta}\mathscr{C}^\infty$ and $v=\pi_{\perp,\pm}w\in\pi_{\perp,\pm}\left(\cap_{s'\in\mathbb{R}}\tilde{\mathscr{W}}^{0,s'}\right)=\pi_{\perp,\pm}\mathscr{S}$. By projecting the equation $(B_{\pm,b,V^h}+Q_{A,L,V^h}-z)w=U_{\pm,\theta}u_-$ written in $\mathscr{S}$ with the projections $\pi_{0,\pm}$ and $\pi_{\perp,\pm}$, we obtain:

$$\pi_{0,\pm}(Q_{A,L,V^h}+\gamma_\pm-z)u+\pi_0(\frac{1}{b}\beta_\pm+\gamma_\pm)v=U_{\pm,\theta}u_-,$$

and
$$\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)u+\pi_{\perp,\pm}(B_{\pm,b,V^h}-z)v=0.$$

By taking $f_+=-U_{\pm,\theta}^{-1}u$ and by noticing $(\frac{1}{b}\beta_\pm+\gamma_\pm)u\in\tilde{\mathscr{W}}^{0,s-3/2}$, with $|s|\leq s_{max}-3/2$, we deduce

$$\pi_{0,\pm}(Q_{A,L,V^h}+\gamma_\pm-z)u-\pi_{0,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)(\overline{B_{\pm,b,V^h,\perp}}^{s-3/2}-z)^{-1}(\frac{1}{b}\beta_\pm+\gamma_\pm)u=U_{\pm,\theta}u_-.$$

This means exactly that $f_+=U_{\pm,\theta}^{-1}u\in\mathscr{C}^\infty\subset H^{s-1/2}$ satisfies $E_{-+}f_+=u_-$ in $H^{s-5/2}$. But the formula $f_+=(E_{-+})_r^{-1}u_-=U_{\pm,\theta}^{-1}\pi_0(\overline{B_{\pm,b,V^h}+Q_{A,L,V^h}}^{s'}-z)^{-1}U_{\pm,\theta}u_-$ for all $s'\in\mathbb{R}$, proves that this right-inverse $(E_{-+})_r^{-1}$ actually belongs to $\mathscr{L}(\mathscr{C}^\infty;\mathscr{C}^\infty)$.

With the dual statements of **1)** and **2)** this means also that the formal adjoint $E'_{-+}\in\mathscr{L}(H^{-s-1/2};H^{-s-5/2})\subset\mathscr{L}(\mathscr{C}^\infty;\mathscr{E}')$ admits the right-inverse $(E'_{-+})_r^{-1}u_-=U_{\pm,\theta}\pi_0(\overline{B'_{\pm,b,V^h}+Q_{A,L,V^h}}^{-s'}-\bar{z})^{-1}U_{\pm,\theta}$ for all

$s' \in \mathbb{R}$ and $(E'_{-+})_r^{-1} \in \mathscr{L}(\mathscr{C}^\infty;\mathscr{C}^\infty)$. Duality implies that $(E_{-+})_\ell^{-1} = [(E'_{-+})_r^{-1}]' \in \mathscr{L}(\mathscr{E}';\mathscr{E}')$ is a left-inverse of $E_{-+} \in \mathscr{L}(\mathscr{C}^\infty;\mathscr{D}')$. With the formula

$$[(E'_{-+})_r^{-1}]' = \left[U_{\pm,\theta}^{-1}(\overline{B'_{\pm,b,V^h} + Q_{A,L,V^h}}^{s'} - \bar{z})^{-1}U_{\pm,\theta}\right]' = U_{\pm,\theta}^{-1}(\overline{B_{\pm,b,V^h} + Q_{A,L,V^h}}^{-s'} - z)^{-1}U_{\pm,\theta}\,, \tag{3.4.1.10}$$

the regularizing property of $(E_{-+})_\ell^{-1} \in \mathscr{L}(H^{s'};H^{s'+2/3})$ and the identification $(E_{-+})_\ell^{-1}\big|_{\mathscr{C}^\infty} = (E_\pm)_r^{-1}$ are straightforward.
4) When $|s| \le s_{max} - 3/2$ and $\mathbb{R}\mathrm{e}\,z \le \frac{1}{24b^2}$, the first formula of **2)** implies

$$(EB_{Q,z} + E_+U_{\pm,\theta}^{-1}\pi_0)u = u \quad \text{in } \tilde{\mathscr{W}}^{0,s-3/2}\,, \tag{3.4.1.11}$$

$$(E_-B_{Q,z} + E_{-+}U_{\pm,\theta}^{-1}\pi_0)u = 0 \quad \text{in } H^{s-5/2}\,, \tag{3.4.1.12}$$

for all $u \in \tilde{\mathscr{W}}^{0,s}$. With $|s-5/2| \le |s| + 5/2 \le s_{max} - 3/2$, and **3)**, the equality (3.4.1.12) becomes

$$U_{\pm,\theta}^{-1}\pi_0 u = -(E_{-+})_\ell^{-1}E_-B_{Q,z}u \quad \text{in } H^{s-5/2+2/3}\,.$$

for $z \notin \sigma(B_{\pm,b,V^h} + Q_{A,L,V^h})$.
Put in the equality (3.4.1.11) we obtain

$$(E - E_+(E_{-+})_\ell^{-1}E_-)(B_{Q,z})u = u \quad \text{in } \tilde{\mathscr{W}}^{0,s-5/2+2/3-1}\,,$$

when $z \notin \sigma(B_{\pm,b,V^h} + Q_{A,L,V^h})$ and $u \in \tilde{\mathscr{W}}^{0,s}$. Applied to $u = (\overline{B_{\pm,b,V^h} + Q_{A,L,V^h}}^{s-2/3} - z)^{-1}v \in \tilde{\mathscr{W}}^{0,s}$ for $v \in \tilde{\mathscr{W}}^{0,s-2/3}$, we obtain

$$(E - E_+(E_{-+})_\ell^{-1}E_-) = (\overline{B_{\pm,b,V^h} + Q_{A,L,V^h}}^{s-2/3} - z)^{-1} \text{in } \mathscr{L}(\tilde{\mathscr{W}}^{0,s-2/3};\tilde{\mathscr{W}}^{s-5/2-1/3}).$$

and we know that the right-hand side is a meromorphic function of z in $\mathbb{C}$.
The condition $|s| \le s_{max} - 4 \le s_{max} - 5/2 - 2/3$ allows to replace $s - 2/3$ by s, by noticing $s + 2/3 - 5/2 - 1/3 \ge s - 3$, with

$$(E - E_+(E_{-+})^{-1}E_-) = (\overline{B_{\pm,b,V^h} + Q_{A,L,V^h}}^{s} - z)^{-1}$$

as an $\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s-3})$-valued meromorphic function in $\left\{z \in \mathbb{C}, \mathbb{R}\mathrm{e}\,z < \frac{1}{24b^2}\right\}$. □

3.4.2 Quantitative comparisons of truncated resolvents

After setting

$$\delta_{B,\Delta,z} = (B_{\pm,b,V^h} + Q_{A,L,V^h} - z)^{-1} - U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1} \tag{3.4.2.1}$$

we consider the finite rank operators

$$\delta_{B,\Delta,z} \circ Q_{A,L',V^h} \quad \text{and} \quad Q_{A,L',V^h} \circ \delta_{B,\Delta,z}\,,$$

where $L' \ge L \ge 1$ will be fixed later.

Proposition 3.4.2. *Let $L \ge 1$ be the uniform constant of Proposition 3.3.18 and fix $L' \ge 1$. For all $s \in \mathbb{R}$, there exist constants $C_s \ge 1$, determined by s such that the condition $C_s \max(b, Ab, A^{-1}) \le 1$ implies the inequalities*

$$\|\delta_{B,\Delta,z} \circ Q_{A,L',V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \le C_s \frac{bA^{\frac{3}{2}}}{1 + A^{-1}\sqrt{|\mathrm{Im}z|}} \tag{3.4.2.2}$$

$$\|Q_{A,L',V^h} \circ \delta_{B,\Delta,z}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \le C_s \frac{bA^{\frac{3}{2}}}{1 + A^{-1}\sqrt{|\mathrm{Im}z|}} \tag{3.4.2.3}$$

for all $z \in \mathbb{C}$ such that $|\mathrm{Re}z| \le \frac{A^2}{2}$.

Proof. For a given $s\in\mathbb{R}$ we fix $s_{\max}\geq |s|+10$ so that the estimates of Proposition 3.4.1 and the expressions of E,E_-,E_+ and E_{-+} given in the proof make sense for the Sobolev exponent s replaced by $s_2\in[s,s+6]$. Actually for $|s_2|\leq s_{\max}-4$ and $\mathrm{Re} z\leq \frac{A^2}{2}\leq \min(\frac{A^2}{2},\frac{1}{24b^2})$, the equality (3.4.1.5) gives

$$\delta_{B,\Delta,z}\circ Q_{A,L',V^h}=(E-E_+(E_{-+})_\ell^{-1}E_-)Q_{A,L',V^h}-U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}\,,$$

as an equality of $\mathcal{L}(\tilde{\mathcal{W}}^{0,s_2};\tilde{\mathcal{W}}^{0,s_2-3})$-valued meromorphic functions. Actually because $Q_{A,L',V^h}\in\mathcal{L}(\mathcal{S}';\mathcal{S})$ while $\delta_{B,\Delta,z}\in\mathcal{L}(\mathcal{S},\mathcal{S})$ when $\mathrm{Re} z\leq\frac{A^2}{2}$, $z\not\in\mathrm{Spec}(\overline{B_{\pm,b,V^h}+Q_{A,L,V^h}}^{s_2})$, the left-hand side as well as the final term belong to $\mathcal{L}(\mathcal{S}';\mathcal{S})$. The above equality can therefore be extended to an equality of $\mathcal{L}(\tilde{\mathcal{W}}^{0,s_2};\tilde{\mathcal{W}}^{0,s_2})$-valued meromorphic function. Actually with $\mathrm{Re} z\leq A^2/2$, z is not in $\mathrm{Spec}(\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h})$ and the final term is holomorphic. Owing to $EQ_{A,L,V^h}=E\pi_{0,\pm}=0$ and with the expressions (3.4.1.7) of E_+ and (3.4.1.8) of E_-, we obtain

$$\begin{aligned}\delta_{B,\Delta,z}\circ Q_{A,L',V^h} &= -E_+(E_{-+})_\ell^{-1}E_-Q_{A,L',V^h}-U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}\\ &= (I)+(II),\end{aligned}$$

where

$$\begin{aligned}(I) &= -U_{\pm,\theta}[(E_{-+})_\ell^{-1}+(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}]U_{\pm,\theta}^{-1}Q_{A,L',V^h}\\ &= -U_{\pm,\theta}(E_{-+})_\ell^{-1}[\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z+E_{-+}](\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}\,,\\ (II) &= (\overline{B_{\pm,b,V^h,\perp}}^{s_2'}-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)U_{\pm,\theta}(E_{-+})_\ell^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}.\end{aligned}$$

and s_2' is any other exponent such that $|s_2'|\leq s_{\max}$.
By combining Bismut's formula (3.2.5.23), recalled here,

$$U_{\pm,\theta}^{-1}[\pi_{0,\pm}(\gamma_\pm-\beta_\pm\alpha_\pm^{-1}\beta_\pm)\pi_{0,\pm}]U_{\pm,\theta}=\frac{1}{2}\Delta_{V^h,1}\,,$$

with the expression (3.4.1.9) of E_{-+}, (I) becomes

$$(I)=U_{\pm,\theta}(E_{-+})_\ell^{-1}U_{\pm,\theta}^{-1}\pi_{0,\pm}(III)U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}\,,$$

where

$$(III)=\beta_\pm\alpha_\pm^{-1}\beta_\pm-(\frac{1}{b}\beta_\pm+\gamma_\pm)(\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm),$$

and $|s_2''|\leq s_{\max}$. The above operator can be rewritten

$$\begin{aligned}(III) &= (\frac{1}{b}\beta_\pm+\gamma_\pm)\pi_{\perp,\pm}[b^2\alpha_\pm^{-1}-(\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)^{-1}]\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)\\ &\qquad -b(\beta_\pm+b\gamma_\pm)\alpha_\pm^{-1}\pi_{\perp,\pm}\gamma_\pm-b\gamma_\pm\alpha_\pm^{-1}\beta_\pm\\ &= (\frac{1}{b}\beta_\pm+\gamma_\pm)\alpha_\pm^{-1}\pi_{\perp,\pm}[b^2(\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)-\alpha_\pm](\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)\\ &\qquad -b(\beta_\pm+b\gamma_\pm)\alpha_\pm^{-1}\pi_{\perp,\pm}\gamma_\pm-b\gamma_\pm\alpha_\pm^{-1}\beta_\pm\\ &= \underbrace{(\frac{1}{b}\beta_\pm+\gamma_\pm)\alpha_\pm^{-1}\pi_{\perp,\pm}[b\beta_\pm+b^2(\gamma_\pm-z)](\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)}_{(III')}\\ &\qquad \underbrace{-b(\beta_\pm+b\gamma_\pm)\alpha_\pm^{-1}\pi_{\perp,\pm}\gamma_\pm-b\gamma_\pm\alpha_\pm^{-1}\beta_\pm}_{-(III.3)}\,.\end{aligned}$$

Let's rewrite (III') as the sum of $(III.1)$ and $(III.2)$ where

$$\begin{aligned}(III.1) &= (\frac{1}{b}\beta_\pm+\gamma_\pm)\alpha_\pm^{-1}\pi_{\perp,\pm}[b\beta_\pm+b^2\gamma_\pm](\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm)\\ (III.2) &= -zb^2(\frac{1}{b}\beta_\pm+\gamma_\pm)\alpha_\pm^{-1}(\overline{B_{\pm,b,V^h,\perp}}^{s_2''}-z)^{-1}\pi_{\perp,\pm}(\frac{1}{b}\beta_\pm+\gamma_\pm).\end{aligned}$$

The operator $(III.1)$ can be depicted by

$$\tilde{\mathscr{W}}^{0,s}\xleftarrow[\frac{1}{b}\beta_\pm+\gamma_\pm]{}\tilde{\mathscr{W}}^{2,s+1}\xleftarrow[\alpha_\pm^{-1}\pi_{\perp,\pm}]{}\tilde{\mathscr{W}}^{0,s+1}\xleftarrow[b\beta_\pm+b^2\gamma_\pm]{}\tilde{\mathscr{W}}^{0,s+\frac{5}{2}}\xleftarrow[(B_{\pm,b,V^h,\perp}-z)^{-1}\pi_{\perp,\pm}]{}\tilde{\mathscr{W}}^{0,s+\frac{5}{2}}\xleftarrow[\frac{1}{b}\beta_\pm+\gamma_\pm]{}\tilde{\mathscr{W}}^{2,s+\frac{7}{2}}$$

and "$-(III.2)/zb^2$" is depicted by

$$\tilde{\mathscr{W}}^{0,s}\xleftarrow[\frac{1}{b}\beta_\pm+\gamma_\pm]{}\tilde{\mathscr{W}}^{2,s+1}\xleftarrow[\alpha_\pm^{-1}\pi_{\perp,\pm}]{}\tilde{\mathscr{W}}^{0,s+1}\xleftarrow[(B_{\pm,b,V^h,\perp}-z)^{-1}\pi_{\perp,\pm}]{}\tilde{\mathscr{W}}^{0,s+1}\xleftarrow[\frac{1}{b}\beta_\pm+\gamma_\pm]{}\tilde{\mathscr{W}}^{2,s+2}.$$

Combining the previous decomposition with, on one side

$$\forall s_1,s_2\in\mathbb{R},\quad \beta_\pm+b\gamma_\pm\in \mathrm{OpS}_\Psi^{\frac{3}{2}}(X^h,\mathscr{E}_\pm^h)\cap\mathscr{L}(\tilde{\mathscr{W}}^{s_1+2,s_2+1};\tilde{\mathscr{W}}^{s_1,s_2})$$

with semi-norms uniformly bounded with respect to $b\in(0,1]$, and on the other side

$$\forall s\in\mathbb{R},\quad \|(B_{\pm,b,V^h,\perp}-z)^{-1}\pi_{\perp,\pm}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}\leq 24b^2$$

due to Proposition 3.3.14 as soon as $\mathrm{Re}z\leq\frac{1}{24b^2}$, implies

$$\|(III.1)\|_{\mathscr{L}(\tilde{\mathscr{W}}^{2,s+\frac{7}{2}};\tilde{\mathscr{W}}^{0,s})}\leq bC_s\quad\text{and}\quad\|(III.2)\|_{\mathscr{L}(\tilde{\mathscr{W}}^{2,s+2};\tilde{\mathscr{W}}^{0,s})}\leq C_s|z|b^2.$$

We claim that

$$\|(III.3)\|_{\mathscr{L}(\tilde{\mathscr{W}}^{2,s+1};\tilde{\mathscr{W}}^{0,s})}\leq C_sb.$$

Now we decompose (I) as

$$(I)=(I.1)+(I.2)-(I.3)$$

With $(I.*)$ depicted by

$$\tilde{\mathscr{W}}^{0,s}\xleftarrow[U_{\pm,\theta}(E_{-+})_\ell^{-1}U_{\pm,\theta}^{-1}\pi_{0,\pm}]{}\tilde{\mathscr{W}}^{0,s}\xleftarrow[(III.*)]{}\tilde{\mathscr{W}}^{2,s'}\xleftarrow[U_{\pm,\theta}(\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}]{}\tilde{\mathscr{W}}^{0,s},\tag{3.4.2.4}$$

where the choice of s' will depend on the cases for indexed by $*$.

$$\|U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{2,s'})}=\frac{d}{2}\|(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}\tilde{Q}_{A,L',V^h}\|_{\mathscr{L}(H^s;H^{s'})}.$$

With $\tilde{Q}_{A,L',V^h}=A^2\chi(\frac{C_d+\Delta_{V^h,1}}{(L'A)^2})$, the inequality (3.3.3.6) gives

$$\|\tilde{Q}_{A,L',V^h}u\|_{H^{s'}}\leq C_{L',s,s'}A^2A^{(s'-s)_+}\|u\|_{H^s};$$

The inequality 3.3.3.7 gives

$$\|(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}\|_{\mathscr{L}(H^{s_2};H^{s_2})}\leq\frac{4}{A^2+2|\mathrm{Im}z|}$$

when $\frac{1}{A}\leq\frac{1}{C_{s'}}$ and $C_{s'}\geq 1$ large enough, and

$$\|(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}\tilde{Q}_{A,L',V^h}u\|_{\mathscr{L}(H^s;H^{s'})}\leq C_{s,s'}\frac{A^{(s'-s)_+}}{1+2|\mathrm{Im}z|A^{-2}}.$$

We conclude that

$$\|U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h}+\tilde{Q}_{A,L,V^h}-z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L',V^h}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{2,s'})}\leq C_{s,s'}\frac{A^{(s'-s)_+}}{1+2|\mathrm{Im}z|A^{-2}}.$$

For the left arrow, the expression (3.4.1.10) of E_{-+}^{-1} combined with the subelliptic estimate of Proposition 3.3.18 (and Proposition 3.3.1) gives

$$\|(E_{-+})_\ell^{-1}\|_{\mathcal{L}(H^s;H^s)}\leq\frac{C_s}{A^2(1+b\sqrt{|\mathrm{Im}\ z|})}.$$

We can now conclude for the norm estimate of $I.*$ decomposed as (3.4.2.4):

— For $(I.*)=(I.1)$, we take $s'=s+\frac{7}{2}$ and we obtain

$$\begin{aligned}\|(I.1)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})} &\leq C_s\frac{A^{-2}}{1+b\sqrt{|\mathrm{Im}z|}}\|(III.1)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s+\frac{7}{2}})}\frac{A^{\frac{7}{2}}}{1+2|\mathrm{Im}z|A^{-2}}\\ &\leq \frac{C_s}{(1+b\sqrt{|\mathrm{Im}z|})(1+2|\mathrm{Im}z|A^{-2})}bA^{\frac{3}{2}}.\end{aligned}$$

— For $(I,*)=(I.2)$, we take $s'=s+2$ and we obtain

$$\|(I.2)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}\leq C_s\frac{|z|b^2}{(1+b\sqrt{|\mathrm{Im}z|})(1+2|\mathrm{Im}z|A^{-2})}.$$

— For $(I,*)=(I.3)$, we take $s'=s+1$ and we obtain

$$\|(I.3)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}\leq C_sA^{-2}\|(III.3)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s+1})}A\leq C_s\frac{b}{A(1+b\sqrt{|\mathrm{Im}z|})(1+2|\mathrm{Im}z|A^{-2})}.$$

we proved

$$\|(I)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}\leq C_s\frac{bA^{\frac{3}{2}}+b^2|z|+\frac{b}{A}}{(1+b\sqrt{|\mathrm{Im}z|})(1+2|\mathrm{Im}z|A^{-2})}.$$

The operator (II) is depicted by

$$\tilde{\mathcal{W}}^{0,s}\xleftarrow{(\overline{B_{\pm,b,V^h,\perp}}^s-z)^{-1}\pi_{\perp,\pm}}\tilde{\mathcal{W}}^{0,s}\xleftarrow{\frac{1}{b}\beta_\pm+\gamma_\pm}\tilde{\mathcal{W}}^{2,s+1}\xleftarrow{U_{\pm,\theta}(E_{-+})_\ell^{-1}U_{\pm,\theta}^{-1}}\tilde{\mathcal{W}}^{2,s+1}\xleftarrow{Q_{A,L',V^h}}\tilde{\mathcal{W}}^{0,s}.$$

The same arguments as above lead to

$$\|(II)\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}\leq C_s\frac{Ab}{(1+b\sqrt{|\mathrm{Im}z|})(1+2|\mathrm{Im}z|A^{-2})}.$$

The largest upper bound is $C_s\frac{bA^{\frac{3}{2}}+b^2|z|}{(1+b\sqrt{|\mathrm{Im}z|})(1+2|\mathrm{Im}z|A^{-2})}$) obtained for the term (I).
Under the condition $|\mathrm{Re}z|\leq A^2$ we have the inequaity

$$\frac{|z|b^2}{(1+|\mathrm{Im}z|A^{-2})(1+b\sqrt{|\mathrm{Im}z|})}\leq A^2b^2+\frac{\sqrt{|\mathrm{Im}z|}b}{1+2|\mathrm{Im}z|A^{-2}}.$$

This leads to

$$\|\delta_{B,\Delta,z}\circ Q_{A,L',V^h}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}\leq C_s\frac{bA^{\frac{3}{2}}+b\sqrt{\mathrm{Im}(z)}}{1+|\mathrm{Im}z|A^{-2}}\leq C_s'\frac{bA^{\frac{3}{2}}}{1+A^{-1}\sqrt{|\mathrm{Im}z|}}.$$

Finally the estimate (3.4.2.3) for $Q_{A,L',V^h}\delta_{B,\Delta,z}$ is obtained by taking the adjoints with z replaced by $\overline{z}$ and $B_{\pm,b,V^h}$ replaced by its formal adjoint $B'_{\pm,b,V^h}$ which has the same properties as $B_{\pm,b,V^h}$.
□

For all $z \notin \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h})$, we define the intermediate operators

$$
\begin{aligned}
M_{B,z} &= I_{\tilde{\mathscr{W}}^{0,s}} - (B_{\pm,b,V^h} + Q_{A,L,V^h} - z)^{-1} Q_{A,L,V^h} \\
M_{\Delta,z} &= I_{\tilde{\mathscr{W}}^{0,s}} - U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1} U_{\pm,\theta}^{-1} Q_{A,L,V^h} \\
\tilde{M}_{\Delta,z} &= I_{H^s} - (\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1} \tilde{Q}_{A,L,V^h}
\end{aligned}
$$

while the other ordered products are recovered by taking the formal adjoints

$$
\begin{aligned}
M'_{B',\bar{z}} &= I_{\tilde{\mathscr{W}}^{0,s}} - Q_{A,L,V^h}(B_{\pm,b,V^h} + Q_{A,L,V^h} - z)^{-1} \\
M'_{\Delta,\bar{z}} &= I_{\tilde{\mathscr{W}}^{0,s}} - Q_{A,L,V^h} U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1} U_{\pm,\theta}^{-1} \\
\tilde{M}'_{\Delta,\bar{z}} &= I_{H^s} - \tilde{Q}_{A,L,V^h}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}.
\end{aligned}
$$

Lemma 3.4.3. *For $s \in \mathbb{R}$, there is a constant $C_s \geq 1$ depending on s and for all $z \in \mathbb{C}$ the inequality holds*

$$
\begin{aligned}
\|\tilde{M}_{\Delta,z}^{-1}\|_{\mathscr{L}(H^s;H^s)} &\leq C_s(1 + \frac{A^2}{\mathrm{dist}(z, \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}) \\
\|M_{\Delta,z}^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} &\leq C_s\left(1 + \frac{A^2}{\mathrm{dist}(z, \mathrm{Spec}(\Delta_{V^h,1}))}\right)
\end{aligned}
\tag{3.4.2.5}
$$

for all $z \notin \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}) \cap \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h})$.
The more accurate conditions

$$
\begin{aligned}
&|\mathrm{Re} z| \leq \frac{A^2}{2} \quad , \quad bA^4 \mathrm{dist}\,(z, \frac{1}{2}\mathrm{Spec}\,(\Delta_{V^h,1}) \leq \frac{1}{C_s}, \\
&C_s \max(b, Ab, A^{-1}) \leq 1,
\end{aligned}
$$

suffice for the uniform estimates

$$
\|M_{\Delta,z}^{-1}\delta_{B,\Delta,z} Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} + \|M_{B,z}^{-1}\delta_{B,\Delta,z} Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq \frac{1}{2} \tag{3.4.2.6}
$$

$$
\|M_{B,z}^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq C_s\left(1 + \frac{A^2}{\mathrm{dist}(z, \mathrm{Spec}(\Delta_{V^h,1}))}\right) \tag{3.4.2.7}
$$

$$
\|(M'_{B',\bar{z}})^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq C_s\left(1 + \frac{A^2}{\mathrm{dist}(z, \mathrm{Spec}(\Delta_{V^h,1}))}\right). \tag{3.4.2.8}
$$

Proof. A straightforward computation gives

$$
\tilde{M}_{\Delta,z} = (\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}(\frac{1}{2}\Delta_{V^h,1} - z).
$$

We deduce that the operator $\tilde{M}_{\Delta,z}$ is invertible when $z \notin \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})$ and the inverse is given by

$$
\begin{aligned}
\tilde{M}_{\Delta,z}^{-1} &= (\frac{1}{2}\Delta_{V^h,1} - z)^{-1}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z) \\
&= I + \tilde{Q}_{A,L,V^h}(\frac{1}{2}\Delta_{V^h,1} - z)^{-1}.
\end{aligned}
$$

The functional calculus for the self adjoint operator $\frac{1}{2}\Delta_{V^h,1}$ yields

$$\|\tilde{M}_{\Delta,z}^{-1}\|_{\mathscr{L}(H^s;H^s)} \leq C_s\left(1+\frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right).$$

The operator $M_{\Delta,z}$ is actually invertible when $z \notin \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})$ for the following reason:

$$\begin{aligned} M_{\Delta,z} &= \pi_{\perp,\pm} + \pi_{0,\pm}\left[I_{\ker(\alpha_\pm)} - U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}Q_{A,L,V^h}\right]\pi_{0,\pm} \\ &= \pi_{\perp,\pm} + U_{\pm,\theta}\tilde{M}_{\Delta,z}U_{\pm,\theta}^{-1} \end{aligned}$$

and its inverse equals

$$\begin{aligned} M_{\Delta,z}^{-1} &= \pi_{\perp,\pm} + U_{\pm,\theta}\tilde{M}_{\Delta,z}^{-1}U_{\pm,\theta}^{-1} \\ &= I_{\tilde{\mathscr{W}}^{0,s}} + U_{\pm,\theta}\left[\tilde{Q}_{A,L,V^h}(\frac{1}{2}\Delta_{V^h,1} - z)^{-1}\right]U_{\pm,\theta}^{-1}. \end{aligned}$$

The estimate (3.4.2.5) of $M_{\Delta,z}^{-1}$ follows.
Another computation gives

$$M_{B,z} = M_{\Delta,z}(I - M_{\Delta,z}^{-1}\delta_{B,\Delta,z}Q_{A,L,V^h})$$

we deduce that it is invertible as soon as $\|M_{\Delta,z}^{-1}\delta_{B,\Delta,z}Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} < 1$ and the norm of its inverse is given by

$$\|M_{B,z}^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq \frac{\|M_{\Delta,z}^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}}{1-\|M_{\Delta,z}^{-1}\delta_{B,\Delta,z}Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}}.$$

Applying inequality (3.4.2.2) yields

$$\begin{aligned} \|M_{\Delta,z}^{-1}\delta_{B,\Delta,z}Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} &\leq C_s\left(1+\frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)\left(\frac{bA^{\frac{3}{2}}}{1+A^{-1}\sqrt{|\mathrm{Im}z|}}\right) \\ &\leq \begin{cases} C_s\frac{bA^{\frac{7}{2}}}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))} & \text{if } |\mathrm{Im}z| \leq A^2 \\ C_s\frac{bA^{\frac{5}{2}}}{\sqrt{|\mathrm{Im}z|}} & \text{if } |\mathrm{Im}z| \geq A^2 \end{cases} \end{aligned}$$

Conditions $|\mathrm{Im}z| \geq A^2$ and $|\mathrm{Re}z| \leq \frac{A^2}{2}$ ensure

$$\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})) \geq |\mathrm{Im}z| \geq \frac{1}{2}|z| \geq \frac{1}{2}\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})). \qquad (3.4.2.9)$$

The condition $C_s\max(Ab,b,A^{-1}) \leq 1$ and 3.4.2.9 allow us to give a sufficient condition for

$$\|M_{\Delta,z}^{-1}\delta_{B,\Delta,z}Q_{A,L,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} < \frac{1}{2}$$

which is $b\frac{A^4}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))} \leq \frac{1}{4C_s}$. Finally the estimate $M'_{B',\bar{z}}$ is obtained by taking the adjoints with z replaced by $\bar{z}$ and $B_{\pm,b,V^h}$ replaced by its formal adjoint $B'_{\pm,b,V^h}$, which has the same properties as $B_{\pm,b,V^h}$. □

3.4.3 Quantitative comparison of resolvents

When $\mathrm{Re} z \leq \frac{A^2}{2}$, Proposition 3.3.18 and the subelliptic estimate (3.3.1.1) for $B_{\pm,b,V^h}+A^2\pi_{0,\pm}-z$ say that z belongs to the resolvent set of $\overline{B_{\pm,b,V^h}+Q_{A,L,V^h}}^s$ and

$$\begin{aligned} B_{\pm,b,V^h}-z &= (I-Q_{A,L,V^h}(B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1})(B_{\pm,b,V^h}+Q_{A,L,V^h}-z) \\ &= M'_{B',\bar{z}}(B_{\pm,b,V^h}+Q_{A,L,V^h}-z) \quad \text{in } \mathscr{L}(D(\overline{B_{\pm,b,V^h}}^s);\tilde{\mathscr{W}}^{0,s}) \quad (3.4.3.1)\\ \text{and} \quad B_{\pm,b,V^h}-z &= (B_{\pm,b,V^h}+Q_{A,L,V^h}-z)(I-(B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1}Q_{A,L,V^h}) \\ &= (B_{\pm,b,V^h}+Q_{A,L,V^h}-z)M_{B,z} \quad \text{in } \mathscr{L}(\tilde{\mathscr{W}}^{0,s};D((B^{*,s}_{\pm,b,V^h})')) \quad (3.4.3.2) \end{aligned}$$

Because Corollary 3.3.3 says that $\overline{B_{\pm,b,V^h}+C_s}^s$ is maximal accretive, we focus on the case $|\mathrm{Re} z| \leq \frac{A^2}{2}$ under the condition $A \geq C_s \geq 1$.

Proposition 3.4.4. *For $s \in \mathbb{R}$ there exists $C_s \geq 1$ such that the conditions $C_s \max(Ab, b, A^{-1}) \leq 1$ and*

$$|\mathrm{Re} z| \leq \frac{A^2}{2} \quad , \quad \mathrm{dist}(z, \mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})) \geq C_s b A^4 ,$$

imply the inequalities

$$\begin{aligned} &\|\big((B_{\pm,b,V^h}-z)^{-1}-U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}U^{-1}_{\pm,\theta}\big)\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \\ &\qquad \leq \left[\left(1+\frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)^2 bA^{-\frac{1}{2}}+A^{-2}\right]\frac{C_s}{1+b\sqrt{|\mathrm{Im} z|}}. \quad (3.4.3.3) \end{aligned}$$

Proof. Let us decompose $u \in \tilde{\mathscr{W}}^{0,s}$ as $u = u_{low}+u_{high}$ with

$$\begin{cases} u_{low} &= U_{\pm,\theta}\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2})U^{-1}_{\pm,\theta}u = \frac{1}{A^2}Q_{A,L',V^h}u \quad \text{with} \quad L'=2L \\ u_{high} &= u-u_{low} \end{cases}.$$

The upper bound on the norm of the operator $\mathfrak{D} = \big((B_{\pm,b,V^h}-z)^{-1}-U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}U^{-1}_{\pm,\theta}\big)$ will be obtained by considering separately its action on the two pieces u_{high} and u_{low}.
For u_{high} we have

$$\|\mathfrak{D}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}} \leq \|(B_{\pm,b,V^h}-z)^{-1}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}}+\|U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}U^{-1}_{\pm,\theta}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}} .$$

The second term of the right-hand side is bounded by

$$\begin{aligned} \|U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}U^{-1}_{\pm,\theta}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}} &= \|U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2}))U^{-1}_{\pm,\theta}u\|_{\tilde{\mathscr{W}}^{0,s}} \\ &\leq \|(\frac{1}{2}\Delta_{V^h}-z)^{-1}(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2}))\|_{\mathscr{L}(H^s;H^s)}\|u\|_{\tilde{\mathscr{W}}^{0,s}} . \end{aligned}$$

The choice of the cut-off function χ and $\tilde{Q}_{A,L,V^h} = A^2\chi\left(\frac{C_d+\Delta_{V^h,1}}{(LA)^2}\right)$ ensure the equality

$$(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2})) = (\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}\circ(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2}))$$

where the two factors satisfy

$$\begin{aligned} &\|(\frac{1}{2}\Delta_{V^h,1}+\tilde{Q}_{A,L,V^h}-z)^{-1}\|_{\mathscr{L}(H^s;H^s)} \leq \frac{4}{A^2+2|\mathrm{Im} z|} \\ \text{and} \quad &\|(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2}))\|_{\mathscr{L}(H^s;H^s)} \leq C_s \end{aligned}$$

respectively according to inequality (3.3.3.6) and inequality (3.3.3.7). We have proved

$$\|(\frac{1}{2}\Delta_{V^h}-z)^{-1}(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2}))\|_{\mathscr{L}(H^s;H^s)}\leq\frac{C_s}{A^2+2|\mathrm{Im}z|}\leq\frac{C_s}{A^2}\frac{1}{1+b\sqrt{|\mathrm{Im}z|}}\,.$$

According to the formula (3.4.3.1), the inverse of $B_{\pm,b,V^h}-z$ equals

$$\begin{aligned}(B_{\pm,b,V^h}-z)^{-1} &= (B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1}M'^{-1}_{B',\bar z}\\ &= (B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1}\left(M'^{-1}_{\bar z}+[M'^{-1}_{B',\bar z}-M'^{-1}_{\bar z}]\right)\\ &= (B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1}\left(I+M'^{-1}_{B',\bar z}\left[Q_{A,L,V^h}\delta_{B,\Delta,z}\right]\right)M'^{-1}_{\bar z}\,.\end{aligned}$$

Therefore, the subelliptic estimate given in Proposition 3.3.18 for $B_{\pm,b,V^h}+Q_{A,L,V^h}-z$ when $|\mathrm{Re}z|\leq\frac{A^2}{2}$,

$$\|(B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}\leq\frac{C_s}{A^2(1+b\sqrt{|\mathrm{Im}z|})}\,,$$

and the inequality (3.4.2.6) imply

$$\begin{aligned}\|(B_{\pm,b,V^h}-z)^{-1}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}} &\leq C_s\frac{1}{A^2(1+b\sqrt{|\mathrm{Im}z|})}\|M'^{-1}_{\bar z}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}}\\ &= C_s\frac{1}{A^2(1+b\sqrt{|\mathrm{Im}z|})}\|u_{high}\|_{\tilde{\mathscr{W}}^{0,s}}\,.\end{aligned}$$

We have proved

$$\|\mathfrak{D}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}}\leq\frac{C_s}{A^2}\frac{1}{1+b\sqrt{|\mathrm{Im}z|}}\,.$$

The last inequality comes from the fact that $Q_{A,L,V^h}u_{high}=U_{\pm,\theta}\chi(\frac{C_d+\Delta_{V^h,1}}{(LA)^2})(1-\chi(\frac{C_d+\Delta_{V^h,1}}{(2LA)^2}))U^{-1}_{\pm,\theta}u=0$ implies $M'^{-1}_{\bar z}u_{high}=(I-Q_{A,L,V^h}U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z+Q_{A,L,V^h})^{-1}U^{-1}_{\pm,\theta})^{-1}u_{high}=u_{high}$.
For the u_{low}-component, $u_{low}=\frac{1}{A^2}Q_{A,L',V^h}u$, the formula (3.4.3.2) for $B_{\pm,b,V^h}$ and the analogous one for $\frac{1}{2}\Delta_{V^h,1}$ give

$$\begin{aligned}\mathfrak{D} &= M^{-1}_{B,z}(B_{\pm,b,V^h}+Q_{A,L,V^h}-z)^{-1}-U_{\pm,\theta}\tilde M^{-1}_{\Delta,z}(\frac{1}{2}\Delta_{V^h,1}+\tilde Q_{A,L,V^h}-z)^{-1}U^{-1}_{\pm,\theta}\\ &= (M_{B,z})^{-1}\delta_{B,\Delta,z}+\left[(M_{B,z})^{-1}-U_{\pm,\theta}\tilde M^{-1}_{\Delta,z}U^{-1}_{\pm,\theta}\right]U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}+\tilde Q_{A,L,V^h}-z)^{-1}U^{-1}_{\pm,\theta}\end{aligned}\quad(3.4.3.4)$$

Owing to

$$(M_{B,z})^{-1}\delta_{B,\Delta,z}u_{low}=\frac{1}{A^2}(M_{B,z})^{-1}\delta_{B,\Delta,z}Q_{A,L',V^h}u\,,$$

the inequality (3.4.2.2) combined with the inequality (3.4.2.8) imply that the first term of (3.4.3.4) applied to u_{low} is bounded by

$$\|(M_{B,z})^{-1}\delta_{B,\Delta,z}u_{low}\|_{\tilde{\mathscr{W}}^{0,s}}\leq\frac{C_s}{A^2}\left(1+\frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)\left(\frac{bA^{\frac{3}{2}}}{1+A^{-1}\sqrt{|\mathrm{Im}z|}}\right)\|u\|_{\tilde{\mathscr{W}}^{0,s}}\,.$$

On $\ker(\alpha_\pm)=\mathrm{Ran}(U_{\pm,\theta})$, we know

$$U_{\pm,\theta}\tilde M^{-1}_{\Delta,z}U^{-1}_{\pm,\theta}=M^{-1}_{\Delta,z}\,.$$

The second term of (3.4.3.4) thus equals

$$\Big[(M_{B,z})^{-1} - U_{\pm,\theta}\tilde{M}_{\Delta,z}^{-1}U_{\pm,\theta}^{-1}\Big]U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}$$
$$= \Big[(M_{B,z})^{-1} - M_{\Delta,z}^{-1}\Big]U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}.$$

Further computations show that the right-hand side is

$$\begin{aligned}
&\Big[(M_{B,z})^{-1} - M_{\Delta,z}^{-1}\Big]U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}\\
&\quad = M_{B,z}^{-1}\Big[M_{\Delta,z} - M_{B,z}\Big]M_{\Delta,z}^{-1}U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}\\
&\quad = M_{B,z}^{-1}\Big[M_{\Delta,z} - M_{B,z}\Big]U_{\pm,\theta}\tilde{M}_{\Delta,z}^{-1}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}\\
&\quad = M_{B,z}^{-1}\Big[\delta_{B,\Delta,z}Q_{A,L,V^h}\Big]\Big(U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} - z)^{-1}U_{\pm,\theta}^{-1}\Big).
\end{aligned}$$

By using again inequality (3.4.2.7) and Proposition 3.4.2 for the right-hand side, the above operator is estimated by

$$\Big\|\Big[(M_{B,z})^{-1} - U_{\pm,\theta}\tilde{M}_{\Delta,z}^{-1}U_{\pm,\theta}^{-1}\Big]U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} + \tilde{Q}_{A,L,V^h} - z)^{-1}U_{\pm,\theta}^{-1}\Big\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}$$
$$\leq C_s\left(1 + \frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)\left(\frac{bA^{\frac{3}{2}}}{1 + A^{-1}\sqrt{|\mathrm{Im} z|}}\right)\frac{1}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}.$$

By adding the two terms we obtain

$$\|\mathfrak{D}u_{low}\|_{\tilde{\mathscr{W}}^{0,s}} \leq C_s\left(1 + \frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)^2\left(\frac{bA^{-\frac{1}{2}}}{1 + A^{-1}\sqrt{|\mathrm{Im} z|}}\right).$$

By summing the two upper bounds for $\|\mathfrak{D}u_{high}\|_{\tilde{\mathscr{W}}^{0,s}}$ and $\|\mathfrak{D}u_{low}\|_{\tilde{\mathscr{W}}^{0,s}}$, we get

$$\|\mathfrak{D}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \leq C_s\left[\left(1 + \frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)^2\left(\frac{bA^{-\frac{1}{2}}}{1 + A^{-1}\sqrt{|\mathrm{Im} z|}}\right) + \frac{1}{A^2}\frac{1}{1 + b\sqrt{|\mathrm{Im} z|}}\right],$$

which can be simplifed into (3.4.3.3) owing to $b \leq A^{-1}$. □

3.5 Spectral consequences

This final section actually ends the proof of Theorem 3.1.3 and its various statements are picked from Proposition 3.5.1, Proposition 3.5.3 and Proposition 3.5.6. A simple translation is obtained after recalling the unitary equivalences

$$\begin{aligned}
B_{\pm,b,\frac{V}{h}} = B_{\pm,(Q,g,\frac{V}{h},b)} &\longleftrightarrow \frac{1}{h^2}B_{\pm,(Q^h,g^h,V^h,b')} = \frac{1}{h^2}B_{\pm,b',V^h} \quad b' = \frac{b}{h},\\
\Delta_{V,h} = \Delta_{(Q,g,V,h)} &\longleftrightarrow \Delta_{(Q^h,g^h,V^h,1)} = \Delta_{V^h,1}\\
\frac{z}{h^2} \in \mathrm{Spec}(B_{\pm,b,\frac{V}{h}}) &\longleftrightarrow z \in \mathrm{Spec}(B_{\pm,b',V^h}) \quad b' = \frac{b}{h},\\
\tilde{\mathscr{W}}_h^{s_1,s_2}(X,\mathscr{E}_\pm) &\longleftrightarrow \tilde{\mathscr{W}}_{h=1}^{s_1,s_2}(X^h,\mathscr{E}_\pm^h).
\end{aligned}$$

Once this is fixed the first statement **a)** of Theorem 3.1.3 is a corollary of Proposition 3.5.1. The second statement **b)** of Theorem 3.1.3 is a rewriting of Proposition 3.5.6 of which the proof strongly relies on the Hodge structure of restricted operator $B_{\pm,b,V^h}\big|_{E_{\pm,b,V^h}}$ and where the hermitian form $\langle\ ,\ \rangle_r\Big|_{(E_{\pm,b,V^h})^2}$ is positive definite by the PT-symmetry argument checked in the proof of Proposition 3.5.1. Finally the third statement **c)** of Theorem 3.1.3 about the semigroup expansion is a transcription of Proposition 3.5.3.

3.5.1 Rough estimates

The data of our problem are the spectrum of the semiclassical Witten Laplacian $\mathrm{Spec}(\Delta_{V,h}) = \mathrm{Spec}(\Delta_{V^h,1})$ and the parameters $b,h \in]0,1]$. We introduced the additional parameter $A \geq 1$ and recall the condition

$$C_s \max(Ab, b, A^{-1}) \leq 1\,.$$

We recall, according to Definition 3.1.1, that the parameter $\varrho_h \in]0,1]$ measures a spectral gap for $\Delta_{V,h}$ according to

$$\mathrm{Spec}(\frac{1}{2}\Delta_{V,h}) \cap [0,\varrho_h] \subset [0, e^{-\frac{c}{h}}] \subset [0, \frac{\varrho_h}{2}] \tag{3.5.1.1}$$

$$\text{and} \qquad \mathrm{Spec}(\frac{1}{2}\Delta_{V,h}) \cap]\varrho_h, +\infty[\subset [4\varrho_h, +\infty[\tag{3.5.1.2}$$

for all $h \in]0,1]$. Remember as well the notations

$$\mathcal{N}_{\pm}^{(p)}(V) = \mathrm{rank}\ 1_{[0,\varrho_h]}(\frac{1}{2}\Delta^{(p)}_{V,h}) \quad \text{and} \quad \mathcal{N}_{\pm}(V) = \sum_{p=0}^{d} \mathcal{N}_{\pm}^{(p)}(V)\,,$$

where the $\pm$ sign refers to the choice of the line bundle $F_+ = Q \times \mathbb{C}$ or $F_- = (Q \times \mathbb{C}) \otimes \mathbf{or}_Q$.
Making an accurate use of the quantitative comparison of the resolvents in Proposition 3.4.4 requires the identification of different areas in the complex plane, presented in the picture below, and of which the accurate definitions are given just after.

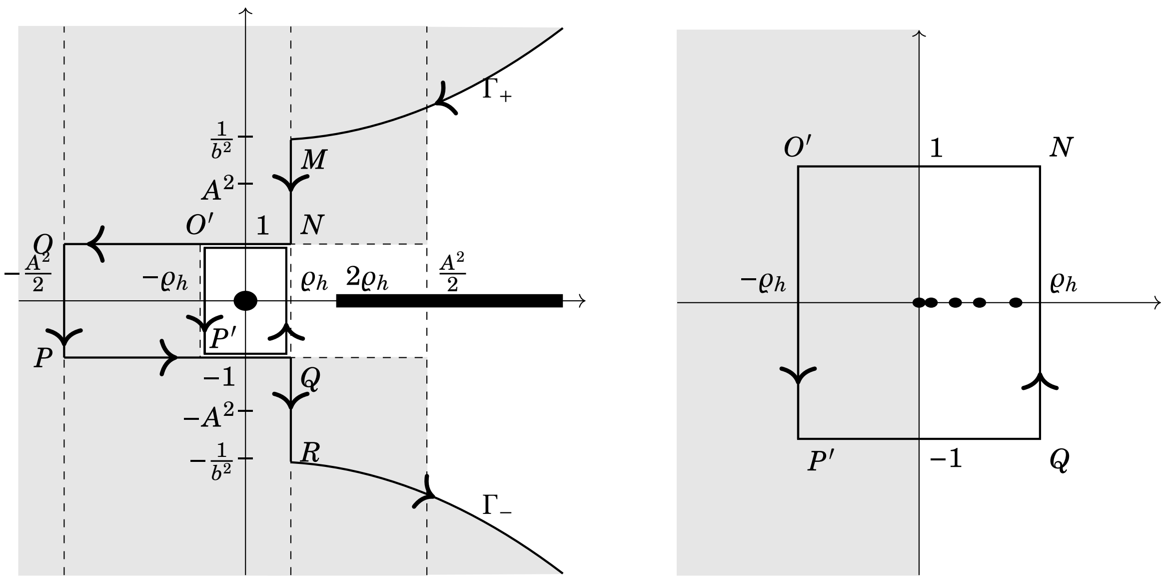

Figure 3.5.1 – The left-hand side summarizes the spectral localization and the shape of contours deduced from the analysis in $|\mathrm{Re}z| \le \frac{A^2}{2}$. In this picture $\mathrm{Spec}(\frac{1}{2}\Delta_{\mathrm{V}^h,1}) = \mathrm{Spec}(\frac{1}{2}\Delta_{V,h})$ is represented by the black circle around 0 and the thick real half-line $[2\varrho_h, +\infty[$.
The picture on the right-hand side is zoomed into the region $|\mathrm{Re}z| \le \varrho_h$ and $|\mathrm{Im}\, z| \le 1$. The small circles represent the eignvalues of $B_{\pm,b,V^h}$ in the region $|\mathrm{Re}z| \le \varrho_h$.
In both pictures the gray area represents a part of the resolvent set of $B_{\pm,b,V^h}$.

The curves $\Gamma_\pm$ are defined by

$$\Gamma_\pm = \left\{z \in \mathbb{C}, \quad \pm[1 + (\mathrm{Re}z - \varrho_h)^2] = b^2 \mathrm{Im}\, z\right\}. \tag{3.5.1.3}$$

The points M, N, Q, R are on the line $\mathrm{Re}z = \varrho_h$ with the respective imaginary parts $\frac{1}{b^2}, 1, -1, -\frac{1}{b^2}$. The points O, O' (resp. P, P') have an imaginary part equal to 1 (resp. -1) and real part equal to $-\frac{A^2}{2}$ and $-\varrho_h$.
We will use the oriented contours $\Gamma = \Gamma_+ + \Gamma_0 + \Gamma_-$ with $\Gamma_0 = \left\{z \in \mathbb{C}, |z - \varrho_h| = \frac{1}{b^2}, \mathrm{Re}z \le \varrho_h\right\}$, $\Gamma_+ + [MNOPQR] + \Gamma_-$, $\Gamma_+ + [MNO'P'QR] + \Gamma_-$, $\Gamma_+ + [MR] + \Gamma_-$ and $[NO'P'Q]$.

The main results of this section are gathered in the two following propositions.

Proposition 3.5.1. *There exists $C_0 \ge 1$ such that $A = C_0$ and $2C_0^5 b \le \varrho_h$ implies the following properties.*

a) *The sets $\left\{\mathrm{Re}z \le \frac{A^2}{2} \text{ and } |\mathrm{Im}\, z| \ge 1\right\}$, $\left\{\mathrm{Re}z \le -\varrho_h\right\}$, $[N,Q] = \left\{\mathrm{Re}z = \varrho_h, |\mathrm{Im}\, z| \le 1\right\}$ and the one partly delimited by $\Gamma_\pm$, $\left\{\mathrm{Re}z \ge \varrho_h \text{ and } 1 + (\mathrm{Re}z - \varrho_h)^2 \le b^2 |\mathrm{Im}\, z|\right\}$, are all contained in the resolvent set of $B_{\pm,b,V^h}$. The union of these sets (the gray area in the left-hand side picture of Figure 3.5.1) contains all the oriented contours listed above.*

b) *If $E^{(p)}_{\pm,b,V^h}$ is the characteristic space, which is the range of*

$$\pi_{E^{(p)}_{\pm,b,V^h}} = \frac{1}{2i\pi} \int_{NOPQ} (z - B^{(p)}_{\pm,b,V^h})^{-1} dz = \frac{1}{2i\pi} \int_{NO'P'Q} (z - B^{(p)}_{\pm,b,V^h})^{-1}\, dz$$

then $\mathcal{N}_+^{(p)} := \dim E^{(p)}_{+,b,V^h} = \mathcal{N}_+^{(p)}(V)$ and $\mathcal{N}_-^{(p)} := \dim E^{(p)}_{-,b,V^h} = \mathcal{N}_-^{(p-d)}(V)$.

c) *When $r : X^h \to X^h$ is the involution defined by $r(q,p) = (q,-p)$ and r^* denotes its action on $\mathscr{S}(X^h;\mathscr{E}^h_\pm)$, according to Definition 3.2.4, then r^* is a unitary involution of $L^2(X^h;\mathscr{E}^h_\pm)$ such that $r^* B_{\pm,b,V^h}(r^*)^{-1} = r^* B_{\pm,b,V^h} r^* = B'_{\pm,b,V^h}$.*

d) *The hermitian form $(u,v) \mapsto \langle u\,,\,v\rangle_r = \langle u\,,\,r^* v\rangle_{L^2}$, of Definition 3.2.4 is positive definite on $E_{\pm,b,V^h}$ and $B_{\pm,b,V^h}\big|_{E_{\pm,b,V^h}}$ is self-adjoint and positive for the scalar product $\langle\ ,\ \rangle_r$. The eigenvalues of $B_{\pm,b,V^h}$ in the disc $\{z \in \mathbb{C}, |z| \le \varrho_h\}$ actually belong to $[0,\varrho_h[$. Additionally on $E_{\pm,b,V^h}$ we have the equivalence of norms*

$$\forall u \in E_{\pm,b,V^h}\,,\quad \sqrt{1 - C_0 b^2}\|u\|_{L^2} \le \|u\|_r \le \|u\|_{L^2}\,. \tag{3.5.1.4}$$

Definition 3.5.2. The eigenvalues of $B^{(p)}_{\pm,b,V^h}$ lying in $[0,\varrho(h)]$ will be denoted by $(\lambda^{(p)}_{\pm,j})_{1\le j\le \mathscr{N}^{(p)}_\pm}$ with the associated $\langle\ ,\ \rangle_r$-orthonormal basis of eigenvectors $(u^{(p)}_{\pm,j})_{1\le j\le \mathscr{N}^{(p)}_\pm}$.

Proposition 3.5.3. *For every $s \in \mathbb{R}$ there exists $C_s \ge 1$ such that taking $A = C_s$ with the condition $\varrho_h \ge b C_s^5$ implies that the semigroup $(e^{-tB_{\pm,V^h}})_{t>0}$ admits for every $t > 0$ the following convergent integral representation*

$$e^{-tB^{(p)}_{\pm,b,V^h}} = \frac{1}{2i\pi}\int_\Gamma \frac{e^{-tz}}{z - B^{(p)}_{\pm,b,V^h}}\,dz = \underbrace{\frac{1}{2i\pi}\int_{NO'P'Q} \frac{e^{-tz}}{z - B^{(p)}_{\pm,b,V^h}}\,dz}_{(I)} + \underbrace{\frac{1}{2i\pi}\int_{\Gamma_+ + [MR] + \Gamma_-} \frac{e^{-tz}}{z - B^{(p)}_{\pm,b,V^h}}\,dz}_{(II)}\,.$$

In the above formula the first term equals

$$(I) = \sum_{i=1}^{\mathscr{N}^{(p)}_\pm} e^{-t\lambda^{(p)}_{\pm,j}} |u^{(p)}_{\pm,j}\rangle\langle r^* u^{(p)}_{\pm,j}|$$
$$\text{with}\qquad \|u^{(p)}_{\pm,j}\|_{\tilde{\mathscr{W}}^{0,s}} = \|r^* u^{(p)}_{\pm,j}\|_{\tilde{\mathscr{W}}^{0,s}} \le C_s\,,$$

and the second term satisfies

$$\|(II)\|_{\mathscr{L}(\tilde{\mathscr{W}}^s;\tilde{\mathscr{W}}^s)} \le \frac{1 + t^{-1}}{b^2} e^{-t\varrho_h}\,.$$

The proofs of Proposition 3.5.1 and Proposition 3.5.3 actually rely on the two following lemmas. The first one is an application of Proposition 3.4.4 with the specific geometric partition of $\mathbb{C}$ represented in Figure 3.5.1.

Lemma 3.5.4. *For any $s \in \mathbb{R}$, there exists $C_s \ge 1$ such that $A \ge C_s$ and $\frac{\varrho_h}{2} \ge C_s b A^4$ imply that the norms of*

$$(B_{\pm,b,V^h} - z)^{-1} \quad,\quad \mathfrak{D}_z = (B_{\pm,b,V^h} - z)^{-1} - U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1} - z)^{-1} U^{-1}_{\pm,\theta}\,,$$

have upper bounds given by the following table. Because $B_{\pm,b,V^h}$ and $U_{\pm,\theta}\frac{1}{2}\Delta_{V^h,1}U^{-1}_{\pm,\theta}$ preserve the total degree $p \in \{0,\ldots,2d\}$, $(B_{\pm,b,V^h} - z)^{-1}$ and $\mathscr{D}_z$ can be respectively replaced by

$$(B^{(p)}_{\pm,b,V^h} - z)^{-1} \quad,\quad \mathfrak{D}^{(p)}_z = (B^{(p)}_{\pm,b,V^h} - z)^{-1} - U_{\pm,\theta}(\frac{1}{2}\Delta^{(p-\frac{d}{2}\pm\frac{d}{2})}_{V^h,1} - z)^{-1} U^{-1}_{\pm,\theta}\,.$$

Proof. The conditions $A \ge C_s$ and $\frac{\varrho_h}{2} \ge C_s b A^4$ with $\varrho_h \le 1$ ensure the validity of the hypotheses of Proposition 3.4.4:

$$C_s \max(Ab, b, A^{-1}) \le 1 \quad\text{and}\quad \operatorname{dist}(z, \operatorname{Spec}(\frac{1}{2}\Delta_{V^h,1})) \ge C_s b A^4$$

as soon as $\operatorname{dist}(z, \operatorname{Spec}(\frac{1}{2}\Delta_{V^h,1})) \ge \frac{\varrho_h}{2}$.

Sets	$\|\mathfrak{D}_z\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}$	$\|(B_{\pm,b,V^h}-z)^{-1}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})}$	Label
$\{\mathrm{Re} z \leq -\frac{A^2}{2}\}$		$\frac{4}{A^2}$ for $s=0$	**(1)**
$\{\lvert\mathrm{Re} z\rvert \leq \frac{A^2}{2}$ and $\lvert\mathrm{Im}\, z\rvert \geq 1\}$	$C_s(bA^{\frac{7}{2}}+A^{-2})$	$C_s(\frac{1}{\lvert\mathrm{Im}\, z\rvert}+bA^{\frac{7}{2}}+A^{-2})$	**(2)**
$\left\{\begin{pmatrix} \mathrm{Re} z \leq -\varrho_h \\ \text{or } \mathrm{Re} z = \varrho_h \end{pmatrix} \text{ and } \lvert\mathrm{Im}\, z\rvert \leq 1\right\}$	$C_s\left(A^{-2}+\frac{bA^{\frac{7}{2}}}{\varrho_h^2/4+\lvert\mathrm{Im}\, z\rvert^2}\right)$	$C_s\left(A^{-2}+\frac{1+bA^{\frac{7}{2}}}{\varrho_h^2/4+\lvert\mathrm{Im}\, z\rvert^2}\right)$	**(3)**
$\left\{\mathrm{Re}\, z = 0 \text{ and } \lvert\mathrm{Im}\, z\rvert \geq \frac{1}{b^2}\right\}$	$C_s\frac{A^{-2}+bA^{-\frac{1}{2}}}{b\sqrt{\lvert\mathrm{Im}\, z\rvert}} \leq \frac{1}{8b\sqrt{\lvert\mathrm{Im}\, z\rvert}}$	$\frac{1}{\lvert\mathrm{Im}\, z\rvert}+\frac{1}{8b\sqrt{\lvert\mathrm{Im}\, z\rvert}} \leq \frac{1}{4b\sqrt{\lvert\mathrm{Im}\, z\rvert}}$	**(4)**

Table 3.5.1 – Resolvent norm estimates

(1) This line is only concerned with the case $s=0$. By Corollary 3.3.3 there is constant $C_0 \geq 1$ such that

$$C_0 + B_{\pm,b,V^h}$$

is accretive as soon as $C_0 b \leq 1$ and $h \in]0,1]$. Take $z \in \{\mathrm{Re} z \leq -\frac{A^2}{2}\}$,

$$\begin{aligned}
&\|(B_{\pm,b,V^h}-z)u\|_{L^2}^2\\
&\quad = \|(B_{\pm,b,V^h}-i\mathrm{Im}\, z)u\|_{L^2}^2 + |\mathrm{Re}\, z|^2\|u\|_{L^2}^2 + 2(-\mathrm{Re} z)\,\mathrm{Re}\langle B_{\pm,b,V^h}u,u\rangle\\
&\quad = \|(B_{\pm,b,V^h}-i\mathrm{Im}\, z)u\|_{L^2}^2 + \left[|\mathrm{Re}\, z|^2 - 2C_0|\mathrm{Re} z|\right]\|u\|_{L^2}^2 + 2|\mathrm{Re} z|\,\underbrace{\mathrm{Re}\langle (C_0+B_{\pm,b,V^h})u,u\rangle}_{\geq 0}\\
&\quad \geq \left[|\mathrm{Re}\, z|^2 - 2C_0|\mathrm{Re} z|\right]\|u\|_{L^2}^2\\
&\quad \geq \frac{A^2}{2}\left(\frac{A^2}{2}-2C_0\right)\|u\|_{L^2}^2
\end{aligned}$$

When $\frac{A^2}{8} \geq C_0$, we obtain

$$\|(B_{\pm,b,V^h}-z)u\|_{L^2} \geq \frac{A^2}{4}\|u\|_{L^2}.$$

(2) For $z \in \{|\mathrm{Re} z| \leq \frac{A^2}{2}$ and $|\mathrm{Im}\, z| \geq 1\}$ we know

$$\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})) \geq |\mathrm{Im} z| \geq 1 \geq \frac{\varrho_h}{2}$$

and

$$\|U_{\pm,\theta}(\frac{1}{2}\Delta_{V^h,1}-z)^{-1}U_{\pm,\theta}^{-1}\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})} \leq \frac{1}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))} \leq \frac{1}{|\mathrm{Im}\, z|} \leq 1. \qquad (3.5.1.5)$$

Proposition 3.4.4 gives for a proper choice of $C_s \geq 1$ the inequality

$$\begin{aligned}
\|\mathfrak{D}_z\|_{\mathcal{L}(\tilde{\mathcal{W}}^{0,s};\tilde{\mathcal{W}}^{0,s})} &\leq \frac{C_s}{1+b\sqrt{|\mathrm{Im}\, z|}}\left[A^{-2}+\left(1+\frac{A^2}{\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))}\right)^2 bA^{-\frac{1}{2}}\right]\\
&\leq 4C_s[A^{-2}+bA^{7/2}]. \qquad (3.5.1.6)
\end{aligned}$$

The upper bound for $(B_{\pm,b,V^h}-z)^{-1}$ is deduced at once from (3.5.1.5) and (3.5.1.6).

(3) The following inequality holds for $z \in \{(\mathrm{Re} z \leq -\varrho_h$ or $\mathrm{Re} z = \varrho_h)$ and $|\mathrm{Im}\, z| \leq 1\}$

$$\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))^2 \geq \frac{\varrho_h^2}{4} + |\mathrm{Im}\, z|^2,$$

with the detailed cases:

— if $\mathrm{Re}z \le -\varrho_h$ then $\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))^2 \ge |z|^2 \ge \varrho_h^2 + |\mathrm{Im}z|^2 \ge \frac{\varrho_h^2}{4} + |\mathrm{Im}z|^2$;

— if $\mathrm{Re}z = \varrho_h$ then the hypothesis (3.5.1.1) ensure that $\mathrm{dist}(z,\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1}))^2 \ge (\frac{\varrho_h}{2})^2 + |\mathrm{Im}\ z|^2$.

Applying Proposition 3.4.4 gives

$$\|\mathfrak{D}_z\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \le C_s \left[A^{-2} + \frac{bA^{7/2}}{\varrho_h^2/4 + |\mathrm{Im}\ z|^2}\right].$$

(4) If $z \in \{\mathrm{Re}z =$ and $|\mathrm{Im}\ z| \ge \frac{1}{b^2}\}$ the distance to the spectrum is bounded by

$$\mathrm{dist}(z, \frac{1}{2}\Delta_{V^h,1}) \ge |\mathrm{Im}\ z| \ge \frac{1}{b^2}.$$

Proposition 3.4.4, with $Ab \le 1$, implies

$$\|\mathfrak{D}_z\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \le C_s \frac{A^{-2} + bA^{-\frac{1}{2}}}{b\sqrt{|\mathrm{Im}\ z|}} \le \frac{1}{8b\sqrt{|\mathrm{Im}\ z|}},$$

by choosing again A and $\frac{1}{b}$ large enough. Finally

$$\|(B_{\pm,b,V^h} - z)^{-1}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})} \le \frac{1}{|\mathrm{Im}z|} + \frac{1}{8b\sqrt{|\mathrm{Im}\ z|}} \le \frac{b^2 + \frac{1}{8}}{b\sqrt{|\mathrm{Im}\ z|}} \le \frac{1}{4b\sqrt{|\mathrm{Im}\ z|}}.$$

□

The second lemma is a variation of [Nie]-Proposition 2.15, which was itself inspired from the article of Hérau-Hitrik-Sjöstrand [HHS].

Lemma 3.5.5. *Let $(B,D(B))$ be a closed densely defined operator in a separable Hilbert space $\mathscr{H}$, such that $(1+B)^{-1}$ is compact, so that Spec(B) is discrete, and $D(B) = D(B^*)$. Assume that there exists a unitary involution $U^* = U^{-1} = U$ such that $U^*BU = B^*$. Then the spectrum Spec(B) is invariant by complex conjugation.*
If additionally there exist $\gamma > 0$, $\varepsilon \in]0,\frac{1}{4}[$, an orthogonal projection $\Pi_0 = \Pi_0^*$ and a bounded contour Γ, symmetric w.r.t $z \to \bar{z}$, such that:

— *$\Pi_0 U = U\Pi_0 = \Pi_0$;*

— *the real part $\mathrm{Re}B = \frac{B+B^*}{2}$ is non negative and $\mathrm{Re}B \ge \gamma(1-\Pi_0) - \varepsilon\gamma$;*

— *with $\Pi_\Gamma = \frac{1}{2\pi i}\int_\Gamma (z-B)^{-1}\, dz$,*

$$\mathrm{ReTr}\left[\frac{1}{2\pi i}\int_\Gamma B(z-B)^{-1}\, dz\right] = \mathrm{Tr}\,[B\Pi_\Gamma] \le \varepsilon\gamma; \tag{3.5.1.7}$$

then the following properties hold:

— *The form $\langle u,v\rangle_U = \langle u,Uv\rangle$ is a hermitian positive definite form on $E_\Gamma = \mathrm{Ran}\,\Pi_\Gamma$.*

— *The norms $\|u\|_U = \sqrt{\langle u,Uu\rangle}$ and $\|u\|$ are equivalent*

$$\forall u \in E_\Gamma, \quad \sqrt{1-4\varepsilon}\|u\| \le \|u\|_U \le \|u\|. \tag{3.5.1.8}$$

— *The restricted operator $B\big|_{E_\Gamma} = B\Pi_\Gamma\big|_{E_\Gamma} = \Pi_\Gamma B\big|_{E_\Gamma}$ is self-adjoint and non negative for the scalar product $\langle\ ,\ \rangle_U$.*

— *The vector space E_Γ admits a basis of eigenvectors of B, $(e_1,\ldots,e_N)$, orthonormal for the scalar product $\langle\ ,\ \rangle_U$.*

— *For all $z \in \mathbb{C}$ inside the contour Γ, the inequality*

$$\|(z-B)^{-1}\big|_{E_\Gamma}\|_{\mathscr{L}(E_\Gamma;E_\Gamma)} \leq \frac{1}{\sqrt{1-4\varepsilon}\,\mathrm{dist}(z,\{\lambda_1,\ldots,\lambda_N\})},$$

holds with the initial norm $\|\ \|$ on E_Γ.

— *The "distance" $\vec{d}(E_\Gamma, \mathrm{Ran}\,\Pi_0) = \|(1-\Pi_0)\Pi_\Gamma\|_{\mathscr{L}(\mathscr{H})}$ is bounded by $\sqrt{2\varepsilon}$ and Π_0 is an isomorphism from E_Γ to $\Pi_0 E_\Gamma$.*

Proof. The PT-symmetry property $B^* = U^* B U$ implies

$$(z-B^*)^{-1} = (z-U^*BU)^{-1} = U^*(z-B)^{-1}U$$

whenever one of the resolvent exists, so that

$$\overline{\mathrm{Spec}(B)} = \mathrm{Spec}(B^*) = \mathrm{Spec}(B)$$

and the spectrum $\mathrm{Spec}(B)$ is symmetric with respect to the real axis. The accretivity of B gives $\mathrm{Spec}(B) \subset \{z \in \mathbb{C},\ \mathrm{Re} z \geq 0\}$. Another consequence is that if the integration contour Γ is symmetric w.r.t the real axis and if $z \to f(z)$ is a holomorphic function satisfying $\overline{f(\overline{z})} = f(z)$ then

$$f_\Gamma(B)^* = \frac{-1}{2i\pi}\int_\Gamma \frac{f(\bar{z})}{(\bar{z}-B^*)}\,d\bar{z} = U^*\left[\frac{1}{2i\pi}\int \frac{f(z)}{(z-B)}\,dz\right]U = U^* f_\Gamma(B) U,$$

and in particular

$$\mathrm{Tr}\,[f_\Gamma(B)] = \frac{1}{2}\left(\mathrm{Tr}\,[f_\Gamma(B)] + \mathrm{Tr}\,[U^* f_\Gamma(B) U]\right) = \mathrm{Re}\mathrm{Tr}\,[f_\Gamma(B)] \in \mathbb{R}.$$

Therefore the condition (3.5.1.7) makes sense (take $f(z) = z$) when there are eigenvalues of B with small real parts and multiplicities that are not too large. On the space $E_\Gamma = \mathrm{Ran}\,\Pi_\Gamma$, the form $(u,v) \mapsto \langle u, v\rangle_U = \langle u, Uv\rangle$ is a hermitian form and it is a scalar product when $\langle u, u\rangle_U > 0$ for any nonzero $u \in E_\Gamma$.

When $u \in E_\Gamma$ with $\|u\| = 1$, it can be completed into an orthonormal basis $(e_1 = u, e_2, \ldots, e_N)$ of E_Γ for the scalar product $\langle\ ,\ \rangle$. We have

$$\begin{aligned}
\varepsilon\gamma \geq \mathrm{Tr}\,(B\Pi_\Gamma) &= \mathrm{Re}\left(\sum_{j=1}^N \langle e_j, Be_j\rangle\right) \\
&= \sum_{j=1}^N \langle e_j, \mathrm{Re}Be_j\rangle \geq \langle u, \mathrm{Re}Bu\rangle \geq \gamma\|(1-\Pi_0)u\|^2 - \varepsilon\gamma,
\end{aligned}$$

and

$$\|(1-\Pi_0)u\|^2 < 2\varepsilon = 2\varepsilon\|u\|^2. \tag{3.5.1.9}$$

Now compute

$$\begin{aligned}
\langle u, Uu\rangle &= \langle u, \Pi_0 Uu\rangle + \langle u, (1-\Pi_0)Uu\rangle \\
&= \|\Pi_0 u\|^2 + \langle (1-\Pi_0)u, U(1-\Pi_0)u\rangle \\
&\geq \|\Pi_0 u\|^2 - \|(1-\Pi_0)u\|^2 = \|u\|^2 - 2\|(1-\Pi_0)u\|^2 \geq (1-4\varepsilon)\|u\|^2 > 0.
\end{aligned}$$

This proves that the hermitian form $\langle\ ,\ \rangle_U$ is positive definite on E_Γ and the equivalence of norms comes at once.

Let B_Γ be the restriction of B to E_Γ, which is a finite dimensional Hilbert space with the scalar product $\langle\ \ \rangle_U$ (and $\langle\ ,\ \rangle$). For $u,v\in E_\Gamma$, the series of equalities

$$\begin{aligned}\langle u\,,B_\Gamma v\rangle_U=\langle u\,,U(B\Pi_\Gamma)v\rangle &= \langle u\,,(B\Pi_\Gamma)^*Uv\rangle=\langle (B\Pi_\Gamma)u\,,Uv\rangle\\ &= \langle B_\Gamma u\,,v\rangle_U\,.\end{aligned}$$

says that B_Γ is self-adjoint on $(E_\Gamma,\langle\ ,\ \rangle_U)$. The two statements follow for the scalar product $\langle\ ,\ \rangle_U$ and the norm $\|\ \|_U$, are consequences.
The equivalence of the norms $\|\ \|_U$ and $\|\ \|$ gives the upper bound on $\|(z-B)^{-1}\big|_{E_\Gamma}\|_{\mathscr{L}(E_\Gamma;E_\Gamma)}$.
Finally the estimate on $\vec{d}(E_\Gamma;\mathrm{Ran}\,\Pi_0)$ is due to (3.5.1.9) □

Proof of Proposition 3.5.1. This is concerned essentially with the localization of the spectrum, which does not depend on $s\in\mathbb{R}$ for the closed realization $\overline{B^s_{\pm,b,V^h}}$ in $\tilde{\mathscr{W}}^{0,s}(X;\mathscr{E}_\pm)$. Therefore we focus on the case $s=0$ in particular while applying Lemma 3.5.4.
The conditions $A=C_0$ and $\varrho_h\geq 2C_0^5 b$, for $C_0\geq 1$, large enough, imply the conditions

$$A\geq C_0\quad\text{and}\quad \frac{\varrho_h}{2}\geq C_0bA^4$$

of Lemma 3.5.4.
a) The first three complex domains considered in **a)** are covered by unions of (1),(2) and (3) in Lemma 3.5.4. For the last domain $\left\{\mathrm{Re}z\geq\varrho_h \text{ and } 1+(\mathrm{Re}z-\varrho_h)^2\leq b^2|\mathrm{Im}\,z|\right\}$ we start with

$$\begin{aligned}\|(B_{\pm,b,V^h}-z)u\|_{\mathscr{L}(L^2;L^2)} &\geq \|(B_{\pm,b,V^h}-i\mathrm{Im}\,z)u\|_{L^2}-|\mathrm{Re}z|\|u\|_{L^2}\\ &\geq \left(4b\sqrt{|\mathrm{Im}\,z|}-|\mathrm{Re}z|\right)\|u\|_{L^2}\,.\end{aligned}$$

We conclude by noticing that $b\sqrt{|\mathrm{Im}\,z|}\geq\sqrt{1+(\mathrm{Re}z-\varrho_h)^2}$ implies

$$\begin{aligned}&b\sqrt{|\mathrm{Im}\,z|}\geq|\mathrm{Re}z-\varrho_h|\geq|\mathrm{Re}z|-1\\ \text{and}\quad &|\mathrm{Re}z|\leq 1+b\sqrt{|\mathrm{Im}\,z|}\leq 2b\sqrt{|\mathrm{Im}\,z|}.\end{aligned}$$

We have actually proved $\|(B_{\pm,b,V^h}-z)^{-1}\|_{\mathscr{L}(L^2;L^2)}\leq\frac{1}{2b\sqrt{|\mathrm{Im}\,z|}}$ in this domain.
b) Let us consider now the operator

$$\mathfrak{R}^{(p)}=\frac{1}{2i\pi}\int_{NO'P'Q}\mathfrak{D}_z^{(p)}\,dz$$

which is the difference between the projection $\pi_{E^{(p)}_{\pm,b,V^h}}$ and the orthogonal projection $\pi_h^{(p)}=U_{\pm,\theta}\tilde{\pi}_h^{(p)}U_{\pm,\theta}^{-1}$ with

$$\tilde{\pi}_h^{(p)}=\frac{1}{2i\pi}\int_{NO'P'Q}(z-\frac{1}{2}\Delta_{V^h,1}^{(p-\frac{d}{2}\pm\frac{d}{2})})^{-1}\,dz\,.$$

The following bounds are consequences of Table 3.5.1 of Lemma 3.5.4 for $s=0$:

$$\begin{aligned}\|\int_{NO'\cup P'Q}\mathfrak{D}_z^{(p)}\,dz\|_{\mathscr{L}(L^2;L^2)} &\leq 8C_0\varrho_h(A^{-2}+bA^{7/2})\leq 8C_0A^{-2}+4A^{-1/2}\qquad (3.5.1.10)\\ &\leq \frac{8}{C_0}+\frac{4}{\sqrt{C_0}}\,,\\ \|\int_{O'P'\cup QN}\mathfrak{D}_z^{(p)}\,dz\|_{\mathscr{L}(L^2;L^2)} &\leq 2C_0A^{-2}+2C_0bA^{7/2}\int_{-1}^{1}\frac{1}{(\varrho_h/2)^2+t^2}\,dt\qquad (3.5.1.11)\\ &\leq 2C_0(A^{-2}+2A^{7/2}\varrho_h^{-1}\pi)\leq 2C_0A^{-2}+A^{-1/2}2\pi\\ &\leq \frac{2}{C_0}+\frac{2\pi}{\sqrt{C_0}}\,,\end{aligned}$$

where we used $C_0 b A^4 \leq \frac{\varrho_h}{2} \leq \frac{1}{2}$, and finally $A = C_0$ for the last upper bounds. With $C_0 \geq 1$ large enough we have proved

$$\|\pi_{E^{(p)}_{\pm,b,V^h}} - \pi_h^{(p)}\|_{\mathscr{L}(L^2;L^2)} < 1\,.$$

With $\|\pi_h^{(p)}\|_{\mathscr{L}(L^2;L^2)} \leq 1$,we obtain

$$\|(1-\pi_{E^{(p)}_{\pm,b,V^h}})\pi_h^{(p)}\|_{\mathscr{L}(L^2;L^2)} = \|(\pi_{E^{(p)}_{\pm,b,V^h}} - \pi_h^{(p)})\pi_h^{(p)}\|_{\mathscr{L}(L^2;L^2)} < 1\,,$$

and $\pi_{E^{(p)}_{\pm,b,V^h}} : \mathrm{Ran}\,\pi_h^{(p)} \to E^{(p)}_{\pm,b,V^h}$ is one to one and $\mathscr{N}_\pm^{(p)} = \dim E^{(p)}_{\pm,b,V^h} \geq \mathscr{N}_\pm^{(p-\frac{d}{2}\pm\frac{d}{2})}(V)$.
With $\|(1-\pi_h^{(p)})\|_{\mathscr{L}(L^2;L^2)} \leq 1$, we obtain

$$\|(1-\pi_h^{(p)})\pi_{E^{(p)}_{\pm,b,V^h}}\|_{\mathscr{L}(L^2;L^2)} = \|(1-\pi_h^{(p)})\pi_{E^{(p)}_{\pm,b,V^h}}(\pi_{E^{(p)}_{\pm,b,V^h}} - \pi_h^{(p)})\|_{\mathscr{L}(L^2;L^2)} < 1\,,$$

and $\pi_h^{(p)} : E^{(p)}_{\pm,b,V^h} \to \mathrm{Ran}\,\pi_h^{(p)}$ is one to one and $\mathscr{N}_\pm^{(p)} = \dim E^{(p)}_{\pm,b,V^h} \leq \mathscr{N}_\pm^{(p-\frac{d}{2}\pm\frac{d}{2})}(V)$, which is finite and independent of $h \in]0,h_0]$.
c) It comes from Bismut identification of $B_{\pm,b,V^h}$ as a Hodge type operator for the $\langle\,,\,\rangle_r$ hermitian form, recalled in (3.2.5.14):

$$B_{\pm,b,V^h} = 2(\delta_{\pm,b,V^h} + \delta^{*,r}_{\pm,b,V^h})^2 : \mathscr{S}(X^h;\mathscr{E}^h_\pm) \to \mathscr{S}'(X^h;\mathscr{E}^h_\pm)\,,$$

which implies $r^* B_{\pm,b,V^h}(r^*)^{-1} = r^* B_{\pm,b,V^h} r^* = B'_{\pm,b,V^h}$.
d) We apply Lemma 3.5.5 with $B = C_0 + B_{\pm,b,V^h}, U = r^*, \Pi_0 = \pi_{0,\pm}$ and the translated contour $\Gamma = C_0 + NO'P'Q$. Let us check the assumption of Lemma 3.5.5 while specifying the values of $\gamma > 0$ and $\varepsilon \in]0,\frac{1}{4}[$:

- The equality $\Pi_0 U = U\Pi_0 = \Pi_0$ comes from the fact that $r^*\alpha_\pm(r^*)^{-1} = \alpha_\pm$.
- The real part $\mathrm{Re}B = C_0 + \frac{1}{b^2}\alpha_\pm + \mathrm{Re}\gamma_{\pm,V^h}$ is non negative owing to the accretivity of $C_0 + B_{\pm,b,V^h}$ in Corollary 3.3.3. The inequality $\mathrm{Re}B \geq \gamma(1-\Pi_0) - \varepsilon\gamma$ is obtained by the same integration by parts computations as in Proposition 3.3.4 and for the accretivity of Proposition 3.3.1. Let us compute

$$\begin{aligned}
\langle u, \mathrm{Re}Bu\rangle_{L^2} &\geq C_0\|u\|^2_{L^2} + \frac{1}{b^2}\langle u, \alpha_\pm u\rangle - \|\gamma_{\pm,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};\tilde{\mathscr{W}}^{-1,0})}\|u\|^2_{\tilde{\mathscr{W}}^{1,0}}\\
&\geq C_0\|u\|^2_{L^2} + \frac{1}{b^2}\langle u, \alpha_\pm u\rangle - \|\gamma_{\pm,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};\tilde{\mathscr{W}}^{-1,0})}(\frac{d}{2}\|\Pi_0 u\|^2_{L^2} + \|(1-\Pi_0)u\|^2_{\tilde{\mathscr{W}}^{1,0}})\\
&\geq (C_0 - d\|\gamma_{\pm,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};\tilde{\mathscr{W}}^{-1,0})})\|u\|^2_{L^2} + (\frac{1}{b^2} - \|\gamma_{\pm,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};\tilde{\mathscr{W}}^{-1,0})})\|(1-\Pi_0)u\|^2_{L^2}\\
&\geq \frac{C_0}{2}\|u\|^2_{L^2} + \frac{1}{2b^2}\|(1-\Pi_0)u\|^2_{L^2}\,,
\end{aligned}$$

by fixing C_0 larger than $2d\|\gamma_{\pm,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};\tilde{\mathscr{W}}^{-1,0})}$ and by using

$$\frac{1}{b^2} \geq \frac{1}{b} \geq 2C_0A^4 \geq 2C_0 \geq 2\|\gamma_{\pm,V^h}\|_{\mathscr{L}(\tilde{\mathscr{W}}^{1,0};\tilde{\mathscr{W}}^{-1,0})}\,.$$

With this constraint on $C_0 \geq 1$, we have proved

$$\langle u, \mathrm{Re}Bu\rangle \geq \gamma\|(1-\Pi_0)u\|_{L^2} \quad \text{with } \gamma = \frac{1}{2b^2}\,.$$

— For the upper bound of the trace $\mathrm{Tr}[B\Pi_\Gamma]$ we use the notations

$$\dim E_{\pm,b,V^h} = \mathcal{N}_\pm = \sum_{p=0}^{2d} \mathcal{N}_\pm^{(p)} = \sum_{p'=0}^{d} \mathcal{N}_\pm^{(p')}(V) = \dim \mathrm{Ran}\, \tilde{\pi}_h\,.$$

We write:

$$\begin{aligned}
&\mathrm{Re}\mathrm{Tr}\left[\frac{1}{2i\pi}\int_{C_0+NO'P'Q} B(z-B)^{-1}\,dz\right]\\
&\quad = C_0\mathcal{N}_\pm + \mathrm{Re}\mathrm{Tr}\left[\frac{1}{2i\pi}\int_{NO'P'Q} z(z-B_{\pm,b,V^h})^{-1}\,dz\right]\\
&\quad = C_0\mathcal{N}_\pm + \mathrm{Re}\mathrm{Tr}\left[\frac{1}{2i\pi}\int_{NO'P'Q} z\mathfrak{D}_z\,dz\right] + \mathrm{Re}\mathrm{Tr}[\pi_h]\\
&\quad \leq C_0\mathcal{N}_\pm + \frac{C_0}{2\pi}\mathcal{N}_\pm(2A^{-2}+8bA^{7/2}\varrho_h^{-1}\frac{\pi}{2}) + \mathcal{N}_\pm\frac{\varrho_h}{2}\\
&\quad \leq 3C_0\mathcal{N}_\pm\\
&\quad \leq \varepsilon\gamma\,,
\end{aligned}$$

with $\gamma = \frac{1}{2b^2}$ like above and $\varepsilon = 6C_0\mathcal{N}_\pm b^2$ which belongs to $]0,\frac{1}{4}[$ when $b^2 < \frac{1}{24C_0\mathcal{N}_\pm}$. Remember that $\mathcal{N}_\pm = \dim\mathrm{Ran}\,\tilde{\pi}_h$ does not depends on $h \in]0,h_0]$.

The three above points ensure that all the hypotheses of Lemma 3.5.5 are fulfilled. Therefore $\langle\,,\rangle_r$ is hermitian positive definite form on Ran $\Pi_\Gamma = E_{\pm,b,V^h}$ and the equivalence of norms (3.5.1.4) is a straightforward consequence of (3.5.1.8).

The space $E_{\pm,b,V^h}$ is a finite dimensional subspace of $\mathcal{S}(X^h;\mathcal{E}^h_\pm)$ endowed with the positive definite hermitian form $\langle\ ,\ \rangle_r$. Additionally because $\delta_{\pm,b,V^h}B_{\pm,b,V^h} = B_{\pm,b,V^h}\delta_{\pm,b,V^h}$ on $\mathcal{S}(X^h;\mathcal{E}^h_\pm)$ and the same holds when $\delta_{\pm,b,V^h}$ is replaced by $\delta^{*,r}_{\pm,b,V^h}$, we deduce that $\delta_{\pm,b,V^h}$ and $\delta^{*,r}_{\pm,b,V^h}$ send $E_{\pm,b,V^h}$ into itself. Thus formula (3.2.5.14) implies that $B_{\pm,b,V^h}\big|_{E_{\pm,b,V^h}} = [(\delta_{\pm,b,V^h}+\delta^{*,r}_{\pm,b,V^h})\big|_{E_{\pm,b,V^h}}]^2$ is the square of a self-adjoint operator on $(E_{\pm,b,V^h},\langle\ ,\ \rangle_r)$. Therefore $B_{\pm,b,V^h}\big|_{E_{\pm,b,V^h}}$ is a self-ajoint non negative operator for $\langle\ ,\ \rangle_r$ and its eigenvalues are non-negative. □

Proof of Proposition 3.5.3. The expression of (I) is a consequence of Proposition 3.5.1 because it says that $B^{(p)}_{\pm,b,V^h}\big|_{E^{(p)}_{\pm,b,V^h}}$ is diagonalizable. Additionally the L^2 dual basis of $(u^{(p)}_{\pm,j})_{1\leq j\leq\mathcal{N}^{(p)}_\pm}$ in $E^{(p)}_{\pm,b,V^h}$ is $(r^*u^{(p)}_{\pm,j})_{1\leq j\leq\mathcal{N}^{(p)}_\pm}$ because

$$\langle u^{(p)}_{\pm,i}, u^{(p)}_{\pm,j}\rangle_r = \langle r^*u^{(p)}_{\pm,i}, u^{(p)}_{\pm,j}\rangle_{L^2} = \delta_{ij}\,,$$

while $r^*B_{\pm,b,V^h}r^* = B^{(p)\prime}_{\pm,b,V^h}$ implies that $r^*u^{(p)}_{\pm,j}$ is an eigenvector of $B^{(p)\prime}_{\pm,b,V^h}$.

Let us check the uniform bound of $\|u^{(p)}_{\pm,j}\|_{\tilde{\mathcal{W}}^{0,s}}$. For $s=0$ it comes from (3.5.1.4) with

$$\|u^{(p)}_{\pm,j}\|_{L^2} \leq \frac{1}{\sqrt{1-C_0b^2}}\|u^{(p)}_{\pm,j}\|_r = \frac{1}{\sqrt{1-C_0b^2}} \leq 2\,.$$

For $s>0$ we use the equation

$$[B_{\pm,b,V^h} + A^2\pi_{0,\pm}]u^{(p)}_{\pm,j} = \lambda^{(p)}_{j,\pm}u_{\pm,j} + A^2\pi_{0,\pm}u^{(p)}_{\pm,j}$$

where the subelliptic estimate of Proposition 3.3.1 implies

$$C_0^{16/9}\|u^{(p)}_{j,\pm}\|_{\tilde{\mathcal{W}}^{0,s+2/9}} = A^{16/9}\|u^{(p)}_{j,\pm}\|_{\tilde{\mathcal{W}}^{0,s+2/9}} \leq C(\lambda^{(p)}_{j,\pm}+A^2)\|u^{(p)}_{\pm,j}\|_{\tilde{\mathcal{W}}^{0,s}} \leq C(1+C_0^2)\|u^{(p)}_{\pm,j}\|_{\tilde{\mathcal{W}}^{0,s}}$$

by choosing $A=C_0$ like in Proposition 3.5.1 . A bootstrap argument leads to

$$\|u_{j,\pm}^{(p)}\|_{\tilde{\mathscr{W}}^{0,2/9k}} \leq [C(1+C_0^{2/9})]^k \|u_{\pm,j}^{(p)}\|_{L^2} \leq 2[C(1+C_0^{2/5})]^k$$

for every $k\in\mathbb{N}$ and the general result for $s>0$ follows by interpolation.
The integral

$$\frac{1}{2i\pi}\int_\Gamma \frac{e^{-tz}}{z-B_{\pm,b,V^h}^{(p)}}\,dz$$

converges if and only if the two integrals $\int_{\Gamma_\pm} e^{-t\mathrm{Re}z}\|(z-B_{\pm,b,V^h}^{(p)})^{-1}\|\,|dz|$ converge. With the parametrization of $\Gamma_\pm$ given by $z=u+\varrho_h\pm i(\frac{1}{b^2}(1+u^2))$ for $u\in[0,+\infty[$, we obtain

$$\begin{aligned}
\int_{\Gamma_\pm} e^{-t\mathrm{Re}z}\|(z-B_{\pm,b,V^h}^{(p)})^{-1}\|\,|dz| &\leq \int_0^{+\infty}\left\|\frac{e^{-tz}}{z-B_{\pm,b,V^h}^{(p)}}\right\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}\left|1\pm 2i\frac{u}{b^2}\right|du\\
&\leq \int_0^{+\infty}\frac{e^{-t(u+\varrho_h)}}{4b\sqrt{b^{-2}(1+u^2)}}\left|1\pm i\frac{2u}{b^2}\right|du\\
&\leq \frac{e^{-t\varrho_h}}{2b^2}\int_0^{+\infty}e^{-tu}\,du\\
&\leq \frac{e^{-t\varrho_h}}{2b^2}\frac{1}{t}.
\end{aligned}$$

The integral over the line segment MR is estimated by

$$\begin{aligned}
&e^{t\varrho_h}\int_{-\frac{1}{b^2}}^{\frac{1}{b^2}}\left\|\frac{e^{-tz}}{z-B_{\pm,b,V^h}}\right\|_{\mathscr{L}(\tilde{\mathscr{W}}^{0,s};\tilde{\mathscr{W}}^{0,s})}du\\
&\quad\leq C_s\int_{1\leq|u|\leq\frac{1}{b^2}}(|u|^{-1}+bA^{7/2}+A^{-2})du+C_s\int_{|u|\leq 1}(A^{-2}+\frac{1+bA^{7/2}}{\varrho_h^2/4+u^2})du\\
&\quad\leq 2C_s(-2\ln b+(2bA^{7/2}+A^{-2})(\frac{1}{b^2}-1))+C_s2(A^{-2}+\frac{1}{\varrho_h}(1+bA^{7/2})\frac{\pi}{2})\leq\frac{1}{b^2}
\end{aligned}$$

by using $C_sbA^4\leq\frac{\varrho_h}{2}\leq\frac{1}{2}$, $C_sbA^{7/2}=\frac{C_sbA^4}{A^{1/2}}\leq\frac{1}{10}$ for $A\geq 1$ large enough, $b^2|\ln(b)|\leq\frac{1}{10C_s}$ and $\frac{C_s}{\varrho_h}\leq\frac{1}{bA^4}\leq\frac{1}{b}\leq\frac{1}{10b^2}$ for $b>0$ small enough.
By conclude by adding the two upper bound for $\int_{\Gamma_+}+\int_{\Gamma_-}$ and $\int_{[MR]}$. □

3.5.2 Hodge type structure and accurate spectral estimates

We prove now the accurate comparison of $\mathrm{Spec}(B_{\pm,b,V^h})\cap\{z\in\mathbb{C},|z|\leq\varrho_h\}\subset[0,\varrho_h[$ and $\mathrm{Spec}(\frac{1}{2}\Delta_{V^h,1})\cap[0,\varrho_h]=\mathrm{Spec}(\frac{1}{2}\Delta_{V,h})\cap[0,e^{-\frac{c}{h}}]$.

Proposition 3.5.6.
Let $(\lambda_j^{(p)})_{\substack{1\leq j\leq\mathscr{N}_\pm^{(p)}\\0\leq p\leq 2d}}$ be the eigenvalues of $B_{\pm,b,V^h}$ contained in $[0,\varrho_h[$ and let $(\tilde{\lambda}_j^{(p)}(V))_{\substack{1\leq j\leq\mathscr{N}_\pm^{(p)}(V)\\0\leq p\leq d}}$ be the eigenvalues of $\frac{1}{2}\Delta_{V^h,1}$ contained in $[0,e^{-\frac{c}{h}}]$. There exists $C_0\geq 1$ such that for all $A\geq C_0$ and with the additional condition $1\geq\varrho_h\geq C_0A^4b\geq C_0^5b$, the eigenvalues are compared according to

$$\forall p\in\{0,\ldots,2d\},\forall j\in\{1,\ldots,\mathscr{N}_\pm^{(p)}\},\quad (1+C_0A^{-1/2})^{-1}\tilde{\lambda}_j^{(p-\frac{d}{2}\pm\frac{d}{2})}(V)\leq\lambda_j^{(p)}\leq(1+C_0A^{-1/2})\tilde{\lambda}_j^{(p-\frac{d}{2}\pm\frac{d}{2})}(V),$$

where we recall that $\mathscr{N}_\pm^{(p)}=\mathscr{N}_\pm^{(p-\frac{d}{2}\pm\frac{d}{2})}$ vanishes when $p-\frac{d}{2}\pm\frac{d}{2}\not\in\{0,\ldots,d\}$.

This result relies on the Hodge type structure of $2B_{\pm,b,V^h}\big|_{E_{\pm,b,h}}$ and $\Delta_{V^h,1}\big|_{\mathrm{Ran}1_{[0,\varrho_h]}(\Delta_{V^h,1})}$ and the identification of eigenvalues of those operators with the squares of singular values by following the strategy of [HKN][Lep][LNV2] and other related works. We start with three lemmas.

Lemma 3.5.7. *Let E be a finite dimensional Hilbert space with the hermitian positive definite form $(\ ,\ )$. Let $\mathbf{d}$ be an operator such that $\mathbf{d}\circ\mathbf{d}=0$ and set $\Delta=(\mathbf{d}+\mathbf{d}^*)^2=\mathbf{d}\mathbf{d}^*+\mathbf{d}^*\mathbf{d}$, where $\mathbf{d}^*$ is the adjoint of $\mathbf{d}$ for the scalar product $(\ ,\ )$. The E admits the orthogonal decomposition*

$$E=\ker(\Delta)\overset{\perp}{\oplus}\mathrm{Ran}\,\mathbf{d}\overset{\perp}{\oplus}\mathrm{Ran}\,\mathbf{d}^*=\ker(\mathbf{d})\overset{\perp}{\oplus}\mathrm{Ran}\,\mathbf{d}^*=\ker(\mathbf{d}^*)\overset{\perp}{\oplus}\mathrm{Ran}\,\mathbf{d}.$$

The eigenvalues of Δ are the squares of the singular values of $\mathbf{d}$ and equivalently $\mathbf{d}^*$. More precisely there exists an orthonormal basis $(u_j)_{1\leq j\leq \dim E}$ such that $(u_j)_{1\leq j\leq N}$ (resp. $(u_j)_{N+1\leq j\leq 2N}$) is an orthonormal basis of $\mathrm{Ran}\,\mathbf{d}^*$ (resp. $\mathrm{Ran}\,\mathbf{d}$) and $(u_j)_{2N+1\leq j\leq \dim E}$ is an orthonormal basis of $\ker\Delta$ and

$$\forall j\in\{1,\ldots,N\}\,,\quad \mathbf{d}u_j=\mu_j u_{j+N}\quad \mu_j>0\,.$$

We will set $\mu_j=0$ for $j\notin\{1,\ldots,N\}$ and write, with an abuse of notation $\mathbf{d}u_j=\mu_j u_{j+N}$ for all $j\in\{1,\ldots,\dim E\}$.

Proof. It suffices to notice $\Delta=0\oplus^\perp\mathbf{d}\mathbf{d}^*\oplus^\perp\mathbf{d}^*\mathbf{d}$ in the decomposition $E=\ker(\Delta)\oplus^\perp\mathrm{Ran}\,\mathbf{d}\oplus^\perp\mathrm{Ran}\,\mathbf{d}^*$. Then one takes for $(u_j)_{1\leq j\leq N}$ an orthonormal eigenbasis of $\mathbf{d}^*\mathbf{d}\big|_{\mathrm{Ran}\,\mathbf{d}}$ with $\mathbf{d}^*\mathbf{d}u_j=\mu_j^2u_j$ and to set $u_{j+N}=\frac{1}{\mu_j}\mathbf{d}u_j$. □

Definition 3.5.8. In a finite dimensional Hilbert space $(E,(\ ,\ ))$, a basis $\mathscr{B}=(v_j)_{1\leq j\leq \dim E}$ is ε-orthonormal for $\varepsilon\in]0,1[$ if $\|((v_j,v_k))_{1\leq j,k\leq \dim E}-\mathrm{Id}_{\mathbb{C}^{\dim E}}\|\leq\varepsilon$.
The function $\tau:\sqcup_{n=1}^{\infty}]0,1[^n\to]0,+\infty[$ is defined by $\tau(\varepsilon_1,\ldots,\varepsilon_n)=\prod_{k=1}^n\frac{1+\varepsilon_k}{1-\varepsilon_k}$.

The following lemma is extracted from Proposition 5.4 in [LNV2].

Lemma 3.5.9. *Let $\mathscr{B}=(u_j)_{1\leq j\leq \dim E}$ (resp. $\mathscr{B}'=(v_j)_{1\leq j\leq \dim E}$) be an ε_1- (resp. ε_2-) orthonormal basis of the Hilbert space $(E,(\ ,\ ))$ for $\varepsilon_1,\varepsilon_2\in]0,1[$. For $B\in\mathscr{L}(E)$, let $(\mu_j(B))_{1\leq j\leq \dim E}$ (resp. $(\mu_j(\tilde{B}))_{1\leq j\leq \dim E}$) denote the singular values of B (resp. of the matrix $\tilde{B}=((v_k,Bu_j))_{1\leq j,k\leq \dim E}$), in the usual decreasing order. Then*

$$\forall j\in\{1,\ldots,\dim E\}\,,\quad \tau(\varepsilon_1,\varepsilon_2)^{-1/2}\tilde{\mu}_j\leq\mu_j\leq\tau(\varepsilon_1,\varepsilon_2)^{1/2}\tilde{\mu}_j\,.$$

Proof of Proposition 3.5.6. We start the proof for B_{+,b,V^h} and the case of B_{-,b,V^h} will be recovered in the end by a Poincaré duality argument.
We do not distinguish the form degree here and recall that the number of eigenvalues of $\frac{1}{2}\Delta_{V^h,1}$ in $[0,e^{-\frac{c}{h}}]$ and of B_{+,b,V^h} in $[0,\varrho_h]$ are equal to

$$\mathscr{N}_+=\sum_{p=0}^{2d}\mathscr{N}_+^{(p)}=\sum_{p=0}^{d}\mathscr{N}_+(V)=\mathscr{N}_+(V)\,.$$

Let us set $\tilde{\pi}_h=1_{[0,e^{-\frac{c}{h}}]}(\frac{1}{2}\Delta_{V^h,1})=1_{[0,2e^{-\frac{c}{h}}]}(\Delta_{V^h,1})$ and $\pi_h=U_{+,\theta}\tilde{\pi}_hU_{+,\theta}^*$. Because $\Delta_{V^h,1}=(d_{V^h,1}+d_{V^h,1}^*)^2$, Lemma 3.5.7 tells us that there is an orthonormal basis $(\tilde{u}_j)_{1\leq j\leq\mathscr{N}_+}$ such that $\tilde{u}_j$ belongs to Ran $(d_{V^h,1}^*\big|_{\mathrm{Ran}\,\tilde{\pi}_h})$ (resp. Ran $(d_{V^h,1}\big|_{\mathrm{Ran}\,\tilde{\pi}_h})$) for $1\leq j\leq N$ (resp. $N+1\leq j\leq 2N$) and

$$d_{V^h,1}\tilde{u}_j=\begin{cases}\tilde{\mu}_j\tilde{u}_{j+N} & \text{if } 1\leq j\leq N\\ 0 & \text{otherwise}\,,\end{cases}$$

where $\tilde{\mu}_j$, $1 \leq j \leq N$, are the non zero singular values of $d_{V^h,1}\big|_{\mathrm{Ran}\,\tilde{\pi}_h}$. With the abuse of notation of Lemma 3.5.9, it is summarized by $d_{V^h,1}\tilde{u}_j = \tilde{\mu}_j \tilde{u}_{j+N}$ for all $j \in \{1,\ldots,\mathcal{N}_+\}$ with $\tilde{\mu}_j = 0$ for $j > N$.
Let $\pi_{E_{+,b,V^h}}$ be the spectral projector associated with B_{+,b,V^h} given by

$$\pi_{E_{+,b,V^h}} = \frac{1}{2i\pi}\int_{NO'P'Q}(z - B_{+,b,V^h})^{-1}\,dz$$

like in Proposition 3.5.1.
We consider $\mathscr{B}' = (v_j)_{1\leq j\leq \mathcal{N}_+}$ with

$$v_j = \pi_{E_{+,b,V^h}}[U_{+,\theta}\tilde{u}_j],$$
$$\text{and}\qquad U_{+,\theta}\tilde{u}_j = \frac{e^{-\mathscr{H}}}{\pi^{d/4}}[\tilde{u}_j](q)\quad,\quad \mathscr{H} = \frac{|p|_q^2}{2}.$$

Let us compute the scalar products $\langle v_j, v_{j'}\rangle_r$:

$$\begin{aligned}\langle v_j, v_{j'}\rangle_r &= \langle \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde{u}_j), \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde{u}_{j'})\rangle_r \overset{\pi^{*,r}_{E_{+,b,V^h}}=\pi_{E_{+,b,V^h}}}{=} \langle U_{+,\theta}\tilde{u}_j, \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde{u}_{j'})\rangle_r\\ &= \langle U_{+,\theta}\tilde{u}_j, r^*\pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde{u}_{j'})\rangle \overset{r^*U_{+,\theta}=U_{+,\theta}}{=} \langle U_{+,\theta}\tilde{u}_j, \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde{u}_{j'})\rangle\\ &= \delta_{j,j'} + \langle U_{+,\theta}\tilde{u}_j, [\pi_{E_{+,b,V^h}} - \pi_h](U_{+,\theta}\tilde{u}_{j'})\rangle.\end{aligned}$$

But from (3.5.1.10) and (3.5.1.11) we deduce

$$\|\pi_{E_{+,b,V^h}} - \pi_h\| \leq 10C_0A^{-2} + (4+2\pi)A^{-1/2} \leq 30A^{-1/2}.$$

We deduce that $\mathscr{B}'$ is an ε-orthonormal basis of E_{+,b,V^h} with $\varepsilon = 30\mathcal{N}_+A^{-1/2} \in]0,1[$ for $A \geq C_0$ large enough.
Remember $\mu_0 = \hat{f}_i \wedge \mathbf{i}_{f^i}$ introduced in Definition 3.2.4. We now consider the basis $\mathscr{B} = (u_j)_{1\leq j\leq\mathcal{N}_+}$ with

$$u_j = \pi_{E_{+,b,V^h}}[e^{-\mu_0}U_{+,\theta}\tilde{u}_j] = v_j + \pi_{E_{+,b,V^h}}[(e^{-\mu_0}-1)U_{+,\theta}\tilde{u}_j].$$

From $(e^{-\mu_0}-1) = \sum_{k=1}^d \frac{(-1)^k}{k!}(\hat{f}_i\wedge\mathbf{i}_{f^i})^k$ and $\mathrm{Ran}\,U_{+,\theta} = \mathrm{Ran}\,\pi_{0,+}$ we deduce

$$(e^{-\mu_0}-1)U_{+,\theta} = (1-\pi_{0,+})(e^{-\mu_0}-1)U_{+,\theta}.$$

We write now

$$\begin{aligned}\|\pi_{E_{+,b,V^h}}(1-\pi_{0,+})\| &= \|(1-\pi_{0,+})^*\pi^*_{E_{+,b,V^h}}\| = \|(1-\pi_{0,+})r^*\pi_{E_{+,b,V^h}}r^*\|\\ &= \|r^*(1-\pi_{0,+})\pi_{E_{+,b,V^h}}r^*\| = \|(1-\pi_{0,+})\pi_{E_{+,b,V^h}}\|\end{aligned}$$

but we proved in Proposition 3.5.1 after Lemma 3.5.5 the upper bound

$$\|(1-\pi_{0,+})\pi_{E_{+,b,V^h}}\| = \vec{d}(E_{+,b,V^h}, \mathrm{Ran}\,\pi_{0,+}) \leq \sqrt{2C_0'\mathcal{N}_+b^2}.$$

Remember also the equivalence of norms $\frac{1}{2}\|u\| \leq \|u\|_r \leq 2\|u\|$ for $u \in E_{\pm,b,V^h}$. Therefore there exists a constant $C_{0,\mathcal{N}_+} \geq 1$ such that $\mathscr{B}$ (and $\mathscr{B}'$) are ε-orthornormal bases of $(E_{+,b,V^h}, \langle\ ,\ \rangle_r)$ with $\varepsilon = C_{0,\mathcal{N}_+}A^{-1/2} \in]0,1[$ for $A \geq 2C^2_{0,\mathcal{N}_+}$.
We now apply Lemma 3.5.7 to

$$2B_{+,b,V^h}\big|_{E_{+,b,V^h}} = (\delta_{+,b,V^h}\big|_{E_{+,b,V^h}})(\delta^{*,r}_{+,b,V^h}\big|_{E_{+,b,V^h}}) + (\delta^{*,r}_{+,b,V^h}\big|_{E_{+,b,V^h}})(\delta_{+,b,V^h}\big|_{E_{+,b,V^h}}),$$

where we recall (3.2.5.15)

$$\delta_{+,b,V^h} = e^{-\mu_0} e^{-\mathscr{H}-V^h}(K_b d^X K_b^{-1}) e^{+\mathscr{H}+V^h} e^{\mu_0}.$$

Because $\delta_{+,b,V^h} B_{+,b,V^h} = B_{+,b,V^h}\delta_{+,b,V^h}$ on $\mathscr{S}(X^h;\mathscr{E}_+^h)$ and $E_{\pm,b,V^h}\subset\mathscr{S}(X^h;\mathscr{E}_+^h)$ we know actually

$$\begin{aligned} &\delta_{+,b,V^h}\pi_{E_{+,b,V^h}} = \pi_{E_{+,b,V^h}}\delta_{+,b,V^h} = \pi_{E_{+,b,V^h}}\delta_{+,b,V^h}\pi_{E_{+,b,V^h}} \\ \text{and}\quad &\delta^{*,r}_{+,b,V^h}\pi_{E_{+,b,V^h}} = \pi_{E_{+,b,V^h}}\delta^{*,r}_{+,b,V^h} = \pi_{E_{+,b,V^h}}\delta^{*,r}_{+,b,V^h}\pi_{E_{+,b,V^h}}. \end{aligned}$$

We now compute $\langle v_{j'}, \delta_{+,b,V^h} u_j\rangle_r$:

$$\begin{aligned} \langle v_{j'}, \delta_{+,b,V^h} u_j\rangle_r &= \langle \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde u_{j'}), \delta_{+,b,V^h}\pi_{E_{+,b,V^h}}(e^{-\mu_0}U_{+,\theta}\tilde u_j)\rangle_r \\ &= \langle \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde u_{j'}), \delta_{+,b,V^h}(e^{-\mu_0}U_{+,\theta}\tilde u_j)\rangle_r \\ &= \langle \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde u_{j'}), e^{-\mu_0}e^{-\mathscr{H}-V^h}(K_b d^X K_b^{-1})e^{+\mathscr{H}+V^h}U_{+,\theta}\tilde u_j\rangle_r \\ &= \langle \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde u_{j'}), e^{-\mu_0}U_{+,\theta}[d_{V^h,1}\tilde u_j]\rangle_r \\ &= \tilde\mu_j\langle \pi_{E_{+,b,V^h}}(U_{+,\theta}\tilde u_{j'}), e^{-\mu_0}U_{+,\theta}[\tilde u_{j+N}]\rangle_r \\ &= \tilde\mu_j\langle U_{+,\theta}\tilde u_{j'}, r^* e^{-\mu_0}U_{+,\theta}[\tilde u_{j+N}]\rangle + \tilde\mu_j R_{j,j',h} \\ &= \tilde\mu_j\langle U_{+,\theta}\tilde u_{j'}, r^* e^{-\mu_0}U_{+,\theta}[\tilde u_{j+N}]\rangle + \tilde\mu_j R_{j,j',h} \\ &= \tilde\mu_j\langle e^{-\lambda_0} r^* U_{+,\theta}\tilde u_{j'}, U_{+,\theta}[\tilde u_{j+N}]\rangle + \tilde\mu_j R_{j,j',h} \\ &= \tilde\mu_j\delta_{j',j+N} + \tilde\mu_j R_{j,j',h} \end{aligned}$$

where we used $e^{-\lambda_0}r^*U_{+,\theta} = e^{-\lambda_0}r^*\pi_{0,+}U_{+,\theta} = U_{+,\theta}$ in the last identity, where $\tilde\mu_j = 0$ for $j>N$ and where

$$|R_{j,j',h}| = \left|\langle[\pi_{E_{+,b,V^h}} - \pi_h](U_{+,\theta}\tilde u_{j'}), e^{-\mu_0}U_{+,\theta}\tilde u_{j+N}\rangle_r\right| \le 2\|[\pi_{E_{+,b,V^h}} - \pi_h](U_{+,\theta}\tilde u_{j'})\| \le 60A^{-1/2}.$$

By Gaussian elimination like in [Lep] or equivalently by changing the basis $\mathscr{B}'$ into an other ε-orthonormal basis of $(E_{+,b,V^h}, \langle\ ,\ \rangle_r)$ with $\varepsilon = C_{0,\mathscr{N}_+}A^{-1/2}$, we deduce with a possibly enlarged constant $C_{0,\mathscr{N}_+}$ that the singular values, μ_j of $\delta_{+,b,V^h}\big|_{E_{+,b,V^h}}$ satisfy:

$$\forall j\in\{1,\ldots,\mathscr{N}_+\},\quad (1+C_{0,\mathscr{N}_+}A^{-1/2})^{-1}\tilde\mu_j \le \mu_j \le (1+C_{0,\mathscr{N}_+}A^{-1/2})\tilde\mu_j$$

We deduce the comparison between eigenvalues $(\lambda_j)_{1\le j\le\mathscr{N}_+}$ of $B_{\pm,b,h}$ and $(\tilde\lambda_j)_{1\le j\le\mathscr{N}_+}$ or $\frac{1}{2}\Delta_{V^h,1}$ is deduced from $\lambda_j = \frac{1}{2}\mu_j^2$ and $\tilde\lambda_j = \frac{1}{2}\tilde\mu_j^2$, by doubling the constant $C_0\ge 1$.
Finally, still in the case of B_{+,b,V^h}, the degree can be followed by splitting the orthonormal basis $(\tilde u_j)_{1\le j\le\mathscr{N}_+}$ according to the degree $p\in\{0,\ldots,d\}$ with $\{\tilde u_j, 1\le j\le\mathscr{N}_+\} = \cup_{p=0}^d\{\tilde u_{j_k}^{(p)}, , 1\le k\le \mathscr{N}_+^{(p)}(V)\}$. In particular we notice that $\tilde u_j = \tilde u_{j_k}^{(p)}$ implies that the total degree of u_j and v_j equals p (in this +-case).

Let us give some details for the Poincaré duality argument which gives the result in the $-$ case. Attention must be paid on the choice of the Thom form, the unitary map $U_{-,\theta}$, when Q is not orientable and $F = Q\times\mathbb{C} \neq (Q\times\mathbb{C})\otimes\mathbf{or}_Q$. Let us split the basis $\tilde u_{j,V^h}$ constructed in the + case according to the degree $p\in\{0,\ldots,d\}$, for the potential and let $\star_Q$ (resp. $\star_X$) denote the Hodge star operator on Q (resp X which is orientable). We construct the basis $\mathscr{B}$ (resp. $\mathscr{B}'$) with the vectors

$$\begin{aligned} &u_j^{(d+p)} = \pi_{E^{(d+p)}_{-,b,V^h}} e^{-\lambda_0}U_{-,\theta}[\star_Q\ \tilde u^{(d-p)}_{j,-V^h}] = \pi_{E^{(d+p)}_{-,b,V^h}}\star_X\ e^{-\mu_0}U_{+,\theta}[\tilde u^{(d-p)}_{j,-V^h}], \\ \text{resp.}\qquad &v_{j'}^{(d+p)} = \pi_{E^{(d+p)}_{-,b,V^h}} U_{-,\theta}[\star_Q\ \tilde u^{(d-p)}_{j,-V^h}] = \pi_{E^{(d+p)}_{-,b,V^h}}\star_X\ U_{+,\theta}[\tilde u^{(d-p)}_{j,-V^h}]. \end{aligned}$$

The $\frac{C_0}{A^{1/2}}$-orthonormality of $\mathscr{B}$ and $\mathscr{B}'$ is easily deduced from the + case. Then we must compute

$$\begin{aligned}
\delta^{*,r}_{-,b,V^h} u_j^{(d+p)} &= e^{-\lambda_0} e^{-\mathscr{H}+V^h} K_b d^{*,X} K_b^{-1} e^{\mathscr{H}-V^h} e^{\lambda_0} u_j^{(d+p)} \\
&= \star_X\, e^{-\mu_0} e^{-\mathscr{H}+V^h} K_b \star_X^{-1} d^{*,X} \star_X K_b^{-1} e^{-\mathscr{H}+V^h} U_{+,\theta}[\tilde{u}^{(d-p)}_{j,-V^h}] \\
&= (-1)^{(d-p+1)} \star_X\, e^{-\mu_0} e^{-\mathscr{H}+V^h} K_b d^X K_b^{-1} e^{\mathscr{H}-V^h} U_{+,\theta}[\tilde{u}^{(d-p)}_{j,-V^h}] \\
&= (-1)^{(d-p+1)} \star_X e^{-\mu_0} U_{+,\theta}[d_{-V^h,1}\, \tilde{u}^{(d-p)}_{j,-V^h}] \\
&= (-1)^{(d-p+1)} \tilde{\mu}^{(d-p)}_{j,-V^h} \star_X e^{-\mu_0} U_{+,\theta}[\tilde{u}^{(d-p+1)}_{k(j),-V^h}].
\end{aligned}$$

By taking the $\langle\,,\,\rangle_r$ scalar product with $v_{j'}^{(d+p')}$ like in the + case, we obtain

$$\langle v_{j'}^{(d+p')}, \delta^{*,r}_{-,b,V^h} u_j^{(d+p)} \rangle_r = (-1)^{d-p+1} \delta_{p',p-1} \delta_{j',k(j)} \tilde{\mu}^{(d-p)}_{j,-V^h} + \tilde{\mu}^{(d-p)}_{j,-V^h} \mathscr{O}(A^{-1/2}).$$

We deduce that the singular values of $\delta^{*,r}_{-,b,V^h}\big|_{E^{(d+p)}_{-,b,V^h}}$ are comparable, with an $(1+\mathscr{O}(A^{-1/2}))$ factor, with the singular values of $d_{-V^h,1} : \mathrm{Ran}\, 1_{[0,2\varrho_h]}(\Delta^{(d-p)}_{F^+,-V^h,1}) \to \mathrm{Ran}\, 1_{[0,2\varrho_h]}(\Delta^{(d-p+1)}_{F^+,-V^h,1})$, where we recall that $\Delta_{F_+,-V^h,1}$ acts on $\mathscr{C}^\infty(Q, \Lambda T^*Q \otimes \mathbb{C})$. The eigenvalues of $B^{(d+p)}_{-,b,V^h}$ are thus comparable with an $(1+\mathscr{O}(A^{-1/2}))$ factor, with the eigenvalues of $\frac{1}{2}\Delta^{(d-p)}_{F_+,-V^h,1}$ which by Poincaré duality for the Witten Laplacian are the eigenvalues of $\frac{1}{2}\Delta^{(p)}_{F_-,+V^h,1}$, where we recall that $\Delta_{F_-,V^h,1}$ acts on $\mathscr{C}^\infty(Q, \Lambda T^*Q \otimes \mathbb{C} \otimes \mathbf{or}_Q)$. □

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